# Relational Patterns, Partiality, and Set Lifting in Modal Semantics

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January 26, 2021

**Abstract**

We articulate a relational understanding of modality, and show it at work in a survey of pre-order models, set-lifting, and modal languages.[1]

## 1 Modality as a relational pattern

People often demand to know from me "what a possible world is". How can they understand modal logic, if I do not tell them this basic simple thing first? When I hear such questions, alarm bells go off in my mind.

**What and how questions.** As a student, I was raised in the spirit of Evert Beth, an erudite philosopher and creative logician. His magnum opus is "Foundations of Mathematical Logic" (Beth, 1959), and a chapter that impressed me is that on 'Aristotle's Theory of Science' with its focus on "What Is" questions. For Beth, this agenda became obsolete with the rise of modern science, where the deep insights came from asking "How" questions about how Nature functions. Beth felt that, despite the efforts of Russell and his generation in bringing together science and philosophy, Aristotle's essentialist mindset still held sway widely. However this may be, ever since my student days, I have felt attracted to a philosophical "How" approach, as in David Lewis' famous saying that to know what a meaning *is*, one must know what a meaning *does* (Lewis, 1970).

[1]In the 1960s, Saul Kripke opened the doors to modern modal logic, and I was one of a generation of students entering this wonderland of philosophical ideas pursued with mathematical clarity. Over the years, many further influences have shaped the field, coming also from linguistics and computer science, and I am a representative of the resulting 'Amsterdam style' in modal logic, cf. (Blackburn, de Rijke & Venema, 2001), (van Benthem, Blackburn & Wolter, eds., 2006). But whatever agenda one lives by, one constant in the field is our shared indebtedness to Saul. It is a pleasure to make this small offering in his honor.

**Abstraction.** To me, possible worlds semantics is a Carnapian conceptual framework, and worlds are objects in abstract modal structures. When you use them to represent things, they can be just anything you like: points in time, situations, deals of cards, people. Abstraction is a cognitive talent and a driver in intellectual history. Being able to 'let go' of pre-set interpretations is an intellectual talent that fosters understanding at the level where it thrives best.[2]

**An interactive relational perspective.** Still, one can say more about worlds, even at an abstract level. Consider the relational models $\mathbf{M} = (W, R, V)$ introduced by Kanger and Kripke. Critics see the accessibility relations $R$ between worlds as an ad-hoc device for modeling some logics that would lack completeness theorems otherwise. My own view is the opposite: the relational structure is the heart of the matter. A possible world may have some internal structure of local facts, but its truly modal behavior is its *pattern of interaction* with other worlds. To paraphrase Zellig Harris' famous dictum about word meanings, "to understand a world, look at the company it keeps". This reflects the message of modern logic promoted by Russell and his generation. Objects are not just monads determined by their properties; equally important are the binary and other relations encoding their relationships to other objects: hence the change from the traditional Aristotelean logic of properties to a Fregean logic of predicates.[3]

I would go even further, and say that the modal content of a model *is* its relational pattern: modality lives 'in the air', not inside the worlds – it is team play. These modal patterns arise in many settings, leading to such different families as modal logics of information and knowledge, time and action, space, metaphysics, and more as the scope of modal logic keeps growing.

**Invariant patterns and modal languages.** But the relational perspective, once taken, is pervasive. In particular, a modal model is just one representative of the relational 'pattern' that I mentioned. The real structural pattern is the equivalence class of that model under the appropriate structural *invariance relation*. There are many fundamental invariance relations in mathematics and science, and logic follows suit, with relations such as isomorphism, homomorphism, or for basic modal logic: bisimulation (Blackburn, de Rijke & Venema, 2001).[4] This takes us to a next aggregation level: possible worlds models do not live alone, but in a modal universe of models, and the identity of a model is also determined by its relations to those other models.

[2]One can of course legitimately criticize an abstract semantics for fit to some concrete application. Such 'criticisms of fit' are in fact one engine of progress in the field.

[3]Saying that worlds are determined by how they relate to other worlds may sound circular. But think of Category Theory, where categories describe objects and morphisms, not through internal structure, but in a pattern of external interactions. Since Kripke's pre-order models for intuitionistic logic and modal S4 are categories, this analogy is perfectly felicitous, and modal logic is mathematically sound. However, I admit there may be legitimate metaphysical worries. If an object is determined by its interactions with other objects, changing its context will automatically change the object: this weathervane behavior may be considered problematic.

[4]The definition of bisimulation (Blackburn, de Rijke & Venema, 2001) reflects what we said above about worlds: matched worlds only need to agree on a few thin atomic properties; the modal structure is in the back-and-forth conditions on available relational steps.

In this setting, *modal languages* enter, poorer or richer, offering perspicuous ways of expressing invariant relational structure of models. There is an extensive literature on pattern invariance and definability in matching modal languages (Goranko & Otto, 2006). Thus, there is an equally important syntactic side to modal logic, which consists in the study of different languages for describing different relational patterns up to different levels of expressive power, depending on how structurally discerning we make the invariance relation.

These modal languages offer a generic perspective going beyond peculiarities of specific models whose complete structure may be infinitely complex. We can describe given models up to some level of syntactic complexity, using normal forms, modal Scott sentences, or even the drastic information reduction of the modal method of filtration (van Benthem & Bezhanishvili, 2020). I will not emphasize this important generic function in what follows, but fine-structure and genericity is what all languages, logical or natural, are for.[5] Also, modal syntax offers characteristic patterns of reasoning, and modal syntactic patterns, too, are a major strength of the framework that I will ignore here.

**Modal logics.** Finally, different modal languages induce different logical systems of valid inference forms, often axiomatizable in modal proof systems.[6] Of course, our relational perspective also applies to these languages and logics. Much about their identity is revealed through *translations* into their neighbors: other languages and logics, and this, too, will be a theme in what follows.

I apologize for my obsessive relational context-oriented stance at many levels, but to me, awareness of modal structure is really turtles all the way down.[7] Even so, I admit that the relational view can be challenged. There are also forces inside modal logic that locate the essence in a monad-style view of worlds. Appendix II contains further thoughts on the role of modal languages, and on design of modal languages for all the three relational levels mentioned here.

**Content of this paper.** The preceding themes become more concrete with some history of ideas. I will discuss partial possibilities, set-lifting and language design, since these topics relate to recent challenges to modal logic coming from situation theory, truthmaker semantics, or hyper-intensionality generally. As we shall see, these challenges have an interesting past, and their implementation involves doing more, rather than less, modal logic, requiring richer relational patterns than the sparse models of the founding generation. The presentation to follow is a light historical survey with commentary, and it can hopefully be appreciated even if one does not share, or rejects, my patternist stance.

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[5] One may wonder what makes a *language* modal, as opposed to any language for relational structures, say first- or second-order. Common answers are decidable validity or perspicuous variable-free notation for proof theory, but the border-line between modal and just any logical languages remains a bit elusive. In fact, switching perspectives is often helpful, with the 'standard translation' from modal into first-order languages as a prime example.

[6] The above expressive diversity is not the usual deductive diversity of different 'modal logics' over the same language, say: K, S4, or S5. Deductive diversity is best understood as reflecting different restrictions one can make on the relational patterns one is interested in.

[7] If you see *Coalgebra* (Jacobs, 2016) as the fresh young face of modal logic, then turtles all the way down, not bottom-up construction, is the non-wellfounded essence of modality.

## 2 Intuitionistic logic and modal S4

Let us start with the classic paper "Semantic Considerations on Intuitionistic Logic I" (Kripke, 1963). Our further themes will unfold from there.

**Basic language and semantics.** Kripke models for the language of intuitionistic propositional logic are triples $\mathbf{M} = (W, \leq, V)$ where $(W, \leq)$ is a reflexive transitive pre-order, and the valuation $V$ assigns sets of points in $W$ that are *upward closed*: if $s \in V(p)$ and $s \leq t$, then $t \in V(p)$. One can also restrict attention to *partial orders*, but this is not essential to what follows. The truth definition explains when a formula $\varphi$ is true at a point $s$ in a model $\mathbf{M}$ (written $\mathbf{M}, s \models \varphi$), with the following central recursive clauses:

$\mathbf{M}, s \models \varphi \wedge \psi$ iff $\mathbf{M}, s \models \varphi$ and $\mathbf{M}, s \models \psi$

$\mathbf{M}, s \models \varphi \vee \psi$ iff $\mathbf{M}, s \models \varphi$ or $\mathbf{M}, s \models \psi$

$\mathbf{M}, s \models \neg\varphi$ iff for no $t \geq s$, $\mathbf{M}, t \models \varphi$

$\mathbf{M}, s \models \varphi \to \psi$ iff for all $t \geq s$, if $\mathbf{M}, s \models \varphi$, then $\mathbf{M}, s \models \psi$

With this understanding of the logical operators, all formulas denote upward-closed sets of points. This semantics fits intuitionistic propositional logic: a formula is universally valid iff it is provable in Heyting's standard proof system.

**Geometry of information and inquiry.** While the completeness theorem shows 'fit' of validity and provability,[8] a broader virtue in this analysis is the *conceptual framework* as a style of thinking. We can now analyze modal axioms or proof principles in terms of an easily visualized 'geometry of information', or, viewing models as possible histories of investigation, the 'geometry of inquiry'.[9]

**The package.** Many decisions went into making intuitionistic logic the system of reasoning here. There was a choice of language (the vocabulary of propositional logic), type of models, constraints put on the ordering (reflexivity and transitivity), special properties demanded of propositions (upward-closure), and specific truth conditions for the logical operations linking language and models. Like in many modal semantics, this is a finely tuned conceptual machinery with lots of interlocking moving parts that could easily have been set differently. For instance, classical logic emerges by changing the clause for disjunction from immediate to 'cofinal choice', and requiring denotations $X$ of formulas to satisfy upward-closure plus *regularity*: "if $\forall t \geq s\, \exists u \geq t : u \in X$, then $s \in X$.[10]

[8] Tighter connections between semantic structures and proofs themselves are the 'full completeness' results in more category-theoretic frameworks (Abramsky & Jagadeesan, 1994).

[9] For a concrete example, consider the *Disjunction Principle* for intuitionistic logic: if $\varphi \vee \psi$ is provable, then so is $\varphi$ or $\psi$. A semantic proof for this takes pointed countermodels for $\varphi$ and $\psi$, forms their disjoint union, and puts the distinguished points under a new root: in the resulting model, the disjunction is refuted. In fact, the semantic content of the Disjunction Principle is *precisely* the model glueing property of joint rooting, but a detailed correspondence perspective on modal derivation rules would be a subject for a more technical paper.

[10] In topological terms, this is a restriction to the Boolean algebra of regular open sets. One can also think of a Sure Thing Principle: "if you see that $\varphi$ is inevitable, accept it right now".

**Structures.** Though a very simple mathematical structure, around 1960, pre-orders were the right choice at the time. In fact, they are special cases of the topological semantics for intuitionistic and modal logics which had been known since the 1930s,[11] but specialization was progress. Preorders are also close in spirit to Beth tree models for intuitionistic logic (Beth, 1956), but their simplicity won the day.[12] In fact, far beyond intuitionistic logic, Kripke's models offer a setting where many logics can be defined and compared, as we shall see.

**Two states in the same land: the modal logic S4.** Intuitionistic logic famously shares the land of pre-orders with classical propositional modal logic, where the induced logic is S4. With the latter, the style of thinking is different. The classical understanding of the Boolean operations is retained, but the new order structure $\leq$ is described by matching universal and existential modalities, making S4 an 'explicit' classical counterpart to the 'implicit' non-classical intuitionistic logic on the same models (van Benthem, 2019).

Comparisons between the two neighbors can run in various ways. In terms of structural invariance relations, the language of modal S4 has a natural and well-known fit with the ubiquitous notion of bisimulation, whereas an invariance analysis of intuitionistic logic requires more care, involving entangled directed simulation relations (Paterson, 1997), (Olkhovikov, 2013).

**Translations.** Surely the most famous connection between the two great pre-order logics is one of syntactic *translation*. Gödel 1933 translated intuitionistic propositional logic faithfully into modal S4, and Kripke's truth definition presented above is a semantic transposition of this recursive translation. The translation might be seen as a case of embedding a poorer system into a richer one, since modal S4 does not just govern reasoning about upward-persistent formulas $\Box\varphi$, but also about non-persistent formulas $\Diamond\varphi$. Yet a surprise was in store. Answering a question of Dana Scott, Fernandez 2006 gave a converse translation, embedding S4-validity faithfully into intuitionistic validity – with an important correction in Goré & Thomson 2019. If one truly knows a logic through its neighbors, as we have suggested, all this shows how relations can be delicate – especially, given the lack of a systematic theory of translatability (and not to forget: non-translatability) between logical languages and systems. We know much less than we would like about this modal meta-realm.[13]

**Conclusion.** Pre-orders with an intuitive reading as information states or histories of inquiry offer a rich playground for modeling and comparing existing logics. This marriage is a happy one: in modal logic since the 1960s, intuitionistic and classical logics have often been studied in tandem, to mutual benefit. In the following sections, we will expand this co-existence to more logics.

[11] Preorders match special 'Alexandroff topologies' (van Benthem & Bezhanishvili, 2007).

[12] Of course, these issues are never settled once and for all. Dummett and van Dalen have argued that Beth models are superior for some foundational purposes, and an attractive mathematical perspective taking them on board is found in Bezhanishvili & Holliday 2019.

[13] Two recent dissertations discussing the landscape of translations between modal logics are French 2011, Kocurek 2018. General model-theoretic perspectives on relative interpretability can be found in van Benthem & Pearce 1984, Barrett & Halvorsen, 2016.

One can also view this section as adding *fine-structure* to classical models. Points in an intuitionistic model are stages toward a complete classical valuation, persistence holds for assertions that are definitely true, while modal S4 adds assertions about possibilities that may be present for a while and then drop out.[14] The inclusion order embodies this fine-structure of partial approximation, and what its presence leads to is more, rather than less modal logic.

# 3 Languages and logics on information models

**Semantics or language design.** Once on the level playing field of pre-orders, further aspects of information and inquiry can be studied. This represents a shift in direction from what is suggested by the term 'possible worlds *semantics*'. A semantics is usually given for some existing language, which sets the standards for what proposed models are meant to achieve. In this sense, we have discussed semantics for an intuitionistic language and its associated reasoning practice, or for a language with modal expressions of, say, knowledge or necessity. But one can also take proposed structures as primary, and ask what sort of language would best bring out important features of reasoning in this setting. Both directions occur in modal logic, with new language design perhaps more prominent in logics of time and action. We will continue in the latter mode, but in our story, the two directions are not mutually exclusive or opposed.

*Note*: Henceforth, we will drop the term 'world' and use the neutral 'point'.

**A richer modal information logic.** Our first offering is a system proposed in van Benthem 1996. One natural structure in pre-orders are *suprema* (lowest upper bounds) and *infima* (greatest lower bounds) of states. On partial orders, suprema and infima will be unique, but one need not assume this in general. Here, a supremum of points $s$ and $t$ can be seen as modeling a 'merge' of two information states, or alternatively, as a meeting point where two lines of inquiry come together. This suggests adding two binary modalities $\langle \sup \rangle \varphi$ and $\langle \inf \rangle \varphi$ to the basic modal language, with the following interpretation:

$\mathbf{M}, s \models \langle \sup \rangle \varphi\, \psi$ iff there exist two points $t, u$ such that
(a) $s$ is a supremum of $t, u$, (b) $\mathbf{M}, t \models \varphi$ and (c) $\mathbf{M}, u \models \psi$.

and likewise for $\langle inf \rangle \varphi \psi$. In this language, we can define the standard modalities in both directions. For instance, $\langle \leq \rangle \varphi$ is equivalent to $\langle inf \rangle \varphi \top$.

The new binary modalities satisfy counterparts of familiar modal principles, including distribution over disjunctions, giving the system a natural feel.

As for further principles, we did not assume that suprema and infima always exist in our models. Doing so will create a semi-lattice or lattice structure that

[14]This is like in formal learning theory (Kelly, 1996), where we only know the complete truth in a completed history, while knowledge and belief play on the way. Note that this form of knowledge is *procedural*: it refers to a current process of inquiry, and thus differs from more standard static conceptions in epistemic logic of knowledge as a range of options. Van Benthem 2009 discusses Kripke's models extensively from both epistemic perspectives.

validates new principles beyond the minimal logic. However, these can still be analyzed by modal frame correspondence techniques, as many principles in this setting have so-called 'Sahlqvist syntax' (Blackburn, de Rijke & Venema, 2001). An example of an additional principle is Associativity

$$\langle \text{sup} \rangle (\langle \text{sup} \rangle \varphi\, \psi)\, \alpha \to \langle \text{sup}\, \rangle \varphi \langle \text{sup} \psi \rangle\, \alpha$$

This enforces existence of enough suprema in the model to reorder arbitrary combinations of the form $sup(sup(s,t),u)$ to $sup(s, sup(t,u))$. While this seems a harmless requirement that helps computation, such existential assumptions can increase the complexity of validity: some basic modal logics of associative binary modalities are undecidable (Kurucz, Németi, Sain & Simon, 1995).

Perhaps surprisingly, not much is known about this simple modal information logic mildly extending the familiar S4. For instance, no complete axiomatization has been found yet. In Appendix 1, we give an axiomatization using an extra device: nominals that name specific points. This will show at least how one can work in practice with a modal language like the one presented here.[15]

Another and perhaps even more striking open problem about our simple extension of S4 is the following: Is basic modal information logic *decidable*?

**Categorial logics.** Viewing points in our models as information pieces also suggests further modalities, such as the following implication, stating that combining with any evidence for the antecedent yields evidence for the consequent:

$\mathbf{M}, s \models \varphi \Rightarrow \psi$ iff for each point $u$ with $\mathbf{M}, u \models \varphi$ and each point $t$ that is a supremum of $s$ and $u$, it holds that $\mathbf{M}, t \models \psi$.

This implication satisfies the basic principles of categorial and relevant implication (Bimbó & Dunn, 2008) (Kurtonina & Moortgat, 2009). It can also be seen as introducing a binary modality for the ternary supremum relation $z = sup(x,y)$ from the perspective of the point $x$ rather than $z$, as with $\langle \text{sup} \rangle$, leading to a more elegant overall modal logic design (Venema, 1992).

**Updates and matching information conditionals.** Here is another perspective on the pre-order semantics of Section 2. We follow the line of thinking in van Benthem 1989 who proposed generalizing from Brouwer's 'idealized mathematician' to ordinary cognitive agents. Moving upward along the ordering $\leq$, such agents arrive at more information about the actual world, or moving downward, they backtrack and give up things they thought they knew.

These informational movements correspond to natural *epistemic actions.* Updates with new information $\varphi$ take us from a current point to points higher up in the $\leq$ order where $\varphi$ is true, perhaps, according to a common intuition about updating, to *closest* points where $\varphi$ is true. Conversely, what may be called downdates take us to closest points lower in the order where $\varphi$ is false.

[15] In that same appendix, we will also briefly discuss modal axiomatizations of the above-mentioned semi-lattices and lattices, as an instance of 'lifting' equational theories of algebraic structures to modal logics– in the spirit of Section 5 below.

The dynamic perspective of information update in pre-order models at once suggests the introduction of a matching new modality to the usual vocabulary:

$\mathbf{M}, s \models [+\varphi]\psi$ iff for each point $t \geq s$ that is closest to $s$ among the points satisfying $\varphi$ in the model, $\mathbf{M}, t \models \psi$.

Downdate modalities are defined in a completely analogous manner referring to closest points $T$ below $s$ in the information ordering $\leq$.

The resulting logic is a simple abstract theory of information update and retraction, comparable in scope with belief revision theory (Gärdenfors & Rott, 1995), or dynamic-epistemic logic (Baltag & Renne, 2016). The update modalities also connect with *conditional logic*, being a Lewis-style variant of intuitionistic implication. With this natural modal system, we are again in largely uncharted territory, except for some forays in van Benthem 2018. The complete conditional update logic of pre-orders remains to be determined.

**Structures and invariances.** Our examples raise the question what structures are most natural. Should we specialize from pre-orders to *lattices*, where suprema and infima always exist, or to distributive lattices? Computer science has a long tradition in relevant abstract models of information. In particular, Scott's *Domain Theory* (Abramsky & Jung, 1994) offers a rich account of information structures as DCPOs: partial orders where all suprema of directed subsets exist, over which abstract computation can take place.[16]

In conjunction with this, there is also the issue of which structural invariance relations match the extended languages introduced just now. For a modal information theory over lattices, a bisimulation analogue of 'Scott continuity' might be the answer. For our conditional logic of updates and downdates, some adaptation of known bisimulations for conditional logic may be appropriate.

**Digression: partial logic.** This may be a good point to mention a further striking feature of some logical theories of information: a distinction between *positive* and *negative* information, and matching this, a simultaneous and mutually recursive use of 'support' and 'rejection' of formulas at pointed models in modal semantics. Examples of this approach are the 'data semantics' of Veltman, 1984 or the 'polarity-based' situation semantics of Barwise & Perry 1983, and of course, there is also the long three- (or higher-)valued tradition in intuitionistic and modal logic (Wansing, 1993).

There is something appealing to the positive/negative distinction, even in the restricted setting of mathematical inquiry. The modal-style negation $\neg\varphi$ in an intuitionistic model tells us that the current space of inquiry will never lead to support for $\varphi$: as in some indirect cultures, negation is signaled by silence. But one can also have a concrete counter-example to $\varphi$ available right now, representing a stronger form of negation as '*certified* exclusion'.

[16] I will not say much about Domain Theory in this paper, but it is rather surprising to me how little impact this fundamental and elegant theory of information and computing developed in the 1970s has had in philosophical logic or even in modal logic.

With a partial approach, logical operators split into variants, and so do notions of consequence, in ways that are well-known from the literature (Blamey, 2002). Such a framework also allows us to separate positive support from negative support in new ways. An example is the truthmaker semantics of Fine 2014 and later publications. It uses complete distributive lattices as semantic structures, and then interprets a propositional language with clauses like the following. An information state $s$ supports a disjunction $\varphi \vee \psi$ as saying that $s$ is the infimum of states $t$ supporting $\varphi$ and $u$ supporting $\psi$. This is the attractive idea of a disjunction as a 'mixture' that has a distinguished history. Next, an information state $s$ rejects $\varphi \vee \psi$ if it rejects both $\varphi$ and $\psi$: i.e., the classical view of refuting both disjuncts, without adding rejection of mixtures.

**Translations.** Despite the diversity of richer partial logics over preorders or lattices, many connections remain under translation. For instance, faithful translations from many-valued partial logics into classical true/false logics have long been folklore. Van Benthem 2018 presents a translation from Fine's truthmaker semantics into the above modal information logic with a double recursion trick that goes back to Gilmore in the 1950s. Many further translations no doubt exists, although it would also be of interest to show negative results for a change, such as the non-translatability of the above two extended modal logics of information, the one for suprema/infima and that for update conditionals.

**Conclusion.** Preorder models and lattices support richer languages than those of intuitionistic logic or modal S4, and provide a vehicle for modal analyses of information combination or update. While this opens up an area of investigation, it is surprising how easy it is to define new simple modal systems for which open problems arise at once. The unknown starts right at our doorstep.

# 4 Possibility semantics

**Motivations for partializing.** One way of viewing the preceding sections is that of modalizing the structure of an approximation or growth relation $\leq$. Let us now enrich the perspective, and start from a classical modal logic whose accessibility relation can stand for anything from epistemic uncertainty to time steps or process actions. How can we partialize this sort of semantics?

Systems to this effect were proposed around 1980. The motivations in Humberstone 1981 for a partialized modal semantics were philosophical, such as approximating over-committed possible worlds in the classical sense by partial situations, or basing knowledge and belief on partial 'possibilities'. The motivations in van Benthem 1981 for partializing the semantics for the language of first-order logic were mathematical, namely, removing the inelegant non-canonicity in the usual Henkin-type completeness argument which needs an arbitrary choice of a maximally consistent extension, and also removing the appeals to non-constructive principles like the Axiom of Choice in standard model theory. These possibility semantics did not catch on widely in the ensuing decades, though rediscoveries have occurred, cf. Rumfitt 2015.

**Possibility semantics for modal logic.** Consider the language of propositional modal logic, with a universal modality $\Box$ and an existential modality $\Diamond$. Modal models are now tuples $\mathbf{M} = (W, R, \leq, V)$ where $\leq$ is a preorder and $R$ is a binary relation subject to the following two constraints, for all points $s, t, u$:

(a) if $s \leq t \, R \, u$, then $\exists v : \, s \, R \, v \leq u$,

(b) if $s \, R \, t$, then $\exists u \geq s \, \forall v \geq u \, \exists w \geq t \, : v \, R \, w$.

Here the 'compatibility relation' $x \, C \, y$ holds if $\exists x : \, z \leq x \wedge z \leq y$.

We also stipulate that the sets $V(p)$ assigned by the valuation to proposition letters are upward-closed and regular w.r.t. $\leq$ (cf. Section 2 for this notion).

Now the key truth conditions run as follows:

$\mathbf{M}, s \models \varphi \wedge \psi$ iff $\mathbf{M}, s \models \varphi$ and $\mathbf{M}, s \models \psi$

$\mathbf{M}, s \models \varphi \vee \psi$ iff for all $t \geq s$, there is a $u \geq t$ with $\mathbf{M}, u \models \varphi$ or $\mathbf{M}, u \models \psi$

$\mathbf{M}, s \models \neg\varphi$ iff for no $t \geq s$, $\mathbf{M}, t \models \varphi$

$\mathbf{M}, s \models \Box\varphi$ iff for all $t$ with $sRt$, $\mathbf{M}, t \models \varphi$

The above two perhaps somewhat technical-looking conditions on possibility models belong to a family of possible connections between the accessibility relation $R$ and the partial extension relation $\leq$ in the literature. They are there to ensure that the following assertion becomes true. The denotations of any formula of the language are upward-closed and regular.[17]

The minimal modal logic K is the complete theory of possibility semantics in this sense. First, each standard modal model is a possibility model if we take the identity relation for $\leq$, and in that special case, standard modal semantics coincides with possibility semantics. Thus, non-theorems of K are refultable on possibility models. But also, given any pointed possibility model for a modal formula $\varphi$, we can turn this model into a classical modal model for $\varphi$ by restricting to the submodel of points that 'decide' all subformulas of $\varphi$.[18]

**Concerns and considerations.** Now all this invites discussion. The presented semantics looks somewhat engineered, so have we just managed to replace something simple (standard modal semantics) by something more complex (possibility semantics) for no better reason than ideological purity? One mathematical benefit of this semantic move is that the current more complex set-up allows for completeness proofs that are much more constructive than the usual ones, involving no appeals to set-theoretic equivalents of the Axiom of Choice like Zorn's Lemma. A philosophical benefit is that one can now model functional intuitions of modal notions that would not work in standard semantics, such as an agent's belief referring to just one possibility instead of a set of worlds.

[17] The intuitionistic modal logic of Božić & Došen 1984 uses only the first condition.

[18] For details, cf. the Filtration Lemma for possibility models in Holliday 2021. I thank Wes Holliday for removing some epicycles in my thinking about this completeness issue.

Another point to note is that the 'packaging' in this setting (more complex than in Kripke's semantics for intuitionistic logic) can be modified by shifting functions between the truth conditions, assumptions on denotations of formulas and structural constraints on the underlying possibility frames. In particular, no particular modal logic of possibility is favored by the possibilities setting, since there is a variety of natural meanings for the logical operations.[19]

**Modern themes in possibility logic.** A comprehensive development of the paradigm has been given by Holliday and his co-workers, cf. the wide-ranging recent survey Holliday 2021. This includes both philosophical themes, establishing connections to epistemology and philosophy of language, and mathematical ones, such as choice-free representation theorems in algebraic logic (Bezhanishvili & Holliday, 2020), or prospects for a choice-free model theory.[20] A general theme that returns here is *translation*. Extending the spirit of Gödel's translation of intuitionistic logic into modal S4, van Benthem, Bezhanishvili & Holliday 2017 shows how many modal possibility logics can be translated faithfully into bimodal logics over possibility frames, thus providing a richer modal language where many different logics of possibility and accessibility coexist.

**Set-based models.** One theme in possibility semantics points ahead at Section 5 below, namely the existence of special models whose points are sets. A good method for developing intuitions about possibility models is through the following construction. One takes a standard relational model $\mathbf{M} = (W, R, V)$ and considers the powerset of $W$ with the inclusion relation and the natural lifting of the accessibility relation $R$ to sets:

$$XRY \text{ iff } \forall s \in X\, \forall t \,\text{with}\, Rst:\ t \in Y$$

Setting an appropriate valuation for proposition letters as sets of sets yields a model $poss(\mathbf{M})$ that satisfies the above conditions for a possibility model.[21]

Now a simple but important *Possibilization Lemma* due to Holliday states that for all propositional modal formulas $\varphi$,

$$poss(\mathbf{M}), X \models \varphi \text{ iff } \mathbf{M}, s \models \varphi \text{ for all}\, s \in X.$$

Thus, these models have what might be called a 'supervaluation' flavor. In particular, for each formula $\varphi$, there is a unique weakest possibility (largest set) that supports $\varphi$. General possibility models can then be seen as an abstraction out of these structures, where one must decide which properties of set-possibilities make sense in general, and which are just special technical features of sets.

Reversing this abstraction process, there is an issue of representing abstract possibility frames as concrete set models. This is not a simple topic, and the best available results are only partial, cf. Harrison-Trainor 2017.

[19] For instance, intuitionistic disjunction is immediate choice, classical disjunction is eventual choice, and the earlier-mentioned 'mixture' interpretation of disjunction makes sense too.

[20] This line continues the filter-based approach to ultraproducts and model-theoretic definability results based on these that was initiated in van Benthem 1981.

[21] Many semantics of this sort impose valuations of special forms guaranteeing closure of denotations for atomic propositions under subsets, or unions, or yet other properties.

**Temporal dynamics.** Possibility models may contain information states about more than one object: states can be incompatible in many ways. A concrete form of modal possibility semantics occurs with *dynamical systems* where the relation $R$ is a transition function $F$ that takes states of the system to other states. This leads to new issues. For instance, from an informational viewpoint, *continuous* functions are important, as these approximate function values through appropriate approximations of their arguments. In the setting of pre-orders, this just means that the function has to be *monotonic* for inclusion:

if $x \leq y$, then $F(x) \leq F(y)$.

This can be viewed as an abstract notion of *computability*, an obvious dynamic addition to the logical study of static information structures.

Just as an illustration, the following axiom for continuity occurs widely in the modal literature on topological systems (Kremer & MInts, 2007):

$$\langle F\rangle[\leq]\varphi \rightarrow [\leq]\langle F\rangle\varphi$$

This commutation principle is not valid in the above general modal possibility logic, but it singles out an important application area.[22]

**Conclusion** The informational view of modal models combines naturally with accessibility relations that can stand for a wide range of notions, including dynamical systems and computation using information structure for some purpose. It is somewhat surprising that modal-style analysis of these natural dynamic companions to widely studied static notions has taken so long.

# 5 Set lifting

We noted in the preceding section that one concrete way of creating partial possibilities models is by set lifting standard semantics. Set lifting keeps emerging with a certain frequency in the logical literature as a means of providing richer partialized semantic structures, but it does not seem to have a well-known established tradition. In this section, we survey a few manifestations.

**Temporal interval logic.** Around 1980, a number of authors proposed changing the usual Prior-style semantics of temporal logics in terms of durationless points to temporal interval structures where logical languages could be interpreted directly at sets of points. Kamp 1979 presented a discourse semantics for temporal expressions in natural language where discourse creates event structures, where events have intervals as their running times. Van Benthem 1979 proposed both abstract and set-lifted interval semantics for temporal logic as representing ontologically better motivated primitives, more in the tradition of mereology. Interestingly, as an independent discovery around the same time, intervals were also proposed in the AI literature on 'common sense physics'

[22] A similar modal perspective applies to Domain Theory, but there, two notions of continuity occur, our monotonicity version and 'Scott continuity' (Abramsky & Jung, 1994).

(Hayes, 1979), (Allen, 1983), as providing a model for databases for maintaining temporal information that would admit of simpler computation than models closer to standard mathematical point structures.

Here is a sample of an interval tense logic, cf. Cresswell 1977, Halpern & Shoham 1986 for further samples. An *interval model* $\mathbf{M} = (T, <, \mathbf{I}, V)$ consists of an ordered set $(T, <)$ with a family $\mathbf{I}$ of subsets of $T$, and a valuation $V$ for proposition letters. There may be further constraints, say, making intervals *convex sets* $X$ satisfying $\forall s, t \in X : \forall u : s < u < t \Rightarrow u \in X$. The following truth definition lifts the usual semantics of temporal logic to intervals as indices of evaluation, while adding a modality the new relation of inclusion. We only display the major clauses, which reflect intuitions about temporal statements in natural language when the primitive components of time have duration:

$$
\begin{array}{lll}
\mathbf{M}, X \models \varphi \wedge \psi & \text{iff} & \mathbf{M}, X \models \varphi \text{ and } \mathbf{M}, X \models \psi \\
\mathbf{M}, X \models \neg\varphi & \text{iff} & \text{for no } Y \subseteq X,\ \mathbf{M}, Y \models \varphi \\
\mathbf{M}, X \models \varphi \vee \psi & \text{iff} & \text{there exist } Y, X \text{ with } X = Y \cup Z \\
 & & \text{such that } \mathbf{M}, Y \models \varphi \text{ and } \mathbf{M}, Z \models \psi \\
\mathbf{M}, s \models F\varphi & \text{iff} & \text{for some } Y >> X,\ \mathbf{M}, Y \models \varphi \\
 & & \text{with } Y >> X \text{ for } \forall x \in X,\ y \in Y : x < y \\
\mathbf{M}, X \models [\downarrow]\varphi & \text{iff} & \text{for all } Y \subseteq X,\ \mathbf{M}, Y \models \varphi
\end{array}
$$

Typically, standard logical notions split now. For instance, the new disjunction is the merge disjunction of modal information logic, but the language can also define the cofinal choice disjunction of possibility semantics (reading $Y \subseteq X$ as a counterpart of the earlier $x \leq y$). But there are also differences with possibility semantics, since the temporal setting brings its own features, such as the special set lifting $X << Y$ to total precedence of intervals.

Another difference are new constraints on intervals that have no counterpart at all for durationless temporal points, or for possibilities as usually conceived. One intriguing candidate is 'Homogeneity' (van Benthem, 1983): all intervals are isomorphic, a view of time where the large reflects in the small.

Also differently from earlier logics, the denotations of propositions are not required to have the special properties of upward-closure or regularity. Upward closure is downward closure in the interval subset order, and this only holds for special temporal assertions such as "being alive". Instead, interval logic offers a general theory which includes reasoning about propositions with many different kinds of temporal behavior coming, e.g., from well-known linguistic classifications of verb classes into states, activities, or accomplishments.

**Other set liftings.** Set lifting is also common in other areas, especially computer science, witness the theory of power domains (Plotkin, 1976), concurrent dynamic logic (Peleg, 1985), or event structures (Winskel, 1989). Brink 1992 is an excellent survey of both mathematical theory and a range of applications from computation to verisimilitude in the philosophy of science.

In a more logical setting, a modern example of set lifting is the logic of *dependence* of Väänänen 2007 where the language of first-order logic is interpreted on sets of assignments. This lifts the propositional connectives in the same way as we saw for temporal interval logic, while also retaining earlier persistence constraints. But crucially, the new richer structure is also exploited in other ways, adding new forms of quantification as well as dependence atoms making the functional dependence structure in sets of assignments explicit.

A recent instance of set lifting is the 'bilateral semantics' of Aloni 2019 for classical modal logic which redraws the boundaries of semantics and pragmatics in order to deal with delicate linguistic phenomena such as free choice permissions. Unlike possibility semantics, this semantics uses support and rejection clauses on sets of classical worlds, lifting the accessibility relation in new ways. Beyond persistence, another constraint on propositions in this semantics is that the sets supporting (or rejecting) a formula are closed under unions.[23]

Thus, motivations for set lifting, old and new, are diverse: what is common to many cases is the richer pattern of interpretation for basic logical operations.

**Theory of complex algebras.** There exists an unfortunately little-known mathematical theory of set lifting. Given any algebra $\mathbf{A}$, define the *complex algebra* $C(\mathbf{A})$ as the power set of the set of objects in $\mathbf{A}$, lifting the operations $f$ as follows, where we just display the case of a binary operation for simplicity:

$$X\, C(f)\, Y = \{xfy \mid x \in X, y \in Y\}$$

An early issue investigated at this level of generality is which equations valid for the original algebras remain valid as they stand for their lifted versions. A theorem in Gautam 1957 that this is rare, being the case only if the terms in the equations have only single occurrences of variables. Further theory of complex algebras from a general modal perspective can be found in Goldblatt 1989. As for more recent developments, Hodkinson, Mikulás & Venema 2001 is a modern source using game techniques to axiomatize varieties of complex algebras in a perspicuous manner when the original algebras have a computably enumerable theory. Incidentally, the theory of complex algebras covers both full power set algebras and subalgebras generated by suitably closed subfamilies of sets.

To understand what complex algebras do, and how they relate to the topics discussed earlier, it helps to think of a concrete example: complex Boolean algebras. Moving to a power set, or a suitable subfamily thereof, we find lifted operations that one might call the *inner Booleans* of the complex algebra:

$$X + Y = \{x + y \mid x \in X, y \in Y\}$$
$$X.Y = \{x.y \mid x \in X, y \in Y\}$$
$$-X = \{-x \mid x \in X\}.$$

[23] One of Aloni's systems does not just set lift, but combines interpretation on worlds and on sets of worlds. This joint way of proceeding seems closer to the two-sorted approach in terms of 'objects' and 'types' in Barwise & Seligman 1995, and it might also suggest a generalization of modal neighborhood semantics (Pacuit, 2017) to combined point- and set-languages.

Note that inner Boolean disjunction is precisely the set-lifted disjunction of temporal interval logics or dependence logics. In addition to inner Booleans, there are also *outer Booleans*, as one can also interpret disjunction or negation in the usual way via their corresponding set-theoretic operations. This duplication of available meanings for logical operations extends to the basic constants. For instance, the zero of Boolean algebra splits into an inner Boolean $\{\emptyset\}$ and an outer Boolean $\emptyset$ standing for the empty family of sets.[24]

Viewed more abstractly, the binary modality $\langle \text{inf} \rangle$ of Section 3 is a generalized counterpart to inner Boolean disjunction, and it also illustrates concretely how one can think of power algebras as supporting *modal operators*.

An intriguing feature of complex algebra that might extend to its modern manifestations are natural, yet hard open problems. For instance, no purely modal axiomatization is known for complex Boolean Algebra, though Goranko & Vakarelov 1999 give one by adding *nominals* from hybrid logic that name specific objects of the original algebra. Also open is the complex algebra version of a problem posed earlier in Section 3: *Is complex Boolean Algebra decidable?* This has been unresolved for several decades now, and it might pose a real challenge to existing modal methods for establishing decidability.

**Set lifting and abstract semantics** While set lifting is an elegant way of creating richer partialized versions of given algebras, one can also move on to more abstract versions. For instance, in temporal interval logic, an alternative approach treats intervals as primitive objects ('chunks of time') with two relations: one for temporal precedence, and one reflecting the extended nature of intervals: inclusion as in the above, overlap (Wiener, 1914) (Röper, 1980), or yet others. This leads to issues of representation of abstract interval or event models by concrete set structures that go back to Russell's views on the interface of science and common sense (Russell, 1926). Van Benthem 1983 has an extensive representation theory connecting the two views of time both ways.[25] Likewise, in possibility semantics, we noted the issue of set construction versus abstract structures that adopt some, but not necessarily all properties that hold for set-theoretic reasons. It is likely that similar abstract structures will emerge for the more recent manifestations of set lifting we have mentioned.

**Conclusion** Set-lifting is a general way of creating richer models for classical systems that keeps returning in semantics. The connection between set lifting, which imports specific structure of power sets, and more general abstract structures is intriguing, and there is room for a general representation theory behind its many manifestations. This may well call for a new phase of mathematization in modal logic going beyond the first wave of the 1970s.

[24]This splitting of constants is common in hyperintensional semantics, cf. Leitgeb 2019.

[25]An interesting analysis of Russell's views of 'public time' as a limiting point structure arising out of growing interval-based 'private times' is given in Thomason 1989 using an inverse limit construction that might make sense for all the partial semantics we have considered.

## 6 Conclusion

The introduction to this paper stated my commitment to relational structure as the locus of modality. How one views the semantics is of course just one aspect of modal logic, and there are further mantras of the Amsterdam School, concerning modal languages and complexity, that I have refrained from reciting. I also stressed abstract realms as a good place to work, facilitating pleasant and unforced walks from one logical system to another, borrowing ideas across.

The case history presented here of partialized relational structures may have shown that modal language and logic design is alive and well, and that, when all is said and done, current hyperintensional semantics may not be a threat to, but rather a part of modal logic. Another point was the coming together of influences from fields as different as philosophy, mathematics, and computer science when one takes a broader view of the development of modal ideas.

At the start of this paper, I objected to questions like "What are possible worlds?" as being intellectually misguided. Nevertheless, despite having used the word "modal" many times in this text, I admit there is a reasonable question of "What makes topics modal?" Even so, I have not offered an answer, let alone a definition. Overall, I feel that broad history of ideas is more revealing of a field than a priori definition and boundary setting.[26] And there is not even one such history. If you wish, modal logic is a Sea of Stories (Rushdie, 1990).

Which brings me to a final question from my students which I find totally to the point. "Your lecture was nice, but: does it *always* have to be modal logic?" To that question, I do have a definite answer: "Not necessarily".

**Acknowledgments** I thank Wes Holliday for thoughtful comments and firm corrections.

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[26] Cf. van Benthem 2006 for a history of ideas approach to modern logic in general.

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# A Axiomatizing modal information logic

While axiomatic proof systems for the sparse languages of K or S4 are well-known, working with the richer languages introduced in this paper may be much less familiar. This appendix sketches a Henkin-style heuristics for finding an axiomatization of the modal information logic of Section 3. The method uses two mild additional devices: *nominals* that name individual points, and an *existential modality* $E\varphi$ saying that $\varphi$ is true in at least one point in the current model. We only treat the case of the key modality $\langle \text{inf} \rangle \varphi\, \psi$, which we will write for convenience in infix notation as $\varphi + \psi$. The axiomatization wil be found in the course of a step-by-step heuristic canonical model-style analysis.

(a) Assume the axioms and proof rules of the minimal modal logic for a binary $\varphi + \psi$ (Blackburn, de Rijke & Venema, 2001) and all modalities defined in what follows. Also, take all standard proof principles for nominals and the existential modality. We will add further axioms and rules as needed.

(b) Define a unary modality $\langle\uparrow\rangle\varphi$ as $\varphi + \top$. To make its matching abstract order $\leq$ in a canonical Henkin model reflexive, add an axiom $\varphi \to \varphi + \top$, and to make it transitive, add $(\varphi + \top) + \top \to \varphi + \top$.[27]

(c) In the canonical model of all maximally consistent sets, we want to (c1) name worlds, (c2) introduce witnesses for the two arguments of the abstract existential binary modality $\varphi\ + \psi$, and (c3) make the abstract ternary relation $Rz, xy$ for $\langle \text{inf} \rangle$ in a standard model completeness proof coincide with the real infimum relation $inf(x, y) = z$ with respect to the binary order $\leq$ in canonical models induced by the modality $\langle\uparrow\rangle$ in the usual way.

(c1) To name worlds, we use the standard hybrid proof rule

If $\vdash p \to \varphi$, where the nominal $p$ does not occur in $\varphi$, then $\vdash \varphi$

(c2) To have witnesses for the components, we take the hybrid proof rule

If $\vdash (E(k \wedge \varphi) \wedge E(m \wedge \psi) \wedge (k + m)) \to \alpha$, where the nominals $k, m$ do not occur in $\varphi$, then $\vdash \varphi + \psi \to \alpha$

(c3) will be dealt with presently.

(d) Define $\Sigma \leq \Delta$ on maximally consistent sets if $\alpha \in \Delta$ implies $\langle\uparrow\rangle\alpha \in \Sigma$.

(e) We show that, for any three max-cons sets, if $k + m \in \Sigma, k \in \Delta, m \in \Gamma$, then $\Sigma$ is an inf of $\Delta$ and $\Gamma$ in the just-defined ordering $\leq$. By our construction of the canonical model, we can assume that $\Sigma$ is named by some nominal $n$.

(e1) First, assume that $\Sigma \leq \Delta$. Consider any $\alpha \in \Delta$: we then have the formula $E(k \wedge\ \alpha)$ present in any max.cons set. At this point, add the following valid schema as an axiom (in case this is not already derivable):

$(k + m) \wedge E(k \wedge \alpha) \to \alpha + \top$

[27] This is a form of associativity, though not as strong as full associativity for $\varphi + \psi$.

The consequent, which equals $\langle\uparrow\rangle\alpha$, is then in $\Sigma$. The same analysis works for $\Sigma \leq \Gamma$, now also adding an obvious axiom for symmetry of infima:

$\varphi + \psi \rightarrow \psi + \varphi$

(e2) Next let $\Sigma$' be a lower bound of $\Delta$ (named by the nominal $k$) and $\Gamma$ (named by $m$) in the order $\leq$. We must show that $\Sigma$' $\leq \Sigma$. By the definition of $\leq$, we have $\langle\uparrow\rangle k, \langle\uparrow\rangle m \in \Sigma$'. Now

$((\langle\uparrow\rangle k \wedge \langle\uparrow\rangle m \wedge E(n \wedge (k+m))) \rightarrow \langle\uparrow\rangle n$

is valid in our models, so we put it as one more axiom. Therefore, $\langle\uparrow\rangle n \in \Sigma$'. Using this, it is easy to see that, for any formula $\alpha$, if $\alpha \in \Sigma$, then $\langle\uparrow\rangle\alpha \in \Sigma$'.

This concludes the list of principles that will make the logic complete. [28]

As a test in using the above hybrid calculus, the reader may want to derive the simple validity $n \rightarrow (n+n)$ using the above principles.

As for stronger proof systems, using nominals, a simple extra axiom makes $\leq$ a partial order. One can also impose, say, full associativity for +, or add Aloni's condition $\varphi + \varphi \rightarrow \varphi$ as a special kind of pre-orders. The analysis also extends to possibility logic with modalities $\Box, \Diamond$ with their own accessibility relation $R$ by supplying axioms that match natural constraints on $R, \leq$ such as $\langle\uparrow\rangle\Diamond\varphi \rightarrow \Diamond\varphi$ for $x \leq y\, R\, z \Rightarrow x\, R\, z$ and $\Diamond\langle\uparrow\rangle\varphi \rightarrow \Diamond\varphi$ for $x\, R\, y \leq z \Rightarrow x\, R\, z$.

The question remains if there is a purely modal complete proof system without hybrid gadgets. This raises some interesting issues of complex algebra as in Section 5. One can axiomatize the pure modal logic of set-lifted algebraic structures satisfying simple equations as in Gautam 1957, but the problem is whether the interesting richer structures in information models are of this kind.

# B Modal and metamodal languages

**Modal languages and focus on single worlds.** Modal languages are usually 'local', describing the structure of a model from the vantage point of one current world. This may be one source of the idea that possible worlds encode all the information we need about relational models. The local design allows for a simple analysis of completeness with canonical models where points are maximally consistent sets of formulas, while a straightforward choice of accessibility relation makes sure that the different local descriptions match up correctly. Canonical models may be another source of the idea that all modal structure is present in the worlds themselves: unlike in professional soccer, here, a collection of overpaid self-centered stars makes a good team. To see whether these features of locality really threaten relationalism, let us give a few more details.

Modal formulas $\varphi$ describe properties of points in models up to worlds reachable at a distance matching the modal depth of $\varphi$. To describe models in full

[28]This analysis leaves out some easy details for dealing with the existential modality.

depth, we need an *infinitary modal language* with infinite conjunctions and disjunctions of sets of formulas. For each pointed model $\mathbf{M}, s$ there is an infinitary modal formula (a 'modal Scott sentence') which holds in exactly those models $\mathbf{N}, t$ that have a bisimulation with $\mathbf{M}, s$. The construction of modal Scott sentences describes, inductively through the ordinals, using a suitable $\Diamond, \Box$ pattern, which types of points are reachable at preceding ordinal description levels.[29]

In a sense, then, reading off the modal properties of a single point tells us the structure of a whole model, insofar as reachable from that point. But these descriptions would not be true without the relational pattern, so this seems no more surprising than that pools surrounded by trees reflect those trees.

*Coda.* Even so, the canonical model construction for modal logic remains intriguing. Nothing similar works for the standard semantics of first-order logic. The integers have just one type (maximally consistent theory) with one free variable (all integers are connected by order automorphisms), and the integers are not easily retrieved by a uniform construction on this single type.

A related way in which the modal language is special shows in the method of *filtration* for proving decidability. Given a finite set of formulas and a relational model, we can extract the types of worlds with respect to just that finite set, and turn them into a model respecting the truth values of the selected formulas. While filtration can be defined for first-order logic (van Benthem & Bezhanishvili, 2020), it seldom yields models for the relevant formulas, and it is even undecidable when this happy coincidence occurs. Having said this, the greater canonicity of modal-style completeness proofs for first-order logic in possibility semantics (Section 4) shows there is more to be understood here.

**Modal logics for meta-purposes.** The two higher levels of relationalism mentioned in the Introduction, too, fall within the realm of modal logic.

First, it was suggested that abstract patterns are best understood by looking at models in their various relations to other models. In fact, there is a large variety of logics with modalities interpreted in the following pattern:

$\mathbf{M}, s \models [\bullet]\varphi$ iff $\mathbf{N}, s \models \varphi$ for all models $\mathbf{N}$ (perhaps from some relevant family of models) standing in some relation $R^\bullet$ to $\mathbf{M}$

Relevant relations $R^\bullet$ include those of being a submodel, homomorphic image, bisimulation image, filtration, and the like. Barwise & van Benthem 1999 proposed this setting for a study of 'entailment along a relation' between models as a generalized form of consequence involving transfer between models. There are broad families of such logics, including dynamic-epistemic logics, where shifts between models correspond to information updates (Baltag & Renne, 2016), logics for graph changes occurring in computation or games, and logics that formalize part of the model theory of modal logic (van Benthem & Bezhanishvili, 2020). Such languages for model change can be studied with standard techniques from modal logic, adapted to the new setting.

[29]For finite models, Scott sentences can even be defined in propositional dynamic logic.

A second suggestion in the Introduction was that translations between languages and logics provide another relational structure that can itself be profitably modalized. Such modalizations have indeed happened in the specialized setting of arithmetical proof systems, witness the 'interpretability logic' surveyed in Japaridze & de Jongh 1998, many of whose principles also make sense for general translations and relative interpretations between arbitrary theories. Still more generally, one can also introduce modalities inside one language and proof system that refer to principles statable and provable in another language via some translation mechanism. This sort of natural combination of different systems is central to the 'combining logics' framework of Gabbay 1999.