THE EXTENT OF CONSTRUCTIVE GAME LABELLINGS

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ABSTRACT. We define a notion of combinatorial labellings, and show that $\mathbf{\Delta}^0_2$ is the largest boldface pointclass in which every set admits a combinatorial labelling.

1. Introduction

Labellings are at the core of the theory of infinite games. We can see the game solutions of finite perfect information games due to Zermelo's "backward induction technique" from [Zer13] and the extension to infinite games with open payoff by Gale and Stewart in [GalSte53] as proofs by labellings. These proofs are gems of game theory and we can arguably call proofs of this kind **constructive determinacy proofs**. Such proofs, in particular those using the Cantor-Bendixson method, were investigated by Büchi and Landweber in their seminal paper on games and finite automata [BücLan69]. Büchi describes his fascination with constructive determinacy proofs:[1]

> "The [constructive] proof '*actually presents*' a winning strategy. The [nonconstructive] proofs do no such thing; all you know at the end is existence of a winning strategy."[2]

Although Büchi offers a general idea of what it means for a determinacy proof to be constructive, he doesn't give specific criteria. In this paper, we develop a notion of *combinatorial labelling* that we consider

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[1]The term "constructive determinacy" is used by Gurevich in [Gur90] whereas Büchi more technically writes "CB-proof of determinacy" where "CB" refers "Cantor-Bendixson".

[2][Büc83, p. 1171]; italics in the original.

to be a possible formalization of "constructive proofs" (even though it does not coincide with Büchi's notion, see below) : a game that is analyzed by a labelling using only the combinatorial structure of the payoff set and no additional information (*cf.* § 4 for details).

The main result of this paper is Theorem 6.5:

> The pointclass $\mathbf{\Delta}^0_2$ is the largest boldface pointclass in which every set has a combinatorial labelling.

The history of set-theoretic game theory has seen determinacy proofs using increasingly more complicated arguments, *e.g.*, Davis' argument for the $\mathbf{\Sigma}^0_2$ games [Dav63], Wolfe's argument for the $\mathbf{\Sigma}^0_3$-games [Wol55], Paris' argument for the $\mathbf{\Sigma}^0_4$-games [Par72], and in general Martin's inductive proof for Borel games [Mar75, Mar85]. Harvey Friedman proved [Fri70/71] that the increase in complexity is not avoidable: higher determinacy proofs cannot be done in second order number theory and essentially need higher type objects. However, Büchi conjectured that there is a constructive proof of Borel determinacy [Büc83, Problem 1]; this suggests that his notion of "constructive" is at least far more liberal than our notion of a combinatorial labelling.

The paper does not assume any specialized knowledge in set theory, descriptive set theory or logic. We just assume naïve set theory and provide all necessary definitions and notation in Section 2. Sections 3 and 4 define the notions of sound and constructive labellings, respectively, and discuss their basic properties. Our definitions are motivated by the Gale-Stewart labelling which is discussed in a very general version in Section 5. Finally, in Section 6, we define a labelling for sets in the Hausdorff difference hierarchy over the open sets, proving our main result, Theorem 6.5. In an appendix, we connect the material of this paper to results on subsystems of second order arithmetic defined via determinacy axioms. The authors would like to thank Michael Möllerfeld for discussions about the proof theory of weak determinacy axioms that were important for writing the appendix.

2. Notation & Definitions

We shall use the usual notation from set-theoretic game theory. The two players will be called Player I and Player II, and their possible moves (from a countable set of moves) will be represented by natural numbers. The games under consideration will run for ω rounds, so that the set of runs is the set ω^ω of infinite sequences of natural numbers. This set is naturally endowed with the product topology and as a topological space it is homeomorphic to the set of irrational numbers. We therefore call its elements **real numbers**.

Alternatively, ω^ω can be seen as the set of branches through a tree. Let $\omega^{<\omega}$ be the set of finite sequences of natural numbers. A subset T of $\omega^{<\omega}$ is called a **tree** if it is closed under initial segments, i.e., if $s \subseteq t \in T$, then $s \in T$. For each tree T, we define its set of branches

$$[T] := \{x \in \omega^\omega \,;\, \forall n \in \omega (x \upharpoonright n \in T)\}$$

where $x \upharpoonright n$ is the finite initial segment of x of length n. Note that $[\omega^{<\omega}] = \omega^\omega$. If $s \in \omega^{<\omega}$, then $T_s := \{t \,;\, t \subseteq s \vee s \subseteq t\}$ is a tree, and we write $[s] := [T_s] = \{x \in \omega^\omega \,;\, s \subset x\}$. The sets $[s]$ form a basis of the product topology on ω^ω. We use the notation (s) to denote the set of finite extensions of s, so $(s) := \{u \in \omega^{<\omega} \,;\, s \subseteq u\}$. If x is a finite or infinite sequence of natural numbers, then we denote by $s^\frown x$ the **concatenation of** s **with** x. If $s, t \in \omega^{<\omega}$ and $t = s^\frown\langle k\rangle$ for some $k \in \omega$, then we say that t is a **successor** of s. If $A \subseteq \omega^\omega$, then define $A_{\llcorner s \lrcorner} := \{x \in \omega^\omega : s^\frown x \in A\}$. The **length** of s is denoted by $\mathrm{lh}(s)$.

The product topology on ω^ω has the property that a set $A \subseteq \omega^\omega$ is open or $\mathbf{\Sigma}^0_1$ if and only if there is a set $S \subseteq \omega^{<\omega}$ such that $x \in A \Leftrightarrow \exists s \in S\, (s \subset x)$. Similarly, a set $A \subseteq \omega^\omega$ is closed or $\mathbf{\Pi}^0_1$ if and only if there is a tree $T \subseteq \omega^{<\omega}$ such that $A = [T]$. The $\mathbf{G}_\delta$ or $\mathbf{\Pi}^0_2$ sets are the countable intersections of open sets, and the $\mathbf{F}_\sigma$ or $\mathbf{\Sigma}^0_2$ sets are the countable unions of closed sets. The class of sets that are both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$ is denoted by $\mathbf{\Delta}^0_2$.

A class of subsets of ω^ω is called a **boldface pointclass** if it is closed under continuous preimages; the classes $\mathbf{\Sigma}^0_1$, $\mathbf{\Pi}^0_1$, $\mathbf{\Delta}^0_2$, $\mathbf{\Sigma}^0_2$, and $\mathbf{\Pi}^0_2$ are examples of boldface pointclasses. If Γ is a pointclass, we call $\breve{\Gamma} := \{A \,;\, \omega^\omega \backslash A \in \Gamma\}$ its **dual**. Thus, *e.g.*, $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ are duals of each other. For a pointclass Γ, we define $\forall^{\mathbb{R}}\Gamma := \{\forall^{\mathbb{R}} A \,;\, A \in \Gamma\}$ where $\forall^{\mathbb{R}} A := \{x \in \omega^\omega \,;\, \forall y \in \omega^\omega\, (y \oplus x \in A)\,\}$ with

$$(y \oplus x)(n) := \begin{cases} y(k) & \text{if } n = 2k, \\ x(k) & \text{if } n = 2k+1. \end{cases}$$

As usual, we call an ordinal α **even** (**odd**) if it is of the form $\lambda + 2n$ ($\lambda + 2n + 1$) for some limit ordinal λ and some natural number n. For a sequence $\langle A_\gamma \,;\, \gamma < \alpha\rangle$, we define the **Hausdorff difference**, $\mathrm{Diff}(\langle A_\gamma \,;\, \gamma < \alpha\rangle)$, to be the set

$$\left\{ x \in \bigcup_{\gamma<\alpha} A_\alpha \,;\, \min\{\gamma \,;\, x \in A_\gamma\} \text{ has different parity from } \alpha \right\}.$$

If $A = \mathrm{Diff}(\langle A_\gamma \,;\, \gamma < \alpha\rangle)$, we call $\langle A_\gamma \,;\, \gamma < \alpha\rangle$ a **presentation of** A. In general, the presentation of a set need not be unique.

Let $\mathbf{\Gamma}$ be a boldface pointclass and let α be a countable ordinal. The **Hausdorff difference classes** are defined as follows: $A \in \alpha$-$\mathbf{\Gamma}$ if there is an increasing sequence $\langle A_\gamma\,;\,\gamma < \alpha\rangle$ of sets in $\mathbf{\Gamma}$ such that $A = \mathrm{Diff}(\langle A_\gamma\,;\,\gamma < \alpha\rangle)$.

For a set in the Hausdorff difference hierarchy over the open sets, we define by recursion its **canonical presentation**:

Say that $A \in \alpha$-$\mathbf{\Sigma}^0_1$ and α is odd, then we define

$$C_0 := \bigcup\{[s]\,;\,[s] \subseteq A\},$$

$$C_\beta := \bigcup\{[t]\,;\,[t] \subseteq (\omega^\omega \backslash A) \cup \bigcup_{\delta<\beta} C_\delta\} \text{ (for odd } \beta\text{), and}$$

$$C_\beta := \bigcup\{[s]\,;\,[s] \subseteq A \cup \bigcup_{\delta<\beta} C_\delta\} \text{ (for even } \beta\text{).}$$

If α is even, the roles of even and odd ordinals are interchanged in the obvious way:

$$C_0 := \bigcup\{[s]\,;\,[s] \subseteq \omega^\omega \backslash A\},$$

$$C_\beta := \bigcup\{[t]\,;\,[t] \subseteq (\omega^\omega \backslash A) \cup \bigcup_{\delta<\beta} C_\delta\} \text{ (for even } \beta\text{), and}$$

$$C_\beta := \bigcup\{[s]\,;\,[s] \subseteq A \cup \bigcup_{\delta<\beta} C_\delta\} \text{ (for odd } \beta\text{).}$$

It is not difficult to see that the canonical presentation is itself a presentation of A. Furthermore, the canonical presentation does not require any choice, it is canonical in the sense that it is directly definable from the set A.

The following theorem expresses the Hausdorff difference classes in terms of the arithmetical hierarchy. A proof can be found in [Kec95, Theorem (22.27)].

Theorem 2.1 (Hausdorff-Kuratowski)**.** $\bigcup_{\alpha<\omega_1} \alpha$-$\mathbf{\Sigma}^0_1 = \mathbf{\Delta}^0_2$.

After these preliminaries on descriptive set theory, we move to the games that we discuss in this paper: We consider infinite perfect information games with two players and payoff sets contained in ω^ω. All games of length ω in which both players choose from an at most countable set of possible moves can be modeled as games of this sort.

In our games, players will always move in turn: Player I moves in even-numbered rounds, Player II in odd-numbered rounds. We denote the set of all finite sequences of even or odd length with M_{I} and M_{II}, respectively. We define $\mu(s)$ to be the parity of the length of s, *i.e.*, $\mu(s) = 0$ if and only if s has even length. If necessary, we could give up our moving conventions by changing the sets M_{I} and M_{II} and the function μ.

A game is described by giving a payoff set A. The game starts at position $\varnothing$. At each position s, the player whose turn it is plays an element n of ω. The next position of the game is $s^\frown\langle n\rangle$. An element of ω^ω, i.e. a real number, is produced after ω rounds. If this real number is an element of A, Player I has won, if not, then Player II has won.

A **strategy for Player I** is a function $\sigma : M_{\mathrm{I}} \to \omega^{<\omega}$ such that $\sigma(s)$ is a successor of s. Similarly, a **strategy for Player II** is a function $\tau : M_{\mathrm{II}} \to \omega^{<\omega}$ such that $\tau(s)$ is a successor of s. If σ is a strategy for Player I and τ is a strategy for Player II, we denote by $\sigma * \tau$ the unique element of ω^ω that is produced if Player I follows σ and Player II follows τ.

We call a strategy σ for Player I **winning** if for all counterstrategies τ, we have $\sigma * \tau \in A$. Similarly, a strategy τ for Player II is **winning** if for all counterstrategies σ, we have $\sigma * \tau \notin A$. Clearly, at most one player can have a winning strategy, in which case the set A is called **determined**.

Alternatively, a strategy can be viewed as a tree. If σ is a strategy for Player I, we can define a tree T_σ by $\varnothing \in T_\sigma$ and

$$s^\frown\langle n\rangle \in T_\sigma \ :\Longleftrightarrow\ s \in M_{\mathrm{II}} \vee (s \in M_{\mathrm{I}} \wedge \sigma(s) = s^\frown\langle n\rangle).$$

We define an analogous tree T_τ if τ is a strategy for Player II. If σ is winning for Player I, then $[T_\sigma] \subseteq A$; similarly, if τ is winning for Player II, then $[T_\tau] \cap A = \varnothing$. We call a tree T a **strategic tree for Player I** (**Player II**) if there is a winning strategy σ for Player I (Player II) such that $T = T_\sigma$.

For a position $s \in \omega^{<\omega}$, consider the variant of the game beginning at s. An s**-strategy for Player I** is a function $\sigma : (s) \cap M_{\mathrm{I}} \to \omega^{<\omega}$ such that $\sigma(u)$ is a successor of u. We define an s**-strategy for Player II** in the analogous way.

If σ is an s-strategy for Player I, we may define a tree T_σ by $t \in T_\sigma$ for $t \subseteq s$ and

$$t^\frown\langle n\rangle \in T_\sigma \ :\Longleftrightarrow\ t \in (s) \cap M_{\mathrm{II}} \vee (t \in (s) \cap M_{\mathrm{I}} \wedge \sigma(t) = t^\frown\langle n\rangle).$$

We say that T is an s**-strategic tree for Player I** if there is a winning s-strategy σ such that $T = T_\sigma$. The notion of an s**-strategic tree for Player II** is defined in the analogous way.

3. Labellings I: Soundness

We say that $\mathbf{L} = \langle L_{\mathrm{I}}, <_{\mathrm{I}}, L_{\mathrm{II}}, <_{\mathrm{II}}\rangle$ is a **labelling system** if L_{I} and L_{II} are disjoint sets, $<_{\mathrm{I}}$ is a well-ordering on L_{I}, and $<_{\mathrm{II}}$ is a well-ordering on L_{II}. The elements of L_{I} are called **I-labels** and the elements of L_{II} are called **II-labels**. We shall sometimes write $\mathbf{L}$ for the set $L_{\mathrm{I}} \cup L_{\mathrm{II}}$. We call any partial function $\ell : \omega^{<\omega} \to \mathbf{L}$ a **labelling**.

Fix a labelling ℓ and a position s. We say that an s-strategy σ for Player I is **ℓ-good** if it satisfies the following property: if $t \in \mathrm{dom}(\sigma)$ and there exists a $j \in \omega$ such that $t^\frown\langle j\rangle \in \mathrm{dom}(\ell)$ and $\ell(t^\frown\langle j\rangle)$ is a I-label, then $\ell(\sigma(t))$ is the $<_{\mathrm{I}}$-least element of the set $\{\ell(t^\frown\langle j\rangle)\,;\, j \in \omega\} \cap L_{\mathrm{I}}$. In other words, if there are I-labelled successors of t, then $\sigma(t)$ is a I-labelled successor with the smallest possible label. The Player II case is handled analogously.

Letting A be the payoff set, we say that ℓ is **A-sound at** s if either $\ell(s)$ is a I-label and every ℓ-good s-strategy for Player I is winning, or if $\ell(s)$ is a II-label and every ℓ-good s-strategy for Player II is winning.

If the context is clear, we sometimes write "good" and "sound" instead of "ℓ-good" and "A-sound."

Proposition 3.1. *Let $A \subseteq \omega^\omega$ and $s \in \omega^{<\omega}$. Then Player I has a winning s-strategy if and only if there is a labelling ℓ such that $\ell(s) \in L_{\mathrm{I}}$ and ℓ is A-sound at s. Similarly, Player II has a winning s-strategy if and only if there is a labelling ℓ such that $\ell(s) \in L_{\mathrm{II}}$ and ℓ is A-sound at s.*

Proof. We just prove the statement for Player I.

"$\Leftarrow$" Suppose there is a labelling ℓ such that $\ell(s)$ is a I-label and ℓ is sound at s. It suffices to define a good s-strategy σ for Player I. For $t \in (s) \cap M_{\mathrm{I}}$, if there are I-labelled successors of t, let $\sigma(t) := t^\frown\langle j\rangle$ where j is the smallest natural number such that $\ell(t^\frown\langle j\rangle)$ is the $<_{\mathrm{I}}$-least element of the set $\{\ell(t^\frown\langle k\rangle)\,;\, k \in \omega\} \cap L_{\mathrm{I}}$. If there are no I-labelled successors of t, then the value of $\sigma(t)$ is irrelevant.

"$\Rightarrow$" Let σ be the winning s-strategy for Player I. Consider a labelling system with one I-label, 0, and let $\ell(v) := 0$ if $v = \sigma(u)$ for some $u \in \mathrm{dom}(\sigma)$. To show that ℓ is sound at s, it sufficies to show that any good s-strategy for Player I is winning. But this is immediate since σ is the only such strategy. $\square$

Proposition 3.2. *For any $A \subseteq \omega^\omega$, A is determined if and only if there is an A-sound labelling at $\varnothing$.*

Proof. Take $s = \varnothing$ and apply Proposition 3.1. $\square$

One feature of the Gale-Stewart proof (*cf.* Theorem 5.3) is that the labelling produced by it is sound at every position s, not only at $\varnothing$. Thus, it is natural to consider labellings that satisfy this stricter condition. We say that a labelling is **globally A-sound** if it is A-sound at every $s \in \omega^{<\omega}$. Note that every globally sound labelling must be total. Proposition 3.2 becomes false if we consider globally A-sound labellings instead of A-sound labellings at $\varnothing$, but the result still holds classwise for boldface pointclasses.[3]

Proposition 3.3. *Suppose* $\mathbf{\Gamma}$ *is a boldface pointclass. Then, using* $\mathsf{AC}_\omega(\mathbb{R})$*, the following are equivalent:*

(1) *Every set in* $\mathbf{\Gamma}$ *is determined.*
(2) *For every set* $A \in \mathbf{\Gamma}$*, there is a labelling that is globally* A*-sound.*

Proof. "$2 \Rightarrow 1$" Follows immediately from Proposition 3.2.

"$1 \Rightarrow 2$" Since $\mathbf{\Gamma}$ is boldface, the sets $A_{\llcorner s \lrcorner}$ for any $s \in \omega^{<\omega}$ are in $\mathbf{\Gamma}$ as well, hence determined. Thus for every s, the set $\{\sigma\,;\,\sigma$ is a winning s-strategy for Player I$\} \cup \{\tau\,;\,\tau$ is a winning s-strategy for Player II$\}$ is non-empty. Using $\mathsf{AC}_\omega(\mathbb{R})$, let ν_s be an element of this nonempty set for each $s \in \omega^{<\omega}$.

Fix an enumeration $\{s_i\,;\,i \in \omega\}$ of $\omega^{<\omega}$. Let $L_{\mathrm{I}} := \{i\,;\,\nu_{s(i)}$ is winning for Player I$\}$ and let $L_{\mathrm{II}} := \{i\,;\,\nu_{s(i)}$ is winning for Player II$\}$. For each i, let T_i be the strategic tree corresponding to $\nu_{s(i)}$. Let $n \in \omega$, and suppose ℓ_{n-1} has been defined (by convention, we let $\ell_{-1} := \varnothing$). Let

$$\ell_n(t) := \begin{cases} \ell_{n-1}(t) & \text{if } \ell_{n-1}(t) \text{ is defined,} \\ n & \text{if } t \in T_n \end{cases}$$

It is not difficult to see that $\ell := \bigcup_{n\in\omega} \ell_n$ is globally A-sound.

□

The use of the Axiom of Choice in Proposition 3.3 can be replaced by assuming more determinacy:

Proposition 3.4. *If* $\mathbf{\Gamma}$ *is a boldface pointclass, and every set in* $2\text{-}\breve{\mathbf{\Gamma}}$ *is determined, then every set* $A \in \mathbf{\Gamma}$ *has a globally* A*-sound labelling.*

Proof. Let A be a given set in $\mathbf{\Gamma}$. Define the following auxiliary game. Player I plays a finite sequence s, Player II answers with a single 0/1 bit b. After that, both Player I and Player II play natural numbers consecutively and produce x. Player I wins if

$$s^\frown x \in A \text{ and } b = 0, \text{ or}$$
$$s^\frown x \notin A \text{ and } b = 1.$$

Clearly, the payoff set of this game is $2\text{-}\breve{\mathbf{\Gamma}}$, so by $2\text{-}\breve{\mathbf{\Gamma}}$-determinacy we know that one of the players has a winning strategy. But $2\text{-}\breve{\mathbf{\Gamma}}$-determinacy implies $\mathbf{\Gamma}$-determinacy, so we know that Player I cannot

[3]Suppose that X is a nondetermined set (using AC). Then clearly, Player I has a winning strategy for $X^* := \{z\,;$ either $z(0) = 0$ or there is some $x \in X$ such that $z = \langle 1\rangle^\frown x\}$, and so X^* is determined. But there cannot be an X^*-sound labelling at $\langle 1\rangle$ by Proposition 3.1.

have a winning strategy in this game: no matter what s he plays, the game restricted to s is determined, and Player II can pick the winning rôle. Consequently, Player II has a winning strategy τ in the auxiliary game. But the function mapping s to the final segment of τ after Player I has played s and Player II has answered with $\tau(s)$ is a choice function for the set of winning strategies for A_s, and the proof of Proposition 3.3 needs only this. □

As a consequence of Proposition 3.4, we get that if $\mathbf{\Gamma}$ is a boldface pointclass such that $2\text{-}\breve{\mathbf{\Gamma}} \subseteq \mathbf{\Gamma}$, then $\mathbf{\Gamma}$-determinacy implies the existence of globally sound labellings for all sets in $\mathbf{\Gamma}$. This condition is satisfied for the classes $\mathbf{\Delta}^0_n$, $\mathbf{\Delta}^1_n$ and the class of projective sets.

4. Labellings II: Combinatorial Labellings

The equivalence results of Section 3 have shown that the notion of a sound labelling doesn't seem add anything beyond the notion of determinacy. In particular, we have no idea how to construct the labellings without already knowing the winning strategy. This defeats the main purpose of labelling: namely, constructing a winning strategy.

One of the main differences between the Gale-Stewart labelling and the labellings from the proofs of Propositions 3.1 and 3.3 is that the Gale-Stewart labelling is determined by the combinatorial structure of the game tree. For positions s and t, if $[s] \cap A$ and $[t] \cap A$ have the same structure, then s and t will receive the same label. In other words, at a position s, the Gale-Stewart labelling at s is determined by the structure of the game tree below s.

In this section, we shall formalize what we mean by "have the same structure". We begin with some background information about bisimulations. As usual, if $\mathbf{G} = \langle G, E_G \rangle$ and $\mathbf{H} = \langle H, E_H \rangle$ are directed graphs, then we call a relation $R \subseteq G \times H$ a **bisimulation** if the following conditions ("back and forth") hold:

$$\forall g, g^* \in G\ \forall h \in H \left(\begin{array}{l} \text{if } \langle g, h \rangle \in R \ \&\ \langle g, g^* \rangle \in E_G \text{ then there is an} \\ h^* \in H \text{ such that } \langle g^*, h^* \rangle \in R \ \&\ \langle h, h^* \rangle \in E_H \end{array} \right)$$
$$\forall g \in G\ \forall h, h^* \in H \left(\begin{array}{l} \text{if } \langle g, h \rangle \in R \ \&\ \langle h, h^* \rangle \in E_H \text{ then there is a} \\ g^* \in G \text{ such that } \langle g^*, h^* \rangle \in R \ \&\ \langle g, g^* \rangle \in E_G \end{array} \right)$$

This is the ordinary notion of bisimulation for graphs well-known from modal logic. Let $s \in \omega^{<\omega}$. We can view (s) as a directed graph E such that $\langle u, v \rangle \in E :\Leftrightarrow v$ is a successor of u. If $A \subseteq \omega^\omega$ and R is a bisimulation between (s) and (t), we say that R is **A-preserving** if for every $x, y \in \omega^\omega$, the following holds:

if $\forall n \in \omega\ [R(s^\frown(x{\restriction}n), t^\frown(y{\restriction}n))]$, then $s^\frown x \in A$ if and only if $t^\frown y \in A$.

Let $s, t \in \omega^{<\omega}$ such that $s \in M_{\mathrm{I}} \leftrightarrow t \in M_{\mathrm{I}}$. We say that s and t are **A-bisimilar** if there is an A-preserving bisimulation R between (s) and (t) such that $\langle s, t\rangle \in R$. Furthermore, we say that a labelling ℓ is **A-combinatorial** if $\ell(s) = \ell(t)$ for any two A-bisimilar nodes s and t. In other words, ℓ is combinatorial if it doesn't distinguish between any two bisimilar nodes.

Proposition 4.1. *The labellings constructed in the proof of Propositions 3.1 and 3.3, respectively, are not in general A-combinatorial.*

Proof. Let $A := \omega^\omega$, let $s = \varnothing$, let σ be any strategy for Player I (thus winning, since $A = \omega^\omega$), and let ℓ be defined as in the proof of "$\Rightarrow$" of Proposition 3.1. For all natural numbers $n \in \omega$, the positions $\langle n\rangle$ are mutually A-bisimilar. But $\ell(\sigma(\varnothing)) = 0$, and $\ell(\langle n\rangle)$ is undefined for all $\langle n\rangle \neq \sigma(\varnothing)$. Therefore, the labelling ℓ is not A-combinatorial.

A similar argument shows that the labelling constructed in the proof of $1 \Rightarrow 2$ of Proposition 3.3 is not in general A-combinatorial.

□

The proof of Proposition 4.1 hints at the fact that good labellings like the Gale-Stewart labelling are more closely associated with quasi winning strategies than with winning strategies.

Proposition 4.2. *There is a $\mathbf{\Sigma}^0_2$ set A such that no A-sound labelling at $\varnothing$ is A-combinatorial.*

Proof. Define A as follows:

$$x \in A \;:\Longleftrightarrow\; \exists n \forall k \geq n\, (x(k) = 0).$$

It is clear that A is $\mathbf{\Sigma}^0_2$ and that Player II has a winning strategy. Note the following key fact:

($*$) For every $s, t \in \omega^{<\omega}$, $A_{\llcorner s \lrcorner} = A_{\llcorner t \lrcorner}$.

Suppose that ℓ is A-combinatorial. It will be shown that ℓ is not A-sound at $\varnothing$. If $\varnothing$ is unlabeled, then we are done. If $\ell(\varnothing) \in L_{\mathrm{I}}$, then we are done by Proposition 3.1. Suppose $\ell(\varnothing) \in L_{\mathrm{II}}$. Since ℓ is combinatorial, it follows from ($*$) that $\ell(u) = \ell(v) \in L_{\mathrm{II}}$ for all u, $v \in M_{\mathrm{II}}$. Therefore, any strategy τ for Player II is ℓ-good. In particular, the strategy $\tau(s) := s^\frown\langle 0\rangle$ is ℓ-good for Player II. But τ is not winning for Player II: let σ be the strategy for Player I defined by $\sigma(s) := s^\frown\langle 0\rangle$, then $\sigma * \tau \notin A$. It follows that ℓ is not A-sound at $\varnothing$. □

We remark that the set A from the proof of Proposition 4.2 is (or can be identified with) the set of finite sequences of integers.

Let us summarize the results of Sections 3 and 4: By Propositions 3.3 or 3.4, the notion of soundness corresponds to abstract existence of a winning strategy (the *ontic* answer). We are in this paper more interested in concretely constructing a strategy (the *epistemic* answer) which corresponds to our notion of being combinatorial.

5. Gale-Stewart labellings

In this section, we describe the Gale-Stewart procedure in a general setting and develop some of the techniques we'll use later. This procedure goes back to Gale and Stewart [GalSte53] and their famous theorem of open determinacy.

Let ℓ be a labelling. Let $\mathbf{L}$ be a labelling system that includes all of the labels used in ℓ and in addition, new labels

$$Z_{\mathrm{I}} := \{z_{\mathrm{I}}^{\eta}\,;\,\eta < \omega_1\} \subseteq L_{\mathrm{I}}, \text{ and}$$

$$Z_{\mathrm{II}} := \{z_{\mathrm{II}}^{\eta}\,;\,\eta < \omega_1\} \subseteq L_{\mathrm{II}},$$

with well-orderings $<_{\mathrm{I}}$ and $<_{\mathrm{II}}$ with the following properties:

- for all $z \in L_{\mathrm{I}} \cap \mathrm{ran}(\ell)$ and all $z^* \in Z_{\mathrm{I}}$, we have $z <_{\mathrm{I}} z^*$,
- for all $z \in L_{\mathrm{II}} \cap \mathrm{ran}(\ell)$ and all $z^* \in Z_{\mathrm{II}}$, we have $z <_{\mathrm{II}} z^*$, and
- $z_{\mathrm{I}}^{\alpha} <_{\mathrm{I}} z_{\mathrm{I}}^{\beta}$ if and only if $z_{\mathrm{II}}^{\alpha} <_{\mathrm{II}} z_{\mathrm{II}}^{\beta}$ if and only if $\alpha < \beta$.

We define the Gale-Stewart closure of ℓ by transfinite recursion:

Let $\ell^0 := \ell$, and let $\ell^{\lambda} := \bigcup_{\alpha<\lambda} \ell^{\alpha}$ for limit ordinals λ. For successor ordinals $\alpha+1$, we define $\ell^{\alpha+1}$ as follows:

For each node $s \in \mathrm{dom}(\ell^{\alpha})$, let $\ell^{\alpha+1}(s) = \ell^{\alpha}(s)$. For each node s unlabelled in ℓ^{α}, we consider the following four cases:

- **(I-I)** If $s \in M_{\mathrm{I}}$ and there is an immediate successor t of s such that $\ell^{\alpha}(t) \in L_{\mathrm{I}}$, then let $\ell^{\alpha+1}(s) := z_{\mathrm{I}}^{\alpha}$.
- **(I-II)** If $s \in M_{\mathrm{I}}$ and all immediate successors of s are already labelled by II-labels, then we $\ell^{\alpha+1}(s) := z_{\mathrm{II}}^{\alpha}$.
- **(II-I)** If $s \in M_{\mathrm{II}}$ and all immediate successors of s are already labelled by I-labels, then we $\ell^{\alpha+1}(s) := z_{\mathrm{I}}^{\alpha}$.
- **(II-II)** If $s \in M_{\mathrm{II}}$ and there is an immediate successor t of s such that $\ell^{\alpha}(t) \in L_{\mathrm{II}}$, then we $\ell^{\alpha+1}(s) := z_{\mathrm{II}}^{\alpha}$.

If none of the four conditions **(I-I)**, **(I-II)**, **(II-I)**, and **(II-II)** is satisfied, we do not label s.

Note that the sequence of labellings is monotone: if $\alpha \leq \beta$, then $\ell^{\alpha} \subseteq \ell^{\beta}$. Since there are only countably many nodes in $\omega^{<\omega}$, there is a countable ordinal ξ such that $\ell^{\xi+1} = \ell^{\xi}$. We call the fixed point labelling ℓ^{ξ} the **Gale-Stewart closure of ℓ with Z_{I} and Z_{II}** and denote it by $\mathrm{GSC}(\ell, Z_{\mathrm{I}}, Z_{\mathrm{II}})$. We call a labelling ℓ **Gale-Stewart closed** if $\ell = \mathrm{GSC}(\ell, Z_{\mathrm{I}}, Z_{\mathrm{II}})$.

For every $s \in \mathrm{dom}(\mathrm{GSC}(\ell, Z_{\mathrm{I}}, Z_{\mathrm{II}}))$, we call the least β such that $s \in \mathrm{dom}(\ell^{\beta})$ the **Gale-Stewart index** of s, or simply the **index** of s. Note that the index of s, if defined, is either 0 or a successor ordinal.

Observation 5.1. *Let $\ell^* = \mathrm{GSC}(\ell, Z_{\mathrm{I}}, Z_{\mathrm{II}})$ and let $s \in \mathrm{dom}(\ell^*)$. If the index of s is a successor ordinal $\alpha + 1$, then $\ell^*(s) = z^{\alpha}_{\mathrm{I}}$ or $\ell^*(s) = z^{\alpha}_{\mathrm{II}}$.*

Lemma 5.2. *Suppose that ℓ is globally A-sound. Then for any $s, t \in \omega^{<\omega}$, if s and t are A-bisimilar, then either $\ell(s) \in L_{\mathrm{I}}$ and $\ell(t) \in L_{\mathrm{I}}$ or $\ell(s) \in L_{\mathrm{II}}$ and $\ell(t) \in L_{\mathrm{II}}$.*

Proof. Let $R \subseteq (s) \times (t)$ witness that s and t are A-bisimilar. We prove the case for $\ell(s) \in L_{\mathrm{I}}$. By Proposition 3.1, Player I has a winning s-strategy σ_s. Define $f : (s) \to (t)$ as follows by recursion. Let $f(s) := t$. Fix $n \geq \mathrm{lh}(s)$, let $u \in (s)$ such that $\mathrm{lh}(u) = n$, and suppose that $f(u)$ has been defined. Inductively, we may assume that $\langle u, f(u) \rangle \in R$. If v is a successor of u, define $f(v) := f(u)^\frown\langle k \rangle$, where k is least such that $\langle v, f(u)^\frown\langle k \rangle \rangle \in R$.

Similarly, we define $g : (t) \to (s)$ such that $g(t) = s$ and $\langle g(u), u \rangle \in R$ for all $u \in (t)$. The function $\sigma_t : (t) \cap M_{\mathrm{I}} \to {}^{<\omega}\omega$ defined by

$$\sigma_t(u) := f(\sigma_s(g(u)))$$

is a winning t-strategy for Player I. By Proposition 3.1, it follows that $\ell(t) \in L_{\mathrm{I}}$. $\square$

The notion of the Gale-Stewart closure and Lemma 5.2 give the usual proof of the Gale-Stewart Theorem:

Theorem 5.3. *Every open set A admits a globally A-sound labelling that is A-combinatorial.*

Proof. Let A be an open set and let 0 be a I-label. Define the following partial labelling: $\ell(s) := 0$ if and only if $[s] \subseteq A$. Let $\ell^* := \mathrm{GSC}(\ell, Z_{\mathrm{I}}, Z_{\mathrm{II}})$. Let 1 be a II-label which is greater than every label in Z_{II}, and define

$$\ell^{**}(s) := \begin{cases} \ell^*(s) & \text{if } s \in \mathrm{dom}(\ell^*) \\ 1 & \text{otherwise.} \end{cases}$$

Let $L_{\mathrm{I}} := \{0\} \cup Z_{\mathrm{I}}$ and $L_{\mathrm{II}} := Z_{\mathrm{II}} \cup \{1\}$. The function ℓ^{**} is total. We claim that ℓ^{**} is *(1)* globally A-sound and *(2)* A-combinatorial.

(1) Suppose $\ell^{**}(s) \in L_{\mathrm{I}}$, let σ be an ℓ^{**}-good s-strategy for Player I, and let τ be any s-strategy for Player II. To show that ℓ^{**} is A-sound at s, it must be shown that $\sigma * \tau \in A$. Let $x = \sigma * \tau$. We shall show the following fact:

($*$) for any $n \geq \mathrm{lh}(s)$, if $\ell^{**}(x{\restriction}n) \in Z_{\mathrm{I}}$, then $\ell^{**}(x{\restriction}(n+1)) <_I \ell^{**}(x{\restriction}n)$.

Let $n \geq \mathrm{lh}(s)$ and suppose $\ell^{**}(x{\restriction}n) \in Z_{\mathrm{I}}$. It follows that the index of $x{\restriction}n$ is a successor ordinal $\alpha+1$, $\ell^{**}(x{\restriction}n) = z_{\mathrm{I}}^{\alpha}$, and we either in case **(I-I)** or **(II-I)** of the construction. If we are in case **(I-I)**, then there is a successor t of $x{\restriction}n$ such that $\ell^{**}(t) \in L_{\mathrm{I}}$ and $\ell^{**}(t) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha}$. Since σ is good, it follows that $\ell^{**}(\sigma(x{\restriction}n)) \leq_{\mathrm{I}} \ell^{**}(t) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha}$. This shows that $\ell^{**}(x{\restriction}(n+1)) <_I \ell^{**}(x{\restriction}n)$. If we are in case **(II-I)**, the proof is similar. This completes the proof of ($*$).

Therefore, there exists an $N \geq \mathrm{lh}(s)$ such that $\ell^{**}(x{\restriction}N) = 0$, since otherwise there would be an infinite strictly decreasing sequence of ordinals. It follows that $x \in A$.

Now, suppose that $\ell^{**}(s) \in L_{\mathrm{II}}$, let τ be an ℓ^{**}-good s-strategy for Player II, and let σ be any s-strategy for Player I. Since the labelling ℓ has no II-labels, it follows that $\ell^{*}(t) \notin Z_{\mathrm{II}}$ for all t and $\ell^{**}(s) = 1$.

Again letting $x = \sigma * \tau$, we show the following:

($**$) for any $n \geq \mathrm{lh}(s)$, $\ell^{**}(x{\restriction}n) = 1$.

We argue by induction. Suppose $n \geq \mathrm{lh}(s)$ and $\ell^{**}(x{\restriction}n) = 1$. If $\ell^{**}(x{\restriction}n) \in M_{\mathrm{I}}$, then we must have that no successor of $x{\restriction}n$ got a I-label, since otherwise $x{\restriction}n$ would have gotten a I-label by **(I-I)**. It follows that $\ell^{**}(x{\restriction}(n+1)) = 1$. If $\ell^{**}(x{\restriction}n) \in M_{\mathrm{II}}$, then there must be a successor of $x{\restriction}n$ that did not get a I-label, since otherwise $x{\restriction}n$ would have gotten a I-label by **(II-I)**. Since τ is good, it follows that $\ell^{**}(x{\restriction}(n+1)) = 1$. This completes the proof of ($**$), from which it follows that $x \notin A$.

(2) Suppose ℓ^{**} is not A-combinatorial. Call a position s **bad** if there is a t that is A-bisimilar to s such that $\ell^{**}(s) \neq \ell^{**}(t)$. Suppose for contradiction that there are bad positions. It is immediate that the bad positions cannot all be II-labelled, since there is only one II-label in ℓ^{**}. Choose s such that $\ell^{**}(s)$ is $<_{\mathrm{I}}$-least among the set $\{\ell^{**}(u)\,;\, u \text{ is bad}\} \cap L_{\mathrm{I}}$, and let t witness that s is bad. By Lemma 5.2, $t \in L_{\mathrm{I}}$. Therefore, $\ell^{**}(s) <_{\mathrm{I}} \ell^{**}(t)$ by minimality of s. Furthermore, it is clear that the index of s is a successor ordinal $\alpha+1$.

Case 1: If $s \in M_{\mathrm{I}}$, then we are in case **(I-I)**. Therefore, there is some $j \in \omega$ such that $\ell^{**}(s^\frown\langle j\rangle) \in L_{\mathrm{I}}$, so $\ell^{**}(s^\frown\langle j\rangle) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha} = \ell^{**}(s)$. Because s and t are A-bisimilar, there is k such that $s^\frown\langle j\rangle$ and $t^\frown\langle k\rangle$ are A-bismilar. But now the minimality of s implies that $\ell^{**}(s^\frown\langle j\rangle) = \ell^{**}(t^\frown\langle k\rangle)$, so $\ell^{**}(t^\frown\langle k\rangle) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha}$. By **(I-I)** we have that $\ell^{**}(t) \leq z_{\mathrm{I}}^{\alpha} = \ell^{\alpha+1}(s)$, contradicting the choice of t.

Case 2: If $s \in M_{\mathrm{II}}$, then we are in case **(II-I)**, so for all $j \in \omega$, $\ell^{**}(s^\frown\langle j\rangle) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha}$. Again, by minimality of s, it follows that for all $k \in \omega$, $\ell^{**}(t^\frown\langle k\rangle) <_{\mathrm{I}} z_{\mathrm{I}}^{\alpha}$. Therefore, $\ell^{**}(t) \leq_{\mathrm{I}} z_{\mathrm{I}}^{\alpha} = \ell^{**}(s)$, contradicting the choice of t. □

6. Games with payoffs in the Hausdorff hierarchy

We shall use the general Gale-Stewart construction of Section 5 to prove that sets in the Hausdorff difference hierarchy have labellings which respect bisimilarity.

Let $A \in \alpha\text{-}\mathbf{\Sigma}^0_1$ and let $\langle C_\gamma\,;\,\gamma < \alpha\rangle$ be the canonical presentation of A. We shall construct an increasing sequence $\langle \ell_\gamma\,;\,\gamma \leq \alpha\rangle$ of labellings which are Gale-Stewart closed.

For each $\gamma \leq \alpha$, we introduce two sets of labels $Z^\gamma_{\mathrm{I}} := \{z^{\gamma,\eta}_{\mathrm{I}}\,;\,\eta < \omega_1\}$ and $Z^\gamma_{\mathrm{II}} := \{z^{\gamma,\eta}_{\mathrm{II}}\,;\,\eta < \omega_1\}$. In addition to that, we add two special labels $z^{\gamma,*}_{\mathrm{I}}$ and $z^{\gamma,*}_{\mathrm{II}}$ for each ordinal $\gamma \leq \alpha$.

Let

$$L_{\mathrm{I}} := \bigcup_{\gamma\leq\alpha} Z^\gamma_{\mathrm{I}} \cup \{z^{\gamma,*}_{\mathrm{I}}\} \text{ and } L_{\mathrm{II}} := \bigcup_{\gamma\leq\alpha} Z^\gamma_{\mathrm{II}} \cup \{z^{\gamma,*}_{\mathrm{II}}\} \cup \{\bullet\}$$

with

$$z^{\gamma,\eta}_{\mathrm{I}} <_{\mathrm{I}} z^{\gamma',\eta'}_{\mathrm{I}} \text{ if and only if } \gamma < \gamma' \vee (\gamma = \gamma' \,\&\, \eta < \eta'),$$

$$z^{\gamma,\eta}_{\mathrm{I}} <_{\mathrm{I}} z^{\gamma',*}_{\mathrm{I}} <_{\mathrm{I}} z^{\gamma',\eta}_{\mathrm{I}} \text{ for all } \gamma < \gamma' \text{ and all } \eta < \omega_1,$$

$$z^{\gamma,\eta}_{\mathrm{II}} <_{\mathrm{I}} z^{\gamma',\eta'}_{\mathrm{II}} \text{ if and only if } \gamma < \gamma' \vee (\gamma = \gamma' \,\&\, \eta < \eta'), \text{ and}$$

$$z^{\gamma,\eta}_{\mathrm{II}} <_{\mathrm{II}} z^{\gamma',*}_{\mathrm{II}} <_{\mathrm{II}} z^{\gamma',\eta}_{\mathrm{II}} \text{ for all } \gamma < \gamma' \text{ and all } \eta < \omega_1,$$

$$z^{\gamma,\eta}_{\mathrm{II}} <_{\mathrm{II}} \bullet \text{ for all } \gamma \text{ and } \eta.$$

Let $\ell_0 := \varnothing$. Now suppose that some ordinal $\gamma < \alpha$ is given and that ℓ_ξ has been constructed for all $\xi < \gamma$.

Limit Case: If γ is a limit ordinal, let $\ell^*_\gamma := \bigcup_{\xi<\gamma} \ell_\xi$, and then $\ell_\gamma := \mathrm{GSC}(\ell^*_\gamma, Z^\gamma_{\mathrm{I}}, Z^\gamma_{\mathrm{II}})$.

Successor Case: If $\gamma = \xi + 1$ is a successor, then define ℓ^*_γ as follows:

$$\ell^*_\gamma(s) := \begin{cases} \ell_\xi(s) & \text{if } s \in \mathrm{dom}(\ell_\xi), \\ z^{\gamma,*}_{\mathrm{I}} & \text{if } s \notin \mathrm{dom}(\ell_\xi) \,\&\, [s] \subseteq A_\xi \,\& \\ & \quad \xi \text{ and } \alpha \text{ have different parity, or} \\ z^{\gamma,*}_{\mathrm{II}} & \text{if } s \notin \mathrm{dom}(\ell_\xi) \,\&\, [s] \subseteq A_\xi \,\& \\ & \quad \xi \text{ and } \alpha \text{ have the same parity.} \end{cases}$$

Then set $\ell_\gamma := \mathrm{GSC}(\ell^*_\gamma, Z^\gamma_{\mathrm{I}}, Z^\gamma_{\mathrm{II}})$.

This procedure produces a partial labelling ℓ_α. As in the proof of Theorem 5.3, we totalize ℓ_α by setting

$$\ell(s) := \begin{cases} \ell_\alpha(s) & \text{if } s \in \mathrm{dom}(\ell_\alpha) \\ \bullet & \text{otherwise.} \end{cases}$$

Observation 6.1. *If $\ell(s) = z_{\mathrm{I}}^{\gamma,\eta}$ or $\ell(s) = z_{\mathrm{II}}^{\gamma,\eta}$ then $s \in \mathrm{dom}(\ell_\gamma)\backslash\mathrm{dom}(\ell^*_\gamma)$. If $\ell(s) = z_{\mathrm{I}}^{\gamma,*}$ or $\ell(s) = z_{\mathrm{II}}^{\gamma,*}$ then $\gamma = \xi + 1$ and $s \in \mathrm{dom}(\ell^*_\gamma)\backslash\mathrm{dom}(\ell_\xi)$.*

Proof. This observation can easily be verified by checking the construction. □

Lemma 6.2. *The labelling ℓ is globally A-sound.*

Proof. The proof of soundness is analogous to the proof of soundness in Theorem 5.3. We begin with the case for $\ell(s) \in L_{\mathrm{I}}$. Let σ_s be a good s-strategy for Player I, and let τ_s be any s-strategy for Player II. It must be shown that $x = \sigma_s * \tau_s \in A$. The following fact can be proved as before, this time using Observation 6.1:

($*$) for any $n \geq \mathrm{lh}(s)$, if $\ell(x \upharpoonright n) = z_{\mathrm{I}}^{\gamma,\eta}$, then $\ell(x \upharpoonright (n+1)) <_{\mathrm{I}} \ell(x \upharpoonright n)$.

Furthermore, the following statement follows easily from the definitions:

($**$) for any $n \geq \mathrm{lh}(s)$, if $\ell(x \upharpoonright n) = z_{\mathrm{I}}^{\gamma,*}$, then $\ell(x \upharpoonright (n+1)) \leq_{\mathrm{I}} \ell(x \upharpoonright n)$

It follows by ($*$) and ($**$) that there exists a $\gamma = \eta + 1$ and an $N \geq \mathrm{lh}(s)$ such that for all $m \geq N$, $\ell(x \upharpoonright m) = z_{\mathrm{I}}^{\gamma,*}$, since otherwise there would exist an infinite strictly decreasing sequence of ordinals. Therefore, $x \in A$.

The case for $\ell(s) \in L_{\mathrm{II}}$ is similar, with a slight modification. Instead of (*) and (**), we get "for any $n \geq \mathrm{lh}(s)$, if $\ell(x \upharpoonright n) = z_{\mathrm{II}}^{\gamma,\eta}$, then $\ell(x \upharpoonright (n+1)) <_{\mathrm{II}} \ell(x \upharpoonright n)$" and "for any $n \geq \mathrm{lh}(s)$, if $\ell(x \upharpoonright n) = z_{\mathrm{II}}^{\gamma,*}$ or $\ell(x \upharpoonright n) = \bullet$, then $\ell(x \upharpoonright (n+1)) \leq_{\mathrm{II}} \ell(x \upharpoonright n)$", and can show that there is a $\gamma = \eta + 1$ and an $N \geq \mathrm{lh}(s)$ such that for all $m \geq N$, $\ell(x \upharpoonright m) = z_{\mathrm{I}}^{\gamma,*}$ or for all $m \geq N$, $\ell(x \upharpoonright m) = \bullet$. It follows that $x \notin A$. □

Before proceeding with the proof that ℓ respects A-bisimilarity, we need a lemma.

Lemma 6.3. *Let $\langle C_\beta\,; \beta < \alpha\rangle$ be the canonical presentation of a set $A \in \alpha$-$\mathbf{\Sigma}^0_1$. If $[s] \subseteq C_\beta$ and s and t are A-bisimilar then $[t] \subseteq C_\beta$.*

Proof. We give a proof for α odd, by induction on β. Using the fact that s and t are A-bisimilar, we may let $g : (t) \to (s)$ be defined as in the proof of Lemma 5.2. Let $\hat{g} : [t] \to [s]$ be defined by $\hat{g}(x) := \bigcup \{g(u)\,;\, t \subseteq u \subset x\}$.

Base case. Suppose $[s] \subseteq C_0$, so $[s] \subseteq A$. To show that $[t] \subseteq C_0$, it suffices to show that $[t] \subseteq A$. Let $y \in [t]$. Then $\hat{g}(y) \in [s] \subseteq A$, so $\hat{g}(y) \in A$ and thus $y \in A$.

Inductive case. We provide the proof for odd β. Suppose $[s] \subseteq C_\beta$, so $[s] \subseteq (\mathbb{N}^\omega \backslash A) \cup \bigcup_{\delta<\beta} C_\delta$. To show that $[t] \subseteq C_\beta$, it suffices to show that $[t] \subseteq (\omega^\omega \backslash A) \cup \bigcup_{\delta<\beta} C_\delta$. Let $y \in [t]$. Then $\hat{g}(y) \in [s]$, so one of the following two cases holds.

Case 1: $\hat{g}(y) \in \omega^\omega \backslash A$. Then it is immediate that $y \in \omega^\omega \backslash A$.

Case 2: $\hat{g}(y) \in \bigcup_{\delta<\beta} C_\delta$. Let $\delta < \beta$ such that $\hat{g}(y) \in C_\delta$. C_δ is open, so let $s' \supseteq s$ such that $\hat{g}(y) \in [s'] \subseteq C_\delta$. Let $u \in \omega^{<\omega}$, $t \subseteq u \subset y$, such $g(u) \supseteq s'$. We have that u and $g(u)$ are A-bisimilar and $[g(u)] \subseteq C_\delta$, so $[u] \subseteq C_\delta$ by induction. Therefore $y \in C_\delta$.

□

Lemma 6.4. *The labelling ℓ is combinatorial.*

Proof. As in the proof of Theorem 5.3, call a position s **bad** if there is a position t such that s and t are A-bisimilar and $\ell(s) \neq \ell(t)$. We show that there are no bad positions.

Case 1. Suppose there is a bad s such that $\ell(s) \in L_{\mathrm{I}}$. Choose s such that $\ell(s)$ is $<_{\mathrm{I}}$-least in the set $\{\ell(s)\,;\, s \text{ is bad }\} \cap L_{\mathrm{I}}$, and let t witness that s is bad. By Lemma 5.2, $\ell(t) \in L_{\mathrm{I}}$, so $\ell(s) <_{\mathrm{I}} \ell(t)$ by minimality of s.

Subcase 1a: $\ell(s) = z_{\mathrm{I}}^{\gamma,*}$. By Observation 6.1, $\gamma = \xi + 1$ and $s \in \mathrm{dom}(\ell^*_\gamma) \backslash \mathrm{dom}(\ell_\xi)$. Therefore $[s] \subseteq C_\xi$ and ξ and α have different parity. By Lemma 6.3, it follows that $[t] \subseteq C_\xi$ and therefore $\ell(t) \leq_{\mathrm{I}} z_{\mathrm{I}}^{\gamma,*}$, a contradiction.

Subcase 1b: $\ell(s) = z_{\mathrm{I}}^{\gamma,\eta}$. We argue as in the proof of Theorem 5.3.

Case 2. All the bad positions are II-labelled. Choose s such that $\ell(s)$ is $<_{\mathrm{II}}$-least in the set $\{\ell(s)\,;\, s \text{ is bad }\}$, and let t witness that s is bad. It must be the case that $\ell(s) \neq \bullet$, so we may argue as in Case 1.

□

Theorem 6.5. *The pointclass $\mathbf{\Delta}^0_2$ is the largest boldface pointclass in which every set has a globally sound combinatorial labelling.*

Proof. Lemmas 6.2 and 6.4 show that all sets in $\mathbf{\Delta}^0_2$ have a globally sound labelling that respects bisimilarity. Let $\mathbf{\Gamma}$ be any boldface pointclass containing $\mathbf{\Delta}^0_2$. Any boldface pointclass that is a proper superset of $\mathbf{\Delta}^0_2$ contains either all $\mathbf{\Sigma}^0_2$ sets or all $\mathbf{\Pi}^0_2$ sets. Let $A \in \mathbf{\Sigma}^0_2$ be given by Proposition 4.2, then either A or $\omega^\omega \setminus A$ witnesses that $\mathbf{\Gamma}$ has a set without the desired property. □

Appendix

In this appendix, we briefly situate our Theorem 6.5 within the tradition of proof-theoretic results on determinacy axioms.

Proof theorists are interested in determining the logical strength of weak determinacy statements. This is naturally connected to the question of complexity of strategies: Let $\Gamma \subseteq \Gamma^*$. If there is a Γ game whose winning strategies are not in Γ^*, then the set of all Γ^* sets is a model in which Γ determinacy fails. Consequently, the theory of this model is not enough to prove Γ determinacy. For example, for Σ^0_1 games, winning strategies are in general definable over $\mathbf{L}_{\omega_1^{\mathrm{CK}}}$, but not necessarily elements of it.

For the pointclasses in this paper, the corresponding reverse mathematics results are due to [Ste77, Tan90, Tan91]:[4]

Theorem A (Steel-Tanaka)**.** *In the base theory* ACA_0*, we can compute the proof-theoretic strength of (lightface) determinacy axioms as follows:*

- *The determinacy of all (lightface)* Σ^0_1 *games has the strength of* ATR_0*,*
- *the determinacy of all (lightface)* Δ^0_2 *games has the strength of* Π^1_1-TR_0 *(already* 2-Σ^0_1*-determinacy gives more strength than* ATR_0*), and*
- *the determinacy of all (lightface)* Σ^0_2 *games implies* Σ^1_1-MI *(the converse holds in the base theory* ATR_0*.*

The proof-theoretic systems ATR_0, Π^1_1-TR_0 and Σ^1_1-MI are quite different in strength, separating Σ^0_1 determinacy, Δ^0_2 determinacy and Σ^0_2 determinacy.

Our result Theorem 6.5 can also be seen as a computation of complexities of strategies, but our notion of complexity ("being combinatorial") does not distinguish between Σ^0_1 sets and Δ^0_2 sets. In fact, the proofs that the Gale-Stewart labelling for Σ^0_1 sets (Theorem 5.3) and its extension to Δ^0_2 sets (Lemma 6.4) are combinatorial are remarkably similar in their use of techniques, so from a set theoretic point of view the different strength of these determinacy axioms may come as a surprise.

[4]We do not define any of the subsystems of second order arithmetic here and point the reader to [Sim99].

The reason for this is similar to the difference between Proposition 3.2 and Proposition 3.4: In order to analyse an open game, it is enough to compute the label at $\varnothing$ and the labelling function does not have to exist as a completed object. In order to carry out the iteration in the Δ^0_2 case, the labellings in the iteration must exist as completed objects, so some extra strength is needed (as in Proposition 3.4, for example).

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