6 Appendix: Notation and definitions

6.1 Notations for sequences. 2^ω is the set of infinite binary sequences. If $x \in 2^\omega$, then $x(n)$ is the intial segment of x of length n, and x_n is the n^{th} term (also called coordinate) of x. The mapping $T: 2^\omega \to 2^\omega$ (called the left shift) is defined by $(Tx)_n = x_{n+1}$. x is used consistently as a variable over 2^ω; ξ always denotes a variable over $(2^\omega)^\omega$.

$2^{<\omega}$ is the set of all finite binary sequences. An finite binary sequence is alternatively called a *word* or a *string*. The length of a word w is denoted $|w|$. 2^n is the set of all strings w such that $|w| = n$. If $m \le |w|$, then $w(m)$ is the initial segment of w of length m, and w_m is the m^{th} term of w. If v is an initial segment of w, we write $v \subseteq w$; if $v \subseteq w$ and $v \neq w$, we write $v \subset w$. The empry string is denoted <>.

6.2 Topology on 2^ω. If B is a set, 1_B denotes the characteristic function of B. Let $2 = \{0,1\}$ have the discrete topology and form the product topology on 2^ω. The open sets in this topology are then unions of *cylinders* [w] defined by $[w] := \{x \in 2^\omega | \, x(|w|) = w\}$. If $S \subseteq 2^{<\omega}$, then the open set generated by S, namely $\{x \in 2^\omega | \, \exists w \in S \, (x(|w|) = w)\}$, is denoted [S]. The topology on spaces of the form $(2^\omega)^m$ is constructed analogously.

For any subset A contained in 2^ω, Cl(A) denotes the closure of A, and Int(A) the interior of A. The *boundary* of A, denoted ∂A, is defined to be $\partial A := \mathrm{Cl}(A) - \mathrm{Int}(A)$.

The Borel σ-algebra on 2^ω is the smallest σ-algebra containing the open sets in 2^ω. Elements of this algebra are called *Borel sets.*

6.3 Measures on 2^ω. A measure on the Borel σ-algebra is completely determined by its values on the cylinders. We shall consider *probability measures* only, i.e. measures μ for which $\mu(2^\omega) = 1$. Now let $(p_n)_n$, where $p_n \in [0.1]$, be a sequence of reals. This sequence determines a product measure on 2^ω, denoted $\prod_n (1 - p_n, p_n)$ and defined as

$$\prod_n (1-p_n, p_n)\,[w] = \prod_{k=1}^{|w|} \bar{p}_k, \text{ where } \bar{p}_k := p_k \text{ if } w_k = 1 \text{ and } \bar{p}_k := 1-p_k \text{ otherwise.}$$

One product measure on 2^ω occurs so often that it isgiven a special name: $\lambda = (\frac{1}{2}, \frac{1}{2})^\omega$.

λ is the image of the Lebesgue measure on the unit interval under the natural map and will also be called Lebesgue measure.

The following relationships among probability measures μ and ν are of special importance.
- μ is *singular* with respect to ν (denoted: $\mu \perp \nu$) if there exists a Borel set A such that $\mu A = 1$ and $\nu A = 0$.
- μ is *absolutely continuous* with respect to ν (denoted: $\mu << \nu$) if for all Borel sets A such that $\nu A = 0$, also $\mu A = 0$.
- μ and ν are *equivalent* (denoted: $\mu \approx \nu$) if $\mu << \nu$ and $\nu << \mu$.

Let $(\mu_n)_n$ be a sequence of measures. We say that μ_n *converges weakly* to ν if for all Borel sets A such that $\nu \partial A = 0$, $\mu_n A$ converges to νA. The Portmanteau theorem [4] states (among else) that weak convergence is equivalent to convergence on the cylinders. We say that μ_n *converges strongly* to ν if for all Borel sets A, $\mu_n A$ converges to νA.

6.4 Computability We shall take as primitive the notion of an algorithm operating on natural numbers, which yields as output natural numbers. It is understood that an algorithm need not terminate on every input. A *partial recursive function* f: $\omega \to \omega$ is a function which can be computed by an algorithm. With this intuitive description it is more or less clear that there exists an effective procedure which associates to each partial recursive function a natural number, its *Gödelnumber*. A *recursive function* is a partial recursive function which is in fact total. More formal definitions of (partial) recursive function and Gödelnumber are possible; see Rogers [86] and Soare [92]. The connection between the informal concept of an algorithm and the formal definition of a partial recursive function is provided by *Church's Thesis*, which states that every algorithm computes a partial recursive function.

Usually one does not formally verify that an apparently recursive function is indeed recursive; one exhibits an algorithm which computes the function and Church's Thesis is invoked to guarantee that the function is in fact recursive. We shall do likewise. We must, however, warn the reader that in constructing algorithms we freely use classical logic; as a consequence, proving the existence of a recursive function need not mean that we can lay our hands on it.

Although we defined partial recursive functions to have the natural numbers as domain and range, this restriction is not as severe as may seem, since many objects can be coded into the natural numbers. In particular, this is true for $\mathbb{Q}$ and $2^{<\omega}$. The following concepts thus make sense. A function f: $\omega \to \mathbb{R}$ is called *computable* if there exists a recursive function g: $\omega \times \omega \to \mathbb{Q}$ such that for all n,k: $|f(n) - g(n,k)| < 2^{-k}$. A measure μ on is computable if there exists a recursive function g: $2^{<\omega} \times \omega \to \mathbb{Q}$ such that for all w, n: $|\mu[w] - g(w,n)| < 2^{-k}$

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We shall often use the *arithmetical hierarchy* for subsets of ω and of 2^{ω}. We say that $A \subseteq \omega^k$ is *recursive* if its characteristic function is a recursive function. Starting from the recursive sets, we can define increasingly complex subsets of ω^k using quantification over ω. A is *recursively enumerable or* Σ_1 if there exists a recursive $B \subseteq \omega^{k+1}$ such that

$$A = \{u \in \omega^k \mid \exists n\,(<n,u> \in B)\}.$$

A is $\prod_1$ if A^c is Σ_1. In general, A is Σ_n if there exists a $B \subseteq \omega^{k+1}$ such that B is $\prod_{n-1}$ and

$$A = \{u \in \omega^k \mid \exists n\,(<n,u> \in B)\};$$

A is $\prod_n$ if A^c is Σ_n. Note that $\prod_n$ sets A can be written as

$$A = \{u \in \omega^k \mid \forall n\,(<n,u> \in B)\},$$

for some Σ_{n-1} set B. A is called Δ_n if it is both Σ_n and $\prod_n$. This is the arithmetical hierarchy for subsets of ω^k. (In the textbooks the Σ, $\prod$ and Δ usually have superscripts "0", to indicate quantification over natural numbers. Since we shall never quantify over sequences, we have dropped the superscripts.)

We now generalize the concept of recursiveness to spaces of the form $\omega^k \times (2^\omega)^m$. Roughly, a relation $R \subseteq \omega \times 2^\omega$ is *recursive* if for each natural number n and each x, the truth value of R(n,x) can be computed using only a finite piece of x; similarly for relations in $\omega^k \times (2^\omega)^m$.

A subset A of $\omega^k \times (2^\omega)^m$ is Σ_1 if there exists a recursive relation B in $\omega^{k+1} \times (2^\omega)^m$ such that

$$A = \{<\bar{n},\bar{x}> \in \omega^k \times (2^\omega)^m \mid \exists j\, B(j,\bar{n},\bar{x})\}.$$

A $\prod_1$ set is the complement of a Σ_1 set. The reader can now copy the defintions of Σ_n, $\prod_n$ and Δ_n from the corresponding definitions for subsets of ω^k.

We now specialize the preceding definition to the case that subsets of $(2^\omega)^m$ are defined using recursive relations and quantification over natural numbers. Let A be of the form

$$A = \{\bar{x} \in (2^\omega)^m \mid R(\bar{n},\bar{x})\},$$

for some recursive relation R. It follows from the intuitive explanation of recursiveness and the compactness of $(2^\omega)^m$ that A is of this form is A is clopen. The clopen sets will also be called Σ_0 sets. It is easily verified that Σ_1 sets are open and that $\prod_1$ sets are closed. The converse is of course false, as a cardinality argument shows.

6.5 Ergodic Theory A measure μ on 2^ω is called *stationary* if for all Borel sets A, $\mu T^{-1}A = \mu A$, where T is the left shift defined in 7.1. A measure μ is *ergodic* if for all Borel sets A: $T^{-1}A = A$ implies that μA is either 0 or 1. The single most important fact about stationary measures is the

Ergodic theorem (see [82]) Let μ be a stationary measure on 2^ω, f: $2^\omega \to \mathbb{R}$ integrable. Then

$$f^*(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(T^k x)$$

exists μ-a.e., f^* is T-invariant and $\int f d\mu = \int f^* d\mu$. In addition, if μ is ergodic then f^* is constant μ-a.e.

We say that a measure μ on 2^{ω} is *strongly mixing* if for all Borel sets A, B: $\mu(T^{-n}A \cap B)$ converges to $\mu A \cdot \mu B$.