Representable Forests and Diamond Systems

MSc Thesis (*Afstudeerscriptie*)

written by

Damiano Fornasiere
(born 20th of June 1997 in Udine, Italy)

under the supervision of **dr. Nick Bezhanishvili** and **dr. Tommaso Moraschini**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:
16th of April 2021

Members of the Thesis Committee:
dr. Benno van den Berg
dr. Nick Bezhanishvili (co-supervisor)
prof. dr. Vincenzo Marra
dr. Tommaso Moraschini (co-supervisor)
prof. dr. Yde Venema (chair)

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Representable Forests and Diamond Systems

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

We study the classical problem of representing partially ordered sets as prime spectra. A poset is said to be Priestley (resp. Esakia) representable if it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra). We study this problem by restricting the attention to two classes of posets: forests, i.e., disjoint union of trees, and diamond systems, a class that includes the order duals of forests. This class has been introduced recently in order to characterize the varieties of Heyting algebras whose profinite members are profinite completions.

We provide a characterization of Priestley and Esakia representable diamond systems. As Priestley representable posets are closed under order duals, this yields a new proof of Lewis' description of Priestley representable forests. While a classification of arbitrary Esakia representable forests remains open, the main result of this thesis gives a full description of the well-ordered ones. Moreover, we investigate the Esakia representability of countable forests and provide two forbidden configurations of Esakia representable countable forests. We also prove a number of facts about Priestley and Esakia topologies on arbitrary posets. In particular, we identify some properties of Priestley (resp. Esakia) topologies that revolve around infinite chains.

Acknowledgements

I wish to thank my supervisors, Nick Bezhanishvili and Tommaso Moraschini, for their devoted supervision and for having proposed this research topic in the first place. Nick supported and advised me several times throughout the whole Master: as a student, as a TA and as a supervisee. During the writing of the thesis, he patiently helped me improve every single paragraph. Working with Tommaso has been truly inspiring, thanks to his mathematical elegance and clarity of thought. In addition to this, he solicitously assisted and encouraged me at every step of this work.

I extend my thanks to Benno van den Berg, Vincenzo Marra and Yde Venema for having accepted to be part of the Thesis Committee. I would especially like to thank Yde for his guidance as my academic mentor at the MoL.

I owe a special word of thanks to Lara Mattiussi who, in these rather overloaded months, found the best way to represent a forest, as you can see on the frontispiece.

Contents

CHAPTER 1

Introduction

This thesis studies the *representability problem* for partially ordered sets. The problem asks which partially ordered sets are isomorphic to the prime spectrum (that is, the poset of prime filters) of a bounded distributive lattice or of a Heyting algebra. Accordingly, a poset is said to be *Priestley* (resp. *Esakia*) *representable* if it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra).

Priestley duality (Priestley, 1970) establishes a dual equivalence between the category of bounded distributive lattices and the category of Priestley spaces. Analogously, Esakia duality (Esakia, 1974) establishes a dual equivalence between the category of Heyting algebras and the category of Esakia spaces. The idea underlying these dualities is that the poset of prime filters of a bounded distributive lattice (resp. Heyting algebra) can be endowed with a topology turning it into a Priestley (resp. Esakia) space. Therefore, the problem of understanding which posets are Priestley (resp. Esakia) representable reduces to the study of posets which can be turned into a Priestley (resp. Esakia) space.

The Priestley representability for posets was first raised by Chen and Grätzer in (Chen & Grätzer, 1969). In (Grätzer, 1971) we can find the following question, which appears as Problem 34, on page 156[1]:

> *"Characterize the poset of prime ideals of a distributive lattice $\mathbb{L}$ under the additional assumption that $\mathbb{L}$ has a minimum, a maximum, or both. If $\mathbb{L}$ has a maximum, then every chain of prime ideals has a supremum; if $\mathbb{L}$ has a minimum, then every chain of prime ideals has an infimum. Are these the only additional conditions?"*

[1] In this chapter most of the quotations are a faithful report from the original source. However, for the sake of readability, slight modifications have been made, in order to keep the notation coherent with the rest of the thesis.

A filter F of a lattice $\mathbb{L}$ is a said to be prime if it is proper and if $x \vee y \in F$, then $x \in F$ or $y \in F$. Equivalently, a filter F of $\mathbb{L}$ is prime when its complement is an ideal. Similarly, an ideal I of $\mathbb{L}$ is said to be prime if its complement is a filter. Because of this, the posets of prime filters and of prime ideals of $\mathbb{L}$ are dually isomorphic. Therefore, Grätzer's problem is essentially equivalent to the one that we have introduced above.

Observe also that Grätzer already recognizes a necessary condition for a poset to be Priestley representable: its nonempty chains must have infima and suprema. From now on we will refer to this property by C1.

In (Kaplansky, 1970) Kaplansky formulated a problem that turned out to be equivalent to the representability problem of Grätzer. More specifically, Kaplansky asked to characterize the poset of prime ideals of a commutative unital ring. We can find this question at the very beginning of (Kaplansky, 1970), on page 5.

> *"We conclude this section with some remarks on the set of prime ideals in a ring R. It seems reasonable to think of the partial ordering on it as its first, basic structure. Question: can an arbitrary partially ordered set be the partially ordered set of prime ideals in a ring? There is a first negative answer, which is fairly immediate: every chain [ed. of prime ideals] has a least upper bound and a greatest lower bound."*

The spectrum of a commutative unital ring is the poset of its prime ideals. The spectra of bounded distributive lattices and the spectra of commutative rings with unit are the same, up to isomorphism (see, e.g, (Priestley, 1994)). Actually, a stronger result holds. It follows from (Hochster, 1969) that spectral spaces are topological spaces which are homeomorphic to the set of prime ideals of a commutative unital ring endowed with the Zariski topology. In (Stone, 1938) Stone proved a representation of distributive lattices in terms of spectral spaces. As it happens, the categories of Priestley spaces and of spectral spaces are isomorphic (Cornish, 1975). Because of this, Kaplansky's question turns out to be equivalent to the one of Grätzer. Returning back to our problem, in (Kaplansky, 1970) we can find the following:

> *"We return to the partially ordered set of prime ideals. It does have another (perhaps slightly unexpected) property: between any two elements we can find a pair of "immediate neighbors".*

The property Kaplansky is referring to can be phrased as follows: given a poset $(P, \leqslant)$ and two elements $x < y$, there are x_1 and y_1 in P such that $x \leqslant x_1 < y_1 \leqslant y$, and no other $z \in P$ is strictly between x_1 and y_1. We will denote this condition by C2. A poset which satisfies C2 is said to have *enough gaps*. Unfortunately, Kaplansky's suggestions stop at this point, and he ends the section by saying:

> *"I do not know of any further conditions that the spectrum of a ring has to satisfy. In other words, it is conceivable that if a partially ordered set satisfies the conclusions of Theorems 9 and 11* [2] *then it is isomorphic to the partially ordered set of prime ideals in some commutative ring."*

As a matter of fact, C1 and C2 are not sufficient for a poset to be Priestley representable. This was already observed by Lewis in (Lewis, 1971), who attributes the discovery to Hochster. Nevertheless, this is the case if we restrict the attention to chains, i.e., linearly ordered posets. More precisely, it follows from (Balbes, 1971) and (Lewis, 1971) that a chain is Priestley representable if and only if it satisfies C1 and C2.

This positive result suggests to study posets which arise as simple "combinations" of chains. For example, in (Lewis, 1971) Lewis studied trees, i.e. posets with a minimum whose principal downsets are chains. Lewis showed that a tree is Priestley representable if and only if it satisfies C1 and C2. Moreover, since Priestley representable posets are closed under disjoint unions (see Proposition 3.14), this result characterizes Priestley representable forests (i.e., disjoint unions of trees) as well. Positive results have been obtained also in (Joyal, 1971), (Speed, 1972). Joyal and Speed proved that the class of Priestley representable posets coincides with the class of profinite posets. However, this result does not provide an internal characterization of Priestley representable posets, as there is no internal characterization of profinite posets. Another abstract characterization appears in (Davey, 1973) but, again, it does not give an insight on the order-theoretic features that a representable poset has to satisfy.

The problem of characterizing Esakia representable posets can be found in the appendix of (Esakia, 2019), an English translation of the volume first published in 1985. Reporting directly from Esakia's book:

> *"We conclude the section by quoting (Grätzer, 1978): "Investigate further the poset of prime ideals of a distributive lattice $\mathbb{L}$" (Problem II.4) and "Characterize the poset of prime ideals of a distributive lattice $\mathbb{L}$ under the additional assumption that $\mathbb{L}$ has a unit and/or a zero" (Problem II.5).*
>
> *It is tempting to replace $\mathbb{L}$ by $\mathbb{H}$*[3] *in those quotations and suggest this as a new problem to the reader."*

Since every Heyting algebra is, in particular, a bounded distributive lattice, Grätzer's and Esakia's questions are related. In particular, every Esakia representable poset is Priestley representable. However, the converse does not hold (see, e.g., Example 3.20). An important difference between the classes of

[2] The theorems that he his referring to imply the just-mentioned properties: i.e. Priestley representable posets must have chains with infima and suprema and enough gaps.

[3] The notation $\mathbb{H}$ refers to a Heyting algebra, as opposed to an arbitrary bounded distributive lattice.

Priestley and Esakia representable posets is that the former is closed under order duals, while the latter is not.

Spectra of Heyting algebras are interesting also from a logical point of view. For example, Gödel algebras coincide with Heyting algebras whose spectrum is a forest (Horn, 1969). The spectra of Gödel algebras have been recently studied in (Aguzzoli, Gerla, & Marra, 2008) and (Bezhanishvili, Bezhanishvili, Moraschini, & Stronkowski, 2021).

The aim of this thesis is to contribute to the representability problem. In particular, we will focus on *forests* and on *diamond systems*, a generalization of their order duals. We shall prove that both well-ordered forests (forests whose chains are well-ordered) and diamond systems are Priestley and Esakia representable if and only if they satisfy C1 and C2. Thus, we extend our understanding of Priestley and Esakia representable posets to the class of well-ordered forests and diamond systems.

In order to do so, we shall first prove some order-theoretic properties that a Priestley (resp. Esakia) representable poset must satisfy. For example, we will prove (Proposition 3.25) that the infima (resp. suprema) of infinite descending (resp. ascending) chains cannot be isolated in any Priestley topology. We will also strengthen this result (Proposition 3.26) by showing that if such infima (resp. suprema) are minimal (resp. maximal), then any open set of a Priestley topology must contain a nontrivial downset (resp. upset). Both results extend to Esakia topologies as well. We will also show that there are posets of height 2 and width 2 which satisfy C1 and C2 but are not Priestley representable (Example 3.23).

In (Lewis & Ohm, 1976) the authors attribute to Hochster the discovery that there is a third condition – which they call H – that a Priestley representable poset must satisfy: any family of principal upsets (resp. downsets) whose intersection is empty admits a finite subfamily whose intersection is empty. The paper provides an example of a poset which satisfies C1 and C2, but not H. We will generalize H into a condition C3 in Proposition 3.21.

Next we will show some order-theoretic configurations which involve infinite descending chains that are forbidden in an Esakia representable poset (Theorems 3.30 and 3.32).

In view of these results, we will be able to study the Priestley (resp. Esakia) representability of the above mentioned classes of forests and diamond systems. The latter class was introduced in (Bezhanishvili et al., 2021), in order to solve the problem of whether each profinite Heyting algebra is isomorphic to the profinite completion of some Heyting algebra. A Heyting algebra is said to be a *profinite completion* if it is isomorphic to the limit of the inverse system of finite homomorphic images of some Heyting algebra. In (Bezhanishvili et al., 2021) it is proved that there is a largest variety `DHA` of Heyting algebras whose profinite members are profinite completions. The posets underlying the Esakia duals of these Heyting algebras are of a special form: they all are diamond systems. Moreover, the Heyting algebra of upsets of a diamond system always belongs to this variety. Intuitively, diamond systems are a generalization of the order duals of forests, whose width is allowed to be 2.

As already announced above, this leads to one of the main result of the thesis. We will prove (Theorem 4.9) that a diamond system is Esakia representable if and only if it satisfies C1 and C2, and that it is Priestley representable if and only if it is Esakia representable. Moreover, we will show (Theorem 4.32) how to use this result in order to simplify the proof of the main theorem of (Bezhanishvili et al., 2021). Because every Heyting algebra $\mathbb{H}$ in $\mathtt{DHA}$ is the dual of a diamond system, knowing which diamond systems are Esakia representable will allow us to directly prove that if $\mathbb{H}$ is profinite then it is a profinite completion of some Heyting algebra.

The Priestley representation of diamond systems implies the Priestley representation of their order duals (Proposition 3.19) and, in particular, of forests. However, in view of Theorems 3.30 and 3.32, the Esakia representability of forests is much harder to tackle. The difficulty in understanding which forests are Esakia representable lies in the fact that arbitrary forests might have infinite descending chains. Accordingly, we will develop the machinery necessary to address the problem of Esakia representable well-ordered forests, that is, forests with no infinite descending chains. This is the main contribution of the thesis: a well-ordered forest is Esakia representable if and only if it satisfies C1 and C2 (Theorem 4.36).

Finally, we will show that the case of countable forests is quite peculiar on its own, for every compact Hausdorff space which is countable must have an isolated point (Theorem 2.35). This will allow us to provide two new classes of non-Esakia representable countable forests (Theorems 4.53 and 4.55). Intuitively, these results show that in a countable Esakia representable forest no point can be the infimum of two incomparable infinite descending chains. Moreover, given an infinite descending chain, its points cannot be infima of certain infinite descending chains.

In summary, the main contributions of this thesis are the following:

- The characterization of Priestley and Esakia representable well-ordered forests and the description of two forbidden configurations of Esakia representable countable forests.
- The characterization of Priestley and Esakia representable diamond systems and a simplification of the main proof of (Bezhanishvili et al., 2021) via this result.
- A number of results about possible Priestley (resp. Esakia) topologies on a poset and the description of two order-theoretic forbidden configurations of Esakia representable posets.

- An example of a poset of height and width 2 which satisfies C1 and C2 but is not Priestley representable.

The remaining chapters are structured as follows. In chapter 2 we review the necessary preliminaries on orders, lattices, topological spaces, and on Priestley and Esakia dualities. Section 3.1 of chapter 3 collects the proofs of some well-known facts about Priestley (resp. Esakia) representable posets. Then, section 3.2 provides some new results on Priestley (resp. Esakia) representable posets, along with two forbidden configurations of Esakia representable posets, which will be used consequently.

In chapter 4 we will use the results obtained in the previous chapters in order to characterize the classes of Priestley (resp. Esakia) representable diamond systems (section 4.1) and well-ordered forests (section 4.3). In section 4.2 we will apply the characterization of Esakia representable diamond systems in order to simplify the main proof of (Bezhanishvili et al., 2021). Section 4.4 will describe two classes of non-Esakia representable countable forests. The thesis ends in chapter 5 with a summary of the results and a discussion of possible directions with the representability problem.

CHAPTER 2

Preliminaries

In this chapter we review the basic facts that are relevant for the problems considered in this exposition. We assume familiarity with the basic concepts of category theory, such as categories, morphisms, functors and natural transformations.

In section 2.1 we recall the notions of partially ordered sets, lattices, Heyting algebras and topological spaces and discuss some of their properties. Section 2.2 considers Stone, Priestley and Esakia dualities. We will not provide proofs, but will only discuss how to construct the functors establishing these dual equivalences.

2.1 Orders, lattices and topological spaces

We refer to (Sankappanavar & Burris, 1981) and (Davey & Priestley, 2002) for an introduction to orders and lattices. We start with the definition of poset.

Definition 2.1. Let P be a set and $\leqslant \subseteq P \times P$ a binary relation on it. The pair $\mathbb{P} = (P, \leqslant)$ is said to be a *partially ordered set* –from now only simply a *poset*– whenever $\leqslant$ satisfies the following:

- *reflexivity*: $x \leqslant x$, for every $x \in P$;
- *antisymmetry*: If $x \leqslant y$ and $y \leqslant x$, then $x = y$, for every $x, y \in P$;
- *transitivity*: If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$, for every $x, y, z \in P$.

The elements $x \in P$ will be also called the *elements/points/nodes* of the poset $\mathbb{P}$, and the relation $\leqslant$ will be called the *order/ordering* of $\mathbb{P}$.

Definition 2.2. Let $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \leqslant_{\mathbb{Q}})$ be two posets. A map $f : P \to Q$ is said to be *order-preserving* if, for every $x, y \in P$, $x \leqslant_{\mathbb{P}} y$ implies $f(x) \leqslant_{\mathbb{Q}} f(y)$.

Posets and order-preserving maps form a category, which we denote by **Pos**. From now on, we will often avoid the subscripts of the form $\leqslant_{\mathbb{P}}$ if confusion does not arise. For instance, the previous definition could have been stated as follows: a map $f : P \to Q$ is order-preserving whenever $x \leqslant y$ implies $f(x) \leqslant f(y)$ for every $x, y \in P$.

Given a poset $(P, \leqslant)$ and two elements $x, y \in P$, we will often write $x < y$ as a shorthand for $x \leqslant y$ and $x \neq y$. Moreover, we are going to use the following notational conventions:

$$[x, y] = \{z \in P \mid x \leqslant z \leqslant y\} \quad (x, y) = \{z \in P \mid x < z < y\}$$

$$(x, y] = \{z \in P \mid x < z \leqslant y\} \quad [x, y) = \{z \in P \mid x \leqslant z < y\}.$$

Given a poset $\mathbb{P}$ and a subset $X \subseteq P$, the relation $\leqslant_{X\times X}$, defined as the restriction of $\leqslant$ to $X \times X$, makes the pair $(X, \leqslant_{X\times X})$ a poset. We will refer to $\leqslant_{X\times X}$ as the *induced* ordering of $\mathbb{P}$ onto X, or simply the *induced ordering* on X, when $\mathbb{P}$ is clear from the context. Moreover, for the sake of readability, we will often make use of the simpler shorthand $(X, \leqslant)$. We will say that the pair $(X, \leqslant)$ is a *subposet* of $\mathbb{P}$.

Definition 2.3. Given a poset $\mathbb{P}$, we will refer to $\mathbb{P}^\partial = (P, \geqslant)$ as the *order-dual* of $\mathbb{P}$, where $x \geqslant y$ if and only if $x \leqslant y$.

Definition 2.4. Let $\{(P_i, \leqslant_i)\}_{i\in I}$ be a collection of posets and consider their disjoint union $P := \bigsqcup_{i\in I} P_i = \{(x, i) \mid x \in P_i, i \in I\}$. We can equip P with the ordering $\leqslant$ defined as:

$$\bigcup_{i\in I} \{((x, i), (y, i)) \mid x_i \leqslant_i y_i\}.$$

The pair $(P, \leqslant)$ is a poset and it is called the *sum* of the P_i's.

Definition 2.5. A poset $\mathbb{P}$ is said to be *linearly ordered* if $x \leqslant y$ or $y \leqslant x$ for every $x, y \in P$. Linearly ordered posets are also called *chains*. On the other hand, a poset $\mathbb{P}$ is said to be an *antichain* if $x \nleqslant y$ and $y \nleqslant x$ for every pair of elements of P. Two such elements are said *incomparable* or *parallel*, in symbols $x \parallel y$.

Among the subsets of a poset $\mathbb{P}$, some are of special interest.

Definition 2.6. Let $\mathbb{P}$ be a poset. A subset $X \subseteq P$ is said to be:

1. *Downward closed* or a *downset* of $\mathbb{P}$ if for every $x \in X$ and $y \in P$, if $y \leqslant x$ then $y \in X$, for every $y \in P$;

2. *The downward closure* of a subset $Y \subseteq P$ in $\mathbb{P}$ if

$$X = \{x \in P \mid \text{ there is some } y \in Y \text{ such that } x \leqslant y\}.$$

Similarly, we say that X is *upward closed*, or an *upset*, if it is a downset of $\mathbb{P}^\partial$ and we say that X is the *upward closure* of a subset $Y \subseteq P$ in $\mathbb{P}$ if X is the downward closure of Y in $\mathbb{P}^\partial$. The downward (resp. upward) closure of a singleton $\{x\}$ for some $x \in P$ is simply denoted by $\downarrow x$ (resp. $\uparrow x$); we refer to it as *principal* downset (resp. upset).

Definition 2.7. A poset $\mathbb{P}$ is said to be *image finite* if $\uparrow x$ is finite for every $x \in X$.

Definition 2.8. Let $\mathbb{P}$ be a poset and $n \in \mathbb{N}$. The poset $\mathbb{P}$ is said to be of *width at most* n if $\uparrow x$ does not contain any antichain of $n+1$ elements, for every $x \in P$.

Definition 2.9. An element $x \in P$ of a poset $\mathbb{P}$ is said to be a:

1. *minimum* of $\mathbb{P}$ if $x \leqslant y$ for every $y \in P$;
2. *minimal element* of $\mathbb{P}$ if $y \nless x$ for any $y \in P$;
3. *lower bound* of a subset $X \subseteq P$ in $\mathbb{P}$ if $x \leqslant y$ for every $y \in X$.

Similarly, we will say that x is a *maximum* of $\mathbb{P}$ (resp. maximal element of $\mathbb{P}$, resp. upper bound of $X \subseteq P$ in $\mathbb{P}$) whenever x is the minimum of $\mathbb{P}^\partial$ (resp. minimal element of $\mathbb{P}^\partial$, resp. lower bound of $X \subseteq P$ in $\mathbb{P}^\partial$). The set of minimal (resp. maximal) elements of a poset $\mathbb{P}$ will be denoted by $\min(\mathbb{P})$ (resp. $\max(\mathbb{P})$).

It might be worth mentioning that a minimum and a maximum, whenever they exist, are unique. However, this is not the case for minimal (resp. maximal) elements, nor it is for lower (resp. upper) bounds. Thus, one can safely introduce the notation $\bot$ for *the* minimum of $\mathbb{P}$, and $\top$ for *the* maximum of $\mathbb{P}$, whenever they exist. When $\mathbb{P}$ has both a maximum and a minimum it is said to be *bounded*.

Let X be a subset of a poset $\mathbb{P}$, and consider the poset of the lower (resp. upper) bounds of X. If it has a maximum (resp. a minimum), such maximum (resp. minimum) is unique, and we refer to it as *the infimum* (resp. *supremum*) of X in $\mathbb{P}$, in symbols $\inf X$ (resp. $\sup X$).

Definition 2.10. A poset $\mathbb{P}$ is said to be *complete* if $\inf X$ and $\sup X$ exist in $\mathbb{P}$ for every $X \subseteq P$.

Definition 2.11. A poset $\mathbb{L} = (L, \leqslant)$ is said to be a *lattice* if both $\inf\{x, y\}$ and $\sup\{x, y\}$ exist, for every $\{x, y\} \subseteq L$.

Given a lattice $\mathbb{L}$, we shall often write $x \wedge y$ or $x \vee y$ in place of $\inf\{x, y\}$ or $\sup\{x, y\}$. We should mention that a lattice can be presented either as a poset whose binary infima and suprema exist, or purely by algebraic means, as a tuple $(L, \wedge, \vee)$, where $\wedge, \vee : L^2 \to L$ are such that, for any $x, y, z \in L$, the following holds:

$$x \wedge x = x \quad x \vee x = x$$

$$x \wedge y = y \wedge x \quad x \vee y = y \vee x$$
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad x \vee (y \vee z) = (x \vee y) \vee z$$
$$x \vee (x \wedge y) = x \quad x \wedge (x \vee y) = x$$

The binary operations $\wedge$ and $\vee$ will also be called *meet* and *join*, respectively. If the lattice is complete we usually use the symbols $\bigwedge$ and $\bigvee$ in order to denote arbitrary infima (meets) and suprema (joins). In the same fashion, a bounded lattice can be introduced as a tuple $(L, \wedge, \vee, \bot, \top)$ where $(L, \wedge, \vee)$ is a lattice and $\bot$ and $\top$ satisfy the following conditions: for every $x \in L$, $x \vee \top = \top$ and $x \wedge \bot = \bot$. Let us observe that if a lattice is defined using the algebraic notation, one can recover the underlying partial order $\leqslant$ defined as follows: $x \leqslant y$ if and only if $x \wedge y = x$ or, equivalently, if and only if $x \vee y = y$.

Definition 2.12. Let $\mathbb{L}$ and $\mathbb{L}'$ be two lattices. A map $f : L \to L'$ is said to be a *lattice homomorphism* if, for every $x, y \in L$, we have $f(x \vee y) = f(x) \vee f(y)$ as well as $f(x \wedge y) = f(x) \wedge f(y)$. If the lattices $\mathbb{L}$ and $\mathbb{L}'$ are bounded, we say that a lattice homomorphism is *bounded* if it satisfies $f(\bot) = \bot$ and $f(\top) = \top$.

Remark 2.13. Every chain is a lattice: given two elements $x, y \in C$, without loss of generality we have $x \leqslant y$, which implies $x \wedge y = x$ and $x \vee y = y$.

In this thesis, we will deal with lattices which validate one of the following equations:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice validates one of those two equations if and only if it validates both of them. This leads to the following definition.

Definition 2.14. A lattice $\mathbb{L}$ is said to be *distributive* if, for every $x, y, z \in L$, the equation:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds.

In the following chapters we will extensively work with bounded distributive lattices and bounded lattice homomorphism. They form a category which we denote by **BDL**.

Definition 2.15. A bounded distributive lattice $\mathbb{H}$ is said to be a *Heyting algebra* if and only if, for every $x, y \in H$, there is an element $x \to y \in H$ such that for all $z \in H$ it holds:

$$z \wedge x \leqslant y \quad \text{if and only if} \quad z \leqslant x \to y.$$

For every $x \in H$, we shall write $\neg x$ for the unique element $x \to \bot$.

Definition 2.16. Let $\mathbb{H}$ and $\mathbb{H}'$ be two Heyting algebras and f a bounded lattice homomorphism between them. The morphism f is said to be a *Heyting homomorphism* if, for all $x, y \in H$, we have $f(x \to y) = f(x) \to f(y)$.

Heyting algebras and Heyting homomorphisms form a category, which we denote by **HA**.

Remark 2.17. Every linearly ordered bounded lattice $\mathbb{L}$ is a Heyting algebra. In fact, for every $x, y \in L$ we have:

$$x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{if } y \leqslant x \end{cases}.$$

Remark 2.18. Let $\mathbb{P}$ be a poset and $\mathsf{Up}(\mathbb{P})$ the collection of upsets of $\mathbb{P}$. The poset $(\mathsf{Up}(\mathbb{P}), \subseteq)$ is a distributive lattice and it can be endowed with a Heyting algebra structure.

Definition 2.19. A Heting algebra $\mathbb{B}$ is said to be a *Boolean algebra* if, for every $x \in B$, the equation $x \vee \neg x = 1$ holds. Equivalently, a Boolean algebra is a tuple $(B, \wedge, \vee, \bot, \top, \neg)$ where $(B, \wedge, \vee, \bot, \top)$ is a bounded distributive lattice and moreover it holds $x \vee \neg x = \top$ for any $x \in B$.

Bounded lattice homomorphisms already preserve $\neg$ if the underlying lattice is a Boolean algebra. Accordingly, a Boolean homomorphism between Boolean algebras simply is a bounded lattice homomorphism. We call **BA** the category of Boolean algebras and Boolean homomorphisms.

Heyting and Boolean algebras are specially noteworthy in logic. In order to recall why, let **IPC** denote the intuitionistic propositional calculus, and **CPC** the classical propositional calculus. The two following theorems are well-known.

Theorem 2.20. *The propositional intuitionistic calculus **IPC** is sound and complete with respect to the class of Heyting algebras.*

Theorem 2.21. *The propositional classical calculus **CPC** is sound and complete with respect to the class of Boolean algebras.*

We conclude the preliminaries on lattices, Heyting and Boolean algebras by recalling the notions of filters and ideals of lattices.

Definition 2.22. Let $\mathbb{L}$ be a lattice. A *filter* F is a nonempty subset of L such that, for all $x, y \in L$, the following conditions hold:

1. F us upward closed;

2. If $x \in F$ and $y \in F$, then $x \wedge y \in F$.

Dually, an *ideal* I of $\mathbb{L}$ is a filter of $\mathbb{L}^\partial$. A filter F (resp. an ideal I) is said to be *proper* if $F \neq L$ (resp. $I \neq L$).

Observe that if F is a filter on a lattice $\mathbb{L}$, then $\uparrow x$ is a filter of $\mathbb{L}$ and $\downarrow x$ is an ideal of $\mathbb{L}$, for every $x \in L$. They are called *principal filters* and *principal ideals* respectively. Moreover, if F is a filter on a Boolean algebra $\mathbb{B}$, then $\{\neg x \mid x \in F\}$ is an ideal of $\mathbb{B}$.

For a lattice $\mathbb{L}$, let us denote by $\mathcal{F}$ the set of proper filters of $\mathbb{L}$, ordered by inclusion. A filter F of $\mathbb{L}$ is said to be *maximal* if it is maximal in $\mathcal{F}$. Moreover, F is said to be *prime* if its complement relative to L is an ideal or, equivalently, if F is proper and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. In the same way, we can introduce the notions of maximal and prime ideal. Maximal and prime filters may very well differ in an arbitrary lattice $\mathbb{L}$, but they actually coincide if $\mathbb{L}$ is a Boolean algebra.

Let us mention an important property on prime filters, which we will be used consequently.

Proposition 2.23. *Let $\mathbb{L}$ be a bounded distributive lattice and $\{F_i \mid i \in I\}$ a non empty $\subseteq$-chain of prime filters on it, i.e. $F_i \subseteq F_j$ or $F_j \subseteq F_i$ for every $i, j \in I$. Then, both $\bigcap_{i \in I} F_i$ and $\bigcup_{i \in I} F_i$ are prime filters.*

We can now introduce topological spaces. We refer to (Engelking et al., 1977) for a wealth of information on general topology.

Definition 2.24. Let X be a set. A *topology* τ_X on X is a subset of $\mathcal{P}(X)$ closed under binary intersections and arbitrary unions, and it contains the empty set $\emptyset$ as well as the whole X. In this case, the pair (X, τ) is called a *topological space*.

The elements of τ will be called *open* subsets of X, moreover we will refer to their set-theoretical complements as *closed* subsets of X. An open set which is also closed will be called *clopen*.

Definition 2.25. Let (X, τ) and (Y, σ) be two topological spaces. A map $f : X \to Y$ is said to be *continuous* if $f^{-1}(U) \in \tau$ whenever $U \in \sigma$.

If there is a continuous bijection between two topological spaces, whose inverse is continuous as well, we say that those spaces are *homeomorphic*.

Definition 2.26. A subset $\mathcal{B} \subseteq \tau$ is said to be a *base* of τ if the closure of $\mathcal{B}$ under arbitrary unions is exactly τ. Equivalently, $\mathcal{B}$ is a base for τ if every open of τ can be written as a union of elements of $\mathcal{B}$.

Definition 2.27. Let (X, τ) be a topological space. A subset $\mathcal{S} \subseteq \tau$ is said to be a *subbase* of τ if the closure of $\mathcal{S}$ under finite intersections is a base for τ. Equivalently, $\mathcal{S}$ is a subbase for τ if every open of τ can be written as a union of finite intersections of elements of $\mathcal{S}$.

Given a set X, there might be more than one topology on it. If τ and σ are two topologies on X, we will say that τ is *coarser* or *smaller* than σ when $\tau \subseteq \sigma$; vice versa we will say that τ is *finer* or *bigger* than σ if $\sigma \subseteq \tau$. Observe that there are a smallest and a largest topology on any set X, namely $\{\emptyset, X\}$ and $\mathcal{P}(X)$, respectively. They are usually called the *trivial* and the *discrete* topology on X, respectively. If $x \in X$ is such that $\{x\} \in \tau$, we say that x is *isolated*.

Definition 2.28. Let (X, τ) be a topological space and Y a subset of X. The following collection of subsets of Y is a topology on Y, and it will be called the *induced topology* of (X, τ) on Y:

$$\tau_Y := \{U \cap Y \mid U \in \tau\}.$$

When considering topological spaces, we will often consider two kinds of properties: separation and connectedness properties. Let us summarize the ones that we will be using frequently.

Definition 2.29 (Separation properties)**.** A topological space (X, τ) is said to be:

1. T_0 or *Kolmogorov* if, for every distinct points $x, y \in X$ there exists $U \in \tau$ such that $x \in U$ and $y \notin U$, or viceversa;

2. T_1 or *Fréchet* if, for every distinct points $x, y \in X$ there are $U, V \in \tau$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $y \notin V$;

3. T_2 or *Hausdorff* if, for every distinct points $x, y \in X$ there are $U, V \in \tau$ such that $x \in U$, $y \notin U$, $y \in V$, $y \notin V$ and moreover $U \cap V = \emptyset$.

These separation properties are listed in ascending order of inclusion of classes of spaces satisfying them.

Definition 2.30 (Connectedness properties)**.** A topological space (X, τ) is said to be:

1. *Connected* if the only clopens of τ are $\emptyset$ and X;

2. *Disconnected* if it is not connected;

3. *Totally separated* if every pair of distinct points can be separated by two disjoint opens whose union is the whole X;

4. *Totally disconnected* if the greatest subsets of X which cannot be written as union of smaller disjoint opens are the singletons $\{x\} \subseteq X$.

We should mention one last property of topological spaces.

Definition 2.31. A topological space (X, τ) is said to be *compact* if, for every collection of opens $(U_i)_{i \in I}$ such that $X = \bigcup_{i \in I} U_i$, there is some finite $F \subseteq I$ such that $X = \bigcup_{i \in F} U_i$. In this case, we will also say that the topology τ is compact.

A family of opens $(U_i)_{i \in I}$ such that $X = \bigcup_{i \in I} U_i$ will also be called a *covering* of X. If a covering admits a finite subcovering we will also say that such covering can be *finitized*.

We conclude this section by mentioning the following important facts, which will be useful throughout this thesis.

Proposition 2.32. *Let (X, τ) be a compact topological space. If $C \subseteq X$ is a closed subset of X, then (C, τ_C) is a compact topological space.*

Theorem 2.33 (Alexander Subbase Theorem)**.** *Let (X, τ) be a topological space and $\mathcal{B}$ a subbase of τ. Then, τ is compact if and only if every covering of X by means of opens of $\mathcal{B}$ can be finitized.*

Theorem 2.34 (Tychonoff's Theorem)**.** *For every collection of compact spaces, their product space equipped with the product topology is compact.*

Theorem 2.35 ((Semadeni, 1971). Thm. 8.5.4)**.** *Let (X, τ) be a compact T_2 topological space. If (X, τ) is countable, then it has an isolated point.*

2.2 Duality theory

We will start our review of duality theory by recalling Stone's celebrated duality for Boolean algebras, which asserts that the category for Boolean algebras is dually equivalent to the category of Stone spaces. Analogously, bounded distributive lattices and Heyting algebras admit a categorical duality via Priestley and Esakia spaces, respectively. The purpose of this section is to overview these dualities. We assume some familiarity with the categorical notions of functors and natural transformations; two standard references to the subject are (Awodey, 2010) and (Mac Lane, 2013).

For every category $\mathcal{C}$, we denote by $\mathsf{Id}_\mathcal{C}$ the identity functor $\mathsf{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ which maps every object and every morphism into itself. Moreover, given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, we denote by GF their natural composition.

Definition 2.36. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, in symbols $\mathcal{C} \cong \mathcal{D}$, whenever there exist a functor $F : \mathcal{C} \to \mathcal{D}$ which is:

- *fully faithful*: for any pair of objects $x, y \in \mathsf{ob}(\mathcal{C})$, the assignment $F : \mathsf{hom}_\mathcal{C}(x, y) \to \mathsf{hom}_\mathcal{D}(F(x), F(y))$ is bijective;

- *essentially surjective*: for any object $y \in \mathsf{ob}(\mathcal{D})$ there is an object $x \in \mathsf{ob}(\mathcal{C})$ such that $F(x)$ and y are isomorphic in $\mathcal{D}$.

Equivalently, $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\varepsilon : FG \to \mathsf{Id}_{\mathcal{C}}$ and $\eta : \mathsf{Id}_{\mathcal{D}} \to GF$.

Definition 2.37. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *dually equivalent* if $\mathcal{C} \cong \mathcal{D}^{\mathrm{op}}$ or, equivalently, $\mathcal{C}^{\mathrm{op}} \cong \mathcal{D}$.

It follows from Stone's seminal paper (Stone, 1936) that the category $\mathbf{BA}$ is dually equivalent to the category of the now-called Stone spaces.

Definition 2.38. A (X, τ) topological space is called a *Stone space* provided that it is compact, T_0 and it has a base of clopens sets.

We denote by $\mathbf{Stone}$ the category of Stone spaces and continuous functions. Stone spaces exhibit some properties which we shall summarize. As an example, they are totally disconnected and totally separated. Two standard references on the subject are (Stone, 1936) and (Davey & Priestley, 2002).

Proposition 2.39. *Let (X, τ) be a topological space. The following are equivalent:*

1. *(X, τ) is a Stone space;*
2. *(X, τ) is compact, T_2 and totally disconnected;*
3. *(X, τ) is compact and totally separated;*
4. *(X, τ) is compact, T_2 and it is homeomorphic to the limit of an inverse system of finite T_1 spaces;*
5. *(X, τ) is homeomorphic to the limit of an inverse systems of finite discrete spaces.*

Let us briefly review Stone representation theorem. Given a Stone space (X, τ), the lattice

$$(\mathsf{Clop}(X), \cap, \cup, \smallsetminus, \emptyset, X)^{1}$$

induced by the poset $(\mathsf{Clop}(X), \subseteq)$ is a Boolean algebra. Moreover, a continuous function $f : X \to Y$ between two topological spaces (X, τ) and (Y, σ) induces an assignment f^{-1} between $\mathcal{P}(Y)$ and $\mathcal{P}(X)$. Then, observe that $f^{-1}(V) \in \tau$ for any $V \in \sigma$ and furthermore, if $V^c \in \tau$ we also deduce $f^{-1}(V^c) = (f^{-1}(V))^c \in \tau$. This proves that if V is clopen, then so it is $f^{-1}(V)$. In other words, f^{-1} restricts to a map $\mathsf{Clop}(Y) \to \mathsf{Clop}(X)$. One can also check that this restriction is a Boolean algebra homomorphism.

In other words, we have a functorial assignment Clop from from $\mathbf{Stone}$ into $\mathbf{BA}^{\mathrm{op}}$. Stone representation theorem tells us something more: the functor Clop is part of an equivalence of categories.

[1] Here the symbol $\smallsetminus$ denotes the set-theoretical complement with respect to $\mathsf{Clop}(X)$.

Let $\mathbb{B}$ be a Boolean algebra, and denote by $\mathsf{Spec}(\mathbb{B})$ its set of prime filters. For any $x \in B$, we can consider the prime filters that contain x. Formally, there is an assignment $\varphi : B \to \mathsf{Spec}(\mathbb{B})$ defined by

$$\varphi(x) = \{F \in \mathsf{Spec}(\mathbb{B}) \mid x \in F\}.$$

It turns out that φ is an isomorphism of $\mathbf{BA}$ between $\mathbb{B}$ and $\mathsf{Spec}(\mathbb{B})$. Indeed, the poset $(\mathsf{Spec}(\mathbb{B}), \subseteq)$ is a Boolean algebra, and moreover the following hold for every $x, y \in B$:

$$\begin{array}{rcl}
\varphi(\bot) & = & \emptyset; \\
\varphi(\top) & = & \mathsf{Spec}(\mathbb{B}); \\
\varphi(x \wedge y) & = & \varphi(x) \cap \varphi(y); \\
\varphi(x \vee y) & = & \varphi(x) \cup \varphi(y); \\
\varphi(\neg x) & = & B \smallsetminus \varphi(x).
\end{array}$$

This implies that $\{\varphi(x) \mid x \in B\}$ is closed under binary intersections and complements, and moreover it contains $\emptyset$ and $\mathsf{Spec}(\mathbb{B})$. This means that we may generate a topology τ on $\mathsf{Spec}(\mathbb{B})$. This topology makes $(\mathsf{Spec}(\mathbb{B}), \tau)$ a Stone space. For instance, observe that we have already found a basis of clopens, namely $\{\varphi(x) \mid x \in B\}$. Finally, for every homomorphism $f : \mathbb{B} \to \mathbb{B}'$, the inverse image f^{-1} restricts to $f^{-1} : \mathsf{Spec}(\mathbb{B}') \to \mathsf{Spec}(\mathbb{B})$ and moreover it is continuous. Thus, there is a functorial assignment Spec from $\mathbf{BA}$ to $\mathbf{Stone}$.

The functors Clop and Spec yield the following duality.

Theorem 2.40 (Stone duality)**.** *The categories* $\mathbf{BA}$ *and* $\mathbf{Stone}$ *are dually equivalent.*

The second duality we are going to revisit was discovered by Priestley, see (Priestley, 1970) and (Priestley, 1984). This duality states that the category $\mathbf{BDL}$ of bounded distributive lattices is dually equivalent to a category of certain ordered Stone spaces. She called these spaces *totally order disconnected Stone spaces*, but they now bring her name.

Definition 2.41. A topological ordered space $\mathbb{X} = (X, \tau, \leqslant)$ is said to be a *Priestley space* if τ is compact and, in addition, for every $x, y \in X$ such that $x \nleqslant y$ there is some clopen upset U containing x but not y. This separation property is called the *Priestley separation axiom*.

Definition 2.42. Assume $\mathbb{X}$ and $\mathbb{Y}$ are two Priestely spaces and let $f : X \to Y$ be a map between their underlying sets. The map f is said to be a *Priestley morphism* if it is continuous and order-preserving.

Priestley spaces and Priestley morphisms form a category $\mathbf{Pries}$. Let us summarise some useful properties of Priestley spaces (they are well-known, for a reference see, for instance, (Bezhanishvili, Bezhanishvili, Gabelaia, & Kurz, 2010)).

Proposition 2.43. *Let* $\mathbb{X} = (X, \tau, \leqslant)$ *be a Priestley space. Then, the following hold:*

1. *(X, τ) is a Stone space;*
2. *For every closed $F \subseteq X$, both $\downarrow F$ and $\uparrow F$ are closed. Moreover, both $\downarrow F$ and $\uparrow F$ are Priestley subspaces of $\mathbb{X}$, when endowed with the induced order and the induced topology of $\mathbb{X}$.*

Corollary 2.44. *Let $\mathbb{X} = (X, \tau, \leqslant)$ be a Priestley space. Then, for every $x \in X$, both $\uparrow x$ and $\downarrow x$ are closed as well as Priestley subspaces of $\mathbb{X}$.*

It is easy to see that $(\mathsf{ClopUp}(\mathbb{X}), \cap, \cup, \emptyset, X)$ is a bounded distributive lattice. Moreover, any continuous map f between Priestley spaces gives rise to a dual homomorphism between their induced lattices, as in the case of Stone spaces. That is, there is a functor $\mathsf{ClopUp} : \mathbf{Pries} \to \mathbf{BDL}$.

Once again, the difficult part is to construct the functor inverse to ClopUp. The complication lies in the fact that, if we look at the spectrum of a bounded distributive lattice $\mathbb{L}$, in symbols $\mathsf{Spec}(\mathbb{L})$, the set $\{\varphi(x) \mid x \in L\}$ is no more necessarily closed under complements. This is a problem because we are looking for a basis of clopens which, by definition, must be closed under set-theoretical complements.[2] This problem can be solved by considering the following set

$$\{\varphi(x) \mid x \in L\} \cup \{\varphi(x)^c \mid x \in L\}$$

which is closed under complement by construction. On the other hand, it is not necessarily closed under finite intersections, but we can still take it as a *subbase* for a topology τ on $\mathsf{Spec}(\mathbb{L})$. The triple $(\mathsf{Spec}(\mathbb{L}), \tau, \subseteq)$ is a Priestley space, and the assignment Spec is a functor which is inverse of ClopUp.

Summing up, Priestley proved the following theorem.

Theorem 2.45 (Priestley duality)**.** *The categories* **BDL** *and* **Pries** *are dually equivalent.*

Finally, let us review Esakia duality for Heyting algebras. This duality was discovered by Esakia, and was presented in (Esakia, 1974), see also (Esakia, 2019). It states that **HA** is dually equivalent to the category of what Esakia called *hybrids*, and they are now known as *Esakia spaces*.

Definition 2.46. A Priestley space $\mathbb{X}$ is said to be an *Esakia space* if $\downarrow U$ is open whenever U is open. Sometimes we will just say that $\mathbb{X}$ is Esakia.

Definition 2.47. Assume $\mathbb{X}$ and $\mathbb{Y}$ are two Esakia spaces and let $f : X \to Y$ be a map between their underlying sets. f is said to be an *Esakia morphism* if it is a Priestley morphism and, in addition, for every $x \in X$, $y \in Y$, if $f(x) \leqslant y$ then there is some $z \in X$ such that $x \leqslant z$ and $f(z) = y$.

[2] It might be worth mentioning that the assignment φ actually gives rise to a duality, namely the duality between bounded distributive lattices and spectral spaces. See, e.g., (Stone, 1938) or (Johnstone, 1982).

A map f between topological ordered spaces which satisfies the third condition of an Esakia morphism is sometimes called a *bounded morphism*, or a *p-morphism*, in analogy with the bounded (resp. p-morphisms) of modal logic (see (Blackburn, de Rijke, & Venema, 2002)). Esakia spaces and bounded morphisms form a category, which will be denoted by **Esa**.

The next proposition shows two useful characterizations of Esakia spaces. A standard reference on the subject is (Esakia, 2019).

Proposition 2.48. *Let $\mathbb{X}$ a Stone space. Then, the following are equivalent:*

1. *$\mathbb{X}$ is Esakia;*
2. *$\mathbb{X}$ is Priestley and $\downarrow U$ is clopen whenever U is clopen;*
3. *$\uparrow x$ is closed for each $x \in X$ and $\downarrow U$ is clopen whenever U is clopen.*

Another important property of Esakia spaces is the next one.

Proposition 2.49. *Let $\mathbb{X} = (X, \leqslant, \tau)$ be an Esakia space. The set of maximal elements $\max(\mathbb{X})$ is closed in τ.*

It is not difficult to see that the topological ordered space $(\mathsf{Spec}(\mathbb{H}), \tau, \subseteq)$, where τ is generated as in the Priestley case, is an Esakia space. On the other hand, if we start with an Esakia space $\mathbb{X} = (X, \tau, \leqslant)$ and we want to turn $(\mathsf{ClopUp}(\mathbb{X}), \cap, \cup, \emptyset, X)$ into a Heyting algebra, we should be able to find a Heyting implication $\rightarrow$ on it. In order to do this, let U and V be two clopen upsets. It is possible to prove that $(\downarrow(U \smallsetminus V))^c$ satisfies the properties of a Heyting implication. Moreover, since $\mathbb{X}$ is an Esakia space, we have that $\downarrow(U \smallsetminus V)$ is a clopen downset, and thus its complement is a clopen upset. This now suffices to state the last theorem of this section.

Theorem 2.50 (Esakia duality)**.** *The categories* **HA** *and* **Esa** *are dually equivalent.*

This concludes the review on preliminaries. We are now ready to start our investigation of the Priestley and Esakia representability problems.

CHAPTER 3

The representability problem

This chapter collects some important facts about Priestley (resp. Esakia) representable posets. We will employ them extensively throughout chapter 4.

The purpose of the first section is to review some standard techniques used to study the representability problem. Therefore, we provide self-contained proofs of some results that are already known.

In the second section we prove some technical facts on Priestley (resp. Esakia) representable posets. In particular, we focus on points of a poset which cannot be isolated for any Priestley (resp. Esakia) topology. Moreover, we generalize Hochster's condition H and we provide two forbidden configurations of Esakia representable posets. To the best of our knowledge, the results that we discuss in the second section have not appeared in the literature.

3.1 First properties

Before studying the spectra of bounded distributive lattices and of Heyting algebras, one might first wonder which posets are isomorphic to the prime spectrum of a Boolean algebra. This problem is easier to resolve, because the prime filters of a Boolean algebra $\mathbb{B}$ are maximal with respect to the set of proper filters of $\mathbb{B}$. In particular, they are pairwise incomparable. Thus, a poset isomorphic to the prime spectrum of a Boolean algebra must be an antichain. In view of Stone representation (2.40), the converse reduces to the problem of studying which sets could carry a Stone topology.

Theorem 3.1. *Every set X can be endowed with a Stone topology.*

Proof. We have two cases: either X is finite or not. In the former case, the discrete topology $\mathcal{P}(X)$ is a Stone topology on X. In fact, it is compact, T_0 (every singleton is open) and every open is clopen, thus it has a basis of clopens. If, on the other hand, X is infinite, X is in particular nonempty, hence there exists some $x_0 \in X$. Then, consider the topology τ generated by the base $\mathsf{FinCofin}(x_0)$ defined as:

$$\{U \subseteq X \mid x_0 \notin U \text{ and } |U| < \aleph_0\} \cup \{U \subseteq X \mid x_0 \in U \text{ and } |U^c| < \aleph_0\}$$

That this defines a base is clear: the intersection of two cofinite (resp. finite) subsets of X containing x_0 (resp.
not containing x_0) is again cofinite (resp. finite) and does not contain x_0 (resp. does contain x_0). Moreover, observe that this base consists of clopens by definition. We should mention that this topology is essentially the one-point compactification of the discrete set $X \setminus \{x_0\}$.

Then, τ is T_0: if $x \neq y$, we have two cases. Either $x_0 = x$ or not. In the former case, both $\{y\}$ and $\{y\}^c$ are opens. They are disjoint and they contain y and x respectively. In the latter case, the same reasoning holds for the subsets $\{x\}$ and $\{x\}^c$.

Finally, τ is compact. For, any open covering $\bigcup_{i\in I} U_i$ must include an open set U which contains x_0. By definition, this open set covers the whole X but finitely many $x_1, \ldots, x_n$. In fact, the open set U must be a union of basic opens, thus x_0 already belongs to one of such basic opens and, by definition, such basic open must be cofinite. Then, for each x_i there is an open set U_{x_i}, and hence $U \cup \bigcup_{0<i\leqslant n} U_{x_i}$ is a finite subcovering of $\bigcup_{i\in I} U_i$.

□

The cases of Priestley spaces and Esakia spaces are much harder to tackle and, in fact, they are still open. However, there is an immediate result that we should mention: every *finite* poset is Esakia representable, and hence Priestley representable.

Proposition 3.2. *Every finite poset* $(P, \leqslant)$ *is uniquely Esakia representable.*

Proof. Let $\mathcal{P}(P)$ be the discrete topology on P and consider $(P, \leqslant, \mathcal{P}(P))$. We claim that it is an Esakia space: compactness holds because P is finite; the Priestley separation axiom holds because if $x \nleqslant y$ then ${\downarrow}y$ is a clopen downset containing y but not x and the Esakia condition holds trivially because every set is open.

Moreover, this topology is unique, because the only T_1 topology on a finite set is the discrete one. □

Corollary 3.3. *Every finite poset* $(P, \leqslant)$ *is uniquely Priestley representable.*

Proof. This follows immediately from the proof of the previous proposition.

□

We will now introduce an important concept for a poset.

Definition 3.4. Given a poset $(P, \leqslant)$, a pair $(x, y) \in P \times P$ is said to be a *gap* of P if $x < y$ and $[x, y] = \{x, y\}$. A poset $(P, \leqslant)$ is said to have *enough gaps* if, for every $x, y \in P$ such that $x < y$, there is a gap (x_1, y_1) such that $x \leqslant x_1 < y_1 \leqslant y$.

Perhaps, the best-known facts about a Priestley representable poset $\mathbb{P}$ are the following two:

C1 The nonempty chains of $\mathbb{P}$ have infima and suprema in $\mathbb{P}$;

C2 $\mathbb{P}$ has *enough gaps*.

These conditions have already been introduced in (Grätzer, 1971) and (Kaplansky, 1970). In order to prove that a Priestley representable poset satisfies C1 and C2, we make use of Priestley duality, i.e., Theorem 2.45.

Proposition 3.5 (Condition C1)**.** *Let $\mathbb{P}$ be a Priestley representable poset. Then, every nonempty chain $C \subseteq P$ has infumum and supremum.*

Proof. In view of Priestley duality we can assume $\mathbb{P}$ to be the poset of prime filters of some bounded distributive lattice $\mathbb{L}$, since $\mathbb{P}$ is Priestley representable; in particular the ordering of $\mathbb{P}$ is the set theoretical inclusion $\subseteq$ between prime filters. Then, let $C \subseteq P$ be a nonempty chain of $\mathbb{P}$. In view of Proposition 2.23, both $\bigcap C$ and $\bigcup C$ are prime filters of $\mathbb{L}$, hence they belong to P. Clearly, these are the infimum and the supremum of C in $\mathbb{P}$, respectively. □

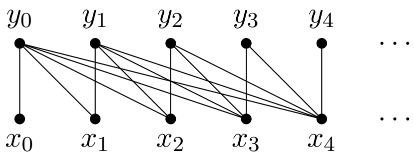
Proposition 3.6 (Condition C2)**.** *Any Priestley representable poset $\mathbb{P}$ has enough gaps.*

Proof. Let $\mathbb{P}$ be a Priestley representable space. This means that $\mathbb{P}$ can be thought as the poset of prime filters of a bounded distributive lattice ordered by inclusion. Accordingly, let $F \subset G$ two prime filters. Let us consider the poset of chains of prime filters between F and G, ordered by inclusion. This poset is non empty, because $\{F, G\}$ belongs to it. Then, every chain of this poset has an upper bound: the union of chain of chains between F and G is still a chain of prime filters between F and G and clearly it extends every other chain between F and G. Hence, Zorn's lemma guarantees that there is a maximal chain $\{H_i \mid i \in I\}$ of prime filters between F and G. Now, $F \subset G$ means that there is some $x \in G \smallsetminus F$. Define $G_1 := \bigcap_{x \in H_i} H_i$ and $F_1 := \bigcap_{x \notin H_i} H_i$. Due to Proposition 2.23, G_1 and F_1 are prime filters. Moreover, we have $F \subseteq F_1 \subset G_1 \subseteq G$, because F_1 does not contain x while G_1 does. Finally, the pair (F_1, G_1) is a gap, because any prime filter between F_1 and G_1 would extend the chain $\{H_i \mid i \in I\}$, against its maximality. □

Observe that, since every Esakia space is, in particular, a Priestley space (Definition 2.46), an Esakia representable poset has to satisfy C1 and C2 as well.

The condition of having enough gaps is less straightforward than C1, and it suggests that the problem of Priestley (resp. Esakia) representability is a rather complex one. This is also manifested by the fact that C1 and C2 are not the only conditions that a Priestley (resp. Esakia) representable poset must satisfy. The following example appears in (Lewis & Ohm, 1976), where its discovery is attributed to Hochster.

Example 3.7. Let $\mathbb{P} = (P, \leqslant)$ a poset whose universe P is equal to $X \cup Y$, where X and Y are two denumerable disjoint sets, $X = \{x_n \mid n \in \mathbb{N}\}$ and $Y = \{y_n \mid n \in \mathbb{N}\}$ and the ordering $\leqslant$ is defined as follows: $x \leqslant y$ if and only if $x = y$ or $x = x_n$ for some $n \in \mathbb{N}$ and $y = y_m$ for some $m \leqslant n$. See the picture below.[1]



Clearly $\mathbb{P}$ has complete chains and it has enough gaps. However, we claim that it is not Priestley representable. For the sake of contradiction, let us suppose there is a Priestley topology on it. As we have already observed, it must contain the complement of every principal downset. In particular, each $(\downarrow y_n)^c$ is open, and moreover the equation

$$P = \bigcup_{n \in \mathbb{N}} (\downarrow y_n)^c$$

holds. For, $y_m \notin \downarrow y_n$ for every $n \neq m$. Moreover, for every $n > m$ it holds $x_m \notin \downarrow y_n$. We thus have an infinite open covering of P. Observe that this covering does not have any finite subcovers. In fact, for any finite union $\downarrow y_{n_1} \cup \cdots \cup \downarrow y_{n_m}$ there is some $k > \max\{y_{n_1}, \ldots, y_{n_m}\}$ and thus $y_k \notin \downarrow y_{n_i}$ for $i \leqslant m$.

This contradicts the fact that every Priestley topology is compact, thus implying that the above poset is not Priestley representable.

The poset that we have just discussed is not representable because of the following reason: there is a collection $\mathcal{C}$ of principal downsets whose intersection is empty but $\mathcal{C}$ does not have a finite subcollection whose intersection is empty. We will see how to generalize this condition in the next section (see Proposition 3.21). Notice also that such poset has height 2 but is not image finite. We should mention that there are also image finite posets which are not Priestley representable. More specifically, in Example 3.23 we will provide a poset of height 2 and width 2 which satisfies C1 and C2 but is not Priestley representable.

[1] As a convention, we will not draw reflexive nor transitive arrows.

Nevertheless, there are classes of Priestley (resp. Esakia) representable posets which are characterized by C1 and C2. For example, it follows from (Balbes, 1971) that the class of nonempty Priestley representable chains is the class of chains which satisfy C1 and C2. Even more is true: Priestley and Esakia representable chains coincide. Recall also that the empty chain is also Priestley (resp. Esakia) representable, because all finite posets are.

Theorem 3.8. *A nonempty chain* $(C, \leqslant)$ *is Priestley representable if and only if it satisfies C1 and C2.*

Proof. We already know that every Priestley representable chain must have enough gaps. Moreover, it has to be complete, it must satisfy C1. As for the other direction, consider the topology τ generated by the union of the two following sets:

$$\{{\downarrow}g_1 \mid \exists g_2(g_2 \in C \text{ and } (g_1, g_2) \text{ is a gap})\};$$

$$\{{\uparrow}g_2 \mid \exists g_1(g_1 \in C \text{ and } (g_1, g_2) \text{ is a gap})\}.$$

Let us show that τ is a Priestley topology on C.

Compactness: In view of Alexander's subbase theorem, it suffices to show that the subbase we have defined is compact. As such, let $(U_i)_{i \in I}$ be a covering of C by means of subbase opens. By assumption C is complete, therefore there are $\inf C \leqslant \sup C$. If $\inf C = \sup C$ there is nothing to prove, otherwise let (g_1, g_2) be a gap between them, which exist since has C has enough gaps. Then, there must be two open sets U_1 and U_2 containing g_1 and g_2 respectively. If $U_1 = {\downarrow}g_3$ *and* $U_2 = {\uparrow}g_4$ or $U_1 = {\uparrow}g_3$ *and* $U_2 = {\downarrow}g_4$ then in both cases $U_1 \cup U_2$ already covers the whole C, and we are done.

On the other hand, if $U_1 = {\downarrow}g_3$ *and* $U_2 = {\downarrow}g_4$ (respectively, $U_1 = {\uparrow}g_3$ *and* $U_2 = {\uparrow}g_4$) without loss of generality we may assume $g_3 \leqslant g_4$. Now, if $g_4 = \sup C$ (resp. $g_3 = \inf C$) there is nothing to prove, otherwise consider a gap (g_5, g_6) between g_4 and $\sup C$ (resp. $\inf C$ and g_3). The just described reasoning applies to (g_5, g_6) as well. Either this sequence stops after finitely many steps, or we end up with ω-many downsets (resp. upsets). In other words, we obtain a chain whose supremum g must exist by completeness. Let U be an open set containing g: observe that it cannot be $U = {\uparrow}g$ because this would mean that (h, g) is a gap for some $h < g$, but g comes as a limit of an infinite ascending chain. Thus, either $U = {\uparrow}h$ for some $h < g$ or $U = {\downarrow}h$ for some $g < h$. In the former case, let g_n a member of the sequence whose limit is g such that $h < g_n$. Then, ${\downarrow}\, g_n \cup {\uparrow}h$ covers the whole C. In the latter case, every open set ${\downarrow}g_n$ is included in ${\downarrow}h$ and we can consider a gap between h and $\sup C$ and proceed as before. The other limit cases are analogous to what we have just described for ω. In conclusion, τ is compact.

Priestley separation: Suppose $x \nleqslant y$ for some $x, y \in C$. Notice that C being a chain implies $y < x$. Thus, there is a gap (g_1, g_2) between them. Then, from $C = {\downarrow}g_1 \cup {\uparrow}g_2$ and ${\downarrow}g_1 \cap {\uparrow}g_2$ it follows ${\uparrow}g_1 = ({\downarrow}g_2)^c$. Whence, ${\uparrow}g_1$ is a clopen upset containing x but not y. □

Corollary 3.9. *A chain $(C, \leqslant)$ is Priestley representable if and only if it is Esakia representable.*

Proof. Every Esakia space is, in particular, a Priestley space. Hence, it suffices to show that, for chains, being Priestley representable implies being Esakia representable. Accordingly, suppose $(C, \leqslant)$ is a Priestley representable chain. Let us show that the topology τ that we have defined in the proof of Proposition 3.8 is an Esakia topology. Let U be an open in τ. From the definition of τ used in the previous proof it follows that U is the union of a finite intersection of subbase opens. Recall that $\downarrow$ commutes with arbitrary unions, therefore it suffices to show that the downset of any finite intersection of subbase open sets is again an open set. In order to see this, consider the following set:

$$\downarrow({\downarrow}g_1 \cap \cdots \cap {\downarrow}g_n \cap {\uparrow}h_1 \cap \cdots \cap {\uparrow}h_m)$$

where every g_i is part of a gap (g_i, g_i') and every h_i is part of a gap (h_i', h_i). Because both n and m are finite, without loss of generality we might assume g_1 and h_1 to be the maximum and the minimum among the g_i's and the h_i's, respectively.

If $g_1 < h_1$, we deduce ${\downarrow}g_1 \cap {\uparrow}h_1 = \emptyset$, and clearly ${\downarrow}\emptyset = \emptyset$.

Otherwise, since C is a chain, we deduce $h_1 \leqslant g_1$ and thus ${\downarrow}({\downarrow}g_1 \cap {\uparrow}h_1) = {\downarrow}([h_1, g_1]) = {\downarrow}g_1$. This concludes the proof. □

The characterization of Priestley (resp. Esakia) representable chains implies that the class of Priestley (resp. Esakia) representable posets is not elementary. Recall that a class $\mathcal{C}$ of similar structures (in our case, a class of posets) is said to be elementary if there is a first order theory whose class of models coincide with $\mathcal{C}$.

Corollary 3.10. *The class of Priestley (resp. Esakia) representable posets is not elementary.*

Proof. Elementary classes are closed under ultraproducts. In order to see this, let $\mathcal{C} = \mathrm{Th}(\mathcal{C})$ be an elementary class and $\mathbb{M}_i$ a collection of models of $\mathrm{Th}(\mathcal{C})$, indexed by $i \in I$. Then, if U is an ultraproduct on I, it holds $\prod_U \mathbb{M}_i \models \varphi$ for every $\varphi \in \mathrm{Th}(\mathcal{C})$, because of Łoś Theorem. However, it is known that the ultraproduct of complete chains need not be complete. □

Before proceeding, we shall provide a method for constructing concrete examples of complete chains with enough gaps.

Remark 3.11. Let $(C, \leqslant)$ be a complete chain. Then, we build a complete chain with enough gaps $(C^*, \preceq)$ by replacing every point $x \in C$ with a gap (x_1, x_2), in such a way that $x_1 \prec x_2$ and whenever $x \leqslant y$ then $x_1 \prec x_2 \preceq y_1 \prec y_2$.

The chain $(C^*, \preceq)$ will have enough gaps by construction and it is complete because so is $(C, \leqslant)$.

Since we have focused on chains, we shall mention a result on antichains.

Proposition 3.12. *There are Priestley (resp. Esakia) representable posets of arbitrary width.*

Proof. Let κ be a cardinal and consider the set $X_\kappa := \kappa \sqcup \{\bot\}$ ordered by $\leqslant_\kappa := \{(\bot, \lambda) \mid \lambda \in \kappa\}$. This defines a poset of cardinality κ. Moreover, the family of subsets of X_κ

$$\{U \subseteq \kappa \mid 0 \notin U \text{ and } |U| < \aleph_0\} \cup \{U \subseteq \kappa \mid 0 \in U \text{ and } |U^c| < \aleph_0\}$$

induces an Esakia topology on it. Compactness and the Esakia condition are immediate as well. Then, if $x \nleqslant y$ we have two cases: if $x = 0$ then $U = \kappa \smallsetminus \{\bot, y\}$ is a clopen upset containing x but not y (because $x \nleqslant y$ implies $x \neq \bot$); otherwise so it is $U = \{x\}$ for the same reason. □

Priestley and Esakia representable chains coincide, but this is not true for every class of posets. We will see at the end of this section that there are Priestley representable posets which are not Esakia representable. However, before showing the differences between the classes of Priestley and Esakia representable spaces, we shall mention two similarities: they are both closed under *arbitrary disjoint unions* and *finite ordered sums*.

That the class of Priestley spaces is closed under disjoint unions was already noticed by Lewis and Ohm in (Lewis & Ohm, 1976). They did not gave a proof but, for the sake of completeness, we shall provide one. First, we need an observation.

Observation 3.13. Let $\mathbb{P}$ be a Priestley (resp. Esakia) representable poset. If P is nonempty then the set of maximal elements of $\mathbb{P}$ is nonempty.

Proof. We can appeal to Zorn's lemma: let C be a nonempty subchain of $\mathbb{P}$. Since $\mathbb{P}$ is Priestley representable we know that C has a supremum in $\mathbb{P}$. Therefore, Zorn's lemma implies that the set of maximals of $\mathbb{P}$ is nonempty. □

Proposition 3.14. *Let $(P_i, \leqslant_i)_{i\in I}$ a collection of posets. If each $(P_i, \leqslant_i)$ is Esakia representable then so it is $(\bigsqcup_{i\in I} P_i, \leqslant)$.*

Proof. Let τ_i be an Esakia topology on P_i, which exists by assumption, and denote by P the disjoint union of the P_i's.

Without loss of generality we may assume $I \neq \emptyset$, otherwise there is nothing to prove. Moreover, we can assume $P_i \neq \emptyset$ for every i. Let us observe that for all $i \in I$ it holds $\max P_i \neq \emptyset$, in view of the previous Observation. Then, since I is nonempty, we can consider P_0 and a maximal point $x_0 \in P_0$.

Declare that a subset $U \subseteq P$ is open if and only if the two following hold:

1. $U = \bigcup_{i\in J} U_i$ for some $J \subseteq I$ and $U_i \in \tau_i$;
2. If $x_0 \in U$ then there is some $J' \subseteq J$ cofinite in I such that $\bigcup_{j\in J'} P_j \subseteq U$ and $U_j \neq \emptyset$ for every $j \in J'$.

Call τ the family of subsets of P that we have just defined and let us prove that it is an Esakia topology.

Compactness: Any open covering of P must contain an open set U such that $x_0 \in U$. Then, U covers already the entire P up to finitely many P_i's, because of the second condition that defines τ. However, a covering of P_i by means of open sets of τ is, in particular, an open covering of P_i by means of open sets in τ_i, thanks to the first condition defining τ. But each τ_i is compact, therefore we can conclude.

Priestely separation: Assume $x \nleqslant y$ for some $x, y \in P$. Then, there are $i, j \in I$ such that $x \in P_i$ and $y \in P_j$.

- If $i = j$, since $(P_i, \leqslant_i, \tau_i)$ is an Esakia space there is a clopen upset $U_i \in \tau_i$ such that $x \in U_i$ but $y \notin U_i$. If $i = 0$ and $x_0 \in U_0$ then set

  $$U := U_0 \cup \bigcup_{k\in I\smallsetminus\{0\}} P_k.$$

  Otherwise, consider $U := U_i$.

  In both cases, both U and U^c are opens. For, if $i = 0$ and $x_0 \in U_0$ then $U^c = P_0 \smallsetminus U_0$. Otherwise, $U^c = (P_0 \smallsetminus U_0) \cup \bigcup_{k\neq 0} P_k$. Moreover, in both cases U is an upset because so is U_i.

- If $i \neq j$, then one among them is non-zero, we might assume without loss of generality that $i \neq 0$. Thus, P_i is a clopen set both upward and downward closed containing x but not y.

Esakia condition: Fix some $U = \bigcup_{i\in J} U_i$ for $U_i \in \tau_i$ and consider $\downarrow U$.

Recall that $\downarrow$ commutes with arbitrary unions, hence $\downarrow U = \bigcup_{i\in J} \downarrow U_i$. Now, each $\downarrow U_i$ belongs to its respective τ_i, because each $(P_i, \leqslant_i, \tau_i)$ is an Esakia space.

Moreover, if $x_0 \in \downarrow U$ then, since x_0 was chosen to be maximal in P_0, it must be $x_0 \in U$ already, and hence J is cofinite in I because U is an open set by assumption. □

Corollary 3.15. *Let $(P_i, \leqslant_i)_{i\in I}$ a collection of posets. If each $(P_i, \leqslant_i)$ is Priestley representable then so it is $(\bigsqcup_{i\in I} P_i, \leqslant)$.*

Proof. This follows from the proof of the previous proposition. □

Another well-known closure property is mentioned on page 85 of (Esakia, 2019)). There it is said that the class of Esakia representable posets is closed under *finite ordered sums*. Let us explain what does this mean: let $(P, \leqslant)$ a poset and let $(P_x, \leqslant_x)$ be a poset for every $x \in P$. The *ordered sum* of the $(P_x, \leqslant_x)$ is the poset $(P^*, \leqslant^*)$ whose universe is $\bigsqcup_{x\in P} P_x$ and the relation $\leqslant^*$ is defined as follows:

$$a \leqslant^* b \iff (a, b \in P_x \text{ and } a \leqslant_x b) \text{ or } (a \in P_x, b \in P_y \text{ and } x < y)$$

for some $x, y \in P$.

Observe that we have used the first letters of the latin alphabet $a, b, c, \dots$ for the elements of P^* and of each P_x, while we used the letters $x, y, z, \dots$ for the elements of P.

Proposition 3.16. *Let $(P, \leqslant)$ be a finite poset and $\{(P_x, \leqslant_x) \mid x \in P\}$ a collection of (possibly infinite) Esakia representable posets. Then, its ordered sum $(P^*, \leqslant^*)$ is Esakia representable.*

Proof. Because every $(P_x, \leqslant_x)$ is Esakia representable there is a topology τ_x on P_x for every $x \in P$ which makes $(P_x, \leqslant_x, \tau_x)$ an Esakia space. Then, consider the topology τ^* on P^* generated by

$$\mathcal{B} = \bigcup_{x\in P} \tau_x \cup \{\emptyset\}.$$

This defines a base because if $U, V \in \mathcal{B}$ we either have $U, V \in \tau_x$ or not. In the former case, $U \cap V \in \tau_x$ and hence $U \cap V \in \mathcal{B}$. In the latter case, $U \cap V = \emptyset \in \mathcal{B}$.

Compactness: let $(U_i^*)_{i\in I}$ be an open covering of P^*. As usual, we may assume this covering to consists of open sets from the base. In other words, $\bigcup_{i\in I} U_i^* = P^*$. In particular, for every $x \in P$ there is some $I_x \subseteq I$ such that $P_x \subseteq \bigcup_{i\in I_x} U_i^*$. Every such U_i must belong to τ_x by definition, and since τ_x is a compact topology, without loss of generality, we may assume I_x to be finite for every $x \in P$. But P is finite as well, hence our covering is made by a finite unions of finite open sets, i.e. it is finite itself.

Priestley separation: Assume $a \nleqslant b$. We have two cases: either $a, b \in P_x$ for some $x \in P$ or not.

1. In the former case, since $(P_x, \leqslant_x, \tau_x)$ is an Esakia space, there is a clopen upset $U_x \in \tau_x$ containing a but not b. Then, the upset generated by U_x within $(P^*, \leqslant^*)$ is a clopen upset of τ^* containing a but not b. In fact, $\uparrow U_x = U_x \cup \bigcup_{x<y} P_y$ and we have the following equalities:

$$(\uparrow U_x)^c = (U_x)^c \cap \bigcap_{x<y} (P_y)^c = (P_x \smallsetminus U_x) \cup \bigcup_{z<x} P_x.$$

2. In the latter case, it holds $a \in P_x$ and $b \in P_y$ for two distinct elements x and y of P. By definition, we deduce $x \nleqslant y$. Hence, $\bigcup_{z\leqslant y} P_z$ is an open downset containing y but not x. It is also closed since its complement is $\bigcup_{z\nleqslant y} P_z$.

Esakia condition: Let U be an open of τ^*. By definition, U is union of basic open sets of the form $U_x \in \tau_x$.

Recall that $\downarrow$ commutes with arbitrary unions. This means that in order to show that $\downarrow U$ is an open set, it suffices to show that each $\downarrow U_x$ belongs to τ.

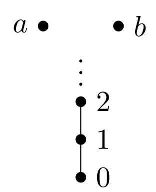
This is the case, since $\downarrow U_x = (\downarrow U_x \cap P_x) \cup \bigcup_{y<x} P_y$ and $\downarrow U_x \cap P_x \in \tau_x$ because $(P_x, \leqslant_x, \tau_x)$ is an Esakia space. □

Corollary 3.17. *Let $(P, \leqslant)$ be a finite poset and $\{(P_x, \leqslant_x) \mid x \in P\}$ a collection of (possibly infinite) Priestley representable posets. Then, its ordered sum $(P^*, \leqslant^*)$ is Priestley representable.*

Proof. This follows from the proof of the previous proposition. □

We can notice that since $(P, \leqslant)$ is finite it is, in particular, Esakia (whence Priestley) representable. Accordingly, one could wonder whether the previous proposition holds if we ask the poset $(P, \leqslant)$ to be Esakia (resp. Priestley) representable, rather than just finite. This is not the case, as the following example shows.

Example 3.18. Consider the ordered structure of the ordinal $\omega + 1$, i.e. $\mathbb{P} = (\omega + 1, \in)$. We already know that $\mathbb{P}$ is Esakia (whence Priestley) representable, because it is a complete chain with enough gaps. Moreover, for every $n \in \omega$ we define $(P_n, \leqslant_n)$ to be a different copy of the one element poset $(\{n\}, \{(n, n)\})$ while we let $(P_\omega, \leqslant_\omega)$ be the two elements antichain $(\{a, b\}, \{(a, a), (b, b)\})$. Then, consider the poset $(P^*, \leqslant^*)$, whose universe P^* consists of the set $\omega \cup \{a, b\}$ and a and b are above every natural number and moreover $a \parallel b$; see picture below.



We can observe that $(P^*, \leqslant^*)$ is not Priestley (hence not Esakia) representable. For, the chain $\omega \subseteq P^*$ does not have a supremum, since its set of upper bounds does not have a least element.

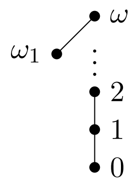
At this point, we shall mention the differences between Priestley and Esakia representable posets. The underlying idea is that Priestley spaces are closed under order duals, but this is not the case for Esakia spaces. For, the Esakia condition is asymmetrical with respect to the ordering of the poset, due to the requirement that the downset of an open set must be an open set as well.

Proposition 3.19. *A poset $\mathbb{P}$ is Priestely representable if and only if so is its order-dual $\mathbb{P}^\partial$.*

Proof. It suffices to prove that if $(\mathbb{P}, \tau)$ is a Priestley space, then so is $(\mathbb{P}^\partial, \tau)$. Accordingly, observe that a topology is compact on $(P, \leqslant)$ if and only if it is such on $(P, \geqslant)$. Moreover, $x \ngeqslant y$ if and only if $x \nleqslant y$ and an open is a $\leqslant$-upset if and only if U^c is $\geqslant$-upset, and this implies that two points satisfy the Priestley separation axiom on $\mathbb{P}$ if and only if they satisfy it in $\mathbb{P}^\partial$. □

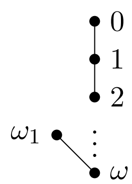
As we have mentioned, this is not the case for Esakia representable posets, as the next example shows.

Example 3.20. Let $\mathbb{P}$ be the poset whose universe is $\mathbb{N} \cup \{\omega, \omega_1\}$ and the ordering $\leqslant$ is defined to be the usual ordering of the natural numbers together with the reflexive closure of $\{(n, \omega) \mid n \in \mathbb{N}\} \cup \{(\omega_1, \omega)\}$, see the picture below.



Then, let τ be the topology generated by the basis $\mathsf{FinCofin}(\omega)$, which is the set of finite subsets not containing ω together with the cofinite ones containing it. It is easy to see that $(\mathbb{P}, \tau)$ is an Esakia (whence Priestley) space. For, any open covering can, without loss of generality, be taken to contain only basic open sets, one of which must contain ω and hence it is cofinite. This implies that τ is a compact topology. Moreover, the Priestley separation axiom holds because if $x \nleqslant y$ then $y \neq \omega$ and thus $\uparrow x \cup \uparrow n$ for $n > m$ if $y = m$ is a clopen upset containing x but not y. Finally, for the Esakia condition, let U be a clopen. If $\omega \in U$ then $\downarrow U = P$ which is clopen. Otherwise, $\downarrow U$ is finite and it does not contain ω.

Let us now consider the order dual $\mathbb{P}^\partial$ of $\mathbb{P}$, i.e. the poset whose universe is $\mathbb{N} \cup \{\omega, \omega_1\}$ and the ordering $\leqslant$ is the reflexive closure of the following: $\{(n, m) \mid m \in n\} \cup \{(\omega, m) \mid m \in \mathbb{N}\} \cup \{(\omega, \omega_1)\}$, see the picture below.



In view of Proposition 3.19, we know that $\mathbb{P}^\partial$ is Priestley representable. However, we claim that it is not Esakia representable. In order to see this, let us proceed by contradiction, assuming that there is an Esakia topology on it. Clearly, $\omega_1 \nleqslant 0$, therefore there is a clopen upset U containing ω_1 but not 0. In particular, U cannot contain ω nor any natural number, i.e. $U = \{\omega_1\}$. Now, because we have supposed our topology to satisfy the Esakia condition, it must be that $\downarrow U = \{\omega_1, \omega\}$ is an open set as well. But then, consider $\downarrow U \cup \bigcup (\downarrow n)^c$. It is an open covering of $\mathbb{P}$, because every $\downarrow n$ must be a closed set, and moreover for every m there is some n such that $m \in n$ and $m \in (\downarrow n)^c$. That is, we have found an infinite covering of $\mathbb{P}$. Clearly, it does not have any finite subcovers. For, ω does not belong to any open of the covering apart from $\downarrow U$, and for any finite union of the form $(\downarrow n_1)^c \cup \cdots \cup (\downarrow n_m)^c$ there is k such that $\max\{n_1, \ldots, n_m\} \in k$ and thus $k \notin (\downarrow n_1)^c \cup \cdots \cup (\downarrow n_m)^c$. This is a contradiction, because every Esakia space is, in particular, compact.

The non-Esakia representable poset of the above example will play an important role in the following section, hence let us call it $\mathbb{P}_0$.

3.2 Necessary conditions

In the previous section we have reviewed some of the best known properties of Priestley and Esakia representable posets. This section is devoted to the presentation of some results which, to the best of our knowledge, do not appear in the literature. Some of them are rather technical, but we will employ most of them extensively in Chapter 4.

Recall that Example 3.7 shows that there is a poset which satisfies C1 and C2 but is not Priestley representable. As observed by Lewis and Ohm in (Lewis & Ohm, 1976), the poset described in this example is not Priestley (resp. Esakia) representable because it has a collection $\mathcal{C}$ of principal downsets whose intersection is empty, but $\mathcal{C}$ does not have a finite subcollection whose intersection is empty.

This property can be generalized as follows.

Proposition 3.21 (Condition C3)**.** *Let $\mathbb{P}$ be a Priestley representable poset, and let $\mathcal{U} \subseteq \mathcal{P}(P)$ be the least family of subsets of P that contains:*

1. *Every principal upset of $\mathbb{P}$;*
2. *Every principal downset of $\mathbb{P}$;*

and it is closed under finite unions, arbitrary intersections, generated upsets and generated downsets. Then, if $\{U_i \mid i \in I\}$ is a family of elements of $\mathcal{U}$ whose intersection is empty, there exists a finite subfamily whose intersection is also empty.

Proof. Let τ be a topology such that $(\mathbb{P}, \tau)$ is a Priestley space.

Observe that the family $\mathcal{U}$ consists of closed subsets of $(\mathbb{P}, \tau)$. For, in view of Corollary 2.44 we know that the principal upsets and principal downsets of $\mathbb{P}$ are closed. Then, the set of closed subsets of $\mathbb{P}$ is closed under finite unions and arbitrary intersections. Moreover, in view of Proposition 2.43, generated upsets and downsets of closed sets are closed sets as well.

Therefore, the above mentioned property is a reformulation of compactness by means of closed subsets of P. □

Proposition 3.22 (Condition C3 for Esakia representable posets)**.** *Let $\mathbb{P}$ be an Esakia representable poset, and let $\mathcal{U} \subseteq \mathcal{P}(P)$ be the least family of subsets of P that contains*

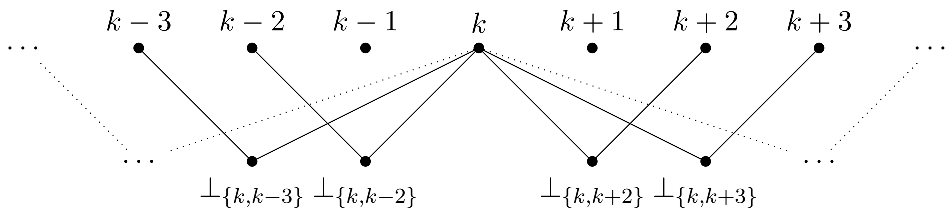
1. *every principal upset of $\mathbb{P}$;*
2. *every principal downset of $\mathbb{P}$;*
3. *the set of maximals $\max(\mathbb{P})$ of $\mathbb{P}$;*

and it is closed under finite unions, arbitrary intersections, generated upsets, generated downsets and complements of generated downsets of complements. Then, if $\{U_i \mid i \in I\}$ is a family of elements of $\mathcal{U}$ whose intersection is empty, there exists a finite subfamily whose intersection is also empty.

Proof. The proof is analogous to that of the proof of Proposition 3.21. Just recall that for every Esakia space, its set of maximals is closed (Proposition 2.49). Moreover, in any Esakia space the space the downset of every open set is an open set as well, i.e. the complement of the downset of a complement of a closedm set, is a closed set. □

Let us provide an application of the previous proposition. We will use it in order to show that there is a poset of height 2 and width 2 which satisfies C1 and C2 but it is not Priestley (resp. Esakia) representable, because it does not satisfy C3.

Example 3.23. Consider the poset $\mathbb{P}$ depicted below.



It is defined starting from the set of integers $\mathbb{Z}$. Then, for every $k \in \mathbb{Z}$ and $n+1 < k$, we consider a point $\perp_{\{n,k\}}$. We define:

$$P = \mathbb{Z} \cup \{\perp_{\{n,k\}} \mid k \in \mathbb{Z}, n+1 < k\}.$$

The ordering $\leqslant$ on P is defined as follows:

$$x \leqslant y \iff \text{either } x = y \text{ or } (x = \perp_{\{n,k\}} \text{ and } y \in \{n,k\}).$$

Observe that $\mathbb{P}$ has height two and width two. Moreover, it satisfies C1 and C2. In (Bezhanishvili et al., 2021) it was shown that $\mathbb{P}$ is not Esakia representable. Even more is true: we will prove that it does not satisfy C3, and hence it is not

Priestley representable either. In order to see this, consider, for every $k \in \mathbb{Z}$, the upset $\uparrow\{\downarrow k\}$ of its principal downset $\downarrow k$. We claim that $\bigcap_{k\in\mathbb{Z}} \uparrow\{\downarrow k\} = \emptyset$. In order to see this, let us do some observations.

1. For every $k \in \mathbb{Z}$, it holds $\downarrow k = \{k\} \cup \{\bot_{\{n,k\}} \mid n + 1 < k\}$. This follows immediately from the definition of $\leqslant$.

2. For every $k \in \mathbb{Z}$ it holds $\uparrow\{\downarrow k\} = \{\bot_{\{n,k\}} \mid n+1 < k\} \cup (\mathbb{Z} \smallsetminus \{k - 1, k + 1\})$. In view of the previous point, it suffices to prove the following:

$$\uparrow\{\downarrow k\} \cap \mathbb{Z} = \mathbb{Z} \smallsetminus \{k - 1, k + 1\}.$$

($\subseteq$) If h is above some $\bot_{\{k,n\}}$, for $n + 1 < k$, it must be either $h = k$ or $h = n$. In both cases we have $h \in \mathbb{Z} \smallsetminus \{k - 1, k + 1\}$ because $k \notin \{k - 1, k + 1\}$ and $n + 1 < k$ implies $n \notin \{k - 1, k + 1\}$.

($\supseteq$) Let $h \in \mathbb{Z} \smallsetminus \{k-1, k+1\}$. If $h \neq k$ we deduce $\bot_{\{h,k\}} \leqslant h$. Otherwise, if $h = k$, we have, for instance, $\bot_{\{k,k+2\}} \leqslant k = h$.

We then have:

$$\bigcap_{k\in\mathbb{Z}} \uparrow\{\downarrow k\} = \bigcap_{k\in\mathbb{Z}} \left(\{\bot_{\{n,k\}} \mid n + 1 < k\} \cup (\mathbb{Z} \smallsetminus \{k - 1, k + 1\})\right).$$

However, $\bot_{\{n,k\}}$ does not belong to any $\mathbb{Z} \smallsetminus \{k - 1, k + 1\}$ and, viceversa, if $h \in \mathbb{Z} \smallsetminus \{k - 1, k + 1\}$ then $h \neq \bot_{\{n,k\}}$ for any $n, k \in \mathbb{Z}$. In other words, the following equation holds:

$$\bigcap_{k\in\mathbb{Z}} \uparrow\{\downarrow k\} = \left[\bigcap_{k\in\mathbb{Z}} \{\bot_{\{n,k\}} \mid n + 1 < k\}\right] \cup \left[\bigcap_{k\in\mathbb{Z}} (\mathbb{Z} \smallsetminus \{k - 1, k + 1\})\right].$$

But then, we have:

$$\bigcap_{k\in\mathbb{Z}} \uparrow\{\downarrow k\} \cap \mathbb{Z} = \bigcap_{k\in\mathbb{Z}} \mathbb{Z} \smallsetminus \{k - 1, k + 1\} = \emptyset.$$

as well as:

$$\bigcap_{k\in\mathbb{Z}} \{\bot_{\{n,k\}} \mid n + 1 < k\} = \emptyset.$$

However, there is no finite subfamily of the $\uparrow\{\downarrow k\}$'s whose intersection is empty, because no finite intersection of $\uparrow\{\downarrow k\} \cap \mathbb{Z}$ is empty. In other words, $\mathbb{P}$ does not satisfy C3 and whence, in view of Proposition 3.21, it is not Priestley representable.

Recall that a Priestley (resp. Esakia) space needs to be, in particular, a compact topological space. This means that an infinite Priestley (resp. Esakia) space must have a nonempty set of non-isolated points. For example, in the proof of Proposition 3.12 we were dealing with an infinite poset. In order to turn it into an Esakia space, we have arbitrarily chosen a point x_0 and we constructed a topology in which x_0 is non-isolated.

It is possible to show that there are posets whose ordered structure already determines a set of non-isolated points. In other words, the set of isolated points of a Priestley (resp. Esakia) representable poset cannot be chosen arbitrarily. These points are, for example, the suprema (resp. infima) of infinite ascending (resp. descending) chains.

Assume we have a poset $\mathbb{P}$ which contains a chain $C \subseteq P$. If we want $\mathbb{P}$ to be Priestley (or Esakia) representable, we know that C must have both an infimum and a supremum in $\mathbb{P}$. We have two cases: either $\sup(C)$ (resp. $\inf(C)$) has an *immediate predecessor* (resp. *immediate successor*) in C or not, in the following sense.

Definition 3.24. Let $\sup(C)$ be the supremum of a chain C in a poset $\mathbb{P}$. We say that $\sup(C)$ is a *successor* of C in $\mathbb{P}$ if there is some $x \in C$ such that $x < \sup(C)$ and moreover for every $y \in C$ if $y < \sup(C)$ then $x \leqslant y$. Otherwise, we will say that $\sup(C)$ is a *limit* of C in $\mathbb{P}$. Similarly, $\inf(C)$ is said to be a *predecessor* of C in $\mathbb{P}$ if it is a successor of $\mathbb{P}^\partial$, *limit* otherwise.

Observe that if a point is a limit of a chain, such chain must be infinite. The terminology "limit" reminds us about the limit points of a topology. This can be made precise by the following.

Proposition 3.25. *Let $\mathbb{P}$ be a Priestley space, $C \subseteq P$ a nonempty chain and U an open set of $\mathbb{P}$ such that $\sup(C) \in U$. If $\sup(C)$ is a limit of C in $\mathbb{P}$, then there is some $x \in C \smallsetminus \{\sup(C)\}$ such that $[x, \sup(C)] \subseteq U$.*

Proof. Let C be as in the hypothesis. We claim that there is some $x < \sup(C)$ such that $x \in C \cap U$. Suppose not. Recall that $(\uparrow x)^c$ is an open set for every $x \in C \cap U$, in view of Corollary 2.44.

Then, for every $y \notin \bigcup_{x \in C} (\uparrow x)^c \cup U$ it holds $\sup(C) < y$. Hence, there is a clopen upset U_y containing y but not $\sup(C)$. Observe that this means that no $x \in C$ belongs to any U_y. In other words, the following union

$$\bigcup_{x \in C} (\uparrow x)^c \cup \bigcup_{x<y} U_y \cup U$$

is an infinite covering of P – because $\sup(C)$ is a limit of C – with no finite subcovers – because no x belongs to $U \cup \bigcup_{x<y} U_y$. This contradicts the fact that $\mathbb{P}$ is compact.

We can now prove the statement of the proposition. It follows from the claim that we have just proved that there is some $x \in U \cap (C \smallsetminus \{\sup(C)\})$. Suppose, towards contradiction, that there is no $z \in C \smallsetminus \{\sup(C)\}$ such that $[z, \sup(C)] \subseteq U$. Consider the chain $\{z \in C \mid z \in U \text{ and } x < z\}$. The supremum $\bar{z}$ of this set must exist, and we have two cases: either $\bar{z} < \sup(C)$ or $\bar{z} = \sup(C)$.

1. In the former case, let $U_{\bar{z}}$ be a clopen upset containing $\sup(C)$ but not $\bar{z}$. Consider now the open $U \cap U_{\bar{z}}$: it contains $\sup(C)$ by construction, but no $z \in C \smallsetminus \{\sup(C)\}$ belongs to it, in contradiction with what we have showed at the beginning of this proof.

2. In the latter case, since there is no $z \in C \smallsetminus \{\sup(C)\}$ such that $[z, \sup(C)] \subseteq U$, there must be some $x_1 \in [x, \sup(C)]$ which is not in U. Hence, in particular, $x < x_1$.

   Therefore, there is a clopen upset U_{x_1} containing x_1 but not x.

   Consider the open $U \cap U_{x_1}$. It must hold $[x_1, \sup(C)] \nsubseteq U \cap U_{x_1}$. Thus, there is some $x_2 \in [x_1, \sup(C)]$ not in $U \cap U_{x_1}$. We can then proceed recursively in the same way. In conclusion, we have found an infinite covering of $[x, \sup(C)]$ with no finite subcovers, and this is a contradiction because $[x, \sup(C)] = {\uparrow}x \cap {\downarrow}\sup(C)$ is a closed subset of a compact space, whence compact.

□

Corollary 3.26. *Let $\mathbb{P}$ be a Priestley space, $C \subseteq P$ a nonempty chain and U and open set of $\mathbb{P}$. If $\inf(C)$ is a limit of C in $\mathbb{P}$, then there is $x \in C \smallsetminus \{\inf(C)\}$ such that $[\inf(C), x] \subseteq U$.*

Proof. The proof is analogous to the proof of the previous proposition. □

In the case where $\sup(C)$ is maximal, Proposition 3.25 can be strengthened as follows.

Proposition 3.27. *Let $\mathbb{P}$ a Priestley space, $C \subseteq P$ a nonempty chain and U an open set of $\mathbb{P}$. If $\sup(C) \in \max(P)$ and $\sup(C)$ is a limit of C in $\mathbb{P}$, then there is some $x \in C \smallsetminus \{\sup(C)\}$ such that ${\uparrow}x \subseteq U$.*

Proof. In view of Priestley's representation theorem, we may assume $\mathbb{P}$ to be the poset of prime filters of a bounded distributive lattice $\mathbb{L}$. Thus, $\inf(C)$ is a prime filter of $\mathbb{L}$, call it y. Since U is open, it is union of basic clopens which, in turn, are intersections of subbase opens of the form $\varphi(a)$ or $\varphi(b)^c$ for $a, b \in L$. Therefore, there are $a, b \in L$ such that $y \in \varphi(a) \cap \varphi(b)^c$ or, in other words, $a \in y$ and $b \notin y$. Now, since we have assumed y to be maximal, there is some $d \in y$ such that $d \wedge b = \bot$. For, if not, we could generate a proper filter from $y \cup \{b\}$ and then, since every proper filter can be extended to a maximal one, we would have a a prime filter z such that $y \cup \{b\} \subseteq z$, against the maximality of y.

We can then observe that, because the ordering of $\mathbb{P}$ is given by $\subseteq$ and y is a limit of the chain C, it must be

$$y = \bigcup C.$$

In order to see this, consider both inclusions. The inclusion from right to left is clear, because if $c \in z$ for some prime filter $z \in C$, then since $z \subseteq y$ we deduce $c \in y$. On the other hand, the union $\bigcup C$ is a prime filter, in view of Proposition 2.23. But y is a limit of C, meaning that C is infinite ascending, therefore $z \subset \bigcup C$ for every $z \in C$. But then, the minimality of y among the upper bounds of C implies $C \subseteq \bigcup C$.

Therefore, as $d \in y$, there is $x_1 \in C$ (thus $x_1 \subset y$)such that $d \in x_1$ and moreover, because $d \wedge b = \bot$, no filter above x_1 can contain b, i.e. $\uparrow x_1 \subseteq \varphi(b)^c$.

Then, remember that $a \in y$ and thus, for the same reason as before, we deduce that there is some $x_2 \in C$ (thus $x_2 \subset y$) such that $a \in x_2$, and thus every filter extending x_2 contains a, i.e. $\uparrow x_2 \subseteq \varphi(a)$.

Finally, since C was a chain, we either have $x_1 \subseteq x_2$ or $x_2 \subseteq x_1$, i.e. either $\uparrow x_2 \subseteq \varphi(a) \cap \varphi(b)^c \subseteq U$ or $\uparrow x_1 \subseteq \varphi(a) \cap \varphi(b)^c \subseteq U$, as desired. □

Corollary 3.28. *Let $\mathbb{P}$ a Priestley space, $C \subseteq P$ a chain and U an open set of $\mathbb{P}$. If $\inf(C) \in \min(P)$ and $\inf(C)$ is a limit of C in $\mathbb{P}$, then there is $\inf(C) < x$ such that $\downarrow x \subseteq U$.*

Proof. The proof is analogous to the proof of the previous Proposition. □

In the previous section we have proved that there is a Priestley representable poset $\mathbb{P}_0$ which is Priestley representable but not Esakia representable, as shown by Example 3.20. The universe of $\mathbb{P}_0$ was $\mathbb{N} \cup \{\omega, \omega_1\}$ and its ordering was given by the reversed ordering of the natural numbers, with ω at the bottom of the poset and ω_1 above ω but incomparable to every $n \in \mathbb{N}$. We conclude this chapter by proving that this example can be generalized. Before proving this, we give an important definition, which will be useful in the following chapters as well.

Definition 3.29. Let $(P, \leqslant)$ be a poset and C a chain in $\mathbb{P}$. A point $x \in P$ is said to be a *ramifying point* of C provided that there is some $c \in C$ such that $c < x$ and moreover $x \parallel c'$ for every $c' \in (\uparrow c \cap C) \setminus \{c\}$. The upset $\uparrow x$ is said to be *ramification* of C in $\mathbb{P}$. Moreover, the chain C is said to *ramify* if it has a ramification. In the same way, if $c \in C$ has a ramification point above it, c is said to *ramify*.

Theorem 3.30. *Let $(P, \leqslant)$ be a poset such that there is an order embedding $f : \mathbb{P}_0 \to \mathbb{P}$. If the three following conditions hold*

1. $f(\omega) = \inf\{f(n) \mid n \in \mathbb{N}\}$;
2. $(f(\omega), f(0))$ *is a chain with no ramifications in $\mathbb{P}$;*
3. $[f(0), f(\omega))$ *is a chain with no ramifications in $\mathbb{P}^\partial$;*

then $(P, \leqslant)$ is not Esakia representable.

Proof. First, let us denote by x_n the image of $f(n)$, and remember that f being an order embedding means $x_n < x_m$ if and only if $m < n$. Moreover, we will denote by x_ω the image of ω and similarly for x_{ω_1}. For the sake of contradiction, suppose there is a topology τ which makes $(P, \leqslant, \tau)$ an Esakia space. We will now find an infinite open covering with no finite subcovers, thus proving that τ is not compact, which is a contradiction.

1. $(\downarrow x_0 \cup \uparrow x_0)^c$ is open because we have assumed $(P, \leqslant, \tau)$ to be an Esakia space, hence the principal downsets of $\mathbb{P}$ as well as its principal upsets are closed sets. Moreover, the Esakia condition implies that $U := \downarrow((\downarrow x_0 \cup \uparrow x_0)^c)$ is an open set.

   Observe that $(x_\omega, x_0) \cup \uparrow x_0 = U^c$. For, $y \in U$ if and only if $y \leqslant z$ for some $z \parallel x_0$. Then, let us prove both inclusions.

   ($\subseteq$) Consider a point $y \in (x_\omega, x_0)$ and some z be such that $y \leqslant z$. Suppose $z \parallel x_0$. Since $x_\omega < y$, we deduce $x_\omega < z$. Then, because $z \parallel x_0$, it holds $z \notin (x_\omega, x_0)$. In other words, z is a ramifying point of (x_ω, x_0), which is a contradiction. This means that every z above y is not parallel to x_0, i.e. $y \in U^c$.

   If $y \in \uparrow x_0$ then, for all $z \geqslant y$, we deduce $z \geqslant y \geqslant x_0$, i.e. $y \in U^c$.

   ($\supseteq$) Suppose $y \in U^c$, i.e. for all z, if $y \leqslant z$ then $z \leqslant x_0$ or $x_0 \leqslant z$. In particular, $x_0 \leqslant y$ or $y < x_0$. The former case implies $y \in \uparrow x_0$, while in the latter case we claim that $x_\omega < y$, i.e. $y \in (x_\omega, x_0)$. We have two possibilities: either $y \leqslant x_\omega$ or not. In the former case, because $x_\omega < x_{\omega_1}$ by hypothesis, we deduce $y < x_{\omega_1}$ for $x_{\omega_1} \parallel x_0$, a contradiction with $y \in U^c$. Therefore it must be $y \nleqslant x_\omega$. Finally, since $[x_0, x_\omega)$ does not have ramifications in $\mathbb{P}^\partial$, it cannot be $x_\omega \parallel y < x_0$. In other words, it must be $y \in (x_\omega, x_0)$.

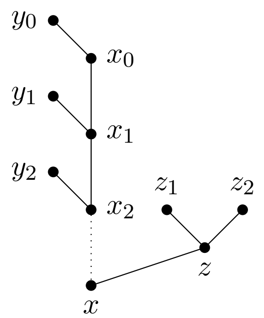
2. Consider $x_0 \nleqslant x_1$. Because τ is an Esakia topology, in particular it satisfies the Priestely separation axiom, meaning that there is an clopen upset V_0 containing x_0 but not x_1. As V_0 is an upset, it can't be a superset of $[x_\omega, x_1]$ but it is a superset of the whole $\uparrow x_0$ and thus, thanks to the previous point, we deduce $(x_\omega, x_0)^c \subseteq U \cup V_0$.

   Proceed recursively: $x_i \nleqslant x_{i+1}$, therefore there is a clopen upset V_i such that $[x_\omega, x_{i+1}] \nsubseteq V_i$ and $(x_\omega, x_{i+1})^c \subseteq U \cup \bigcup_{i+1 \leqslant j} V_j$.

The points 1. and 2. together give us an infinite covering T with no finite subcovers, hence proving the theorem. □

The previous theorem proves that for every Esakia space, if the infimum $\inf(C)$ of an infinite descending chain C ramifies, then there is no subchain of C whose infimum coincides with $\inf(C)$ but its elements do not ramify. In other words, if $\mathbb{P}$ is an Esakia representable poset and C is an infinite descending chain of $\mathbb{P}$, either $\inf(C)$ does not ramify, or C ramifies "very often". As it turns out, a stronger result holds. Let us introduce it with an example.

Example 3.31. Consider the tree depicted below.



Observe that it satisfies C1, C2 and it doesn't satisfy the hypothesis of Theorem 3.30. However, we claim that it is not Esakia representable. For the sake of contradiction, suppose there is an Esakia topology τ on $\mathbb{T}$ and consider z_1 and z_2. They are incomparable, therefore, because of the Priestley separation axiom, there are two clopen downsets U_1 and U_2, containing z_1 but not z_2 and z_2 but not z_1 respectively. Then, consider the following set:

$$V := U_1 \cap {\downarrow}(U_2 \setminus U_1).$$

Observe that V is open: for, U_1 is open and so it is $U_2 \setminus U_1$, because $U_2 \setminus U_1 = U_2 \cap U_1^c$ and U_1 is clopen. Then, because of the Esakia condition ${\downarrow}(U_2 \setminus U_1)$ must be open and then the intersection of two opens is open as well. Observe also that $z \in V$, because U_1 is a downset containing z_1 and $z \leqslant z_1$, moreover ${\downarrow}(U_2 \setminus U_1)$ is a downset containing z_2 and $z \leqslant z_2$.

Finally, we claim that there is no maximal element v that belongs to V. This is because if v is maximal and $v \in U_1$ then $v \notin U_2 \setminus U_1$, and since every v is maximal, it holds $v \in {\downarrow}(U_2 \setminus U_1)$ if and only if $v \in U_2 \setminus U_1$. In particular, no y_i belongs to V.

Then, the downset ${\downarrow}x_0$ must be closed, i.e. $({\downarrow}x_0)^c$ is open. Whence, the set $V' := V \cap ({\downarrow}x_0)^c$ is open. Then, notice that $z \in V'$ because we have observed that $x \in V$ and moreover $x \nleqslant x_0$, but no y_i belongs to V' (because no y_i belongs to V) and no x_i belongs to V' (because no x_i belongs to $({\downarrow}x_0)^c$). Therefore, $x \in {\downarrow}V'$ but $x_i \notin {\downarrow}V'$ for every $i \in \mathbb{N}$, contradicting Corollary 3.26.

The previous example can be generalized as follows.

Theorem 3.32. *Let $(P, \leqslant)$ be a poset and $x, y \in P$ such that $(x, y]$ is a chain whose infimum is x. Suppose the following conditions hold:*

1. *The cardinality of every ramification from $(x, y]$ is bounded by some $n \in \mathbb{N}$;*
2. *There is a ramification from x whose width is strictly greater than n.*

Then, $(P, \leqslant)$ is not Esakia representable.

Proof. By contradiction, suppose there is an Esakia topology τ on P. By hypothesis, there is a ramifying point $z > x$ and there are $z_0, \dots, z_n$ such that $z < z_i$ and $z_i \parallel z_j$ for every $i, j \leqslant n$ and $j \neq i$. If $n = 0$ we are under the hypothesis of Theorem 3.30 and hence there is nothing else to prove. Otherwise, assume $n > 0$.

The topology τ satisfies the Priestley separation axiom, therefore there is a clopen downset V_{01} containing z_0 but not z_1. In the same way, there are clopen downsets $V_{0,i}$ containing z_0 but not z_i, for $0 < i \leqslant n$. Consider the following set:

$$U_0 := \bigcap_{0<i\leqslant n} V_{0i}.$$

Since it is a finite intersection of clopen downsets, it is a clopen downset itself. Moreover, $z_0 \in U_0$ but no $z_i \in U_0$, for any $0 < i \leqslant n$. In the same fashion, we can consider U_i, defined analogously for every $i \leqslant n$.

Then, we define the following:

$$U := \bigcap_{0\leqslant i\leqslant n} \downarrow \left(U_i \smallsetminus \left(\bigcup_{j\neq i} U_j \right) \right)$$

Observe that U is a clopen downset. For, every $\bigcup_{j\neq i} U_j$ is a clopen set, because it is a finite union of clopen sets. Therefore,

$$U_i \smallsetminus \left(\bigcup_{j\neq i} U_j \right) = U_i \cap \left(\bigcup_{j\neq i} U_j \right)^c$$

is a clopen set as well. Then, since τ is an Esakia topology, the following is a clopen downset:

$$\downarrow \left(U_i \smallsetminus \left(\bigcup_{j\neq i} U_j \right) \right)$$

Finally, the intersection of finitely many clopen downsets is a clopen downset as well.

Let us prove the following facts: $z \in U$ and no point of any ramification of $(x, y]$ belongs to U. The first observation is clear: for every $i \leqslant n$, it holds $z_i \in U_i$ and $z_i \notin U_j$ for any $j \neq i$, therefore $z_i \in U_i \smallsetminus \left(\bigcup_{j\neq i} U_j \right)$. Then, since $z < z_i$, and we can reason in the same way for every $i \leqslant n$, we deduce $z \in U$.

Then, in order to prove that $w \notin U$, for every w that belongs to some ramification of $(x, y]$, suppose there is some $w \in U$. This can happen if and only if there are $w_0, \dots, w_n$ such that $w_i \in U_i \smallsetminus \left(\bigcup_{j\neq i} U_j \right)$, for every $i \leqslant n$. In particular, the w_i's are all distinct, but this is impossible since the cardinality of every ramification of $(x, y]$ is bounded by n.

Summing up, $z \in U$ and no point of any ramification of $(x, y]$ belongs to U. In order to conclude, intersect U with $(\downarrow y)^c$ and consider the following set:

$$U^* := \downarrow (U \cap (\downarrow y \cup \uparrow y)^c) .$$

The set U^* is an open set, because (a) U is an open set; (b) $(\downarrow y \cup \uparrow y)^c$ the complement of a closed set, whence an open set; (c) in an Esakia space the downset of every open set is an open set.

Observe also that $x \in U^*$, because $z \in U$, $z \nleqslant y$ and $x \leqslant z$. However, we claim $(x, y] \cap U^* = \emptyset$. For, take some $v \in (x, y]$ and some w such that $v \leqslant w$. If w is either below or above y then $v \in \downarrow y \cup \uparrow y$, thus implying $v \notin U^*$. Otherwise, if $w \parallel y$, since $v \leqslant w$ and $w \in (x, y]$, we have that w is a ramifying point of $(x, y]$, and we have just proved that it cannot belong to U. In conclusion, no point w above v can be both in U and in $(\downarrow y \cup \uparrow y)^c$.

Hence, we have shown that there is an open U^* which contains the limit point of the chain $(x, y]$ but it does not contain any point of the chain, thus contradicting Proposition 3.26. This concludes the proof.

□

Theorems 3.30 and 3.32 will be particularly relevant for the next chapter, where we are going to study the Esakia representability of forests.

CHAPTER 4

Forests and diamond systems

This chapter is devoted to the study of Priestley and Esakia representability of two classes of posets: forests and diamond systems.

In the first section we characterize the Priestley (resp. Esakia) representable diamond systems by proving that a diamond system is Priestley (resp. Esakia) representable if and only if it satisfies C1 and C2. In the second section we will see how to apply this result in order to simplify the main proof of (Bezhanishvili et al., 2021). In particular, we will study which profinite Heyting algebras are profinite completions of some Heyting algebra.

In the third section we move to the study of forests. In view of Theorems 3.30 and 3.32, the Esakia representability of forests which have infinite descending chains is highly non-trivial. Accordingly, in the third section we characterize the Esakia representable well-ordered forests, i.e., forests which do not have infinite descending chains. We prove that a well-ordered forest is Esakia representable if and only if it satisfies C1 and C2. This is the main result of this thesis. In the fourth section we suggest how to proceed in the case of countable forests. In particular, we provide two new forbidden configurations of Esakia representable countable forests.

4.1 Diamond systems

In section 3.1 we showed that the class of Priestley (resp. Esakia) representable nonempty chains can be characterized as the class of complete chains with enough gaps. We summarized the properties that a representable poset must satisfy in section 3.2. Motivated by the positive results on chains, it is natural to study posets which arise as simple "combinations" of chains. For example, Lewis investigated in (Lewis & Ohm, 1976) the Priestley representability of forests, i.e., posets whose principal downsets are chains.

Definition 4.1. A poset $\mathbb{P} = (P, \leqslant)$ is said to be a *forest* if $\downarrow x$ is a chain for every $x \in P$.

Lewis proved that a forest is Priestley representable if and only it satisfies C1 and C2. In this section, we will be dealing with a generalization of the class of the order duals of forests, as we shall explain below.

Definition 4.2. A poset $\mathbb{P}$ is said to be a *root system* if its order dual $\mathbb{P}^\partial$ is a forest.

In view of Proposition 3.19, Lewis' characterization of Priestley representable forests solves the problem of Priestley representable root systems as well. We will generalize the work of Lewis by characterizing the class of Priestley representable diamond systems, an extension of the class of root systems. Diamond systems have been introduced in (Bezhanishvili et al., 2021), in order to answer the question of which profinite Heyting algebras are profinite completions. We will delve into this problem in Section 4.2.

Definition 4.3. A poset $\mathbb{P}$ is said to satisfy the *three point rule* if whenever $x, y, z \in P$ are distinct elements of P such that $x \parallel y$, then $x \leqslant z$ implies $y \leqslant z$.

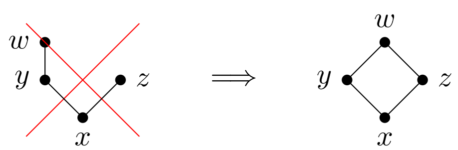
Definition 4.4. A poset $\mathbb{P}$ is said to be *upward directed* (resp. *downward directed*) if whenever $x, y \in P$, there is $z \in P$ such that $x, y \leqslant z$ (resp. $z \leqslant x, y$).

Definition 4.5 ((Bezhanishvili et al., 2021), Definition 4.1)**.** A poset $\mathbb{P}$ is called a *diamond system* if it satisfies the following conditions:

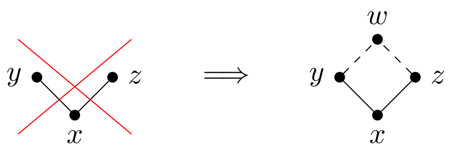
1. Each principal upset $\uparrow x$ satisfies the three point rule;
2. $\mathbb{P}$ has width at most two;
3. Each principal upset $\uparrow x$ is upward directed;
4. For every $\bot, x, y, z, v, \top \in P$, if $\bot \leqslant x, y \leqslant z, v \leqslant \top$, there is a $w \in P$ such that $x, y \leqslant w \leqslant z, v$.

Intuitively, diamond systems are a generalization of root systems whose width is allowed to be two. In order to see this, let us explain what the conditions 1, 3 and 4 of a diamond system imply, with the help of some figures. In the following pictures, the left hand side depicts how a diamond system cannot look like, while the image at the right hand side shows how it should.

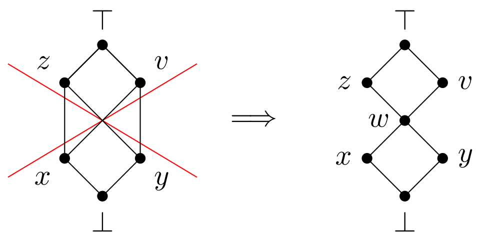
The first condition says that every $\uparrow x$ must satisfy the three point rule. That is, whenever two incomparable points y and z are above some x, the points y and z must share every successor.



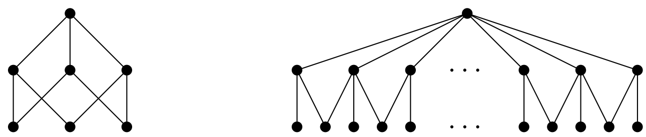
The third condition may already be more familiar than the others. In fact, upward (resp. downward) directed posets are well-known. However, here we do not require the whole $\mathbb{P}$ to be upward directed, but just every principal upset $\uparrow x$; see the picture below.



Finally, the fourth condition expresses that whenever there might be a diamond-like shape in our poset, there must be one. Formally, whenever we have two pairs of points $\{x, y\}, \{z, v\}$ such that these two pairs are bounded from below and from above by some $\bot$ and $\top$, and both z and v are above both x and y, then there is some w in between the pairs; see the picture below.



Observe that the notation $\bot$ and $\top$ does not mean to imply that $\mathbb{P}$ is bounded. This is just a convenient notation that suggests that the subposet $\{x, y, z, v\}$ is bounded by $\bot$ and $\top$ within $\mathbb{P}$. On the other hand, examples of diamond systems are depicted below.



As a matter of fact, in the picture above we can view three examples of diamond systems: the one depicted at the left, the one drawn at the right, and their disjoint union. In fact, every diamond system is a disjoint union of simpler diamond systems, as we explain below.

Definition 4.6. A *path* in a poset $(P, \leqslant)$ is a finite tuple $(x_1, \ldots, x_n)$ of elements of P such that for all $i < j \leqslant n$, either $x_i \leqslant x_j$ or $x_j < x_i$). We call x_1 the starting point of the path and x_n the final point of the path, and we say that there is a path from x_1 to x_n. A poset $(P, \leqslant)$ is said to be *connected* if, for every $x, y \in P$, there is a path from x to y.

Observation 4.7. Every diamond system is a disjoint union of connected diamond systems.

We have already mentioned that every root system is a diamond system. We will now prove this fact.

Proposition 4.8. *A root system is a diamond system whose width is at most one.*

Proof. Let $\mathbb{P}$ be a root system. Since $\uparrow x$ is a chain for every $x \in P$, we know that $\mathbb{P}$ has width one, thus, in particular, at most two. Moreover, every $\uparrow x$ satisfies the three point rule vacuously and it is upward directed, since every two elements above x are comparable. Finally, if $\bot \leqslant x, y \leqslant z, v \leqslant \top$, it means that $x, y \in \uparrow\bot$, thus $x \leqslant y$ or $y \leqslant x$. Without loss of generality assume the former. Then, $x, y \leqslant y \leqslant z, v$. □

In (Bezhanishvili et al., 2021) it was shown that every image finite diamond system can be extended to an Esakia representable diamond system. The principal aim of this section is to characterize the class of *all* Priestley (resp. Esakia) representable diamond systems. More explicitly, we are going to prove the following result.

Theorem 4.9. *A diamond system $\mathbb{P}$ is Esakia representable if and only if it satisfies C1 and C2.*

Recall that a Priestley representable poset must satisfy C1 and C2, and that an Esakia representable poset is, in particular, Priestley representable. Whence, we get, as a corollary of Theorem 4.9, that a diamond system is Priestley representable if and only if it satisfies C1 and C2.

In order to prove Theorem 4.9 we need some technical lemmas, which will help us define Esakia topologies on diamond systems.

Definition 4.10. Let $\mathbb{P}$ be a poset and $x, y \in P$. The element x is said to be an *immediate predecessor* of x if y if the pair (x, y) is a gap. In this case, y is called an *immediate successor* of y.

Recall from Definition 2.5 that we use $x \parallel y$ as a shorthand for $x \nleqslant y$ *and* $x \nleqslant y$. In this case, we say that x and y are *incomparable*, or *parallel*.

Lemma 4.11. *Let $\mathbb{P}$ be a diamond system which satisfies C1. For every $x, y \in P$, if $x \parallel y$ and there is some $z \in P$ such that $z \leqslant x, y$, then there is $m \leqslant x, y$ which is an immediate predecessor both of x and of y.*

Proof. We will proceed by appealing to Zorn's lemma. Consider the subset of $\mathbb{P}$ with universe $X = \{z \in P \mid z \leqslant x, y\}$ and the partial order on it induced by $\leqslant$. By assumption X is nonempty. Let $C \subseteq X$ a chain of X. By assumption, there exists $\sup C \in P$. Observe that by definition of sup, we deduce $\sup C \leqslant x, y$. Therefore, $\sup(C) \in X$. Hence, we can use Zorn's lemma to deduce that there is some $m \leqslant x, y$ maximal of X.

We shall prove that m is the desired immediate predecessor of x and y. First, observe that since $m \leqslant x, y$ and $x \parallel y$, we have $m < x, y$. Then, suppose

$m < x' \leqslant x$. We claim that $x' \parallel y$. For, since $x' \leqslant x \parallel y$, we deduce $y \nleqslant x'$, otherwise we would have $y \leqslant x$. Moreover, $x' \leqslant y$ would imply $x' \leqslant x, y$. Since $m < x'$, this contradicts the maximality of m in X. This establishes the claim.

Hence, the elements x', x and y are all different, and such that $x' \parallel y$ and $x' \leqslant x$. If $x' < x$, i.e. $x' \neq x$, the three point rule would imply $x \leqslant y$, a contradiction. Thus, we can conclude that $x' = x$ and, therefore, that m is an immediate predecessor of x.

A similar argument shows that m is also an immediate predecessor of y. □

Definition 4.12. Let $\mathbb{P}$ be a diamond system. An element $x \in P$ is called a *predecessor* if it is maximal or it has an immediate successor. We denote by $\mathcal{P}$ the set of predecessors of $\mathbb{P}$.

Lemma 4.13. *Let $\mathbb{P}$ be a diamond system which satisfies C1 and C2. If $x \in P$ is a non maximal predecessor, then one of the following conditions hold:*

1. $\uparrow x = \{x\} \cup \uparrow y$ *for some y immediate successor of x;*
2. $\uparrow x = \{x\} \cup \uparrow\{y, z\}$ *for $y \parallel z$ immediate successors of x.*

Proof. Let x be as in the hypothesis. Since x is a non maximal predecessor of $\mathbb{P}$, there is an immediate successor y of x. Then, assume that the first condition does not hold, i.e., there is some $z \in P$ such that $x < z$ and $y \nleqslant z$. Observe that $y \parallel z$. For, it cannot be $z < y$ because $x \neq z \neq y$ and since y is an immediate predecessor of x it holds $[x, y] = \{x, y\}$. We have two cases: either z is an immediate successor of x or not. We want to show that the latter is impossible. If z is not an immediate successor of x, there is some w such that $x < w < z$. We claim that $w \parallel y$. For, if $y \leqslant w$ then $y \leqslant w < z$, a contradiction. Moreover, it cannot be $w < y$ because, in this case, $x < w < y$, against the assumption that y is an immediate successor of x. This establishes the claim.

Therefore, we have $x < w, y$ and $y \parallel w$. Because $\uparrow x$ satisfies the three point rule and $w < z$ we deduce $y < z$, which is a contradiction. Therefore, z is an immediate successor of x. The fact that $\uparrow x$ has width at most two yields $\uparrow x = \{x\} \cup \uparrow\{y, z\}$, thus meaning that the second condition holds. □

Lemma 4.14. *Let $\mathbb{P}$ be a connected diamond systems which satisfies C1. $\mathbb{P}$ has a maximum.*

Proof. Let $\max(\mathbb{P})$ be the set of maximal elements of $\mathbb{P}$. We claim that it is nonempty. Let C be a nonempty subchain of $\mathbb{P}$. Since $\mathbb{P}$ is satisfies C1 C has a supremum in $\mathbb{P}$. Therefore, Zorn's lemma implies that the set of maximals of $\mathbb{P}$ is nonempty.

Then, let m_1 and m_2 be two maximals of $\mathbb{P}$. We have two cases: either m_1 and m_2 have a common predecessor or not.

In the former case, let it be x. Then, $\uparrow x$ is upward directed, and thus there is a point m_3 such that $m_1, m_2 \leqslant m_3$. But m_1 and m_2 belong to $\max(\mathbb{P})$, hence $m_1 = m_2 = m_3$.

Since $\mathbb{P}$ is connected, it is easy to see by induction on the (finite) path connecting m_1 and m_2 that the latter case can be reduced to the former. □

We are now ready to start with the proof of Theorem 4.9. Fix a diamond system $\mathbb{P}$. We already know that satisfying C1 and C2 is a necessary condition for $\mathbb{P}$ to be Esakia representable. Let us show that it is sufficient as well. It is easy to see that every diamond system which satisfies C1 and C2 is a disjoint union of connected diamond systems which satisfy C1 and C2. Thus, in view of Proposition 3.14, it suffices to prove that every connected component of a diamond system is Esakia representable. Accordingly, we can suppose $\mathbb{P}$ to be nonempty and connected. Moreover, we assume $\mathbb{P}$ to satisfy C1 and C2. We will define an Esakia topology on it. Denote by $\bot_{\{y,z\}}$ an immediate predecessor (guaranteed to exist by the Lemma 4.11) of any two $y, z \in P$ such that $y \parallel z$ and $\downarrow y \cap \downarrow z \neq \emptyset$. Then, consider the topology τ generated by the union of the two following sets:

$$\{\downarrow x \mid x \in \mathcal{P} \text{ and } \forall y, z \in P (x \neq \bot_{\{y,z\}})\}$$

$$\{(\downarrow x)^c \mid x \in \mathcal{P} \text{ and } \forall y, z \in P (x \neq \bot_{\{y,z\}})\}$$

We will prove that τ is a compact topology which satisfies the Priestley separation axiom as well as the Esakia condition.

Proposition 4.15. *The topology τ on $\mathbb{P}$ is compact.*

Proof. For the sake of contradiction, let us assume that $(P, \leqslant, \tau)$ is not a compact topological space. That is, there exists a covering $\mathcal{C}$ with no finite subcovers. In view of Alexander's subbase theorem (Theorem 2.33), we might assume the opens of $\mathcal{C}$ to belong to the subbase of τ. That is, they are of the form $\downarrow x$ or $(\downarrow x)^c$ for some $x \in \mathcal{P}$ and $x \neq \bot_{\{x_1,x_2\}}$ for any $x_1, x_2 \in P$.

We claim that we can construct a sequence $(x_\alpha)_{\alpha \in Ord}$ which satisfies the two following conditions:

1. For every $\alpha \in \beta$ it holds $x_\beta < x_\alpha$;

2. For every α it holds $(\downarrow x_\alpha)^c \in \mathcal{C}$.

This would be a contradiction, because it would imply that P does not inject in any ordinal number. Let us construct this sequence recursively.

In view of Lemma 4.14, $\mathbb{P}$ must contain a greatest element, say x_0. Such x_0 must belong to an open set of the covering. If the open set is of the form $\downarrow x_1$, then $x_1 = x_0$ (because x_0 is maximal). Hence, $\downarrow x_0 = P$ in contradiction with the assumption that $\mathcal{C}$ has no finite subcovers. Whence, the open set containing

x_0 it is of the form $(\downarrow x_1)^c$. Observe that $x_1 < x_0$, because x_0 is the maximum of $\mathbb{P}$ and $x_1 \neq x_0$. Without loss of generality we may assume $(\downarrow x_0)^c = \emptyset \in \mathcal{C}$, and thus we have found the first two elements of the sequence.

Suppose to have found $\alpha + 1$ elements of the sequence. That is, we have $(x_\beta)_{\beta<\alpha+1}$ which satisfy conditions 1 and 2. Consider the element x_α. As in the previous case, there must be an open set from the covering which contains x_α. If this open is of the form $\downarrow y$ then $x_\alpha \leqslant y$ and hence $(\downarrow x_\alpha)^c \cup \downarrow y = P$, a contradiction. Thus, the open set containing x_α is of the form $(\downarrow y)^c$ for some $y \in P$. Consider $\downarrow x_\alpha \cap \downarrow y$. If it is empty, then $P = (\downarrow x_\alpha \cap \downarrow y)^c = (\downarrow x_\alpha)^c \cup (\downarrow y)^c$ and we reach, once again, a contradiction. Otherwise, there exists some $z \leqslant x_\alpha, y$. We have two subcases: either $x_\alpha \parallel y$ or not.

- In the former case, without loss of generality, we may assume z to be the immediate predecessor of both x_α and y, thanks to Lemma 4.11. Moreover, we can assume it to be of the form $\perp_{\{x_\alpha,y\}}$. Then, z belongs to an open set from $\mathcal{C}$ as well.

  If this open set is equal to $\downarrow w$ for some $w \in P$ then, because $z = \perp_{\{x_\alpha,y\}}$ we deduce $z \neq w$.

  Observe that either $x_\alpha \leqslant w$ or $y \leqslant w$. In order to see this, remember that (a) $z \leqslant w$, (b) z is an immediate predecessor of x_α and y, (c) $x_\alpha \parallel y$ and (d) $\mathbb{P}$ has width at most two.

  Then, in either case, $\downarrow x_\alpha \cap \downarrow y \subseteq \downarrow z$, and thus $(\downarrow x_\alpha)^c \cup (\downarrow y)^c \cup \downarrow w = P$, a contradiction.

  If, on the other hand, the open set containing z is of the form $(\downarrow w)^c$ for some w, we have two subcases: $z \parallel w$ or not. We shall prove that in both cases either we reach a contradiction, or we can assume that $w < x_\alpha, y$. If the latter case holds, we set $x_{\alpha+1} := w$.

  ▷ If $z \parallel w$, we want to prove that $w < x_\alpha, y$. Consider $\downarrow x_\alpha \cap \downarrow y \cap \downarrow w$. If this intersection is empty, we deduce that the following holds:

  $$(\downarrow x_\alpha)^c \cup (\downarrow y)^c \cup (\downarrow w)^c = P.$$

  Whence, we have a contradiction. Otherwise, let a be below x_α, y and w. Because $\mathbb{P}$ has width at most two, we have $w \nparallel x_\alpha$ or $w \nparallel x_\alpha$. Suppose, without loss of generality, $w \nparallel x_\alpha$. This implies $x_\alpha > w$, because $w \parallel z$.

  Then, look at y and w. Since $w \parallel z$ we have $y \nleqslant w$. If $w \nleqslant y$, by the three point rule we obtain $x_\alpha \nparallel y$, a contradiction. Therefore, $w \leqslant y$.

  In other words, we are left to find a finite subcover of $\downarrow w$, since we have just proved that $w < x_\alpha, y$ and thus $\downarrow x_\alpha \cap \downarrow y \subseteq \downarrow w$. Thus, we can set $x_{\alpha+1} := w$.

▷ If $z \not\parallel w$, it follows from $z \nleqslant w$ that $w < z \leqslant x_\alpha, y$. Accordingly, we can set $x_{\alpha+1} := w$.

- If $x_\alpha \not\parallel y$, it follows from $x_\alpha \notin \downarrow y$ that $y < x_\alpha$ and therefore we can set $x_{\alpha+1} := y$.

At limit steps, suppose we have built a sequence of the form $(x_\alpha)_{\alpha<\gamma}$ for some limit ordinal γ. By assumption, such sequence satisfies conditions 1 and 2.

Because $\mathbb{P}$ satisfies C1, it exists $x := \inf\{x_\alpha \mid \alpha < \gamma\}$. As in the previous cases, there must be an open set of $\mathcal{C}$ that contains x.

- If this open set is of the form $\downarrow y$ it must be $x < y$, since x is not a predecessor (it is a limit of an infinite sequence). In this case, we claim that there is some $\alpha < \gamma$ such that $x_\alpha \leqslant y$. In order to see this, suppose that $x_\alpha \nleqslant y$ for every $\alpha < \gamma$. It cannot be $y \leqslant x_\alpha$ for every $\alpha < \gamma$, because otherwise we would deduce $x_\alpha \leqslant x = \inf\{x_\alpha \mid \alpha < \gamma\}$, in contradiction with $x < y$. Whence, there is some x_α such that $y \nleqslant x_\alpha$. Without loss of generality, we may assume $y \nleqslant x_\beta$ for every $\alpha \leqslant \beta < \gamma$. Since we have assumed also $x_\alpha \nleqslant y$ we obtain $x_\alpha \parallel y$.

  Then, we claim $y \parallel x_\beta$ for every $x_\beta \in \{x_\beta \mid \alpha \leqslant \beta < \gamma\}$. We already know that it is not the case that $x_\beta \leqslant y$, for any $\alpha \leqslant \beta < \gamma$. For the other direction, if $y \leqslant x_\beta$ for some x_β. This yields $y \leqslant x_\alpha$, a contradiction.

  Finally, let $x_\beta \in \{x_\beta \mid \alpha \leqslant \beta < \gamma\}$. Then, we have (a) $x \leqslant y, x_\alpha$, (b) $y \parallel x_\beta$ and (c) $x_\beta \leqslant x_\alpha$. Thus, the three point rule implies $y \leqslant x_\alpha$, a contradiction. This establishes the claim.

  Whence, there is some $\alpha < \gamma$ such that $x_\alpha \leqslant y$. Therefore, we have $\downarrow(x_\alpha)^c \cup \downarrow y = P$, against the fact that (P, τ) was supposed not to be compact.

- If, on the other hand, the open set containing x is of the form $(\downarrow y)^c$, we have two cases: either $y < x$ or $y \parallel x$.

  ▷ If $y < x$ we have $y < x_\alpha$ for every $\alpha < \gamma$. For, if $m(n) = n$ then $x_{\omega+1} < x_\omega < x'_n$. Therefore, we can set $x_\gamma := y$.

  ▷ If $y \parallel x$ then we can reason as in the successor case. That is, consider $\downarrow y \cap \downarrow x$. It cannot be empty, because otherwise we would have $P = (\downarrow x)^c \cup (\downarrow y)^c$, in contradiction with the fact that (P, τ) is not compact. Thus, we can consider a common predecessor of x and y of the form $\bot_{\{x,y\}}$. This predecessor must belong to an open from the covering. Since the reasoning is exactly analogous to the one that we have considered in the successor case, we shall not repeat it. Just remember that in any case we will en up finding some z such that $z < x, y$ and $\downarrow z \in \mathcal{C}$. Therefore, we can set $x_\gamma := z$.

Therefore, the limit case is tackled as well. In conclusion, we have proved that P contains an infinite decreasing chain indexed by the class of all ordinals, which is impossible. This proves that the topological space $(P, \leqslant, \tau)$ is compact. □

Before proving that $(P, \leqslant, \tau)$ satisfies the Priestley separation axiom, we need two preliminary lemmas.

Lemma 4.16. *Let $\mathbb{P}$ be a connected diamond system which satisfies C1. If $x = \bot_{\{x_1,x_2\}}$ for some x_1 and x_2, then there are no x_3 and x_4 such that $x_i = \bot_{\{x_3,x_4\}}$ for $i \in \{1, 2\}$.*

Proof. Assume, by contradiction, that $x_1 = \bot_{\{x_3,x_4\}}$ for some x_3 and x_4 (the case for x_2 is analogous). First, it follows from Lemma 4.14 that $\mathbb{P}$ has a maximum x_0, since $\mathbb{P}$ is connected and it satisfies C1.

Since $x \leqslant x_2, x_3, x_4$, the width of $\mathbb{P}$ is at most two and $x_3 \parallel x_4$, we deduce $x_2 \not\parallel x_3$ and $x_2 \not\parallel x_4$. However, it cannot be $x_3 \leqslant x_2$ or $x_4 \leqslant x_2$, because otherwise we would deduce $x_1 \leqslant x_2$, in contradiction with $x_1 \parallel x_2$. Therefore, it holds $x_2 < x_3, x_4$. Now, recall that $x \leqslant x_1, x_2$, and thus we have $x \leqslant x_1, x_2 \leqslant x_3, x_4 \leqslant x_0$ (where x_0 is the maximum of $\mathbb{P}$). Hence, it follows from the definition of diamond system that there is some z such that $x_1, x_2 < z < x_3, x_4$, in contradiction with the fact that x_1 is an immediate predecessor of x_3 and x_4.

Since we can use a similar argument for x_2, this establishes the claim. □

Lemma 4.17. *Let $\mathbb{P}$ be a connected diamond system which satisfies C1. If $x \in P$ is part of a gap (x, y) then there is no chain $C \subseteq P$ such that $x = \inf(C)$ and $x \notin C$.*

Proof. Recall from Lemma 4.14 that $\mathbb{P}$ has a maximum x_0, since $\mathbb{P}$ is connected and it satisfies C1.

By contradiction, assume $x = \inf(C) \notin C$ for some $x \in P$ and some infinite descending chain $C \subseteq P$. Moreover, assume that x is part of a gap (x, y). Observe that $c \nleqslant y$ for any $c \in C$. Otherwise, since C is infinite descending, the pair (x, y) would not be a gap. It cannot be $y \leqslant c$ for all C either. Otherwise we would have $y \leqslant \sup(C) = x$, which is a contradiction. Whence, there is some $\bar{c} \in C$ such that $y \parallel \bar{c}$. Then, we claim $y \parallel c$ for every $c \in C \cap {\downarrow}\bar{c}$. We already know that it cannot be $c \leqslant y$. For the other direction, if $y \leqslant c$ we would deduce $y \leqslant \bar{c}$, a contradiction.

Finally, let $c \in C \cap {\downarrow}\bar{c}$ be such that $c < \bar{c}$. Such c exists because C is infinite descending. Then, we have (a) $x \leqslant y, \bar{c}$, (b) $y \parallel c$ and (c) $c \leqslant \bar{c}$. Thus, the three point rule implies $y \leqslant \bar{c}$, a contradiction. This concludes the proof. □

Proposition 4.18. *The topological ordered space $(P, \leqslant, \tau)$ satisfies the Priestley separation axiom.*

Proof. Assume $x \nleqslant y$ for some x and y in P. If $y \in \mathcal{P}$ and $y \neq \bot_{\{z_1,z_2\}}$ for any $z_1, z_2 \in P$, then $({\downarrow}y)^c$ is a clopen upset containing x but not y by construction. Otherwise, we have two cases: either $y \notin \mathcal{P}$ or $y = \bot_{\{z_1,z_2\}}$ for some $z_1, z_2 \in P$.

- In the former case, y must be the limit of a descending sequence $(y_\alpha)_\alpha$. Then, there is some y_α such that $x \nleqslant y_\alpha$. In fact, y cannot be below every y_α, because in this case y would be a lower bound of the y_α's and thus $x \leqslant y$, since of $y = \inf_\alpha y_\alpha$.

  We have two subcases: either $y_\alpha \leqslant x$ or $y_\alpha \parallel x$.

  ▷ Assume $y_\alpha \leqslant x$. Without loss of generality we may assume $y_\alpha < x$, otherwise just consider $y_{\alpha+1}$ and x. Then, because $\mathbb{P}$ satisfies C2, there is a gap (g_1, g_2) between y_α and x. In view of Lemma 4.17, g_1 is not the limit of an infinite descending sequence, since it is part of the gap (g_1, g_2). That is, $g_1 \in \mathcal{P}$.

  If $g_1 \neq \bot_{\{z_1,z_2\}}$ for any z_1 and z_2, then $\downarrow g_1$ is a clopen downset containing y but not x.

  Assume, on the other hand, there are some z_1 and z_2 such that $\bot_{\{z_1,z_2\}} = g_1$. It follows from Lemma 4.16 that both z_1 and z_2 are not of the form $\bot_{\{z_3,z_4\}}$, for any z_3 and z_4. Moreover, at least one of z_1 and z_2 is not equal to x, because $z_1 \parallel z_2$ and hence $z_1 \neq z_2$.

  Without loss of generality, $z_1 \neq x$. Clearly, it cannot hold $x \leqslant z_1$, because $g_1 \leqslant z_1$ and $g_1 \leqslant x$. Similarly to the proof of Lemma 4.17, we can show, using the three point rule, that z_1 cannot be the limit of an infinite descending chain, since we have $z_1 \parallel z_2$, $g_1 \leqslant z_1, z_2$.

  This proves, together with the fact that $z_1 \neq \bot_{\{z_3,z_4\}}$, that $z_1 \in \mathcal{P}$. Therefore, $\downarrow z_1$ is a clopen downset containing y but not x.

  ▷ Assume $y_\alpha \parallel x$. If there is some $\beta \geqslant \alpha$ such that $y_\beta < \alpha$ we can proceed as in the previous case. Otherwise, since x cannot be below every y_β, without loss of generality we may assume y_α be such that for every $y_\beta \leqslant y_\alpha$ it holds $y_\beta \parallel x$.

  Therefore, consider $y_{\alpha+1}$ and a gap (g_1, g_2) between $y_{\alpha+1}$ and y_α. Because $(y_\alpha)_\alpha$ is infinite descending, without loss of generality we may assume g_1 and g_2 to be part of the chain, otherwise just extend it. In view of our previous observation, it holds $g_1, g_2 \parallel x$.

  If $g_1 \neq \bot_{\{z_1,z_2\}}$ for any z_1 and z_2, it holds $g_1 \in \mathcal{P}$, since g_1 is part of a gap and thus we can use Lemma 4.17. Hence, $\downarrow g_1$ is a clopen downset containing y but not x.

  On the other hand, assume $g_1 = \bot_{\{z_1,z_2\}}$ for some z_1 and z_2. Thanks to Lemma 4.16, not z_1 nor z_2 is of the form $\bot_{\{z_3,z_4\}}$ for any z_3 and z_4. Moreover, we can reason as above in order to deduce that not z_1 nor z_2 is the limit of an infinite descending chain. In other words, $z_2 \in \mathcal{P}$.

  Notice that it must be $g_2 \leqslant z_1$ or $g_2 \leqslant z_2$, because the width of $\mathbb{P}$ is at most two, and g_1 is an immediate predecessor of g_2. Without loss of generality assume $g_2 \leqslant z_2$.

We claim $g_2 = z_2$. For the sake of contradiction, assume $g_2 < z_2$.

Consider g_2 and z_1. It cannot be $z_1 \leqslant g_2$ because this would imply $z_1 \leqslant z_2$, in contradiction with $z_1 \parallel z_2$. On the other hand, it cannot be $g_2 \leqslant z_1$ because $g_1 < g_2$ and g_1 is an immediate predecessor of z_1 and z_2. In other words, $g_2 \parallel z_1$.

But then, we have $g_1 \leqslant g_2, z_1$, $z_1 \parallel z_2$ and $g_2 < z_2$. Thus, the three point rule implies $z_1 \leqslant z_2$, a contradiction.

Thus, we have proved $z_2 = g_2$.

Recall that $z_2 = g_2 \in \mathcal{P}$ and that z_2 is not the limit of an infinite descending chain. As before, since $g_2 \leqslant y_\alpha$, we may assume g_2 to be part of the chain $(y_\alpha)_\alpha$. Thus, in particular, $x \parallel g_2$. Whence, ${\downarrow}g_2$ is a clopen downset containing y but not x.

- In the latter case, assume $y = \bot_{\{z_1,z_2\}}$.

  It follows from Lemma 4.16 that $z_1, z_2 \neq \bot_{\{z_3,z_4\}}$ for any $z_3, z_4 \in P$. If $x \not\leqslant z_i$, for $i \in \{1, 2\}$, we have two cases: either $z_i \in \mathcal{P}$ or not. In the former case, $({\downarrow}z_i)^c$ is a clopen upset containing x but not y; in the latter case, we can proceed as in the previous case in order to separate x and y.

  If, on the other hand, $x \leqslant z_1, z_2$, we claim that there is a maximal lower bound $\bar{x}$ of z_1 and z_2 such that $x \leqslant \bar{x}$ and $\bar{x} \parallel y$. This follows from the fact that $y = \bot_{\{z_1,z_2\}}$ and $x \not\leqslant y$, together with the proof of Lemma 4.11. Notice that ${\downarrow}\bar{x} \cap {\downarrow}y$. For, if $\bot \leqslant \bar{x}, y \leqslant z_1, z_2 \leqslant x_0$, we deduce that there is some w such that $\bar{x}, y \leqslant w \leqslant z_1, z_2$, against the maximality of y.

  Therefore, consider the clopen $({\downarrow}\bar{x})^c \cap {\downarrow}y$. It contains y but not x, and we claim that it is a downset. In order to see this, assume $w \leqslant v \in ({\downarrow}\bar{x})^c \cap {\downarrow}y$. Then, $w \leqslant y$ and thus it cannot be $w \leqslant \bar{x}$. This concludes the proof.

□

Proposition 4.19. *The topological ordered space $(P, \leqslant, \tau)$ satisfies the Esakia condition.*

Proof. We want to show that ${\downarrow}U$ is open whenever U is open. In order to do this, it suffices to show that the downset of every basic open is open, because every open is union of basic opens and $\downarrow$ commutes with arbitrary unions. In turn, every basic open is a finite intersection of subbase opens. Accordingly, suppose $U = {\downarrow}x_1 \cap \cdots \cap {\downarrow}x_n \cap ({\downarrow}y_1)^c \cap \cdots \cap ({\downarrow}y_m)^c$. If $U = \emptyset$ then ${\downarrow}U = \emptyset$ which is an open, and there is nothing else to show. Therefore, we may assume $U \neq \emptyset$. Then, because and the width of $\mathbb{P}$ is at most two, it holds ${\downarrow}x_1 \cap \cdots \cap {\downarrow}x_n = {\downarrow}x_i \cap {\downarrow}x_j$ for some $i, j \in \{1, \ldots, n\}$. There are two cases: either $x_i \leqslant x_j$ or not. Consider first the former case. The latter case will be analogous. Since we have assumed $U \neq \emptyset$ it holds, in particular, ${\downarrow}x_i \cap {\downarrow}x_j \neq \emptyset$, i.e. x_i and x_j have a common lower bound and thus a maximal such of the form $\bot_{\{x_i,x_j\}}$. Moreover, observe that $(\downarrow y_1)^c \cap \cdots \cap (\downarrow y_m)^c = (\downarrow y_1 \cup \cdots \cup \downarrow y_m)^c = (\downarrow\{y_1,\ldots,y_m\})^c$. We then claim:

$$\downarrow(\downarrow x_1 \cap \downarrow x_2 \cap (\downarrow\{y_1,\ldots,y_m\})^c) = \downarrow x_1 \cap \downarrow x_2$$

($\subseteq$) Assume $z \leqslant \bar{x}$ for some $\bar{x} \in \downarrow x_1 \cap \downarrow x_2 \cap (\downarrow\{y_1,\ldots,y_m\})^c$. In particular $z \leqslant \bar{x} \leqslant x_1, x_2$, i.e. $z \in \downarrow x_1 \cap \downarrow x_2$.

($\supseteq$) If $z \leqslant x_1, x_2$, we want to show that z is below something which is in $\downarrow x_1 \cap \downarrow x_2 \cap (\downarrow\{y_1,\ldots,y_m\})^c$.

The point $\bot_{\{x_i,x_j\}}$ does the job: $\bot_{\{x_i,x_j\}} \in \downarrow x_1 \cap \downarrow x_2$ and we claim $\bot_{\{x_i,x_j\}} \notin \downarrow\{y_1,\ldots,y_m\}$.

In order to see this suppose, by contradiction, that $\bot_{\{x_i,x_j\}} \leqslant y$ for some $y \in \{y_1,\ldots,y_m\}$. Because $\mathbb{P}$ has width at most two, it must be $x_i \not\parallel y$ or $x_j \not\parallel y$. Without loss of generality we may assume the former case.

If $y \geqslant x_i$ we claim: $U = \downarrow x_1 \cap \downarrow x_2 \cap (\downarrow\{y_1,\ldots,y_n\})^c = \emptyset$. For, if $w \leqslant x_1, x_2$ then $w \leqslant y$ and thus $w \notin (\downarrow\{y_1,\ldots,y_n\})^c$.

On the other hand, suppose $y < x_i$. Then, it cannot be $y \geqslant x_j$. Hence we have two subcases:

- If $y \leqslant x_j$ we then have $y \leqslant x_i, x_j$, and since $\bot_{\{x_i,x_j\}}$ is a maximal lower bound of x_i and x_j and $\bot_{\{x_i,x_j\}} \leqslant y$ we deduce $\bot_{\{x_i,x_j\}} = y$. But this is a contradiction with the definition of the subbase of the topology.
- If $y \parallel x_j$, consider $\bot_{\{x_i,x_j\}}$. We have $\bot_{\{x_i,x_j\}} \leqslant y, x_j$, $y \parallel x_j$ and $y < x_i$. The three point rule implies $x_j \leqslant x_i$, in contradiction with $x_i \parallel x_j$.

This concludes the proof of the case $x_i \parallel x_j$, since we have shown that $\downarrow U = \downarrow x_i \cap \downarrow x_j$, that is, $\downarrow U$ can be written as a finite intersection of open sets.

If $x_i \not\parallel x_j$, without loss of generality we may assume $x_i \leqslant x_j$. Then, the open set U is of the form: $U = \downarrow x_i \cap (\downarrow\{y_1,\ldots,y_m\})^c$. We can reason as in the previous case in order to show that:

$$\downarrow(\downarrow x_i \cap (\downarrow\{y_1,\ldots,y_m\})^c) = \downarrow x_i.$$

As before, the inclusion from left to right is clear. As for the inclusion from right to left, let z be such that $z \leqslant x_i$. We claim that there is some $\bar{x} \in \downarrow x_i \cap (\downarrow\{y_1,\ldots,y_m\})^c)$ such that $z \leqslant \bar{x}$. We claim that x_i already does the job. For, $x_i \leqslant x_i$ and if $x_i \leqslant y$ for some $y \in \{y_1,\ldots,y_m\}$ then $U = \emptyset$, a contradiction.

In conclusion, we have shown that $\downarrow U$ is an open set if so is U. □

The three propositions that we have just proved together complete the proof of Theorem 4.9. In addition, we obtain the following corollaries.

Corollary 4.20. *A diamond system is Priestley representable if and only if it satisfies C1 and C2.*

Proof. Priestley representable diamond systems have nonempty complete chains and enough gaps. Moreover, we have just proved that diamond systems with complete chains and enough gaps are Esakia representable, thus Priestley representable. □

Corollary 4.21. *A root system is Esakia representable if and only if it satisfies C1 and C2.*

Proof. Esakia representable diamond systems satisfy C1 and C2. Moreover, since every root system is a diamond system, we know that satisfying C1 and C2 implies being Esakia representable. □

Corollary 4.22. *A root system is Priestley representable if and only if it satisfies C1 and C2.*

Proof. Priestley representable root systems satisfy C1 and C2. Moreover, we have just noticed that root systems which satisfy C1 and C2. are Esakia representable, thus Priestley representable. □

Corollary 4.23. *A forest is Priestley representable if and only if it satisfies C1 and C2.*

Proof. The previous corollary establishes that Priestley representable root systems coincide with root systems which it satisfy C1 and C2. Then, forests are the order duals of root systems, thus we can use Proposition 3.19 in order to conclude. □

4.2 Profiniteness

In this section we present an application of the Esakia representability of diamond systems to a problem of profinite Heyting algebras. In (Bezhanishvili & Morandi, 2009) it was left as an open question whether every profinite Heyting algebra is a profinite completion. Let us recall what a profinite Heyting algebra is.

Definition 4.24. A Heyting algebra is said to be *profinite* if it is isomorphic to the limit of an inverse system of finite Heyting algebras.

Given a Heyting algebra $\mathbb{H}$, one could look at the inverse system of its finite homomorphic images. The limit of this inverse system is denote by $\hat{\mathbb{H}}$.

Definition 4.25. A Heyting algebra is said to be a *profinite completion* if it is isomorphic to $\hat{\mathbb{H}}$, for some Heyting algebra $\mathbb{H}$.

By definition, every profinite completion of a Heyting algebra is, in particular, a profinite Heyting algebra. The converse is not true, as it has been recently established in (Bezhanishvili et al., 2021). The paper shows that there are profinite Heyting algebras that are not isomorphic to the profinite completion of any Heyting algebra. More generally, the authors proved that there is a largest variety, $\mathtt{DHA}$, whose profinite members are profinite completions. Recall that a class of similar algebras is said to be a variety if it is closed under homomorphic images, subalgebras and products or, equivalently, if it can be axiomatized by a set of equations (Birkhoff, 1935). We should now clarify what the variety $\mathtt{DHA}$ is.

Definition 4.26. A Heyting algebra $\mathbb{H}$ is said to be a *diamond Heyting algebra* if its dual poset $\mathsf{Spec}(\mathbb{H})$ is a diamond system. The class of diamond Heyting algebras forms a variety, which is denoted by $\mathtt{DHA}$.

As announced, the main result of (Bezhanishvili et al., 2021) is the following theorem.

Theorem 4.27 ((Bezhanishvili et al., 2021), Thm. 6.1)**.** *Let* $\mathtt{V}$ *be a variety of Heyting algebras. The profinite members of* $\mathtt{V}$ *are profinite completions if and only if* $\mathtt{V}$ *is a subvariety of* $\mathtt{DHA}$*.*

The characterization of Esakia representable diamond systems that we have provided in Theorem 4.9 implies a simpler proof of Theorem 4.27. The aim of this section is to show this proof. In order to do this, we first need to mention two results. Let $\mathbb{P}$ be a poset and let us denote by $\mathsf{Up}(\mathbb{P})$ the collection of upsets of $\mathbb{P}$ ordered by inclusion.

Theorem 4.28 ((Bezhanishvili & Bezhanishvili, 2008), Thm. 3.6)**.** *A Heyting algebra* $\mathbb{H}$ *is profinite if and only if it is isomorphic to* $\mathsf{Up}(\mathbb{P})$ *for some image finite poset* $\mathbb{P}$*.*

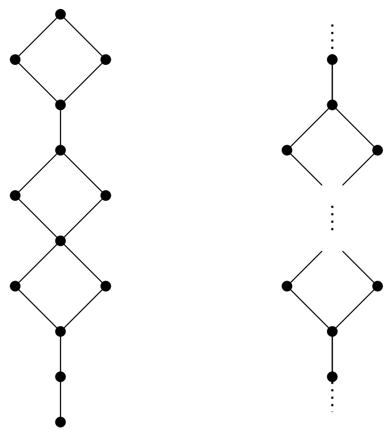
Recall Definition 2.7: a poset $\mathbb{P}$ is said to be image finite if $\uparrow x$ is finite for every $x \in P$. Then, given a poset $\mathbb{P}$, we denote by $\mathbb{P}_{\text{fin}}$ the *image finite part of* $\mathbb{P}$, that is the subposet of $\mathbb{P}$ whose universe is $P_{\text{fin}} = \{x \in P \mid \uparrow x \text{ is finite}\}$.

Theorem 4.29 ((Bezhanishvili & Bezhanishvili, 2008), Thm. 4.7)**.** *A Heyting algebra* $\mathbb{H}$ *is isomorphic to a profinite completion* $\hat{\mathbb{G}}$ *of some Heyting algebra* $\mathbb{G}$ *if and only if it is isomorphic to* $\mathsf{Up}(\mathbb{P}_{fin})$ *for some Esakia representable poset* $\mathbb{P}$*.*

This shows why the problem of Esakia representability is connected to the study of profiniteness. Before proceeding, let us introduce the notion of *diamond sequence*.

Definition 4.30. A diamond system is said to be a *diamond sequence* if it is downward directed.

As in the case of diamond systems, diamond sequences have been introduce in (Bezhanishvili et al., 2021). Typical examples of diamond sequences are depicted below.



Theorem 4.27 consists of two implications, both addressed in (Bezhanishvili et al., 2021). We are going to focus on the direction from right to left, i.e., we are going to show that a profinite diamond Heyting algebra is a profinite completion of some diamond Heyting algebra. Our proof relies on Theorem 4.9, and it is much simpler than the original one. This is to suggest that the study of representability is not just interesting *per se*, but it can also be instructive, in the sense that knowing the structure of Esakia topologies can be beneficial for other problems. Before proving the announced theorem, let us state the following.

Proposition 4.31 ((Bezhanishvili et al., 2021), Corollary 4.8)**.** *Let $\mathbb{P}$ be an image finite poset. If $\mathsf{Up}(\mathbb{P})$ is a diamond algebra, then $\mathbb{P}$ is a diamond system.*

We can now prove the following theorem.

Theorem 4.32. *Every profinite diamond Heyting algebra is a profinite completion.*

Proof. Let $\mathbb{H}$ be a profinite diamond Heyting algebra. In view of Theorem 4.28, we know that $\mathbb{H}$ is a profinite Heyting algebra if and only if $\mathbb{H} \cong \mathsf{Up}(\mathbb{P})$ for some image finite poset $\mathbb{P}$. In view of Theorem 4.29, it is sufficient to prove that $\mathbb{P} = \mathbb{X}_{\mathrm{fin}}$ for an Esakia representable poset $\mathbb{X}$. This is what we are going to do, and Theorem 4.9 will play a crucial role. First, observe that Proposition 4.31 implies that $\mathbb{P}$ is a image finite diamond system. Then, we will explicitly define the poset $\mathbb{X}$ as follows.

Let $\mathcal{D}_\infty$ be the set of infinite diamond sequences in $\mathbb{P}$ which are maximal with respect to inclusion. For every diamond sequence $D \in \mathcal{D}_\infty$ we consider a new point $\perp_D$. Then, we define the universe X of $\mathbb{X}$ to be the following union:

$$P \cup \{\perp_D \mid D \in \mathcal{D}_\infty\}.$$

The ordering $\leqslant_{\mathbb{X}}$ on X is defined as follows:

$$x \leqslant_{\mathbb{X}} y \iff (x = y) \text{ or } (x \leqslant_{\mathbb{P}} y) \text{ or } (y \in D \text{ for some } D \in \mathcal{D}_\infty \text{ and } x = \bot_D).$$

Observe that $\mathbb{X}$ has width at most two. We claim that $\mathbb{P} = \mathbb{X}_{\text{fin}}$ and that $\mathbb{X}$ is a diamond system which satisfies C1 and C2. This would imply by 4.29 that $\mathbb{P}$ is the image finite part of a diamond system $\mathbb{X}$ which is Esakia representable, as established by Theorem 4.9, thus allowing us to conclude.

First, we prove $\mathbb{P} = \mathbb{X}_{\text{fin}}$. The inclusion from left to right is clear: we have $P \subseteq X$ and moreover $\mathbb{P}$ is image finite. As for the other direction, assume $\uparrow x$ is image finite and $x \in X$. It cannot be $x = \bot_D$ for some $D \in \mathcal{D}_\infty$. For, if $D \in \mathcal{D}_\infty$ then D is infinite, and therefore $\uparrow \bot_D$ cannot be finite. Whence, $x \in P$.

In order to see that $\mathbb{X}$ is a diamond system, it suffices to observe that every $\bot_D$ is a minimal element of $\mathbb{X}$, since D is a maximal diamond sequence. Thus, it follows $\uparrow\bot_D = \{\bot_D\} \cup D$ for every $D \in \mathcal{D}$. Therefore, it is easy to verify that all the properties of a diamond system are satisfied by $\mathbb{X}$.

Finally, we show that $\mathbb{X}$ satisfies C1 and C2.

1. Let $C \subseteq X$ be a nonempty chain of $\mathbb{X}$. If $C \cap P = \emptyset$, it must be $C = \{\bot_D\}$ for some $D \in \mathcal{D}$. In this case we have $\sup(\{\bot_D\}) = \inf(\{\bot_D\}) = \bot_D \in X$. On the other hand, suppose we have $C \cap P \neq \emptyset$. Since the $\bot_D$'s are minimal elements of $\mathbb{X}$ (thus pairwise incomparable) we have $C \subseteq C^* \cup \{\bot_D\}$ for at most one $D \in \mathcal{D}$, and C^* a nonempty chain of P. Because $\mathbb{P}$ is image finite, we deduce that it exists $\sup(C^*)$ and that it belongs to $C^* \subseteq P \subseteq X$. Clearly, we have $\sup(C) = \sup(C^*)$. As for the infimum of C, either it exists in $\mathbb{P}$ (and thus in $\mathbb{X}$) or not. In the latter case, C is a maximal infinite chain of $\mathbb{P}$, and thus it can be extended to a maximal diamond sequence $D \in \mathcal{D}$. Then, by construction, $\bot_D$ is the infimum of C in $\mathbb{X}$.

2. The fact that $\mathbb{P}$ is image finite means that $\uparrow x$ is finite for every $x \in P$, in particular $\mathbb{P}$ has enough gaps. We claim that this implies that so it is $\mathbb{X}$. For, if $x < y$ for some $x, y \in X$, either $x, y \in P$, and thus there is a gap in between them, or $x = \bot_D < y$, but this means that x is limit of an infinite diamond sequence of $\mathbb{P}$. Therefore, there is some $x_1 \in P$ such that $\bot_D \leqslant x_1 < y$, and thus there is a gap between x_1 and y and so between x and y.

We can now appeal to Theorem 4.9 in order to deduce that $\mathbb{X}$ is Esakia representable. In conclusion, we have $\mathbb{H} \cong \mathsf{Up}(\mathbb{P}) = \mathsf{Up}(\mathbb{X}_{\text{fin}})$ where $\mathbb{X}$ is an Esakia representable poset. Therefore, in view of Theorem 4.29, $\mathbb{H}$ is isomorphic to a profinite completion of some Heyting algebra. $\square$

This concludes our discussion on diamond systems. We now move to the study of representable well-ordered forests.

4.3 Well-ordered forests

In section 4.1 we have seen that it is possible to characterize the class of root systems which are Esakia representable. This result, in view of Proposition 3.19, gives a characterization of the Priestley representable forests, i.e., disjoint union of trees. However, we know that we cannot infer that the same class is Esakia representable, as established by Example 3.20. The main issue that arises when considering an arbitrary forest is that it may contain infinite descending chains, which are problematic, as established by Theorems 3.30 and 3.32.

Accordingly, we proceed in two ways: first, we study the Esakia representability of forests which do not admit infinite descending chains; secondly we consider forests whose infinite descending chains are "well-behaved", in the sense that they do not satisfy the hypothesis of Theorems 3.30 or 3.32. The first direction is the one that we are going to explore in this section, while the latter will be discussed in section 4.4.

Recall that, given an ordinal α and a chain $(C, \leqslant)$, we say that $(C, \leqslant)$ has order type α is $(C, \leqslant)$ is order isomorphic to $(\alpha, \in)$.

Definition 4.33. A forest $\mathbb{P}$ is said to be *well-ordered* if the order type of the chain ${\downarrow}x$ is an ordinal number, for every $x \in P$. Equivalently, a forest is well-ordered if it does not have infinite descending chains.

As in the case of diamond systems, we can appeal to Proposition 3.14 in order to reduce the problem to the study of well-ordered trees, i.e., the connected components of a forest.

Definition 4.34. A forest $\mathbb{T} = (T, \leqslant)$ is said to be a *tree* if it is connected.

Let us introduce some terminology.

Definition 4.35. Let $\mathbb{T} = (T, \leqslant)$ be a well-ordered tree.

- We denote by $o(x)$ the order type of ${\downarrow}x$, for every $x \in T$;
- A subset $T_\beta \subseteq T$ is said to be the β^{th} *level of* T if $T_\beta = \{x \in T \mid o(x) = \beta\}$;
- We define the subtree $\mathbb{T}_{\leqslant\beta}$ of $\mathbb{T}$ as the subposet of $\mathbb{T}$ whose universe is $T_{\leqslant\beta} = \bigcup_{\gamma\leqslant\beta} T_\gamma$;
- We define the *height* $h(\mathbb{T})$ of $\mathbb{T}$ as $h(\mathbb{T}) = \sup\{o(x) + 1 \mid x \in T\}$.

Our goal is to characterize Esakia representable well-ordered forest. More precisely, the rest of the section is devoted to the proof of the following theorem, which is the main result of this thesis.

Theorem 4.36. *A well-ordered tree $\mathbb{T}$ is Esakia representable if and only it satisfies C1.*

Observe that in the assumptions of Theorem 4.36 does not appear the request that $\mathbb{T}$ has enough gaps. We will see in Corollary 4.39 why this is the case. Notice also that we do not need to require the the chains of $\mathbb{T}$ to have infima, because $\mathbb{T}$ is well-ordered. Thus, every nonempty chain of $\mathbb{T}$ already has an infumum in $\mathbb{T}$.

We will prove Theorem 4.36 by defining a topology τ on $\mathbb{T}$ by transifinite recursion. More specifically, for every $\mathbb{T}_{\leqslant\beta}$ we will define a topology τ_β turning $(\mathbb{T}_{\leqslant\beta}, \tau_\beta)$ into an Esakia space. In order to do this, we first need some useful lemmas on well-ordered trees as well as some definitions.

Lemma 4.37. *Every nonempty well-ordered tree $\mathbb{T}$ has a least element.*

Proof. Let $x \in T$ and consider ${\downarrow}x$. Since $\mathbb{T}$ is a tree, ${\downarrow}x$ is a chain. Moreover, because $\mathbb{T}$ is well ordered ${\downarrow}x$ has a least element, say x_0. Now, let x and y be two distinct points of T and consider ${\downarrow}x \cap {\downarrow}y$. We claim that the least element x_0 of ${\downarrow}x$ and the least element y_0 of ${\downarrow}y$ coincide. In order to see this, first observe that ${\downarrow}x \cap {\downarrow}y \neq \emptyset$. For, suppose ${\downarrow}x \cap {\downarrow}y = \emptyset$. This implies $x \parallel y$, because otherwise we would have either $x \in {\downarrow}x \cap {\downarrow}y$ or $y \in {\downarrow}x \cap {\downarrow}y$. But from the fact that $\mathbb{T}$ is a tree and $x \parallel y$ we deduce that x and y cannot have common successors, thus implying that $\mathbb{T}$ is not connected, which is a contradiction. Therefore, it holds ${\downarrow}x \cap {\downarrow}y \neq \emptyset$. In particular, ${\downarrow}x \cap {\downarrow}y$ is a chain of $\mathbb{T}$, since $\mathbb{T}$ is a tree, and it has a least element, which must coincide with x_0 and y_0, i.e. $x_0 = y_0$. That is, x_0 is the least element of every ${\downarrow}y$ for every $y \in P$, i.e. x_0 is the least element of $\mathbb{T}$. □

Lemma 4.38. *Let $\mathbb{T}$ be a well-ordered tree and $x, y \in T$. If $x < y$ then there exists an immediate successor of x.*

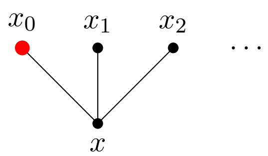
Proof. For the sake of contradiction, suppose not. That is, there is no $z \in T$ such that $[x, z] = \{x, z\}$. This means that $[x, y]$ is not well-ordered, and consequently neither is ${\downarrow}y$, a contradiction. □

Corollary 4.39. *Every well-ordered tree $\mathbb{T}$ has enough gaps.*

Proof. Suppose $x < y$ for some $x, y \in T$. Because $x < y$, the point x cannot be maximal, hence it has an immediate successor $x_1 \leqslant y$. Then, (x, x_1) is a gap between x and y. □

This corollary tells us that one of the necessary conditions for the Esakia representability is always satisfied by any well-ordered tree. Therefore, as the Theorem 4.36 states, we are going to prove that a well-ordered tree is Esakia representable if and only if its nonempty chains have suprema (we don't need to ask the infima to exist, as we are considering well-ordered trees). Before starting with the proof of Theorem 4.36, it might be worth providing an intuitive idea of the construction we are going to do.

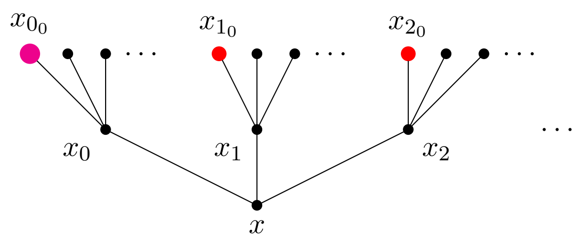
Example 4.40. Recall Proposition 3.12, which states that there are Esakia representable posets of arbitrary large width. As an example of this, we have topologized every tree of height one, by choosing an immediate successor x_0 of the root x. Then, we have considered the topology $\mathsf{FinCofin}(x_0)$ on it. If we want to topologize a tree $\mathbb{T}$ of height two, the idea is to topologize its first level as explained in the proof of Proposition 3.12, and then lift up the opens of the first level to the second level. The following depicts $\mathbb{T}_{\leqslant 1}$.



We have coloured x_0 in red in order to remember that there is an Esakia topology τ_1 on $\mathbb{T}_{\leqslant 1}$ such that x_0 is a limit point in τ_1. In particular, every open set of $\mathbb{T}_{\leqslant 1}$ such that x_0 belongs to it, contains the whole upset $\uparrow x$ but finitely many points of $\mathbb{T}_{\leqslant 1}$.

Inspired by this, one can try to mimic the definition of τ_1 to obtain a topology τ_2 on $\mathbb{T}_{\leqslant 2}$: for every x_i choose some x_{i_0} among its immediate successors. Then, say that a subset $U \subseteq \mathbb{T}_{\leqslant 2}$ is an open set if whenever some x_{i_0} belongs to it, then U contains the whole upset $\uparrow x_i$ but finitely many points in $\uparrow x_i$.

This defines a topology which, however, is not compact. This is because $\bigcup_i \uparrow x_i \cup \{x\}$ is an open covering of $\mathbb{T}_{\leqslant 2}$ with no finite subcovers. What we can do is to choose some *special* point among the x_{i_0}'s, in the same way that we have chosen a special successor of each x_i. Then, we can impose that an open containing this special point contains all but finitely many x_{i_0}'s. The most natural choice is to make x_{0_0} special. See the picture below.



Summarizing, we say that $U \in \tau_2$ if and only if the two following conditions are satisfied.

1. If $x_{i_0} \in U$ then U contains all but finitely many immediate successors of x_i;

2. If $x_{0_0} \in U$ then there are at most finitely many $i \in \mathbb{N}$ such that $\uparrow x_i \smallsetminus \{x_i\} \nsubseteq U$.

As it turns out, this construction gives rise to an Esakia topology for every tree of finite height (as the proof of Theorem 4.36 will show), but it does not tell what happens in the case of height ω or higher. This is more difficult to explain at an intuitive level, and the remaining part of this section is aimed at answering this question in a precise way.

Let us start the proof of Theorem 4.36. We shall use the following notational convention. Let $\mathbb{T}$ be a well-ordered tree such that $h(\mathbb{T}) = \alpha$. Then, for each $\beta \leqslant \alpha$ and $X \subseteq T_{\leqslant\beta}$, we will denote by $\uparrow^{\beta} X$ the set $\uparrow X \cap T_{\leqslant\beta}$, that is:

$$\uparrow^{\beta} X := \{x \in T_{\leqslant\beta} \mid \text{ there is } y \in X \text{ such that } y \leqslant x\}.$$

Let us reckon that the same notational distinction is not needed for $\downarrow X$ if $X \subseteq T_{\leqslant\beta}$. In fact, $\downarrow X = \downarrow X \cap T_{\leqslant\beta}$. Keeping this in mind, let us recursively define a topology τ_β on $\mathbb{T}_{\leqslant\beta}$ for each $\beta \leqslant \alpha$. It will turn out that each $(\mathbb{T}_{\leqslant\beta}, \tau_\beta)$ is an Esakia space, thus proving the theorem.

- $\tau_0 := \mathcal{P}(T_0)$;
- If $\beta = \gamma + 1$ and τ_γ is defined, without loss of generality we may assume $T_{\gamma+1}$ to be nonempty, otherwise we would be done already. Let us denote by P_γ the following subset of T_γ:

  $$\{x \in T_\gamma \mid \text{ there is } y \in T_\beta \text{ such that } x < y)\}.$$

  Observe that we can well-order the successors in $T_{\gamma+1}$ of each $x \in P_\gamma$; we will denote the least successor of $x \in P_\gamma$ (with respect to this well-order) by x_0. Because $T_{\leqslant\gamma+1}$ is a tree, if $x_0 = x'_0$ then $x = x'$, so the notation x_0 is unambiguous. Also, let us denote by $S_{\gamma+1}$ the set of "non-first" successors in $T_{\gamma+1}$, i.e.

  $$S_{\gamma+1} = \{y \in T_{\gamma+1} \mid \text{ for all } x \text{ it holds } y \neq x_0\}.$$

  Then, τ_β will be the topology generated by the base defined as follows: $U \subseteq T_{\leqslant\gamma+1}$ is in the base if and only if U satisfies one of the following:

  1. $U = \{y\}$ for some $y \in S_{\gamma+1}$;
  2. or $U = \downarrow x$ for some $x \in P_\gamma$;
  3. or $U = \big(V \cup \uparrow^{\beta} (V \cap T_\gamma)\big) \smallsetminus (\downarrow Z)$ for $V \in \tau_\gamma$ and $Z \subseteq P_\gamma \cup S_{\gamma+1}$ is finite.

  From now on, we will refer to opens of the first (resp. second, resp. third) kind according to this definition.

- If $\beta = \bigcup_{\gamma<\beta} \gamma$ and each τ_γ is defined, we let τ_β be the topology generated by the base defined to contain the opens of the form

$$V \cup \uparrow^{\beta} (V \cap T_\gamma),$$

for some $\gamma < \beta$ and $V \in \tau_\gamma$.

First, we should prove why our definition gives rise to bases, as opposed to subbases only. However, before answering this question, it is worth lingering one moment on a lemma. It guarantees that the open sets of the third kind remain open sets when restricted to lower levels. This lemma will play an important role when proving that we actually have defined bases for topologies. Moreover, it will be useful when proving the compactness of each τ_β.

Lemma 4.41. *For every $\gamma < \beta \leqslant \alpha$ and $\delta \in [\gamma+1, \beta]$, if $V \in \tau_\gamma$, then it holds $\big(V \cup \uparrow^{\beta} (V \cap T_\gamma)\big) \cap T_{\leqslant\delta} \in \tau_\delta$.*

Proof. In order to prove this, it suffices to notice that for every $\delta \in [\gamma+1, \beta]$ the following holds:

$$(V \cup \uparrow^{\beta}(V \cap T_\gamma)) \cap T_{\leqslant\delta} = V \cup \uparrow^{\delta}(V \cap T_\gamma))$$

and $V \cup \uparrow^{\delta}(V \cap T_\gamma))$ is an open set from the base of τ_δ for every δ.

□

We are now ready to see why the bases defined for the successor and the limit cases are closed under binary (whence finite) intersections.

In both cases the empty set is in the base: if τ_γ is defined for $\gamma < \beta$, then $\emptyset \in \tau_\gamma$ and thus $\emptyset \cup \uparrow(\emptyset \cap T_\gamma) = \emptyset \in \tau_\beta$. Then, for the successor case it is clear that intersecting an open of the first (resp. second) kind with any other open in the base will return either the empty set or an open of the first (resp. second) kind. As for the third case, observe the following:

$$\Big[\Big(V \cup \uparrow^{\beta} (V \cap T_\gamma)\Big) \smallsetminus (\downarrow Z)\Big] \cap \Big[\Big(V' \cup \uparrow^{\beta} (V' \cap T_\gamma)\Big) \smallsetminus (\downarrow Z')\Big] =$$

$$= \Big[\Big[\Big(V \cup \uparrow^{\beta} (V \cap T_\gamma)\Big)\Big] \cap \Big[\Big(V' \cup \uparrow^{\beta} (V' \cap T_\gamma)\Big)\Big]\Big] \smallsetminus (\downarrow Z \cup \downarrow Z').$$

Notice that, $\downarrow Z \cup \downarrow Z' = \downarrow (Z \cup Z')$ and moreover $Z \cup Z'$ is finite. In addition, we have:

$$\Big[\Big(V \cup \uparrow^{\beta} (V \cap T_\gamma)\Big)\Big] \cap \Big[\Big(V' \cup \uparrow^{\beta} (V' \cap T_\gamma)\Big)\Big] =$$

$$= (V \cap V') \cup (V \cap \uparrow^{\beta} (V' \cap T_\gamma)) \cup (V' \cap \uparrow^{\beta} (V \cap T_\gamma)) \cup (\uparrow^{\beta} (V \cap V' \cap T_\gamma))$$
$$= (V \cap V') \cup \uparrow^{\beta} (V \cap V' \cap T_\gamma).$$

and since $V \cap V' \in \tau_\gamma$ we are done.

For the limit case, let $V \cup \uparrow^{\beta} (V \cap T_\gamma)$ and $V' \cup \uparrow^{\beta} (V' \cap T_\delta)$ be such that $V \in \tau_\gamma$ and $V' \in \tau_\delta$. We may assume $\gamma \leqslant \delta < \beta$ and, in view of Lemma 4.41, we know that $U := (V \cup {\uparrow}^{\beta}(V \cap T_\gamma)) \cap T_{\leqslant\delta} = [V \cup {\uparrow}^{\delta}(V \cap T_\gamma)] \in \tau_\delta$ as well as $U' := [V' \cup {\uparrow}^{\delta}(V' \cap T_\delta)] \in \tau_\delta$. Because $U \cap U' \in \tau_\delta$, in order to conclude it suffices to show that following holds:

$$[V \cup {\uparrow}^{\beta}(V \cap T_\gamma)] \cap [V' \cup {\uparrow}^{\beta}(V' \cap T_\delta)] = U \cap U' \cup [{\uparrow}^{\beta}(U \cap U' \cap T_\delta)].$$

This is actually the case because $[{\uparrow}^{\beta}(U \cap U' \cap T_\delta)] = {\uparrow}^{\beta}(V \cap V' \cap T_\delta)$.

We shall now give more insights on the structure of the basic opens. In particular, we should stress that at each successor case we have chosen some "special" successor point x_0 of each $x \in P_\gamma$. Of course, whenever possible, we can proceed and choose some x_{00}. If this process continues we will obtain a chain whose supremum must exist, because we have assumed that the nonempty chains of $\mathbb{T}$ have suprema. Likewise, whenever possible, we must choose a special successor of this limit and proceed again in this fashion. This sequence of choices must stop at some point, either because we have reached a point with no successors or because we have reached the top of the tree. We could then call the last chosen node $x_{\bar{0}}$. We will now formally define this procedure, while Lemma 4.42 will show what makes $x_{\bar{0}}$ even more "special".

Take $\gamma \leqslant \beta \leqslant \alpha$ and suppose $x \in T_\gamma$. Consider the assignment $f_x : [\gamma, \beta] \to T_{\leqslant\beta}$, defined by recursion as follows:

$$\begin{aligned} f_x(\gamma) &:= x \\ f_x(\varepsilon+1) &:= \begin{cases} (f_x(\varepsilon))_0 & \text{if } f(\varepsilon) \in P_\varepsilon \\ f_x(\varepsilon) & \text{otherwise} \end{cases} \\ f_x(\textstyle\bigcup_{\varepsilon<\delta} \varepsilon) &:= \sup_{\varepsilon<\delta} f_x(\varepsilon). \end{aligned}$$

Denote by $x_{\bar{0}}$ the element $f_x(\beta)$; notice also that $f_x(\beta) \in \max T_{\leqslant\beta}$.

Lemma 4.42. *For every $\gamma \leqslant \beta \leqslant \alpha$, if $x \in T_\gamma$ and U is a basic open set of τ_β such that $x_{\bar{0}} \in U$, then there are at most finitely incomparable $y_1, \ldots, y_n$ such that$x < y_i$ for all $i \leqslant n$ and ${\uparrow}^{\beta}(y_1, \ldots, y_n) \nsubseteq U$.*

Proof. We proceed by induction on the unique δ such that $\beta = \gamma + \delta$.

If $\delta = 0$ it means that $\gamma = \beta$, i.e., $x = x_{\bar{0}} \in \max T_{\leqslant\beta}$. For, from the definition of x_0 it follows that $x_0 \leqslant x$. Thus, there are no y_i's strictly above it.

If $\delta = \varepsilon + 1$, then $\beta = \gamma + \varepsilon + 1$ is a successor ordinal as well and thus, by definition of τ_β, the open U must be of the form $\big(V \cup {\uparrow}^{\gamma+\varepsilon+1}(V \cap T_{\gamma+\varepsilon})\big) \smallsetminus {\downarrow} Z$ for some $V \in \tau_{\gamma+\varepsilon}$ and some finite $Z \subseteq P_{\gamma+\varepsilon} \cup S_{\gamma+\varepsilon+1}$. Consider $f_x(\gamma+\varepsilon) \in V$. Since $x_{\bar{0}}$ belongs to U but not to $T_{\leqslant\gamma+\varepsilon}$, the element $f_x(\gamma+\varepsilon) \in V$ must belong to $V \cap T_\gamma$. Therefore, since the ordinal difference between $\gamma+\varepsilon$ and γ is ε, i.e., strictly smaller than $\delta = \varepsilon+1$, we can use appeal to the inductive hypothesis. By inductive hypothesis there are at most finitely many incomparable $y_1, \ldots, y_n$ all strictly above x such that ${\uparrow}^{\gamma+\varepsilon}(y_1, \ldots, y_n) \nsubseteq V$. Then, as Z is finite as well, we can deduce that the same holds for $f_x(\beta) = x_{\bar{0}}$ and $(V \cup {\uparrow}^{\beta}(V \cap T_{\gamma+\varepsilon})) \smallsetminus {\downarrow} Z$.

Finally, if $\delta = \bigcup_{\varepsilon<\delta} \varepsilon$, we deduce that $\beta = \gamma + \delta = \bigcup_{\varepsilon<\delta}(\gamma + \varepsilon)$, i.e., β is a limit as well. Therefore, the open U must be of the form $V \cap {\uparrow}^{\beta}(V \cap T_\eta)$ for some $\eta < \beta$ and $V \in \tau_\eta$. If $\eta \geqslant \gamma$, by definition we have $f_x(\eta) \leqslant f_x(\beta)$ and thus it must be $f_x(\eta) \in V$. We can then use the inductive hypothesis in order to conclude as in the successor case. Otherwise, pick any $\zeta \in [\gamma + 1, \beta]$. By definition $f_x(\zeta) \leqslant f_x(\beta)$ and thus, since $\eta < \zeta$, it holds $f_x(\zeta) \in {\uparrow}^{\beta}(V \cap T_\eta)$. Now, due to Lemma 4.41, we know that $U \cap T_{\leqslant\zeta} \in \tau_\zeta$. We can then use the inductive hypothesis in order to conclude. □

The previous lemmas put into light some strong properties of the topologies we are working with; we are going to rely on them in order to finally move to the most difficult part of the proof: showing compactness of each τ_β. The cornerstone for this will be the next proposition, which tells us that if we cover the whole set of maximals of $T_{\leqslant\beta}$ with some opens, then we can cover it with just finitely many such and, moreover, these finitely many opens are already covering the whole $T_{\leqslant\beta}$ up to finitely many downsets.

Proposition 4.43. *For each $\beta \leqslant \alpha$, whenever there is a collection of opens $(U_i)_{i\in I}$ of τ_β such that $\max T_{\leqslant\beta} \subseteq \bigcup_{i\in I} U_i$, then there exists a finite $F \subseteq I$ s.t. $\max T_{\leqslant\beta} \subseteq \bigcup_{i\in F} U_i$ and for each $Z \subseteq T_{\leqslant\beta}$, if ${\downarrow} Z \not\subseteq \bigcup_{i\in F} U_i$, then Z is finite.*

Proof. By induction on β.

- If $\beta = 0$ there is nothing to do;
- If $\beta = \gamma + 1$, let $(U_i)_{i\in I}$ be a covering of $\max T_{\leqslant\beta}$ by means of open sets of τ_β. Without loss of generality, we may assume that these open sets all belong to the basis of τ_β, thus implying that they all must be open sets of the first or of the third kind. However, since, $T_\beta \neq \emptyset$, not all the open sets can be of the first kind, because we have to cover the x_0's for $x \in P_\gamma$ as well. Thus, some of the U_i's are of the form $(V \cup {\uparrow}(V \cap T_\gamma)) \smallsetminus ({\downarrow} Z)$ for $V \in \tau_\gamma$ and $Z \subseteq P_\gamma \cup S_{\gamma+1}$ at most finite. Moreover, every point in $\max T_{\leqslant\beta} \setminus T_\beta$ must belong to an open that already is in τ_γ, because of the definition of τ_β.

  In other words, for every x_0 we have an open V such that $x \in V$ and the same holds for every $y \in \max T_{\leqslant\beta} \setminus T_\beta$. That is, there is a collection $(V_j)_{j\in J}$ of open sets of τ_γ that covers the whole $\max T_{\leqslant\gamma}$ and thus, by inductive hypothesis, we may assume J to be finite and the collection $(V_j)_{j\in J}$ to already cover $T_{\leqslant\gamma} \setminus {\downarrow} Z'$, for some Z' at most finite. Now, for each U_i that comes as a superset of some $V_j \cap T_\gamma$, because of the finiteness of J we can extract a finite subset $F \subseteq I$ such that $(U_i)_{i\in F}$ covers all the $x_0 \in T_\beta$. Moreover, each such U_i contains all but finitely many successors of every $x \in T_\gamma$, meaning that we need no more than finitely many opens of the first kind. Finally, as F is finite and so is each Z, the covering $(U_i)_{i\in F}$ leaves out at most finitely many downsets, as Z' was finite as well.

- If $\beta = \bigcup_{\gamma<\beta} \gamma$, let $(U_i)_{i\in I}$ be a covering of $\max T_{\leqslant\beta}$ by means of opens τ_β. As in the successor case, we may assume the U_i to be basic opens. We claim that there is already some $\gamma < \beta$ such that $\uparrow^\beta(\max T_{\leqslant\gamma}) \subseteq \bigcup_{i\in I} U_i$. If not, for each $\gamma < \beta$ there is $x_\gamma \in \uparrow^\beta \max T_{\leqslant\gamma}$ such that for all $i \in I$ it holds $x_\gamma \notin U_i$. Consider the following two cases: either the sequence $(x_\gamma)_{\gamma<\beta}$ has a $<$-subchain of height $|\beta|$ or not. In the former case, called (a) below, for the sake of notational readability, we may assume the subchain to be the original chain itself.

  (a) If the sequence $(x_\gamma)_{\gamma<\beta}$ is a $<$-chain, i.e., $x_\delta < x_\gamma$ whenever $\delta < \gamma$, then, by assumption it has a supremum $x := \sup\{x_\gamma \mid \gamma < \beta\} \in T_\beta$. Therefore, $x \in U_i$ for some $i \in I$, and it follows from the definition of τ_β that U_i must be of the form $V \cup \uparrow(V \cap T_{\tilde{\gamma}})$ for some $\tilde{\gamma} < \beta$. But $T_{\leqslant\beta}$ is a tree, thus implying that $x_\gamma \in U_i$ for all $\gamma > \tilde{\gamma}$, contradiction.

  (b) If $(x_\gamma)_{\gamma<\beta}$ has no subchains of height $|\beta|$, then it must contain a subsequence of cardinality $|\beta|$ of incomparable elements. As before, for the sake of notational readability we may assume the sequence $(x_\gamma)_{\gamma<\beta}$ to be made of incomparable elements already. Let us consider the chain $\bigcap_{\gamma<\beta} \downarrow x_\gamma$. By assumption, there exists $\sup\left(\bigcap_{\gamma<\beta} \downarrow x_\gamma\right)$, call it x. Observe that $x < x_\gamma$ for each $\gamma < \delta$, as $x_\gamma \parallel x_{\gamma'}$ for all $\gamma, \gamma' < \beta$. Moreover, as x is the *greatest* element below all of the x_γ's, we deduce that for each $x' > x$ there exists some γ such that $x_\gamma \in \uparrow x \setminus \uparrow x'$. This allows us to reach the desired contradiction: for, consider $x_{\overline{0}} \in \max T_{\leqslant\beta}$ and some $i \in I$ s.t. $x \in U_i$. Then, because of Lemma 4.42, there are at most finitely many upsets above x not contained in U_i. But β is a limit ordinal (whence not finite), and thus there must be some $\gamma < \beta$ such that $x_\gamma \in U_i$, contradicting our hypothesis.

  We have thus proved that the there is some $\gamma < \beta$ such that $\max(\mathbb{T}_{\leqslant\gamma})$ are entirely covered by the collection $(U_i)_{i\in I}$. Moreover, because of Lemma 4.41, each U_i, when intersected with $T_{\leqslant\gamma}$, belongs to τ_γ, thus implying that we can use the inductive hypothesis on $T_{\leqslant\gamma}$ and conclude.

□

As announced, we can now prove that each τ_β is a compact topology. As a matter of fact, we are almost done: every open covering of $T_{\leqslant\beta}$ must cover $\max T_{\leqslant\beta}$ in the first place and thus, as highlighted by the previous proposition, we can extract a finite subcovering which leaves out at most finitely many principal downsets. The next lemma will show how to conclude the proof of compactness.

Lemma 4.44. *For every $\beta \leqslant \alpha$, $\downarrow x \subseteq T_{\leqslant\beta}$ and $U \in \tau_\beta$, if the order type of $\downarrow x$ is a limit ordinal and $x \in U$, then there is some $z < x$ such that $[z, x] \subseteq U$.*

Proof. By induction on β.

If $\beta = 0$ there is nothing to prove.

If $\beta = \gamma + 1$, let $x \in U$. By definition of τ_β, U is a union of some opens of the first/second/third kind and thus $x \in U$ means that x already belongs to a basic open. The order type of $\downarrow x$ is a limit ordinal, therefore $x \in T_{\leqslant \gamma}$ and hence U cannot be an open of the first kind. If $U = \downarrow z$ for some $z \in S_{\gamma+1}$ we are done, so let us consider $U = (V \cup \uparrow^\beta (V \cap T_\gamma)) \smallsetminus (\downarrow Z)$ for some $V \in \tau_\gamma$ and $Z \subseteq P_\gamma \cup S_{\gamma+1}$ at most finite. Because $x \in T_{\leqslant \gamma}$ it must be $x \in V \in \tau_\gamma$ and since the order type of $\downarrow x$ is the same in $T_{\leqslant \gamma}$ and in $T_{\leqslant \gamma+1}$ we can use the inductive hypothesis and deduce that $[z, x] \subseteq V$ for some $z < x$. Because of the presence of $\downarrow Z$ in U, it might be that $z \notin U$, but as Z is finite and x is a limit ordinal (hence $\downarrow x$ is infinite) there must be some $z' < x$ such that $[z', x] \subseteq U$.

If $\beta = \bigcup_{\gamma < \beta} \gamma$ we might assume U to be a basic open of the form $V \cup \uparrow^\beta (V \cap T_\gamma)$ for $V \in \tau_\gamma$ and $\gamma < \beta$. If $x \in V$ we can use the inductive hypothesis as in the successor case, if $x \in \uparrow^\beta (V \cap T_\gamma) \smallsetminus V$ then, because $V \subseteq T_{\leqslant \gamma}$, we deduce $h(x) > \gamma$ and since $\uparrow^\beta (V \cap T_\gamma)$ is an upset there must be some $z < x$ such that $[z, x] \subseteq U$. □

Proposition 4.45. *For every $\beta \leqslant \alpha$, the topology τ_β is compact.*

Proof. We proceed as described above: let $(U_i)_{i \in I}$ an open covering of $T_{\leqslant \beta}$, extract from it a covering of $\max T_{\leqslant \beta}$ which, thanks to Proposition 4.43, might be assumed to be finite and to leave out at most finitely many principal downsets $\downarrow z$. Each such downset can then be finitely covered by appealing to Lemma 4.44. □

This concludes the proof of the compactness of each τ_β. The two following propositions will show that these topologies are, in fact, Esakia topologies as desired. First, let us show that every τ_β satisfies the Priestley separation axiom.

Proposition 4.46. *For every $\beta \leqslant \alpha$ and $x, y \in T_{\leqslant \beta}$, if $x \nleqslant y$ then there is clopen upset $U \in \tau_\beta$ such that $x \in U$ and $y \in U^c$.*

Proof. We proceed by induction on β.

If $\beta = 0$ there is nothing to prove.

If $\beta = \gamma + 1$ then either $y \in T_{\leqslant \gamma}$ or not.

- In the former case, choose some $x' \in T_{\leqslant \gamma} \cap \downarrow x \cap (\downarrow y)^c$. Such x' exists: if $x \in T_{\leqslant \gamma}$ take $x' = x$; if $x \in T_{\gamma+1}$ then, since $h(x) > h(y)$, there is some x' in $T_{\leqslant \gamma}$ below x not below y. By the inductive hypothesis there is a clopen upset $V \in \tau_\gamma$ such that $x' \in V$ and $y \notin V$. Then, set $U := V \cup \uparrow^\beta (V \cap T_\gamma)$ and observe that U and $(T_{\leqslant \gamma} \smallsetminus V) \cup \uparrow^\beta ((T_{\leqslant \gamma} \smallsetminus V) \cap T_\gamma)$ are the complement of each other in $T_{\leqslant \gamma+1}$ -because $T_{\leqslant \gamma+1}$ is a tree. Moreover, they are open by definition and U is an upset by construction. Finally, as U is an upset containing x' and $x' \leqslant x$, U contains x as well.

- If, on the other hand, $y \in T_{\gamma+1}$, we have two more cases that need to be considered.

  If $y \in S_{\gamma+1}$ then $\downarrow y$ is an open downset non containing x. This holds because $\downarrow y = \{y\} \cup \downarrow z$, where z is the immediate predecessor of y, and moreover $y \in S_{\gamma+1}$ as well as $z \in P_\gamma$. In order to conclude it suffices to show that $\downarrow y$ is closed. In order to see this, notice that $(\downarrow y)^c = (T_{\leqslant\gamma} \cup \uparrow^\beta(T_{\leqslant\gamma} \cap T_\gamma)) \smallsetminus \downarrow y$, i.e. $(\downarrow y)^c$ is a basic open of the third kind.

  Finally, if $y = z_0$ for some $z \in P_\gamma$, either $x > z$ or not. In the former case, $\{x\}$ is a clopen upset: it is open by definition, moreover it is closed because its complement can be written as $[(T_{\leqslant\gamma} \cup (T_{\leqslant\gamma} \cap T_\gamma)) \smallsetminus (\downarrow x)] \cup \downarrow z$, i.e. union of two basic opens, whence open. Otherwise, if $x \not> z$, it must hold $x \not\leqslant z$, because $x \not\leqslant y$. Hence, we can use the same argument that we have detailed in the case $y \in T_{\leqslant\gamma}$, by choosing $x' \in T_{\leqslant\gamma} \cap \downarrow x \cap (\downarrow z)^c$.

If $\beta = \bigcup_{\gamma<\beta} \gamma$ let γ_1 and γ_2 be the unique ordinals such that $x \in T_{\gamma_1}$ and $y \in T_{\gamma_2}$. Let γ be defined as $\gamma := \max\{\gamma_1, \gamma_2\}$ and consider the tree $\mathbb{T}_{\leqslant\gamma}$. By inductive hypothesis, there is a clopen upset $V \in \tau_\gamma$ such that $x \in V$ but $y \notin V$. Then, consider $U := V \cup \uparrow^\beta(V \cap T_\gamma)$. By definition of τ_β, we have $U \in \tau_\beta$. Observe also that U is an upset. In order to conclude, we will show that U is a closed set, i.e., U^c is an open set of τ_β. In order to see that $U^c \in \tau_\beta$, observe that $U^c = (V \cup \uparrow^\beta(V \cap T_\gamma))^c = V^c \cap (\uparrow^\beta(V \cap T_\gamma))^c$. Then we claim that the following equality holds:

$$V^c \cap (\uparrow^\beta(V \cap T_\gamma))^c = (T_{\leqslant\gamma} \smallsetminus V) \cup \uparrow^\beta(T_{\leqslant\gamma} \smallsetminus V \cap T_\gamma).$$

($\subseteq$) Let $x \notin V$. Let δ be the unique ordinal such that $x \in T_\delta$. We have two cases: either $\delta \leqslant \gamma$ or $\delta > \gamma$. In the former case, we can conclude that $T_{\leqslant\gamma} \smallsetminus V$. In the latter case, there is some $y \in T_\gamma$ such that be such that $y \leqslant x$. Since $x \notin (\uparrow^\beta(V \cap T_\gamma))^c$ we deduce that either $y \notin V$ or $y \notin T_\gamma$. But we have assumed $y \in T_\gamma$, whence $y \notin V$. In other words, $y \in (T_{\leqslant\gamma} \smallsetminus V) \cap T_\gamma$, whence $x \in \uparrow^\beta((T_{\leqslant\gamma} \smallsetminus V) \cap T_\gamma)$.

($\supseteq$) First, assume $x \in T_{\leqslant\gamma} \smallsetminus V$. Because $V \subseteq T_{\leqslant\gamma}$ we deduce $x \notin V$. Moreover, let y be such that $y \leqslant x$. We want to show that either $y \notin V$ or $y \notin T_\gamma$. In order to see this, assume $y \in T_\gamma \cap V$. But V is an upset and $y \leqslant x$. This implies $x \in V$, in contradiction with $x \in T_{\leqslant\gamma} \smallsetminus V$.

On the other hand, suppose $x \in \uparrow^\beta((T_{\leqslant\gamma} \smallsetminus V) \cap T_\gamma)$. That is, there is some $y \leqslant x$ such that $y \in (T_{\leqslant\gamma} \smallsetminus V) \cap T_\gamma$. We have two cases: either $x = y$ or $y < x$. In the former case, we deduce $x \notin V$ and $x \notin \uparrow^\beta(V \cap T_\gamma)$, since $V \subseteq T_{\leqslant\gamma}$. In the latter, we have $x \notin V$ becase $V \subseteq T_{\leqslant\gamma}$ and $x \in T_\delta$ for some $\delta > \gamma$. In order to see that $x \notin \uparrow^\beta(V \cap T_\gamma)$ assume there is some z be such that $z \leqslant x$ and $z \in V \cap T_\gamma$. Because $\mathbb{T}$ is a tree, y is the unique element below y which belongs to T_γ. But, by assumption, we have $y \notin V$, a contradiction. This establishes the claim.

□

It is left to prove that every τ_β satisfies the Esakia condition.

Proposition 4.47. *For every $\beta \leqslant \alpha$, if $U \in \tau_\beta$ then ${\downarrow}U \in \tau_\beta$.*

Proof. By induction on β.

If $\beta = 0$ there is nothing to do.

Assume $\beta = \gamma + 1$ and $U \in \tau_\beta$. Recall that U can be written as union of opens from the basis and that $\downarrow$ commutes with arbitrary unions. Thus, it suffices to show that the downset of every basic open is an open as well.

The downset of $\{y\}$ for any $y \in S_{\gamma+1}$ is equal to $\{y\} \cup {\downarrow}x$, where $x \in P_\gamma$ is the immediate predecessor of y. This is an union of two basic opens, whence open. Then, the downset of an open of the second kind already is open by definition. As for the downset of a basic open of the form $(V \cup {\uparrow}^\beta(V \cap T_\gamma)) \smallsetminus ({\downarrow}Z)$, notice that:

$${\downarrow}((V \cup {\uparrow}^\beta(V \cap T_\gamma))) = {\downarrow}V \cup {\downarrow}({\uparrow}^\beta(V \cap T_\gamma)) = {\downarrow}V \cup {\uparrow}^\beta({\downarrow}V \cap T_\gamma).$$

where the first equality follows from the fact that $\downarrow$ commutes with unions. As for the second equality, the inclusion from right to left is clear. As for the other, assume $x \leqslant y$ for some $y \in T_{\leqslant\beta}$ such that $y \geqslant z$ for some $z \in V \cap T_\gamma$. If $x = y$ then $x \in {\uparrow}^\beta({\downarrow}V \cap T_\gamma)$ and we are done. If $x < y$ then, as $T_{\leqslant\beta}$ is a tree, we either have $x \leqslant z$ or $z \leqslant x$. In the former case, $x \in {\uparrow}^\beta(V \cap T_\gamma) = {\uparrow}^\beta({\downarrow}V \cap T_\gamma)$. In the latter case, $x \in {\downarrow}V$, and hence we are done as well.

This allows us to conclude because $\downarrow V \in \tau_\beta$ by inductive hypothesis, and thus we have proved that the downset of a basic open of the third kind is an open of the third kind itself (taking the downset of $(V \cup {\uparrow}^\beta(V \cap T_\gamma)) \smallsetminus ({\downarrow}Z)$ will leave out at most finitely many downsets from ${\downarrow}Z$).

In the case where β is a limit ordinal, we can repeat the observations that we have done in the case of open sets of the third kind, since the basic open sets of τ_β for β ordinal are of the same form of the open sets of the third kind. This concludes the proof. □

This concludes our discussion on the proof of Theorem 4.36. In fact, we have proved that for every well-ordered tree $(T, \leqslant)$ whose nonempty chains have suprema, we can define a topology τ making $(T, \leqslant, \tau)$ a compact space, which satisfies the Priestley separation axiom and such that τ is closed under generated downsets. In view of Proposition 3.14, this implies that a well-ordered forest is Esakia representable if and only if its nonempty chains have suprema.

4.4 Countable forests

In the previous sections we have characterized the class of Priestley representable forests and we have acknowledged that, in view of Proposition 3.30, the Esakia representability of the same class is much more difficult to tackle. Accordingly, we have studied the Esakia representability of the class of well-ordered forests and we have discovered that it coincides with the class of well-ordered forests whose nonempty chains have suprema. The Esakia representability of arbitrary forests is still an open problem. In this section we suggest how to proceed in the case of countable forests. For, in the countable case we can make use of Theorem 2.35 in order to find some more forbidden configurations. Moreover, restricting our attention to countable forests might be convenient because they cannot ramify "too much", as we shall explain soon. As usual, let us appeal to Proposition 3.14 in order to restrict our attention to trees.

Definition 4.48. The *infinite binary tree* is the poset $\mathbb{T}_{\mathbf{2}}$ whose universe consists of finite sequences of 0's and 1's and it is ordered by end-extension.

Observation 4.49. Let $\mathbb{T}$ be a countable Esakia representable tree. Then, there is no order embedding of $\mathbb{T}_{\mathbf{2}}$ into $\mathbb{T}$.

Proof. Suppose there is such an embedding. Then, there are $2^{\aleph_0}$ infinite distinct chains in $\mathbb{T}$, whose suprema must exist, because $\mathbb{T}$ is Esakia representable. Hence, there are at least $2^{\aleph_0}$ distinct points in $\mathbb{T}$, a contradiction. □

This observation tells us that an Esakia representable tree $\mathbb{T}$ which is countable does not ramifies too much, in the sense that there is no point $x \in T$ such that every successor of x has a two incomparable successors. In other words, we have a condition on the principal upsets of $\mathbb{T}$. Likewise, let us consider the principal downsets of $\mathbb{T}$, i.e., the chains of $\mathbb{T}$.

Observation 4.50. Let $\mathbb{T}$ be an Esakia representable countable tree. Then, there is no order embedding from the poset $\mathbb{Q}$ of rational numbers into $\mathbb{T}$.

Proof. Suppose there is such an embedding $f : \mathbb{Q} \to \mathbb{T}$. Because $\mathbb{T}$ is Esakia representable, it must satisfy C1. In particular, for every subset $X \subseteq \mathbb{Q}$ both $\sup(f[X])$ and $\inf(f[X])$ must exist in $\mathbb{T}$. Whence, there are at least $2^{\aleph_0}$ distinct points in $\mathbb{T}$, a contradiction. □

The previous observation tells us that there is no order embedding from $\mathbb{Q}$ into the chains of $\mathbb{T}$. The linear orders in which $\mathbb{Q}$ does not embed are called *scattered*. There following theorem, due to Hausdorff, characterizes the class of scattered orders of cardinality less than $\aleph_\alpha$, for a regular cardinal $\aleph_\alpha$.

Theorem 4.51 ((Hausdorff, 1908), Thm. 12)**.** *Let $\aleph_\alpha$ be a regular cardinal. The scattered linear orders of cardinality strictly less than $\aleph_\alpha$ form a ring whose basis consists of all ordinals strictly smaller than ω_α and their order duals.*

This theorems tells us that the chains of a countable Esakia representable poset are obtained as linear combinations of ordinals of cardinality $\aleph_0$ and of their duals. This is useful for our purposes because, given a countable tree $\mathbb{T}$, a first test to see whether it is *not* Esakia representable is too look at its chains. Moreover, since we still do not know which countable trees are Esakia representable, knowing how their chains should look like might help us define a topology on them.

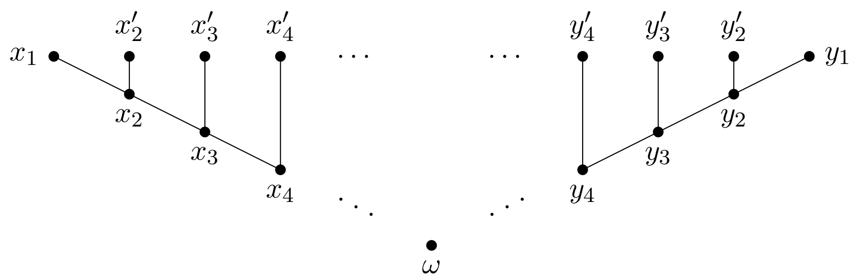
Let us proceed in the study of Esakia representable countable trees. Theorem 3.30 states that a forest with an infinite descending non-ramifying chain whose infimum ramifies is not Esakia representable, albeit being Priestley representable, in view of Theorem 4.9. Consequently, a non-well-ordered tree $\mathbb{T}$ might be Esakia representable in the two following scenarios: either the infimum of every infinite descending chain does not ramify, or every infinite descending chain ramifies "very often", whenever its infimum ramifies. By "very often" we mean that, if the infimum of an infinite descending chain C of $\mathbb{T}$ ramifies, then the following must hold:

$$\inf(C) = \inf\{x \in C \mid x \text{ ramifies }\}.$$

For, if $\inf(C) < \inf\{x \in C \mid x \text{ ramifies }\}$ then, the chain $[\inf(C), \inf\{x \in C \mid x \text{ ramifies }\}]$ is (a) infinite descending (because so it is C), (b) it does not have ramifications by assumption, and (c) its infimum ramifies, again by assumption. Therefore, $\mathbb{T}$ is not Esakia representable.

In other words, the problem of Esakia representable forests reduces to the study of trees with infinite descending chains whose infimum does not ramifies or, if it does, the set of ramifying points of the chain converges to it. The latter case already is non trivial, as the following example shows.

Example 4.52. Consider the tree-like poset $\mathbb{T}$ depicted below. It does have two infinite descending chains whose infima ramify, but we are not under the hypothesis of Proposition 3.30 because the infimum of the set of ramifying points of both chains coincides with the infimum of the chain itself. Nonetheless, we claim that such poset is not Esakia representable. In order to show this, we will make use of Theorem 2.35, which says that every compact, T_2 and countable topological space has an isolated point.



By construction, the universe of $\mathbb{T}$ is defined as follows:

$$T = \underbrace{\{x_n \mid n \in \mathbb{N}\} \cup \{x'_n \mid n \in \mathbb{N}\}}_{X} \cup \underbrace{\{y_n \mid n \in \mathbb{N}\} \cup \{y'_n \mid n \in \mathbb{N}\}}_{Y} \cup \{\omega\}.$$

Suppose there is an Esakia topology τ on $\mathbb{T}$. The topology induced by τ on $\max(\mathbb{T})$ is compact and T_2: for, the former holds because $\max(\mathbb{T})$ is closed, see item 2 of Proposition 2.43, and thanks to Proposition 2.32 we know that a closed subset of a compact space is compact as well; moreover $\max(\mathbb{T})$ is T_2 because so it is τ. Since $\max(\mathbb{T})$ is countable we can apply Proposition 2.35 in order to deduce that there is an isolated point in $\max(\mathbb{T})$, i.e. an open $U \in \tau$ such that, without loss of generality, $U \cap \max(\mathbb{T}) = \{x'_n\}$ for some $n \in \mathbb{N}$. Because τ is Esakia it must hold ${\downarrow}U \in \tau$. Observe that ${\downarrow}U \cap Y = \emptyset$ and $\omega \in {\downarrow}U$. Therefore, we have:

$$T = {\downarrow}U \cup \bigcup_{n \in \mathbb{N}} ({\downarrow}y_n)^c$$

because no $x \in X$ is below any y_n, $\omega \in {\downarrow}U$ and for every y_n and y'_n there is $m \geqslant n$ such that not y_n nor y'_n belong to $({\downarrow}y_m)^c$. As usual, this defines a covering of T, because every ${\downarrow}y_n$ must be closed. However, this covering does not have any finite subcover: for, for every $\{y_{n_1}, \ldots, y_{n_m}\}$ there is $k \geqslant \max\{n_1, \ldots, n_m\}$ and thus $y_k \leqslant y_{n_1}, \ldots, y_{n_m}$. Moreover, $y_k \notin {\downarrow}U$ because ${\downarrow}U \cap Y = \emptyset$. This is in contradiction with the supposed compactness of τ, thus implying that $\mathbb{T}$ is not Esakia representable.

In the previous example Theorem 2.35 played a crucial role. One can observe that we were able to use that theorem in virtue of the denumerability of the poset we have considered. This fact suggests that the Esakia representability of countable forests might differ from the Esakia representability of the uncountable ones. Moreover, it proves that the countable case already exhibits some peculiar features, which we should investigate. Accordingly, let $\mathbb{T}$ be an Esakia representable countable tree. So far we have proved two conditions: one on principal upsets and one on principal downsets, respectively. Moreover, we have seen an example of a non-trivial countable forests which is not Esakia representable, although it does not fall under the forbidden configurations that we have encountered so far. The peculiar feature of that example is that it has a point which is an infimum of two incomparable infinite descending chains. This example can be generalized. Once again, our result relies on Theorem 2.35.

Theorem 4.53. *Let $\mathbb{T}$ be a countable tree such that there is a point $x \in T$ and two nonempty chains C, D of $\mathbb{T}$ such that:*

1. *$x \notin C \cup D$;*
2. *$x = \inf(C) = \inf(D)$;*

3. *there are no $c \in C$, $d \in D$ such that ${\downarrow}c \cup {\downarrow}d$ is a chain.*

Then, $\mathbb{T}$ is not Esakia representable.

Proof. We proceed by contraposition: we will show that if $\mathbb{T}$ is Esakia representable then, for every $x \in T$ and nonempty chains C and D which satisfy items 1 and 2 above, there are $c \in C$ and $d \in D$ such that ${\downarrow}c \cup {\downarrow}d$ is a chain. Accordingly, let τ be an Esakia topology on $\mathbb{T}$ and x as in the hypothesis. In view of Corollary 2.44, we know that ${\uparrow}x$ is a Priestley subspace of $\mathbb{T}$. Analogously to what we have done in Example 4.52, the set $\max({\uparrow}x) = \max(\mathbb{T}) \cap {\uparrow}x$ is a compact T_2 and countable subspace of $\mathbb{T}$. Therefore, we can apply Theorem 2.35, in order to deduce that $\max({\uparrow}x)$ as an isolated point y, i.e. there is a clopen $U \in \tau$ such that $U \cap \max({\uparrow}x) = \{y\}$. Then, consider the following set.

$$V := {\downarrow}(({\downarrow}(U^c))^c).$$

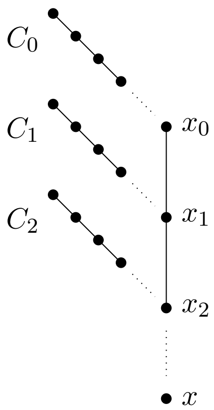
The set V is clopen because so it is U and moreover τ is an Esakia topology. We then claim that the following holds:

$$V \cap {\uparrow}x = {\downarrow}y \cap {\uparrow}x.$$

In order to see this, suppose $z \in V$ and $x \leqslant z$. This means that $z \leqslant w$ for some w not in ${\downarrow}(U^c)$, i.e. $w \leqslant v$ implies $v \in U$. But $U \cap \max({\uparrow}x) = \{y\}$, hence $x \leqslant z \leqslant w \leqslant y$. For the other direction, assume z to be such that $x \leqslant z \leqslant y$. Clearly, we have $y \notin {\downarrow}(U^c)$, because y is maximal and hence the only element above it, i.e. y itself, is in U. Observe that, since $x \notin C \cup D$, $x = \inf(C) = \inf(D)$ and U is open, we can use Corollary 3.26 in order to deduce that there are $c \in C$, $d \in D$, such that $c, d \in U$. Moreover, since $x \leqslant c, d$, from the above display it follows $c, d \leqslant y$. In other words, ${\downarrow}c \cup {\downarrow}d \subseteq {\downarrow}y$, and because ${\downarrow}y$ is a chain so it is ${\downarrow}c \cup {\downarrow}d$. □

The previous theorem exhibits another configuration that countable Esakia representable forests must avoid. There is at least one other class of countable forests whose members are not Esakia representable. Let us introduce it by means of an example.

Example 4.54. Consider the tree depicted in the figure below.



Observe that it does not satisfy the hypothesis of Theorems 3.30 and 3.32, nor of Theorem 4.53. However, we claim that it is not Esakia representable. The next theorems implies this claim.

Theorem 4.55. *Let $\mathbb{T}$ a countable tree. Assume there is an infinite descending chain of distinct elements $(x_n)_{n\in\mathbb{N}}$, and infinitely many chains C_n such that the following conditions hold:*

1. *For every $n \in \mathbb{N}$ it holds $x_n \notin C_n$ and $x_n = \inf(C_n)$;*
2. *If $n \neq m$ there are no comparable $y \in C_n$ and $z \in C_m$.*

Then, $\mathbb{T}$ is not Esakia representable.

Proof. Assume $\mathbb{T}$ to be as in the hypothesis, and suppose that there is an Esakia topology τ on it. Then, there must exits $x = \inf(\{x_n \mid n \in \mathbb{N}\})$. As in Proposition 4.53, we know that $\max(\uparrow x)$ is a compact Hausdorff and countable subspace of $\mathbb{T}$, hence it has an isolated point y, and there is a clopen $U \in \tau$ such that $U \cap \uparrow x = \downarrow y \cap \uparrow x$. Observe that $x_n \leqslant y$ for some $n \in \mathbb{N}$. For, if not, since $U \cap \uparrow x = \downarrow y \cap \uparrow x$ we deduce that no x_n belongs to U, contradicting Corollary 3.26, because U is open. Wherefore, let x_n be such that $x_n \leqslant y$. Once, again, since $U \cap \uparrow x = \downarrow y \cap \uparrow x$, we deduce that $x_{n+1} \in U$, because $x \leqslant x_{n+1} \leqslant x_n \leqslant y$. Our goal is to prove $U \cap C_{n+1} = \emptyset$, which is in contradiction with Corollary 3.26. In order to do so, let $c \in C_{n+1}$. We claim $c \parallel x_n$.

1. First, observe that $c \nleqslant x_n = \inf(C_n)$, otherwise every element in C_n would be comparable with c, against the second hypothesis of the theorem.
2. Secondly, assume $x_n < c$. This means $x_n \in \downarrow C_{n+1}$. But then, because $x_{n+1} \notin C_{n+1}$ and $x = \inf(C_{n+1})$, we deduce that there is $c' \in C_{n+1}$ such that $x_{n+1} < c' < x_n$. But this implies that c' is comparable with every element in C_n, which is a contradiction.

We then claim $U \cap C_{n+1} = \emptyset$. In order to see this, suppose $c \in C_{n+1} \cap U$. Then, we have $x < c \in U$, and thus $x < c \leqslant y$, because $U \cap {\uparrow}x = {\downarrow}y \cap {\uparrow}x$. But this implies $c \not\parallel x_n$, because $x_n \leqslant y$ and ${\downarrow}y$ is a chain. This is in contradiction with what we have just shown, therefore $U \cap C_{n+1} = \emptyset$, contradicting Corollary 3.26. This concludes the proof. □

Theorems 4.53 and 4.55 show that the Esakia representability of countable forests exhibits some peculiar features. It is still to fully understand how this problem differs from the Esakia representability of uncountable forests. This concludes our discussion on the representability problem.

CHAPTER 5

Conclusion

In this thesis we studied the Priestley and Esakia representability of partially ordered sets by restricting the attention to some special classes of posets. In particular, our main result is the characterization of Priestley and Esakia representable well-ordered forests and diamond systems. We showed that a well-ordered forest is Esakia representable if and only if its nonemtpy chains have suprema. Moreover, we proved that a diamond system is Esakia representable if and only if its nonempty chains have suprema and infima and it has enough gaps. We referred to these properties by C1 and C2, respectively. From this it follows that a well-ordered forest (resp. diamond system) is Priestley representable if and only if it is Esakia representable.

We also identified some properties of Priestley (resp. Esakia) topologies that revolve around infinite chains. For example, if the infimum of an infinite descending chain is minimal, then any open set of a Priestley (resp. Esakia) topology containing it must contain a nontrivial downset. The order dual version of this proposition is also true. We also showed that, for every Esakia space, the cardinalities of the ramifications of an infinite descending chain C are bounded by the cardinality of the ramifications of $\inf(C)$. This proves that there are some forbidden order-theoretic configurations for the Esakia representable posets. Finally, we discussed how to address the case of countable forests. In particular, we proved that an Esakia representable countable forests cannot have points which are limits of two incomparable infinite descending chains. Moreover, if an Esakia representable forest is countable, its infinite descending chains cannot consist of infima of certain infinite descending chains.

Below we discuss some directions for future work on the representability problem. In particular, we propose some classes of posets for which it is natural to study this problem, especially in view of the results of this thesis.

- Example 3.23 shows that there are posets of height 2 and width 2 which satisfy C1 and C2 but are not Priestley representable. It is still unknown if there are posets of finite height (resp. finite height and finite width) which satisfy C1, C2 and C3 but are not Priestley representable. The condition C3 was introduced in Proposition 3.21, and it generalizes Hochster's condition H, as it appears in (Lewis & Ohm, 1976). More generally, we propose to study the representability problem for the class of posets with finite height (resp. finite height and finite width).

- Diamond systems are a generalization of root systems with width at most 2. Priestley (resp. Esakia) representable systems are characterized in Section 4.1. We suggest to find an extension $\mathcal{C}$ of the class of diamond systems whose poset are allowed to have width at most n, for $2 < n$, and study the representability problem for the class $\mathcal{C}$.

- In Section 4.4 we studied the Esakia representability of countable forests. Because of Theorems 2.35, 4.53 and 4.55, it seems that the Esakia representability of countable forests might differ from the representability of arbitrary forests. The characterization of Priestley (resp. Esakia) representable countable and arbitrary forests is still an open question and we suggest it as future work.

- We know that the class of Priestley representable posets is closed under disjoint unions, finite ordered sums and order duals. On the other hand, the class of Esakia representable posets is not closed under order duals, and none of these classes is closed under arbitrary ordered sums. We leave it as an open problem to find other closure properties for these classes.

References

Aguzzoli, S., Gerla, B., & Marra, V. (2008). Gödel algebras free over finite distributive lattices. *Ann. Pure Appl. Logic*, *155*(3), 183–193.

Awodey, S. (2010). *Category theory* (2nd ed.). USA: Oxford University Press, Inc.

Balbes, R. (1971). On the partially ordered set of prime ideals of a distributive lattice. *Canadian Journal of Mathematics*, *23*(5), 866–874.

Bezhanishvili, G., & Bezhanishvili, N. (2008). Profinite heyting algebras. *Order*, *25*(3), 211–227.

Bezhanishvili, G., Bezhanishvili, N., Gabelaia, D., & Kurz, A. (2010). Bitopological duality for distributive lattices and heyting algebras.

Bezhanishvili, G., Bezhanishvili, N., Moraschini, T., & Stronkowski, M. (2021). Profiniteness and representability of spectra of heyting algebras. *Submitted manuscript, available online on the AirXiv*.

Bezhanishvili, G., & Morandi, P. J. (2009). Profinite heyting algebras and profinite completions of heyting algebras.

Birkhoff, G. (1935). On the structure of abstract algebras. *Proceedings of the Cambridge*.

Blackburn, P., de Rijke, M., & Venema, Y. (2002). *Modal logic*. Cambridge University Press.

Chen, C., & Grätzer, G. (1969). Stone lattices. ii. structure theorems. *Canadian Journal of Mathematics*, *21*, 895–903.

Cornish, W. H. (1975). On h. priestley's dual of the category of bounded distributive lattices. *Matematički Vesnik*, *12*(60), 329–332.

Davey, B. A. (1973). A note on representable posets. *algebra universalis*, *3*(1), 345–347.

Davey, B. A., & Priestley, H. A. (2002). *Introduction to lattices and order*. Cambridge university press.

Engelking, R., et al. (1977). General topology.

Esakia, L. (1974). Topological kripke models. In *Doklady akademii nauk* (Vol. 214, pp. 298–301).

Esakia, L. (2019). *Heyting algebras: Duality theory* (Vol. 50). Springer.

Grätzer. (1978). General lattice theory. *Birkhauser Verlag, Basel*.

Grätzer, G. (1971). *Lattice theory. first concepts and distributive lattices*. W. H. Freeman and Company, San Francisco.

Hausdorff, F. (1908). Grundzüge einer theorie der geordneten mengen. *Mathematische Annalen*, *65*(4), 435–505.

Hochster, M. (1969). Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, *142*, 43–60.

Horn, A. (1969). Free l-algebras. *The Journal of Symbolic Logic*, *34*(3), 475–480.

Johnstone, P. T. (1982). *Stone spaces* (Vol. 3). Cambridge university press.

Joyal, A. (1971). Spectral spaces and distributive lattices-preliminary report. In *Notices of the american mathematical society* (Vol. 18, p. 393).

Kaplansky, I. (1970). *Commutative rings [bad ocr]* (revised edition ed.). Allyn and Bacon.

Lewis, W. J. (1971). The spectrum of a ring as a partially ordered set.

Lewis, W. J., & Ohm, J. (1976). The ordering of $\mathtt{Spec}(r)$. *Canadian Journal of Mathematics*, *28*(4), 820–835.

Mac Lane, S. (2013). *Categories for the working mathematician* (Vol. 5). Springer Science & Business Media.

Priestley, H. A. (1970). Representation of distributive lattices by means of ordered stone spaces. *Bulletin of the London Mathematical Society*, *2*(2), 186–190.

Priestley, H. A. (1984). Ordered sets and duality for distributive lattices. In *North-holland mathematics studies* (Vol. 99, pp. 39–60). Elsevier.

Priestley, H. A. (1994). Spectral sets. *Journal of Pure and Applied Algebra*, *94*(1), 101–114.

Sankappanavar, H. P., & Burris, S. (1981). A course in universal algebra. *Graduate Texts Math*, *78*.

Semadeni, Z. (1971). *Banach spaces of continuous functions* (Vol. 1). PWN Polish Scientific Publishers.

Speed, T. P. (1972). Profinite posets. *Bulletin of the Australian Mathematical Society*, *6*(2), 177–183.

Stone, M. H. (1936). The theory of representation for boolean algebras. *Transactions of the American Mathematical Society*, *40*(1), 37–111.

Stone, M. H. (1938). Topological representations of distributive lattices and brouwerian logics. *Časopis pro pěstování matematiky a fysiky*, *067*(1), 1-25. Retrieved from http://eudml.org/doc/27235