Canonical Formulas for the Lax Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

We develop the method of canonical formulas for the lax logic. This is an intuitionistic modal logic that formalises nuclei of pointless topology and has applications in formal hardware verification. We show that all extensions of the lax logic can be axiomatised by *lax canonical formulas*. We give a dual description of lax canonical formulas by extending generalised Esakia duality for nuclear implicative semilattice homomorphisms. We go on to generalise lax canonical formulas to introduce steady logics, a class of lax logics that is structurally very similar to subframe logics for intermediate logics, for example, they all have the finite model property and are generated by classes closed under subframes. We look at translations of intermediate logics into lax logic and show a number of preservation results. In particular, we prove a lax analogue for the Dummett-Lemmon conjecture that the least modal companion of each Kripke-complete intermediate logic is Kripke-complete.

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Chapter 1

Introduction

Intermediate logics are consistent logics that extend the intuitionistic propositional calculus (IPC). Uniformly axiomatising intermediate logics has been a significant problem since their structure is quite complicated. Zakharyaschev introduced canonical formulas and showed that every intermediate logic can be axiomatised by them [50]. Later he generalised this approach to axiomatise all transitive modal logics [51]. The essence of the method of canonical formulas is a two-step procedure:

1. Characterise every formula with a finite number of refutation patterns.
2. Encode the refutation patterns into formulas: *canonical formulas*.

Consequently, for each formula this method generates semantically equivalent canonical formulas, and thus every logic is axiomatised by them. Moreover, it does so in a uniform manner and this provides a framework which can be used to solve a wide range of problems. In many ways canonical formulas have been used to characterise how intermediate logics relate to their modal companions, e.g., Zakharyaschev used them to give a positive answer to the Dummett-Lemmon conjecture that the least modal companion of every Kripke-complete intermediate logic is Kripke-complete [51], and for providing a new proof for the Blok-Esakia Theorem [17, Theorem 9.66]. However, Zakharyaschev's approach to canonical formulas heavily relies on the dual structure of finitely generated Heyting algebras and K4-algebras [17, Chapter 9], and it is unclear how to generalise it to other logics, e.g., intuitionistic modal logics.

In [5], Bezhanishvili and Bezhanishvili gave an algebraic counterpart of Zakharyaschev's frame-theoretic method of canonical formulas. Algebraically, the method relies on locally finite reducts. For the first step, the refutation patterns are given by a collection of algebras obtained by local finiteness of the reduct. That is, we need a locally finite reduct such that we can expand the finite algebras of the reduct into algebras of the full type. In the intuitionistic case this locally finite reduct is given by the join-free reduct, which is locally finite by Diego's celebrated theorem [21]. The second step is similar to the construction of Jankov formulas [32], but instead of encoding the full structure only the structure of locally finite reduct is encoded completely. The extent to which the remaining structure is encoded is based on some parameters. The frame-theoretic and the algebraic approach to canonical formulas are linked via generalised Esakia duality. This duality gives a dual description of (bounded) implicative semilattice homomorphisms between Heyting algebras. For more details on the algebraic method of canonical formulas see [6, 7, 8].

The algebraic method suggests that we can export the method of canonical formulas to other settings using locally finite reducts. Indeed, [7] developed an alternative class of canonical formulas for intermediate logics using the implication-free reduct, and [14] introduced canonical formulas for substructural logics. Canonical formulas for intuitionistic modal logic in general are still beyond reach since we do not know about a suited locally finite reduct. However, the recent work [9] by Bezhanishvili et al. proved that the variety of nuclear implicative semillatices is locally finite. This is exactly the variety of the disjunction-free reducts of nuclear (Heyting) algebras, which provide sound and complete semantics for lax logics [25, 28].

Lax logic is an intuitionistic modal logic that has properties of both classic necessitation and possibility. It is a very naturally occurring logic; lax logic is connected to many fields in mathematics and computer science, e.g., lax modalities are studied in a proof-theoretic setting [1, 18], they are known as *nuclei* in pointless topology [34, Section II 2], and lax logic coincides with the logic of Moggi's computational lambda calculus [3, 43]. These and other applications are discussed in more detail in Chapter 3. In short, there is significant interest in lax logic.

Furthermore, [11] showed that nuclear algebras can be represented by subframes on their dual Esakia spaces. This indicates that there is a relation between lax logics and subframe logics. Fine developed the theory of subframe logics in the setting of transitive modal logics [26], which was generalised to cofinal subframe logics and the intermediate setting in [52]. Herein, Zakharyaschev showed that all subframe logics are axiomatised by subframe formulas, which are special instances of canonical formulas. Moreover, in the intermediate setting cofinal subframe logics are exactly the logics axiomatised by disjunction-free formulas. Algebraically, subframe formulas are exactly those canonical formulas which encode only the structure of the locally finite reduct.

In this thesis we will develop the method of canonical formulas for the lax logic. We will use the recent discovery of Diego's Theorem for nuclear implicative semilattices [9] to find finite refutation algebras, and then encode them into lax canonical formulas. We show that every lax logic is axiomatised by lax canonical formulas, and moreover, we will set up a framework that allows us to characterise lax logics that are axiomatised with a restricted syntax with special instances of lax canonical formulas. We will give a dual description of lax canonical formulas by extending generalised Esakia duality to (bounded) nuclear implicative semilattice homomorphisms. Moreover, we introduce *steady logics* – lax analogues of subframe logics, and give a number of examples of steady logics. Contrasting intermediate subframe logics, steady logics are not the lax logics axiomatised by disjunction-free formulas, but by disjunction-free lax canonical formulas that only partially encode the modal structure of algebras. Accordingly, we call these formulas *steady formulas* and show that a canonical version of them can be used to axiomatise all lax logics, making them an alternative to lax canonical formulas. Furthermore, we consider translations from intermediate logics to lax logics, inspired by [10] and [31, Section 6.5]. We then prove some preservation results for these translations, which will underpin the usefulness of (steady) canonical formulas. In particular, we prove an analogue of the Dummett-Lemmon conjecture that the least modal companion of each Kripke-complete intermediate logic is complete, i.e., we will prove that for each Kripke-complete intermediate logic L the least lax logic containing L is Kripke-complete.

Summarising, the main contributions of this thesis are:

- The method of canonical formulas for the lax logic.
- A dual description of (bounded) nuclear implicative semilattice homomorphisms between nuclear algebras.
- The introduction of steady and cofinal steady logics – classes of lax logics that structurally are analogues of subframe and cofinal subframe logics.
- Several examples of steady and cofinal steady logics, including their geometric refutation patterns.
- A number of preservation theorems for the least lax extension of intermediate logics, including specifically, preservation of Kripke-completeness.

The thesis is structured as follows. Chapter 2 is a preliminary chapter built around intermediate logics. We introduce them syntactically, and discuss semantics in the form of Kripke frames and Heyting algebras. Besides, we recall Esakia duality and some relevant notions from universal algebra. In Chapter 3 we discuss lax logics. First, there is a historical overview of lax logic and related topics in the literature. We then look at the formal definition and examine nuclear algebras. Next, we discuss a few Kripke-style semantics that have been used for lax logic, and their descriptive counterparts. Lastly, we discuss some translations of intermediate logics into lax logics. Chapter 4 recalls canonical formulas for intermediate logics and then we extend them for the lax case. We then extend generalised Esakia duality to generalised nuclear Esakia duality. In Chapter 5 we will recall subframe logics, introduce steady logics and steady canonical formulas, and prove some preservation results for the translations discussed in Chapter 3.

Chapter 2

Intermediate Logics

In this chapter we go over notation, definitions, and logical concepts used throughout the thesis. While this is primarily a preliminary chapter it serves a secondary purpose in introducing the theory around intermediate logics, justifying its name. Intermediate logics play an important role in this thesis since lax logics are a modal expansion of them. We also state some relevant results of universal algebra and duality theory. All results are well known and prominent in the literature so we state them without proofs.

We assume a naive understanding of sets, classes, relations, functions, order theory, and topology. In particular, we assume familiarity with the standard set theoretic notation, partial orders, lattices, and some simple topological notions, for example, Haussdorff spaces. Furthermore, the material we cover is introduced in a lax manner. That is, we often simplify concepts to suit the narrative. For more elaborate and precise information we refer to [16] for universal algebra and lattices, [17] for intermediate logics, and [24] for Esakia duality.

In Section 2.1 we introduce superintuitionistic logics and their language. We define them as sets of formulas extending the intuitionistic propositional calculus closed under modus ponens and uniform substitution. Section 2.2 defines algebraic and Kripke semantics for intermediate logics. Besides, the section acts as a preliminary for concepts and definitions from universal algebra. Next, Section 2.3 recalls Esakia duality and the functors it uses. We also briefly go over our conventions for topological notation. Finally, in Section 2.4 we define relevant classes of intermediate logics and discuss how they relate.

2.1 Syntax

The languages in this thesis are built from a countable set Prop of infinitely many propositional variables, which we usually denote with Latin letters $p, q, r, \dots \in \mathsf{Prop}$. We define the *propositional language* $\mathcal{L}_\mathsf{p}$ for intuitionistic logic in the usual way, i.e., $\mathcal{L}_\mathsf{p}$ is defined using the following grammar:

$$\mathcal{L}_{\mathsf{p}} \ni \varphi ::= p \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi,$$

for any $p \in \mathsf{Prop}$. Elements of $\mathcal{L}_{\mathsf{p}}$, denoted with Greek letters $\varphi, \psi, \chi, \ldots$, are called *formulas.* The connectives $\neg, \top$, and $\leftrightarrow$ are the usual abbreviations, i.e.,

$$\begin{aligned} \neg\varphi &:= \varphi \to \bot, \\ \top &:= \neg\bot, \text{ and} \\ \varphi \leftrightarrow \psi &:= (\varphi \to \psi) \wedge (\psi \to \varphi). \end{aligned}$$

For any formula φ we write $\varphi(p_1, \ldots, p_n)$ to denote explicitly that $p_1, \ldots, p_n \in \mathsf{Prop}$ are the only propositional variables occurring in φ. We view *substitutions* as the act of replacing propositional variables in a formula with formulas. That is, given a formula $\varphi(p_1, \ldots, p_n)$ and formulas $\psi_1, \ldots, \psi_n$, we write $\varphi(\psi_1, \ldots, \psi_n)$ for the formula obtained by substituting the formula ψ_i for each occurrence of p_i in φ.

The formulas used in the construction of a formula φ according to the grammar above and the formula φ itself are *subformulas* of φ. Then for any connective f of arity n we call a formula *f-free* if f does not occur in φ, i.e., if there is no $\psi_1, \ldots, \psi_n$ such that $f(\psi_1, \ldots, \psi_n)$ is a subformula of φ. If a formula φ is f-free for all $f \in \mathcal{G} \subseteq \mathcal{F}$ then we say that φ is *$\mathcal{G}$-free.* We also write $(f_1, \ldots, f_n)$-free for $\mathcal{G}$-free if $\mathcal{G} = \{f_1, \ldots, f_n\}$.

To characterise intermediate logics we make use of the familiar derivation rules *modus ponens* and *uniform substitution*, defined respectively as:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\text{MP}) \qquad\qquad \frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)}\ (\text{US})$$

Roughly, any set of formulas that is closed under these rules is some kind of propositional logic. We are interested in the logics that contain the usual intuitionistic axioms.

Definition 2.1.1. A set $\mathsf{L} \subseteq \mathcal{L}_{\mathsf{p}}$ closed under (MP) and (US) is a *superintuitionistic logic* (si-logic) if it contains the formulas:

- $p \to (q \to p)$,
- $(p \to (q \to p)) \to ((p \to q) \to (p \to r))$,
- $p \wedge q \to p$,
- $p \wedge q \to q$,
- $p \to p \vee q$,
- $p \to q \vee p$,
- $(p \to q) \to ((q \to r) \to (p \vee q) \to r)$, and
- $\bot \to p$. $\dashv$

The smallest si-logic is known as the *intuitionistic propositional calculus* (IPC), and the smallest si-logic containing $\neg\neg p \to p$ is the *classical propositional calculus* (CPC). Given a si-logic L and a formula φ, we call φ a *theorem* of L, denoted $\mathsf{L} \vdash \varphi$, if $\varphi \in \mathsf{L}$. Given two si-logics L and M, we say M is an *extension* of L if $\mathsf{L} \subseteq \mathsf{M}$. For any set $\Gamma \subseteq \mathcal{L}_{\mathsf{p}}$

and si-logic L we define $\mathsf{L} \oplus \Gamma$ as the smallest extension M of L such that $\Gamma \subseteq \mathsf{M}$. If $\Gamma = \{\varphi\}$ is a singleton we usually write $\mathsf{L} \oplus \varphi$. We also say a si-logic L is *axiomatisable* by a set $\Gamma \subseteq \mathcal{L}_\mathsf{p}$ if $\mathsf{L} = \mathsf{IPC} \oplus \Gamma$, and L is *finitely axiomatisable* if there is a finite Γ that axiomatises it. We will return to this property in Section 2.4. If a logic is axiomatisable by f-free ($\mathcal{G}$-free) formulas we call the logic *f-free* (*$\mathcal{G}$-free*).

A si-logic L is called *consistent* or *inconsistent* if $\mathsf{L} \nvdash \bot$ or $\mathsf{L} \vdash \bot$, respectively. It follows that every inconsistent si-logic contains all formulas. From now on we call consistent si-logics *intermediate logics*. Their name is justified by the fact that CPC is the largest consistent si-logic. Hence, any intermediate logic extends IPC and is contained in CPC.

Definition 2.1.2. A si-logic L is an *intermediate logic* if $\mathsf{L} \subseteq \mathsf{CPC}$. $\dashv$

We write Λ_{IPC} for the collection of all intermediate logics. It is well known that Λ_{IPC} forms a complete lattice. In the rest of this thesis if it is clear from the context we are dealing with intermediate logics we will just refer to them as *logics*.

We have now seen the syntactical definition of si-logics. In the next section we give the semantic counterpart of these logics.

2.2 Heyting algebras and Kripke frames

We use this section to acquaint ourselves with some standard semantics for intermediate logics: Heyting algebras and Kripke frames. Besides, we also use this occasion to present the algebraic jargon used in this thesis. For more information on universal algebra we refer to [16].

Universal algebra

Universal algebra is a prominent ingredient of this thesis. Its main objective is the study of *algebraic structures* for some fixed set $\mathcal{F}$ of *function symbols*. We also call $\mathcal{F}$ an *algebraic type*. For us it suffices to think of $\mathcal{F}$ as a subset of $\{\wedge, \vee, \to, \top, \bot, \Box\}$. Every symbol $f \in \mathcal{F}$ has an *arity* $\sigma(f)$ which determines what arity the operation should have in the algebraic structure. If $\sigma(f) = 0$ we call f a *constant*, e.g., $\top$ and $\bot$ are constants. *$\mathcal{F}$-algebras* are non-empty sets $A, B, C, \ldots$ with associated operations that correspond to the function symbols in $\mathcal{F}$. For example, if $f \in \mathcal{F}$ and A is an $\mathcal{F}$-algebra then there is some implicit function $f_A : A^{\sigma(n)} \to A$. We usually abuse notation slightly and drop the subscript, i.e., instead of writing $f^A(a_1, \ldots, a_{\sigma(f)})$ we write $f(a_1, \ldots, a_{\sigma(f)})$ for $f \in \mathcal{F}$ and $a_1, \ldots, a_{\sigma(f)} \in A$. Besides, we refer to $\mathcal{F}$-algebras simply as algebras if $\mathcal{F}$ is clear from context. Moreover, we do this for anything that is prefixed by $\mathcal{F}$ in this way. An overview of all algebraic types relevant to this thesis is in Table 2.2.1.

Algebras give rise to semantics using *valuations* and *equations*. Equations are made out of *terms*. We define terms recursively over the set of propositional variables Prop using the function symbols in $\mathcal{F}$. That is, Term is the smallest set such that $\mathsf{Prop} \subseteq \mathsf{Term}$, and if $f \in \mathcal{F}$ and $\varphi_1, \ldots, \varphi_{\sigma(f)} \in \mathsf{Term}$ then $f(\varphi_1, \ldots, \varphi_{\sigma(f)}) \in \mathsf{Term}$. For intermediate logics we simply think of Term as the set of formulas $\mathcal{L}_{\mathsf{p}}$. However, there is small discrepancy involving $\top$. We will assume $\top \in \mathcal{F}$ but recall that in the language we have defined it using $\bot \to \bot$.

Name	Algebraic type
Implicative semillatice	$\mathcal{I} = \{\wedge, \top, \to\}$
Bounded implicative semillatice	$\hat{\mathcal{I}} = \{\wedge, \top, \bot, \to\}$
Heyting algebra	$\mathcal{H} = \{\wedge, \vee, \top, \bot, \to\}$
Nuclear implicative semillatice	$\mathcal{J} = \{\wedge, \top, \to, \Box\}$
Bounded nuclear implicative semillatice	$\hat{\mathcal{J}} = \{\wedge, \top, \bot, \to, \Box\}$
Nuclear (Heyting) algebra	$\mathcal{N} = \{\wedge, \vee, \top, \bot, \to, \Box\}$

Table 2.2.1: Overview of relevant algebraic signatures

As one would expect, an equation simply is a pair of two terms. We usually write $\varphi \approx \psi$ for the equation consisting of the terms φ and ψ. A *valuation* on an algebra A is a function $v : \mathsf{Prop} \to A$ from variables to elements of A. Since terms are recursively defined over the function symbols, which correspond to the operations associated with the algebra, we can naturally extend $\hat{v} : \mathsf{Term} \to A$ to a function from all terms to elements in A, i.e., $\hat{v}(p) := v(p)$ for $p \in \mathsf{Prop}$ and $\hat{v}(f(\varphi_1, \ldots, \varphi_{\sigma f}) := f(\hat{v}(\varphi_1), \ldots, \hat{v}(\varphi_{\sigma f}))$ for all $f \in \mathcal{F}$. We simply write v for both v and $\hat{v}$ from now on.

To see whether an algebra A *validates* an equation $\varphi \approx \psi$ we have to check whether $v(\varphi) = v(\psi)$ for all valuations $v : \mathsf{Prop} \to A$. Observe that if $|A| = 1$ then A validates all equations. For that reason such algebras are called *trivial*. Similarly, we will call classes of algebras *non-trivial* if they contain at least one algebra that is not trivial.

If an equation $\varphi \approx \psi$ is valid on A we write $A \vDash \varphi \approx \psi$. Nonetheless, we will not deal explicitly with such equations in the remainder of this thesis. Namely, from now on we assume $\top \in \mathcal{F}$, and we only really pay attention to equations of the form $\varphi \approx \top$. Then we write $A \vDash \varphi$ for $A \vDash \varphi \approx \top$, which means $A \vDash \varphi$ iff $v(\varphi) = \top$ for all valuations v on A. Furthermore, for a set $\Gamma \subseteq \mathsf{Term}$ we write $A \vDash \Gamma$ if $A \vDash \varphi$ for all $\varphi \in \Gamma$ and for a class C of algebras we write $\mathsf{C} \vDash \Gamma$ and $\mathsf{C} \vDash \varphi$ if for all $A \in \mathsf{C}$ we have respectively $A \vDash \Gamma$ and $A \vDash \varphi$.

Given a set of equations Γ, we can select the class of all algebras validating Γ, i.e., $\{A \mid A \vDash \Gamma\}$. We call this the *class of* Γ*-algebras.* Such a class is called an *equational class.* Conversely, given a class C of algebras we can select a set of equations $\{\varphi \approx \psi \mid \mathsf{C} \vDash \varphi \approx \psi\}$ that is validated in all algebras of the class, also known as the *theory* of C. However, we are more interested in the formulas that are validated in C. We denote this set $\mathsf{Logic}(\mathsf{C}) := \{\varphi \in \mathcal{L} \mid \mathsf{C} \vDash \varphi\}$, and call it the *logic* of C.

Suppose now we have some relation R on algebras such that ARB and $A \vDash \varphi$ implies $B \vDash \varphi$ for all equations φ. We call such relations *truth-preserving.* It follows that every equational class is closed under truth-preserving relations. It is a famous result by Birkhoff that equational classes correspond to so-called *varieties*, which are normally defined using three such relations: *homomorphic images* (**H**), *subalgebras* (**S**), and *direct products*[1] (**P**). Specifically, a class C of algebras is a variety if it is closed under **H**, **S** and **P**.

Theorem 2.2.2 (Birkhoff [16, Theorem 11.9])**.** A class C of algebras is a variety iff it is an equational class.

The exact characterisation of **P** is not important in the remainder of this thesis but the others require some elaboration. This requires the notion of *homomorphisms*, and these and derived notions play a crucial role in this thesis. Formally, an $\mathcal{F}$*-homomorphism* is a function $h : A \to B$ between algebras such that

$$h(f(a_1, \ldots, a_{\sigma(f)})) = f(h(a_1), \ldots, h(a_{\sigma(f)})) \tag{$\blacktriangle$}$$

for all $f \in \mathcal{F}$ and all $(a_1, \ldots, a_{\sigma(f)}) \in A^{\sigma(f)}$. That is, a homomorphism is a function that is *compatible* with the functions symbols. Sometimes we have functions that satisfy Equation $\blacktriangle$ only for some $f \in \mathcal{F}$ and $D \subseteq A^{\sigma(f)}$. In that case we say that they are f*-compatible over* D. This notion can naturally be extended to relations, i.e., a relation $R \subseteq A \times B$ is f-compatible iff $a_i R b_i$ for all $1 \leq i \leq \sigma(f)$ implies $f(a_1, \ldots, a_n) R f(b_1, \ldots, b_n)$. Furthermore, we call a homomorphism an *embedding* if it is an injection, and an *isomorphism* if it is a bijection. Given algebras A and a class C of algebras, we have

- $A \in \mathbf{I}(\mathsf{C})$ iff A is *isomorphic* to some $B \in \mathsf{C}$ iff there exists an isomorphism from A to some $B \in \mathsf{C}$;
- $A \in \mathbf{H}(\mathsf{C})$ iff *homomorphic image* of some $B \in \mathsf{C}$ iff there exists a onto homomorphism from some $B \in \mathsf{C}$ to A;
- $A \in \mathbf{S}(\mathsf{C})$ iff A is a *subalgebra* of some $B \in \mathsf{C}$ iff $A \subseteq B$ for some $B \in \mathsf{C}$ and the identity map from A to B is an embedding.

For every class C of algebras there exists a least variety containing C. In fact, by Tarski's Theorem this variety is given by $\mathbf{HSP}(\mathsf{C})$, see [16, Theorem 9.5].

Moreover, some algebras are more fundamental than others. Birkhoff found that the general building blocks of algebras are *subdirectly irreducible* (s.i.) algebras. Technically, an algebra is s.i. iff it is non-trivial and not the subdirect product of other algebras. A more useful characterisation involves *congruences*. Congruences are simply equivalence relations on algebras that are compatible with the function symbols. An algebra is s.i. iff there is a second largest congruence on it. S.i. algebras form the building blocks of varieties; every variety is generated by its s.i. members. The following theorem due to Birkhoff puts it in a nutshell.

Theorem 2.2.3 ([16, Corollary 9.7])**.** Let C be variety. Then

$$\mathsf{C} = \mathbf{HSP}(\{A \in \mathsf{C} \mid A \text{ is s.i.}\}).$$

[1] Actually, **P** has to be seen as a relation between a family of algebras $\mathcal{A}$ and an algebra B, i.e., $\mathcal{A}\mathbf{P}B$ iff B is the direct product of $\mathcal{A}$. Then **P** is truth-preserving in the sense that $\mathcal{A} \vDash \varphi$ implies $B \vDash \varphi$.

An important property of varieties is *local finiteness*. An algebra is locally finite iff its finitely generated subalgebras are finite. A variety is locally finite iff every algebra in the variety is locally finite. Equivalently, a variety is locally finite iff its finitely generated algebras are finite.

This concludes our short excursion through the world of universal algebra. Next, we use this algebraic framework to define semantics for intermediate logics using Heyting algebras.

Heyting algebras

A *Heyting algebra* A is a $\mathcal{H} = \{\wedge, \vee, \top, \bot \rightarrow\}$-algebra such that

- $(A, \wedge, \vee, \top, \bot)$ is a bounded (distributive) lattice, and
- $a \wedge b \leq c$ iff $a \leq b \rightarrow c$ for all $a, b, c \in A$.

A *Heyting homomorphism* is a $\mathcal{H}$-homomorphism between Heyting algebras. The class of all Heyting algebras is an equational class, and therefore a variety. For an equational definition see [16, Chapter II §1]. It is well known that Heyting algebras give the algebraic semantics for IPC. Namely, the class of IPC-algebras is exactly the class of $\mathcal{H}$-algebras that consists of all Heyting algebras. Given a logic L, we say a class C of Heyting algebras is *sound* with respect to L iff $\mathsf{L} \vdash \varphi$ implies $\mathsf{C} \vDash \varphi$ for all formulas φ, and conversely C is *complete* with respect to L iff $\mathsf{C} \vDash \varphi$ implies $\varphi \in \mathsf{L}$ for all formulas φ. Every intermediate logic L is sound and complete with respect to the class of L-algebras and for every non-trivial class C of Heyting algebras we have that $\mathsf{Logic}(\mathsf{C})$ is an intermediate logic. In case C is trivial we have that $\mathsf{Logic}(\mathsf{C}) = \mathcal{L}_{\mathsf{p}}$.

Theorem 2.2.4 ([17, Theorem 7.8])**.** Every intermediate logic L is sound and complete with respect to the class of L-algebras.

Combining the previous theorem and Theorem 2.2.3 we have that every intermediate logic is determined by its s.i. Heyting algebras. There is an interesting characterisation of s.i. Heyting algebras. Namely, for a Heyting algebra A there is a correspondence between filters on A and congruences on A. Besides, $A \setminus \{\top\}$ has a largest element iff A has a least non-trivial filter. Usually, we call the largest element of $A \setminus \{\top\}$ the *second largest element* of A and denote it $s \in A$. Consequently, we can characterise s.i. Heyting algebras by the existence of such a second largest element, i.e., a Heyting algebra A is s.i. iff A has a second largest element.

An important fact about Heyting algebras follows from Diego's Theorem [21]. Namely, the variety of $\hat{\mathcal{I}}$-reducts[1] of Heyting algebras is locally finite. In other words, we can bound the size of finitely generated $\hat{\mathcal{I}}$-algebras.

[1]Reecall from Table 2.2.1 that $\hat{\mathcal{I}} = \{\wedge, \top, \bot, \rightarrow\}$ is the algebraic type of bounded implicative semillatices.

Theorem 2.2.5. There exists a function $c : \mathbb{N} \to \mathbb{N}$ such that if A is a Heyting algebra and $B \subseteq A$ is a finite subset then the $\hat{\mathcal{I}}$-algebra generated by B is at most of size $c(|B|)$.

The local finiteness of $\hat{\mathcal{I}}$-reducts of Heyting algebras gives us a powerful method to find finite algebras that refute some formula φ for any Heyting algebra that refutes φ. Moreover, the finite algebra will be an $\hat{\mathcal{I}}$-subalgebra of the original algebra.

Lemma 2.2.6. Let B be a (s.i.) Heyting algebra such that $A \nvDash \varphi$. Then there is a finite (s.i.) Heyting algebra such that A is a $\hat{\mathcal{I}}$-subalgebra of B and $A \nvDash \varphi$.

Proof. (Sketch, see [8, Lemma 4.6]). Let Γ be the set of subformulas of φ and v the valuation on A that refutes φ. Then we generate an $\hat{\mathcal{I}}$-algebra A with $v[\Gamma] = \{v(\varphi) \mid \varphi \in \Gamma\}$. A is finite by Theorem 2.2.5. Moreover, we can extend finite $\hat{\mathcal{I}}$-algebras to Heyting algebras. In particular, we define $a \vee_A b := \bigwedge\{c \in A \mid a, b \leq c\}$. It follows that $a \vee_A b$ coincides with $a \vee b$ if $a \vee b \in v[\Gamma]$. Consequently, $A \vDash \varphi$. ■

The previous lemma is sometimes called the *Selective Filtration Lemma*. This is motivated by the fact that frame-theoretically it corresponds with a selective filtration. Recall that *filtrations* are a standard tool in intuitionistic and modal logic to prove completeness with respect to finite models, see [17, Section 5.3]. There exist two filtration methods for intuitionistic logic: *selective filtration* and *standard filtration*. Algebraically, these filtrations correspond to using the $\hat{\mathcal{I}}$-reduct and the reduct of bounded distributive lattices of Heyting algebras respectively. For more details on filtrations algebraically and how they relate to their frame-theoretic counterparts we refer to [12].

Kripke frames

Another prominent semantics for intermediate logics is given by Kripke frames. An (*intutionistic*) *Kripke frame* is just a partial order. Given a partial order $(X, \leq)$ we usually leave $\leq$ implicit and simply write X. Hence, when we say X is a partial order then we assume there is some implicit partial ordering $\leq$ on it. Whenever necessary we distinct partial orders by a subscript. That is, if X and Y are partial orders we occasionally write $\leq_X$ and $\leq_Y$ to refer to their orderings. An *upset* is a subset $Y \subseteq X$ such that $x \in Y$ and $x \leq y$ implies $y \in Y$. The set of all upsets of X is denoted by $\mathrm{Up}(X)$, and for any subset $Y \subseteq X$ we define $\uparrow Y \in \mathrm{Up}(X)$ as the smallest upset containing Y. If $Y = \{x\}$ is a singleton we usually write $\uparrow x$. *Downsets*, $\mathrm{Down}(X)$, $\downarrow Y$ and $\downarrow x$ are defined similarly.

Since Kripke frames are partial orders, saying "X is a Kripke frame" means the same as saying "X is partial order." We will only use the former when we actually use the Kripke semantics on X. Similarly to algebraic semantics, a formula is valid on a Kripke frame iff it holds for all valuations. A valuation on a Kripke frame X is a function $v : \mathsf{Prop} \to \mathrm{Up}(X)$ from propositional variables to upsets. For $x \in X$ we define $X, x \Vdash_v \varphi$

recursively:

$$
\begin{aligned}
X, x \Vdash_v p &\iff x \in v(p)\\
X, x \Vdash_v \bot &\iff \text{never}\\
X, x \Vdash_v \varphi \wedge \psi &\iff X, x \Vdash_v \varphi \text{ and } X, x \Vdash_v \psi\\
X, x \Vdash_v \varphi \vee \psi &\iff X, x \Vdash_v \varphi \text{ or } X, x \Vdash_v \psi\\
X, x \Vdash_v \varphi \to \psi &\iff X, y \Vdash_v \varphi \text{ implies } X, y \Vdash_v \psi \text{ for all } x \leq y
\end{aligned}
$$

When X and v are clear from context we write $x \Vdash \varphi$ for $X, x \Vdash_v \varphi$. We write $X \Vdash \varphi$ iff for all valuations v on X and $x \in X$ and we have $X, x \Vdash_v \varphi$. Given a class C of Kripke frames and a set Γ of formulas, we write $\mathsf{C} \Vdash \Gamma$ iff for all $X \in \mathsf{C}$ and $\varphi \in \Gamma$ we have $X \Vdash \varphi$. As usual, we write $\mathsf{C} \Vdash \varphi$ or $X \Vdash \Gamma$ if respectively $\Gamma = \{\varphi\}$ or $\mathsf{C} = \{X\}$ is a singleton. Analogously to algebras, we define $\mathsf{Logic}(\mathsf{C}) := \{\varphi \in \mathcal{L}_\mathsf{p} \mid \mathcal{C} \Vdash \varphi\}$ as the *logic* of C. Conversely, given a logic L, the class of L-*frames* is defined as the class of frames validating L, i.e., $\{X \mid X \Vdash \mathsf{L}\}$. Soundness and completeness for classes of Kripke frames is defined as it is for Heyting algebras.

$\mathsf{Logic}(\mathsf{C})$ is an intermediate logic for every non-empty class of Kripke frames. However, there exist intermediate logics L that are not complete with respect to the class of L-frames. These logics are called *Kripke-incomplete*. For an example see [17, Section 6.5]. Not very surprisingly, we call a logic *Kripke-complete* iff it is sound and complete with respect to the class of its frames, more on this could be found in Section 2.4.

An alternative way to look at the semantics of Kripke frames is to look at the algebra given by the upsets of a frame. It is well known that this is a Heyting algebra of the same logic. Precisely, given a Kripke frame X, we have that the *complex algebra* $X^{\mathrm{Up}} := (\mathrm{Up}(X), \cap, \cup, X, \varnothing, \to)$ is a (complete) Heyting algebra, where for each $U, V \in \mathrm{Up}(X)$ we define

$$U \to V := X \setminus {\downarrow}(U \setminus V).$$

Then for each formula φ have $X \Vdash \varphi$ iff $X^{\mathrm{Up}} \vDash \varphi$, i.e., $\mathsf{Logic}(X) = \mathsf{Logic}(X^{\mathrm{Up}})$. Consequently, X is an L-frame iff X^{Up} is an L-algebra.

The fact that Kripke semantics is the same as interpreting formulas on the Heyting algebra of upsets indicates that Kripke semantics is persistent. Namely, we call any semantics $\Vdash$ on a poset X *persistent* iff $x \Vdash \varphi$ and $x \leq y$ implies $y \Vdash \varphi$.

Conversely, given a Heyting algebra we can not generally find a Kripke frame with the same logic. This discrepancy will be addressed in the next section.

2.3 Esakia duality

In the previous section we saw that not every logic is complete with respect to a class of Kripke frames; while every Kripke frame X has a corresponding Heyting algebra X^{Up}, the converse does not hold. That is, there are Heyting algebras A such that for all

Kripke frames X we have $\mathsf{Logic}(A) \neq \mathsf{Logic}(X)$. Fortunately, there is a way to restrict valuations such that every algebra has a *dual*. Before we get into more details let us clarify what we mean by *duality*.

When we say duality we mean dual equivalence of some categories. In this case, we want to show the connection between Heyting algebras and Esakia spaces. However, for a complete understanding of this connection we would need to dive into category theory. Fortunately, for us it suffices to have a naive grasp of the matter. For more details we refer to [38].

The duality that we will establish in this section is called *Esakia duality* and is due to Esakia [23, 24].

Esakia spaces

Given a topological space (X, τ), we usually leave the topology implicit. To refer to the open, closed, and clopen sets of some space (X, τ) we write $\mathrm{Op}(X)$, $\mathrm{Cl}(X)$, and $\mathrm{Clop}(X)$, respectively. In addition, if X is also a partial order, we write $\mathrm{ClUp}(X)$ for derivatives like $\mathrm{Cl}(X) \cap \mathrm{Up}(X)$.

An *Esakia space* $(X, \tau, \leq)$ is a compact, Hausdorff, zero-dimensional space (X, τ) and a partial order $(X, \leq)$ such that

- ${\uparrow}x \in \mathrm{Cl}(X)$ for all $x \in X$;
- $U \in \mathrm{Clop}(X)$ implies ${\downarrow}U \in \mathrm{Clop}(X)$.

As with partial orders we usually just write "X is an Esakia space" and leave the order and the topology implicit. We will use the following facts about Esakia spaces freely throughout this thesis. The proofs can be found in [24].

Lemma 2.3.1. Let X be an Esakia space.

(1) If $Y \in \mathrm{Cl}(X)$, then ${\uparrow}Y \in \mathrm{Cl}(X)$ and ${\downarrow}Y \in \mathrm{Cl}(X)$.

(2) If $Y \in \mathrm{ClUp}(X)$, then Y is an Esakia space with the subspace topology and $\leq_X$ restricted to Y.

(3) If $x \not\leq y$, then there exists $U \in \mathrm{ClopUp}(X)$ such that $x \in U$ and $y \notin U$.[1]

(4) If $Y \in \mathrm{Clop}(X)$, then there exist $U_1, \dots, U_n, V_1, \dots, V_n \in \mathrm{ClopUp}(X)$ such that $Y = \bigcup_{i=1}^{n} U_i \setminus V_i$.

For every Esakia space X its clopen upsets form a Heyting algebra. Explicitly, we have that $X^* = (\mathrm{ClopUp}(X), \cap, \cup, X, \varnothing, \to)$ is a Heyting algebra, where $\to$ is defined as in the previous section. Conversely, from every Heyting algebra A we can generate an Esakia space A_* as follows: take the set of prime filters $\mathrm{Pf}(A)$, order it by inclusion, and take

[1] This is known as Priestley separation axiom since it is a defining property of Priestley spaces.

topology determined by the basis $\{\widehat{a} \setminus \widehat{b} \mid a, b \in A\}$, where $\widehat{a} := \{x \in \mathrm{Pf}(A) \mid a \in x\}$ is the set of prime filters containing a for each $a \in A$.

Moreover, composing these constructions forms an identity up to isomorphism. That is, $(A_*)^*$ is isomorphic to A, and X is isomorphic to $(X^*)_*$ in the category of Esakia spaces and their morphisms. Morphisms between Esakia spaces are appropriately called *Esakia morphisms*. A map $f : X \to Y$ between Esakia spaces is an Esakia morphism iff

- $U \in \mathrm{Clop}(Y)$ implies $f^{-1}[U] \in \mathrm{Clop}(X)$, and
- $\uparrow f(x) = f[\uparrow x]$ for all $x \in X$.

Recall, that a function $f : X \to Y$ between partial orders is called a *p-morphism* iff it satisfies $\uparrow f(x) = f[\uparrow x]$ for all $x \in X$. Hence, an Esakia morphism is a continuous p-morphism. Esakia spaces are *isomorphic* iff there exists a bijective Esakia morphism between them.

For a fleshed out duality between the categories of Esakia spaces and Heyting algebras, we also have to define $(\_)_*$ and $(\_)^*$ on (homo)morphisms. The correspondence between Heyting homomorphisms and Esakia morphisms is given by their inverses: for each Heyting homomorphism $h : A \to B$ and Esakia morphism $f : X \to Y$, we have that $h_* := h^{-1}$ is an Esakia morphism from B_* to A_* and $f^* := f^{-1}$ is a Heyting homomorphism from Y^* to X^*. Consequently, h is onto (one-to-one) iff h_* is one-to-one (onto) and similarly for f and f^*.

Theorem 2.3.2 (Esakia duality)**.** The functors $(\_)_*$ and $(\_)^*$ establish a dual equivalence between the category of Heyting algebras and Heyting homomorphisms and the category of Esakia spaces and Esakia morphisms.

A *dual* (*space*) of a Heyting algebra A is an Esakia space X isomorphic to A_*, and likewise a *dual* (*algebra*) of an Esakia space X is a Heyting algebra A such that A is isomorphic to X^*. It follows that X and A are each others dual iff X^* is isomorphic to A (or A_* is isomorphic to X) since the composition of the functors is an identity up to isomorphism.

Esakia duality constitutes a useful link between the setting of Heyting algebras and setting of Esakia spaces. The proofs can be found in [24].

Theorem 2.3.3. Let A and B be Heyting algebras and X and Y their respective duals.

(1) $B \in \mathbf{H}(A)$ iff Y is isomorphic to $Z \in \mathrm{ClUp}(X)$;

(2) $B \in \mathbf{S}(A)$ iff there exists an onto Esakia morphism $f : X \to Y$;

(3) A is s.i. iff X is *strongly rooted*, which means there is some $r \in X$ such that $\uparrow r = X$ and $\{r\} \in \mathrm{Clop}(X)$.

From the duality it is clear how we should interpret formulas on Esakia spaces to assign them the logic of their dual. The bluntest way to put it is that we interpret formulas on their dual algebras. Alternatively, we can interpret formulas pointwise as we did for

Kripke frames. The only difference is that valuations have to go to clopen upsets. That is, a valuation on an Esakia space X is a function $v : \mathsf{Prop} \to \mathrm{ClopUp}(X)$. We then define $X, x \Vdash_v \varphi$ and the derived notions exactly like we did for Kripke frames.

If an Esakia spaces is finite then $\mathrm{ClopUp}(X) = \mathrm{Up}(X)$. It follows that finite Esakia spaces and finite Kripke frames coincide. Whence, we usually do not differentiate between the two; we will simply call them *finite frames*.

As usual, given a logic L, we define the class of L-*spaces* as $\{X \mid X \Vdash \mathsf{L}\}$ and the logic corresponding to a class of Esakia spaces C as $\mathsf{Logic}(\mathsf{C}) := \{\varphi \in \mathcal{L}_\mathsf{p} \mid \mathsf{C} \Vdash \varphi\}$. From the duality it follows that $\mathsf{Logic}(\mathsf{C})$ is an intermediate logic for each non-empty class of Esakia spaces, and that each intermediate logic is complete with respect to its spaces.

Theorem 2.3.4. Every intermediate logic L is sound and complete with respect to the class of L-spaces.

In the next section we look at ways to characterise different classes of intermediate logics.

2.4 Properties of intermediate logics

We can separate intermediate logics into different classes by certain properties. For instance, some logics are Kripke-incomplete while others are Kripke-complete. In this section we introduce a couple of properties of logics and discuss how they relate.

We consider four classes of intermediate logics in this thesis: *tabular* logics, logics with the *finite model property*, *Kripke-complete* logics, and *finitely axiomatisable* logics. This gives us some classification of logics in terms of their difficulty.

1. A logic L is *tabular* iff it is the logic a finite algebra, i.e., $\mathsf{L} = \mathsf{Logic}(A)$ for some finite Heyting algebra A. By Esakia duality, L is tabular iff $\mathsf{L} = \mathsf{Logic}(X)$ for some finite frame X. Moreover, being the logic of a finite set of finite algebras is a necessary and sufficient condition for a logic to be tabular.

2. L has the *finite model property* (fmp) iff it is the logic of a class of finite algebras, i.e., $\mathsf{L} = \mathsf{Logic}(\mathsf{C})$ for a class C of finite Heyting algebras. As before, dually we get that L has the fmp iff $\mathsf{L} = \mathsf{Logic}(\mathsf{C})$ for a class of C of finite frames.

3. Recall from Section 2.2 that a logic L is *Kripke-complete* iff L is sound and complete with respect to class of L-frames. Equivalently, L is Kripke-complete iff there exists a class C of Kripke frames such that $\mathsf{L} = \mathsf{Logic}(\mathsf{C})$.

4. In Section 2.1 we stated that L is *finitely axiomatisable* iff $\mathsf{L} = \mathsf{IPC} \oplus \Gamma$ for a finite set of formulas Γ. Note that L is finitely axiomatisable iff L is axiomatised by a single formula. Namely, $\bigwedge \Gamma$ is a formula if Γ is finite.

All these properties give an indication of the difficulty of a logic. The first three tell us something about the semantic complexity, i.e., how difficult is the class of structures we need to consider to decide whether a given formula is a theorem of the logic. In this sense, tabular logics are the easiest since we only have to inspect a single finite frame. Next in this hierarchy come logics with the fmp. Clearly, every tabular logic has the fmp. This inclusion is strict: IPC has the fmp but is not tabular, see [17, Theorems 2.56 and 2.57]. Thus, generally for logics with the fmp we have to inspect all of their finite frames. Since finite Esakia spaces and finite Kripke frames coincide it follows that every logic with the fmp is Kripke-complete. Again, this is a strict inclusion, see for instance [17, Theorem 6.3].

On the other hand, finite axiomatisability tells us something about the syntactic complexity of logics. Interestingly, every tabular logic is finite axiomatisable, see [17, Theorem 12.4]. However, fmp does not provide a finite axiomatisation: the Medvedev logic has the fmp but cannot be finitely axiomatised [41].

This marks the end of this preparatory chapter. We have discussed intermediate logics, their syntax and semantics, the dual equivalence of Heyting algebras and Esakia spaces, and we have covered a few properties of logics that describe complexity. In the next chapter we will decorate this canvas with the lax modality to lay out the fundamentals of *lax logic*.

Chapter 3

Rudiments of Lax Logics

The aim of this chapter is to familiarise ourselves with what we will call lax logics. These are the logics that extend the propositional lax logic. As stated in the introduction, lax logic appears quite frequently in the literature. The term *lax* originated in Mendler's PhD thesis [42] but has mostly gained traction through his shared work with Fairtlough [25]. However, these are certainly not the first works considering this logic. Another prominent work is [28] by Goldblatt, where sound and complete algebraic- and Kripke semantics are introduced for lax logic. We will dive into more details about the history of lax logic in Section 3.1.

There exist multiple semantics for lax logic. In this chapter we will examine algebraic and relational semantics. Algebraic semantics are given by Heyting algebras with *nuclei*, which we will call *nuclear algebras*. The name nuclei for such operations comes from pointless topology, see for instance [34, Section II 2]. Relational semantics for lax logic come in multiple flavors. We will see that lax logic is complete with respect to common Kripke semantics for intuitionistic modal logic, see IK-frames in [47]. Goldblatt [28] uses similar frames with a slightly weaker condition. Later, Fairtlough and Mendler provided novel Kripke semantics using fallible bimodal Kripke frames inspired by a translation into a bimodal logic extending (S4,S4) [25]. On the descriptive side we have *nuclear* spaces, which are Esakia spaces with an appropiate modal relation. Bezhanishvili and Ghilardi showed that such relations on an Esakia space are in a one-to-one correspondence with subframes of the space [11]. This inspired new semantics in the form of *S-spaces* [10, 31].

This chapter is structured as follows. First, in Section 3.1 we give a short exposition of the history of lax logic and related concepts. Then in Section 3.2 lax logics are syntactically defined. Section 3.3 contains algebraic and relational semantics for lax logic. The dual spaces of nuclear algebras are examined in Section 3.4. Finally, Section 3.5 discusses some translations of intermediate logics to lax logics.

3.1 History of lax modalities

The (*propositional*) *lax logic* (PLL) is an intuitionistic modal logic. It features a modal operator that has properties of necessity and possibility in the classical context. That is, it satisfies axioms that are associated with both $\Box$ (necessity) and $\Diamond$ (possibility) in classical modal logic. Specifically, a *lax modality* $\bigcirc$ satisfies the axioms:

$$\bigcirc p \wedge \bigcirc q \to \bigcirc(p \wedge q), \tag{$S_\bigcirc$}$$

$$p \to \bigcirc p, \text{ and} \tag{$CM_\bigcirc$}$$

$$\bigcirc\bigcirc p \to \bigcirc p. \tag{$C4_\bigcirc$}$$

and respects the rule of *regularisation*:

$$\frac{\varphi \to \psi}{\bigcirc\varphi \to \bigcirc\psi}\ (\mathrm{R})$$

In classical modal logic, $S_\bigcirc$ is an elementary axiom of $\Box$, and attributing $C4_\bigcirc$ to $\Box$ results in dense[1] frames[2]. However, $CM_\bigcirc$ may be considered unnatural with respect to $\Box$. Indeed, adding it to a normal modal logic results in a very simple class of frames. Namely, only frames with an accessibility relation that is a subset of the diagonal will be such that $\Box$ satisfies $CM_\bigcirc$. Conversely, attributing $CM_\bigcirc$ and $C4_\bigcirc$ to $\Diamond$ give us the classes of reflexive and transitive frames, respectively. These types of frames are often considered in modal logic. Nevertheless, $S_\bigcirc$ becomes problematic for $\Diamond$: a frame satisfies $S_\bigcirc$ with $\Diamond$ iff for each world there is at most a single accessible world. In addition, if the frame is reflexive we get that the accessibility relation has to be the identity, i.e., the logic is axiomatised by $p \leftrightarrow \Diamond p$. Thus, in the context of classical modal logic the combination of these axioms would be a bit odd for either modality.

This is probably the reason that it is common to use symbol $\bigcirc$ instead of the more traditional $\Box$ or $\Diamond$. This notation is due to Fairtlough and Mendler [25]. While most studies approach lax modalities from the perspective of possibility, we will argue in this chapter that lax modalities should be seen as an intuitionistic version of necessity, and we will switch to $\Box$ in the next section.

Historically, lax modalities have been studied since (at least) the 1950's with appearances under several names in various fields. The following is by no means an exhaustive list but it gives a good indication of the wide spread of lax modalities.

- The first steps towards lax modalities may have been [20] in the late 19th century by Dedekind in relation to Galois theory, see also [44, Chapter Closure]. Recall, a *Galois connection* between posets X and Y is a pair of monotone functions $f : X \to Y$, $g : Y \to X$ such that

$$fx \leq y \iff x \leq gy$$

[1] A binary relation R is dense iff xRy implies there exists z such that xRz and zRy.

[2] Classically, a Kripke frame is a pair (W, R) such that W is a set and R is a binary relation on R. Usually, elements of W are called *worlds* and R is called the *accessibility relation*.

for all $x \in X$ and $y \in Y$. This connects to lax modalities in the following sense. Suppose $f : A \to A$ is a function on a Heyting algebra. Then f forms a Galois connection with the identity map from $f[A]$ to A iff

$$fa \leq fb \iff a \leq fb$$

for all $a \in A$. An algebraic interpretation of this equation due to Macnab turns out the be the most succinct condition for lax modalities [40, Theorem 1.3]. In general, the condition is equivalent to f being a closure operator. Lax modalities correspond exactly to those closure operators that preserve meets.

- The first appearance of actual lax modalities was probably in the works by Curry [18, 19] who studied a Gentzen-style calculus with the lax axioms for proof-theoretic purposes. Besides, in proof theory lax modalities are studied in the form of Gödel-Gentzen style translations from IPC into itself [1]. The most notorious translation like this is probably the double negation translation. We will look at more general Gödel-Gentzen style translations in Section 3.5. Recently, van den Berg showed that lax modalities can also be used to obtain a Kuroda-style translation from IPC into itself [4].
- In the study of pointless topology, Dowker and Papert introduced specific congruences on complete Heyting algebras [22], which turned out to correspond to lax modalities [39, Chapter 9]. These congruences are the lattice congruences that are compatible with arbitrary joins. In other words, each equivalence class has a greatest element. Later they also appear as functions on lattices under the name nuclei [34, 45].
- In topos theory they first popped up under the name of *Lawvere-Tierney topologies* [28, 36], but are now more commonly called *local operators* [35, Section A4.4]. Sometimes they are also named *j-operators* [27] because traditionally the letter j was used to denote them.
- Mendler was the first to call them *lax* modalities in his PhD thesis [42], which provides formal methods for the design of correct computer hardware. His shared work [25] with Fairtlough is noteworthy. It investigates several proof-theoretic properties of lax logic and provides a sound and complete Kripke-style semantics for it. Its reading of $\bigcirc\varphi$ is "φ holds for some constraint." This appears to be a very existential reading of this modality, and it modelled their semantics accordingly, as we will see in Section 3.3.
- In category theory and computer science they are known as (*strong*) *monads*. In [3] the lax logic appears as the logic of Moggi's computational lambda calculus [43]. Here a proposition p denotes a type and $\bigcirc p$ is interpreted as "a computation of type p," whence it indicates the "possibility of a value of type p." Again, we arrived at quite an existential reading of the modality.
- Macnab calls them *modal operators* in a study of their algebraic properties on

Heyting algebras [2, 39, 40], and Goldblatt called them *local operators* in his crucial work [28]. Goldblatt's reading of $\bigcirc\varphi$ is "φ is locally true." This can be seen as a more universal reading of $\bigcirc$, i.e., something is locally true if it is true at all near points. Moreover, Goldblatt was the first to give complete algebraic- and Kripke-style semantics for lax logic.

- Lax modalities are also closely related to subframe logics. Bezhanishvili and Ghilardi showed that nuclear algebras are in a one-to-one correspondence with subframes of Esakia spaces [11]. This allowed Holliday [30] to develop Beth semantics for inquisitive intuitionistic logic which can also be used for lax logic. Furthermore, Bezhanishvili, Bezhanishvili and Ilin showed in [10] that lax logics are related to subframisations of intermediate logics. In fact, this relation provides translations of intermediate logics into lax logic which we will discuss in Section 3.5. This connection between lax logics and subframe logics is also discussed in detail in [31, Section 6.5].

Thus, lax modalities are wide-spread throughout branches of mathematics and computer science, which contrasts their at first glance unnatural axioms. In the next sections we will discuss lax logics formally.

3.2 Syntax of lax logics

We define lax logics akin to intermediate logics. The difference is that we add a modality with the lax axioms. Thus, to obtain the language of lax logics we only have to add a modality to the propositional language $\mathcal{L}_{\mathsf{p}}$. We will use $\Box$ instead of $\bigcirc$ to denote the lax modality. Choosing a traditional symbol over the unorthodox one is simply a stylistic choice. Moreover, picking $\Box$ over $\Diamond$ is justified by the $\mathrm{S}_{\bigcirc}$ axiom. It is standard that $\Box$ satisfies this axiom, even in the context of intuitionistic modal logic [47]. Consequently, as we will see in the next section, semantically the modality corresponds with the interpretation of $\Box$ in modal logic. Formally, the *modal language* $\mathcal{L}_{\Box}$ is defined using the following grammar:

$$\mathcal{L}_{\Box} \ni \varphi ::= p \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

Recall the difference between si-logics and intermediate logics: a si-logic is intermediate if it is contained in CPC. We can make a similar distinction for lax logics. However, it is a little more subtle. For now let us define lax logics.

Definition 3.2.1. A set $\mathsf{L} \subseteq \mathcal{L}_{\Box}$ closed under (MP), (US), and (R) is a *lax logic* if it contains all the formulas of Definition 2.1.1, and the formulas:

$$\Box p \wedge \Box q \to \Box(p \wedge q) \tag{S}$$

$$p \to \Box p \tag{CM}$$

$$\Box\Box p \to \Box p \tag{C4}$$

$\dashv$

We let PLL be the smallest lax logic. Furthermore, we use the same terminology and notation for lax logics as we have used for intermediate logics in the previous chapter, e.g., a theorem of a lax logic is a formula that is contained in the lax logic, and $\mathsf{L} \oplus \Gamma$ is the smallest lax logic extending L and containing Γ.

PLL relates to lax logics as IPC relates to si-logics, i.e., PLL is the smallest lax logic. However, what is the analogue of CPC? There are multiple lax logics containing CPC. The greatest consistent lax logic would be $\mathsf{PLL} \oplus \{\neg\neg p \to p, \neg\Box\bot\}$. This is a trivial lax logic in the sense that we can derive $p \leftrightarrow \Box p$. In fact, we call any lax logic that contains $p \leftrightarrow \Box p$ a *trivial* lax logic. The smallest trivial lax logic is $\mathsf{PLL} \oplus \Box p \to p$. Moreover, we call a lax logic *classical* if it contains $\neg\neg p \to p$, i.e., if it contains CPC. We are mostly interested in lax logics that are neither trivial nor classical. We will dub these *proper* lax logics. Finally, we are also interested in lax logics containing $\neg\Box\bot$. These are sometimes called *dense* lax logics [30], but we call them *serial* lax logics instead since $\neg\Box\bot$ is known as the seriality axiom in classical modal logic. These classifications are summarised in Table 3.2.2.

L is	iff
trivial	$\mathsf{L} \vdash \Box p \to p$
classical	$\mathsf{L} \vdash \neg\neg p \to p$
serial	$\mathsf{L} \vdash \neg\Box\bot$
proper	$\mathsf{L} \nvdash (\neg\neg p \vee \Box p) \to p$

Table 3.2.2: Characterising lax logics

We can compare lax logics and intermediate logics since the modal language extends the propositional language. We say that a lax logic M is an extension of an intermediate logic L iff $\mathsf{L} \subseteq \mathsf{M}$. Moreover, we can think of $\mathsf{PLL} \oplus \mathsf{L}$ as the least lax extension of L. Intuitively, we take L and add a lax modality to it. We will discuss this way of *laxifying* intermediate logics in more detail in Section 3.5.

In the introduction of chapter we called lax logics intuitionistic modal logics. These come in several flavors since $\Diamond$ and $\Box$ are not interdefinable intuitionistically. One can extend the syntax with $\Box$, $\Diamond$, or both, see [47]. We have defined lax logics as of the former kind. A set $\mathsf{L} \subseteq \mathcal{L}_\Box$ is then an *intuitionistic* (*normal*) *modal logic* if it is closed under the rules (MP), (US), *necessitation* (N), and contains the K-axiom: $\Box(p \to q) \to (\Box p \to \Box q)$. Every lax logic is such a set. Indeed, (N) is defined as

$$\frac{\varphi}{\Box\varphi}\ (\mathrm{N})$$

and that lax logics are closed under it follows immediately from (MP), (US), and CM. Besides, that lax logics contain K can be derived from (R), (US), CM, and S, compare for instance with [17, Exercise 3.1]. Conversely, we can derive S and closure under (R) in every intuitionistic modal logic containing CM and C4. Hence, lax logics are nothing more than intuitionistic modal logics containing CM and C4.

3.3 Lax semantics

In this section we give algebraic semantics for lax logics in the form of nuclear algebras. Moreover, we discuss some interesting properties of nuclear algebras. Then we look at a few Kripke-style semantics for lax logic and compare them.

Nuclear algebras

We have seen in Section 2.2 that algebraic semantics for intermediate logics are given by varieties of Heyting algebras. Since lax logics are enrichments of intermediate logics with a lax modality, it is intuitive that algebraic semantics for lax logics are given by Heyting algebras with an appropriate additional operator. It is, an operator that respects S, CM, C4, and (R). We call such operators *nuclei* and the corresponding algebras are called *nuclear algebras*. As we stated in the introduction of this chapter, calling such operators nuclei is standard in pointless topology.

Definition 3.3.1. A unary function $j : A \to A$ on a lattice A is a *nucleus* iff

$$ja \wedge jb = j(a \wedge b) \qquad \text{(multiplicativity)}$$
$$a \leq ja \qquad \text{(inflationary)}$$
$$jja = ja \qquad \text{(idempotent)}$$

for all $a, b \in A$. $\dashv$

An alternative characterisation of nuclei is due to Macnab, which says a function j on a Heyting algebra A is a nucleus iff $a \to jb = ja \to jb$ for all $a, b \in A$ [40, Theorem 1.3]. Since nuclei are multiplicative it follows that they are *monotone*, i.e., $a \leq b$ implies $ja \leq jb$ for all $a, b \in A$.

Definition 3.3.2. A *nuclear algebra* is an $\mathcal{N} = \{\wedge, \vee, \top, \bot, \to, \Box\}$-algebra A such that A is a Heyting algebra and $\Box$ is a nucleus on A. $\dashv$

Below we recall some identities that are satisfied by nuclear algebras. Proofs can be found in [40, Theorem 1.2].

Lemma 3.3.3. Let A be a nuclear algebra. Then

(1) $\Box(a \vee b) = \Box(\Box a \vee \Box b)$

(2) $a \to \Box b = \Box a \to \Box b = \Box(\Box a \to \Box b)$

for all $a, b \in A$.

The semantics for nuclear algebras are defined as expected. Goldblatt showed that PLL is sound and complete with respect to all nuclear algebras [28, Section 6]. Besides, nuclear algebras form an equational class just as Heyting algebras. Namely, the class of PLL-algebras is exactly the variety of $\mathcal{N}$-algebras that corresponds to nuclear algebras. Moreover, by the standard Lindenbaum-Tarski construction each lax logic is sound and complete with respect to its variety. Conversely, it is easy to see that $\mathsf{Logic}(\mathsf{C})$ is a lax logic for every class C of nuclear algebras.

Theorem 3.3.4. Every lax logic L is sound and complete with respect to the class of L-algebras.

Algebraically, nuclear algebras are a very simple enrichment of Heyting algebras. For instance, to decide whether a nuclear algebra is s.i. we only have to look at the Heyting reduct. That is, the characterisation of s.i. nuclear algebras is the same as the characterisation of s.i. Heyting algebras.

Lemma 3.3.5. A nuclear algebra A is s.i. iff A has a second largest element.

Proof. By [29, Proposition 1.1], A is s.i. iff A has an opremum, i.e., there exists $s \in A \setminus \{\top\}$ such that for all $a \in A \setminus \{\top\}$ there exists an integer n such that $\bigwedge_{i=0}^{n} \Box^i a \leq s$, where $\Box^i a$ is defined recursively as $\Box^0 a := a$, and $\Box^{i+1} a := \Box\Box^i a$. Since $a \leq \Box a$ for all $a \in A$ we have $\bigwedge_{i=0}^{n} \Box^i a = a$ for all integers n and $a \in A$. Consequently, oprema correspond with second largest elements in nuclear algebras. ∎

Alternatively, Heyting algebras are s.i. iff they have a least non-unital filter. The same holds for modal filters[1] of Heyting algebras with modal operators. Then the previous lemma is a consequence of the fact that each filter on nuclear algebra is also a modal filter.

We will work towards giving an alternative proof of the fact that PLL has the fmp, using the recent result in [9] that the variety of bounded nuclear implicative semilattices is locally finite. That PLL has the fmp was first shown in [28], and later in [25], both frame-theoretically. We denote the algebraic type of bounded nuclear implicative semilattices by $\hat{\mathcal{J}} = \{\wedge, \top, \bot, \to, \Box\}$.

Theorem 3.3.6 ([9, Theorem 7.12])**.** There exists a function $c : \mathbb{N} \to \mathbb{N}$ such that if A is a nuclear algebra and $B \subseteq A$ is a finite subset then the $\hat{\mathcal{J}}$-algebra generated by B is at most of size $c(|B|)$, i.e., the variety of $\hat{\mathcal{J}}$-reducts of nuclear algebras is locally finite.

In Section 2.2 we saw that the local finiteness of $\hat{\mathcal{I}}$-reducts of Heyting algebras gives the Selective Filtration Lemma. We can now generalise this lemma to the lax case. Consequently, we will call it the *Nuclear Selective Filtration Lemma.*

Lemma 3.3.7 (Nuclear Selective Filtration)**.** Let B be a (s.i.) nuclear algebra such that $A \nvDash \varphi$. Then there is a finite (s.i.) nuclear algebra such that A is a $\hat{\mathcal{J}}$-subalgebra of B and $A \nvDash \varphi$.

Proof. Let $v : \mathsf{Prop} \to B$ be the valuation that refutes φ on B and let A_0 be the set containing $\{v(\psi) \mid \psi \text{ is a subformula of } \varphi\}$ and the second largest element s of B if it

[1] A modal filter is a filter F such that $a \in F$ implies $\Box a \in F$.

exists. Next, let A be the $\hat{\mathcal{J}}$-algebra generated by A_0. Then A is finite by Theorem 3.3.6. Moreover, A is a Heyting algebra with

$$a \vee_A b := \bigwedge\{c \in A \mid a, b \leq c\}$$

for all $a, b \in A$. In fact, it is a nuclear algebra since $\Box$ is a nucleus. Furthermore, from the definition of $\vee_A$ it is easy to see that $a \vee_B b \leq a \vee_A b$ for all $a, b \in A$, and whenever $a \vee_B b \in A$ we have $a \vee_A b = a \vee_B b$. Since $A_0 \subseteq A$ we can restrict the range of v to A. Then $v(\varphi)_A = v(\varphi)_B \neq \top_B = \top_A$. Whence, $A \nvDash \varphi$. Finally, if s is the second largest element of B then it is also the second largest element of A. Thus, if B is s.i. so is A. ■

Theorem 3.3.8. PLL has the fmp.

Proof. Suppose $\mathsf{PLL} \nvdash \varphi$. Since PLL is complete with respect to its algebras there exists a nuclear algebra B such that $B \nvDash \varphi$. By Nuclear Selective Filtration, there exists a finite nuclear algebra A such that $A \nvDash \varphi$. ■

In fact, Nuclear Selective Filtration gives a more general result. Let $\mathcal{F} \subseteq \mathcal{N}$ be some algebraic subtype of nuclear algebras. We say that a class C of nuclear algebras is *closed under $\mathcal{F}$-subalgebras* iff for all nuclear algebras A and B we have that $A \in \mathsf{C}$ and B is an $\mathcal{F}$-subalgebra of A implies that $B \in \mathsf{C}$.

Theorem 3.3.9. Let L be a lax logic and $\mathcal{F} \subseteq \hat{\mathcal{J}}$ an algebraic subtype of nuclear bounded implicative semillatices. If the class of L-algebras is closed under $\mathcal{F}$-subalgebras then L has the fmp.

Proof. Following the proof of Theorem 3.3.8 it suffices to show that any $\hat{\mathcal{J}}$-subalgebra of some L-algebra is an L-algebra but this follows immediately from closure under $\mathcal{F}$-subalgebras since $\mathcal{F} \subseteq \hat{\mathcal{J}}$. ■

Since fmp is a very strong property, closure under $\hat{\mathcal{J}}$-subalgebras characterises a very well-behaved class of lax logics. Indeed, we will see in the next section that for these logics the considered Kripke semantics coincide.

An interesting aspect of nuclear algebras is that the fixpoints of the nucleus form a Heyting algebra. For any nuclear algebra A, we define $A_\Box$ as the set of $\Box$-fixpoints. Since $\Box$ is idempotent we have $A_\Box = \{\Box a \mid a \in A\}$.

Proposition 3.3.10. Let A be nuclear algebra. Then $A_\Box = (A_\Box, \wedge, \vee_\Box, \to, \top, \Box\bot)$, where $a \vee_\Box b = \Box(a \vee b)$ for all $a, b \in A_\Box$ is a Heyting algebra.

Proof. It follows from Lemma 3.3.3 that $A_\Box$ is closed under $\vee_\Box$ and $\to$. That $A_\Box$ is closed under $\wedge$ is given by multiplicativity of $\Box$. ■

The fact that the $\Box$-fixpoints of nuclear algebras form Heyting algebras is related to the notion of S-spaces which we encounter in the next section. Namely, we can see nuclear algebras as pairs (A, B) of Heyting algebras such that $B = A_\Box$ for some nucleus $\Box$ on A. That is, there is a one-to-one correspondence between nuclear algebras and such pairs. This can be seen as follows. Obviously, for every nuclear algebra we can take the pair $(A, A_\Box)$. Conversely the set of fixpoints uniquely determines a nucleus:

Proposition 3.3.11. Let A be a nuclear algebra. Then for all other nuclei $\boxdot : A \to A$ we have

$$A_\Box = A_\boxdot \text{ iff } \Box = \boxdot.$$

Proof. Right-to-left is obvious. For the other direction, take some $a \in A$. Then $\Box a, \boxdot a \in A_\Box$ since

$$\{a \in A \mid a = \Box a\} = \{\Box a \mid a \in A\}$$

and similarly for $\boxdot$. Hence, $\boxdot\Box a = \Box a$ and $\Box \boxdot a = \boxdot a$. But then

$$\Box a \wedge \boxdot a = \boxdot\Box a \wedge \boxdot a = \boxdot(\Box a \wedge a) = \boxdot\Box a = \Box a,$$

which means $\Box a \leq \boxdot a$. The converse we get in the same way. ■

Of course, when we defined the pairs we assumed there is a nucleus. Equivalently, we can demand $B \subseteq A$ and that the inclusion map $i : B \to A$ has a left adjoint that preserves meets and top. For more details, see [10, Section 5] or [31, Section 6.5.2].

Kripke-style semantics

In this subsection we compare four Kripke-style semantics found in or derived from the literature which are sound and complete for PLL. Some of these connections were already discussed in [10, Section 5]. We expand this analysis by covering the non-descriptive setting. First, looking at lax logic from the perspective of intuitionistic modal logic we get the following characterisation of frames.

Definition 3.3.12 (IK-frames)**.** An *IK-frame* (validating PLL) is a structure $(X, \leq, R)$ such that (i) $(X, \leq)$ is a partial order, (ii) $\leq \circ R \circ \leq \; = R$, (iii) $R \subseteq \; \leq$, and (iv) $R \subseteq R^2$. $\dashv$

We will follow our routine of leaving structural components implicit and simply denote IK-frames $(X, \leq, R)$ by X. Note that by dropping conditions (iii) and (iv) we get the standard frames in intuitionistic modal logic literature, see [46, 47, 48]. Conditions (iii) and (iv) will us soundness with respect to PLL. Henceforth, we call a binary relation R on a partial order *lax* if it satisfies conditions (ii), (iii) and (iv).

The semantics of IK-frames naturally extend intuitionistic Kripke semantics. Indeed, formulas without modality are interpreted on valuations $v : \mathsf{Prop} \to \mathrm{Up}(X)$ as in the Kripke semantics of Section 2.2. Finally, the clause for $\Box$ is given by

$$X, x \Vdash_v \Box\varphi \iff xRy \text{ implies } X, y \Vdash_v \varphi$$

for all $x \in X$. Derived notions such as $X \Vdash \varphi$ are defined as expected.

Lemma 3.3.13. For any IK-frame X we have $X \vDash \mathsf{PLL}$.

Proof. It suffices to show that X validates CM and C4. First, suppose $x \Vdash p$ and $x \leq yRz$. Then $y \leq z$ by (iii), and thus $x \leq z$. Whence $z \Vdash p$ since $v(p)$ is an upset. Thus, $y \Vdash \Box p$, and therefore $x \Vdash p \to \Box p$. Next, suppose $x \Vdash \Box\Box p$ and $x \leq yRz$. Then $x \leq yRz \leq z$, and so by (ii), xRz. Then by (iii), xR^2z, which means $z \Vdash p$. We can conclude $y \Vdash \Box p$ and $x \Vdash \Box\Box p \to \Box p$. ∎

Thus far we have merely identified a sound class of frames. In short, we will prove completeness of this semantics by reducing the frames used by Goldblatt into IK-frames. IK-frames can be seen as nuclear algebras in the following way. Recall that X^{Up} is the complex algebra of the partial order X. Then let $\Box_R U := \{x \in X \mid R[x] \subseteq U\}$ for all $U \in \mathrm{Up}(X)$. Then it is not hard to check that $\Box_R$ is a nucleus on X^{Up}. Thus, for any IK-frame X we have that X^{Up} with $\Box_R$ is a nuclear algebra. Moreover, it is a nuclear algebra of the same logic.

Goldblatt defined very similar frames in [28]. They satisfy the same conditions except (ii). Goldblatt's frames instead satisfy a slightly weaker version.

Definition 3.3.14 (Goldblatt frames)**.** A *Goldblatt frame* is a structure $(X, \leq, R)$ such that (i) $(X, \leq)$ is a partial order, (ii) $\leq \circ R \subseteq R$, (iii) $R \subseteq \leq$, and (iv) $R \subseteq R^2$. ⊣

Goldblatt semantics is identical to the previously defined semantics on IK-frames. Furthermore, the complex algebra corresponding to a Goldblatt frame is determined in the same way as for an IK-frame. Indeed, Goldblatt's (ii) is sufficient for X^{Up} to be closed under $\Box_R$. The stronger assumption on IK-frames is result of completeness results for intuitionistic modal logics [46]. When we take the dual spaces of the Heyting reduct of nuclear algebras and define R as we would in standard modal duality then the space will satisfy IK-(ii). Consequently, assuming it does not limit the strength of the semantics. Indeed, every Goldblatt frame has an IK-frame with the same logic.

Lemma 3.3.15. The semantics of (1) Goldblatt frames, and (2) IK-frames are persistent.

Proof. (1) Suppose $x \Vdash \varphi$ and $x \leq y$. We prove by induction on the complexity of φ. Only the $\Box$ case is relevant, thus suppose $x \Vdash \Box\varphi$ and yRz. Then $x \leq yRz$, so by Goldblatt-(ii), xRz. Whence, $z \Vdash \varphi$. We can conclude $y \Vdash \Box\varphi$.

(2) Since IK-(ii) implies Goldblatt-(ii) every IK-frame is a Goldblatt frame. Besides, the semantics follow the same clauses. Consequently, IK-frame semantics is also persistent. ∎

Lemma 3.3.16. Let $X = (X, \leq, R)$ be a Goldblatt frame. Then $X' = (X, \leq, R')$ with $R' := \leq \circ R \circ \leq$ is an IK-frame such that $X \Vdash \varphi$ iff $X' \Vdash \varphi$ for all formulas φ.

Proof. That X' is an IK-frame follows by definition of R'. For the second part we will prove the stronger claim that $X, x \Vdash \varphi$ iff $X', x \Vdash \varphi$ for all $x \in X$ and for all formulas φ (and all valuations). We prove by induction on the complexity of φ. Only the $\Box$ case is relevant. Let $x \in X$. First, suppose $X, x \Vdash \Box\varphi$ and $xR'z$, Then xRy implies $X, y \Vdash \varphi$, and by $xR'z$ there exist $x', z' \in X$ such that $x \leq x'Rz' \leq z$. By Goldblatt-(ii) it follows that xRz'. Whence, $X, z' \Vdash \varphi$. By persistence of Goldblatt semantics we get that $X, z \Vdash \varphi$. From the IH, we obtain $X', z \Vdash \varphi$. Therefore, $X', x \Vdash \Box\varphi$. For the converse, suppose $X', x \Vdash \Box\varphi$ and xRz. Then $xR'y$ implies $X', y \Vdash \varphi$. Clearly, $x \leq xRz \leq z$, whence $xR'z$. Ergo, $X', z \Vdash \varphi$, and by IH $X, z \Vdash \varphi$. Thus, $X, x \Vdash \Box\varphi$. ■

Goldblatt showed that PLL is sound and complete with respect to Goldblatt frames [28, Section 6]. Hence, it follows from Lemmas 3.3.13 and 3.3.16 that IK-frames provide a sound and complete semantics for PLL. Lemma 3.3.16 gives an even stronger result. They determine the exact same class of logics:

Corollary 3.3.17. A lax logic is complete with respect to a class of Goldblatt frames iff it is complete with respect to a class of IK-frames.

Proof. If X is a Goldblatt frame validating L such that $X \nVdash \varphi$ then there is an IK-frame X' that also validates L and refutes φ by Lemma 3.3.16. For the converse we have that each IK-frame is a Goldblatt frame. ■

It is not a new observation that IK-frames and Goldblatt frames are equally expressive in the context of intuitionistic modal logic, see for instance the discussion in [37, Section 2.1].

We will now focus on the more unorthodox semantics for lax logics: *S-frames* and *FM-frames*. The "S" in S-frame stands for subframe. Our S-frame semantics is a non-descriptive version of the semantics given for S-spaces in [10, 31]. Note that S-spaces are called S-frames in [10]. We instead call them S-spaces to differentiate them from their non-descriptive counterpart. We will discuss S-spaces in more detail in the next section. In the Kripke setting we can simply see them as (intuitionistic) frames with a designated subset.

Definition 3.3.18 (S-frames [10])**.** An *S-frame* is a structure $(X, \leq, S)$ such that (i) $(X, \leq)$ is a partial order, and (ii) $S \subseteq X$. ⊣

S-frame semantics is yet another extension of intuitionistic Kripke semantics. Hence, again only the clause for $\Box$ is relevant. This time it is given by

$$X, x \Vdash_v \Box\varphi \iff x \leq y \text{ and } y \in S \text{ implies } X, y \Vdash_v \varphi$$

for all $x \in X$. Algebraically, we interpret formulas on the complex algebra X^{Up} with the nucleus given by $\Box_S U := \{x \in X \mid {\uparrow}x \cap S \subseteq U\}$.

S-spaces are related to the one-to-one correspondence between nuclear spaces and subframes which we will see in the next section. Practically, we can induce a suited lax relation on an Esakia space given a well-behaved subspace of an Esakia space and vice versa. In the non-descriptive case we can use this way to induce relations to construct IK-frames from S-frames.

Definition 3.3.19. Given a partial order $(X, \leq)$ and $S \subseteq X$. Define a relation $S_{\between} \subseteq X^2$ as

$$x \, S_{\between} \, y \text{ iff there exists } s \in S \text{ such that } x \leq s \leq y$$

for all $x, y \in X$. ⊣

The intuition behind this definition is that we define the smallest lax relation which has S as its reflexive points. If we put this relation on the partial order underlying an S-frame we get an IK-frame of the same logic.

Lemma 3.3.20. If $X_S = (X, \leq, S)$ is an S-frame then $X_R = (X, \leq, S_{\between})$ is (1) an IK-frame such that (2) $X_S \Vdash \varphi$ iff $X_R \Vdash \varphi$ for all formulas φ.

Proof. (1) Clearly, $S_{\between} \subseteq \leq \circ S_{\between} \circ \leq$. For the converse suppose $x \leq x' \, S_{\between} \, y' \leq y$. Then there exists $s \in S$ such that $x' \leq s \leq y'$. Hence, $x \leq s \leq y$, which means $x_1 \, S_{\between} \, y_2$. Thus, $S_{\between} = \leq \circ S_{\between} \circ \leq$. Next, suppose $x \, S_{\between} \, y$. Then there exists $s \in S$ such that $x \leq s \leq y$. Hence, $x \leq y$. Thus, $S_{\between} \subseteq \leq$. Finally, suppose $x \, S_{\between} \, y$. Then there exists $s \in S$ such that $x \leq s \leq y$. Hence, $x \leq s \leq s$ and $s \leq s \leq y$. Therefore, $x \, S_{\between} \, s \, S_{\between} \, y$, i.e., $S_{\between} \subseteq S_{\between}{}^2$. Hence, $S_{\between}$ is lax, which means X_R is an IK-frame.

(2) The proof goes as in Lemma 3.3.16. Hence, it boils down to showing $X_S, x \Vdash \Box\varphi$ iff $X_R, x \Vdash \Box\varphi$ for all $x \in X$. That is,

$$x \leq y \text{ and } y \in S \text{ implies } X_S, y \Vdash \varphi \iff x \, S_{\between} \, y \text{ implies } X_R, y \Vdash \varphi.$$

Suppose the former and $x \, S_{\between} \, y$. Then there exists $z \in S$ such $x \leq z \leq y$. Whence, $X_S, z \Vdash \varphi$. By IH, $X_R, z \Vdash \varphi$, and by persistence we have $X_R, y \Vdash \varphi$. Conversely, suppose $X_R, x \Vdash \Box\varphi$, $x \leq y$, and $y \in S$. Clearly, $x \leq y \leq y$, which means $x \, S_{\between} \, y$. Therefore, $X_R, y \Vdash \varphi$, and $X_S, y \Vdash \varphi$ by IH. ■

Remark 3.3.21. A proof for Lemma 3.3.20(1) can also be extracted from [11, Theorem 28]. ⊣

Corollary 3.3.22. If a lax logic is complete with respect to S-frames then it is complete with respect to (1) IK-frames, and (2) Goldblatt frames.

Proof. It suffices to show (1) since every IK-frame is a Goldblatt frame. Thus, suppose X_S is an S-frame validating L such that $X_S \nVdash \varphi$. Then there is an IK-frame X_R that also validates L and refutes φ by Lemma 3.3.20. ■

The converse is problematic since, as opposed to nuclear spaces, IK-frames do not generally have the reflexive points that determine the lax relation. Hence, it is hard to turn them into S-frames. We leave reducing infinite IK or Golblatt frames into equivalent S-frame as an open problem. For finite frames we can find a positive answer since it coincides with descriptive case.

Definition 3.3.23. For any binary relation $R \subseteq X^2$ on a set X we define $R_\triangle := \{x \in X \mid xRx\}$ as the R-reflexive points. $\dashv$

Lemma 3.3.24. If $X_R = (X, \leq, R)$ is a finite IK-frame then $X_S = \{X, \leq, R_\triangle\}$ is an S-frame such that $X_R \Vdash \varphi$ iff $X_S \Vdash \varphi$ for all formulas φ.

Proof. Clearly, $R_\triangle \subseteq X$ so X_S is an S-frame. The remainder boils down to showing

$$xRy \iff \text{there exists } s \in R_\triangle \text{ such that } x \leq s \leq y.$$

Suppose xRy. Since $R \subseteq R^2$ there is some $z' \in X$ such that $xRz'Ry$. But then there is $z'' \in X$ such that $xRz'Rz''Ry$. Since we can repeat this argument infinitely and X is finite we must eventually find some $z \in R_\triangle$ such that $xRzRy$. Conversely, we have $x \leq sRs \leq y$, so by IK-(ii) we get xRy. ■

The final semantics of this subsection is perhaps the most unique. *FM semantics* is named after Fairtlough and Mendler, who introduced it in [25]. While it is still an extension to intuitionistic Kripke frames, it requires fallible worlds and evaluates $\Box$ with an universal and extensional quantifier.

Definition 3.3.25 (FM-frames)**.** An *FM-frame* is a structure $(X, \leq, R, F)$ such that (i) $(X, \leq)$ is a partial order, (ii) (X, R) is a partial order (iii) $R \subseteq \leq$, and (iv) $F \in \mathrm{Up}(X)$. $\dashv$

Elements $f \in F$ are called *fallible*. While the semantic clauses are mostly intuitionistic, there are some subtleties. We only consider valuations $v : \mathsf{Prop} \to \mathrm{Up}(X)$ such that $F \subseteq v(p)$ for all $p \in \mathsf{Prop}$. Then we define $x \Vdash \bot$ iff $x \in F$, which is standard when working with fallible worlds. Algebraically, this corresponds to interpreting formulas on the Heyting algebra of upsets containing F. Finally, the clause for $\Box$ is given by

$$x \Vdash \Box\varphi \iff x \leq y \text{ implies there exists } z \in R[y] \text{ such that } z \Vdash \varphi$$

That is, the nucleus we use is defined by $\Box_\leq \Diamond_R U$ for $U \in \mathrm{Up}(X)$ such that $F \subseteq U$, where $\Diamond_R U = R^{-1}[U]$.

Remark 3.3.26. The clause for $\Box$ in FM-frames is probably exported from a Gödel-Gentzen style translation of lax logic into the bimodal logic $(\mathsf{S4}, \mathsf{S4}) \oplus \Box_1 p \to \Box_2 p$, see [25, Section 5]. $\dashv$

Fairtlough and Mendler showed that FM semantics is sound and complete for PLL [25, Theorems 3.3 and 4.4]. In [10] it was discussed how to turn S-spaces into FM-frames. We can use the same approach to turn S-frames into FM-frames.

Lemma 3.3.27. Let $X_S = (X, \leq, S)$ be an S-frame and $w \notin X$. Then $X_F = (X \cup \{w\}, \leq_F, R, \{w\})$, where

$$\begin{aligned} x \leq_F y &\iff x \leq y \text{ or } y = w \\ xR_F y &\iff x = y \text{ or, } x \notin S \text{ and } y = w, \end{aligned}$$

is an FM-frame such that $X_S \Vdash \varphi$ iff $X_F \Vdash \varphi$ for all formulas φ.

Proof. That X_F is an FM-frame is clear. Furthermore, valuations X_S on and X_F are in a one-to-one correspondence. We only have to add or remove w from the images of the valuations. We prove by induction on the complexity of φ that $X_S, x \Vdash \varphi$ iff $X_F, x \Vdash \varphi$ for all for $x \in X$ and valuations v. The only interesting case is $\Box$.

- Suppose $X_S, x \Vdash \Box\varphi$. Then for all $y \in S$ such that $x \leq y$ we have $y \Vdash \varphi$. Suppose $x \leq y$. We have to show there exists $z \in X_F$ such that $yR_F z$ and $X_F, z \Vdash \varphi$. If $y \in S$ then we have $X_S, y \Vdash \varphi$ and by the IH $X_F, y \Vdash \varphi$. If $y \notin S$ then we have $yR_F w$ and $X_F, w \Vdash \varphi$ vacuously since $w \in F$. We can conclude $X_F, x \Vdash \Box\varphi$.
- Suppose $X_F, x \Vdash \Box\varphi$. Suppose $x \leq y$ and $y \in S$. Then there is $z \in X_F$ such that $yR_F z$ and $X_F, z \Vdash \varphi$. Since $y \in S$ we have $R_F[y] = \{y\}$. Thus, $X_F, y \Vdash \varphi$ and by the IH, $X_S, y \Vdash \varphi$. Therefore, $X_S, x \Vdash \Box\varphi$. ■

Corollary 3.3.28. Every lax logic that is complete with respect to a class of S-frames is complete with respect to a class of FM-frames.

Proof. Let L be a lax logic. For any X_S is an S-frame that validates L we have an equivalent FM-frame X_F by Lemma 3.3.27. Hence, if $X_S \nVdash \varphi$ then $X_F \nVdash \varphi$. ■

We leave the general case of the other direction as an open question. However, we will show the finite case.

Lemma 3.3.29. If $X_F = (X, \leq, R, F)$ is a finite FM-frame then there is an S-frame X_S such that $X_F \Vdash \varphi$ iff $X_S \Vdash \varphi$ for all formulas φ.

Proof. Let $Y = X \setminus F$ and $S = \{x \in Y \mid x \in X \setminus {\downarrow}(X \setminus R^{-1}[U \cup F]) \text{ implies } x \in U\}$. Clearly $X_S = (Y, S)$ is an S-frame. Observe that $\max(Y) \subseteq S$. Again valuations on X_S and X_F are in a one-to-one correspondence. We prove by induction on the complexity of φ that $X_S, x \Vdash \varphi$ iff $X_F, x \Vdash \varphi$ for all for $x \in X$ and valuations v. The only interesting case is $\Box$.

- Suppose $X_S, x \Vdash \Box\varphi$. Then for all $y \in S$ such that $x \leq y$ we have $y \Vdash \varphi$. Suppose $x \leq y$. We have to show there exists $z \in X_F$ such that yRz and $X_F, z \Vdash \varphi$. If $y \in S$ then we have $X_S, y \Vdash \varphi$ and by the IH, $X_F, y \Vdash \varphi$. If $y \notin S$ then there is $U \in \mathrm{Up}(Y)$ such that $y \in X \setminus {\downarrow}(X \setminus R^{-1}[U \cup F])$ and $y \notin U$. It follows that ${\uparrow}y \subseteq R^{-1}[U \cup F]$. In particular, $y \in R^{-1}[U \cup F]$. That is, there is $z \in U \cup F$ such that yRz. Then $x \leq y \leq z$. If $z \in S \cup F$, then we are done vacuously, or by the

IH. If not we can repeat the same argument. Since X is finite we will eventually find $z \in S \cup F$.

- Suppose $X_F, x \Vdash \Box\varphi$. Suppose $x \leq y$ and $y \in S$. Suppose towards a contradiction that $y \notin [\![\varphi]\!] := \{x \in Y \mid X_F, x \Vdash \varphi\}$. Then $[\![\varphi]\!] \in \mathrm{Up}(Y)$. Hence, $y \notin X \setminus {\downarrow}(X \setminus R^{-1}[[\![\varphi]\!] \cup F])$ since $y \in S$. It follows that there exists $z \in X$ such that $y \leq z$ and $z \notin R^{-1}[[\![\varphi]\!] \cup F]$. Then $x \leq y \leq z$ and z has no R-accessible world that satisfies φ. Then $X_F, x \nVdash \Box\varphi$, a contradiction. Hence, $X_F, y \Vdash \varphi$, and by the IH, $X_S, y \Vdash \varphi$. We can conclude $X_S, x \Vdash \Box\varphi$. ■

The reductions covered in this section are summarised in Figure 3.3.30.

$$\begin{tikzcd}
\text{FM-frames} \arrow[r, dashed, bend right, "\text{Lemma 3.3.29}"'] & \text{S-frames} \arrow[l, bend right, "\text{Lemma 3.3.27}"'] \arrow[r, bend right, "\text{Lemma 3.3.20}"'] & \text{IK-frames} \arrow[l, dashed, bend right, "\text{Lemma 3.3.24}"'] \arrow[r, bend right, "\text{identity}"'] & \text{Goldblatt} \arrow[l, bend right, "\text{Lemma 3.3.16}"']
\end{tikzcd}$$

Figure 3.3.30: Reductions between Kripke-style semantics for lax logic. Each arrow represents a procedure to reduce one type of frame into an equivalent frame of the other type. Dashed arrows indicate that the lemma holds for finite frames.

Theorem 3.3.31. Let L be a lax logic. The following are equivalent.

(1) L is complete with respect to finite FM-frames.

(2) L is complete with respect to finite S-frames.

(3) L is complete with respect to finite IK-frames.

(4) L is complete with respect to finite Goldblatt frames.

Proof. (1) $\Rightarrow$ (2). If $X \nVdash \varphi$ is a finite FM-frame validating L, then by Lemma 3.3.29 there exists a finite S-frame $Y \nvDash \varphi$ validating L.

All other cases follow similarly by chasing the diagram in Figure 3.3.30. ■

In the next section we will use this result to show that PLL is complete with respect to the finite frames of each Kripke-style semantics defined this section.

We have seen that usually Kripke semantics are given by an universal interpretation of a modality relation. This cements our interpretation of lax modalities as $\Box$ in a modal setting. Goldblatt frames and IK-frames are both used in this context. However, it is common in the literature to assume the stronger confluence condition on R. Hence, we will use IK-frames in the remainder of this thesis, and simply refer to them as lax frames.

3.4 Nuclear Esakia duality

In this section we will discuss descriptive versions of the semantics of the previous section. We will give two enrichments of Esakia spaces that can be used to represent nuclear algebras: *nuclear spaces* and *S-spaces*. Moreover, we will discuss the equivalence of these representations, as has been shown in [11].

Nuclear spaces

Recall that the dual space of a Heyting algebra is an Esakia space. Since nuclear algebras are Heyting algebras with nuclei it is unsurprising that the dual spaces of nuclear algebras are an enrichment of an Esakia space.

Definition 3.4.1. A *nuclear space* is a structure (X, R) where $R \subseteq X^2$ such that

- X is an Esakia space;
- $U \in \mathrm{ClopUp}(X)$ implies $\Box_R U := \{x \in X \mid R[x] \subseteq U\} \in \mathrm{ClopUp}(X)$;
- $R[x] \in \mathrm{ClUp}(X)$ for all $x \in X$;
- $R \subseteq \leq$;
- $R \subseteq R^2$. ⊣

To continue the tradition of leaving relations and functions of structures implicit we will simple write "X is a nuclear space." We will call any binary relation R on an Esakia space *nuclear* iff it satisfies the last four conditions of Definition 3.4.1. Thus, a nuclear space is an Esakia space paired with a nuclear relation on it. Structures that satisfy the first three conditions of Definition 3.4.1 are called *modal Esakia spaces*. Alternatively, we can define modal Esakia spaces using condition (ii) of IK-frames, i.e., $\leq \circ R \circ \leq = R$. Indeed, we also get the following identities.

Lemma 3.4.2. If X is a modal Esakia space then

(1) $\leq \circ R \circ \leq = \leq \circ R = R \circ \leq = R$.

(2) $\uparrow R[\uparrow x] = \uparrow R[x] = R[\uparrow x] = R[x]$ for all $x \in X$.

Proof. (1) We will only show $\leq \circ R \circ \leq \subseteq R$. All other inclusions follow easily. Suppose $x_1 \leq x_2 R y_1 \leq y_2$. If $y_1 \in R[x_1]$ then we are done since $R[x_1]$ is an upset. Thus, suppose $y_1 \notin R[x_1]$ towards a contradiction. Then $z \not\leq y_1$ for each $z \in R[x_1]$. By the Priestley separation axiom there exists $U_z \in \mathrm{ClopUp}(X)$ such that $z \in U_z$ and $y_1 \notin U_z$. Hence, $R[x_1] \subseteq \bigcup_{z \in R[x_1]} U_z$, and since $R[x_1]$ is closed and X is compact, it follows that there is a finite subcover $\mathcal{U}$ such that $\mathcal{U} = \{U_{z_i} \mid 0 \leq i \leq n\}$ for some $z_1, \ldots, z_n \in R[x_1]$. Then $\bigcup \mathcal{U}$ is a clopen upset such that $y_1 \notin \bigcup \mathcal{U}$. Hence $U := \Box_R(\bigcup \mathcal{U})$ is a clopen upset. Since $x_1 \in U$ we have $x_2 \in U$, which means $R[x_2] \subseteq \mathcal{U}$. But then we have $y_1 \in \bigcup \mathcal{U}$, contradiction.

(2) Follows from (1). ■

Thus, nuclear spaces satisfy all conditions of lax frames. Hence, we can turn them into lax frames by forgetting about the topology.

Definition 3.4.3. A *modal Esakia morphism* $f : X \to Y$ between nuclear spaces is a continuous map such that

- $f[\uparrow x] = \uparrow fx$;
- $f[R[x]] = R[fx]$;

for all $x \in X$. ⊣

Theorem 3.4.4 ([11, Theorem 16])**.** The category of nuclear spaces with modal Esakia morphisms and the category of nuclear algebras with $\mathcal{N}$-homomorphisms are dually equivalent.

The relevant functors are extensions of $(\_)^*$ and $(\_)_*$ from Section 2.3. On functions they work the same as before. On objects, they have to define an additional thing. Namely, the dual of a nuclear algebra is given by the dual of its underlying Heyting algebra and some relation R determined by $\Box$. Precisely, suppose A is nuclear algebra. Then (A_*, R) is given by the Esakia dual A_* of A and R is defined as xRy iff $\Box a \in x$ implies $a \in y$ for all $a \in A$. Conversely, if X is an Esakia space then $\Box_R$ is a nucleus on the clopen upsets of X.

The semantics on nuclear spaces should not come as a surprise. We can just take the semantics on Esakia spaces and add a clause for $\Box$. Evidently, the clause is given by standard universal statement of R-successors of a point, which we have already seen for lax frames, i.e., $x \Vdash \Box\varphi$ iff xRy implies $y \vDash \varphi$. Thus, similar to Esakia spaces and intuitionistic Kripke frames, finite nuclear spaces and finite lax frames coincide. Consequently, we obtain the following completeness result.

Theorem 3.4.5. PLL is sound and complete with respect to finite lax frames, finite Goldblatt frames, finite S-frames, and finite FM frames.

Proof. Suppose $\mathsf{PLL} \nvdash \varphi$. By Theorem 3.3.8, there exists a finite nuclear algebra A such that $A \nvDash \varphi$. By duality, there is a finite nuclear space $X \nvDash \varphi$. Then, X is also a finite lax frame that refutes φ. Hence, PLL is complete with respect to finite lax frames. Completeness with respect to the other classes of finite frames follows by Theorem 3.3.31. ■

Furthermore, by duality we obtain completeness for all lax logics with respect to their nuclear spaces. Given a lax logic L we define the *class of* L*-spaces* as the class of nuclear spaces validating L, and conversely the logic of a class of nuclear spaces is also defined as expected.

Theorem 3.4.6. Every lax logic L is sound and complete with respect to the class of L-spaces.

Proof. Suppose $\mathsf{L} \nvdash \varphi$. By algebraic completeness there exists an L-algebra A that refutes φ. Then by duality, A_* is an L-space that refutes φ. ∎

Since nuclear spaces are lax frames it makes sense to say that nuclear frames are descriptive lax frames. Moreover, since every lax frame is a Goldblatt frame, they also describe Goldblatt frames. In this sense descriptive lax frames and descriptive Goldblatt frames coincide. In the next section we will discuss descriptive S-frames.

S-spaces

We have already shown that S-frames have equivalent lax frames in Lemma 3.3.20 by defining a lax relation $S_{\between}$. This way to induce relations from subsets of posets comes from the correspondence between nuclear algebras and subframes on Esakia spaces. A *subframe* of an Esakia space X is a subset $Y \subseteq X$ such that

- Y is an Esakia space in the induced topology and order, and
- $U \in \mathrm{ClopUp}(Y)$ implies $X \setminus \downarrow(Y \setminus U) \in \mathrm{ClopUp}(X)$.

We will discuss subframes in more detail in Section 5.1. Bezhanishvili and Ghilardi showed in [11] that subframes on an Esakia space are in a one-to-one correspondence with nuclear relations on that space. Concretely, if S is a subframe of X then X is a nuclear space with the relation given by $S_{\between}$, and conversely if X is a nuclear space then its R-reflexive points $R_{\triangle}$ form a subframe. Therefore, we can see nuclear spaces as pairs of Esakia spaces (X, Y) where Y is a subframe of X. We will call these pairs *S-spaces.* This name is due [31]. Just as in S-frames, the "S" in S-spaces stands for subframe.

It is easy to see that the reflexive points of $Y_{\between}$ are exactly Y, i.e., $(Y_{\between})_{\triangle} = Y$. Moreover, for any nuclear space $(R_{\triangle})_{\between} = R$.

Lemma 3.4.7. Let X be a modal Esakia space. Then the following are equivalent

(1) X is nuclear;

(2) There is some $S \subseteq X$ such that $R = S_{\between}$;

(3) $y \in R[x]$ iff there exists $z \in X$ such that $x \leq z \leq y$ and zRz.

Proof. (1) $\Rightarrow$ (2). Let $S = R_{\triangle}$. Suppose xRy. Since $R[x] \in \mathrm{Cl}(X)$, $\downarrow y \in \mathrm{Cl}(X)$ and $R[x] \cap \downarrow y \neq \varnothing$, there exists $z \in \min(R[x] \cap \downarrow y)$. Then since R is dense we must have $xRz'Rz$, but then $z' \leq z$, so $z' \in R[x] \cap \downarrow y$. Since z is minimal we have $z = z'$. Therefore, $z \in S$ and $x \leq z \leq y$. Conversely, suppose $x \leq z \leq y$ and $z \in S$. Then zRz and by Lemma 3.4.2 we get xRy.

(2) $\Rightarrow$ (3). It suffices to show that $z \in S$ implies zRz. By assumption, $R[z] = {\uparrow}(S \cap {\uparrow}z) = {\uparrow}z$. Whence, $z \in {\uparrow}z = R[z]$.

(3) $\Rightarrow$ (1). Suppose xRy. Then there exists $z \in X$ such that $x \leq z \leq y$ and zRz. Hence, $x \leq y$, which means $R \subseteq \leq$. Besides, $x \leq zRz$ implies xRz by Lemma 3.4.2, and similar for zRy. Hence, R is dense. ■

Note that [11] defines nuclear spaces using the condition of Lemma 3.4.7(3) instead of the last two conditions of Definition 3.4.1. While Lemma 3.4.7 gives an indication of the correspondence between subframes and nuclear relations on an Esakia space, we have not shown that $R_\triangle$ is a subframe. The complete proof is in [11] but is too involved and requires concepts outside the scope of this thesis.

Theorem 3.4.8 ([11, Theorem 28])**.** Let X be a nuclear space. There is a one-to-one correspondence between nuclear relations on X and subframes of X. Explicitly, $R_\triangle$ is a subframe for any nuclear relation R, and $S_\between$ is a nuclear relation for any subframe S.

Semantics for S-spaces relate to S-frame semantics in the expected way: we use the same clauses but restrict valuations to clopen upsets.

Corollary 3.4.9. Let $X_R = (X, \leq, R)$ be a nuclear space and $X_S = (X, S)$ the corresponding subframe. Then $X_R, x \Vdash_v \varphi$ iff $X_S, x \Vdash_v \varphi$ for $x \in X$, all formulas φ, and all valuations v.

Proof. By assumption, $S = R_\triangle$ and $R = S_\between$. Let $x \in X$ and $v : \mathsf{Prop} \to \mathrm{ClopUp}(X)$ be some valuation. We prove the claim by induction on the complexity of φ. The only interesting case is $\Box$. Suppose $X_R, x \Vdash \Box\varphi$ and $x \leq y$ such that $y \in S$. Then $x \leq yRy$, and by Lemma 3.4.2, xRy. Thus, $X_R, y \Vdash \varphi$ and by the IH, $X_S, y \Vdash \varphi$, which menas $X_S, x \Vdash \Box\varphi$. Conversely, suppose $X_S, x \Vdash \Box\varphi$ and xRy. By Lemma 3.4.7(3), there exists $s \in R_\triangle = S$ such that $x \leq s \leq y$. Then $X_S, s \Vdash \varphi$. By the IH, $X_R, s \Vdash \varphi$, and by persistence it follows that $X_R, y \Vdash \varphi$. We can conclude $X_R, x \Vdash \Box\varphi$. ■

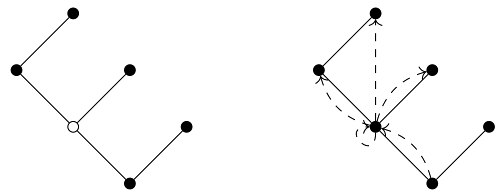
By Theorem 3.4.8 and Corollary 3.4.9 we can think of nuclear spaces and S-spaces and their semantics as the same thing, and can freely move between the two. After this section we will no longer distinguish between the two explicitly. We will now use this correspondence to give a proof of the fact that closed upsets of nuclear spaces induce nuclear spaces, similar to the Esakia case.

Lemma 3.4.10. Let X be a nuclear space. If $Y \in \mathrm{ClUp}(X)$ then Y is a nuclear space in the induced topology and order.

Proof. By the correspondence it suffices to show that $(Y, R_\triangle \cap Y)$ is an S-space. That is, we need to show that $R_\triangle \cap Y$ is a subframe of Y. Since $R_\triangle$ is a subframe of X we have $R_\triangle \in \mathrm{Cl}(X)$, and therefore $R_\triangle \cap Y \in \mathrm{Cl}(Y)$. Suppose $U \in \mathrm{Clop}(Y \cap R_\triangle)$. Then $U = R_\triangle \cap V$ for some $V \in \mathrm{Clop}(Y)$. Then $V = W \cap Y$ for some $W \in \mathrm{Clop}(X)$. Then $W \cap R_\triangle \in \mathrm{Clop}(R_\triangle)$, whence ${\downarrow}(W \cap R_\triangle) \in \mathrm{Clop}(X)$. Then ${\downarrow}(W \cap R_\triangle) \cap Y \in \mathrm{Clop}(Y)$.

Note, $\downarrow_Y U = \downarrow(R_\triangle \cap W \cap Y) \cap Y$. Then it is easy to see that $\downarrow_Y U \subseteq \downarrow(W \cap R_\triangle) \cap Y$. Conversely, suppose $y \in \downarrow(W \cap R_\triangle) \cap Y$. Then $y \in Y$ and there exists $x \in W \cap R_\triangle$ such that $y \leq x$. But Y is an upset, so $x \in Y$. ■

Example 3.4.11 (Drawing conventions of nuclear spaces)**.** Below there are two depictions of a nuclear space. On the left we see it represented by marking the corresponding subframe with white points ($\circ$), and on the right we see the modal relation R drawn (more) explicitly with dashed arrows.



Note that R is transitive, e.g., everything is an R-successor of the root except the most right point. Observe that the white point corresponds exactly with the reflexive point of R, and points are R-accessible from some world iff the white point lies between them. While there might be more succinct ways to draw R explicitly, it is apparent that drawing nuclear spaces as on the left is a more concise and less cluttered way to depict them as opposed to the right. ⊣

Since a subframe is an Esakia space with the induced topology and order, the subframe in an S-space can be seen as an *inner* Esakia space contained in an *outer* Esakia space. Therefore, given an S-space (X, S) we call the Esakia space induced by S the *inner space* of the S-space, and X the *outer space* of the S-space. Moreover, we do the same for nuclear spaces, i.e., $R_\triangle$ is the inner space of a nuclear space X. Algebraically, this relates to the fact that $A_\Box$ is a Heyting algebra for each nuclear algebra A, and it uniquely determines $\Box$.

In this section we have seen two descriptive semantics while there were four Kripke-style semantics in the previous section. The reason for this is two-fold. First, as already mentioned descriptive IK-frames and descriptive Goldblatt frames coincide. Second, we have skipped descriptive FM-frames. Descriptive FM-frames have not been considered in the literature, and we will leave defining them as an open problem. It is likely they can be described with bimodal spaces but such a representation is outside the scope of this thesis. In short, we will work with two descriptive semantics: nuclear semantics and S-space semantics.

3.5 Translations

In this section we will discuss translations from IPC into PLL. Lax modalities that are definable in IPC provide a translation from IPC into IPC via Gödel-Gentzen style translations, see [1]. The most well known being the Gödel-Gentzen negative translation.

In [10], subframisations[1] of intermediate logics were studied. Bezhanishvili, Bezhanishvili and Ilin characterised subframisations by generalising Gödel-Gentzen style translations to arbitrary lax modalities. This translation then maps an intermediate logic L to the lax logic determined by the class of nuclear spaces whose inner spaces validate L. Another translation they use is of a very simple nature; since PLL contains IPC the identity provides a very direct translation from IPC to PLL. This can be seen as translating logics to the lax logic of the class of S-spaces whose outer spaces validate Ł. Thus, these two translations have an interesting connection to nuclear spaces.

First, we will define the *outer space translation.* On formulas this translation is simply defined as the identity from $\mathcal{L}_{\mathsf{p}}$ into $\mathcal{L}_{\Box}$. For logics, we can then define an embedding $(\_)^\bullet$ from the lattice of intermediate logics into the lattice of lax logics as follows. We define the *outer space embedding* $\mathsf{L}^\bullet := \mathsf{PLL} \oplus \{\varphi \in \mathcal{L}_{\Box} \mid \varphi \in \mathsf{L}\}$ for each intermediate logic L. Since $\mathsf{L}^\bullet$ is axiomatised by $\Box$-free formulas membership of the class of $\mathsf{L}^\bullet$-algebras is completely determined by the Heyting reducts of nuclear algebras. Dually, membership of the class of $\mathsf{L}^\bullet$-spaces is determined by the outer spaces. Indeed, we have the following, see also [10, Lemma 6.4(1) and Remark 6.5(1)].

Lemma 3.5.1. Suppose $\varphi \in \mathcal{L}_{\mathsf{p}}$, X a nuclear space, and A a nuclear algebra. Then

(1) $X \Vdash \varphi$ iff $X' \Vdash \varphi$, where X' is the outer space of X.

(2) $A \Vdash \varphi$ iff $A' \Vdash \varphi$, where A' is the Heyting reduct of A.

Proof. Follows trivially since φ is $\Box$-free. ■

Consequently, we have obtained the following theorem.

Theorem 3.5.2. Let L be an intermediate logic, X a nuclear space, and A a nuclear algebra. Then

(1) X is an $\mathsf{L}^\bullet$-space iff X' is an L-space, where X' is the outer space of X.

(2) A is an $\mathsf{L}^\bullet$-algebra iff A' is an L-algebra, where A' is the Heyting reduct of A.

Thus, embedding logics into the lattice of lax logics via the outer space translation gives lax logics that are completely determined by the outer spaces of nuclear spaces. Therefore, we call $\mathsf{L}^\bullet$ the *outer space logic* of L. Naturally, we can ask ourselves whether something similar can be done for inner spaces. Actually, this is precisely what Gödel-Gentzen style translations do. The *inner space* translation $(\_)^\circ : \mathcal{L}_{\mathsf{p}} \to \mathcal{L}_{\Box}$ is defined

[1] The up- and downward subframisations of a logic L are the least subframe logic containing L and the greatest subframe logic contained in L respectively.

recursively as

$$
\begin{aligned}
p^\circ &:= \Box p \\
\bot^\circ &:= \Box \bot \\
(\varphi \wedge \psi)^\circ &:= \varphi^\circ \wedge \psi^\circ \\
(\varphi \vee \psi)^\circ &:= \Box(\varphi^\circ \vee \psi^\circ) \\
(\varphi \to \psi)^\circ &:= \varphi^\circ \to \psi^\circ
\end{aligned}
$$

for all $\varphi \in \mathcal{L}_{\mathsf{p}}$. Observe that this resembles closely the fact that $A_\Box$ is a Heyting algebra for each nuclear algebra A, see Proposition 3.3.10. Note that the inner space translation can equivalently be defined in a Kuroda-style manner, see [4, Proposition 3]. Similar to the outer space translation, the inner space translation incites an embedding from the lattice of intermediate logics into the lattice of lax logics. We define the *outer space embedding* as $\mathsf{L}^\circ := \mathsf{PLL} \oplus \{\varphi^\circ \mid \varphi \in \mathsf{L}\}$ for each intermediate logic L. We get analogues of Lemma 3.5.1 and Theorem 3.5.2, see [10, Lemma 6.4(2) and Remark 6.5(2)] for proofs.

Lemma 3.5.3. Suppose $\varphi \in \mathcal{L}_{\mathsf{p}}$, X a nuclear space, and A a nuclear algebra. Then

(1) $X \Vdash \varphi^\circ$ iff $R_\triangle \Vdash \varphi$.

(2) $A \Vdash \varphi^\circ$ iff $A_\Box \Vdash \varphi$.

Theorem 3.5.4. Let L be an intermediate logic, X a nuclear space, and A a nuclear algebra. Then

(1) X is an L°-space iff $R_\triangle$ is an L-space.

(2) A is an L°-algebra iff $A_\Box$ is an L-algebra.

Thus, the the inner space embedding gives lax logics determined by inner spaces. For more details on the inner and outer space translations we refer to [10] and [31, Section 6.5.2]. We will return to them in Section 5.3.

In this chapter we have discussed lax logics. First, we covered their history and appearances in the literature. Next, we defined them syntactically. Then we gave semantics in the form of nuclear algebras, and a few Kripke-style semantics. Moreover, we gave descriptive counterparts and recalled nuclear duality and the correspondence with S-spaces. Finally, we covered the translations from the propositional language to the modal language arising from subframisations of intermediate logics. In the next chapter we finally enter the setting of canonical formulas.

Chapter 4

Lax Canonical Formulas

We already explained the strengths of canonical formulas in the introduction. In short, they provide powerful axiomatisation methods for intermediate and modal logics, and they demonstrate the structural interrelationships of intermediate logics and their modal companions. As stated before, a key ingredient of canonical formulas is a suited locally finite reduct. The recent proof of Diego's Theorem for nuclear implicative semilattices [9] gives us a satisfactory reduct for nuclear algebras. We will use this result in this chapter to develop canonical formulas for lax logics. We will show that we can axiomatise all lax logics in this manner, and moreover we can classify lax logics with simpler canonical formulas if they are axiomatised with a limited syntax.

In Section 4.1 we will recall canonical formulas for intermediate logics, and in Section 4.2 we will discuss the generalised Esakia duality developed in [5]. Next, in Section 4.3 we construct canonical formulas for lax logic. Section 4.4 introduces generalised nuclear Esakia duality to account for lax canonical formulas dually.

4.1 Intuitionistic case

In this section we recall the details of canonical formulas for intuitionistic logic. For more details we refer to [5, 7, 8] and [17, Chapter 9].

Since we are dealing with many different variants of canonical formulas we will define a general framework to deal with systematically. All canonical formulas in this thesis will be formed as follows.

Definition 4.1.1 (Algebra-based formulas)**.** Let $\mathcal{F} \supseteq \mathcal{I}$ be an algebraic type extending implicative semilattices $\mathcal{I} = \{\wedge, \top, \to\}$, and let A be a finite $\mathcal{F}$-algebra that is an implicative lattice and has a second largest element s. For any $a \in A$ we select a fresh variable $p_a \in \mathsf{Prop}$. Then for any n-ary $f \in \mathcal{F}$ and $D \subseteq A^n$ we define

$$\Gamma^f_D := \bigwedge\{f(p_a, \ldots, p_{a_n}) \leftrightarrow p_{f(a_1,\ldots,a_n)} \mid (a_1, \ldots, a_n) \in D\}.$$

and

$$\Lambda^f_D := \bigwedge\{f(p_a, \ldots, p_{a_n}) \to p_{f(a_1,\ldots,a_n)} \mid (a_1, \ldots, a_n) \in D\}.$$

An *A-based formula* is a formula of the form

$$\big(\Gamma^{\wedge}_{A^2} \wedge \Gamma^{\to}_{A^2} \wedge \bigwedge_{f\in\mathcal{F}} \Gamma^f_{D_f} \wedge \bigwedge_{f\in\mathcal{F}} \Lambda^f_{E_f}\big) \to p_s. \qquad \dashv$$

Thus, an A-based formula is an implication with a big conjunction which encodes the complete $\mathcal{I}$-structure of A and the remaining structure only partially for some parameter sets. The intention of the consequent is to assure that the formula is not prematurely refuted: it forces refuting valuations to be injections. Nevertheless, the exact construction of such formulas is not important. What actually matters is the refutation criterion for such formulas. The *Refutation Lemma* gives refutation criteria for many algebra-based formulas with respect to Heyting algebras and nuclear algebras. It generalises [5, Theorem 5.3 and Corollary 5.15(1)] and we will prove it in essentially the same way. First we show that every A-based formula is refuted on A, shadowing [5, Lemma 5.2]. Indeed, an algebra itself should be the prime example of something which refutes its geometric patterns.

Lemma 4.1.2. If φ is an A-based formula then $A \not\models \varphi$.

Proof. Let Γ be the set of implications such that $\varphi = \bigwedge\Gamma \to p_s$, and let v be a valuation such that $v(p_a) = a$ for all $a \in A$. Suppose $\gamma \in \Gamma$. Then either $\gamma = f(p_{a_1}, \ldots, p_{a_n}) \to p_{f(a_1,\ldots,a_n)}$ or $\gamma = p_{f(a_1,\ldots,a_n)} \to f(p_{a_1}, \ldots, p_{a_n})$. Let $b = f(a_1, \ldots, a_n)$. Then $p_{f(a_1,\ldots,a_n)} = p_b$. Whence, $v(p_{f(a_1,\ldots,a_n)}) = b$. Besides, $v(f(p_{a_1}, \ldots, p_{a_n})) = f(v(p_{a_1}), \ldots, v(p_{a_n}))) = f(a_1, \ldots, a_n) = b$. Thus either way $v(\gamma) = b \to b = \top$. Consequently, $v(\bigwedge\Gamma) = \top$ and $v(p_s) = s \neq \top$, which means $v(\varphi) = \top \to s = s \neq \top$. We can conclude $A \not\models \varphi$. ∎

Observe that the valuation that refutes the algebra-based formula maps propositional variables to the elements that they are labelled by. We call such a valuation *natural*. Thus, the proof of Lemma 4.1.2 shows a stronger result.[1]

Porism 4.1.3. A refutes all A-based formulas with a natural valuation.

Lemma 4.1.4. Let A be a nuclear algebra and $a, b \in A$ such that $a \not\leq b$. Then there exists an s.i. nuclear algebra B and an onto homomorphism $h : A \to B$ such that $h(b) = s$ and $h(a) = \top$, where s is the second largest element of B.

Proof. By [49, Lemma 1], the claim holds for Heyting algebras. But then dually we have some closed upset Y of a nuclear space X. Then by Lemma 3.4.10, Y is also a nuclear space. ∎

[1] We call direct consequences of proofs “porisms” in this thesis.

We are now ready to prove the Refutation Lemma. It is the first of two very technical lemmas in this chapter. However, it will provide a quite useful tool in the remainder of this thesis. The proof is an obvious generalisation of the proof of [5, Theorem 5.3].

Lemma 4.1.5 (Refutation Lemma)**.** Let φ be an A-based formula for some algebra A.

(1) If A is a Heyting algebra then for any Heyting algebra B, we have $B \nvDash \varphi$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding $h : A \to C$ that is

  (i) f-compatible over D_f, and

  (ii) $f(h(a_1), \ldots, h(a_n)) \leq h(f(a_1, \ldots, a_n))$ for all $(a_1, \ldots, a_n) \in D_f \cup E_f$,

  for all $f \in \mathcal{H}$.

(2) If A is a nuclear algebra then for any nuclear algebra B, we have $B \nvDash \varphi$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding $h : A \to C$ which is

  (i) f-compatible over D_f, and

  (ii) $f(h(a_1), \ldots, h(a_n)) \leq h(f(a_1, \ldots, a_n))$ for all $(a_1, \ldots, a_n) \in D_f \cup E_f$,

  for all $f \in \mathcal{N}$.

Proof. The following applies to both (1) and (2). Let Γ be the set of implications such that $\varphi = \bigwedge \Gamma \to p_s$.

($\Rightarrow$) Suppose $B \nvDash \varphi$ with a valuation $v : \mathsf{Prop} \to B$. Then there exists a homomorphic image C of B such that the corresponding homomorphism $h : B \to C$ satisfies $h(v(\bigwedge \Gamma)) = \top$ and $h(v(p_s)) \neq \top$ by Lemma 4.1.4. Since

$$h(v(\bigwedge \Gamma)) = \bigwedge_{\gamma \in \Gamma} h(v(\gamma)) = \top$$

we must have $h(v(\gamma)) = \top$ for all $\gamma \in \Gamma$. Let $g : A \to C$ be defined as $g(a) = h(v(p_a))$ for all $a \in A$. Then for all $\gamma \in \Gamma$ we have $g(\gamma) = h(v(\gamma)) = \top$. We will now show that g is f-compatible over D_f. For simplicity, we will assume $f = \vee$. Thus, suppose $(a, b) \in D_\vee$. Then $(p_a \vee p_b) \to p_{a\vee b} \in \Gamma$. Hence,

$$(ga \vee gb) \to g(a \vee b) = (h(v(p_a)) \vee h(v(p_b))) \to h(v(p_{a\vee b})) = h(v((p_a \vee p_b) \to p_{a\vee b})) = \top.$$

Thus, $(ga \vee gb) \leq g(a \vee b)$. The converse follows similarly since $p_{a\vee b} \to (p_a \vee p_b) \in \Gamma$. Thus, g is $\vee$-compatible over $D_\vee$. We have shown (i), and moreover we have shown that g is $\wedge$-compatible and $\to$-compatible, which means it is an $\mathcal{I}$-homomorphism. Similarly, we can show (ii). Remains to show that g is an embedding. Suppose $a, b \in A$ such that $a \neq b$. Then w.l.o.g. $a \nleq b$. Hence $a \to b \neq \top$, and therefore $(a \to b) \to s = \top$. Then

$$(ga \to gb) \to gs = g((a \to b) \to s) = g(\top) = h(v(\top)) = \top,$$

which menas $ga \nleq gb$, as required.

($\Leftarrow$) Suppose there is a homomorphic image C of B and an $\mathcal{F}$-embedding $h : A \to C$ that satisfies (i) and (ii) for all $f \in \mathcal{F}$. By Porism 4.1.3, a natural valuation $v_A : \mathsf{Prop} \to A$ refutes φ on A. We define a new valuation $v_C := h \circ v_A$, in other words $v_C(p_a) = h(a)$ for all $a \in A$. Suppose that $\gamma \in \Gamma$. There are two cases:

1. $\gamma = p_{f(a_1,\ldots,a_n)} \to f(p_{a_1}, \ldots, p_{a_n})$ and $(a_1, \ldots, a_n) \in D_f$. Then

$$\begin{aligned} v_C(p_{f(a_1,\ldots,a_n)}) &= h(f(a_1, \ldots, a_n)) \\ &= f(h(a_1), \ldots, h(a_n)) && \text{by (i)} \\ &= f(v_C(p_{a_1}), \ldots, v_C(p_{a_n})) \\ &= v_C(f(p_{a_1}, \ldots, p_{a_n})) \end{aligned}$$

   Thus, $v_C(\gamma) = v_C(p_{f(a_1,\ldots,a_n)}) \to v_C(f(p_{a_1}, \ldots, p_{a_n})) = \top$.

2. $\gamma = f(p_{a_1}, \ldots, p_{a_n}) \to p_{f(a_1,\ldots,a_n)}$ and $(a_1, \ldots, a_n) \in D_f \cup E_f$. Then

$$\begin{aligned} v_C(f(p_{a_1}, \ldots, p_{a_n})) &= f(v_C(p_{a_1}), \ldots, v_C(p_{a_n})) \\ &= f(h(a_1), \ldots, h(a_n)) \\ &\leq h(f(a_1, \ldots, a_n)) && \text{by (ii)} \\ &= v_C(p_{f(a_1,\ldots,a_n)}) \end{aligned}$$

   Thus, $v_C(\gamma) = v_C(f(p_{a_1}, \ldots, p_{a_n})) \to v_C(p_{f(a_1,\ldots,a_n)}) = \top$.

Consequently, $v_C(\bigwedge \Gamma) = \top$, and therefore $v_C(\varphi) = \top \to s \neq \top$, which means $C \nvDash \varphi$. Since C is a homomorphic image of B we get $B \nvDash \varphi$. ■

The Refutation Lemma shows that an algebra B refutes an A-based formula iff there is a suited embedding of A into a homomorphic image of B. Canonical formulas are the algebra-based formulas that fully encode the structure of a locally finite reduct.

Definition 4.1.6 (Canonical Formulas)**.** Let A be a finite s.i. Heyting algebra with the second largest element s, and $D \subseteq A^2$. The *canonical formula* $\alpha(A, D, \bot)$ is an A-based formula defined as

$$\alpha(A, D, \bot) := \left(\Gamma^{\wedge}_{A^2} \wedge \Gamma^{\bot}_{A^0} \wedge \Gamma^{\to}_{A^2} \wedge \Gamma^{\vee}_{D}\right) \to p_s. \qquad \dashv$$

Recall that $\hat{\mathcal{I}} = \{\wedge, \top, \bot, \to\}$ is the algebraic type of bounded implicative semilattices, and this is a locally finite reduct of Heyting algebras, see Theorem 2.2.5. Thus, canonical formulas encode the locally finite $\hat{\mathcal{I}}$-reducts of Heyting algebras, but $\vee$ only for some parameter set D. We will now instantiate the Refutation Lemma to obtain the refutation criterion for canonical formulas.

Theorem 4.1.7. Let B be a Heyting algebra and $\alpha(A, D, \bot)$ a canonical formula. Then $B \nvDash \alpha(A, D, \bot)$ iff there exists a homomorphic image C of B and an $\hat{\mathcal{I}}$-embedding $h : A \to C$ which is $\vee$-compatible over D, i.e., $ha \vee hb = h(a \vee b)$ for all $(a, b) \in D$.

Proof. By the Refutation Lemma, $B \nvDash \alpha(A, D, \bot)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding $h : A \to C$ which is

- $\bot$-compatible over A_0, and
- $\vee$-compatible over D.

Evidently, an $\mathcal{I}$-embedding that $\bot$-compatible over A_0 is an $\hat{\mathcal{I}}$-embedding. ■

Using Theorem 4.1.7 and the fact that the $\hat{\mathcal{I}}$-reduct of Heyting algebras is locally finite we are able to represent every formula φ by a conjunction of canonical formulas. Formally speaking every formula is semantically equivalent to a conjunction of canonical formulas. Essentially, we look at the collection of finite s.i. algebras that refute φ that are $n(\varphi)$-generated as $\hat{\mathcal{I}}$-algebras, where $n(\varphi)$ is a bound based on the complexity of φ. For this purpose we will now prove the second technical lemma of this chapter, the *Representation Lemma.* The name is inspired by the fact that for each formula it provides us with an equivalent conjunction of algebra-based formulas. In turn, we obtain a geometric representation of the formula by duality. Illustratively, this can be seen in Tables 5.1.10, 5.2.32, and 5.2.40. However, before we get there we start with an algebraic representation.

Lemma 4.1.8 (Representation Lemma)**.** Let $\mathcal{F}$ be some algebraic subtype of nuclear algebras not containing $\to, \wedge, \vee$, and let $\mathcal{F}' = \mathcal{F} \cup \{\vee\}$.[1]

(1) If $\varphi \in \mathcal{L}_{\mathsf{p}}$ is an $\mathcal{F}'$-free formula then there exists a finite set $\mathscr{C}(\varphi)$ of algebra-based formulas of the form
$$\psi(A) = \bigwedge_{f \in \mathcal{H} \setminus \mathcal{F}'} \Gamma^f_{A^{\sigma(f)}} \to p_s$$
such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ are equivalent for all Heyting algebras B.

(2) If $\varphi \in \mathcal{L}_{\mathsf{p}}$ is an $\mathcal{F}$-free formula Then there exists a finite set $\mathscr{C}(\varphi)$ of algebra-based formulas of the form
$$\psi(A, D) = \big(\Gamma^{\vee}_D \wedge \bigwedge_{f \in \mathcal{H} \setminus \mathcal{F}'} \Gamma^f_{A^{\sigma(f)}}\big) \to p_s$$
such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ are equivalent for all Heyting algebras B.

(3) If $\varphi \in \mathcal{L}_{\Box}$ be an $\mathcal{F}'$-free formula then there exists a finite set $\mathscr{C}(\varphi)$ of algebra-based formulas of the form
$$\psi(A) = \bigwedge_{f \in \mathcal{N} \setminus \mathcal{F}'} \Gamma^f_{A^{\sigma(f)}} \to p_s$$
such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ are equivalent for all nuclear algebras B.

[1] Recall that $\mathcal{H}$ and $\mathcal{N}$ are the algebraic types of Heyting algebras and nuclear algebras.

(4) If $\varphi \in \mathcal{L}_\Box$ be an $\mathcal{F}$-free formula then there exists a finite set $\mathscr{C}(\varphi)$ of algebra-based formulas of the form

$$\psi(A, D) = \big(\Gamma_D^\vee \wedge \bigwedge_{f \in \mathcal{N} \setminus \mathcal{F}'} \Gamma^f_{A^{\sigma(f)}}\big) \to p_s$$

such that $B \models \varphi$ iff $B \models \bigwedge \mathscr{C}(\varphi)$ are equivalent for all nuclear algebras B.

Proof. The only difference between the nuclear and Heyting cases is whether we use the local finiteness of bounded implicative semmilattices, Theorem 2.2.5, or the local finiteness of nuclear bounded implicative semillatices, Theorem 3.3.6. Moreover, all the cases are proved in a similar way. Hence, we will only show one case.

(4) Let Γ be the set of all subformulas of φ and $n = |\Gamma|$. By Theorem 3.3.6 there are only finitely many s.i. nuclear algebras A that are at most n-generated as a bounded nuclear implicative semilattice and refute φ. Denote the set of these algebras $\mathcal{A}$. Given a valuation $v : \mathsf{Prop} \to A$, let $\delta(\Gamma, v) := \{v(\chi_1), v(\chi_2) \mid \chi_1 \vee \chi_2 \in \Gamma\}$. Then we let

$$\mathscr{C}(\varphi) := \{\psi(A, \delta(\Gamma, v)) \mid A \in \mathcal{A} \text{ and } A \text{ refutes } \varphi \text{ with } v\}.$$

Next, suppose B is a nuclear algebra. Suppose there is some $\psi(A, \delta(\Gamma, v)) \in \mathscr{C}(\varphi)$ such that $B \not\models \psi(A, \delta(\Gamma, v))$. By the Refutation Lemma, there exists a homomorphic image C of B and an $\mathcal{I}$-embedding $h : A \to C$ which is f-compatible for all $f \in \mathcal{H} \setminus \mathcal{F}'$, and $\vee$-compatible over $\delta(\Gamma, v)$. Let $v' : \mathsf{Prop} \to C$ be defined as $v' := h \circ v$. We will prove by induction that $v'(\psi) = h(v(\psi))$ for all $\psi \in \Gamma$.

- The atomic case is trivial.
- Let $\psi = \psi_1 \vee \psi_2$.ince $\psi_1 \vee \psi_2 \in \Gamma$ we have $(v(\psi_1), v(\psi_2)) \in \delta(\Gamma, v)$. Then

$$\begin{aligned} h(v(\psi_1 \vee \psi_2)) &= h(v(\psi_1) \vee v(\psi_2)) \\ &= h(v(\psi_1)) \vee h(v(\psi_n)) && \text{since } h \text{ is } \vee\text{-compatible over } \delta(\Gamma, v) \\ &= v'(\psi_1) \vee v'(\psi_n) && \text{by IH} \\ &= v'(\psi_1 \vee \psi_n). \end{aligned}$$

- Let $\psi = f(\psi_1, \ldots, \psi_n)$ for $f \in \mathcal{F} \setminus \{\vee\}$ of arity n. Then $f \notin \mathcal{G}$ since ψ is a subformula of φ and φ is $\mathcal{G}$-free, and therefore h is f-compatible. Then

$$\begin{aligned} h(v(f(\psi_1, \ldots, \psi_n))) &= h(f(v(\psi_1), \ldots, v(\psi_n))) \\ &= f(h(v(\psi_1)), \ldots, h(v(\psi_n))) && \text{since } h \text{ is } f\text{-compatible} \\ &= f(v'(\psi_1), \ldots, v'(\psi_n)) && \text{by IH} \\ &= v'(f(\psi_1, \ldots, \psi_n)). \end{aligned}$$

In particular, we obtain $v'(\varphi) = h(v(\varphi)) \neq \top$ since h is one-to-one. Therefore, $C \not\models \varphi$, which means $B \not\models \varphi$ since validity is preserved through homomorphic images.

Conversely, suppose $B \nvDash \varphi$. Then there exists a s.i. homomorphic image C of B such that $C \nvDash \varphi$ by Lemma 4.1.4. By Nuclear Selective Filtration, there is an n-generated $\hat{\mathcal{J}}$-subalgebra A of C, which is a finite s.i. algebra such that $A \nvDash \varphi$. Then there is a witnessing valuation $v : \mathsf{Prop} \to A$ refuting φ. We get $\psi(A, \delta(\Gamma, v)) \in \mathscr{C}(\varphi)$. Moreover, $h : A \to C$ defined as the identity is a $\hat{\mathcal{J}}$-embedding which is $\vee$-compatible over $\psi(A, \delta(\Gamma, v))$. By the Refutation Lemma $B \nvDash \psi(A, \delta(\Gamma, v))$, which means $B \nvDash \bigwedge \mathscr{C}(\varphi)$, as required. ∎

The Representation Lemma is a generalisation of [5, Theorem 5.7]. Importantly, we use Nuclear Selective Filtration to extend the idea to the lax case. For now, let us apply it to canonical formulas.

Theorem 4.1.9. For any formula φ there exists a finite set of canonical formulas $\mathscr{C}(\varphi)$ such that φ and $\bigwedge \mathscr{C}(\varphi)$ are equivalent.

Proof. We instantiate the Representation Lemma with $\mathcal{F} = \varnothing$. Then we have that $\psi(A, D)$ equals $\alpha(A, D, \bot)$ (modulo exploiting commutativity of $\wedge$). Hence we have the required set of canonical formulas. ∎

The previous theorem gives the desired uniform axiomatisation method.

Theorem 4.1.10. Every intermediate logic is axiomatisable by canonical formulas.

Proof. Let L be a logic. Then it is axiomatised by some set $\Gamma \subseteq \mathcal{L}_{\mathsf{p}}$. By Theorem 4.1.9, for each $\varphi \in \Gamma$ we have a set $\mathscr{C}(\varphi)$ of canonical formulas such that $\bigwedge \mathscr{C}(\varphi)$ and φ are equivalent. Thus, $B \nvDash \varphi$ iff $B \nvDash \alpha(A, D, \bot)$ for some $\alpha(A, D, \bot) \in \mathscr{C}(\varphi)$. Hence, by algebraic completeness, $\mathsf{IPC} \oplus \Gamma = \mathsf{IPC} \oplus \bigcup\{\mathscr{C}(\varphi) \mid \varphi \in \Gamma\}$. ∎

Moreover, a closer inspection of the Representation Lemma shows that we can axiomatise $\bot$-free and/or $\vee$-free formulas with even simpler algebra-based formulas. Namely, we can drop respectively the $\Gamma^{\top}$ and $\Gamma^{\vee}$ conjunction from the canonical formulas. The latter we can already emulate with normal canonical formulas by putting $D = \varnothing$, and we write $\alpha(A, \bot)$ for $\alpha(A, \varnothing, \bot)$. We call formulas of this form *cofinal subframe formulas.* The name is justified by their peculiar connection to cofinal subframe logics, as seen in the next chapter. Historically, cofinal subframe formulas (and subframe formulas) precede canonical formulas as they were used to axiomatise (cofinal) subframe logics. For the $\bot$-free logics we will now define *negation-free canonical formulas.*

Definition 4.1.11 (Negation-free canonical formulas)**.** Let A be a finite s.i. Heyting algebra with the second largest element s, and $D \subseteq A^2$. The *negation-free canonical formula* $\alpha(A, D)$ is an A-based formula defined as

$$\alpha(A, D) := \left(\Gamma^{\wedge}_{A^2} \wedge \Gamma^{\to}_{A^2} \wedge \Gamma^{\vee}_{D}\right) \to p_s. \qquad \dashv$$

Moreover, *subframe formulas* are exactly those negation-free canonical formulas with an empty parameter set, i.e., $\alpha(A) := \alpha(A, \varnothing)$ is called a *subframe formula*.

For all these formulas we can use the Refutation Lemma to obtain refutation criteria, and we can use the Representation Lemma to characterise the relevant $\mathcal{F}$-free formulas.

Theorem 4.1.12. Let B be a Heyting algebra, A a finite s.i. Heyting algebra and $D \subseteq A^2$. Then

(1) $B \nvDash \alpha(A, D)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding from A into C which is $\vee$-compatible over D.

(2) $B \nvDash \alpha(A, \perp)$ iff there exists a homomorphic image C of B and an $\hat{\mathcal{I}}$-embedding from A into C.

(3) $B \nvDash \alpha(A)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding from A into C.

Proof. The Refutation Lemma shows all three claims immediately. ■

Theorem 4.1.13. For every formula φ there exists a finite set $\mathscr{C}(\varphi)$ of

(1) negation-free canonical formulas if φ is $\perp$-free,

(2) cofinal subframe formulas if φ is $\vee$-free,

(3) subframe formulas if φ is $(\perp, \vee)$-free,

such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ for all Heyting algebras B.

Proof. We only show (3). The rest follows similarly. Let $\mathcal{F} = \{\perp\}$. Then in terms of the Representation Lemma, φ is a $\mathcal{F}'$-free formula. Hence, we have a set of algebra-based formulas of the form $\psi(A) = \Gamma^{\rightarrow}_{A^2} \wedge \Gamma^{\wedge}_{A^2} \rightarrow p_s$. But that is exactly the form of the subframe formula $\alpha(A)$. ■

Theorem 4.1.14. Let L be a logic.

(1) L is $\perp$-free iff it is axiomatisable by negation-free canonical formulas;

(2) L is $\vee$-free iff it is axiomatisable by cofinal subframe formulas;

(3) L is $(\perp, \vee)$-free iff it is axiomatisable by subframe formulas.

Proof. We can use the same proof as in Theorem 4.1.10 but instead of Theorem 4.1.9 we use the appropriate part of Theorem 4.1.13. ■

Formulas	Notation	$\Gamma^{\vee}$	$\Gamma^{\perp}$	Axiomatises
canonical	$\alpha(A, D, \perp)$	D	A_0	all logics
negation-free canonical	$\alpha(A, D)$	D	$\varnothing$	$\perp$-free logics
cofinal subframe	$\alpha(A, \perp)$	$\varnothing$	A_0	$\vee$-free logics
subframe	$\alpha(A)$	$\varnothing$	$\varnothing$	$(\perp, \vee)$-free logics

Table 4.1.15: Uniform axiomatisation methods for intermediate logics.

The canonical formulas of this section and they logics the axiomatise are summarised in Table 4.1.15. Logics axiomatised by these special kind of canonical formulas have interesting properties. For instance, cofinal subframe formulas axiomatise cofinal subframe logics, see [17, Section 11.3] and [11]. These are the logics closed under cofinal subframes. We will discuss this in detail in Section 5.1. Algebraically, these logics are closed under $\hat{\mathcal{I}}$-algebras. Moreover, they are exactly the logics generated by classes closed under $\hat{\mathcal{I}}$-algebras.

Theorem 4.1.16 ([8, Theorem 6.10])**.** For a logic L the following are equivalent.

(1) L is axiomatisable by cofinal subframe formulas;

(2) the class of L-algebras is closed under $\hat{\mathcal{I}}$-subalgebras;

(3) L is the logic of a class closed under $\hat{\mathcal{I}}$-subalgebras.

Proof. (Sketch) (1) $\Rightarrow$ (2) is established by the fact that cofinal subframe formulas are $\vee$-free. For (3) $\Rightarrow$ (1) we find the axiomatisation by $\Gamma := \{\alpha(A, \perp) \mid A$ is finite s.i. and $A \nvDash \mathsf{L}\}$. ∎

Similarly, the logics axiomatised by subframe formulas are those closed under $\hat{\mathcal{I}}$-subalgebras.

Theorem 4.1.17 ([8, Theorem 6.9])**.** For a logic L the following are equivalent.

(1) L is axiomatisable by subframe formulas;

(2) the class of L is closed under $\mathcal{I}$-subalgebras;

(3) L is the logic of a class closed under $\mathcal{I}$-subalgebras.

Consequently, being axiomatised by (cofinal) subframe formulas is a sufficient condition for fmp. Namely, a class being closed under $\mathcal{I}$-subalgebras implies that it is closed under using the Selective Filtration.

For logics axiomatised by negation-free canonical formulas we do not have a similar result. While they are also the logics closed under $\perp$-free subalgebras, it is not sufficient for a class to be closed under $\perp$-free subalgebras.

In the next section we will recall the generalised Esakia duality developed in [5] to see how canonical formulas manifest themselves dually.

4.2 Partial Esakia morphisms

The core of the refutation criterion are $\mathcal{I}$-homomorphisms. Dually, $\mathcal{I}$-homomorphisms correspond to particular partial maps between Esakia spaces. Most results in this section are based on [5].

Definition 4.2.1. A *partial Esakia morphism* between Esakia spaces is a partial function $f : X \to Y$ such that

(1) $x \in \mathrm{dom}(f)$ implies $f[\uparrow x] = \uparrow f(x)$.

(2) If $f[\uparrow x] = \uparrow y$ for some $y \in Y$ then $x \in \mathrm{dom}(f)$.

(3) $f[\uparrow x] \in \mathrm{Cl}(Y)$ for all $x \in X$.

(4) $U \in \mathrm{ClopUp}(Y)$ implies $X \setminus \downarrow f^{-1}(Y \setminus U) \in \mathrm{ClopUp}(X)$. $\dashv$

Since $\hat{\mathcal{I}}$-homomorphisms are $\perp$-compatible $\mathcal{I}$-homomorphisms they correspond with particular partial Esakia morphisms.

Definition 4.2.2. A partial Esakia morphism $f : X \to Y$ is

- *cofinal* if $\max(X) \subseteq \mathrm{dom}(f)$;
- *locally cofinal* if $\max(\uparrow \mathrm{dom}(f)) \subseteq \mathrm{dom}(f)$. $\dashv$

We recall some useful facts about partial Esakia morphisms from [5].

Lemma 4.2.3. Let $f : X \to Y$ be a partial Esakia morphism.

1. If $\mathrm{dom}(f) = X$ then f is an Esakia morphism.
2. If $U \in \mathrm{ClopUp}(Y)$ then $f^{-1}[U] = \mathrm{dom}(f) \cap (X \setminus \downarrow f^{-1}[Y \setminus U])$.

Theorem 4.2.4 ([5, Theorem 3.27 and Theorem 3.33])**.**

(1) The category of Heyting algebras and $\mathcal{I}$-homomorphisms is dually equivalent to the category of Esakia spaces and partial Esakia morphisms.

(2) The category of Heyting algebras and $\hat{\mathcal{I}}$-homomorphisms is dually equivalent to the category of Esakia spaces and cofinal partial Esakia morphisms.

The functors that establish the duality are very similar to the functors used in Esakia duality. Hence, we also denote them $(\_)^*$ and $(\_)_*$. On objects the act the same, and on morphisms as follows:

- If $f : X \to Y$ is a partial Esakia morphism then we define $f^* : Y^* \to X^*$ as

$$f^*(U) := X \setminus \downarrow f^{-1}[Y \setminus U]$$

for all $U \in \mathrm{ClopUp}(Y)$.

- Conversely, given an $\mathcal{I}$-homomorphism $h : A \to B$ then define $h_* : B_* \to A_*$ by setting $\mathrm{dom}(h_*) := \{x \in B_* \mid h^{-1}[x] \in A^*\}$ and $h_*(x) := h^{-1}[x]$ for all $x \in \mathrm{dom}(h_*)$.

Dually, $\vee$-compatiblity corresponds with something known as the closed domain condition. The name is due to Zakharyaschev, see [17, Section 9.2].

Definition 4.2.5. Let $f : X \to Y$ be a partial Esakia morphism, and $D \subseteq \mathrm{ClopUp}(Y)^2$. Then f satisfies the *closed domain condition* (CDC) for D if

$$f[\uparrow x] \subseteq U \cup V \text{ implies } f[\uparrow x] \subseteq U \text{ or } f[\uparrow x] \subseteq V.$$

for all $(U, V) \in D$. $\dashv$

Lemma 4.2.6 ([5, Lemma 3.40])**.** If $h : A \to B$ is an $\mathcal{I}$-embedding then h is $\vee$-compatible over $D \subseteq A^2$ iff h_* satisfies (CDC) for $\widehat{D} := \{(\widehat{a}, \widehat{b}) \mid (a, b) \in D\}$.

Now recall that the refutation criterion of canonical and cofinal subframe formulas is given by the particular $\mathcal{I}$-embeddings of homomorphic images. We know that homomorphic images dually correspond to closed upsets. Therefore, we have the following dual reading of the refutation criteria.

Corollary 4.2.7. Let X be an Esakia space, A a finite s.i. algebra, and $D \subseteq A^2$. Then

(1) $X \nvDash \alpha(A, D, \bot)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto cofinal partial Esakia morphism $f : Z \to Y$ that satisfies (CDC) for $\widehat{D}$.

(2) $X \nvDash \alpha(A, D)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto partial Esakia morphism $f : Z \to Y$ that satisfies (CDC) for $\widehat{D}$.

(3) $X \nvDash \alpha(A, \bot)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto cofinal partial Esakia morphism $f : Z \to Y$.

(4) $X \nvDash \alpha(A)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto partial Esakia morphism $f : Z \to Y$.

In fact, for (cofinal) subframe formula we can get ride of the closed upsets as we will see in the next chapter.

Extending subreductions to partial Esakia morphisms

Zakharyaschev used the notion of subreductions in his development of canonical formulas, see [17, Section 9.1]. A partial map $f : X \to Y$ between Esakia spaces is a *subreduction* if it satisfies conditions (1) and (4) of Definition 4.2.1. These conditions are sufficient for f^* to be an $\mathcal{I}$-homomorphism, see below. Subreductions are called (locally) cofinal in the same sense as partial Esakia morphisms. That is, a subreduction $f : X \to Y$ is cofinal iff $\max(X) \subseteq \mathrm{dom}(f)$.

Theorem 4.2.8. Let $f : X \to Y$ be a subreduction between Esakia spaces. Then

(1) f^* is an $\mathcal{I}$-homomorphism.

(2) if f is onto then f^* is an embedding;

(3) if f cofinal then f^* is an $\hat{\mathcal{I}}$-homomorphism.

Proof. (1) is a consequence of [17, Theorem 9.7].

(2) Suppose $f^*[U] = f^*[V]$ for some $U, V \in \mathrm{ClopUp}(Y)$. Suppose $y \in U$. Then there is some $x \in \mathrm{dom}(f)$ such that $fx = y$. Then, $fx \notin f^{-1}[Y \setminus U]$. Hence, $fx \in f^*[U] = f^*[V]$. Therefore, $fx \notin f^{-1}[Y \setminus V]$, which means $fx = y \in V$. We can conclude $U \subseteq V$, and since the converse is symmetric $U = V$.

(3) Since $\max(X) \subseteq \mathrm{dom}(f)$ we have ${\downarrow}\mathrm{dom}(f) = X$. Hence,

$$f^{-1}[\varnothing] = X \setminus {\downarrow} f^{-1}[Y \setminus \varnothing] = X \setminus {\downarrow} f^{-1}[Y] = X \setminus {\downarrow}\mathrm{dom}(f) = \varnothing. \qquad \blacksquare$$

Intuitively, subreductions are a less descriptive version of partial Esakia morphisms. They are particularly useful when working with Kripke frames. In that case we can simply forgot about the topological condition. Then a subreduction can just be seen as a partial p-morphism. Theorem 4.2.8 holds for subreductions in precesily the same way.

Furthermore, we can use duality to extend subreductions to partial Esakia morphisms. This is a useful trick for finding partial Esakia morphisms.

Theorem 4.2.9. If $f : X \to Y$ is a subreduction between Esakia spaces then there is a partial Esakia morphism $f^\star : X \to Y$ such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(f^\star)$ and $f(x) = f^\star(x)$ for all $x \in \mathrm{dom}(f)$.

Proof. By Theorem 4.2.8, $f^* : X \to Y$ is an $\mathcal{I}$-embedding. Let $X' = (X^*)_*$, $Y' = (Y^*)_*$, and $f' = (f^*)_*$. Then $f' : X' \to Y$ is a partial Esakia morphism by duality. Moreover, X and Y are isomorphic to X' and Y', respectively. Thus, we have bijective Esakia morphisms $i : X \to X'$ and $j : Y' \to Y$. Let $f^\star := j \circ f' \circ i$. $\blacksquare$

4.3 Lax case

This section commences our investigation into canonical formulas for lax logic. We will extend the canonical formulas from Section 4.1 with the lax modality. The theory of canonical formulas for lax logics arises naturally. We simply have to add a conjunction describing the $\Box$ structure of the refutation algebras. Thus, recall that $\hat{\mathcal{J}} = \hat{\mathcal{I}} \cup \{\Box\} = \mathcal{N} \setminus \{\vee\}$ is the algebraic type of bounded nuclear implicative semilattices.

Definition 4.3.1 (Lax canonical formulas)**.** Let A be a finite s.i. nuclear algebra with the second largest element s, and $D \subseteq A^2$. The *lax canonical formula* $\alpha(A, D, \Box, \bot)$ is an A-based formula defined as

$$\alpha(A, D, \Box, \bot) := \left(\Gamma^{\wedge}_{A^2} \wedge \Gamma^{\to}_{A^2} \wedge \Gamma^{\bot}_{A^0} \wedge \Gamma^{\Box}_{A} \wedge \Gamma^{\vee}_{D}\right) \to p_s. \qquad \dashv$$

We can immediately apply the Refutation Lemma to lax canonical formulas. We then get $\hat{\mathcal{I}}$-embeddings that are $\Box$-compatible over A^2, i.e., $\hat{\mathcal{J}}$-embeddings.

Theorem 4.3.2. Let B be a nuclear algebra and $\alpha(A, D, \Box, \bot)$ a lax canonical formula. Then $B \nvDash \alpha(A, D, \Box, \bot)$ iff there exists a homomorphic image C of B and a $\hat{\mathcal{J}}$-embedding from A into C which is $\vee$-compatible over D.

Proof. By the Refutation Lemma, $B \nvDash \alpha(A, D, \Box, \bot)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding from A to C which is $\Box$-compatible over A, $\bot$-compatible over A_0, and $\vee$-compatible over D, i.e., a $\hat{\mathcal{J}}$-embedding which is $\vee$ compatible over D. ∎

Similarly, the Representation Lemma gives us canonical representations for each modal formula. However, it should be stressed that this time the Representation Lemma makes use of the fact that the $\hat{\mathcal{J}}$-reducts of nuclear algebras are locally finite. Without this result we would not have been able to state the Representation Lemma as it is, and therefore would not be able to instantiate it in the setting of nuclear algebras for the following theorem.

Theorem 4.3.3. For every formula $\varphi \in \mathcal{L}_\Box$ there exists a finite set $\mathscr{C}(\varphi)$ of lax canonical formulas such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ for all nuclear algebras B.

Proof. Instantiate the Representation Lemma with $\mathcal{F} = \varnothing$. Then we have a finite set $\mathscr{C}(\varphi)$ of algebra formulas of the form

$$\psi(A, D) = \left(\Gamma^{\vee}_{D} \wedge \Gamma^{\to}_{A^2} \wedge \Gamma^{\wedge}_{A^2} \wedge \Gamma^{\bot}_{A^0} \wedge \Gamma^{\Box}_{A}\right) \to p_s.$$

such that φ and $\bigwedge \mathscr{C}(\varphi)$ are equivalent. Then $\psi(A, D)$ equals $\varphi(A, D, \Box, \bot)$ modulo comutativity. Hence, the required set of canonical formulas is given by $\{\varphi(A, D, \Box, \bot) \mid \psi(A, D) \in \mathscr{C}(\varphi)\}$. ∎

Theorem 4.3.4. Every lax logic is axiomatisable by lax canonical formulas.

Proof. Let L be a lax logic. Then there exists some set of formulas Γ such that $\mathsf{L} = \mathsf{PLL} \oplus \Gamma$. Then by Theorem 4.3.3, for each $\varphi \in \Gamma$, there exist an equivalent conjunction of lax canonical formulas $\bigwedge \mathscr{C}(\varphi)$. Consequently, B is an L-algebra iff $B \vDash \alpha(A, D, \Box, \bot)$ for all $\alpha(A, D, \Box, \bot) \in \mathscr{C}(\varphi)$ for all $\varphi \in \Gamma$. Whence, $\mathsf{L} = \mathsf{PLL} \oplus \bigcup\{\mathscr{C}(\varphi) \mid \varphi \in \Gamma\}$. ∎

If in the previous proof L is axiomatised by a finite set of formulas Γ then the produced canonical axiomatisation is also finite. Hence, we have the following porism.

Porism 4.3.5. Every finitely axiomatisable lax logic is axiomatisable by finitely many lax canonical formulas.

Similarly to the intuitionistic case we can axiomatise lax logics axiomatised by disjunction-free and/or negation-free formulas by simpler canonical formulas. Moreover, lax logics axiomatised by $\Box$-free formulas can also be axiomatised by a special kind of canonical formulas. In fact, these formulas coincide with intuitionistic canonical formulas.

Definition 4.3.6 (Special lax canonical formulas)**.** Let A be a finite s.i. nuclear algebra with the second largest element s, and $D \subseteq A^2$.

- The *negation-free lax canonical formula* $\alpha(A, D, \Box)$ is defined as

$$\alpha(A, D, \Box) := \big(\Gamma^{\wedge}_{A^2} \wedge \Gamma^{\to}_{A^2} \wedge \Gamma^{\Box}_{A} \wedge \Gamma^{\vee}_{D}\big) \to p_s$$

- The *disjunction-free lax canonical formula* is defined as $\alpha(A, \Box, \bot) := \alpha(A, \varnothing, \Box, \bot)$.
- The *DN-free lax canonical formula* is defined as $\alpha(A, \Box) := \alpha(A, \varnothing, \Box)$.
- The *lax-free canonical formula* is simply the canonical formula associated with Heyting reduct of A, i.e., $\alpha(A, D, \bot)$. $\dashv$

The definition of lax-free canonical formulas similarly induces definitions for lax-free negation-free canonical formulas and lax-free (cofinal) subframe formulas of nuclear algebras. These are all defined as expected. We can use the Refutation Lemma to determine the refutation criteria for these formulas.

Theorem 4.3.7. Let B be a nuclear algebra. Then

(1) $B \nvDash \alpha(A, D, \Box)$ iff there exists a homomorphic image C of B and a $\mathscr{J}$-embedding from A to C which $\vee$-compatible over D
(2) $B \nvDash \alpha(A, \Box, \bot)$ iff there exists a homomorphic image C of B and a $\hat{\mathscr{J}}$-embedding from A to C.
(3) $B \nvDash \alpha(A, \Box)$ iff there exists a homomorphic image C of B and a $\mathscr{J}$-embedding from A to C.
(4) $B \nvDash \alpha(A, D, \bot)$ iff there exists a homomorphic image C of B and an $\hat{\mathcal{I}}$-embedding from A to C that is $\vee$-compatible over D.
(5) $B \nvDash \alpha(A, \bot)$ iff there exists a homomorphic image C of B and an $\hat{\mathcal{I}}$-embedding from A to C.
(6) $B \nvDash \alpha(A, D)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding from A to C that is $\vee$-compatible over D.
(7) $B \nvDash \alpha(A)$ iff there exists a homomorphic image C of B and an $\mathcal{I}$-embedding from A to C.

The refutation criteria of the previous theorem are summarised in Table 4.3.12. Next, we will apply the Representation Lemma to $\mathcal{F}$-free formulas in $\mathcal{L}_\Box$ to obtain equivalent sets of the relevant type of canonical formulas, and therefore we obtain axiomatisations using these sets. We will skip the proofs since they are just adaptions of the proofs of Theorems 4.3.3 and 4.3.4.

Theorem 4.3.8. For every formula $\varphi \in \mathcal{L}_\Box$ there exists a finite set $\mathscr{C}(\varphi)$ of

(1) negation-free lax canonical formulas if φ is $\bot$-free;
(2) lax-free canonical formulas if φ is $\Box$-free;
(3) lax-free negation-free canonical formulas if φ is $(\Box, \bot)$-free;
(4) disjunction-free lax canonical formulas if φ is $\vee$-free;
(5) DN-free lax canonical formulas if φ is $(\bot, \vee)$-free;
(6) lax-free cofinal subframe formulas if φ is $(\Box, \vee)$-free;
(7) lax-free subframe formulas if φ is $(\Box, \bot, \vee)$-free;

such that $B \vDash \varphi$ iff $B \vDash \bigwedge \mathscr{C}(\varphi)$ for all nuclear algebras B.

Theorem 4.3.9. Let L be a lax logic.

(1) L is $\bot$-free iff it is axiomatisable by negation-free lax canonical formulas;
(2) L is $\Box$-free iff it is axiomatisable by lax-free canonical formulas;
(3) L is $(\Box, \bot)$-free iff it is axiomatisable by lax-free negation-free canonical formulas;
(4) L is $\vee$-free iff it is axiomatisable by disjunction-free lax canonical formulas;
(5) L is $(\bot, \vee)$-free iff it is axiomatisable by DN-free lax canonical formulas;
(6) L is $(\Box, \vee)$-free iff it is axiomatisable by lax-free cofinal subframe formulas;
(7) L is $(\Box, \bot, \vee)$-free iff it is axiomatisable by lax-free subframe formulas.

In Theorem 4.1.16 we saw that $\vee$-free logics are axiomatised by cofinal subframe formulas, i.e., "disjunction-free" canonical formulas. We can generalise this result for lax logics.

Theorem 4.3.10. For a lax logic L, the following are equivalent.

(1) L is axiomatisable by disjunction-free lax canonical formulas;

(2) L is $\vee$-free;

(3) the class of L-algebras is closed under $\hat{\mathcal{J}}$-subalgebras;

(4) L is the logic of a class closed under $\hat{\mathcal{J}}$-subalgebras and homomorphic images.

Proof. This proof is an adaption of the proof in [8, Theorem 6.9].

(1) $\Rightarrow$ (2). Immediate since $\vee$ does not occur in disjunction-free lax canonical formulas.

(2) $\Rightarrow$ (3). Suppose $B \nvDash \mathsf{L}$ and B is a $\hat{\mathcal{J}}$-subalgebra of A. Then $B \nvDash \varphi$ for some $\vee$-free $\varphi \in \mathsf{L}$. Since φ is $\vee$-free we get that $A \nvDash \varphi$, therefore $A \nvDash \mathsf{L}$.

(3) $\Rightarrow$ (4). Immediate since L is the logic of the class of L-algebras by algebraic completeness.

(4) $\Rightarrow$ (1). Let $\Gamma := \{\alpha(A, \Box, \bot) \mid A \nvDash \mathsf{L}$ and A is finite s.i.$\}$. Suppose B is a nuclear algebra such that $B \nvDash \varphi$ for some $\varphi \in \mathsf{L}$. By Lemma 4.1.4 there exists a finite s.i. homomorphic C image of B that refutes φ. Then by Selective Filtration, there is a s.i. $\hat{\mathcal{I}}$-subalgebra A of C that refutes φ. Then $\alpha(A, \Box, \bot) \in \Gamma$ and by the Refutation Lemma, $B \nvDash \alpha(A, D, \Box, \bot)$. Thus, $\mathsf{L} \subseteq \mathsf{PLL} \oplus \Gamma$. Conversely, we have assumed that L is the logic of a class C closed under $\hat{\mathcal{J}}$-subalgebras and homomorphic images. Suppose

towards a contradiction that $B \in \mathsf{C}$ and $B \not\models \Gamma$. Then there is some $\alpha(A, \Box, \bot) \in \Gamma$ such that $B \not\models \alpha(A, \Box, \bot)$. By the Refutation Lemma, there is a homomorphic image C of B and a $\hat{\mathcal{J}}$-embedding from A to C. Then $C \in \mathsf{C}$ and $A \in \mathsf{C}$ because it is closed under $\hat{\mathcal{J}}$-subalgebras and homomorphic images. But then $A \models \mathsf{L}$, whence we cannot have $\alpha(A, \Box, \bot) \in \Gamma$. ■

Theorem 4.3.11. For a lax logic L, the following are equivalent.

(1) L is axiomatisable by DN-free lax canonical formulas;

(2) L is $(\vee, \bot)$-free;

(3) the class of L-algebras is closed under $\mathcal{J}$-subalgebras;

(4) L is the logic of a class closed under $\mathcal{J}$-subalgebras and homomorphic images.

Proof. The proof is similar to the proof of Theorem 4.3.10 ■

We could state similar theorems for lax-free canonical formulas but they simply mirror the intuitionistic case. Hence, we will skip them.

Canonical formula	Corresponding embedding	
$\alpha(A, D, \Box, \bot)$	$\hat{\mathcal{J}}$-embedding	$\vee$-compatible over D
$\alpha(A, D, \Box)$	$\mathcal{J}$-embedding	$\vee$-compatible over D
$\alpha(A, D, \bot)$	$\hat{\mathcal{I}}$-embedding	$\vee$-compatible over D
$\alpha(A, D)$	$\mathcal{I}$-embedding	$\vee$-compatible over D
$\alpha(A, \Box, \bot)$	$\hat{\mathcal{J}}$-embedding	
$\alpha(A, \Box)$	$\mathcal{J}$-embedding	
$\alpha(A, \bot)$	$\hat{\mathcal{I}}$-embedding	
$\alpha(A)$	$\mathcal{I}$-embedding	

Table 4.3.12: Lax canonical formulas versus the embbedings in their refutation criterion. This table can be read as follows. A nuclear algebra B refutes the canonical formulas on the left iff there exists a homomorphic image C of B and an embedding from $A \to C$ that is of the type on the right, see also Theorems 4.3.2 and 4.3.7.

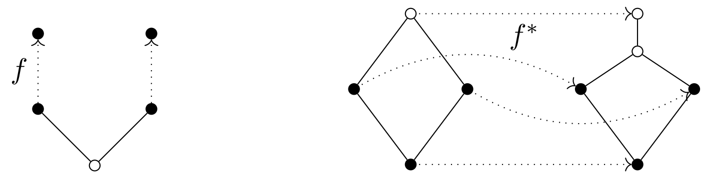
In the next section we will generalise nuclear Esakia duality to give a dual for our canonical formulas. Crucially, we require a dual description of $\hat{\mathcal{J}}$-homomorphisms.

4.4 Modal partial Esakia morphisms

Since $\hat{\mathcal{J}}$-homomorphisms are $\Box$-compatible $\hat{\mathcal{I}}$-homomorphisms, dually the refutation criterion for lax canonical formulas is given by partial Esakia morphisms that respect the R-relations on the spaces in some way. One might expect that it suffices for the partial

Esakia morphism to be a partial p-morphism with respect to R, i.e., $R[fx] = f[R[x]]$ for all $x \in \mathrm{dom}(f)$. However, the following example shows that this condition is too weak.

Example 4.4.1. Consider the partial map $f : X \to Y$ defined in the figure below on the left.[1] Observe, it is a partial Esakia morphism with $R[fx] = f[R[x]]$ for each $x \in \mathrm{dom}(f)$. He



On the right we see the same situation from the dual perspective. There we mark the fixpoints of $\Box$ with white points. Then f^* is an $\hat{\mathcal{I}}$-homomorphism that does is not compatible with $\Box$, whence it is not an $\hat{\mathcal{J}}$-homomorphism. ⊣

The correct condition is a bit peculiar in the sense that it applies to all elements, not just those in the domain. Indeed, when we apply the $(\_)_*$ functor to $\hat{\mathcal{J}}$-homomorphisms, we get partial Esakia morphisms $f : X \to Y$ that satisfy the condition:

$$R[f[\uparrow x]] = f[R[x]]$$

for all $x \in X$. We will call partial Esakia morphisms that satisfy this condition *modal*. Observe that one direction of this condition is given by the expected inclusion.

Lemma 4.4.2. Let $f : X \to Y$ be a partial map between nuclear spaces such that $f[\uparrow x] = \uparrow f(x)$ for each $x \in \mathrm{dom}(f)$. Then

$$R[fx] \subseteq f[R[x]] \text{ for each } x \in \mathrm{dom}(f) \quad \text{iff} \quad R[f[\uparrow x]] \subseteq f[R[x]] \text{ for each } x \in X.$$

Definition 4.4.3. A partial Esakia morphism $f : X \to Y$ between nuclear Esakia spaces is

- *steady* iff $R[fx] \subseteq f[R[x]]$ for each $x \in \mathrm{dom}(f)$;
- *stable* iff $f[R[x] \subseteq Rf[\uparrow x]$ for each $x \in X$;
- *modal* iff it is steady and stable. ⊣

We will now work towards the fact that modal partial Esakia morphisms correspond to $\mathcal{J}$-homomorphisms. Consequently, cofinal modal partial Esakia morphisms will correspond to $\hat{\mathcal{J}}$-homomorphisms. In fact, we will establish a more subtle connection between *steady* and *stable* $\mathcal{I}$-homomorphisms.

Definition 4.4.4. An $\mathcal{I}$-homomorphism $h : A \to B$ between nuclear algebras is

- *steady* if $\Box ha \leq h\Box a$ for all $a \in A$.

[1] Recall the drawing conventions from Example 3.4.11, i.e. a point y is R-accessible from x iff there lies a white point between them.

- *stable* if $h\Box a \leq \Box ha$ for all $a \in A$. $\dashv$

Clearly, an $\mathcal{I}$-homomorphism is a $\mathcal{J}$-homomorphism iff it is steady and stable.

Before we proceed, let us recall a useful fact from generalised Esakia duality [5, Lemma 3.11]

Lemma 4.4.5. Let A, B be Heyting algebras, $h : A \to B$ an $\mathcal{I}$-homomorphism, F a filter of B, and $y \in A_*$. If $h^{-1}(F) \subseteq y$, then there exists $x \in \mathrm{dom}(h_*)$ such that $F \subseteq x$ and $h_*(x) = y$.

Lemma 4.4.6. Let $h : A \to B$ be an $\mathcal{I}$-homomorphism between nuclear algebras.

(1) If h is steady then $h_* : B_* \to A_*$ is steady.

(2) If h is stable then $h_* : B_* \to A_*$ is stable.

(3) If h is a $\mathcal{J}$-homomorphism then $h_* : B_* \to A_*$ is modal.

Proof. (1) Suppose $y \in R[h_* x]$ for some $x \in \mathrm{dom}(h_*)$. Since $\Box ha \leq h\Box a$ for each $a \in A$ we have $h^{-1}\Box^{-1}x \subseteq \Box^{-1}h^{-1}x$. Thus $\Box^{-1}x$ is a filter such that

$$h^{-1}\Box^{-1}x \subseteq \Box^{-1}h^{-1}x \subseteq y.$$

Therefore, by Lemma 4.4.5 there exists $y' \in \mathrm{dom}(h_*)$ such that $\Box^{-1}x \subseteq y'$ and $h_*(y') = y$. Hence, $y' \in R[x]$ and so $h_*(y') = y \in h_*[R[x]]$.

(2) Suppose $y \in h_*[R[x]]$. Then $h^{-1}(x)$ and $\downarrow\{\Box a \mid a \notin y\}$ are respectively a filter and an ideal. Besides, they are disjoint. Indeed, suppose $a' \leq \Box a$ such that $ha' \in x$ and $a \notin y$. Then $ha' \leq h\Box a$ so $h\Box a = \Box ha \in x$. Thus, $ha \in z$ for all $z \in R[x]$, and therefore $a \in y$, contradiction. Thus, by the Prime Filter Theorem there exists $z \in \mathrm{Pf}(A)$ such that $h^{-1}(x) \subseteq z$ and $z \cap \{\Box a \mid a \notin y\} = \varnothing$. It follows that $y \in R[z]$. By Lemma 4.4.5 there exists $z' \in \mathrm{dom}(h_*)$ such that $h_*(z') = z$ and $x \subseteq z'$. Therefore, $y \in R[z] = R[h_*(z')] \subseteq R[h_*[\uparrow x]]$.

(3) Follows from (1), (2), and Lemma 4.4.2. ■

Conversely, as the naming scheme implies, applying $(\_)^*$ to a steady and/or stable partial Esakia morphism results in a steady and/or stable $\mathcal{I}$-homomorphism.

Lemma 4.4.7. Let $f : X \to Y$ be any partial map between Esakia spaces, $V \subseteq X$ and $U \subseteq Y$. Then

(1) $x \in f^*(U)$ iff $f[\uparrow x] \subseteq U$.

(2) $V \subseteq f^*(U)$ iff $f[\uparrow V] \subseteq U$.

Proof. (1) $x \in f^*(U)$ iff $x \in X \setminus \downarrow f^{-1}[Y \setminus U]$ iff $x \notin \downarrow f^{-1}[Y \setminus U]$ iff $\uparrow x \cap f^{-1}[Y \setminus U] = \varnothing$ iff $f[\uparrow x] \cap Y \setminus U = \varnothing$ iff $f[\uparrow x] \subseteq U$.

(2) follows from (1). ■

Lemma 4.4.8. Let $f : X \to Y$ be a partial Esakia morphism between nuclear spaces.

(1) If f is steady then f^* is steady;

(2) If f is stable then f^* is stable;

(3) If f is modal then f^* is a $\mathcal{J}$-homomorphism.

Proof. Suppose $U \in \text{ClopUp}(Y)$. We have

$$x \in \Box_R f^*(U) \iff R[x] \subseteq f^*U \iff f[\uparrow R[x]] \subseteq U \iff f[R[x]] \subseteq U \qquad (\blacktriangledown)$$

and

$$x \in f^*(\Box_R U) \iff f[\uparrow x] \subseteq \Box_R U \iff R[f[\uparrow x]] \subseteq \Box_R U \qquad (\blacktriangle)$$

by Lemma 4.4.7 and Lemma 3.4.2.

(1) Suppose $x \in \Box_R f^*(U)$. Then $f[R[x]] \subseteq U$ by ($\blacktriangledown$). By Lemma 4.4.2 and since f is steady, $R[f[\uparrow x]] \subseteq f[R[x]] \subseteq U$. Then by ($\blacktriangle$) we have $x \in f^*(\Box_R U)$. Whence, $\Box_R f^*(U) \subseteq f^*(\Box_R U)$.

(1) Suppose $x \in f^*(\Box_R U)$. Then by ($\blacktriangle$) $R[f[\uparrow x]] \subseteq \Box_R U$. Since f is stable, $f[R[x]] \subseteq R[f[\uparrow x]] \subseteq \Box_R U$. Then by ($\blacktriangledown$) $x \in \Box_R f^*(U)$. Whence, $f^*(\Box_R U) \subseteq \Box_R f^*(U)$.

(3) Follows from (1) and (2). ∎

Theorem 4.4.9. The following pairs of categories are dually equivalent.

(1) Nuclear algebras with steady $\mathcal{I}$-homomorphisms, and nuclear spaces with steady partial Esakia morphisms;

(2) Nuclear algebras with steady $\hat{\mathcal{I}}$-homomorphisms, and nuclear spaces with cofinal steady partial Esakia morphisms;

(3) Nuclear algebras with $\mathcal{J}$-homomorphisms, and nuclear spaces with modal partial Esakia morphisms;

(4) Nuclear algebras with $\hat{\mathcal{J}}$-homomorphisms, and nuclear spaces with cofinal modal partial Esakia morphisms.

We can now dualise the refutation criteria of all variants of lax canonical formulas in the previous chapter. We leave the lax-free variants implicit since they are identical to Corollary 4.2.7.

Corollary 4.4.10. Let X be a nuclear Esakia space, A a finite s.i. nuclear algebra, and $D \subseteq A^2$. Then

(1) $X \nvDash \alpha(A, D, \Box, \bot)$ iff there exists $Z \in \text{ClUp}(X)$ and an onto modal cofinal partial Esakia morphism $f : Z \to Y$ that satisfies (CDC) for $\widehat{D}$.

(2) $X \nvDash \alpha(A, D, \Box)$ iff there exists $Z \in \text{ClUp}(X)$ and an onto modal partial Esakia morphism $f : Z \to Y$ that satisfies (CDC) for $\widehat{D}$.

(3) $X \nvDash \alpha(A, \Box, \bot)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto modal cofinal partial Esakia morphism $f : Z \to Y$.

(4) $X \nvDash \alpha(A, \Box)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto modal partial Esakia morphism $f : Z \to Y$.

Moreover, dual readings of Theorems 4.3.10 and 4.3.11 give us the following.

Theorem 4.4.11. For a lax logic L, the following are equivalent.

(1) L is axiomatisable by disjunction-free lax canonical formulas;

(2) the class of L-spaces is closed under images of cofinal modal partial Esakia morphisms.

(3) L complete with respect to a class of nuclear spaces that is closed under images of cofinal modal partial Esakia morphisms and closed upsets.

Theorem 4.4.12. For a lax logic L, the following are equivalent.

(1) L is axiomatisable by DN-free lax canonical formulas;

(2) the class of L-spaces is closed under images of modal partial Esakia morphisms.

(3) L complete with respect to a class of nuclear spaces that is closed under images of modal partial Esakia morphisms and closed upsets.

In this chapter we developed the method of canonical formulas for lax logics. First, we recalled canonical formulas for the intuitionistic case and the related generalised Esakia dually. Then we successfully applied the method of canonical formulas to the lax logic: we axiomatised all lax logics with lax canonical formulas. Moreover, we created an adaptable framework which gives special instances of lax canonical axiomatisations for lax logics axiomatised by a restricted syntax of the language. Lastly, we established generalised nuclear Esakia duality to examine lax canonical formulas from the dual perspective. In the next chapter we will put (lax) canonical formulas to work by investigating subframe logics for the lax case, and obtain a number of preservation results for the translations defined in Section 3.5.

Chapter 5

Canonical Formulas at Work

Canonical formulas provide solutions to a wide range of problems for intermediate and transitive modal logics, e.g., the Diego-McKay Theorem that all $\vee$-free intermediate logics have the fmp [50], and the Blok-Esakia Theorem [51]. Moreover, Zakharyaschev extensively used them as tools to investigate modal companions of intermediate logics, obtaining several preservation results [17, Section 9.6], including a proof for the Dummett-Lemmon conjecture. Besides, canonical formulas characterise subframe and cofinal subframe logics – important classes of logics with many good properties such as the fmp.

In this chapter we will extend the method of canonical formulas to lax logics using the toolbox developed in Chapter 4. We will define *steady* and *cofinal steady* lax logics, show that they are structurally similar to subframe and cofinal subframe logics, and give many examples of their geometric axiomatisations. We proceed by proving preservation results for the inner space translation defined in Section 3.5. In particular, we will prove that the least lax extension of each Kripke-complete intermediate logic is Kripke-complete – an analogue of the Dummett-Lemmon conjecture.

In Section 5.1, we briefly discuss subframe logics. We recall the main ingredients which allow them to be axiomatised by subframe formulas. We introduce steady and cofinal steady logics in Section 5.2, and give a number of examples of them. Lastly, Section 5.3 gives proofs for several preservation results for the outer space translation.

5.1 Subframe logics

We have seen in Section 4.1 that logics axiomatised by disjunction-free formulas are axiomatised by cofinal subframe formulas. The name is a result of a correspondence between these formulas and logics whose frames are closed under subframes, see [17, Section 11.3]. Traditionally, a subframe of a Kripke frame (X, R) is another Kripke frame (Y, S) such that $Y \subseteq X$ and $S = R \restriction Y$. For Esakia spaces we have to make sure that subspace topology behaves nicely. It boils down the following.

Definition 5.1.1. If X is an Esakia space then $Y \subseteq X$ is a *subframe of* X iff

(1) $Y \in \mathrm{Cl}(X)$, and

(2) $U \in \mathrm{ClopUp}(Y)$ implies $X \setminus {\downarrow}(Y \setminus U) \in \mathrm{ClopUp}(X)$. $\dashv$

We call a subframe $Y \subseteq X$ *locally cofinal* iff $\max({\uparrow}Y) \subseteq Y$ and *cofinal* iff $\max(X) \subseteq Y$. A class of Esakia spaces C is *closed under subframes* if $X \in \mathsf{C}$ and Y a subframe of X implies $Y \in \mathsf{C}$. Closure under (locally) cofinal subframes is defined similarly.

Definition 5.1.2. A logic L is

- *subframe* iff the class of L-spaces is closed under subframes.
- *cofinal subframe* iff the class of L-spaces is closed under cofinal subframes. $\dashv$

Intuitively, locally cofinal subframes compose cofinal subframes with closed upsets. It is easy to see that in the finite case every locally cofinal subframe is a cofinal subframe of an upset. While this does not hold in general, it is the case that classes of L-spaces closed under locally cofinal subframes and cofinal subframes coincide.

Theorem 5.1.3. A logic L is cofinal subframe iff the class of L-spaces is closed under locally cofinal subframes.

Proof. Suppose the class of L-spaces is closed under cofinal subframes. Suppose $X \Vdash \mathsf{L}$ and Y is a locally cofinal subframe of X. It follows that Y is a cofinal subframe of ${\uparrow}Y$. Since the class of L-spaces is closed under upsets, ${\uparrow}Y \Vdash \mathsf{L}$. Since L is closed under cofinal subframes we can conclude $Y \Vdash \mathsf{L}$. Conversely, every cofinal subframe is locally cofinal. ■

Recall that an Esakia space X refutes a subframe formula $\alpha(A, \bot)$ iff there exists a closed upset Z of X and an onto cofinal partial Esakia morphism from Z to A^*, and similarly for a cofinal subframe formulas. In fact, the closed upsets are superfluous in the refutation criterion.

Lemma 5.1.4 ([8, Lemma 6.2])**.** Let X, Y be finite Esakia spaces and $Z \in \mathrm{ClUp}(X)$. Let A, B, C be finite Heyting algebras and $h : B \to C$ an onto homomorphism.

(1) If $f : Z \to Y$ is an onto (cofinal) partial Esakia morphism then there exists a (cofinal) partial Esakia morphism $g : X \to Y$ such that $f(x) = g(x)$ for all $x \in Z$.

(2) If $g : A \to C$ is an $\mathcal{I}$-embedding ($\hat{\mathcal{I}}$-embedding) then there exists an $\mathcal{I}$-embedding ($\hat{\mathcal{I}}$-embedding) $k : A \to B$ such that $g = h \circ k$.

The previous lemma gives a refinement of the refutation criteria for subframe and cofinal subframe formulas.

Theorem 5.1.5 ([8, Theorem 6.3])**.** Let B be a Heyting algebra, A a finite s.i. Heyting algebra, and X an Esakia space. Then

(1) $B \nvDash \alpha(A)$ iff there exists an $\mathcal{I}$-embedding $h : A \to B$;

(2) $B \nvDash \alpha(A, \bot)$ iff there exists an $\hat{\mathcal{I}}$-embedding $h : A \to B$;

(3) $X \nvDash \alpha(A)$ iff there exists an onto partial Esakia morphism $f : X \to A_*$;

(4) $X \nvDash \alpha(A, \bot)$ iff there exists an onto cofinal partial Esakia morphism $f : X \to A_*$.

Proof. (1) By Selective Filtration, there exists a finite finite $\hat{\mathcal{I}}$-subalgebra B' of B that refutes $\alpha(A)$. By the Refutation Lemma, there is a homomorphic image C of B' and a $\mathcal{I}$-embedding $h : C \to A$. By Lemma 5.1.4, there is a $\mathcal{I}$-embedding $h' : C \to B'$. Hence, C is isomorphic to a $\hat{\mathcal{I}}$-subalgebra of B.

(2) is similar, and (3) and (4) follow from duality. ■

Theorem 5.1.5 shows why it is sufficient for a class to be closed under $\mathcal{I}$-subalgebras ($\hat{\mathcal{I}}$-subalgebras) to be axiomatised by (cofinal) subframe formulas in Theorem 4.1.16 and Theorem 4.1.17. Yet it remains to be seen how partial Esakia morphism relate to subframes. First, let us recall a useful fact from generalised Esakia duality.

Lemma 5.1.6 ([5, Lemma 3.7])**.** Let $f : X \to Y$ be a partial Esakia morphism.

(1) $\mathrm{dom}(f) \in \mathrm{Cl}(X)$.

(2) $\mathrm{dom}(f) \in \mathrm{Clop}(X)$ if Y is finite.

Clearly, every clopen subset of an Esakia space is a subframe. Hence, partial Esakia morphisms into finite frames give raise to subframes by their domain.

Corollary 5.1.7. If $f : X \to Y$ is a (cofinal) partial Esakia morphism and Y is finite then $\mathrm{dom}(f)$ is a (cofinal) subframe of X.

This allows us the following characterisation of subframes logics.

Theorem 5.1.8 ([8, Theorem 6.16])**.** Let L be a intermediate logic. Then the following are equivalent.

(1) L is axiomatisable by cofinal subframe formulas;

(2) L is $\vee$-free;

(3) L is cofinal subframe;

(4) L is generated by a class of Esakia spaces closed under cofinal subframes.

Proof. See the proof of [8, Theorem 6.15]. Importantly, for (4) $\Rightarrow$ (1) we need to show that if L is subframe, X is an L-space, and $X \nVdash \alpha(A, \bot)$ then $A \vDash \mathsf{L}$. By Theorem 5.1.5, there exists an onto cofinal partial Esakia morphism $f : X \to A_*$. Since A is finite, dom(f) is a cofinal subframe of X by Corollary 5.1.7. Whence dom(f) $\Vdash \mathsf{L}$. Since $f : \mathrm{dom}(f) \to A_*$ is an onto partial Esakia morphism $A_* \Vdash \mathsf{L}$, and by duality $A \vDash \mathsf{L}$. ■

Theorem 5.1.9 ([8, Theorem 6.15])**.** Let L be a intermediate logic. Then the following are equivalent.

(1) L is axiomatisable by subframe formulas;

(2) L is $(\bot, \vee)$-free;

(3) L is subframe;

(4) L is generated by a class of Esakia spaces closed under subframes.

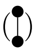
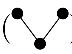
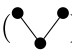
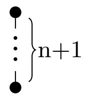
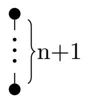
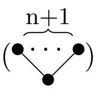
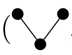
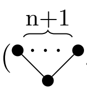
Logic		Subframe axiomatisation
$\mathcal{L}_{\mathsf{p}}$	$=$	$\mathsf{IPC} \oplus \beta(\bullet)$
CPC	$=$	$\mathsf{IPC} \oplus \beta($$)$
LC	$=$	$\mathsf{IPC} \oplus \beta($$)$
LC_n	$=$	$\mathsf{IPC} \oplus \beta($$) \oplus \beta($$)$
BD_n	$=$	$\mathsf{IPC} \oplus \beta($$)$
BW_n	$=$	$\mathsf{IPC} \oplus \beta($$)$
KC	$=$	$\mathsf{IPC} \oplus \beta($$, \bot)$
BTW_n	$=$	$\mathsf{IPC} \oplus \beta($$, \bot)$

Table 5.1.10: (Cofinal) Subframe Logics

We conclude this section with some examples of subframe and cofinal subframe logics.

Definition 5.1.11. Let X be a finite frame, $Y \subseteq X$, and $x \in X$.

- The *depth* of x is the size of the largest chain in $\uparrow x$.
- The *depth* of Y is maximal depth of elements in Y.
- The *width* of x is the size of the largest antichain in $\uparrow x$.
- The *width* of Y is the maximal width of elements in Y.

- The *cofinal width* of x is the size of $\max(\uparrow x)$.
- The *cofinal width* of Y is the maximal cofinal width of elements in Y $\dashv$

Definition 5.1.12. Let $n \geq 1$

- Let LC be the logic all finite chains.
- Let LC_n be the logic the chain of size n.
- Let BD_n be the logic of all finite rooted frames of depth at most n.
- Let KC be the logic of all finite rooted frames that have a largest element.
- Let BW_n be the logic of all finite rooted frames of width at most n.
- Let BTW_n be the logic of all finite rooted frames of cofinal width at most n. $\dashv$

Proofs for the following can be found in [17, Chapter 9].

Theorem 5.1.13. Let $n \geq 1$

(1) The logics $\mathsf{CPC}, \mathsf{LC}, \mathsf{LC}_n, \mathsf{BD}_n, \mathsf{BW}_n$ are subframe logics.

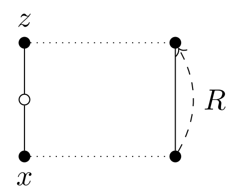
(2) The logics KC and BTW_n are cofinal subframe logics.

The subframe and cofinal subframe formulas axiomatising the logics of Theorem 5.1.13 are depicted in Table 5.1.10. We abuse notation by writing $\alpha(X, \bot)$ and $\alpha(X)$ for $\alpha(X^*, \bot)$ and $\alpha(X^*)$, respectively.

5.2 Steady logics

In this section we will develop lax analogues for subframe and cofinal subframe logics. We saw in the intuitionistic case that cofinal subframe logics are axiomatised by cofinal subframe formulas. Recall that cofinal subframe formulas were those canonical formulas $\alpha(A, D, \bot)$ such that $D = \varnothing$. For lax canonical formulas we called the analogue of these formulas disjunction-free lax canonical formulas. The reason being that these formulas do not induce a natural definition of subframes. First, observe that the class of (finite) nuclear spaces is not closed under taking naive subframes with respect to R. That is, where we simply restrict R to some subset. Namely, the restriction is not generally a lax relation. One might say that lax logic is not a subframe logic in the context of modal Esakia spaces.

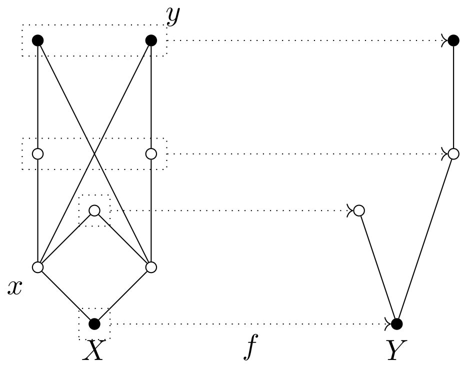
Example 5.2.1. Consider the three element lax frame below.



We have xRz so in the "subframe" $Y = \{x, z\}$ we should have xRz. However, there is no reflexive point between them in the subframe; Y is not a lax frame. This illustrates that restricting R to some subset is not a good way to induce a lax frame. ⊣

Example 5.2.1 might give an indication why subframes for lax logic require more care to be defined. Suppose we define lax subframes using disjunction-free lax canonical formulas. It would be advantageous if lax subframes relate to modal partial Esakia morphisms in the same way as subframes relate to partial Esakia morphisms. For instance, we have seen that clopen domains of onto partial Esakia morphisms are subframes, see Corollary 5.1.7, and this is paramount in the proof of Theorem 5.1.8. However, this can not be the case for any definition of lax subframes.

Example 5.2.2. Consider the lax frames X and Y below. There is an onto cofinal modal partial Esakia morphism $f : X \to Y$. We have $X \Vdash \alpha(\mathrm{dom}(f)^*, \Box, \bot)$, i.e., there is no upset of X that maps onto $\mathrm{dom}(f)$ with a cofinal modal partial Esakia morphism.



Indeed, suppose there is an onto cofinal modal partial Esakia morphism $g : X \to \mathrm{dom}(f)$. Then we must have $R[g[\uparrow x]] = g[R[x]]$. Since x is reflexive we have that $y \in R[x]$. Hence, $gy \in g[R[x]]$. Moreover, since g is onto the only points in $\uparrow x$ that can map to gy are x and y[1], but $y \notin R[y]$, and if $gx = gy$ then g is not a p-morphism. Hence, $y \notin g[R[\uparrow x]]$, a contradiction. Besides, that there is also no upset of X that maps onto $\mathrm{dom}(f)$ follows by the fact that the only irreflexive point that can map to the root of $\mathrm{dom}(f)$ is the root of X.

Now let $\mathsf{L} := \mathsf{PLL} \oplus \alpha(\mathrm{dom}(f)^*, \Box, \bot)$. Then X is an L-frame but $\mathrm{dom}(f)$ is not. Thus, logics axiomatised by disjunction-free lax canonical formulas are not generally closed under finite domains of modal partial Esakia morphisms. ⊣

Hence, either lax subframe logics are not axiomatised by disjunction-free lax canonical formulas or the finite domains of modal partial Esakia morphisms are not necessarily subframes. Either way, defining subframes that correspond with disjunction-free lax canonical formulas becomes *ad-hoc*. We will leave characterising logics axiomatised by disjunction-free lax canonical formulas with some type of subframes an open question. Instead we will turn our attention to S-spaces. Recall, an S-space is a pair (X, S) where X is an Esakia space and S is a subframe of X.

[1] Technically, the other maximal point can map to y but then the argument still works for that point.

The most natural way to think of lax subframes for S-spaces is to intersect the inner space with a subset to form another S-space. That is, if $X_S = (X, S)$ is an S-space then $Y \subseteq X$ should be a lax subframe of X if $Y_S = (Y, S \cap Y)$ is an S-space. Indeed, this will be similar to the definition we will use. Additionally, we require Y to be a subframe of X, i.e., the outer space of Y_S is a subframe of the outer space of X_S.

Definition 5.2.3. Let (X, S) be an S-space. An *S-subframe* of (X, S) is an S-space $(Y, Y \cap S)$ such that Y is a subframe of X. ⊣

Since S-spaces correspond to nuclear spaces we can translate this definition to the following definition for nuclear spaces.

Definition 5.2.4. A *steady subframe* of a nuclear space X is a nuclear Esakia space Y such that

- Y is a subframe of X, and
- $R_Y = ((R_X)_\triangle \cap Y)_\between$, i.e., xR_Yy iff there is $z \in Y$ such that $x \leq z \leq y$ and zR_Xz. ⊣

A steady subframe is (*locally*) *cofinal* if it is a (locally) cofinal subframe of the outer space. Closure under steady subframes is defined as expected. Steady logics are those class of logics that are closed under steady subframes and *contractions*.

Definition 5.2.5. A *contraction* of a nuclear space is X is a nuclear space Y such that $X = Y$ and $(R_Y)_\triangle \subseteq (R_X)_\triangle$. ⊣

Closure under contractions is also defined as expected, i.e., a class C of nuclear spaces is *closed under contractions* if $X \in \mathsf{C}$ and Y is a contraction of X imply $Y \in \mathsf{C}$.

Definition 5.2.6. A lax logic L is

- *steady* iff the class of L-spaces is closed under steady subframes and contractions.
- *cofinal steady* iff the class of L-spaces is closed under cofinal steady subframes and contractions. ⊣

Note that every cofinal steady subframe is a steady subframe. Hence, steady logics are cofinal steady logics. Steady subframes give us an analogue of Corollary 5.1.7 for lax logics. That is, clopen domains of steady[1] partial Esakia morphisms are steady subframes. Indeed, every clopen subset of a nuclear space is a steady subframe.

Lemma 5.2.7. If $Y \in \mathrm{Clop}(X)$ then Y is a steady subframe of X.

Proof. It follows easily that Y is a subframe of X. It remains to be shown that $(Y, Y \cap R_\triangle)$ is an S-space, i.e., that $Y \cap R_\triangle$ is a subframe of Y. Clearly, $R_\triangle \cap Y \in \mathrm{Cl}(Y)$. Next, suppose $U \in \mathrm{Clop}(Y \cap R)_\triangle$. Then $U = R_\triangle \cap V$ for $V \in \mathrm{Clop}(Y) \subseteq \mathrm{Clop}(X)$. Hence,

[1] Recall that a partial Esakia morphism $f : X \to Y$ is steady iff $R[fx] \subseteq f[R[x]]$ for all $x \in \mathrm{dom}(f)$.

$U \in \mathrm{Clop}(R_\triangle)$, and since $R_\triangle$ is a subframe of X, we have ${\uparrow}U \in \mathrm{Clop}(X)$. Therefore, ${\uparrow}U \cap Y \in \mathrm{Clop}(Y)$, as required. ■

Corollary 5.2.8. Let $f : X \to Y$ be a steady partial Esakia morphism.

(1) If $\mathrm{dom}(f) \in \mathrm{Clop}(X)$ then $\mathrm{dom}(f)$ is a steady subframe of X.

(2) If Y is finite then $\mathrm{dom}(f)$ is a steady subframe of X.

Proof. (1) follows immediately from Lemma 5.2.7. For (2), since every steady partial Esakia morphism is a partial Esakia morphism we have $\mathrm{dom}(f) \in \mathrm{Clop}(X)$ by Lemma 5.1.6. Hence, by (1) we have that $\mathrm{dom}(f)$ is a steady subframe. ■

Steady logics are axiomatised by algebra-based formulas that encode the structure of $\Box$ only in the "steady" direction.

Definition 5.2.9. Let A be a finite s.i. nuclear algebra with the second largest element s, and $D \subseteq A^2$. The *cofinal steady canonical formula* $\beta(A, D_\vee, D_\Box, \bot)$ is an A-based formula defined as

$$\beta(A, D_\vee, D_\Box, \bot) := \left(\Gamma^\wedge_{A^2} \wedge \Gamma^\to_{A^2} \wedge \Gamma^\bot_{A^0} \wedge \Delta^\Box_A \wedge \Gamma^\vee_{D_\vee} \wedge \Gamma^\Box_{D_\Box}\right) \to p_s.$$

and the *steady canonical formula* $s\alpha(A, D_\vee, D_\Box)$ is defined as

$$\beta(A, D_\vee, D_\Box) := \left(\Gamma^\wedge_{A^2} \wedge \Gamma^\to_{A^2} \wedge \Delta^\Box_A \wedge \Gamma^\vee_{D_\vee} \wedge \Gamma^\Box_{D_\Box}\right) \to p_s. \qquad \dashv$$

The difference between lax canonical formulas and cofinal steady canonical formulas lies in the conjunctions involving $\Box$. Namely, steady canonical formulas only contain $\Delta^\Box_A$ and $\Gamma^\Box_{D_\Box}$ and not the "full" $\Gamma^\Box_A$.[1] The Refutation Lemma and generalised nuclear Esakia duality gives us the refutation criteria.

Theorem 5.2.10. Let B be a nuclear algebra and $\beta(A, D_\vee, D_\Box, \bot)$ a cofinal steady canonical formula.

(1) Then $B \nvDash \beta(A, D_\vee, D_\Box, \bot)$ iff there exists a homomorphic image C of B and a steady $\hat{\mathcal{I}}$-embedding $h : A \to C$ that is $\vee$-compatible over $D_\vee$ and $\Box$-compatible over $D_\Box$.

(2) Then $B \nvDash \beta(A, D_\vee, D_\Box)$ iff there exists a homomorphic image C of B and a steady $\mathcal{I}$-embedding $h : A \to C$ that is $\vee$-compatible over $D_\vee$ and $\Box$-compatible over $D_\Box$.

For a dual reading we first have to translate $\Box$-compatibility over $D_\Box$.

Definition 5.2.11. Let $f : X \to Y$ be a steady partial Esakia morphism, and $D \subseteq \mathrm{ClopUp}(Y)$. Then f satisfies the *nuclear closed domain condition* (NCDC) for D if

$$R[f[{\uparrow}x]] \subseteq U \text{ implies } f[R[x]] \subseteq U.$$

[1] See the definitions of these conjunctions in Definition 4.1.1.

for all $U \in D$ and all $x \in X$. $\dashv$

Lemma 5.2.12. If $h : A \to B$ is a steady $\mathcal{I}$-homomorphism then h is $\Box$-compatible over $D \subseteq A$ iff h_* satisfies (NCDC) for $\widehat{D} := \{\widehat{a} \mid a \in D\}$.

Proof. This follows from the same argument as in the proofs of Lemmas 4.4.6(2) and 4.4.8(2). ■

This gives us the following reading of the refutation criteria of steady canonical formulas for nuclear spaces.

Theorem 5.2.13. Let X be a nuclear space and $\beta(A, D_\vee, D_\Box, \bot)$ a cofinal steady canonical formula.

(1) Then $X \not\models \beta(A, D_\vee, D_\Box, \bot)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto cofinal steady partial Esakia morphism from Z to A_* that satisfies (CDC) for $\widehat{D_\vee}$ and (NCDC) for $\widehat{D_\Box}$.

(2) Then $X \not\models \beta(A, D_\vee, D_\Box)$ iff there exists $Z \in \mathrm{ClUp}(X)$ and an onto steady partial Esakia morphism from Z to A_* that satisfies (CDC) for $\widehat{D_\vee}$ and (NCDC) for $\widehat{D_\Box}$.

Note, cofinal steady canonical formulas are a generalisation of lax canonical formulas. Indeed, it is easy to see that $\beta(A, D_\vee, A, \bot)$ is equivalent to $\alpha(A, D_\vee, \Box, \bot)$, and similarly for steady canonical formulas and negation-free lax canonical formulas. Consequently, all ($\bot$-free) lax logics are axiomatised by (cofinal) steady canonical formulas.

Theorem 5.2.14. Let L be a lax logic.

(1) L is axiomatisable by cofinal steady canonical formulas.

(2) If L is $\bot$-free then, L is axiomatisable by steady canonical formulas.

Proof. Consider the axiomatisation using (negation-free) lax canonical formulas and translate them into (cofinal) steady canonical formulas by adding $\Delta_A^\Box$ and putting $D_\Box := A$. ■

Cofinal steady logics are those logics that are axiomatised by cofinal steady canonical formulas of the form $\beta(A, \varnothing, \varnothing, \bot)$, and similarly for steady logics and steady canonical formulas. Whence, a (cofinal) steady canonical formula is called a (*cofinal*) *steady formula* iff $D_\vee = D_\Box = \varnothing$, and as usual we denote them by $\beta(A, \bot)$ and $\beta(A)$.

Similarly to subframe formulas, we can get rid of the homomorphic image in the refutation criterion for steady formulas. An $\mathcal{I}$-subalgebra is called steady if the corresponding $\mathcal{I}$-embedding is steady. We will make use of the fact that we can extend subreductions to partial Esakia morphisms, see Theorem 4.2.9. Moreover, if a subreduction is steady then it can be extended to a steady partial Esakia morphism.

Lemma 5.2.15. Let $f : X \to Y$ be a steady subreduction between nuclear spaces. Then

(1) f^* is steady;

(2) there exists a steady partial Esakia morphism $f^\star : X \to Y$ such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(f^\star)$ and $f(x) = f^\star(x)$ for all $x \in \mathrm{dom}(f)$.

Proof. (1) The proof is identical to the proof of Lemma 4.4.8(1).

(2) Follows from Theorem 4.2.9 and (1). ■

Lemma 5.2.16. Let B be a nuclear algebra. Then $B \nvDash \beta(A)$ iff there exists a steady $\mathcal{I}$-embedding from A to B.

Proof. ($\Rightarrow$). Follows directly from the Refutation Lemma.

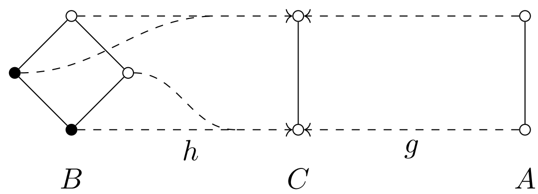
($\Leftarrow$). By Nuclear Selective Filtration, there exists a finite $\hat{\mathcal{J}}$-subalgebra C of B such that $C \nVdash \beta(A)$. We can assume B is finite since any steady $\mathcal{I}$-subalgebra of C is a steady $\mathcal{I}$-subalgebra of B. We proceed by duality. Thus, suppose X is a lax frame and $X \nvDash \beta(A, \bot)$. Then by the Refutation Lemma, there exists $Z \in \mathrm{Up}(X)$ and an onto steady partial Esakia morphism $f : Z \to Y$. Clearly, $f : X \to Y$ is an onto steady subreduction. Then by Lemma 5.2.15(2), we have an onto steady partial Esakia morphism $g : X \to Y$. Dually, we have obtained a steady $\mathcal{I}$-embedding, as required. ■

The dual reading of the refutation criterion for steady formulas already appeared in the previous proof.

Porism 5.2.17. Let X be a nuclear space. Then $X \nvDash \beta(A)$ iff there exists an onto steady partial Esakia morphism from X to A_*.

For cofinal steady formulas we can not get rid of the homomorphic image in the refutation criterion:

Example 5.2.18. Suppose A, B, C are nuclear algebras as depicted below. Then $B \nvDash \beta(A, \bot)$ since $h : B \to A$ is an onto homomorphism, and $g : A \to C$ is a steady $\hat{\mathcal{I}}$-embedding.



However, A cannot be $\hat{\mathcal{I}}$-embedded into B steadily since $\bot_A$ has to be mapped to $\bot_B$ and a $\Box$-fixpoint of B, but $\bot_B$ is not a $\Box$-fixpoint. ⊣

Nonetheless, the refutation criterion is transitive in the sense that refutation of steady formulas is preserved through steady $\hat{\mathcal{I}}$-subalgebras.

Lemma 5.2.19. Suppose B and C are nuclear algebras such that $B \nvDash \beta(A, \perp)$ and B is a steady $\hat{\mathcal{I}}$-subalgebra of C. Then $C \nvDash \beta(A, \perp)$.

Proof. By Nuclear Selective Filration, there exists a finite $\hat{\mathcal{J}}$-subalgebra B' of B that refutes $\beta(A, \perp)$. It follows that B' is a steady subalgebra of C. Consequently, we can assume B is finite. We will proceed by duality, so suppose we have a nuclear space X, a finite lax frame Y, a set $Z \in \mathrm{ClUp}(Y)$, and onto cofinal steady partial Esakia morphisms $f : X \to Y$, $g : Z \to A_*$. Then by Lemma 5.1.6, $\mathrm{dom}(g) \in \mathrm{Clop}(Y)$ since Y is finite . Whence, $f^{-1}[\mathrm{dom}(g)] \in \mathrm{Cl}(X)$, and therefore $\uparrow f^{-1}[\mathrm{dom}(g)] \in \mathrm{ClUp}(X)$. If there exists an onto cofinal steady partial Esakia morphism from $\uparrow f^{-1}[\mathrm{dom}(g)]$ to A_* then $X \nvDash \beta(A, \perp)$ by the Refutation Lemma. Hence, it suffices to show that that $g \circ f$ is a cofinal steady subreduction by Lemma 5.2.15(2).

$$\begin{tikzcd}
Y & Z \arrow[l, hook'] \arrow[r, two heads, "g"] & A_* \\
X \arrow[u, two heads, "f"] & \uparrow f^{-1}[\mathrm{dom}(g)] \arrow[l, hook'] \arrow[r, two heads, "f"'] & \mathrm{dom}(g) \arrow[u, two heads, "g"']
\end{tikzcd}$$

If $x \in \mathrm{dom}(g \circ f)$ then $x \in \mathrm{dom}(f)$ and $fx \in \mathrm{dom}(g)$. Whence, $\uparrow gfx = g[\uparrow fx] = g[f[\uparrow x]]$ and $R[gfx] \subseteq g[R[fx]] \subseteq g[f[R[x]]$ since g and f are steady partial Esakia morphisms. Suppose $U \in \mathrm{Clop}(A_*)$. Then $g^{-1}[U] \in \mathrm{Clop}Y$ since Y is finite. Consequently, $\downarrow f^{-1}[g^{-1}[U]] \in \mathrm{Clop}(X)$ which means $\uparrow f^{-1}[\mathrm{dom}(g)] \cap \downarrow (g \circ f)^{-1}[U] \in \mathrm{Clop}(\uparrow f^{-1}[\mathrm{dom}(g)])$. Hence, $g \circ f$ is a subreduction. Finally, suppose $x \in \max(\uparrow f^{-1}[\mathrm{dom}(g)])$. Then $x \in \max(X)$ since $\uparrow f^{-1}[\mathrm{dom}(g)]$ is an upset. Then $x \in \mathrm{dom}(f)$ since f is cofinal. It follows that $fx \in \max(Y)$ since $x \in \max(X)$, and therefore $fx \in \mathrm{dom}(g)$ since g is cofinal. Thus, $x \in \mathrm{dom}(g \circ f)$, as required. ■

With the previous lemmas we can characterise steady and cofinal steady logics as the logics generated by classes closed under steady subalgebras in the following way.

Theorem 5.2.20. Given a lax logic L, the following are equivalent.

(1) L is axiomatised by cofinal steady formulas;

(2) the class of L-algebras is closed under steady $\hat{\mathcal{I}}$-subalgebras;

(3) L is the logic of a class closed under steady $\hat{\mathcal{I}}$-subalgebras of homomorphic images.

Proof. (1) $\Rightarrow$ (2). Suppose $C \nvDash \mathsf{L}$ and C is a steady $\hat{\mathcal{I}}$-subalgebra of B. Then there exists $\beta(A, \perp) \in \mathsf{L}$ such that $C \nvDash \beta(A, \perp)$. By Lemma 5.2.19, $B \nvDash \beta(A, \perp)$, which means $B \nvDash \mathsf{L}$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Let $\Gamma := \{\beta(A, \bot) \mid A \nvDash \mathsf{L}$ and A is finite s.i.$\}$. We claim that $\mathsf{PLL} \oplus \Gamma = \mathsf{L}$. Suppose $B \nvDash \varphi$ for some $\varphi \in \mathsf{L}$. Then there exists a homomorphic image C of B such that $C \nvDash \varphi$. Moreover, by Nuclear Selective Filtration there is a finite s.i. $\hat{\mathcal{J}}$-subalgebra A of C such that $A \nvDash \varphi$. Therefore, $\beta(A, \bot) \in \Gamma$. By the Refutation Lemma, $B \nvDash \beta(A, \bot)$. Conversely, since L is logic of some class C it is sufficient to show that $B \vDash \Gamma$ for each $B \in \mathsf{C}$. Thus, suppose $B \in \mathsf{C}$ and $B \nvDash \beta(A, \bot)$ for some $\beta(A, \bot) \in \Gamma$. By the Refutation Lemma, A is a steady $\hat{\mathcal{I}}$-subalgebra of a homomorphic image C of B. Whence, $A \in \mathsf{C}$, which means $A \vDash \mathsf{L}$, a contradiction. ■

Theorem 5.2.21. Given a lax logic L, the following are equivalent.

(1) L is axiomatised by steady formulas;

(2) the class of L-algebras is closed under steady $\mathcal{I}$-subalgebras;

(3) L is the logic of a class closed under steady $\mathcal{I}$-subalgebras.

Proof. The proof is similar to the proof of Theorem 5.2.20, but we use Lemma 5.2.16 instead of Lemma 5.2.19. ■

Thus, steady formulas and cofinal steady formulas axiomatise logics of classes closed under steady $\mathcal{I}$-subalgebras and steady $\hat{\mathcal{I}}$-subalgebras respectively. We will now work towards showing that these are exactly steady and cofinal steady logics, i.e., the logics whose classes of spaces are closed (cofinal) steady subframes and contractions. Thus, we will illustrate how steady subalgebras relate to steady subframes. Namely, every steady subframe induces a steady subalgebra on the dual algebras.

Lemma 5.2.22. Let X and Y be nuclear spaces.

(1) If Y is a steady subframe of X then Y^* is a steady $\mathcal{I}$-subalgebra of X^*.

(2) If Y is a cofinal steady subframe of X then Y^* is a steady $\hat{\mathcal{I}}$-subalgebra of X^*.

Proof. (1) Suppose Y is a steady subframe of X. Then the partial identity map $i : X \to Y$ is an onto steady subreduction. By Lemma 5.2.15(2), we can extend i to an onto steady partial Esakia morphism. By duality, Y^* is a steady $\mathcal{I}$-subalgebra of X^*.

(2) Follows in the same way as (1). ■

Conversely, not every steady subalgebra is a steady subframe. For instance, for every Heyting algebra A and nuclei $\Box, \boxdot$ such that $\Box a \leq \boxdot a$ for all $a \in A$ we have that $(A, \Box)$ is a steady subalgebra of $(A, \boxdot)$. However, in terms of steady subframes Y is not a steady subframe of X if they share the same universe but not the same nuclear relation. For this reason we have to use contractions.

Lemma 5.2.23. If $f : X \to Y$ is a steady onto Esakia morphism and Y is finite then there exists a contraction X' of X such that $f : X' \to Y$ is a modal partial Esakia morphism.

Proof. Let $S := f^{-1}[(R_Y)_\triangle]$ is a subframe of X. Since Y is finite $(R_Y)_\triangle \in \mathrm{Clop}(Y)$. Then since f is continuous $S \in \mathrm{Clop}(X)$. It follows easily that S is a subframe of X. Moreover, f is steady $R[f[x]] \subseteq f[R[x]]$. Thus, $S \subseteq (R_X)_\triangle$. Hence, $X_S := (X, S_\emptyset)$ is a contraction of X. Next, that $f : X_S \to Y$ is modal follows from the way we defined S. Suppose $y \in f[R[x]]$. Then $y = fz$ for $z \in R[x]$. Then there exists $z' \in S$ such that $x \leq z' \leq z$. Hence, $fz' \in (R_Y)_\triangle$. Then $fx \leq fz' \leq y$ since f is an Esakia morphism. Thus, $fxRy$, i.e., $y \in R[fx]$. Conversely, if $y \in R[fx]$ then there is $z \in (R_Y)_\triangle$ such that $fx \leq z \leq y$. Since f is an onto Esakia morphism there is $z' \in f^{-1}(z)$ and $y' \in f^{-1}(y)$ such that $x \leq z' \leq y'$. Since $f^{-1}[(R_Y)]_\triangle = S$, we $z' \in S$. Thus, xRy', and therefore $y \in f[R[x]]$. ■

We are now ready to connect (cofinal) steady formulas and (cofinal) steady logics.

Theorem 5.2.24. Let L be a lax logic. The following are equivalent.

(1) L is axiomatisable by cofinal steady formulas.

(2) L is cofinal steady.

(3) L is the logic of class of nuclear spaces closed under cofinal steady subframes, closed upsets, and contractions.

Proof. (1) ⇒ (2). Suppose X is a L-space and Y is a cofinal steady subframe of X. By Lemma 5.2.22, Y^* is a $\hat{\mathcal{I}}$-subalgebra of X^*. By Theorem 5.2.20, the class of L-algebras is closed under steady $\mathcal{I}$-algebras. Hence, Y^* is an L-algebra, and by duality Y is an L-space.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). As before we let $\Gamma := \{\beta(A, \bot) \mid A \nvDash \mathsf{L} \text{ and } A \text{ is finite s.i.}\}$. That $\mathsf{L} \subseteq \mathsf{PLL} \oplus \Gamma$ follows as in Theorem 5.2.20. Conversely, suppose C is the class of nuclear spaces that is complete with respect to L. Suppose towards a contradiction that, $X \in \mathsf{C}$ and $X \nVdash \beta(A, \bot)$ for some $\beta(A, \bot) \in \Gamma$. By the Refutation Lemma, there is $Y \in \mathrm{ClUp}(X)$ and an onto cofinal steady partial Esakia morphism $f : Y \to A_*$. Since A_* is finite we have $\mathrm{dom}(f) \in \mathrm{Clop}(Y)$ by Corollary 5.1.7. Hence, $\mathrm{dom}(f)$ is a steady subframe of Y. Suppose $x \in \max(Y)$. Then $x \in \mathrm{dom}(f)$ since f is cofinal, so $\mathrm{dom}(f)$ is a cofinal steady subframe of Y. Since C is closed under closed upsets and cofinal steady subframes, $\mathrm{dom}(f) \in \mathsf{C}$. By Lemma 4.2.3(1), $f : \mathrm{dom}(f) \to A_*$ is an Esakia morphism. Moreover, it follows that it is onto and steady. By Lemma 5.2.23, there exists a contraction Z of $\mathrm{dom}(f)$ such that $f : Z \to A_*$ is an onto modal Esakia morphism. Since C is closed under contractions, $Z \in \mathsf{C}$. Hence, $Z \Vdash \mathsf{L}$. But then $A_* \Vdash \mathsf{L}$, contradicting $\beta(A, \bot) \in \Gamma$. ■

Theorem 5.2.25. Let L be a lax logic. The following are equivalent.

(1) L is axiomatised by a steady set of DN-free lax canonical formulas.

(2) L is steady.

(3) L is the logic of a class C of nuclear spaces closed under steady subframes and contractions.

Proof. The result follows by the same argument as in the proof of Theorem 5.2.24. ■

Thus, cofinal steady logics are exactly the lax logics axiomatised by steady cofinal formulas and steady logics are the lax logics axiomatised by steady formulas. Algebraically, they correspond to the logics generated by classes closed under steady subalgebras. Hence, they have the fmp.

Theorem 5.2.26. All (cofinal) steady logics have the fmp.

Proof. Suppose L is a cofinal steady logic. By Theorem 3.3.9, it suffices to show that the class of L-algebras is closed under $\hat{\mathcal{J}}$-subalgebras. By Theorem 5.2.20, the class of L-algebras is closed under steady $\hat{\mathcal{I}}$-subalgebras. Since every $\hat{\mathcal{J}}$-subalgebra is a steady $\hat{\mathcal{I}}$-subalgebra it follows that the class of L-algebras is closed under $\hat{\mathcal{J}}$-subalgebras. Thus, all cofinal steady logics have the fmp. Moreover, since all steady logics are cofinal steady, so do steady logics. ■

Thus, steady and cofinal steady logics can be seen as lax analogues of subframe and cofinal subframe logics: the classes of their spaces are closed under subframe-like structures, they are axiomatised by similar formulas, and they all have the fmp.

We conclude this section with some examples of steady and cofinal steady logics. We will review some “homogeneous” and “heterogeneous” (cofinal) steady logics. The reason they are called this becomes apparent in their geometric refutation patterns, see Tables 5.2.32 and 5.2.40. First, we will introduce some very simple homogeneous logics.

Definition 5.2.27. Let $n \geq 1$.

- Let CPC^- be the lax logic of all lax frames of size 1.
- Let LC^- be the lax logic of all finite lax frames that are chains.
- Let LC_n^- be the lax logic of all lax frames that chains of size at most n.
- Let BD_n^- be the lax logic of all finite rooted lax frames of depth at most n.
- Let KC^- be the lax logic of all finite rooted lax frames that have a largest element.
- Let BW_n^- be the lax logic of all finite rooted lax frames of width at most n.
- Let BTW_n^- be the lax logic of all finite rooted lax frames of cofinal width at most n. ⊣

Theorem 5.2.28. Let $n \geq 1$

(1) CPC^-, LC^-, LC_n^-, BD_n^-, and KC^- are steady.

(2) BW_n^- and BTW_n^- are cofinal steady.

Proof. We will prove only that BD_n^- is steady. The other claims are proven similarly, and in fact closely resemble the proof of their intuitionistic counterpart. Let $\mathsf{L} = \mathsf{PLL} \oplus \beta(X^*)$, where $X = \left.\begin{matrix}\bullet\\ \vdots\\ \bullet\end{matrix}\right\}\text{n+1}$, i.e., X is a chain of size $n+1$ and $R_\triangle = \varnothing$. Then logic L is steady since it is axiomatised by steady formulas. We will prove that $\mathsf{BD}_n^- = \mathsf{L}$. Since BD_n^- is the logic of a class of finite lax frames it has the fmp. Moreover, L has the fmp because it is steady. Thus, it suffices to show that

$$Y \Vdash \beta(X^*) \text{ iff } Y \text{ does not contain a chain of size } n+1$$

for all finite rooted lax frames Y. Suppose, $Y \nVdash \beta(X^*)$. Then by Porism 5.2.17, there is onto steady partial Esakia morphism $f : Y \to X$. But then $f^{-1}[X] \subseteq Y$ has depth at least $n+1$. Hence, Y contains a chain of size $n+1$. Conversely, suppose Y contains a chain $y_1 < \cdots < y_{n+1}$. Then we can define $f : Y \to X$ such that f is an onto subreduction. Besides, it is steady since $R_X = \varnothing$. By Lemma 5.2.15(2) and the Refutation Lemma $Y \nVdash \beta(X^*)$.

The steady axiomatisations used in other cases can be seen in Table 5.2.32. ■

These lax logics are very simple in the sense that we have not restricted their inner space in any way. Let us move on to some lax logics where we do put restrictions on the inner spaces. Beforehand, we have to fix some new terminology.

Definition 5.2.29. Let X be a finite lax frame, $Y \subseteq X$, and $x \in X$.

- x is *nuclear* if $x \in R_\triangle$.
- Y is *nuclear* if all its elements are nuclear.
- the *nuclear size* of Y is the cardinality of the largest nuclear subset of Y.
- the *nuclear depth* of x is the size of the largest nuclear chain in $\uparrow x$.
- the *nuclear depth* of Y is the maximal nuclear depth of elements in Y.
- the *nuclear width* of x is the size of the largest nuclear antichain in $\uparrow x$.
- the *nuclear width* of Y is the maximal nuclear width of elements in Y.
- x is $\circ$-*linear* if the nuclear elements of $\uparrow x$ are a chain.
- Y is $\circ$-*linear* if all its elements are.
- x is $\circ$-*cofinal* if $\max(\uparrow x)$ is nuclear.
- Y is $\circ$-*cofinal* if all its elements are.
- Y is $\circ$-*rooted* if Y has a nuclear root.
- Y is $\circ$-*critical* if Y is $\circ$-rooted and $\circ$-cofinal. ⊣

Definition 5.2.30. Let $n \geq 1$.

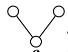
- Let CPC^+ be the lax logic of all finite rooted lax frames of nuclear size 1.

- Let LC^+ be the lax logic of all finite rooted lax frames whose $\circ$-rooted nuclear subsets are chains.
- Let LC_n^+ be the lax logic of all finite rooted lax frames whose $\circ$-rooted nuclear subsets are chains of size at most n.
- Let BD_n^+ be the lax logic of all finite rooted lax frames that have at most a nuclear depth of n.
- Let BW_n^+ be the lax logic of all finite rooted lax frames whose $\circ$-rooted sets have at most a nuclear width of n.
- Let KC^+ be the lax logic of all finite rooted lax frames whose $\circ$-critical subsets have a largest element.
- Let BTW_n^+ be the lax logic of all finite rooted lax frames whose $\circ$-critical subsets have a cofinal width of at most n. $\dashv$

Theorem 5.2.31. Let $n \geq 1$

(1) CPC^+, LC^+, LC_n^+, BD_n^+, and BW_n^+ are steady.

(2) KC^+ and BTW_n^+ are cofinal steady.

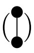
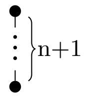
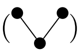
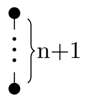
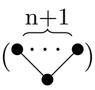
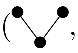
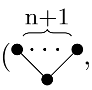
Proof. We will only prove a single case as an illustration. We will show that KC^+ is cofinal steady. In fact, we will prove that $\mathsf{KC}^+ = \mathsf{PLL} \oplus \beta(X^*, \bot)$, where $X =$ . Let r, x, y denote the elements of X such that r is the root. Similarly, to the proof of Theorem 5.2.28 it suffices to show that the $\circ$-critical subsets of a finite rooted lax frame Y have a largest element iff $Y \not\vDash \beta(X^*, \bot)$.

Thus, suppose $Y \not\vDash \beta(X^*, \bot)$. Then by the Refutation Lemma, there exists an upset $Z \in \mathrm{Up}(Y)$ and an onto cofinal steady partial Esakia morphism $f : Z \to X$. Then $\max(f^{-1}(r))$ must contain some nuclear element r' of Y since f is steady. Thus, $\uparrow r'$ is $\circ$-rooted. Moreover, we find nuclear elements $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$ such that $r' \leq x'$ and $r' \leq y'$. Next, since f is cofinal $\max(\uparrow x') \subseteq \mathrm{dom}(f)$. Moreover, $f[\max(\uparrow x')] = \{x\}$. Otherwise, f is not a partial Esakia morphism. We assume $x' \in \max(X)$ for simplicity, and similarly for y'. Since f is steady x' and y' must be nuclear elements. Hence, $\uparrow r'$ does not have a largest element. Moreover, $\uparrow r$ is nuclear cofinal since f is cofinal $f(\max(Z)) \subseteq (R_X)_\triangle$, which means all maximal points of Z must be nuclear. Therefore, $\uparrow r'$ is a $\circ$-critical set with no largest element.

Conversely, suppose there is some $\circ$-critical set with no largest element. Then there are nuclear elements $x', y', r' \in Y$ such that x' and y' are maximal, $r' \leq x'$ and $r' \leq y'$. Let $Z = \uparrow r'$. Then $f : Z \to X$ can be defined to be an onto cofinal steady subreduction. By Lemma 5.2.15(2), we can assume f is an onto steady partial Esakia morphism. By the Refutation Lemma, $Y \not\vDash \beta(X^*, \bot)$, as required.

For the steady axiomatisations used in other cases, see Table 5.2.32. ■

So far all steady logics are closely related to some intuitionistic counterpart by their

Logic		Steady axiomatisation
$\mathcal{L}_\Box$	=	$\mathsf{PLL} \oplus \beta(\bullet)$
CPC^-	=	$\mathsf{PLL} \oplus \beta($$)$
BD_n^-	=	$\mathsf{PLL} \oplus \beta($$)$
LC^-	=	$\mathsf{PLL} \oplus \beta($$)$
LC_n^-	=	$\mathsf{LC}^- \oplus \beta($$)$
BW_n^-	=	$\mathsf{PLL} \oplus \beta($$)$
KC^-	=	$\mathsf{PLL} \oplus \beta($$, \bot)$
BTW_n^-	=	$\mathsf{PLL} \oplus \beta($$, \bot)$

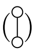
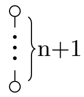
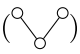
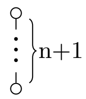
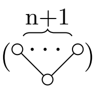
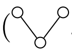
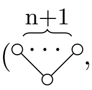
Logic		Steady axiomatisation
NN	=	$\mathsf{PLL} \oplus \beta(\circ)$
CPC^+	=	$\mathsf{PLL} \oplus \beta($$)$
BD_n^+	=	$\mathsf{PLL} \oplus \beta($$)$
LC^+	=	$\mathsf{PLL} \oplus \beta($$)$
LC_n^+	=	$\mathsf{LC}^+ \oplus \beta($$)$
BW_n^+	=	$\mathsf{PLL} \oplus \beta($$)$
KC^+	=	$\mathsf{PLL} \oplus \beta($$, \bot)$
BTW_n^+	=	$\mathsf{PLL} \oplus \beta($$, \bot)$
NCf	=	$\mathsf{PLL} \oplus \beta(\circ, \bot)$
CrS	=	$\mathsf{PLL} \oplus \beta($$, \bot)$

Table 5.2.32: Homogeneous Steady Logics

geometric structure, compare Tables 5.1.10 and 5.2.32. The final examples are of homogeneous (cofinal) steady logics that do not seem to have such a connection.

Definition 5.2.33.

- Let NN be the lax logic of all finite rooted lax frames with no nuclear elements.
- Let NCf be the lax logic of all finite rooted lax frames with no $\circ$-cofinal elements.
- Let CrS be the lax logic of all finite rooted lax frames whose $\circ$-critical sets are singletons. $\dashv$

We can prove the following in the same way as we did the previous theorems. Because of this, we only state the result without proof. As usual, the steady and cofinal steady axiomatisations can be found in Table 5.2.32.

Theorem 5.2.34.

(1) NN is steady.

(2) NCf and CrS are cofinal steady.

This marks the end of examples of homogeneous steady logics. Their steady and cofinal steady axiomatisations are depicted in Table 5.2.32. Again, we abuse notation by writing $\beta(X, \bot)$ and $\beta(X)$ for $\beta(X^*, \bot)$ and $\beta(X^*)$, respectively. We see that these logics are all characterised by refutation frames such that either all points are nuclear or none at all. This is why we call them homogeneous.

We will now turn our attention to heterogeneous steady logics. Contrasting their homogeneous counterpart, these are steady and cofinal steady logics that are not characterised by such "trivial" lax frames where either all or no points are nuclear. First, we will introduce *spans*. A span is a tuple (n, m) of natural numbers. We put an ordering on spans as follows.

$$(n, m) \leq (n', m') \iff n \leq n' \text{ and } m + n \leq m' + n'$$

Definition 5.2.35. Let X be a finite lax frame, $Y \subseteq X$ a subset, $x \in X$.

- x has a *span* of (n, m) if $\uparrow x$ contains an antichain of $n + m$ elements of which at least n are a nuclear.
- Y has a *span* of (n, m) if has an element with span (n, m).
- x has a *cofinal span* of (n, m) if $\max(\uparrow x)$ is an antichain of $n + m$ elements of which at least n are nuclear.
- Y has a *cofinal span* of (n, m) if it has an element with cofinal span (n, m). ⊣

Definition 5.2.36. Let $n \geq 1$.

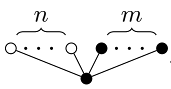
- Let LMx be the lax logic of all finite rooted lax frames such that all nuclear elements are maximal.
- Let LRt be the lax logic of all finite rooted lax frames whose only nuclear element can be the root.
- Let LIC be the lax logic of all finite rooted lax frames whose $\circ$-rooted sets are chains.
- Let LLn be the lax logic of all finite rooted lax frames that are $\circ$-linear.
- Let BIW_n be the lax logic of all finite rooted lax frames with a nuclear width of at most n.
- Let BRW_n be the lax logic of all finite rooted lax frames whose $\circ$-rooted subsets have at most width n.
- Let $\mathsf{BS_{n,m}}$ be the lax logic of all finite rooted lax frames that have a span less than (n, m).
- Let $\mathsf{BIS_{n,m}}$ be the lax logic of all finite rooted lax frames whose $\circ$-rooted subsets have span less than (n, m).
- Let $\mathsf{BCS_{n,m}}$ be the lax logic of all finite rooted lax frames that have a cofinal span less than (n, m).

- Let $\mathsf{BICS_{n,m}}$ be the lax logic of all finite rooted lax frames whose $\circ$-rooted subsets have a cofinal span less than (n, m). $\dashv$

Theorem 5.2.37. Let $n, m \geq 1$.

(1) LMx, LRt, LIC, LLn, BIW_n, BRW_n, $\mathsf{BS_{n,m}}$, and $\mathsf{BIS_{n,m}}$ are steady.

(2) $\mathsf{BCS_{n,m}}$ and $\mathsf{BICS_{n,m}}$ are cofinal steady.

Proof. We will only show that $\mathsf{BIS_{n,m}} = \beta(X^*)$ where $X =$ 

. It suffices to show that

$$Y \text{ has span of at least } (n, m) \text{ iff } Y \not\vDash \beta(X^*).$$

Suppose there is $x \in Y$ an antichain $y_1, \dots, y_n, z_1, \dots, z_m \in \uparrow x$ such that $y_1, \dots, y_n$ are nuclear. Then we can define an onto steady subreduction by mapping x to the root of X, each y_i to a distinct nuclear point of X, and the remaining points of X can be covered by z_i's. By Lemma 5.2.15, there is an onto steady partial Esakia morphism, and by the Refutation Lemma, $Y \not\vDash \beta(X^*, \bot)$. Conversely, suppose $Y \not\vDash \beta(X^*, \bot)$. By Lemma 5.2.16, there is an onto steady partial Esakia morphism $f : Y \to X$. Then there is is some point x in the preimage of the root of X, and since f is steady for each nuclear point in X there is some $y_i \in \uparrow x$ which is nuclear. Similarly, we get z_i's for the remaining points in X. Then $y_1, \dots, y_n, z_1, \dots, z_m$ is an antichain. Hence, x has a span of (n, m).

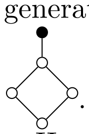
The other steady axiomatisations are shown in Table 5.2.40. ∎

We have shown several examples of steady logics and of steady cofinal logics. Let us end with some lax logics that are not steady.

Definition 5.2.38.

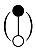
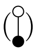
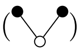
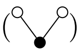
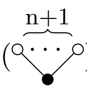
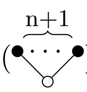
- Let IKC be the lax logic of all finite rooted lax frame whose $\circ$-rooted subsets have a largest nuclear element.
- Let IBTW_n be the lax logic of all finite rooted lax frame whose $\circ$-rooted nuclear sets have at most n maximal elements.
- Let $\mathsf{LS} = \mathsf{PLL} \oplus \neg\Box\bot$. $\dashv$

Lemma 5.2.39. IKC, IBTW_n, are not cofinal steady LS

Proof. The trick for proving the first two is to show their generating class is not closed under cofinal steady subframes. For instance, let $X =$ . Then X is the only $\circ$-rooted subset of X and it has a largest nuclear element. Hence, it is an IKC-frame.

However, X without the second largest element is a cofinal steady subframe of X but now it fails to have a largest nuclear element. The same example works for $\mathsf{IBTW_n}$.

For LS, consider the singleton reflexive frame. Clearly, it is serial. However, the singleton irreflexive frame is a cofinal steady subframe of it and it fails to be serial. ■

Logic		Steady axiomatisation
LMx	$=$	$\mathsf{PLL} \oplus \beta($$)$
LRt	$=$	$\mathsf{PLL} \oplus \beta($$)$
LIC	$=$	$\mathsf{PLL} \oplus \beta($$)$
LLn	$=$	$\mathsf{PLL} \oplus \beta($$)$
BIW_n	$=$	$\mathsf{PLL} \oplus \beta($$)$
BRW_n	$=$	$\mathsf{PLL} \oplus \beta($$)$

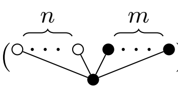
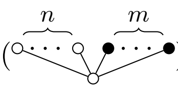
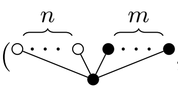
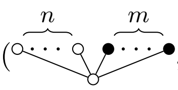
Logic		Steady axiomatisation
$\mathsf{BS_{n,m}}$	$=$	$\mathsf{PLL} \oplus \beta($$)$
$\mathsf{BIS_{n,m}}$	$=$	$\mathsf{PLL} \oplus \beta($$)$
$\mathsf{BCS_{n,m}}$	$=$	$\mathsf{PLL} \oplus \beta($$, \bot)$
$\mathsf{BICS_{n,m}}$	$=$	$\mathsf{PLL} \oplus \beta($$, \bot)$

Table 5.2.40: Heterogeneous Steady Logics

One "problem" with steady and cofinal steady formulas is that they cannot encode the $\Box\bot$-structure of algebras. This is why they fail to axiomatise the logics of the previous theorem. However, we could add this to the formulas and it would fit nicely in our framework. To not overload notation we will not include these formulas in this thesis but we will hint at the fact that all we have to do is add a $\Gamma^{\Box\bot}_{A_0}$ conjunction to the formulas.

In the next section we provide some preservation results for the outer space translation defined in Section 3.5. We will use steady canonical formulas to show the outer space embedding gives incites an interesting connection between intermediate logics and lax logics.

5.3 Preservation in outer space

Canonical formulas are particularly useful since they allow us to assume without loss of generality that all (lax) logics are axiomatised by them. Hence, instead of needing to argue syntactically about exact shapes of formulas we merely need to consider the refutation algebras corresponding to a given (lax) logic. Zakharyaschev used this tactic to obtain various preservation results between intermediate logics and their modal companions, for example, the proof of the Dummett-Lemmon conjecture that the least modal companion of each Kripke-complete intermediate logic is Kripke-complete [51]. In this section, we will obtain a similar result for lax logics. In a sense we continue the investigation of the inner and outer space translations started in [10] and [31, p. 6.5.2] with the method of canonical formulas.

Recall that modal companions of an intermediate logic L are the modal logics M such that

$$\mathsf{M} \vDash \tau(\varphi) \text{ iff } \mathsf{L} \vdash \varphi$$

for all $\varphi \in \mathcal{L}_\mathsf{p}$, where τ is the Gödel-translation, i.e., τ puts $\Box$ in front of implications and propositional variables. In the lax case, there a few ways to think of "lax companions." First, since every lax logic M has an intuitionistic base, we could say that M is a lax companion of L if the $\Box$-free fragment of M coincides with L, i.e., $\mathsf{M} \cap \mathcal{L}_\mathsf{p} = \mathsf{L}$. In this sense we take the identity as our canonical translation. Namely, M would be a companion of L if

$$\mathsf{M} \vdash \varphi \text{ iff } \mathsf{L} \vdash \varphi$$

for all $\varphi \in \mathcal{L}_\mathsf{p}$. Then it follows that the outer space logic[1] $\mathsf{L}^\bullet$ is the least companion of an intermediate logic L. The matching reading of the Dummett-Lemmon conjecture is that the outer space embedding preserves Kripke-completeness. This is, among other things, what we will prove in this section.

Another way to think of lax companions is via the inner space translation. Recall from Section 3.5 that the inner space translation $(\_)^\circ : \mathcal{L}_\mathsf{p} \to \mathcal{L}_\Box$ is recursively defined as:

- $p^\circ = \Box p$, $\bot^\circ = \Box\bot$,
- $(\varphi \vee \psi)^\circ = \Box(\varphi^\circ \vee \psi^\circ)$, and
- $(\varphi * \psi)^\circ = \varphi^\circ * \psi^\circ$ for $* \in \{\wedge, \to\}$.

Evidently, lax companions of an intermediate logic L would be the lax logics M such that

$$M \vdash \varphi^\circ \text{ iff } \mathsf{L} \vdash \varphi$$

for all $\varphi \in \mathcal{L}_\mathsf{p}$. In this sense, Similarly to the outer space case, we find that L° is the smallest companion of an intermediate L. As the name of this section implies, we will investigate the *outer space companions*. We leave similar results for *inner space companions* an open problem.

First, we show that the outer space embedding naturally translates canonical axiomatisations of intermediate logics into steady canonical axiomatisations of their smallest outer space companion. For this purpose we need some ways to induce nuclei on Heyting algebras. Given a Heying algebra A, we let $A_\top$ and A_{Id} denote the nuclear algebras extending A with $\Box$ given respectively by the identity $\Box_{\mathrm{Id}} a = a$ and the constant $\Box_\top a = \top$

[1]Recall from Section 3.5 that $\mathsf{L}^\bullet = \mathsf{PLL} \oplus \mathsf{L}$.

for all $a \in A$. Conversely, given a nuclear algebra A we denote the Heyting reduct of A by A'.

Lemma 5.3.1. Let B be a nuclear algebra and C a Heyting algebra. If C a $\mathcal{H}$-homomorphic image of B', then there exists a nucleus on C such that C is a $\mathcal{N}$-homomorphic image of B.

Proof. This follows immediately from duality since closed upsets of the outer space are also closed upsets of the nuclear space. ■

Lemma 5.3.2. Suppose $\beta(A_1, D_1, \varnothing, \bot)$ and $\beta(A_2, D_2, \varnothing, \bot)$ are steady canonical formulas such that $h : A_1 \to A_2$ is a steady $\hat{\mathcal{I}}$-embedding and $h[D_1] \subseteq D_2$. Then $B \vDash \beta(A_1, D_1, \varnothing, \bot)$ implies $B \vDash \beta(A_2, D_2, \varnothing, \bot)$ for every nuclear algebra B.

Proof. Suppose $B \nvDash \beta(A_2, D_2, \varnothing, \bot)$. By the Refutation Lemma, there exists a homomorphic image C of B and a steady $\hat{\mathcal{I}}$-embedding $g : A_2 \to C$ that is $\vee$-compatible over D_2. It follows that $g \circ h : A_1 \to C$ is a steady $\hat{\mathcal{I}}$-embedding that is $\vee$-compatible over D_1, whence $B \nvDash \beta(A_1, D_1, \varnothing, \bot)$. ■

Theorem 5.3.3. Let $\mathsf{L} = \mathsf{IPC} \oplus \Gamma$ for some set of canonical formulas Γ. Then

$$\mathsf{L}^\bullet = \mathsf{PLL} \oplus \{\beta(A_\top, D, \varnothing, \bot) \mid \alpha(A, D, \bot) \in \Gamma\}.$$

Proof. Let B be a nuclear algebra. Suppose $B \nvDash \beta(A_\top, D, \varnothing, \bot)$ for some $\alpha(A, D, \bot) \in \Gamma$. By the Refutation Lemma, there exists a homomorphic image C of B and a steady $\hat{\mathcal{I}}$-embedding $h : A_\top \to C$ that is $\vee$-compatible over D. Clearly, h is also an $\hat{\mathcal{I}}$-embedding from A to C, and C' is a homomorphic image of B'. By the Refutation Lemma, $B' \nvDash \alpha(A, D, \bot)$. Therefore, $B' \nvDash \mathsf{L}$, and by Theorem 3.5.2, $B \nvDash \mathsf{L}^\bullet$.

Conversely, suppose $B \nvDash \mathsf{L}^\bullet$. By Theorem 3.5.2, $B' \nvDash \mathsf{L}$, Then there is a canonical formula $\alpha(A, D, \bot) \in \Gamma$ such that $B' \nvDash \alpha(A, D, \bot)$. By the Refutation Lemma, there exists a homomorphic image C of B' and an $\hat{\mathcal{I}}$-embedding $h : A \to C$ that is $\vee$-compatible over D. By Lemma 5.3.1, we can extend C to a nuclear algebra that is a homomorphic image of B. Then $h : A_{\mathrm{Id}} \to C$ is a steady $\hat{\mathcal{I}}$-embedding that is $\vee$-compatible over D. By the Refutation Lemma, $B \nvDash \beta(A_{\mathrm{Id}}, D, \varnothing, \bot)$. Since $A_\top$ is a steady subalgebra of A_{Id}, by Lemma 5.3.2, $B \nvDash \beta(A_\top, D, \varnothing, \bot)$. ■

Remark 5.3.4. The translation of a canonical formula $\alpha(A, D, \bot)$ into a steady canonical formula $\beta(A_\top, D, \varnothing, \bot)$ is basically the addition of the conjunction $\Delta^\Box_{A_\top}$ to the antecedent of the original formula. But this is quite a trivial satisfaction requirement with respect to $A_\top$. That is,

$$\Delta^\Box_{A_\top} = \bigwedge_{a \in A} \Box p_a \to p_{\Box a} = \bigwedge_{a \in A} \Box p_a \to p_\top.$$

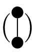
Observe that this evaluates to $\top$ for every valuation that maps $p_\top$ to $\top$. ⊣

Theorem 5.3.3 shows us that the outer lax companions of cofinal subframe logics are cofinal steady logics. Namely, in that case we have $D = \varnothing$ for all $\alpha(A, D, \bot) \in \Gamma$, and therefore $\mathsf{L}^\bullet$ is axiomatised by cofinal subframe formulas. Moreover, we can easily adjust Theorem 5.3.3 to show that intermediate logics axiomatised by negation-free canonical formulas similarly translate into an axiomatisation of their least outer companion with steady canonical formulas. Therefore, we will state this result without proof.

Theorem 5.3.5. Let $\mathsf{L} = \mathsf{IPC} \oplus \Gamma$ for some set of negation-free canonical formulas Γ. Then $\mathsf{L}^\bullet = \mathsf{PLL} \oplus \{\beta(A_\top, D, \varnothing) \mid \alpha(A, D) \in \Gamma\}$.

In turn this shows that the inner space embeddings of the (cofinal) subframe logics in Table 5.1.10 are given by the (cofinal) steady logics in Table 5.2.32.

Theorem 5.3.6. $\mathsf{CPC}^\bullet = \mathsf{CPC}^-$, $\mathsf{BD}_n^\bullet = \mathsf{BD}_n^-$, $\mathsf{LC}^\bullet = \mathsf{LC}^-$, $\mathsf{LC}_n^\bullet = \mathsf{LC}_n^-$, $\mathsf{BW}_n^\bullet = \mathsf{BW}_n^-$, $\mathsf{KC}^\bullet = \mathsf{KC}^-$, and $\mathsf{BTW}_n^\bullet = \mathsf{BTW}_n^-$ for all $n \geq 1$.

Proof. We know that $\mathsf{CPC} = \alpha($$)$. Hence, $\mathsf{CPC}^\bullet = \beta($$)$ by Theorem 5.3.5.

All other cases follow similarly from Tables 5.1.10 and 5.2.32. ■

Let us return to the main goal of this section. We want to prove that the outer space embedding preserves Krikpe-completeness. We now know how logics of the form $\mathsf{L}^\bullet$ are axiomatised by cofinal steady canonical formulas with respect to the canonical axiomatisation of L. By Theorem 5.2.14, all logics are distinguishable by cofinal steady canonical formulas with $D^\Box = A$. Hence we want to show that if $\mathsf{L}^\bullet \nvdash \beta(A, D, A, \bot)$, then there exists an $\mathsf{L}^\bullet$-frame that refutes the formula. We will extract this frame from the Kripke-completeness of L by extending it with a suited lax relation.

Lemma 5.3.7. Let X be an intuitionistic frame and let A be a finite nuclear algebra. If $h : A \to X^{\mathrm{Up}}$ is an $\mathcal{I}$-embedding then there exists an onto subreduction $f : X \to A_*$.

Proof. Let Y be a dual space of A and X' a dual space of X^{Up}. By duality, we have an onto partial Esakia morphism $g : X' \to Y$. Let $i(x) := \{U \in \mathrm{Up}(X) \mid x \in U\}$ for all $x \in X$. It follows that $i(x)$ is a prime filter for all $x \in X$. Whence, $i(x) \in X'$. We define $f : X \to Y$ with $\mathrm{dom}(f) := \{x \in X \mid i(x) \in \mathrm{dom}(g)\}$ and $f(x) := g(i(x))$ for all $x \in \mathrm{dom}(f)$. We need to show $f[{\uparrow}x] = {\uparrow}fx$ for all $x \in \mathrm{dom}(f)$.

($\subseteq$). Suppose $x, y \in \mathrm{dom}(f)$ and $x \leq y$. Then $i(x) \subseteq i(y)$, which means $f(x) = g(i(x)) \subseteq g(i(y)) = f(y)$.

($\supseteq$). Suppose $x \in \mathrm{dom}(f)$ and $f(x) \leq y$. Then then $g(i(x)) \leq y$ so there is $z \in \mathrm{dom}(g)$ such that $i(x) \leq z$ and $g(z) = y$. Since Y is finite ${\uparrow}y, {\uparrow}y \setminus \{y\} \in \mathrm{ClopUp}(Y)$, which means $g^*({\uparrow}y), g^*({\uparrow}y \setminus \{y\}) \in \mathrm{ClopUp}(X')$. Furthermore since $\mathrm{dom}(g)$ is finite, by Lemma 5.1.6, $\mathrm{dom}(g) \in \mathrm{Clop}(X')$. Whence, $g^{-1}({\uparrow}y) = g^*({\uparrow}y) \cap \mathrm{dom}(g) \in \mathrm{Clop}(X')$ and similarly $g^{-1}({\uparrow}y \setminus \{y\}) \in \mathrm{Clop}(X')$, by Lemma 4.2.3(2). Therefore, $g^{-1}(y) = g^{-1}({\uparrow}y) \setminus g^{-1}({\uparrow}y \setminus$

$\{y\}) \in \mathrm{Clop}(X')$. Then by Lemma 2.3.1,

$$g^{-1}(y) = \bigcup_{i=1}^{n} \widehat{U}_i \setminus \widehat{V}_i$$

for some $U_1, V_1, \ldots, U_n, V_n \in \mathrm{Up}(X)$. Consequently, ${\downarrow} g^{-1}(y) = \bigcup_{i=1}^{n} X \setminus (\widehat{U}_i \to \widehat{V}_i)$. Since $i(x) \in {\downarrow} g^{-1}(y)$, there is $U, V \in \mathrm{Up}(X)$ such that $\widehat{U} \setminus \widehat{V} \subseteq f^{-1}(y)$ and $x \in X \setminus (U \to V) = {\downarrow}(U \setminus V)$. Then there is $u \in U \setminus V$ such that $x \leq u$. Hence, $i(u) \in \widehat{U} \setminus \widehat{V} \subseteq g^{-1}(y)$. Whence, $u \in \mathrm{dom}(f)$ and $f(u) = y$. ■

Lemma 5.3.8. Let X be an intuitionistic frame and let Y be a nuclear space. If $f : X \to Y$ is a subreduction, then there exists a lax relation R on X such that f is a modal subreduction.

Proof. Define $R \subseteq X^2$ as $R := f^{-1}[R_\triangle]_\between$. That is, xRy iff there exists $z \in \mathrm{dom}(f)$ such that $x \leq z \leq y$ and $fz \in R_\triangle$. It follows that R is a lax relation. Besides, for all $x \in X$ and $y \in Y$,

$$\begin{aligned} y \in R[f[{\uparrow}x]] &\iff \exists z' \in \mathrm{dom}(f) \text{ such that } x \leq z' \text{ and } fz'Ry \\ &\iff \exists z \in \mathrm{dom}(f) \text{ such that } x \leq z, fz \leq y \text{ and } fz \in R_\triangle \\ &\iff \exists z \in \mathrm{dom}(f) \text{ such that } x \leq z \leq y', fy' = y \text{ and } fz \in R_\triangle \\ &\iff y \in f[R[x]]. \end{aligned}$$

Whence, f is a modal subreduction. ■

Since finite frames can be used to represent all finite algebras we obtain the following from a dual reading of Lemma 5.3.8.

Corollary 5.3.9. Let A be a nuclear Heyting algebra and let B be a finite Heyting algebra. If $h : A \to B$ is an $\mathcal{I}$-embedding then there exists a nucleus on B such that h is a $\mathcal{J}$-embedding.

Lemma 5.3.10. Let L be an intermediate logic. Then $\mathsf{L}^\bullet \nvdash \beta(A, D, A, \bot)$ implies $\mathsf{L}^\bullet \nvdash \beta(A_\top, D, \varnothing, \bot)$.

Proof. Suppose there is some $\mathsf{L}^\bullet$-algebra B such that $B \nvDash \beta(A, D, A, \bot)$. As in the proof of Theorem 5.3.3, it follows that $B' \nvDash \alpha(A, D, \bot)$. Following the other direction of the proof of Theorem 5.3.3, we get $B \nvDash \beta(A_\top, D, \varnothing, \bot)$. ■

We will now prove the lax analogue of the Dummett-Lemmon conjecture, i.e., the least outer space companion of every Kripke complete intermediate logic is Kripke-complete.

Theorem 5.3.11 (Preservation of Kripke-completeness)**.** If L is a Kripke complete intermediate logic, then $\mathsf{L}^\bullet$ is Kripke complete.

Proof. Suppose $\mathsf{L}^\bullet \nvdash \beta(A, D, A, \bot)$. Then by Lemma 5.3.10, $\mathsf{L}^\bullet \nvdash \beta(A_\top, D, \varnothing, \bot)$. By Theorem 5.3.3, $\mathsf{L} \nvdash \alpha(A', D, \bot)$. Hence, there exists an L-frame X such that $X \nVdash \alpha(A', D, \bot)$. By the Refutation Lemma, there exists $Y \in \mathrm{Up}(X)$ and an $\hat{\mathcal{I}}$-embedding $h : A \to Y^*$ that is $\vee$-compatible over D. By Lemma 5.3.7, there is a subreduction $f : Y \to A_*$. Then $f : X \to A_*$ is a subreduction. By Lemma 5.3.8, we can extend X to a lax frame such that f a modal subreduction. By Theorem 4.2.9, $f^* : A \to X^*$ is a $\hat{\mathcal{I}}$-embedding. If follows that f^* is a $\hat{\mathcal{J}}$-embedding since f is a modal subreduction by Lemma 5.2.15(3). By the Refutation Lemma, $X \nVdash \beta(A, D, A, \bot)$. By Theorem 3.5.2 and duality, X is an $\mathsf{L}^\bullet$-frame. ■

Similarly, we can show that the outer space embedding preserves fmp and tabularity.

Theorem 5.3.12 (Preservation of fmp)**.** Suppose L is a intermediate logic with the fmp. Then $\mathsf{L}^\bullet$ has the fmp.

Proof. Suppose $\mathsf{L}^\bullet \nvdash \beta(A, D, A, \bot)$. Again, it follows that $\mathsf{L} \nvdash \alpha(A', D, \bot)$. Hence, there exists a finite L-algebra B such that $B \nvDash \alpha(A', D, \bot)$. By the Refutation Lemma, there is a homomorphic image C of B and an $\hat{\mathcal{I}}$-embedding $h : A' \to C$ that is $\vee$-compatible over D. Then C is finite since B is. By Corollary 5.3.9, we can extend C to a nuclear algebra such that $h : A \to C$ is a $\hat{\mathcal{J}}$-embedding. Since C' is a homomorphic image of B we have $C' \vDash \mathsf{L}$, and therefore by Theorem 3.5.2, $C \vDash \mathsf{L}^\bullet$. Moreover, by the Refutation Lemma, $C \nvDash \beta(A, D, A, \bot)$, as required. ■

Theorem 5.3.13 (Preservation of tabularity)**.** Suppose L is a tabular intermediate logic. Then $\mathsf{L}^\bullet$ is tabular.

Proof. Since L is tabular there exists some finite Heyting algebra B that determines the logic. We claim that $\mathsf{B} := \{C \in \mathbb{NA} \mid C' \in \mathbf{H}(B)\}$ determines $\mathsf{L}^\bullet$. Note that B is finite modulo isomorphisms since B is finite. Suppose $\mathsf{L}^\bullet \nvdash \beta(A, D, A, \bot)$. It follows that $\mathsf{L} \nvdash \alpha(A, D, \bot)$. Hence, $B \nvDash \alpha(A, D, \bot)$. By the Refutation Lemma, there is a homomorphic image C of B such that there is an $\hat{\mathcal{I}}$-embedding $h : A \to C$ that is $\vee$-compatible over D. By Corollary 5.3.9, we can extend C to a nuclear algebra such that $h : A \to C$ is a $\hat{\mathcal{J}}$-embedding. By the Refutation Lemma, $C \nvDash \beta(A, D, A, \bot)$. Moreover, $C \in \mathsf{B}$ since C' is a homomorphic image of B. ■

We have shown that the outer space embedding preserves tabularity, fmp, and Kripke-completeness. Moreover, it follows from Theorem 5.3.3 that it also preserves finite axiomatisations.

Theorem 5.3.14 (Preservation of finite axiomatisations)**.** If a logic L is finitely axiomatisable then $\mathsf{L}^\bullet$ is finitely axiomatisable

Proof. Suppose L is axiomatised by the finite set of canonical formulas Γ. By Theorem 5.3.3, $\mathsf{L}^\bullet$ is axiomatised by $\{\beta(A_\top, D, \varnothing, \bot) \mid \alpha(A, D, \bot) \in \Gamma\}$, which is of the same size as Γ. ■

In this chapter we have displayed the robustness of the method of (steady) canonical formulas for the lax logic. We used them to introduce steady and cofinal steady logics. These logics have good geometric properties and are structurally similar to subframe and cofinal subframe logics. We have given several examples of steady and cofinal steady logics, and shown that they are linked to subframe and cofinal subframe logics via the outer space embedding. Moreover, we have shown that the outer space embedding preserves Kripke-completeness, fmp, and tabularity.

Chapter 6

Conclusion and Future Work

In this thesis we developed the method of canonical formulas for the lax logic. Consequently, we showed that every lax logic is axiomatised by lax canonical formulas. Moreover, we have proven that lax logics axiomatised by disjunction-free formulas are axiomatised by disjunction-free lax canonical formulas, and similarly for lax logics axiomatised by other restricted syntax. Of particular interest are steady canonical formulas that encode the structure of $\Box$ in only one direction. Through them we obtained steady and cofinal steady logics – natural counterparts to subframe and cofinal subframe logics; they are characterised by classes of spaces closed under steady subframes. We also showed that these logic enjoy the fmp. Furthermore, we have used steady canonical formulas to show that the outer space embedding preserves, among other properties, Kripke-completeness. That is, the least lax logic containing a Kripke-complete intermediate logic is Kripke-complete, i.e., we have given a positive answer to a lax analogue of the Dummett-Lemmon conjecture.

We end this thesis with some directions for further work.

- There is a symmetry between intuitionistic $\mathsf{S4}$ ($\mathsf{IS4}$) and the lax logic. They are both characterised by multiplicative idempotent modal operators. The only difference is that for lax logic we have $p \to \Box p$ as an axiom, and for $\mathsf{IS4}$ we assume $\Box p \to p$. This raises the question whether we can develop canonical formulas for $\mathsf{IS4}$. There is no analogue of Diego's Theorem for a suited locally finite reduct. However, we can find finite "stable" subalgebras (in the sense of [13]) using local finiteness of the $\{\wedge, \vee, \top, \bot\}$-reduct. In a way this would echo the fact that both this reduct and the $\hat{\mathcal{I}}$-reduct of Heyting algebras can be used to develop canonical formulas for intermediate logics [7].
- Since lax companions are similar to modal companions one might think of an analogue of the Blok-Esakia Theorem. In fact, for the outer space embedding there is a trivial result like this. It is easy to see that there is an isomorphism between the lattice of extensions of $\mathsf{PLL} \oplus \Box\bot$ and the lattice of intermediate

logics.

- Canonical formulas provide an interesting mechanism for defining "semantic" translations. That is, we can define translations by providing a mechanism that transforms refutation patterns. For instance, the outer space translation is such an embedding. It can be seen as taking intuitionistic frames and turning them into lax frames. Another example are intermediate logics and their smallest modal companion, see [17, Theorem 9.65]. Study of such translations might give very general preservation theorems. Moreover, one might wonder which syntactical translations are semantically definable and vice versa.

- In [15, 33] it is shown how to use the method of canonical formulas (and rules) to obtain a basis for admissible rules and an alternative proof of the decidability of the admissibility problem for IPC and modal logics $\mathsf{K4}$ and $\mathsf{S4}$. We leave it as an open problem whether the canonical axiomatisation of lax logics developed in this thesis could provide a characterisation of admissible rules in PLL and other lax logics.

- As briefly mentioned at the end of Section 5.2, we could construct steady canonical formulas that also encode the $\Box\bot$-structure of algebras. This would allow us not only to characterise a bigger class of steady-like logics, moreover it could possibly provide a method for proving similar preservation results as in Section 5.3 for the inner space embedding.

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