THEORIES WITH TYPE-FREE APPLICATION AND EXTENDED BAR INDUCTION

$$\utilde{\mathbf{T}}(\Finv) \geq \utilde{\mathbf{T}}$$

$$\utilde{\mathbf{APP}} + \mathbf{EAC} \succ \utilde{\mathbf{HA}}$$

$$\utilde{\mathbf{EL}}^{*} + \mathbf{EBI} \equiv_{ar} \utilde{\mathbf{ID}}_{1}$$

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krips repro meppel

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Aan mijn ouders,
broers en zuster

Aan Claire

CONTENTS

The three formulae on the cover are the main theorems of this thesis, rendered in symbolic notation. These theorems can be found in Ch.I, 4.12, Ch.III, 4.21 and Ch.IV, 5.9 respectively.

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INTRODUCTION

1. The main results of this thesis are on axiomatizations of parts of intuitionistic systems, i.e. on the relationships between certain formal systems based on intuitionistic logic. We do not discuss here in any detail the axiomatization of intuitionistic mathematics, but we shall attempt to give sufficient explanations so as to enable the reader without specialized logical knowledge to understand at least the general drift of our work.

   Before giving an outline of the contents of this thesis, we shall present some rough descriptions introducing intuitionistic arithmetic and elementary analysis, realizability, theories with generalized inductive definitions and systems with choice sequences, all of which play an important role in our outline. The reader may, if (s)he wishes, skip to the discussion on bar induction and consult these descriptions when needed.

   a) HA, intuitionistic arithmetic (or Heyting arithmetic), is similar to first order classical arithmetic (Peano arithmetic PA), except that the logic is intuitionistic. Its quantifiers range over N.

   b) EL, elementary analysis, is (roughly) obtained from HA by adding function variables $(a,b,c,\ldots)$ and function quantifiers for functions from N to N. EL* is a slight variant in which the function variables $a,b,c,\ldots$ are replaced by $\alpha,\beta,\gamma,\ldots$.

   c) The notation $\{\cdot\}(\cdot)$ (Kleene brackets) indicates partial recursive function application: $\{x\}(y) \simeq z$ means that the algorithm with code x applied to argument y is defined and yields z .

d) Kleene's realizability is an interpretation which makes systematically explicit the constructive reading of existence ($\exists$) and disjunction ($\vee$), using recursive functions. For example $\forall x \exists y A(x,y)$ is said to be realizable iff there is a recursive function φ with code z such that $A(x,\{z\}(x))$ holds for all x. For arithmetical A, the principle 'A is true iff A is realizable' can be axiomatized by a single schema ECT_0, i.e. we have

$$\underset{\sim}{HA} + ECT_0 \vdash A \leftrightarrow (A \text{ is realizable})$$

and

$$\underset{\sim}{HA} \vdash (ECT_0 \text{ is realizable}).$$

ECT_0 has the form ($\{z\}(x)\downarrow$ means $\{z\}(x)$ is defined)

$$ECT_0 \qquad \forall x(Ax \to \exists y Bxy) \to \exists z \forall x(Ax \to \{z\}(x)\downarrow \wedge B(x,\{z\}(x)));$$

here A is almost negative, i.e. A contains no $\vee$, and $\exists$ only in front of prime formulae.

e) $\underset{\sim}{ID}_1$ is an extension of $\underset{\sim}{HA}$ containing generalized inductive definitions, a typical example of which is the definition of the class $\mathcal{O}$ of the 'recursive ordinals' by Kleene and Church.

f) Finally, certain systems such as $\underset{\sim}{CS}$ will be mentioned: these are extensions of $\underset{\sim}{EL}$ with choice sequences $\alpha,\beta,\gamma,\ldots$ which intuitively may be thought of as ranging over choice sequences. This is expressed in $\underset{\sim}{CS}$ and related systems by the adoption of certain intuitionistic continuity axioms, such as

$$\forall \alpha \exists n A(\alpha,n) \to \forall \alpha \exists m \exists n \forall \beta \in \bar{\alpha} m \, A(\beta,n)$$

(if $\forall \alpha \exists n A(\alpha,n)$, then we can find for each α an n and an initial segment $\bar{\alpha} m$ of α such that for all β with initial segment $\bar{\alpha} m$ $A(\beta,n)$ holds).

Elimination of choice sequences is a method of translating statements of $\underset{\sim}{CS}$ into statements not involving choice variables, i.e. a translation into the 'choice-free' part of $\underset{\sim}{CS}$.

2. *Bar induction.*

Bar induction, implicit already in L.E.J. Brouwer's writings (e.g. [Br27]), is an axiom schema of intuitionistic analysis first formulated explicitly by S.C. Kleene in [KV65], in the following form (D for 'decidable'):

$$\mathrm{BI}_D \qquad \left.\begin{array}{l} \forall\alpha\exists x P(\bar{\alpha}x) \ \wedge \\ \forall n(Pn \vee \neg Pn) \ \wedge \\ \forall n(Pn \to Qn) \ \wedge \\ \forall n(\forall y Q(n*\hat{y}) \to Qn) \end{array}\right\} \to Q\langle\rangle.$$

Here one should think of the n as ranging over codes for finite sequences of natural numbers; $*$ is concatenation, $\hat{y}$ is short for the sequence $\langle y\rangle$ of length 1, an $\langle\rangle$ is the empty sequence.

By the first two premises, the set of sequences n such that $\forall m \succ n\, \neg Pm$ form a well-founded tree (which we think of as growing upwards); the third and fourth hypotheses say that Q holds at the top nodes of this tree and that, if Q holds for all immediate successors $n*\hat{y}$ of a node n, then Q holds for n itself; the conclusion then states that Q holds at the root $\langle\rangle$ of the tree.

Bar induction may be viewed as an induction principle over the 'universal tree' of all finite sequences, ordered by initial segment relation; it is closely related to transfinite induction.

A more general version is (M for monotone):

$$\mathrm{BI}_M \qquad \left.\begin{array}{l} \forall\alpha\exists x P(\bar{\alpha}x) \ \wedge \\ \forall nm(Pn \to P(n*m)) \ \wedge \\ \forall n(Pn \to Qn) \ \wedge \\ \forall n(\forall y Q(n*\hat{y}) \to Qn) \end{array}\right\} \to Q\langle\rangle.$$

BI_M can be reduced to BI_D on assumption of intuitionistic continuity axioms ([HK66]; see also [T77], p.1010sq.). By taking Q equal to P in BI_M, we get

$$\mathrm{BI} \qquad \left.\begin{array}{l} \forall\alpha\exists x P(\bar{\alpha}x) \ \wedge \\ \forall nm(Pn \to P(n*m)) \ \wedge \\ \forall n(\forall y P(n*\hat{y}) \to Pn) \end{array}\right\} \to P\langle\rangle.$$

As observed by R. Grayson [FH79], BI and BI_M are equivalent, since BI_M follows from BI by taking $Pn := \forall m Q(n*m)$ in BI. One may also consider a generalization, where the α range over some *subtree* of the universal tree. In this thesis, we shall consider trees T definable by (essentially) an arithmetical formula, i.e. not containing sequence variables. If T is such a tree (i.e. $T = \{x|A(x)\}$, with no sequence variables in A), then we write $\alpha \in \overline{T}$ for $\forall n(\bar{\alpha}n \in T)$. Thus we obtain the schema EBI:

$$\text{EBI} \qquad \left.\begin{array}{l} \forall\alpha\in\overline{T}\exists x P(\bar{\alpha}x) \wedge \\ \forall nm(n*m\in T \wedge Pn \to P(n*m)) \wedge \\ \forall n \in T(\forall y(n*\hat{y}\in T \to P(n*\hat{y})) \to Pn) \end{array}\right\} \to P\langle\rangle .$$

Classically, EBI is easily seen to follow from BI_D: put $Pn := Qn := (Pn \vee \neg(n\in T))$ in BI_D. Intuitionistically, this is by no means obvious.

Before discussing results on BI and EBI, we shall introduce some notation. We call a theory $\underset{\sim}{T}_2$ extending $\underset{\sim}{T}_1$ *conservative over* $\underset{\sim}{T}_1$ [w.r.t. the set S of formulae] if

$$\text{for all A } [\in S]: \quad \underset{\sim}{T}_2 \vdash A \Rightarrow \underset{\sim}{T}_1 \vdash A \quad .$$

Notation: $\underset{\sim}{T}_2 \succ \underset{\sim}{T}_1$ $[\underset{\sim}{T}_2 \succ_S \underset{\sim}{T}_1]$. For $\succ_{L(HA)}$ we shall write $\succ_{ar}$. If $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ prove the same arithmetical theorems, we say that they are *arithmetically equivalent* and write $\underset{\sim}{T}_1 \equiv_{ar} \underset{\sim}{T}_2$. If $\underset{\sim}{T}_1, \underset{\sim}{T}_2$ only prove the same *negative* (i.e. $\vee$-, $\exists$-free) arithmetical theorems, we write $T_1 \equiv_{ar-} T_2$.

From the work done by Troelstra [T80], it follows that BI and EBI have the same proof-theoretic strength. This is done by proving

(1) $\qquad \underset{\sim\sim}{EL}^* + EBI \equiv_{ar-} \underset{\sim\sim}{ID}_1$;

combining this with $\underset{\sim\sim}{EL}^* + BI \equiv_{ar} \underset{\sim\sim\sim}{IDB}$ ([KT70]; $\underset{\sim\sim\sim}{IDB} = \underset{\sim\sim}{EL} +$ inductively defined neighbourhood functions) and §3.6 of [T80] yields the result. The principal goal of this thesis is to show that we even have

(2) $\qquad \underset{\sim\sim}{EL}^* + EBI \equiv_{ar} \underset{\sim\sim}{ID}_1$,

i.e. *all* arithmetical consequences of EBI hold in $\underset{\sim\sim}{ID}_1$ and vice versa.

3. *Outline of contents and description of methods.*

In the proof of (1) we can distinguish the following steps:

i) EBI is reduced to EBI*, i.e. EBI restricted to trees of the form $\{x \mid \forall i < \mathrm{lth}(x)\ (x)_i \in A\}$. This requires an axiom of partial choice (see 2.5, 2.6 of [T80]), which is derivable from ECT$_0$.

ii) EBI* is reduced to EBI**, i.e. EBI* restricted to trees of the form $\{x \mid \forall i < \mathrm{lth}(x)\ (x)_i \in A\}$, A almost negative. Here ECT$_0$ is needed.

iii) A theory $\mathbf{CS}^*$ is defined, in which EBI** holds.

iv) By an elimination translation, $\mathbf{CS}^* + \mathrm{ECT}_0$ is interpreted in $\mathbf{IDB}^* + \mathrm{ECT}_0$.

v) Using realizability and a result of Sieg on theories of inductive definitions, it is shown that $\mathbf{IDB} + \mathrm{ECT}_0 \equiv_{\mathrm{ar-}} \mathbf{ID}_1$.

If we wish to prove (2) by a sequence of steps analogous to (i) - (v), it seems that it might be useful to have a theory $\mathbf{T}$ containing a choice principle C comparable with ECT$_0$, and which is not merely proof-theoretically equivalent to, but even *conservative* over $\mathbf{HA}$. An example of this is the result by Goodman [Go76]:

$$(3) \qquad \mathbf{HA}^{\omega} + \mathrm{AC} \succ\!\!- \mathbf{HA};$$

here $\mathbf{HA}^{\omega}$ is an extension of $\mathbf{HA}$ with functionals of higher type, and AC is an axiom of choice for all higher types in $\mathbf{HA}^{\omega}$. However, AC is not strong enough to replace ECT$_0$ in the steps (i) and (ii).

In [Be79], M. Beeson gave a proof for (3) using generalized realizability and forcing (the proof is not essentially different from Goodman's proof). Inspection of Beeson's proof shows that in fact all generalized-realizable arithmetical formulae are provable in $\mathbf{HA}$, which suggests that it is possible to find a stronger choice principle (e.g. axiomatizing the realizability Beeson uses) which is still realizable.

A.S. Troelstra suggested the following approach to prove (2): take a theory with an abstract notion of application (in the sense of Feferman's theories in [Fe75], [Fe79]), consider abstract realizability for these theories, find a choice principle axiomatizing it and prove a result analogous to (3) using the Goodman-Beeson method. Then extend that theory to one like

CS* in [T80] which contains EL* + EBI, reduce this theory by means of an elimination translation and show that the resulting theory is arithmetically equivalent to ID_1. Troelstra also suggested to consider a formulation of Feferman's theories in which compound terms are no longer abbreviations, but really belong to the language itself.

The reason why Feferman did not admit compound terms in the language of his formal systems lies probably in the fact that the application is intended to be an abstract version of the so-called Kleene-bracket-application $\{\cdot\}(\cdot)$, which is essentially *partial*. So allowing compound terms yields partial terms - terms which do not automatically refer to existing objects, and for this no provisions have been made in ordinary intuitionistic predicate logic.

A practical way to deal with partial terms and objects is to add an *existence predicate* E to the language, with 'τ exists' or 'τ refers to an existing object' as intended meaning for $E\tau$. This idea is worked out by D.S. Scott in [Sc79]. In this article, he also shows that description terms (terms τ_A signifying 'the unique object satisfying A') can be treated very elegantly in systems with an existence predicate. In chapter I, we discuss descriptions, give a general definition of description operators with which partial functions can be formed and consider the consequences of adding such operators to several logics and the theories based on them. In particular, we give a syntactical proof that adding function symbols for definable partial functions is conservative, also for systems based on intuitionistic logic.

Our investigations of Feferman's systems and the existence predicate led us to the definition of APP^E, a theory with partial application and induction over N. APP^E is a conservative extension of both HA and EL: this makes it appropriate for our purpose. However, when looking at term models for APP^E, we discovered that adding the axiom $\forall xy(Exy)$ (i.e. application is *total*) is conservative for arithmetical formulae. Therefore we defined the theory APP with *total* application (which permits us to drop all references to E). APP is conservative over HA and our starting point for the study of EBI. All this can be found in chapter II.

The definition of realizability for APP is quite straightforward: it is an abstract version of Kleene's realizability for HA. As it is well-known that Kleene's realizability is axiomatized by ECT_0, it will not be a surprise that the realizability of APP is axiomatized by an abstract version of ECT_0, which we call EAC:

EAC $\quad \forall x(Ax \to \exists y B(x,y)) \to \exists f \forall x(Ax \to B(x,fx))$

where A is a negative formula (i.e. contains no $\vee$ or $\exists$).

To show that $\underset{\sim}{APP} + EAC \succ \underset{\sim}{HA}$, we developed our variant of the Goodman-Beeson method to prove (3): we add Hilbert's ε-symbol (a sort of Skolem function) to $\underset{\sim}{APP}$ which makes all arithmetical theorems of $\underset{\sim}{APP} + EAC$ derivable, and use forcing to make the axioms governing ε true.

For an extension of this conservation result to extensions of $\underset{\sim}{APP}$ with inductive definitions, the soundness of both realizability and forcing w.r.t. these extensions is required. It appears that $\underset{\sim}{APP}$ admits a perspicuous treatment of this.

In a digression we show that the method we used for the conservation result on $\underset{\sim}{APP} + EAC$ can also be applied to show $\underset{\sim}{ML}_0 \succ \underset{\sim}{HA}$: $\underset{\sim}{ML}_0$ is the basic part of Martin-Löf's extensional type theory. The natural interpretation of $\underset{\sim}{ML}_0$ in $\underset{\sim}{APP}$ corresponds to an extensional realizability $\underline{e}$, and $\underset{\sim}{ML}_0 \succ \underset{\sim}{HA}$ is obtained via Hilbert's ε and forcing. Unfortunately, we have not found an axiomatization of $\underline{e}$: this is due to the fact that, contrary to ordinary realizability, $\underline{e}$ is not idempotent. This digression on $\underset{\sim}{ML}_0$ ends chapter III.

Now that we know that $\underset{\sim}{APP} + EAC$ is conservative over $\underset{\sim}{HA}$, we are ready for the investigation of EBI. To $\underset{\sim}{APP} + EAC$ we add choice sequences, variables for trees and inductively defined functionals: the result is a theory $\underset{\sim}{T}_1^*$ in which EBI holds. In a number of steps we reduce $\underset{\sim}{T}_1^*$ to $\underset{\sim}{ID}_1$. An important step (corresponding with (iv) above) is done by means of an interpretation which has two equivalent formulations: elimination translation and forcing over a site, i.e. a category with a Grothendieck topology on it. The category involved consists of trees, with the inductively defined functionals as morphisms. As in [KT70], where the elimination translation for $\underset{\sim}{CS}$ is treated in extenso, the soundness proof relies on several closure properties of the set(s) of inductively defined functionals.

The proof method used in all chapters is the method of *interpretations*. A typical situation is: there are two theories $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ with a translation $*$ of formulae of $\underset{\sim}{T}_1$ in formulae of $\underset{\sim}{T}_2$. Now if $*$ is sound, i.e. if

$$\underset{\sim}{T}_1 \vdash A \Rightarrow \underset{\sim}{T}_2 \vdash A^*,$$

we call * an interpretation of $\underset{\sim}{T}_1$ in $\underset{\sim}{T}_2$. If also $\underset{\sim}{T}_1 \supset \underset{\sim}{T}_2$ (i.e. $\underset{\sim}{T}_2 \vdash A \Rightarrow$ $\Rightarrow \underset{\sim}{T}_1 \vdash A$) and if $S = \{A \mid \underset{\sim}{T}_2 \vdash A^* \to A\}$, then we have

$$\underset{\sim}{T}_1 \succ_S \underset{\sim}{T}_2$$

The advantage of the method by interpretation is that the proofs are usually obviously constructive. Often conservation results can also be obtained by model-theoretic methods; but then the reasoning is not always obviously constructive. Forcing as treated here may be seen as a syntactic version of a semantic method; the formalization (i.e. the transformation into a syntactic translation) is needed here to transform a model-construction into a result about formal systems.

4. A preliminary version of Chapter I appeared as Report 82-21, 'Descriptions in mathematical logic', of the Department of Mathematics, University of Amsterdam. Chapter I is also published in Studia Logica, under the same title.

CHAPTER I. DESCRIPTIONS IN MATHEMATICAL LOGIC

§1. Introduction.

1.1. A *description* is a definition of some object by means of a predicate satisfied by exactly one object. If $A(x)$ is such a predicate (i.e. if $\exists!xA(x)$), then we write $Ix.A(x)$ for the object described by $A(x)$. Ix binds the variable x and is called a *descriptor* (or description operator).

1.2. Description operators are almost as old as mathematical logic. Written as $[x\varepsilon]$, $\mathrm{I}\overset{,}{\varepsilon}$, $(℩x)$ or ι_x, they appear in Peano [P89], Frege [Fr93] , Whitehead & Russell [WR10] and Hilbert & Bernays [HB34]. All these authors discuss the well-known problematic aspect of descriptions: what to do with $Ix.A(x)$ if $\exists!xA(x)$ is not (yet) known? We present the three main solutions.

A) Admit $Ix.A(x)$ as a term only in case $\vdash\exists!xA(x)$; this restrictive solution is adopted by Hilbert & Bernays and by Kleene [Kl52].

B) Let $Ix.A(x)$ be the unique x such that $A(x)$ if $\exists!xA(x)$, and something else otherwise. This is the solution of Peano and Frege, also of Bernays [BF58], Quine [Q63] and Scott [Sc67].

C) Explain $Ix.A(x)$ as a 'figure of speech' by giving a contextual definition in which $B(Ix.A(x))$ is replaced by $\exists y(\forall x(A(x)\leftrightarrow x=y)\wedge B(y))$. This approach we find in Whitehead & Russell and in Scott's [Sc79].

1.3. Outline of the rest of this chapter.

In §2 we discuss the cases A, B, C and introduce *function descriptors* (2.6) which slightly generalize Ix. The last three sections are devoted to Scott's variant of C : §3 contains two versions of his logic with existence predicate as described in [Sc79], in §4 we prove that adding function descriptors to a theory based on any of these logics yields a definitional extension, and finally in §5 we consider theories with function quantifiers.

§2. How to handle Ix.A(x).

2.1. Solution A) of 1.2 is of course very safe, but it has the following disadvantages:

i) as $\underset{\sim}{T} \vdash A$ is undecidable for most theories $\underset{\sim}{T}$, we are unable to decide generally whether some expression containing Ix.A is a term (there is a trivial but unelegant solution for theories with a decidable proof-predicate: index Ix.A with the code of a proof of $\exists ! x A$);

ii) it excludes descriptions Ix.A(x) which exist conditionally, i.e. for which we have $\vdash B \to \exists ! x A(x)$.

A mitigated version of A) can be found in Stenlund [St73], [St75]. He presents a natural deduction system extended with prime formulae $t \in I$ (t a term-like expression) with the intended meaning 't is a term' (i.e. t refers to an object), and the rule $\exists ! x A(x) \Rightarrow Ix.A(x) \in I$.

2.2. Solution B) can be rendered by

$$(1) \qquad Ix.A(x) = \begin{cases} \text{the } x \text{ satisfying } A(x) \text{ if } \exists ! x A(x); \\ \text{'something else' if } \neg \exists ! x A(x). \end{cases}$$

Frege [Fr93] and Peano [P89] choose something like $\{x | A(x)\}$ for 'something else', Quine [Q63] works with $\emptyset$, and Scott [Sc67] takes some object * outside the intended domain. The method works rather well for classical theories, but yields an undesired side-effect in the intuitionistic case: as a consequence of (1) we get

$$(B \to \exists ! x A(x)) \to \exists x (B \to A(x))$$

which does not hold intuitionistically. We can sidestep this by weakening (1) to

$$\exists ! x A(x) \to A(Ix.A(x))$$

and restricting the axioms $\forall xA(x) \to A(t)$, $A(t) \to \exists xA(x)$ to I-free terms t : then the meaning of $Ix.A(x)$ is left unspecified as long as $\exists!xA(x)$ is not known. A more systematic elaboration of this idea is described in 2.4.

2.3. Whitehead & Russell [WR10] considered $B(Ix.A(x))$ as an abbreviation of

$$(2) \qquad \exists!xA(x) \wedge \exists y(A(y) \wedge B(y)).$$

As it stands, this is ambiguous, for e.g. $\neg B(Ix.A(x))$ can mean $\neg(\exists!xA(x) \wedge \exists y(A(y) \wedge B(y)))$ or $\exists!xA(x) \wedge \exists y(A(y) \wedge \neg B(y))$, and these formulae are not equivalent. Therefore Whitehead & Russell required the *scope* of a description $Ix.A(x)$ be indicated: this is the context B for which $Ix.A(x)$ is explained as in (2). So we can axiomatize $Ix.A(x)$ by

$$B(Ix.A(x)) \leftrightarrow (\exists!xA(x) \wedge \exists y(A(y) \wedge B(y)))$$

if B is the scope of $Ix.A(x)$.

2.4. A nice and elegant variant of this approach is given by Scott in [Sc79]. He introduces a logical system equipped with a unary predicate E to build formulae Et with the intended meaning ' t exists'; quantification is allowed only over existing terms. Scott's description axiom reads

$$\forall y(y=Ix.A(x) \leftrightarrow \forall x(A(x) \leftrightarrow x=y)).$$

The concept of scope is not needed anymore, for instead of 'the scope of $Ix.A(x)$ is B ' we now can write $B(Ix.A(x)) \wedge E\,Ix.A(x)$.

2.5. Scott describes an elimination translation for descriptions and sketches a proof of the conservativity of adding a descriptor to a theory based on E-logic (logic with predicate E), thereby generalizing the results in [HB34], [Kl52], [St75]. Scott's proof is semantical, based on two facts:

1^o) a completeness proof for E-logic, e.g. relative to Kripke-semantics; the models obtained are Ω-structures for a complete Heyting algebra Ω;

2^o) the construction of a sheaf-completion (sheafification) of the Ω-structure.

In this proof, (2^o) is constructive, but (1^o) not, since the completeness proofs for Kripke-semantics are classical. However, as pointed out to us by A.S. Troelstra, this non-constructive feature can be removed as follows by the use of a more general notion of model:

3^o) first give a completeness proof via the Lindenbaum-algebra construction, for models over a Heyting algebra Δ which is not necessarily closed;

4^o) then transform the model into a model over a complete Heyting algebra Ω by using a constructive method for embedding any Heyting algebra Δ into a complete Heyting algebra Ω preserving the operators $\wedge$, $\vee$, $\rightarrow$, $\perp$ *and* all already existing sup's and inf's (such a method has been given, independently, by R.J. Grayson and I. Moerdijk).

However, this method as it stands is certainly non-elementary: (4^o) in particular uses second order logic with comprehension. Another way of constructivizing the semantical argument as sketched by Scott would be the formalization of the completeness argument in a suitable classical system conservative over the corresponding intuitionistic system for Π^0_2 - sentences; see Smoryński's paper [Sm82].

No doubt this second method can be made to work, but it is very indirect. And it may well be that a closer analysis of the constructive semantical proof outlined above would show us that the non-elementary character of this proof was, proof-theoretically, only apparent. Nevertheless we think it worthwhile to give here an easy and straightforward syntactical argument, which can be formalized in primitive recursive arithmetic.

2.6. Let $A = A(\vec{x},y)$. Sometimes one does not only want the *object* $\mathrm{I}y.A$, but also the (partial) *function* which maps every $\vec{x}$ onto the unique y such that $A(\vec{x},y)$ if this y exists. For this purpose we introduce the *function descriptors* $\exists y(\vec{x})$ which bind the variables y, $\vec{x}$ and are axiomatized by

$$\exists\mathrm{AX} \qquad \forall\vec{x}z(z = (\exists y(\vec{x}).A)\vec{x} \leftrightarrow \forall y(A(\vec{x},y) \leftrightarrow y=z)).$$

Another approach is to add λ - abstraction, axiomatized by

$$\forall x\vec{z}(x = (\lambda\vec{y}.t)\vec{z} \leftrightarrow x = t[\vec{y}:=\vec{z}]):$$

then $\exists y(\vec{x}).A$ can be defined by $\lambda\vec{x}.(\mathrm{I}y.A)$. Besides, λ - abstraction is definable from $\exists y(\vec{x})$ by taking $\lambda\vec{x}.t := (\exists y(\vec{x}).t = y)$.

2.7. REMARK. It is not strictly necessary to use $\dashv$ instead of I, since we may write $Ix.A[\vec{y}:=\vec{t}]$ for $(\dashv x(\vec{y}).A)\vec{t}$; the same holds for adding λ - abstraction. However, working with $\dashv$ (as we shall do in the sequel) has the technical advantage that $\dashv x(\vec{y}).A$ contains no free variables.

§3. Logic with existence predicate.

We present two systems $\underset{\sim}{\mathrm{LE}}$, $\underset{\sim}{\mathrm{LE}}^-$ of intuitionistic predicate logic with existence predicate E, the second of which is equivalent to the version Scott introduced in [Sc79] (see 3.6). Instead of intuitionistic logic we might as well take classical or any intermediate logic. A generalisation to many-sorted logic is straightforward.

3.1. Our language contains predicate symbols E, =, ... (metavariable P) and function symbols (metavariable f). Building terms and formulae goes as usual. We write $\vec{t}$ for a (possibly empty) sequence of terms $t_1,\ldots,t_n$: $P\vec{t}$ stands for $P(t_1,\ldots,t_n)$, $f\vec{t}$ for $f(t_1,\ldots,t_n)$, $\vec{s}=\vec{t}$ for $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \wedge \top$ and $E\vec{t}$ for $Et_1 \wedge \ldots \wedge Et_n \wedge \top$. Besides the axioms and rules for intuitionistic propositional logic, we have

EAX $\quad Et \leftrightarrow \exists x(x=t)$ $\quad$ (x not in t)

=AX $\quad \forall x(x=x) \wedge \forall xyz(x=z \wedge y=z \rightarrow x=y)$

STR
$$\begin{cases} P\vec{t} \rightarrow E\vec{t} \\ Ef\vec{t} \rightarrow E\vec{t} \end{cases}$$

SUB
$$\begin{cases} P\vec{s} \wedge \vec{s}=\vec{t} \rightarrow P\vec{t} \\ Ef\vec{s} \wedge \vec{s}=\vec{t} \rightarrow f\vec{s}=f\vec{t} \end{cases}$$

∀AX $\quad \forall xA \rightarrow A[x:=y]$

∃AX $\quad A[x:=y] \rightarrow \exists xA$

∀-R $\quad \dfrac{A\rightarrow B}{A\rightarrow\forall xB}$ $\qquad$ ∃-R $\quad \dfrac{A\rightarrow B}{\exists xA\rightarrow B}$

(x not free in A) $\qquad$ (x not free in B)

The system thus defined we call $\underset{\sim}{\mathrm{LE}}$.

3.2. The weakening $\mathbf{LE}^-$ of $\mathbf{LE}$ is obtained by taking as quantifier axioms and rules:

$\forall\mathrm{AX}^-$	$\forall xA \wedge Ey \rightarrow A[x:=y]$
$\exists\mathrm{AX}^-$	$A[x:=y] \wedge Ey \rightarrow \exists xA$

$\forall\text{-R}^-$ $\dfrac{A \wedge Ex \rightarrow B}{A \rightarrow \forall xB}$ (x not free in A) $\qquad$ $\exists\text{-R}^-$ $\dfrac{A \wedge Ex \rightarrow B}{\exists xA \rightarrow B}$ (x not free in B)

3.3. LEMMA.

i) $\mathbf{LE} \vdash A \iff \mathbf{LE}^- \vdash \exists xEx \wedge E\vec{y} \rightarrow A$, *where* $\vec{y}$ *are the free variables in* A.

ii) *If* Q *is a prime formula and* x *occurs in* Q *but not in* t, *then* $\mathbf{LE}^- \vdash Q[x:=t] \leftrightarrow \exists x(Q \wedge x=t)$.

iii) *In* $\mathbf{LE}$, $\mathbf{LE}^-$ *we have* $\vdash A[x:=s] \wedge s=t \rightarrow A[x:=t]$, *provided no variables in* s, t *become bound in* A.

PROOF. i) $\Leftarrow$ is trivial, $\Rightarrow$ is proved with induction over the length of a proof of A in $\mathbf{LE}$ (use $\exists\text{-R}^-$ to eliminate Ez with z not in A from the antecedent).

ii) First show $Q[x:=t] \rightarrow Et$ (using STR), then prove $x=t \rightarrow (Q \leftrightarrow Q[x:=t])$ (using SUB); combining this with EAX gives the desired result.

iii) An easy formula induction. Use (ii) to deal with prime formulae. □

3.4. COROLLARY. *Quantification over existing terms is allowed, i.e. we have, in* $\mathbf{LE}$ *and* $\mathbf{LE}^-$:

$$\vdash \forall xA \wedge Et \rightarrow A[x:=t], \qquad \vdash A[x:=t] \wedge Et \rightarrow \exists xA.$$

PROOF. By EAX and AX^- we have $\mathbf{LE}^- \vdash Et \rightarrow \exists y(t=y \wedge (\forall xA \wedge Ey \rightarrow A[x:=y]))$; now apply 3.3.(iii). Similarly for $\exists x$. □

3.5. COROLLARY. *If we add* Et *for all terms* t *(or* $Ef\vec{x}$ *for all function symbols* f*) to* $\mathbf{LE}$, *we get full intuitionistic predicate logic.*

In view of 3.3.(i) we can say that $\mathbf{LE}$ is about inhabited domains, whereas the domain of $\mathbf{LE}^-$ is possibly empty. So $\mathbf{LE}^-$ is more general than $\mathbf{LE}$;

on the other hand, $\mathbf{LE}$ - with its slightly simpler formalism - is preferable as a base for mathematical theories, as these usually have an inhabited domain.

3.6. Scott's logic in [Sc79] ($\mathbf{SL}$ for short; we consider the variant with strictness axioms) has a somewhat different axiomatization, but the same theorems as $\mathbf{LE}^-$:

3.7. LEMMA. $\mathbf{SL} \vdash A \Longleftrightarrow \mathbf{LE}^- \vdash A$.

PROOF. An easy verification. The only non-trivial part is the demonstration that the rule of substitution $A \Rightarrow A[x:=t]$ of $\mathbf{SL}$ is a derived rule of $\mathbf{LE}^-$. □

§4. Conservation results.

In this section we prove that adding function descriptors (see 2.6) to a theory based on $\mathbf{LE}$ or $\mathbf{LE}^-$ yields a definitional extension (theorem 4.12).

4.1. DEFINITION. Let $\mathbf{T}_1$, $\mathbf{T}_2$ be two theories such that $\mathbf{T}_1$ extends $\mathbf{T}_2$, i.e. the language of $\mathbf{T}_2$ is a sublanguage of $\mathbf{T}_1$ and all theorems of $\mathbf{T}_2$ are provable in $\mathbf{T}_1$. Then $\mathbf{T}_1$ is a *definitional extension* of $\mathbf{T}_2$ if there is a mapping $d\colon L(\mathbf{T}_1) \to L(\mathbf{T}_2)$ satisfying

i) d commutes with the logical operators;

ii) if A in the language of $\mathbf{T}_2$, then $\mathbf{T}_2 \vdash A \leftrightarrow d(A)$;

iii) $\mathbf{T}_1 \vdash A \leftrightarrow d(A)$;

iv) $\mathbf{T}_1 \vdash A \Rightarrow \mathbf{T}_2 \vdash d(A)$.

Notation: $\mathbf{T}_1 \geq_d \mathbf{T}_2$ or $\mathbf{T}_1 \geq \mathbf{T}_2$.

Note that $\geq$ is transitive, i.e. $\mathbf{T}_1 \geq_d \mathbf{T}_2 \geq_e \mathbf{T}_3$ implies $\mathbf{T}_1 \geq_{e \circ d} \mathbf{T}_3$.
By (ii), (iv) one has $\mathbf{T}_1 \geq \mathbf{T}_2 \Rightarrow \mathbf{T}_1$ conservative over $\mathbf{T}_2$.

4.2. The proof of theorem 4.12 contains the following steps.

a) First we add one function description $\exists y(\vec{x}).A(\vec{x},y)$ (φ for short) to $\mathbf{LE}$ (or $\mathbf{LE}^-$). For simplicity we assume $\vec{x} = x$, so φ is a one-place function.

b) We generalize (a) to: add φ to a *theory* based on $\mathbf{LE}$ (or $\mathbf{LE}^-$).

c) Then we repeat (b) a finite number of times to obtain the extension of a theory with the function descriptions $\varphi_1,\dots,\varphi_n$, where the defining formula A_j of φ_j only contains φ_i if $i < j$.

d) Finally we turn to the extension $\underset{\sim}{T}(\Finv)$ of a theory with function descriptors, and argue that any subtheory of $\underset{\sim}{T}(\Finv)$ with only finitely many function descriptions is isomorphic to some extension of $\underset{\sim}{T}$ as described under (c).

We successively prove that the extensions described in (a) - (d) are definitional.

4.3. Let $A = A(x,y)$ be a formula of $\underset{\sim}{\mathrm{LE}}$, containing no free variables besides x, y, nor the variable z. We define $\underset{\sim}{\mathrm{LE}}(A,\varphi)$ by adding to $\underset{\sim}{\mathrm{LE}}$ the function symbol φ, the axiom

$$\mathrm{AX}(A,\varphi) \qquad \forall xy(\forall z(A(x,z) \leftrightarrow y=z) \leftrightarrow \varphi x=y)$$

and extending the axioms and rules of $\underset{\sim}{\mathrm{LE}}$ with instances containing φ. So $\underset{\sim}{\mathrm{LE}}(A,\varphi)$ is $\underset{\sim}{\mathrm{LE}}$ plus a function φ which maps x onto the unique y such that $A(x,y)$ if this y exists, and is undefined otherwise. $\underset{\sim}{\mathrm{LE}}^-(A,\varphi)$ is defined similarly.

4.4. To show that $\underset{\sim}{\mathrm{LE}}(A,\varphi) \geq \underset{\sim}{\mathrm{LE}}$, $\underset{\sim}{\mathrm{LE}}^-(A,\varphi) \geq \underset{\sim}{\mathrm{LE}}^-$ hold, we define an interpretation * of $\underset{\sim}{\mathrm{LE}}(A,\varphi)$ into $\underset{\sim}{\mathrm{LE}}$. The effect of * is the elimination of φ by contextual definitions at the prime formula level.

We adopt the following conventions, extending those stated in 3.1. $\vec{x}$ stands for the (possibly empty) sequence of variables $x_1,\dots,x_n$; similarly for $\vec{y}$, $\vec{z}$, $\vec{u}$. $y_1,\dots,y_n$ is a fixed sequence of variables; they are called the y-variables. All formulae B of $\underset{\sim}{\mathrm{LE}}(A,\varphi)$ are supposed to be φ-indexed: this means that all occurrences of φ in B are indexed with positive integers in such a way that in any prime formula Q of B, the indices of occurrences of φ in Q are mutually different, and also different from the indices of the y-variables occurring in Q. So, in general, the φ-indexing of B is not unique: but it will be seen from the definition of * that B^* does not depend (except for renaming of bound variables) on the φ-indexing of B.

$\forall\vec{x}$, $\exists\vec{x}$, $\exists!\vec{x}$ are defined as follows:

$$\forall\vec{x}B \ := \forall x_1\forall x_2\dots\forall x_n B,$$

$$\exists\vec{x}B \ := \exists x_1\exists x_2\dots\exists x_n B,$$

$$\exists!\vec{x}B := \exists\vec{z}\forall\vec{x}(B \leftrightarrow \vec{x}=\vec{z}).$$

So $\exists!\vec{x}B$ means: there is exactly one sequence $x_1,\dots,x_n$ such that B holds. We state some properties of $\exists!\vec{x}$:

4.5. LEMMA.

i) $\exists!\vec{x}B \leftrightarrow \exists\vec{x}B \wedge \forall\vec{x}\vec{z}(B \wedge B[\vec{x}:=\vec{z}] \rightarrow \vec{x}=\vec{z})$.

ii) *Let* $\vec{x}'$ *be some permutation of* $\vec{x}$. *Then* $\exists!\vec{x}B \leftrightarrow \exists!\vec{x}'B$.

iii) *Let* $\vec{x}\vec{u}$ *be the concatenation of* $\vec{x}$ *and* $\vec{u}$. *If the* $\vec{u}$ *do not occur in* B *and the* $\vec{x}$ *not in* C*, then* $\exists!\vec{x}\vec{u}(B \wedge C) \leftrightarrow \exists!\vec{x}B \wedge \exists!\vec{u}C$.

iv) $\exists!\vec{x}B \rightarrow (\exists\vec{x}(B \wedge C) \wedge \exists\vec{x}(B \wedge D) \leftrightarrow \exists\vec{x}(B \wedge C \wedge D))$.

PROOF. Straightforward. For (iii) and (iv), use (i). □

4.6. DEFINITION. First two auxiliary definitions:

$\underline{t} \;= t$ if t φ-free

$\underline{\varphi_i t} = y_i$

$\underline{f\vec{t}} \;= f\underline{\vec{t}}$, where $\underline{\vec{t}}$ abbreviates $t_1,\dots,t_n$;

$\delta(t) \;= 0$ if t φ-free

$\delta(\varphi_i t) = \delta(t) + 1$

$\delta(f\vec{t}) \;= \delta(\vec{t}) = \max(\delta(t_1),\dots,\delta(t_n))$

$\delta(P\vec{t}) \;= \delta(\vec{t})$

$\delta(B\wedge C) = \delta(B\vee C) = \delta(B\rightarrow C) = \max(\delta(B),\delta(C))$

$\delta(\neg B) = \delta(\forall xB) = \delta(\exists xB) = \delta(B)$.

Now we simultaneously define ε and $*$:

$\varepsilon(t) \;= \top$ if t is φ-free

$\varepsilon(\varphi_i t) = A(t,y_i)^* \wedge Et^*$

$\varepsilon(f\vec{t}) \;= \varepsilon(\vec{t})$, where $\varepsilon(\vec{t})$ abbreviates $\varepsilon(t_1) \wedge \dots \wedge \varepsilon(t_n)$;

$(P\vec{t})^* \;= \exists!\vec{y}\varepsilon(t) \wedge \exists\vec{y}(\varepsilon(\vec{t}) \wedge P\underline{\vec{t}})$, where $\vec{y}$ is a sequence of y-variables satisfying: y_i in $\vec{y}$ iff φ_i occurs in $\vec{t}$.

$*$ commutes with all logical operators.

This definition looks circular at first sight, but with induction over $\delta(B)$ one can easily show that it is a good definition (A is φ-free, hence $\delta(A(t,y_i)) = \delta(Et) = \delta(\varphi_i t)-1)$.

4.7. FACTS.

i) B^* is φ-free;

ii) $B^* \leftrightarrow B$ if B φ-free;

iii) $(E\varphi_i t)^* \leftrightarrow \exists! y_i A(t,y_i)^* \wedge (Et)^*$;

iv) $(\varphi_i t{=}x)^* \leftrightarrow \forall y_i (A(t,y_i)^* \leftrightarrow x{=}y_i) \wedge (Et)^* \wedge Ex.$

4.8. LEMMA. $\underset{\sim}{\mathrm{LE}}(A,\varphi) \geq_* \underset{\sim}{\mathrm{LE}}$, $\underset{\sim}{\mathrm{LE}}^-(A,\varphi) \geq_* \underset{\sim}{\mathrm{LE}}^-$.

PROOF. By the definition of $*$ and 4.7.(ii), it suffices to show:

I) $\underset{\sim}{\mathrm{LE}}^-(A,\varphi) \vdash B \leftrightarrow B^*$;

II) $\underset{\sim}{\mathrm{LE}}(A,\varphi) \vdash B \Rightarrow \underset{\sim}{\mathrm{LE}} \vdash B^*$;

III) $\underset{\sim}{\mathrm{LE}}^-(A,\varphi) \vdash B \Rightarrow \underset{\sim}{\mathrm{LE}}^- \vdash B^*$.

In the proofs of I - III which follow, we make the following simplifying assumptions (without essential loss of generality): P and f are unary, $\vec{y}$ are the 'new' y-variables of $\underline{t}$, $\vec{y}'$ those of $\underline{s}$. We also write $A \Rightarrow B$ for: $A \rightarrow B$ derivable in the system under consideration; analogously for $\Longleftrightarrow$.

I): induction over $\delta(B)$.

$\delta(B) = 0$: then B is φ-free. Use 4.7.(ii).

$\delta(B) > 0$: first we show, for all t with $\delta(t) \leq \delta(B)$:

(1) $\underset{\sim}{\mathrm{LE}}^-(A,\varphi) \vdash t = x \leftrightarrow (t{=}x)^*$.

a) t a variable: trivial.

b) $t = fs$. Now $\varepsilon(t) = \varepsilon(s)$, $\underline{t} = f\underline{s}$ and we have

$$\begin{aligned} fs = x &\Longleftrightarrow \exists z(s{=}z \wedge fz{=}x) && \text{(by STR and EAX)}\\ &\Longleftrightarrow \exists z((s{=}z)^* \wedge fz{=}x) && \text{((1) for } t{:=}s)\\ &= \exists z(\exists!\vec{y}\varepsilon(s) \wedge \exists\vec{y}(\varepsilon(s) \wedge \underline{s}{=}z) \wedge fz{=}x)\\ &\Longleftrightarrow \exists!\vec{y}\varepsilon(s) \wedge \exists\vec{y}(\varepsilon(s) \wedge f\underline{s}{=}x)\\ &= \ (fs{=}x)^*. \end{aligned}$$

c) $t = \varphi_i s$: now $\delta(s) = \delta(Es) = \delta(A(s,y_i))$ and $\delta(s) < \delta(t) \le \delta(B)$, so

$$\begin{aligned}
\varphi_i s = x &\iff \forall y_i(A(s,y_i) \leftrightarrow x=y_i) \wedge Es \wedge Ex && \text{(by } AX(A,\varphi))\\
&\iff \forall y_i(A(s,y_i)^* \leftrightarrow x=y_i) \wedge Es^* \wedge Ex && \text{(ind. hyp.)}\\
&\iff (\varphi_i s = x)^* && \text{(by 4.7.(iv)).}
\end{aligned}$$

Now we continue the proof of $B \leftrightarrow B^*$.

B prime: assume $B = Pt$, x not in t. Now

$$\begin{aligned}
Pt &\iff \exists x(Px \wedge x=t) && \text{(by STR and EAX)}\\
&\iff \exists x(Px \wedge (x=t)^*) && \text{(by (1))}\\
&= \exists x(Px \wedge \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge x=\underline{t}))\\
&\iff \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge P\underline{t}) && \text{(by 3.3.(ii))}\\
&= (Pt)^*.
\end{aligned}$$

B not prime: trivial, for * commutes with all logical operators.

II) We only have to look at EAX, STR, SUB and $AX(A,\varphi)$, for * commutes with the logical operators.

EAX:

$$\begin{aligned}
Et^* &= \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge E\underline{t})\\
&\iff \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge \exists x\ x=\underline{t}) && \text{(by EAX)}\\
&\iff \exists x(\exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge x=\underline{t}))\\
&= (\exists x\ x=t)^*.
\end{aligned}$$

STR:

$$\begin{aligned}
Pt^* &= \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge P\underline{t})\\
&\Rightarrow \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge E\underline{t}) && \text{(by STR)}\\
&= (Et)^*.
\end{aligned}$$

The proof of $(Eft)^* \to (Et)^*$ is similar. $(E\varphi_i t)^* \to (Et)^*$ follows from 4.7.(iii).

$$\begin{aligned}
AX(A,\varphi)^* &= \forall xy(\forall z(A(x,z)^* \leftrightarrow (y=z)^*) \leftrightarrow (\varphi_i x=y)^*)\\
&\iff \forall xy(\forall z(A(x,z)^* \leftrightarrow y=z) \leftrightarrow \forall y_i(A(x,y_i)^* \leftrightarrow y=y_i))\\
&\qquad \text{(by 4.7.(iv)),}
\end{aligned}$$

and this is a tautology.

SUB:

$$\begin{aligned}
&(Ps)^* \wedge (s=t)^* =\\
&= \exists ! \vec{y}'\varepsilon(s) \wedge \exists \vec{y}'(\varepsilon(s) \wedge P\underline{s}) \wedge \exists ! \vec{y}\vec{y}'(\varepsilon(s) \wedge \varepsilon(t)) \wedge\\
&\qquad \wedge \exists \vec{y}\vec{y}'(\varepsilon(s) \wedge \varepsilon(t) \wedge \underline{s}=\underline{t})\\
&\Rightarrow \exists ! \vec{y}'\varepsilon(s) \wedge \exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}'(\varepsilon(s) \wedge P\underline{s} \wedge \exists \vec{y}(\varepsilon(t) \wedge \underline{s}=\underline{t}))\\
&\qquad \text{(by 4.5.(iii),(iv))}
\end{aligned}$$

$$\Rightarrow \exists!\vec{y}\varepsilon(t) \wedge \exists\vec{y}(\varepsilon(t) \wedge P\underline{t}) \qquad \text{(by SUB)}$$
$$= (Pt)^*.$$

$(Efs)^* \wedge (s=t)^* \rightarrow (fs=ft)^*$: analogously.

Let $\mathrm{SUB}(\varphi)$ be $E\varphi_i s \wedge s=t \rightarrow \varphi_i s = \varphi_j t$. Before showing $\underset{\sim}{\underline{\mathrm{LE}}} \vdash \mathrm{SUB}(\varphi)^*$, we observe that $Es \wedge s=t \rightarrow Et$ and $A(s,y_i) \wedge s=t \rightarrow A(t,y_i)$ are derivable in $\underset{\sim}{\underline{\mathrm{LE}}}(A,\varphi) - \mathrm{SUB}(\varphi)$ (inspection of the proof of 3.3.(iii); recall that $A(x,y)$ is φ-free), so their $*$-interpretation holds in $\underset{\sim}{\underline{\mathrm{LE}}}$ in virtue of this proof up to here. Now

$$\begin{aligned}
&(E\varphi_i s)^* \wedge (s=t)^* \Longleftrightarrow \\
\Longleftrightarrow\ & \exists!y_i(A(s,y_i))^* \wedge (Es)^* \wedge (s=t)^* && \text{(by 4.7.(iii))}\\
\Rightarrow\ & \exists!y_i(A(s,y_i))^* \wedge (Es)^* \wedge \exists!y_j A(t,y_j)^* \wedge (Et)^* \wedge \\
& \qquad \wedge \exists y_i y_j (A(s,y_i)^* \wedge A(t,y_j)^* \wedge y_i=y_j) \\
\Longleftrightarrow\ & \exists!y_i\varepsilon(\varphi_i s) \wedge \exists!y_j\varepsilon(\varphi_j t) \wedge \\
& \qquad \wedge \exists y_i y_j (A(s,y_i)^* \wedge (Es)^* \wedge A(t,y_j)^* \wedge (Et)^* \wedge y_i=y_j) \\
=\ & (\varphi_i s = \varphi_j t)^*.
\end{aligned}$$

III): completely similar. Use $(Ex)^* = Ex$ to deal with the quantifier rules and axioms of $\underset{\sim}{\underline{\mathrm{LE}}}^-$. □

Now we consider theories.

4.9. DEFINITION. Let $\underset{\sim}{T}$ be a theory based on $\underset{\sim}{\underline{\mathrm{LE}}}$, $A = A(x,y)$ a formula of $\underset{\sim}{T}$ with at most x,y free. Then the extension $\underset{\sim}{T}(A,\varphi)$ of $\underset{\sim}{T}$ is formed by adding to $\underset{\sim}{\underline{\mathrm{LE}}}(A,\varphi)$ all axioms of $\underset{\sim}{T}$ and all instances containing φ of axiom schemes $(\mathcal{A}(A_1,\ldots,A_n)$ *for all* $A_1,\ldots,A_n)$ of $\underset{\sim}{T}$. Similarly for theories based on $\underset{\sim}{\underline{\mathrm{LE}}}^-$.

4.10. LEMMA. $\underset{\sim}{T}(A,\varphi) \geq_* \underset{\sim}{T}$ *(*$\underset{\sim}{T}$ *based on* $\underset{\sim}{\underline{\mathrm{LE}}}$ *or* $\underset{\sim}{\underline{\mathrm{LE}}}^-$*).*

PROOF. Follows directly from lemma 4.8 and from $\mathcal{A}(A_1,\ldots,A_n)^* = \mathcal{A}(A_1^*,\ldots,A_n^*)$. □

4.11. Generalisations.

a) $A = A(x_1,\ldots,x_n,y)$: now φ is an n-place function. The treatment is completely analogous.

b) We can extend $\underset{\sim}{T}' = \underset{\sim}{T}(A,\varphi)$ to $\underset{\sim}{T}'' = \underset{\sim}{T}'(B,\psi)$: here $B = B(\vec{x},y)$ is a formula of $\underset{\sim}{T}'$ and possibly contains φ. Now $\underset{\sim}{T}'' \geq \underset{\sim}{T}' \geq \underset{\sim}{T}$, so $\underset{\sim}{T}'' \geq \underset{\sim}{T}$. This can be repeated a finite number of times to obtain $\underset{\sim}{T}^n = \underset{\sim}{T}(A_1,\ldots,A_n;\ \varphi_1,\ldots,\varphi_n)$, where A_i contains φ_j only if $i > j$;

we then have $\underset{\sim}{T}^{n} \geq_{d} \underset{\sim}{T}$, where d is the composition of n-1 interpretations * as defined in 4.6.

c) More generally, we can add function descriptors $\text{Ǝ}y(\vec{x})$ to $\underset{\sim}{T}$ (see 2.6). The extension $\underset{\sim}{T}(\text{Ǝ})$ is defined in the same way as $\underset{\sim}{T}(A,\varphi)$ in 4.9, and we have

4.12. THEOREM. $\underset{\sim}{T}(\text{Ǝ})$ *is a definitional extension of* $\underset{\sim}{T}$.

PROOF. We use *subtheory* for a restriction of $\underset{\sim}{T}(\text{Ǝ})$ (i.e. its axioms and rules) to some extension of the language of $\underset{\sim}{T}$ with only finitely many function descriptions.

Let $\underset{\sim}{T}_{o}$ be such a subtheory, and let $\text{Ǝ}y(\vec{x}).A_{1},\ldots,\text{Ǝ}y(\vec{x}).A_{n}$ be the function descriptions occurring in $\underset{\sim}{T}_{o}$, ordered according to increasing length (i.e. number of symbols); so $\text{Ǝ}y(\vec{x}).A_{i}$ occurs in A_{j} only if $j > i$.

One straightforwardly verifies that $\underset{\sim}{T}_{o}$ is isomorphic to $\underset{\sim}{T}^{n}$ (as described in 4.11.(b)) by the mapping e induced by $\text{Ǝ}y(x).A_{i} \longmapsto \varphi_{i}$. Now put $c = d \circ e$ (the d from 4.11.(b)) and we get $\underset{\sim}{T}_{o} \geq_{c} \underset{\sim}{T}$.

We also observe that, if the formula B of $T(\text{Ǝ})$ belongs to (the language of) $\underset{\sim}{T}_{o}$ and to another subtheory $\underset{\sim}{T}'_{o} \geq_{c'} \underset{\sim}{T}$ (c' defined in the same way as c), then c(B) and c'(B) are equal modulo renaming of bound variables.

Now we can define an interpretation of $\underset{\sim}{T}(\text{Ǝ})$ into $\underset{\sim}{T}$ by $B \longmapsto c(B)$, where c is the mapping (as described above) of the smallest subtheory containing B into $\underset{\sim}{T}$. It is easily verified that this interpretation satisfies (i) - (iv) of definition 4.1. □

§5. Extensions to systems with function variables.

This final section is devoted to extending theorem 4.12 to theories with quantification over functions. We distinguish two variants, depending on whether the function quantifiers range over total or partial functions. For simplicity only one-place functions are considered.

5.1. We extend the language with function variables $\alpha,\beta,\ldots$. The natural rules and axioms for quantification are:

$\forall_{F}AX$ $\quad\forall\alpha A \rightarrow A[\alpha:=\beta]$

$\exists_{F}AX$ $\quad A[\alpha:=\beta] \rightarrow \exists\alpha A$

$\forall_F$-R $\dfrac{A \to B}{A \to \forall\alpha B}$ $\exists_F$-R $\dfrac{A \to B}{\exists\alpha A \to B}$

(α not free in A) (α not free in B)

The two theories $\mathbf{LEF}_T$, $\mathbf{LEF}_P$ are obtained by adding to $\mathbf{LE}$ these axioms and rules, and also the axiom schema

FAX_T $(\forall x Et(x) \to \exists\alpha\forall x\ \alpha x = t(x)) \wedge \forall\alpha\forall x E\alpha x$

resp.

FAX_P $\exists\alpha\forall xy(t(x) = y \leftrightarrow \alpha x = y)$ (α not in t).

It is clear that in $\mathbf{LEF}_T$ ($\mathbf{LEF}_P$) the function quantifiers range over all (partial) functions definable by a term of the language.

5.2. Let us see what happens when we add Ǝ to $\mathbf{LEF}_T$, $\mathbf{LEF}_P$. Taking $(\text{Ǝ}y(x).A(x,y))x$ for t in FAX_T yields

AC! $\forall x\exists ! yA(x,y) \to \exists\alpha\forall xA(x,\alpha x)$;

in the same way, FAX_P entails APC!, an axiom of unique partial choice:

APC! $\exists\alpha\forall xy(\forall z(A(x,z) \leftrightarrow z = y) \leftrightarrow \alpha x = y)$.

We shall show that AC! resp. APC! axiomatize the extension of $\mathbf{LEF}_T$ resp. $\mathbf{LEF}_P$ with Ǝ - terms.

5.3. THEOREM. $\mathbf{LEF}_T(\text{Ǝ})$ + AC! *is a definitional extension of* $\mathbf{LEF}_T$ + AC!.

PROOF. A straightforward extension of the reasoning in 4.2 - 4.12. To the definition of $\underline{t}$, $\delta(t)$ and $\varepsilon(t)$ we add $\underline{\alpha t} = \alpha\underline{t}$, $\delta(\alpha t) = \delta(t)$, $\delta(\forall\alpha B) = \delta(\exists\alpha B) = \delta(B)$, $\varepsilon(\alpha t) = \varepsilon(t)$.
To extend 4.8, we only have to check $\mathbf{LEF}_T$ + AC! $\vdash$ $(\mathrm{FAX}_T)^*$. We argue as in 4.8 under SUB: in $\mathbf{LEF}_T(\text{Ǝ}) - \mathrm{FAX}_T$ one can derive $Et \to \exists ! z\ z = t$, so by the proof up to here we have $\mathbf{LEF}_T + \mathrm{AC!} - \mathrm{FAX}_T \vdash Et^* \to (\exists ! z\ z = t)^*$.
Now

$$\begin{aligned}(\forall x E t)^* &\Rightarrow (\forall x \exists ! z\ \ z = t)^*\\ &= \forall x \exists ! z(\exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge z = \underline{t}))\\ &\Rightarrow \exists \alpha \forall x(\exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge \alpha x = \underline{t})) \qquad \text{(by AC!)}\\ &= (\exists \alpha \forall x\ \ \alpha x = t)^*\end{aligned}$$

so we have $(\mathrm{FAX}_{\mathrm{T}})^*$ (for $(\forall \alpha \forall x E \alpha x)^* = \forall \alpha \forall x E \alpha x$). □

5.4. THEOREM. $\utilde{\mathrm{LEF}}_{\mathrm{P}}(\Finv) + \mathrm{APC!}$ *is a definitional extension of* $\utilde{\mathrm{LEF}}_{\mathrm{P}} + \mathrm{APC!}$.

PROOF. Analogous to 5.3. To check $\utilde{\mathrm{LEF}}_{\mathrm{P}} + \mathrm{APC!} \vdash (\mathrm{FAX}_{\mathrm{P}})^*$, we observe that $\utilde{\mathrm{LEF}}_{\mathrm{P}}(\Finv) - \mathrm{FAX}_{\mathrm{P}} \vdash t = u \leftrightarrow \forall z(t = z \leftrightarrow z = u)$, so (arguing as in 5.3) $\utilde{\mathrm{LEF}}_{\mathrm{P}} + \mathrm{APC!} - \mathrm{FAX}_{\mathrm{P}} \vdash (t = u)^* \leftrightarrow \forall z(t = z \leftrightarrow z = u)^*$. Now

$$\begin{aligned}(\mathrm{FAX}_{\mathrm{P}})^* &= \exists \alpha \forall x u(\alpha x = u \leftrightarrow t = u)^*\\ &\Longleftrightarrow \exists \alpha \forall x u(\alpha x = u \leftrightarrow \forall z(t = z \leftrightarrow z = u)^*)\\ &= \exists \alpha \forall x u(\alpha x = u \leftrightarrow \forall z(\exists ! \vec{y}\varepsilon(t) \wedge \exists \vec{y}(\varepsilon(t) \wedge \underline{t} = z) \leftrightarrow z = u))\end{aligned}$$

and this is an instance of APC!. □

5.5. REMARK. As a corollary of 5.3, one obtains Kleene's result on the conservativity of adding p-terms (i.e. the so-called Kleene bracket notation $\{e\}(x) \simeq y$) to a two-sorted theory of arithmetic and recursive functions with λ-abstraction and AC! (see [K169]).

CHAPTER II. THE THEORIES APP AND APP^E.

§1. Introduction

1.1. In this chapter, we present two closely related theories, APP and APP^E. Both are one-sorted theories based on intuitionistic logic about a universe of *objects* (among which combinatorial constants and the natural numbers) which can be applied to one another. In APP this application is total, in APP^E partial: to express this, APP^E is equipped with an existence predicate E . In fact, APP^E is just APP based on LE (see Ch.I) instead of ordinary intuitionistic predicate logic.
We establish some properties of APP and APP^E, the most important being that both theories are conservative extensions of HA (intuitionistic arithmetic). Together with its expressive power and flexible character this makes APP an interesting theory for metamathematical investigations (see the next chapters).

1.2. Outline of the rest of this chapter.
In §2 we give the definition of APP and APP^E; some related literature is discussed briefly. We compare APP and APP^E in §3 and prove that all recursive functions are definable in both theories. This is used in §4 to show that APP^E is conservative over HA and EL (elementary intuitionistic analysis). §5 is about term models: the logic-free theories APT($\downarrow$) and APT are presented with which we investigate term reduction for APP^E resp. APP. By formalizing the term model for APP in APP^E we are able to show that APP is conservative over APP^E w.r.t. numerical formulae; from this and §4 it follows that APP is conservative over HA.

§2. The formal systems APP and APP^E.

2.1. DEFINITION of APP.

Constants:	k, s	(projector and substitutor),
	p, p_1, p_2	(pairing and inverses),
	0, S, Pd	(zero, successor and predecessor),
	Δ	(definition by cases).

Variables: $a,b,c,\dots,x,y,z$ (possibly with indices).

Terms: i) all variables and constants are terms;
ii) if σ and τ are terms, then so is $\sigma(\tau)$ (σ *applied to* τ).

Prime formulae: let σ, τ be terms. Then
$\sigma = \tau$ (σ *is equal to* τ)
$\tau \in N$ (τ *is a natural number*)
are prime formulae.

Formulae: built up from prime formulae, using $\wedge$, $\rightarrow$, $\forall$, $\exists$.

Before we give the axioms and rules of APP, we state some abbreviations and conventions.

We write ρ, σ, τ, τ', τ_1, τ_2,... for arbitrary terms. The usual conventions are adopted for dropping superfluous parentheses, so e.g. $\rho\sigma\tau = (\rho(\sigma))(\tau)$. m, n are used for numerical variables, so e.g. $\forall nA$ abbreviates $\forall n(n \in N \rightarrow A)$.

$\top$, $\bot$, $\neg$, $\vee$, $\leftrightarrow$ are defined by

$$\top := (0=0) \qquad\qquad \bot := (0=1)$$

$$\neg A := A \rightarrow \bot$$

$$A \vee B := \exists n((n=0 \rightarrow A) \wedge (\neg n=0 \rightarrow B))$$

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

We also define

$$\langle\sigma,\tau\rangle := p\sigma\tau$$

$$(\tau)_i := p_i\tau \qquad (i = 1,2)$$

$$1,2,3,\ldots := S0,\ S(S0),\ S(S(S0))\ ,\ldots$$

$$(\sigma \neq \tau) := \neg(\sigma = \tau).$$

$\vec{x}$ denotes a sequence of variables $x_1,\ldots,x_n$; similar for $\vec{\tau}$ (terms), $\vec{A}$ (formulae). Substitution: $\sigma[x := \tau]$ $(A[x := \tau])$ is the term (formula) obtained from σ (A) by replacing every (free) x by τ.

Now we give the rules and axioms of $\underset{\sim}{\text{APP}}$.

Logical axioms and rules: we take the following axiomatization of intuitionistic predicate logic with equality.

$$\to\text{AX} \qquad A \to A$$

$$\forall\text{AX} \qquad \forall xA \to A[x := \tau]$$

$$\exists\text{AX} \qquad A[x := \tau] \to \exists xA$$

$$\text{PR1} \qquad \frac{A}{B \to A}$$

$$\text{PR2} \qquad \frac{A \to B \quad B \to C}{A \to C}$$

$$\text{PR3} \qquad \frac{A \quad A \to B}{B}$$

$$\text{PR4} \qquad \frac{A \to B \quad A \to C}{A \to (B \wedge C)}$$

$$\text{PR5} \qquad \frac{(A \wedge B) \to C}{A \to (B \to C)}$$

N.B. The rules PR4, PR5 are double rules, i.e. their 'upside-down' version is also a rule.

$$\forall\text{-R} \qquad \frac{A \to B}{A \to \forall xB} \qquad (\,x \text{ not free in } A\,)$$

$$\exists\text{-R} \qquad \frac{A \to B}{\exists xA \to B} \qquad (\,x \text{ not free in } B\,)$$

$$=\!\text{AX} \qquad \forall x(x=x) \wedge \forall xyz(x=z \wedge y=z \to x=y)$$

$$\text{SUB} \qquad x=y \to zx=zy \wedge xz=yz \wedge (x\in N \to y\in N)$$

Non-logical axioms:

kAX $\qquad kxy = x$

sAX $\qquad sxyz = xz(yz)$

pAX $\qquad p_1(pxy) = x \wedge p_2(pxy) = y$

0AX $\qquad 0 \in N$

SAX $\qquad x \in N \rightarrow Sx \in N \wedge Sx \neq 0$

PdAX $\qquad Pd0 = 0 \wedge (x \in N \rightarrow Pdx \in N \wedge Pd(Sx) = x)$

ΔAX $\qquad x,y \in N \wedge x \neq y \rightarrow \Delta uvxx = u \wedge \Delta uvxy = v$

IND $\qquad A(0) \wedge \forall x(x \in N \wedge A(x) \rightarrow A(Sx)) \rightarrow \forall x \in N\ A(x)$

This completes the definition of APP.

2.2. DEFINITION of APP^E.

Constants, variables, terms: as in APP.

Prime formulae: as in APP, and also $E\tau$ (τ *exists*).

Formulae: as in APP.

Abbreviations: as in APP, and: $\sigma \simeq \tau := E\sigma \vee E\tau \rightarrow \sigma = \tau$.

Logical axioms and rules: APP^E is based on LE (see Ch.I). This means that ∀AX, ∃AX, SUB of APP are replaced by

EAX $\qquad E\tau \leftrightarrow \exists x\ x = \tau$

STR
$$\begin{cases} \sigma = \tau \rightarrow E\sigma \wedge E\tau \\ \tau \in N \rightarrow E\tau \\ E\sigma\tau \rightarrow E\sigma \wedge E\tau \end{cases}$$

SUB
$$\begin{cases} \sigma \in N \wedge \sigma = \tau \rightarrow \tau \in N \\ E\rho\sigma \wedge \sigma = \tau \rightarrow \rho\sigma = \rho\tau \\ E\sigma\rho \wedge \sigma = \tau \rightarrow \sigma\rho = \tau\rho \end{cases}$$

$\forall$AX $\qquad \forall xA \to A[x := y]$

$\exists$AX $\qquad A[x := y] \to \exists xA$

Non-logical axioms: as in APP, but sAX is replaced by

$s\mathrm{AX}^E \qquad \mathrm{E}sxy \wedge sxyz \simeq xz(yz)$

2.3. Some remarks.

APP^E is virtually the same as the 'applicative and inductive part' of Feferman's applicative theories described in [Fe75] and [Fe79] (see also [RT84], a review of these papers). In Feferman's theories, however, compound terms are abbreviations which are explained using the predicate App(x,y,z) with the intended meaning ' x applied to y yields z ': so e.g. $\sigma\tau = \rho$ is inductively defined by $\exists xyz(x=\sigma \wedge y=\tau \wedge z=\rho \wedge \mathrm{App}(x,y,z))$. Following a suggestion by A.S. Troelstra, we combined Feferman's approach with Scott's E-logic (see [Sc79]) and formulated APP^E, where compound terms are no longer abbreviations but an integral part of the language. APP^E has, in common with Feferman's weak theories, a straightforward interpretation in HA via Kleene brackets (see §4); in fact, APP^E may be viewed as an abstract description of Kleene-bracket-application. Going one step further brings us to APP in which application is total and the existence predicate E is no longer needed. In APP we can write down any term we like without bothering about existence. The price we have to pay for this carelessness is a more extensive proof that APP is conservative over HA, using formalization in APP^E of a term model for APP (§5).

§3. Some properties of APP and APP^E.

3.1. In this section we compare APP with APP^E, and show that λ-abstraction and the recursive functions are definable in both theories. But first we note that, by Ch.I, 3.4 we have (recall that APP^E is based on LE):

3.2. LEMMA. $\mathrm{APP}^E \vdash \forall xA \wedge \mathrm{E}\tau \to A[x := \tau],\quad A[x := \tau] \wedge \mathrm{E}\tau \to \exists xA.$

3.3. LEMMA. *In* $\underset{\sim}{\mathrm{APP}}^{E}$ *are derivable:*

$\simeq$AX $\quad \tau \simeq \tau \wedge (\rho \simeq \sigma \wedge \rho \simeq \tau \rightarrow \sigma \simeq \tau)$

SUB($\simeq$) $\quad \sigma \simeq \tau \rightarrow \rho\sigma \simeq \rho\tau \wedge \sigma\rho \simeq \tau\rho \wedge (\sigma \in N \rightarrow \tau \in N) \wedge (E\sigma \rightarrow E\tau)$.

PROOF. If $E\tau$, then $\exists x\ x = \tau$ (by EAX), so $\exists x(x=\tau \wedge x=\tau)$, hence $\exists x(\tau = \tau)$ (by =AX and 3.2), i.e. $\tau = \tau$. So we have $\tau \simeq \tau$ for any term τ. Assume $\rho \simeq \sigma$, $\rho \simeq \tau$, $E\sigma \vee E\tau$. If $E\sigma$, then $\rho = \sigma$ so $E\rho$, hence $\rho = \tau$ and we have (by =AX and 3.2) $\sigma = \tau$; similar if $E\tau$. Thus we get the righthand part of $\simeq$AX.

SUB($\simeq$): analogous. □

3.4. LEMMA. i) $\underset{\sim}{\mathrm{APP}} \vdash \sigma=\tau \rightarrow \rho[x:=\sigma]=\rho[x:=\tau] \wedge (A[x:=\sigma] \leftrightarrow A[x:=\tau])$

ii) $\underset{\sim}{\mathrm{APP}}^{E} \vdash \sigma\simeq\tau \rightarrow \rho[x:=\sigma]\simeq\rho[x:=\tau] \wedge (A[x:=\sigma] \leftrightarrow A[x:=\tau])$

PROOF. i) Assume $\sigma=\tau$. Now $\rho[x:=\sigma]=\rho[x:=\tau]$ is proved using SUB, and $A[x:=\sigma] \leftrightarrow A[x:=\tau]$ by formula induction, using SUB, =AX and $\rho[x:=\sigma]=\rho[x:=\tau]$ if A prime.

ii) The proof is analogous to (ii), reading everywhere $\simeq$ for $=$ and using 3.3. □

3.5. LEMMA. i) $\underset{\sim}{\mathrm{APP}} \vdash A \Rightarrow \underset{\sim}{\mathrm{APP}}^{E} \vdash \forall xyExy \rightarrow A$.

ii) *Let the mapping* $*$: $\underset{\sim}{\mathrm{APP}}^{E} \rightarrow \underset{\sim}{\mathrm{APP}}$ *be given by* $E\tau \longmapsto \top$. *Then*

$$\underset{\sim}{\mathrm{APP}}^{E} \vdash A \Rightarrow \underset{\sim}{\mathrm{APP}} \vdash A^{*}.$$

PROOF. i) First we show

(1) $\quad \underset{\sim}{\mathrm{APP}}^{E} \vdash \forall xyExy \rightarrow E\tau \quad$ for all terms τ

with induction over the complexity of τ.

τ a constant: for any constant c, Ec follows from the axiom on c and STR.

τ a variable, y say: by =AX we have $\forall x\ x = x$ so with $\forall$AX $y = y$, hence Ey (by STR).

$\tau = \tau_1\tau_2$: use $\forall xy\, Exy$ and the induction hypothesis.
Now the result follows from (1) and Ch.I, 3.5.

ii) A straightforward induction over the length of a proof of A in APP^E. □

Now we shall show that in APP and APP^E λ - abstraction, a fixed point operator ϕ, a recursor R and a minimum operator μ are definable. By 3.5.(ii) it suffices to give the proofs only for APP^E. In APP, however, a simpler definition is often possible.

3.6. LEMMA. *For every term* τ*, there is a term* $\lambda x.\tau$ *satisfying*

$$\text{i)}\quad \mathrm{APP}^E \vdash E\lambda x.\tau \wedge (E\sigma \to (\lambda x.\tau)\sigma \simeq \tau[x{:=}\sigma]);$$

$$\text{ii)}\quad \mathrm{APP} \vdash (\lambda x.\tau)\sigma = \tau[x{:=}\sigma].$$

PROOF. i) Induction over the complexity of τ.

a) τ is a constant, or a variable $\neq x$: put $\lambda x.\tau := k\tau$. $E\tau$, so $E\lambda x.\tau$; if $E\sigma$, then $(\lambda x.\tau)\sigma = k\tau\sigma = \tau = \tau[x{:=}\sigma]$.

b) $\tau \equiv x$: put $\lambda x.\tau := skk$. $E\lambda x.\tau$ follows from $s\mathrm{AX}^E$; if $E\sigma$, then $(\lambda x.\tau)\sigma = skk\sigma \simeq k\sigma(k\sigma) = \sigma = \tau[x{:=}\sigma]$.

c) $\tau \equiv \tau_1\tau_2$: put $\lambda x.\tau := s(\lambda x.\tau_1)(\lambda x.\tau_2)$.
By ind. hyp.: $E\lambda x.\tau_1$ and $E\lambda x.\tau_2$, so with $s\mathrm{AX}^E$ and 3.2 we have $E\lambda x.\tau$. If $E\sigma$, then

$$\begin{aligned}(\lambda x.\tau)\sigma &\simeq s(\lambda x.\tau_1)(\lambda x.\tau_2)\sigma\\ &\simeq (\lambda x.\tau_1)\sigma((\lambda x.\tau_2)\sigma)\\ &\simeq \tau_1[x{:=}\sigma]\tau_2[x{:=}\sigma] \qquad \text{(ind. hyp.)}\\ &\simeq \tau_1\tau_2[x{:=}\sigma] \simeq \tau[x{:=}\sigma]\end{aligned}$$

ii) Follows directly from (i), 3.5.(ii) and the fact that $(\sigma \simeq \tau)^*$ is equivalent to $\sigma = \tau$. □

Remark. Note that we can simplify the definition of $\lambda x.\tau$ in APP by adding clauses $\lambda x.\tau := k\tau$, $\lambda x.\tau x := \tau$ if x not in τ. For APP^E, this cannot be done without the risk of losing $E\lambda x.\tau$.

3.7. LEMMA. *(Fixed point construction.) There is a term* ϕ *satisfying*

i) $\quad \mathbf{APP}^E \vdash E\phi x \wedge \phi xy \simeq x(\phi x)y;$

ii) $\quad \mathbf{APP} \vdash \phi xy = x(\phi x)y.$

PROOF. i) Define

$$\chi := \lambda zy.x(zz)y, \qquad \phi := \lambda x.\chi\chi\ .$$

By 3.6.(i) $E\chi$, and

$$\phi x \simeq \chi\chi \simeq (\lambda zy.x(zz)y)\chi \simeq \lambda y.x(\chi\chi)y$$

so $E\phi x$; also

$$\phi xy \simeq x(\chi\chi)y \simeq x(\phi x)y.$$

ii) follows from (i) and 3.5.(ii). □

Remark. $\phi' := \lambda x.\chi'\chi'$ with $\chi' := \lambda z.x(zz)$ also works in $\mathbf{APP}$: we even get $\phi'x = \chi'\chi' = (\lambda z.x(zz))\chi' = x(\chi'\chi') = x(\phi'x)$. However, we do not have $\mathbf{APP}^E \vdash E\phi'x$.

3.8. LEMMA. *(Existence of a recursor.) There is a term* R *satisfying:*

i) $\quad \mathbf{APP}^E \vdash Rxy0 = x \wedge (n \neq 0 \rightarrow Rxyn \simeq yn(Rxy(Pdn)));$

ii) $\quad \mathbf{APP} \vdash Rxy0 = x \wedge (n \neq 0 \rightarrow Rxyn = yn(Rxy(Pdn))).$

PROOF. Define

$$r := \lambda fn.\Delta(kx)(\lambda z.yz(f(Pdz)))n0n, \qquad R := \lambda xy.\phi r\ .$$

Now
$$\begin{aligned} Rxy0 &\simeq \phi r0 \\ &\simeq r(\phi r)0 \\ &\simeq \Delta(kx)(\lambda z.yz(\phi r(Pdz)))000 \\ &\simeq kx0 = x\ ; \end{aligned}$$

if $n \neq 0$, then

$$\begin{aligned} Rxyn &\simeq \phi rn \\ &\simeq r(\phi r)n \\ &\simeq \Delta(kx)(\lambda z.yz(\phi r(Pdz)))n0n \\ &\simeq (\lambda z.yz(\phi r(Pdz)))n \\ &\simeq yn(\phi r(Pdn)) \\ &\simeq yn(Rxy(Pdn)). \end{aligned}$$

ii) follows from (i) and 3.5.(ii). □

Remark. Instead of r we might have taken $r' := \lambda fn.\Delta x(yn(f(Pdn)))n0$ in APP. In APP^E this does not work, for we cannot prove in general that $yn(f(Pdn))$ exists, which we need to apply ΔAX in the proof that $Rxy0 = x$.

Before we turn to the minimum operator, we define the following.

3.9. DEFINITION.

i) $m + n := Rm(kS)n$

ii) $x<y := x,y\in N \wedge \exists n(x+Sn = y)$; $x>y := y<x$

iii) $Adm(f) := \forall n(fn\in N) \vee \exists n(fn=0 \wedge \forall m<n(fm\in N))$

It is easy to verify $m + 0 = m$, $m + Sn = S(m+n)$ and the well-known properties of $<, >$.
Only for f satisfying $Adm(f)$ we can find the least n with $fn = 0$ (if such an n exists); this is a consequence of the fact that we have definition by cases (Δ) only on N.

3.10. LEMMA. *There is a term* μ *satisfying:*

i) $\text{APP}^E \vdash Adm(f) \rightarrow (\mu f=n \leftrightarrow fn=0 \wedge \forall m<n\ fm>0)$

ii) idem for APP.

<u>PROOF</u>. i) Define

$$f^+ := \lambda x.f(Sx)$$

$$M := \lambda xf.\Delta(k0)(\lambda g.S(xg))(f0)0f^+$$

$$\mu := \phi M$$

Now

$$\begin{aligned}
\mu f &\simeq \phi M f \\
&\simeq M(\phi M)f \\
&\simeq M\mu f \\
&\simeq \Delta(k0)(\lambda g.S(\mu g))(f0)0f^+ \\
&\simeq \begin{cases} k0f^+ = 0 & \text{if} \quad f0 = 0 \\ (\lambda g.S(\mu g))f^+ \simeq S(\mu f^+) & \text{if} \quad f0 > 0 \end{cases}
\end{aligned}$$

So

$$f0 \in N \rightarrow (f0{=}0 \wedge \mu f{=}0) \vee (f0{>}0 \wedge \mu f \simeq \mu f^+ + 1)$$

Now we prove (i) with induction over n, assuming $\mathrm{Adm}(f)$.

a) $n = 0$: $\mathrm{Adm}(f) \rightarrow f0 \in N$, so $f0 = 0 \leftrightarrow \mu f = 0$.

b) $n+1$: observe that $\mathrm{Adm}(f)$ implies $(\mathrm{Adm}(f^+) \wedge f0{\in}N) \vee f0{=}0$.

Ind. hyp.: $\mathrm{Adm}(f^+) \rightarrow (\mu f^+ = n \leftrightarrow f^+n{=}0 \wedge \forall m{<}n\ f^+m{>}0)$.

$$\begin{aligned}
\text{Now}\quad \mu f = n{+}1 &\leftrightarrow \mu f \simeq \mu f^+{+}1 = n{+}1 \wedge f0{>}0 \\
&\leftrightarrow \mu f^+ = n \wedge f0{>}0 \\
&\leftrightarrow f^+n = 0 \wedge \forall m{<}n\ f^+m{>}0 \wedge f0{>}0 \\
&\leftrightarrow f(n{+}1) = 0 \wedge \forall m < n{+}1\ fm{>}0.
\end{aligned}$$

ii) follows from (i) and 3.5.(ii). □

<u>Remark</u>. In $\mathbf{APP}$, $M' := \lambda xf.\Delta 0(S(xf^+))(f0)0$ also works. $\mu' := \phi M'$ fails in $\mathbf{APP}^E$, for we cannot prove $\mu f = 0$ if $f0 = 0$, as we do not know $E S(\mu f^+)$.

3.11. THEOREM. *All (general) recursive functions are definable in* $\underset{\sim}{APP}$ *and* $\underset{\sim}{APP}^E$, *in the following sense:*

> *for any* k*-ary recursive* f, *there is a term* τ_f *of* $\underset{\sim}{APP}$ *with* $f(m_1,\ldots,m_k) \simeq n \Rightarrow \underset{\sim}{APP}^{(E)} \vdash \tau_f \underline{m}_1 \ldots \underline{m}_k = \underline{n}$.

Here $\underline{m}$ *is the numeral* $\overbrace{S(S(\ldots(S0)\ldots))}^{m\text{ times}}$.

PROOF. It is obvious that the constant zero function ($k0$), the successor function (S) and the projection functions $I^i_n(\lambda x_1 \ldots x_n \cdot x_i)$ are definable in $\underset{\sim}{APP}$ and $\underset{\sim}{APP}^E$. For closure under composition, recursion and minimalisation we use λ-abstraction, R and μ. Two remarks are to be made:

i) at first sight, R and μ give us only closure under recursion and minimalisation without parameters. For closure with parameters, we use λ-abstraction as follows. Suppose we want to define, given f and g, the function h satisfying

$$h\vec{x}0 \simeq f\vec{x},$$

$$h\vec{x}(n+1) \simeq g\vec{x}n(h\vec{x}n).$$

One readily verifies that $h := \lambda\vec{x}.R(f\vec{x})(\lambda m.g\vec{x}(Pdm))$ works.

ii) The condition Adm(f) in 3.10 is dealt with as follows: if $\min_x[g(\vec{m},x)=0] \simeq n$, then $g(\vec{m},n) \simeq 0 \wedge \forall n'<n\ g(\vec{m},n)>0$, so (by the induction hypothesis) $\mathrm{Adm}(\tau_g\vec{m})$ holds. □

§4. Comparing $\underset{\sim}{APP}^E$ with $\underset{\sim}{HA}$ and $\underset{\sim}{EL}$.

In this section we give embeddings of $\underset{\sim}{HA}$ and $\underset{\sim}{EL}$ in $\underset{\sim}{APP}^E$ and vice versa. As a consequence, we obtain that $\underset{\sim}{APP}^E$ is conservative over $\underset{\sim}{HA}$ and $\underset{\sim}{EL}$.

4.1. The theory $\underset{\sim}{HA}$. We recall that the constants of $\underset{\sim}{HA}$ are 0, S and function symbols for all primitive recursive functions, the prime formulae have the form $s = t$, and the non-logical axioms are IND and the usual axioms for the constants. For a complete definition, see [T73].

We first embed HA in APP^E. Define $^\circ$: HA $\to$ APP^E by

$$0^\circ := 0 \qquad x^\circ := x \quad \text{for all variables } x$$

$$S(t)^\circ := S(t^\circ)$$

$$\phi(t_1,\ldots,t_n)^\circ := \tau_\phi(t_1^\circ)\ldots(t_n^\circ)$$

(ϕ a prim. rec. function symbol, τ_ϕ as in 3.11)

$$(s = t)^\circ := (s^\circ = t^\circ)$$

$^\circ$ commutes with the propositional operators

$$(\forall x A)^\circ := \forall x \in N\ A^\circ$$

$$(\exists x A)^\circ := \exists x \in N\ A^\circ$$

4.2. LEMMA. i) *If* t *is a term of* HA *and* $\vec{x}$ *are its variables, then* $\mathrm{APP}^E \vdash \vec{x} \in N \to t^\circ \in N$.

ii) *If* A *is a formula of* HA *and* $\vec{x}$ *are its variables, then*

$$\mathrm{HA} \vdash A \;\Rightarrow\; \mathrm{APP}^E \vdash \vec{x} \in N \to A^\circ .$$

PROOF. i) An easy induction over the complexity of t .
ii) Induction over the length of a proof of A . For the axioms on the constants of HA we use 3.11; the quantifier axioms and rules follow from (i) and the condition $x \in N$. □

4.3. Now we go the other way round, but instead of HA we take $\mathrm{HA}^* := \mathrm{HA}^E(\Finv)$, where HA^E is HA based on the logic LE plus Et for all terms t of HA, and $\mathrm{HA}^E(\Finv)$ is the extension with $\Finv$, as defined in Ch.I, 4.11 (c). HA^E is obviously conservative over HA, so by Ch.I, 4.12 the same holds for HA^*.

The Kleene bracket notation is defined in HA^* by

$$\{\cdot\}(\cdot) = \lambda xy.\{x\}(y) := \Finv u(x,y).\exists z(Txyz \wedge Uz = u)$$

where T is the Kleene predicate and U the result-extracting function (they satisfy $Txyz \wedge Txyz' \to Uz = Uz'$). For terms t containing only variables and constants of HA and $\{\cdot\}(\cdot)$ we have the so-called Λ-abstraction, which satisfies

$$E\Lambda x.t \wedge \{\Lambda x.t\}(x) \simeq t.$$

This is proved in Kleene [Kl69], Lemma 41. (A proof can be given by repeated use of the s-m-n theorem of recursion theory and induction on the construction of t.)

Now we define $': \mathbf{APP}^E \to \mathbf{HA}^*$.

$$0' := 0 \qquad x' := x \qquad \text{for all variables } x$$

$$S' := \Lambda x.x+1 \qquad Pd' := \Lambda x.x\dot{-}1$$

$$p' := \Lambda xy.j(x,y) \qquad p_i' := \Lambda x.j_i(x) \qquad (i=1,2)$$

(j is some prim. rec. pairing function, with prim. rec. inverses j_1, j_2)

$$k' := \Lambda xy.x$$

$$s' := \Lambda xyz.\{\{x\}(z)\}\{\{y\}(z)\}$$

$$\Delta' := \Lambda uvxy.(u.\overline{sg}|x-y| + v.sg|x-y|)$$

$$(\sigma\tau)' := \{\sigma'\}(\tau')$$

$$(\sigma=\tau)' := (\sigma'=\tau')$$

$$(E\tau)' = (\tau \in N)' := E\tau'$$

$'$ commutes with all logical operators.

4.4. LEMMA. $\mathbf{APP}^E \vdash A \Rightarrow \mathbf{HA}^* \vdash A'$.

PROOF. Straightforward induction over the length of a proof of A in $\mathbf{APP}^E$. The logical rules and axioms are trivial, for $'$ commutes with all logical operators and $\mathbf{APP}^E$, $\mathbf{HA}^*$ are both based on $\mathbf{LE}$. IND is present in both theories, and the axioms for the constants follow from the definition of $'$ and the properties of $\{\cdot\}(\cdot)$ and Λx. □

4.5. LEMMA. i) $\mathbf{HA}^* \vdash t^{\circ\prime} = t$;

ii) $\mathrm{HA}^* \vdash A^{\circ\prime} \leftrightarrow A$.

PROOF. Straightforward inductions over the complexity of t resp. A. In the proof of (ii), we use (i) for the case of A prime. □

4.6. THEOREM. $\mathbf{HA} \vdash A \iff \mathbf{APP}^E \vdash A^\circ$ *for closed* A.

PROOF. $\Rightarrow$ follows from 4.2.(ii), $\Leftarrow$ from 4.4, 4.5.(ii) and the fact that $\mathbf{HA}^*$ is conservative over $\mathbf{HA}$. □

4.7. Now we turn to EL, intuitionistic elementary analysis. This is an extension of HA, obtained by:

- adding variables $a, b, c, d, \dots$ and quantification for functions from N to N, and a recursor R;
- allowing λ - abstraction over numerical terms, axiomatized by $(\lambda x.t)x = t$;
- adding a quantifier-free axiom of choice:

$$\text{QF-AC} \qquad \forall x \exists y A(x,y) \to \exists a \forall x A(x,ax) \qquad \text{(A quantifier-free)}$$

For a complete description of EL we refer to [T73].

Convention. We write $\tau \in (N \Rightarrow N)$ for $\forall n\ \tau n \in N$.

We extend $^{\circ}$ of 4.1 to $^{\circ}$: EL $\to$ APP^{E} as follows:

$$a^{\circ} := a \qquad \text{for all function variables } a$$

$$(Rt\phi)^{\circ} := R(t^{\circ})(\phi^{\circ}) \qquad \text{(the } R \text{ at the right is the same as in 3.8)}$$

$$(\lambda x.t)^{\circ} := \lambda x.(t^{\circ})$$

$$(\phi(t))^{\circ} := (\phi^{\circ})(t^{\circ}) \qquad (\phi \text{ a function term})$$

$$(\forall aA)^{\circ} := \forall a \in (N \Rightarrow N)\ A^{\circ}$$

$$(\exists aA)^{\circ} := \exists a \in (N \Rightarrow N)\ A^{\circ}$$

4.8. LEMMA. i) *If* t *is a numerical term of* EL *and* $\vec{x}, \vec{a}$ *are its free variables, then*

$$\text{APP}^{E} \vdash \vec{x} \in N \wedge \vec{a} \in (N \Rightarrow N) \to t^{\circ} \in N.$$

ii) *If* ϕ *is a function term of* EL *and* $\vec{x}, \vec{a}$ *are its free variables, then*

$$\text{APP}^{E} \vdash x \in N \wedge \vec{a} \in (N \Rightarrow N) \to \phi^{\circ} \in (N \Rightarrow N).$$

iii) *If* A *is a formula of* EL *and* $\vec{x}, \vec{a}$ *are its free variables, then*

$$\text{EL} \vdash A \Rightarrow \text{APP}^{E} \vdash \vec{x} \in N \wedge \vec{a} \in (N \Rightarrow N) \to A^{\circ}.$$

PROOF. (i) and (ii) are proved simultaneously, with induction over the complexity of t resp. ϕ. (iii) is proved as 4.2.(ii). We use 3.6.(i) for the axiom on λx, (i) and (ii) for the quantifier axioms, and for QF-AC we argue as follows. By

$$\begin{array}{lll} r=s & \longmapsto & |r-s| = 0 \\ \neg r=s & \longmapsto & 1 \dot{-} |r-s| = 0 \\ r=s \wedge t=u & \longmapsto & |r-s| + |t-u| = 0 \\ r=s \vee t=u & \longmapsto & |r-s| \cdot |t-u| = 0 \\ r=s \rightarrow t=u & \longmapsto & (1 \dot{-} |r-s|) \cdot |t-u| = 0 \end{array}$$

we reduce QF-AC to $\forall x \exists y t(x,y) = 0 \rightarrow \exists a \forall x t(x,ax) = 0$. Under $^\circ$ this becomes (modulo equivalence) $\forall m \exists n\, t^\circ(m,n) = 0 \rightarrow \exists a \in (N \Rightarrow N)\, \forall m\, t^\circ(m,am) = 0$ and this is derivable in APP^E, as we can take $\mu(\lambda n.t^\circ(m,n))$ for a (one easily checks $\mathrm{Adm}(\lambda n.t^\circ(m,n))$ and $a \in (N \Rightarrow N)$). □

4.9. With the extension of HA to HA^* in 4.3 in mind, we extend EL to a theory EL^* with $(\cdot|\cdot)$, partial continuous function application. In EL^* we have

- equality between function terms $(\phi = \psi)$ and an existence predicate E for both numerical and function terms;
- two-sorted LE as logic;
- Et, $E\phi$ for all terms t, ϕ of EL;
- not function descriptors, but *functor* descriptors $\daleth b(\vec{a})$, so we have new function terms of the form $(\daleth b(\vec{a}).A)(\vec{\phi})$;
- the axiom $a = b \leftrightarrow \forall x\ ax = bx$.

Let EL' be EL + equality between function terms. EL' is conservative over EL (interpret $\phi = \psi$ by $\forall x\ \phi x = \psi x$) and EL^* is conservative over EL' (Ch.I, 4.12), so EL^* is a conservative extension of EL.

We assume coding for n-tuples of natural numbers, written as $<x_1,\ldots,x_n>$, to be defined as usual in EL, and also $\bar{a}x :=$ $<a0,\ldots,a(x-1)>$. We define partial continuous function application $(\cdot|\cdot)$ in EL^* as follows:

$$(\cdot|\cdot) = \lambda ab.(a|b) :=$$

$$:= \daleth c(a,b).\forall x \exists y (a(<x>*\bar{b}y) = cx+1 \wedge \forall z<y\ a(<x>*\bar{b}z) = 0)$$

Kleene proved ([KL69], Lemma 41) that, for any function term ϕ containing only variables and constants of $\mathbf{EL}$ and $(\cdot|\cdot)$ there is a function term $\Lambda'a.\phi$ such that

$$E\Lambda'a.\phi \wedge (\Lambda'a.\phi|a) \simeq \phi \,.$$

See also [T73], p.73-75.

Now we define ": $\mathbf{APP}^E \to \mathbf{EL}^*$ as follows:

$x'' := \iota(x)$ (ι is a fixed injective assignment of variables of $\mathbf{APP}^E$ to function variables of $\mathbf{EL}^*$)

$0'' := \lambda x.0$

$S'' := \Lambda'a.(\lambda x.ax{+}1)$

$Pd'' := \Lambda'a.(\lambda x.ax \dot{-} 1)$

$p'' := \Lambda'ab.(\lambda x.j(ax,bx))$

$p''_i := \Lambda'a.(\lambda x.j_i(ax))$ $(i = 1,2)$

$k'' := \Lambda'ab.a$

$s'' := \Lambda'abc.((a|c)|(b|c))$

$\Delta'' := \Lambda'abcd.(\lambda x(ax.\overline{sg}|cx-dx| + bx.sg|cx-dx|))$

$(\sigma\tau)'' := (\sigma''|\tau'')$

$(\sigma = \tau)'' := (\sigma'' = \tau'')$

$(E\tau)'' := E\tau''$

$(\tau \in N)'' := \exists x \forall y\ (\tau'')y = x$

" commutes with the propositional operators

$(\forall xA)'' := \forall x''A''$

$(\exists xA)'' := \exists x''A''$

4.10. LEMMA. $\mathbf{APP}^E \vdash A \Rightarrow \mathbf{EL}^* \vdash A''$.

PROOF. As for 4.4. □

4.11. LEMMA. *Let* $\vec{x}, \vec{a}, \vec{b}, \vec{c}$ *be sequences of variables of* EL *satisfying* $\vec{x}^{\circ\prime\prime} = \vec{a}$, $\vec{b}^{\circ\prime\prime} = \vec{c}$. *We write* C *for the formula* $\bigwedge_i \forall y\ a_i y = x_i \ \wedge$ $\wedge \bigwedge_i \forall yz(b_i y = (c_i|\lambda x.y)z)$. *Let* t, ϕ, A *be resp. a numerical term, a function term and a formula of* EL, *all with free variables among* $\vec{x}$, $\vec{b}$. *Then:*

i) $\mathrm{EL}^* \vdash C \rightarrow \forall y\ t^{\circ\prime\prime} y = t$;

ii) $\mathrm{EL}^* \vdash C \rightarrow \forall yz\ \phi y = (\phi^{\circ\prime\prime}|\lambda x.y)z$;

iii) $\mathrm{EL}^* \vdash C \rightarrow (A^{\circ\prime\prime} \leftrightarrow A)$.

PROOF. Induction over the complexity of t, ϕ resp. A. (i) and (ii) are proved simultaneously; they are used in the proof of (iii) for the case A prime. □

4.12. THEOREM. *If* A *is a sentence of* EL, *then*

$$\mathrm{EL} \vdash A \iff \mathrm{APP}^E \vdash A^\circ .$$

PROOF. ⇒ follows from 4.8.(iii), ⇐ from 4.10, 4.11.(iii) and the fact that EL^* is conservative over EL. □

§5. Term models for APP^E and APP.

We define in this section two logic-free theories APT(↓) and APT to investigate term reduction and term models for APP^E resp. APP.

5.1. DEFINITION. APT(↓) is the following theory:

Constants: as in APP $(0, S, \mathrm{Pd}, p, p_1, p_2, k, s, \Delta)$.

Terms: the *closed* terms of APP.

Formulae: $\tau\downarrow$ (τ *is in normal form*),

$N\tau$ (τ *is a numeral*),

$\sigma \neq \tau$ (σ *and* τ *are different numerals*),

$\sigma >_1 \tau$ (σ *reduces in one step to* τ),

$\sigma \geq \tau$ (σ *reduces to* τ).

Axioms and rules:

NO $\quad \dfrac{N\tau}{NS\tau}$

$$\frac{N\tau}{S\tau \neq 0} \qquad \frac{N\tau}{0 \neq S\tau} \qquad \frac{\sigma \neq \tau}{S\sigma \neq S\tau}$$

$c\downarrow$ for all constants c

$$\frac{\tau\downarrow}{p\tau\downarrow} \qquad \frac{\sigma\downarrow \;\; \tau\downarrow}{p\sigma\tau\downarrow} \qquad \frac{\tau\downarrow}{k\tau\downarrow} \qquad \frac{\tau\downarrow}{s\tau\downarrow} \qquad \frac{\sigma\downarrow \;\; \tau\downarrow}{s\sigma\tau\downarrow}$$

$$\frac{\tau\downarrow}{\Delta\tau\downarrow} \qquad \frac{\sigma\downarrow \;\; \tau\downarrow}{\Delta\sigma\tau\downarrow} \qquad \frac{\rho\downarrow \;\; \sigma\downarrow \;\; N\tau}{\Delta\rho\sigma\tau\downarrow} \qquad \frac{N\tau}{\tau\downarrow}$$

$$\mathrm{Pd}0 >_1 0 \qquad \frac{N\tau}{\mathrm{Pd}(S\tau) >_1 \tau}$$

$$\frac{\tau\downarrow}{p_1(p\sigma\tau) >_1 \sigma} \qquad \frac{\tau\downarrow}{p_2(p\tau\sigma) >_1 \sigma} \qquad \frac{\tau\downarrow}{k\sigma\tau >_1 \sigma}$$

$$s\rho\sigma\tau >_1 \rho\tau(\sigma\tau) \qquad \frac{\sigma\downarrow \quad N\tau}{\Delta\rho\sigma\tau\tau >_1 \rho} \qquad \frac{\rho\downarrow \;\; \tau\neq\tau'}{\Delta\rho\sigma\tau\tau' >_1 \sigma}$$

$$\frac{\sigma >_1 \tau}{\rho\sigma >_1 \rho\tau} \qquad \frac{\sigma >_1 \tau}{\sigma\rho >_1 \tau\rho} \qquad \tau \geq \tau \qquad \frac{\rho >_1 \sigma \quad \sigma \geq \tau}{\rho \geq \tau}$$

5.2. Conventions. $\sigma \equiv \tau$ means: σ and τ are identical terms. We abbreviate ($\sigma \geq \tau$ and $\tau\downarrow$) to $\sigma \geq \tau\downarrow$.

5.3. LEMMA. *In* APT($\downarrow$) *we have:*

i) $\sigma\tau\downarrow \Rightarrow (\sigma\downarrow$ *and* $\tau\downarrow)$;

ii) $(\sigma\downarrow$ *and* $\sigma \geq \tau) \Rightarrow \sigma \equiv \tau$.

PROOF. i) Inspection of the axioms and rules of APT($\downarrow$).

ii) It is clear that it suffices to show:

it is impossible that $\sigma\downarrow$ and $\sigma >_1 \tau$.

Assume $\sigma\downarrow$, $\sigma >_1 \tau$. Inspection of the axioms and rules learns that the proof of $\sigma >_1 \tau$ ends with the rule $\sigma' >_1 \tau' \Rightarrow \rho\sigma' >_1 \rho\tau'$ or the rule $\sigma' >_1 \tau' \Rightarrow \sigma'\rho >_1 \tau'\rho$. With (i), we now get $\sigma'\downarrow$, $\sigma' >_1 \tau'$. Repeating this argument, we end up with $c\downarrow$, $c >_1 \tau^*$, c some constant - and this is impossible. □

5.4. <u>LEMMA</u>. *(the Church-Rosser property for* APT(↓)).

(1) $\rho \geq \sigma_1$ *and* $\rho \geq \sigma_2 \Rightarrow$ *for some* τ, $\sigma_1 \geq \tau$ *and* $\sigma_2 \geq \tau$.

<u>PROOF</u>. We adapt Rosser's original proof for combinatory logic ([R35]; see also [Ba81], Exercise 7.4.13).
First we define APT(↓)*, which is obtained from APT(↓) by writing everywhere $>_*$ for $>_1$ and adding as new axiom and rule:

$$\tau >_* \tau \qquad \frac{\sigma_1 >_* \tau_1 \quad \sigma_2 >_* \tau_2}{\sigma_1\sigma_2 >_* \tau_1\tau_2}$$

We can interpret APT(↓)* in APT(↓) by reading everywhere $\geq$ for $>_*$, so APT(↓)* is conservative over APT(↓).
Then we prove the so-called Diamond property for $>_*$:

(2) $\rho >_* \sigma_1$ *and* $\rho >_* \sigma_2 \Rightarrow$ *for some* τ, $\sigma_1 >_* \tau$ *and* $\sigma_2 >_* \tau$.

This is done with induction over the length of the proofs of $\rho >_* \sigma_1$ and $\rho >_* \sigma_2$. We treat a typical case: $\rho \equiv k\sigma_1\rho_1$ and the last rule above $\rho >_* \sigma_1$ is $\rho_1{\downarrow} \Rightarrow k\sigma_1\rho_1 >_* \sigma_1$.
There are three possibilities to be distinguished:

i) $\sigma_2 \equiv \rho \equiv k\sigma_1\rho_1$: put $\tau := \sigma_1$.

ii) $\sigma_2 \equiv \sigma_1$: put $\tau := \sigma_1$.

iii) $\sigma_2 \equiv k\sigma'_1\rho'_1$ with $\sigma_1 >_* \sigma'_1$, $\rho_1 >_* \rho'_1$: by $\rho_1{\downarrow}$ and 5.3.(ii) we have $\rho_1 \equiv \rho'_1$, so $k\sigma'_1\rho'_1 >_* \sigma'_1$; hence put $\tau := \sigma'_1$.

Finally, by a well-known argument, (2) implies (1). □

5.5. <u>COROLLARY</u> *(of* 5.4 *and* 5.3.(ii); *uniqueness of normal form).*

$$\sigma \geq \tau_1{\downarrow} \ \textit{and} \ \sigma \geq \tau_2{\downarrow} \Rightarrow \tau_1 \equiv \tau_2.$$

We now state a characteristic property of APT(↓):

5.6. LEMMA. *Let* σ *be a subterm of* τ. *Then*

$$\tau \geq \tau_1\downarrow \;\Rightarrow\; \textit{for some } \sigma_1,\; \sigma > \sigma_1\downarrow .$$

PROOF. Induction over the length of a proof of $\tau \geq \tau_1$.

i) $\tau \equiv \tau_1$. Easy, use 5.3.(i).

ii) $\tau >_1 \tau_2 \geq \tau_1$. A typical case: $\tau \equiv k\tau_2\rho$ and the last rule above $\tau >_1 \tau_2$ is $\rho\downarrow \Rightarrow k\tau_2\rho >_1 \tau_2$. We look at the different positions of σ in τ.

a) $\sigma \equiv \tau$: put $\sigma_1 := \tau_1$.

b) $\sigma \equiv k\tau_2$: put $\sigma_1 := k\tau_1$.

c) $\sigma \equiv k$: put $\sigma_1 := k$.

d) σ is a subterm of τ_2: apply the induction hypothesis.

e) σ is a subterm of ρ : by $\rho\downarrow$ and 5.3.(i) we have $\sigma\downarrow$, so put $\sigma_1 := \sigma$.

Other cases are treated analogously. □

Now we can form a term model for $\underset{\sim}{\mathrm{APP}}^E$.

5.7. DEFINITION.

$$\begin{aligned} T &:= \{\tau \mid \tau \text{ a term of } \underset{\sim}{\mathrm{APT}}(\downarrow)\ \} \\ ET &:= \{\tau \in T \mid \underset{\sim}{\mathrm{APT}}(\downarrow) \vdash \tau\downarrow\ \} \\ NT &:= \{\tau \in T \mid \underset{\sim}{\mathrm{APT}}(\downarrow) \vdash N\tau\ \} \end{aligned}$$

We interpret

$\sigma = \tau$	by	$\exists\rho \in ET(\sigma \geq \rho$ and $\tau \geq \rho)$,
$E\tau$	by	$\exists\rho \in ET(\tau \geq \rho)$,
$\tau \in N$	by	$\exists\rho \in NT(\tau \geq \rho)$,
$\forall x, \exists y$	by	$\forall x \in ET,\ \exists y \in ET$.

5.8. THEOREM. *This interpretation is sound.*

PROOF. Most axioms and rules are easy. We briefly discuss the non-trivial cases:

$\forall xyz(x = z \wedge y = z \rightarrow x = y)$: use 5.5.

$\tau \in N \to E\tau$: recall that $N\tau \Rightarrow \tau\downarrow$ is a rule of $\underset{\sim}{\mathrm{APT}}(\downarrow)$.

$E\sigma\tau \to E\sigma \wedge E\tau$: use 5.6.

$\sigma \in N \wedge \sigma = \tau \to \tau \in N$: use 5.5.

$E\rho\sigma \wedge \sigma = \tau \to \rho\sigma = \rho\tau$: assume $E\rho\sigma$, $\sigma = \tau$, i.e. $\rho\sigma \geq \rho_1\downarrow$, $\sigma \geq \sigma'\downarrow$ $\tau \geq \sigma'\downarrow$. By 5.6: $\rho \geq \rho'\downarrow$, so $\rho\tau \geq \rho'\sigma'$. By 5.5: $\rho\sigma \geq \rho'\sigma' \geq \rho_1\downarrow$, and we conclude $\rho\tau \geq \rho_1\downarrow$, so $\rho\sigma = \rho\tau$.

$E\sigma\rho \wedge \sigma = \tau \to \sigma\rho = \tau\rho$: analogous.

$sxyz \simeq xz(yz)$: if $Es\rho\sigma\tau$, i.e. $s\rho\sigma\tau \geq \tau'\downarrow$, we obtain (by $s\rho\sigma\tau \geq$ $\geq \rho\tau(\sigma\tau)$ and 5.4, 5.5) $\rho\tau(\sigma\tau) \geq \tau'\downarrow$, so $s\rho\sigma\tau = \rho\tau(\sigma\tau)$; on the other hand, if $E\rho\tau(\sigma\tau)$, i.e. $\rho\tau(\sigma\tau) \geq \tau''\downarrow$ we get (by $s\rho\sigma\tau \geq \rho\tau(\sigma\tau)$) $s\rho\sigma\tau \geq \tau''\downarrow$ and again $s\rho\sigma\tau = \rho\tau(\sigma\tau)$. □

5.9. <u>COROLLARY</u>. *Let* σ, τ *be closed terms. Then*

i) $\underset{\sim}{\mathrm{APP}}^E \vdash E\tau \iff (\tau \geq \rho\downarrow$ *for some* $\rho)$;

ii) $\underset{\sim}{\mathrm{APP}}^E \vdash \tau \in N \iff (\tau \geq \rho$ *and* $N\rho$ *for some* $\rho)$;

iii) $\underset{\sim}{\mathrm{APP}}^E \vdash \sigma = \tau \iff (\sigma \geq \rho\downarrow$ *and* $\tau \geq \rho\downarrow$ *for some* $\rho)$.

5.10. <u>REMARKS</u>. i) By 5.9, we may call this interpretation a *free* model.

ii) We can strengthen ΔAX to

$\Delta\mathrm{AX}^+$ $\quad \Delta uvxx = u \wedge (x \neq y \to \Delta uvxy = v)$,

which yields decidability of $=$ for existing objects (for we have $0 \neq 1$, $(x = y \to \Delta 01xy = 0)$ and $(x \neq y \to \Delta 01xy = 1))$ and definition by cases on the universe of all existing objects. A term model for $\underset{\sim}{\mathrm{APP}}^E + \Delta\mathrm{AX}^+$ is obtained as follows: change $\underset{\sim}{\mathrm{APT}}(\downarrow)$ into $\underset{\sim}{\mathrm{APT}}(\downarrow)^+$ by dropping formulae $\sigma \neq \tau$ and the Δ-reduction rules, and adding as new rules

$$\frac{\sigma\downarrow \quad \tau\downarrow}{\Delta\rho\sigma\tau\tau >_1 \rho} \qquad \frac{\rho\downarrow \quad \tau\downarrow \quad \tau'\downarrow}{\Delta\rho\sigma\tau\tau' >_1 \sigma} \qquad \text{for all } \tau,\ \tau' \text{ with } \tau \not\equiv \tau';$$

then prove the Church-Rosser property for $\mathrm{APT}(\downarrow)^+$ (the proof runs analogous to 5.4) and define an interpretation as in 5.7.

Now we set out for a term model of $\underset{\sim}{\mathrm{APP}}$.

5.11. DEFINITION. The theory APT is defined as: APT($\downarrow$) without formulae of the form $\tau\downarrow$ (so several rules and axioms disappear, some rules become axioms).

5.12. LEMMA. *(The Church-Rosser property for* APT*).*
In APT *we have*

$$\rho \geq \sigma_1 \text{ and } \rho \geq \sigma_2 \Rightarrow \text{for some } \tau,\ \sigma_1 \geq \tau \text{ and } \sigma_2 \geq \tau .$$

PROOF. As for 5.4: skip all formulae $\tau\downarrow$. □

5.13. Now we interpret APP as follows (recall the definition of T and NT in 5.7):

$\sigma = \tau$ becomes $\exists\rho \in T\ (\sigma \geq \rho$ and $\tau \geq \rho)$;

$\tau \in N$ becomes $\exists\rho \in NT\ \tau \geq \rho$;

$\forall x$, $\exists y$ become $\forall x \in T$, $\exists y \in T$.

5.14. THEOREM. *This is a sound interpretation.*

PROOF. As for 5.8. We look at some non-trivial axioms:
$\forall xyz(x = z \wedge y = z \rightarrow x = y)$: assume $\rho = \tau$, $\sigma = \tau$, i.e. $\rho \geq \rho'$, $\tau \geq \rho'$ and $\sigma \geq \sigma'$, $\tau \geq \sigma'$. Now, by 5.12, $\rho' \geq \tau'$ and $\sigma' \geq \tau'$ for some τ', so $\rho \geq \tau'$ and $\sigma \geq \tau'$, i.e. $\rho = \sigma$.
$x = y \wedge x \in N \rightarrow y \in N$: assume $\sigma = \tau$, $\sigma \in N$, i.e. $\sigma \geq \tau'$, $\tau \geq \tau'$ and $\sigma \geq \sigma'$, $N\sigma'$. Now, by 5.12, $\sigma' \geq \rho$, $\tau' \geq \rho$ for some ρ; but, by inspection of the rules and axioms of APT, we see that $N\sigma'$ and $\sigma' \geq \rho$ imply $\sigma' \equiv \rho$; so we have $\tau \geq \sigma'$, $N\sigma'$, i.e. $\tau \in N$. □

5.15. We want to use this term model to show that APP is conservative over HA. HA is embedded in APP by the translation * defined in 4.1: observe that A^* is always a formula of APP. We assume APT to be formalized in HA such that
i) for any formula A of APT

(3) $\quad \text{APT} \vdash A \Rightarrow \text{HA} \vdash \ulcorner \text{APT} \vdash A \urcorner$;

ii) the following formalized instance of 5.12 holds:

(4) $$\mathrm{HA} \vdash \ulcorner \mathrm{APT} \vdash \tau \geq \underline{m} \urcorner \wedge \ulcorner \mathrm{APT} \vdash \tau \geq \underline{n} \urcorner \to m = n.$$

It is an easy but tedious affair to show that any reasonable formalisation $\ulcorner \ \urcorner$ makes (3) and (4) true.

5.16. LEMMA. *Let* ϕ *be a prim. rec. function symbol in the language of* HA, *and let* $\tau = \tau_\phi$ *be the corresponding term of* APP *(see* 3.11*). Then*

(5) $$\mathrm{HA} \vdash \phi(\vec{m}) = n \leftrightarrow \ulcorner \mathrm{APT} \vdash \tau\underline{\vec{m}} \geq \underline{n} \urcorner.$$

PROOF. We need the following theorems of APT:

i) $(\lambda x.\tau)\sigma \geq \tau[x:=\sigma]$;

ii) $R\sigma\tau 0 \geq \sigma$, $R\sigma\tau(S\underline{n}) \geq \tau\underline{n}(R\sigma\tau\underline{n})$.

Their derivations run parallel to the proofs of 3.6 resp. 3.7, 3.8. Now we can prove (5) with induction over the definition of ϕ (ϕ is defined using S, λ - abstraction and R , see 3.11). □

5.17. DEFINITION. ^{T}: APP → HA is the formalized version of the interpretation described in 5.13.

5.18. LEMMA. APP $\vdash$ A ⇒ HA $\vdash$ A^T.

PROOF. Formalize 5.14. □

5.19. LEMMA. *Let* $A = A(\vec{n})$ *be a formula of* HA. *Then*

(6) $$\mathrm{HA} \vdash A(\underline{\vec{n}})^{\circ T} \leftrightarrow A(\vec{n})$$

PROOF. Without loss of generality we may assume that the prime formulae of A have the form $\phi(\vec{m}) = n$, ϕ a primitive recursive function symbol. Now we can prove (6) with induction over the logical complexity of A .

A prime: by our assumption, $A = (\phi(\vec{m}) = n)$. Now $(\phi(\underline{\vec{m}}) = \underline{n})^{\circ T} =$ $= (\tau_\phi\underline{\vec{m}} = \underline{n})^T = \ulcorner \exists\rho \in NT(\mathrm{APT} \vdash \tau_\phi\underline{\vec{m}} \geq \rho$ and $\mathrm{APT} \vdash \underline{n} \geq \rho)\urcorner$; this last formula is equivalent to $\ulcorner \mathrm{APT} \vdash \tau_\phi\underline{\vec{m}} \geq \underline{n} \urcorner$, and (6) follows from 5.16.

A not prime: easy. □

5.20. THEOREM. APP *is conservative over* HA, *i.e. if* A *is a sentence of* HA, *then*

$$\mathrm{APP} \vdash A^{\circ} \iff \mathrm{HA} \vdash A.$$

PROOF. ⇐ follows from 4.2.(ii) and 3.5.(ii), ⇒ from 5.18 and 5.19. □

CHAPTER III. THE THEORY APP + EAC.

§1. Introduction.

1.1. EAC (extended axiom of choice) is the following schema:

EAC $\forall x(A(x) \to \exists y B(x,y)) \to \exists f \forall x(A(x) \to B(x,fx))$

A negative (i.e. contains no $\vee$, $\exists$).

In this chapter the theory APP + EAC is considered. We show (among other things) that it is incompatible with classical logic and conservative over HA.

1.2. Outline of the rest of this chapter.

First we consider the relation between EAC and several other schemata (§2). Via APP^{E} some of the results are transferred to HA and EL. In §3 we define realizability, an interpretation of APP into itself which appears to be axiomatized by EAC. The same holds for APP^{E}, and we conclude that realizability in APP^{E} is an abstract version of the well-known realizability interpretations devised by Kleene for HA and EL (see [K145], [K169] and also [T73]).

§4 and §5 are devoted to proving that APP + EAC is conservative over APP w.r.t. arithmetical formulae, and hence over HA. We define APP(ε) by adding Skolem functions ε_A for arithmetical A to APP: now APP(ε) $\vdash A \leftrightarrow \exists x\ x\ \underline{r}\ A$ for arithmetical A, so APP(ε) is conservative over APP + EAC w.r.t. arithmetical formulae. APP(ε) is reduced to APP in §5 by forcing, and the result follows. In §6 we generalize to extensions of $\mathbf{APP}$ with inductive definitions.

A digression is made in §7, where we consider Martin-Löf's basic theory of extensional types $\mathbf{ML}_0$. We interpret $\mathbf{HA}$ in $\mathbf{ML}_0$ and $\mathbf{ML}_0$ in $\mathbf{APP}$; the composition of these interpretations can be extended to an extensional realizability $\underline{e}$, by means of which we show that $\mathbf{ML}_0$ is conservative over $\mathbf{HA}$.

§2. EAC and other schemata.

2.1. We consider several schemata S, and prove either $\mathbf{APP} + \mathrm{EAC} \vdash S$ or $\mathbf{APP} + \mathrm{EAC} + S \vdash \bot$. Most of the results also hold for $\mathbf{APP}^E$ and consequently have their analogue in $\mathbf{HA}$ and $\mathbf{EL}$, via the translations described in Ch.II, §4.

2.2. DEFINITIONS. EAC$^+$, AC, AC$_\vee$, RDC (relativized dependent choices), IP$_N$, IP$^*_N$, IP* (independence of premises), DNS (double negation shift), DEQ (decidable equality) and KS (Kripke's schema) denote the following schemata:

EAC$^+$ $\quad \forall x(A(x) \to \exists y B(x,y)) \to \exists f \forall x(A(x) \to B(x,fx))$

(so EAC$^+$ is EAC without the restriction to negative A)

AC $\quad \forall x \exists y B(x,y) \to \exists f \forall x B(x,fx)$

AC$_\vee$ $\quad \forall x(A(x) \vee B(x)) \to \exists f \forall x((fx = 0 \wedge A(x)) \vee (fx = 1 \wedge B(x)))$

RDC $\quad \forall x(A(x) \to \exists y(A(y) \wedge B(x,y)) \to \forall x(A(x) \to \exists f(f0{=}x \wedge \forall n B(fn,f(n+1))))$

IP$_N$ $\quad (\neg A \to \exists n B(n)) \to \exists n(\neg A \to B(n))$

IP$^*_N$ $\quad (A \to \exists n B(n)) \to \exists n(A \to B(n)) \qquad$ (A negative)

IP* $\quad (A \to \exists x B(x)) \to \exists x(A \to B(x)) \qquad$ (A negative)

DNS $\quad \forall x \neg\neg A(x) \to \neg\neg \forall x A(x)$

DEQ $\quad \forall xy(x = y \vee x \neq y)$

KS $\quad \exists f(\forall n\ fn \in N \wedge (A \leftrightarrow \exists n\ fn = 0))$

2.3. FACT. $\mathrm{EAC}^+ \Rightarrow \mathrm{EAC} \Rightarrow \mathrm{AC} \Rightarrow \mathrm{AC}_\vee$.

2.4. LEMMA. $\underset{\sim}{\mathrm{APP}} + \mathrm{EAC}^+ \vdash \bot$.

PROOF. Take $A(x) := \exists y(xx \neq y)$, $B(x,y) := (xx \neq y)$ in EAC^+, then we get (observing that $\forall x(\exists y(xx \neq y) \to \exists y(xx \neq y))$ is true):

$$\exists f \forall x(\exists y(xx \neq y) \to xx \neq fx).$$

Now put $x := f$, then

$$\begin{aligned} & \exists f(\exists y(ff \neq y) \to ff \neq ff) \\ \Rightarrow\ & \exists f \neg\exists y(ff \neq y) \\ \Rightarrow\ & \exists f \forall y \neg\neg(ff = y) \\ \Rightarrow\ & \exists f(\neg\neg ff = 0 \wedge \neg\neg ff = 1) \\ \Rightarrow\ & \exists f \neg\neg(ff = 0 \wedge ff = 1) \\ \Rightarrow\ & \exists f \neg\neg(0 = 1) \quad \Rightarrow \quad \bot. \end{aligned}$$

□

We shall now show $\underset{\sim}{\mathrm{APP}} + \mathrm{EAC} \vdash \mathrm{RDC}$, IP^*. To derive RDC, we need what could be called a Normal Form Lemma for $\underset{\sim}{\mathrm{APP}} + \mathrm{EAC}$:

2.5. LEMMA. *For any formula* A *of* $\underset{\sim}{\mathrm{APP}}$ *there is a* negative *formula* $A^- = A^-(x)$ (x *not free in* A) *such that*

$$\underset{\sim}{\mathrm{APP}} + \mathrm{EAC} \vdash A \leftrightarrow \exists x A^-(x).$$

PROOF. Formula induction, using the definition of $\vee$, $\neg$ (see Ch.II, 2.1) and the equivalences

(i) $\quad (\exists x A_1(x) \wedge \exists x A_2(x)) \leftrightarrow \exists x(A_1(p_1 x) \wedge A_2(p_2 x))$,

(ii) $\quad (\exists x A_1(x) \to \exists x A_2(x)) \leftrightarrow \exists x \forall y(A_1(y) \to A_2(xy))$,

(iii) $\quad \forall y \exists x A_1(x,y) \leftrightarrow \exists x \forall y A_1(xy,y)$,

(iv) $\quad \exists y \exists x A_1(x,y) \leftrightarrow \exists x A_1(p_1 x, p_2 x)$.

(i), (iv) hold in $\underset{\sim}{\mathrm{APP}}$, (ii) and (iii) require EAC. □

2.6. LEMMA. $\underset{\sim}{\mathrm{APP}} + \mathrm{EAC} \vdash \mathrm{RDC}$.

PROOF. Assume

$$(1) \qquad \forall x(A(x) \to \exists y(A(y) \wedge B(x,y)))$$

By 2.2 there is a negative formula $A^{-}(x,z)$ with

$$(2) \qquad \underset{\sim}{\mathrm{APP}} + \mathrm{EAC} \vdash A(x) \leftrightarrow \exists z A^{-}(x,z),$$

so, combining (1) and (2), we have

$$(3) \qquad \forall xz(A^{-}(x,z) \to \exists yu(A^{-}(y,u) \wedge B(x,y))).$$

Now define

$$(4) \qquad \begin{aligned} A'(x) &:= A^{-}(p_1x, p_2x), \\ B'(x,y) &:= B(p_1x, p_1y), \end{aligned}$$

then, by (3)

$$(5) \qquad \forall x(A'(x) \to \exists y(A'(y) \wedge B'(x,y))).$$

Applying EAC to (5) (observe that A' is negative), we find some g with

$$(6) \qquad \forall x(A'(x) \to (A'(gx) \wedge B'(x,gx)).$$

Assume $A'(x)$ and define $h := \mathrm{R}x(kg)$, then (by Ch.II, 3.8)

$$(7) \qquad h0 = x, \quad h(n+1) = g(hn).$$

From (6) and (7) we obtain, using induction:

$$\forall n B'(hn, h(n+1))$$

so we have

$$\forall x(A'(x) \to \exists h(h0 = x \wedge \forall n B'(hn,h(n+1)))).$$

With (4):

$$\forall xz(A^{-}(x,z) \to \exists h(h0 = pxz \wedge \forall n B(p_1(hn),p_1(h(n+1)))))$$

so, by (2) and putting $f := \lambda n.p_1(hn)$:

$$\forall x(A(x) \to \exists f(f0 = x \wedge \forall n B(fn,f(n+1)))),$$

which is the conclusion of RDC. □

2.7. *LEMMA.* $\mathbf{APP} + \mathrm{EAC} \vdash \mathrm{IP}^{*}$.

PROOF. Assume

$$A \to \exists x B(x),$$

A negative. By EAC:

$$\exists f \forall y(A \to B(fy))$$

where y is some variable not in A, B. Put $y := 0$:

$$\exists f(A \to B(f0))$$

hence (application is total in $\mathbf{APP}$)

$$\exists x(A \to B(x)).$$

□

For IP_N, IP_N^{*} the situation is completely different:

2.8. *LEMMA.* i) $\mathbf{APP} + \mathrm{EAC} + \mathrm{IP}_N \vdash \perp$;

ii) $\mathbf{APP} + \mathrm{AC} + \mathrm{IP}_N^{*} \vdash \perp$.

<u>PROOF</u>. i) We start with the following instance of IP_N:

$$(\neg\neg xx \in N \to \exists n(xx{+}1 = n)) \to \exists n(\neg\neg xx \in N \to xx{+}1 = n).$$

We quantify over x and apply EAC: this is permitted, for $\neg\neg xx \in N \to \exists n(xx{+}1 = n)$ is equivalent to $\neg\neg xx \in N \to xx{+}1 \in N$, a negative formula:

$$\exists f\forall x((\neg\neg xx \in N \to xx{+}1 \in N) \to fx \in N \wedge (\neg\neg xx \in N \to xx{+}1 = fx)).$$

Put $x := f$:

$$\begin{aligned} &\exists f((\neg\neg ff \in N \to ff{+}1 \in N) \to ff \in N \wedge (\neg\neg ff \in N \to ff{+}1 = ff)) \\ \Rightarrow\ &\exists f((\neg\neg ff \in N \to ff{+}1 \in N) \to (ff \in N \wedge ff{+}1 = ff)) \\ \Rightarrow\ &\exists f\neg(\neg\neg ff \in N \to ff{+}1 \in N) \\ \Rightarrow\ &\exists f(\neg\neg ff \in N \wedge \neg ff{+}1 \in N) \\ \Rightarrow\ &\exists f\neg\neg(ff \in N \wedge \neg ff{+}1 \in N) \\ \Rightarrow\ &\neg\neg\bot \Rightarrow \bot . \end{aligned}$$

ii) We start with $xx \in N \to \exists n(xx{+}1 = n)$, which is derivable in $\underset{\sim}{APP}$. By IP_N^* and quantification over x:

$$\forall x\exists n(xx \in N \to xx{+}1 = n).$$

Now we apply AC:

$$\exists f\forall x(fx \in N \wedge (xx \in N \to xx{+}1 = fx)).$$

Put $x := f$:

$$\exists f(ff \in N \wedge (ff \in N \to ff{+}1 = ff)) \Rightarrow \exists f(ff{+}1 = ff \in N) \Rightarrow \bot. \quad \square$$

This and the next lemma show that $\underset{\sim}{APP} + EAC$ is essentially non-classical.

2.9. <u>LEMMA</u>. i) $\underset{\sim}{APP} + DNS + AC_V \vdash \bot$;

ii) $\underset{\sim}{APP} + DEQ + AC_V \vdash \bot$.

PROOF. i) By logic $\forall x \neg\neg(xx = 1 \vee xx \neq 1)$, so with DNS:

$$\neg\neg\forall x(xx = 1 \vee xx \neq 1).$$

By AC_V:

$$\neg\neg\exists f\forall x((fx = 0 \wedge xx = 1) \vee (fx = 1 \wedge xx \neq 1)).$$

Put $x := f$:

$$\neg\neg\exists f((ff = 0 \wedge ff = 1) \vee (ff = 1 \wedge ff \neq 1))$$

and this is a contradiction.

ii) DEQ implies $\forall x(xx = 1 \vee xx \neq 1)$; now proceed as under (i), without $\neg\neg$. □

2.10. COROLLARY. $\underset{\sim}{\mathrm{APP}} \nvdash AC_V$.

PROOF. $\underset{\sim}{\mathrm{APP}} + \mathrm{DEQ}$ is consistent, for $\Delta AX^+ \Rightarrow \mathrm{DEQ}$ and $\underset{\sim}{\mathrm{APP}} + \Delta AX^+$ has a model (see Ch.II, 5.10). □

Finally we combine AC and KS:

2.11. LEMMA. $\underset{\sim}{\mathrm{APP}} + \mathrm{AC} + \mathrm{KS} \vdash \perp$.

PROOF. Take KS with $A := xx \notin N$, and quantify over x :

$$\forall x\exists f(\forall n\ fn \in N \wedge (xx \notin N \leftrightarrow \exists n\ fn = 0)).$$

By AC, we find a g with

(1) $$\forall x(\forall n\ gxn \in N \wedge (xx \notin N \leftrightarrow \exists n\ gxn = 0)).$$

Define $h := \lambda x.\mu(gx)$, then (by Ch.II, 3.10)

(2) $$\forall x(\exists n\ gxn = 0 \leftrightarrow hx \in N),$$

for $\forall n\ gxn \in N$. Now put $x := h$ in (1):

$$\forall n\ ghn = 0 \wedge (hh \notin N \leftrightarrow \exists n\ ghn = 0)$$

and this contradicts (2). □

With exception of 2.7 ($\mathbf{APP} + \mathrm{EAC} \vdash \mathrm{IP}^*$) and 2.9.(ii) ($\mathbf{APP} + \mathrm{DEQ} + \mathrm{AC}_\vee \vdash \vdash \bot$), all results for $\mathbf{APP}$ of this section can be transferred to $\mathbf{APP}^E$:

2.12. LEMMA.

i) $\mathbf{APP}^E + \mathrm{EAC}^+ \vdash \bot$;

ii) $\mathbf{APP}^E + \mathrm{EAC} \vdash \mathrm{RDC}$;

iii) $\mathbf{APP}^E + \mathrm{EAC} + \mathrm{IP}_N \vdash \bot$, $\mathbf{APP}^E + \mathrm{AC} + \mathrm{IP}^*_N \vdash \bot$;

iv) $\mathbf{APP}^E + \mathrm{DNS} + \mathrm{AC}_\vee \vdash \bot$;

v) $\mathbf{APP}^E + \mathrm{AC} + \mathrm{KS} \vdash \bot$.

PROOF. As above. The only modification, concerning the proof of (ii), are
a) read $A_i(\tau) \wedge E\tau$ for $A_i(\tau)$ if τ is a compound term ($i = 1,2$) in the proof of 2.5;
b) replace (4) in 2.6 by $A'(x) := A^-(p_1x, p_2x) \wedge Ep_1x \wedge Ep_2x$, $B'(x,y) := B(p_1x, p_1y) \wedge Ep_1x \wedge Ep_1y$. □

2.13. The interpretations $'$: $\mathbf{APP}^E \to \mathbf{HA}^*$ (Ch.II, 4.3) and $''$: $\mathbf{APP}^E \to \mathbf{EL}^*$ (Ch.II, 4.9) enable us to obtain from lemma 2.12 some results for $\mathbf{HA}$ and $\mathbf{EL}$. To see which schemata in $\mathbf{HA}$, $\mathbf{EL}$ correspond to EAC, we have to find out what happens with negative formulae when going from $\mathbf{APP}^E$ via $\mathbf{HA}^*$, $\mathbf{EL}^*$ to $\mathbf{HA}$, $\mathbf{EL}$.
We claim: negative formulae in $\mathbf{APP}^E$ correspond (modulo logical equivalence) to *almost negative* formulae in $\mathbf{HA}$ and $\mathbf{EL}$. As usual, we call a formula almost negative if it contains no $\vee$, and $\exists$ only in front of prime formulae. To justify our claim, we prove two lemmata.

2.14. LEMMA. *Let* P *be a prime formula of* $\mathbf{APP}^E$, x *not in* P. *Then:*
i) *there is a term* $t = t(x)$ *in* $\mathbf{HA}$ *such that*

(1) $$\mathbf{HA} \vdash P'^{\circ} \leftrightarrow \exists x\; t(x) = 0;$$

ii) *there is a function term* $\phi = \phi(x)$ *of* $\mathbf{EL}$ *such that*

(2) $$\mathbf{EL} \vdash P''^{\circ} \leftrightarrow \forall y \exists x\; \phi(x)y = 0.$$

PROOF. i) Let $P = (\sigma = \tau)$. By applying

$$\sigma = \tau \leftrightarrow \exists x(\sigma = x \wedge \tau = x)$$

$$\sigma\tau = x \leftrightarrow \exists yz(\sigma = y \wedge \tau = z \wedge yz = x)$$

$$\exists x(A \wedge \exists yB) \leftrightarrow \exists xy(A \wedge B) \qquad (\text{ y not in } A)$$

we find $P_1,\ldots,P_n$ with

$$\underset{\sim\sim}{APP}^E \vdash P \leftrightarrow \exists\vec{x}(P_1 \wedge \ldots \wedge P_n)$$

and the P_i equal to $xy = z$, $x = y$ or $x = c$ (c a constant). By Ch.II, 4.4:

$$\underset{\sim}{HA}^* \vdash P' \leftrightarrow \exists\vec{x}(P_1' \wedge \ldots \wedge P_n').$$

Now $(x = y)'^\circ = (x = y)$, $(x = c)'^\circ = (x = c)$, $(xy = z)'^\circ = (\{x\}(y) = z)^\circ$ and this is equivalent to $\exists u(\phi_T(x,y,u) = 0 \wedge Uu = z)$ where ϕ is the primitive recursive characteristic function of Kleene's T-predicate. By Ch.II, 4.3 we get

$$\underset{\sim}{HA} \vdash P'^\circ \leftrightarrow \exists\vec{x}\vec{u}(Q_1 \wedge \ldots \wedge Q_m)$$

where the Q_i are prime formulae of $\underset{\sim}{HA}$. Using

$$s = t \longmapsto |s - t| = 0$$

$$s = 0 \wedge t = 0 \longmapsto s + t = 0$$

$$\exists xyA(x,y) \longmapsto \exists xA(j_1x,j_2x)$$

we find a $t = t(x)$ satisfying (1).

If $P = (E\tau)$ then $P \leftrightarrow \exists x\ x = \tau$, and we proceed as above. $P = (\tau \in N)$ is treated as $P = (E\tau)$, for $(\tau \in N)' = (E\tau)'$.

ii) We need the following two facts:

a) If $\phi = \phi(a)$ is a function term of $\underset{\sim}{EL}^*$, then

(3) $$\mathbf{EL}^* \vdash \forall a \forall y \exists x \forall z \leq y \ \phi(a)z \simeq \phi(f_{\bar{a}x})z;$$

here f_n is a function term satisfying

$$f_n x = \begin{cases} (n)_x & \text{if} \quad x < \text{lth } n, \\ 0 & \text{if} \quad x \geq \text{lth } n. \end{cases}$$

(3) expresses that function terms of $\mathbf{EL}^*$ are continuous in their function parameters.

b) We extend $\mathbf{EL}^*$ conservatively to $\mathbf{EL}^{**}$ by adding Kleene-brackets, in the same way as for $\mathbf{HA}$ (Ch.II, 4.3). Now, if $\phi = \phi(\vec{x})$ is a function term of $\mathbf{EL}^*$ without function variables, then there is a term $t = t(\vec{x},y)$ of $\mathbf{HA}^*$ such that

(4) $$\mathbf{EL}^{**} \vdash \forall \vec{x} y \ \phi(\vec{x})y \simeq t(\vec{x},y).$$

(3) and (4) are proved in a straightforward way by term induction.

Now we prove (ii). Let $P = (\sigma = \tau)$, so $P'' = \forall y(\sigma''y = \tau''y)$. Without loss of generality we assume that a is the only function variable in σ'' and τ'': $\sigma'' = \sigma''(a)$, $\tau'' = \tau''(a)$. By (3):

$$\mathbf{EL}^* \vdash P'' \leftrightarrow \forall y \exists x \exists n(\bar{a}x = n \wedge \sigma''(f_n)y = \tau''(f_n)y).$$

By (4), we can find $s = s(n,y)$, $t = t(n,y)$ in $\mathbf{HA}^*$ such that

$$\mathbf{EL}^{**} \vdash P'' \leftrightarrow \forall y \exists x \exists n(\bar{a}x = n \wedge s(n,y) = t(n,y)).$$

Now we proceed as under (i) to find a ϕ which satisfies (2).

$P = E\tau$: $P'' = \exists a \forall y(ay = \tau''y) \equiv \forall y \exists x(\tau''y = x)$ (for $\mathbf{EL}$ has the axiom QF-AC); now continue as above.

$P = (\tau \in N)$: $P'' = \exists z \forall y(\tau''y = z) \equiv \forall y(\tau''y = \tau''0)$, so this case is reduced to $P = (\sigma = \tau)$, too. □

Now we go the other way round:

2.15. LEMMA. *Let* P *be a prime formula of* $\mathbf{HA}$, *with free variables* x, $\vec{y}$. *Then there is a term* τ *of* $\mathbf{APP}^E$ *such that*

$$\underset{\sim\sim}{APP}^E \vdash \vec{y} \in N \to ((\exists xP)^{\circ} \leftrightarrow \tau \in N).$$

ii) *Let* P *be a prime formula of* $\underset{\sim}{EL}$, *with free variables* $x, \vec{y}, a, \vec{b}$. *Then there are terms* σ, τ *of* $\underset{\sim\sim}{APP}^E$ *such that*

$$\underset{\sim\sim}{APP}^E \vdash \vec{y} \in N \wedge a,\vec{b} \in (N \Rightarrow N) \to ((\exists xP)^{\circ} \leftrightarrow \sigma \in N),$$

$$\underset{\sim\sim}{APP}^E \vdash x,\vec{y} \in N \wedge \vec{b} \in (N \Rightarrow N) \to ((\exists aP)^{\circ} \leftrightarrow \tau \in N).$$

PROOF. Without loss of generality we assume $P = (t = 0)$, so $(\exists xP)^{\circ} = \; = \exists x \in N \; t^{\circ} = 0$. By Ch.II, 4.2.(ii) (soundness of $^{\circ}$) we have

$$\underset{\sim\sim}{APP}^E \vdash \vec{y} \in N \to \forall x \in N(t^{\circ} \in N),$$

so, with Ch.II, 3.10:

$$\underset{\sim\sim}{APP}^E \vdash \vec{y} \in N \to (\exists x \in N \; t^{\circ} = 0 \leftrightarrow \mu(\lambda x.t^{\circ}) \in N).$$

ii) The first part is proved as (i), the second part is reduced to the first by observing that (3) in the proof of lemma 2.14 implies

$$\underset{\sim}{EL} \vdash \exists a \; t(a) = 0 \leftrightarrow \exists n \; t(f_n) = 0.$$

□

.16. COROLLARY. *The negative formulae of* $\underset{\sim\sim}{APP}^E$ *correspond exactly (modulo equivalence) with the almost negative formulae in* $\underset{\sim}{HA}$ *and* $\underset{\sim}{EL}$. *More precisely:*

i) *a formula* A *of* $\underset{\sim}{HA}$ *is almost negative (modulo equivalence) iff there is a negative formula* B *of* $\underset{\sim\sim}{APP}^E$ *with* $\underset{\sim}{HA} \vdash A \leftrightarrow B'^{\circ}$;

ii) *idem for* $\underset{\sim}{EL}$.

PROOF. i) ⇒: replace all subformulae $(\exists xP)^{\circ}$ of A° by $\tau \in N$, according to 2.15.(i). The result we call B. B is negative, and by 2.15.(i) and Ch.II, 4.5.(ii) we obtain $\underset{\sim}{HA} \vdash A \leftrightarrow B'^{\circ}$.

⇐: ' and $^{\circ}$ commute with the logical connectives, so by 2.14.(i) we get B negative ⇒ B'° almost negative.

ii) As (i). □

2.17. *DEFINITION*. i) ECT_0 (extended Church's thesis) is the following schema in $\mathbf{HA}$:

ECT_0 $\quad\forall x(A(x)\to\exists yB(x,y))\to\exists e\forall x(A(x)\to\exists z(Texz\wedge B(x,Uz)))$,

A almost negative.

ii) GC (generalized continuity) is the $\mathbf{EL}$ - schema

GC $\quad\forall a(A(a)\to\exists bB(a,b))\to\exists c\forall a(A(a)\to\exists b(b=(c|a)\wedge B(a,b)))$

A almost negative,

where $b=(c|a)$ abbreviates $\forall x\exists y(c(\langle x\rangle * \bar{a}y)=bx+1\wedge\forall z<y\ c(\langle x\rangle * \bar{a}z)=0)$.

2.18. *LEMMA*. i) $\mathbf{APP}^E+EAC\vdash A\ \Rightarrow\ \mathbf{HA}+ECT_0\vdash A^{\prime\circ}$;

ii) $\mathbf{APP}^E+EAC\vdash A\ \Rightarrow\ \mathbf{EL}+GC\vdash A^{\prime\circ}$.

PROOF. This extension of Ch.II, 4.4 and 4.10 follows from $EAC^{\prime\circ}=ECT_0$, $EAC^{\prime\prime\circ}=GC$. □

We define some other schemata in $\mathbf{HA}$, $\mathbf{EL}$:

2.19. *DEFINITION*. i) ECT_0^+, GC^+ are ECT_0 resp. GC without the restriction to almost negative A.

ii) CT_0, C are ECT_0 resp. GC with $A:=\top$.

iii) RDC_1 is the following schema of $\mathbf{EL}$:

$$RDC_1 \quad \forall a(A(a)\to\exists b(A(b)\wedge B(a,b)))\to$$
$$\to\forall a(A(a)\to\exists c((c)_0=a\wedge\forall nB((c)_n,(c)_{n+1}))$$

where $(c)_n:=\lambda x.c(j(n,x))$.

2.20. *LEMMA*. i) $\mathbf{HA}+ECT_0^+\vdash\bot$;

ii) $\mathbf{EL}+GC^+\vdash\bot$;

iii) $\mathbf{EL}+GC\vdash RDC_1$;

iv) $\mathbf{HA}+ECT_0+IP_N\vdash\bot$;

v) $\mathbf{HA}+CT_0+DNS\vdash\bot$;

vi) $\mathbf{EL}+C+DNS\vdash\bot$.

PROOF. Follows from 2.12, 2.18 and Ch.II, 4.4 and 4.10. For (iii) we need $\forall n\forall y\exists z(c|\lambda x.n)y = z \rightarrow \exists c'\forall n\forall y\ c'_n y = (c|\lambda x.n)y$, and this is derivable with help of QF-AC. □

2.21. REMARKS.

Several of the results of this section are known in the literature, sometimes in a slightly different form:

i) Feferman proves (in [Fe79], IV.10) that $T_0 + AC \vdash \perp$ (the proof is due to Friedman). Now T_0 can be seen as a strengthening of APP^E in which Feferman's AC is comparable with EAC^+, and the proof also works to show $APP^E + EAC^+ \vdash \perp$.

ii) In [Ba73], Barendregt cites a proof by D.S. Scott that the *classical* first order theory of combinatory logic conflicts with AC. The same proof yields APP + AC + classical logic $\vdash \perp$.

iii) Troelstra shows in [T69], 16.3 that KS (even in a weaker version) is incompatible with enumeration principles such as CT_0. In [Be79a], Beeson gives a proof (by Luckhardt) that $C + KS \vdash \perp$: C is a theory related to APP^E in which AC is derivable. The proof is essentially the same as that of 2.11.

iv) $HA + ECT_0^+ \vdash \perp$ was proved in [T73], 3.2.20; there (3.4.14) one also finds a proof of $HA + ECT_0 + IP_N \vdash \perp$, due to Beeson [Be72].

§3. Realizability.

In this section we define an interpretation of APP into itself called realizability, an abstract version of Kleene's recursive realizability for HA (see [Kl45]). Realizability in APP is axiomatized by EAC, and we use this fact to present a syntactically defined class of formulae for which APP + EAC is conservative over APP. The definition of realizability is adapted for APP^E and the results for APP are transferred to APP^E. Finally we turn to HA and EL, via the translations ' and " of Ch.II, §4.

3.1. DEFINITION. $\tau\underline{r}A$ (τ *realizes* A) is defined as follows:

$$\tau\underline{r}P \quad := P \quad \text{for prime } P$$

$$\tau\underline{r}(A \wedge B) := p_1\tau\underline{r}A \wedge p_2\tau\underline{r}B$$

$$\tau\underline{r}(A \rightarrow B) := \forall x(x\underline{r}A \rightarrow \tau x\underline{r}B)$$

$$\tau\underline{r}\forall xA := \forall x(\tau x\underline{r}A)$$

$$\tau\underline{r}\exists xA := p_2\tau\underline{r}(A[x := p_1\tau])$$

3.2. FACTS. i) $\tau\underline{r}A$ is a negative formula;

ii) $(\tau\underline{r}A)[x := \sigma] = \tau[x := \sigma]\underline{r}(A[x := \sigma])$, if x not bound in A nor in $\tau\underline{r}A$.

3.3. THEOREM. APP $\vdash A \Rightarrow$ APP $\vdash \tau\underline{r}A$ *for some term* τ .

PROOF. Induction over the length of a derivation of A in APP. For this one uses the following, which are verified easily (we assume y not in σ, τ):

$\lambda x.x\underline{r}(A \to A)$

$\lambda y.y\tau\underline{r}(\forall xA \to A[x := \tau])$

$\lambda y.p\tau y\underline{r}(A[x := \tau] \to \exists xA)$

$\tau\underline{r}A \Rightarrow \lambda y.\tau\underline{r}(B \to A)$

$\sigma\underline{r}(A \to B)$, $\tau\underline{r}(B \to C) \Rightarrow \lambda y.\tau(\sigma y)\underline{r}(A \to C)$

$\sigma\underline{r}A$, $\tau\underline{r}(A \to B) \Rightarrow \tau\sigma\underline{r}B$

$\sigma\underline{r}(A \to B)$, $\tau\underline{r}(A \to C) \Rightarrow \lambda y.p(\sigma y)(\tau y)\underline{r}(A \to (B \wedge C))$

$\tau\underline{r}(A \to (B \wedge C)) \Rightarrow \lambda y.p_1(\tau y)\underline{r}(A \to B)$, $\lambda y.p_2(\tau y)\underline{r}(A \to C)$

$\tau\underline{r}((A \wedge B) \to C) \Rightarrow \lambda yz.\tau(pyz)\underline{r}(A \to (B \to C))$

$\tau\underline{r}(A \to (B \to C)) \Rightarrow \lambda y.\tau(p_1y)(p_2y)\underline{r}((A \wedge B) \to C)$

$\tau\underline{r}(A \to B) \Rightarrow \lambda yx.\tau y\underline{r}(A \to \forall xB)$

$\tau\underline{r}(A \to B) \Rightarrow \lambda y.\tau[x := p_1y](p_2y)\underline{r}(\exists xA \to B)$

$p(\lambda x.0)(\lambda xyzu.0)\underline{r}$ =AX

$\lambda x.p(p00)(\lambda y.0)\underline{r}$SUB

$0\underline{r}k$AX, sAX

$p00\underline{r}p$AX

$0\underline{r}0$AX

$\lambda x.p00\underline{r}$SAX

$p0(\lambda x.p00)\underline{r}$PdAX

$\lambda x.p00\underline{r}\Delta AX$

$\lambda yu.R(p_1 y)(\lambda xz.(p_2 y)x(p0z))\underline{r}\mathrm{IND}$

□

To be able to show that EAC is realized, we need to know that negative formulae are not affected by $\underline{r}$.

3.4. <u>LEMMA</u>. $\mathrm{APP} \vdash A \leftrightarrow \exists x\ x\underline{r}A \leftrightarrow \forall x\ x\underline{r}A$ *if* A *negative*.

<u>PROOF</u>. Simple, with formula induction. □

3.5. <u>LEMMA</u>. *There is a term* τ *such that*

$$\mathrm{APP} \vdash \tau\underline{r}\mathrm{EAC},$$

i.e. τ *realizes every instance of* EAC.

<u>PROOF</u>. Take

$$\tau := \lambda z.p(\lambda x.p_1(zx0))(\lambda xv.p_2(zx0))$$

and assume $z\underline{r}\forall x(A(x) \to \exists y B(x,y))$ (A negative), i.e.

$$\forall xu(u\underline{r}A(x) \to p_2(zxu)\underline{r}B(x,p_1(zxu))).$$

We put $u := 0$ and use $0\underline{r}A(x) \leftrightarrow \exists v\ v\underline{r}A(x)$ (3.4):

$$\forall xv(v\underline{r}A(x) \to p_2(zx0)\underline{r}B(x,p_1(zx0))),$$

i.e.

$$p(\lambda x.p_1(zx0))(\lambda xv.p_2(zx0))\underline{r}\exists f\forall x(A(x) \to B(x,fx)),$$

and conclude $\tau\underline{r}\mathrm{EAC}$. □

For the axiomatization of $\underline{r}$ we now only need the next lemma.

3.6. <u>LEMMA</u>. $\mathrm{APP} + \mathrm{EAC} \vdash A \leftrightarrow \exists x\ x\underline{r}A$, *for all* A.

<u>PROOF</u>. Formula induction.

A prime, $A = B \wedge C$: trivial resp. easy.

$A = B \to C$: $\exists x\ x\underline{r}(B \to C) = \exists x \forall y (y\underline{r}B \to xy\underline{r}C)$; by EAC (recall that $y\underline{r}B$ is negative) this is equivalent to $\exists y\ y\underline{r}B \to \exists x\ x\underline{r}C$ and we can apply the induction hypothesis.

$A = \forall yB$: $\exists x\ x\underline{r}\forall yB = \exists x \forall y\ xy\underline{r}B$, which is equivalent to $\forall y \exists x\ x\underline{r}B$ (by EAC); now use the induction hypothesis.

$A = \exists yB$: $\exists x\ x\underline{r}\exists yB = \exists x\ p_2 x\underline{r}B[y := p_1 x]$, this is equivalent to $\exists xy\ x\underline{r}B$ and (by the induction hypothesis) to $\exists yB$. □

3.7. THEOREM. $\mathbf{APP} \vdash \exists x\ x\underline{r}A \iff \mathbf{APP} + \mathrm{EAC} \vdash A$.

PROOF. If $\mathbf{APP} \vdash \exists x\ x\underline{r}A$ then also $\mathbf{APP} + \mathrm{EAC} \vdash \exists x\ x\underline{r}A$; now by 3.6 and modus ponens $\mathbf{APP} + \mathrm{EAC} \vdash A$.
On the other hand, if $\mathbf{APP} + \mathrm{EAC} \vdash A$, then $\mathbf{APP} \vdash C \to A$ with C a conjunction of instances of EAC. By 3.3:

(1) $\mathbf{APP} \vdash \forall x (x\underline{r}C \to \tau x\underline{r}A)$ for some τ,

and by 3.5

(2) $\mathbf{APP} \vdash \sigma\underline{r}C$ for some σ;

now (1) and (2) yield $\mathbf{APP} \vdash \exists x\ x\underline{r}A$. □

Theorem 3.7 is the basis for the conservation results we shall prove in this and the next section. A direct consequence of 3.7 is

3.8. LEMMA. $\mathbf{APP} + \mathrm{EAC}$ *is conservative over* $\mathbf{APP}$ *with respect to the class of formulae* $\{A \mid \mathbf{APP} \vdash (\exists x\ x\underline{r}A) \to A\}$.

PROOF. Evident, by 3.7. □

Now we define syntactically a class Γ of formulae of $\mathbf{APP}$ and prove $\Gamma \subset \{A \mid \mathbf{APP} \vdash (\exists x\ x\underline{r}A) \to A\}$. We assume the notions of positive and negative occurrence to be known, and recall that $\vee$ is defined using $\exists$.

3.9. DEFINITION. $\Gamma := \{A \mid$ all negatively occurring subformulae are $\exists$-free$\}$.

3.10. LEMMA. $A \in \Gamma \;\Rightarrow\; \mathrm{APP} \vdash (\exists x\; x\underline{r}A) \to A$.

PROOF. Formula induction.

A prime, $A = B \wedge C$: easy.

$A = B \to C$: let $B \to C \in \Gamma$, then B $\exists$-free and $C \in \Gamma$. Assume $\exists x(x\underline{r}(B \to C))$, i.e. $\exists x \forall y(y\underline{r}B \to xy\underline{r}C)$, so $\exists y(y\underline{r}B) \to \exists z(z\underline{r}C)$. Now (by 3.4) $B \to \exists y(y\underline{r}B)$, so with the induction hypothesis we get $B \to C$.

$A = \forall yB$: let $\forall yB \in \Gamma$, then $B \in \Gamma$. Assume $\exists x(x\underline{r}\forall yB)$, i.e. $\exists x \forall y(xy\underline{r}B)$, so $\forall y \exists x(x\underline{r}B)$. With the induction hypothesis we get $\forall yB$.

$A = \exists yB$: let $\exists yB \in \Gamma$, then $B \in \Gamma$. Assume $\exists x\; x\underline{r}\exists yB$, i.e. $\exists x\; p_2 x\underline{r}B[y := p_1 x]$, so $\exists y \exists x\; x\underline{r}B$. With the induction hypothesis we get $\exists yB$. □

3.11. THEOREM. APP + EAC *is conservative over* APP *with respect to* Γ.

PROOF. Combine 3.8 and 3.10. □

Now we turn to $\mathrm{APP}^{\mathrm{E}}$. The definition of $\underline{r}$ has to be modified in order to ensure $\tau\underline{r}A \to E\tau$: this property of $\underline{r}$ is required in the soundness proof, e.g. to deal with modus ponens (PR3): if $\sigma\underline{r}A$, $\tau\underline{r}(A \to B)$ (i.e. $\forall x(x\underline{r}A \to \tau x\underline{r}B)$) we need $E\sigma$ to conclude $\tau\sigma\underline{r}B$.
Another difference is that we no longer have $A \leftrightarrow \forall x(x\underline{r}A)$ for negative A. This is a consequence of partial application, which makes that we neither have $\forall x\; x\underline{r}(\top \to \top)$, but only for those x with $\forall y\; Exy$. Nevertheless, we can save the essentials of 3.4 and 3.10 by taking $\tau_A\underline{r}A$ instead of $\forall x(x\underline{r}A)$, with τ_A a 'canonical realizer' for A whose definition only depends on the logical form of A.

3.12. DEFINITION. i) For $\mathrm{APP}^{\mathrm{E}}$, $\tau\underline{r}A$ is defined as follows:

$$\begin{array}{lll}
\tau\underline{r}P & := E\tau \wedge P & \text{for prime } P \\
\tau\underline{r}(A \wedge B) & := p_1\tau\underline{r}A \wedge p_2\tau\underline{r}B & \\
\tau\underline{r}(A \to B) & := E\tau \wedge \forall x(x\underline{r}A \to \tau x\underline{r}B) & \\
\tau\underline{r}\forall xA & := \forall x(\tau x\underline{r}A) & \\
\tau\underline{r}\exists xA & := Ep_1\tau \wedge p_2\tau\underline{r}(A[x := p_1\tau]) &
\end{array}$$

ii) We define τ_A, A a negative formula of $\mathrm{APP}^{\mathrm{E}}$, by:

$$\tau_P := 0 \qquad \text{for prime } P$$

$$\tau_{A \wedge B} := p\tau_A\tau_B$$

$$\tau_{A \to B} := \tau_{\forall x B} := \lambda x.\tau_B$$

We collect everything we know about $\underline{r}$ in $\mathbf{APP}^E$ in one theorem.

3.13. THEOREM.

i) $\mathbf{APP}^E \vdash \tau\underline{r}A \to E\tau$;

ii) $\tau\underline{r}A$ *is a negative formula;*

iii) $(\tau\underline{r}A)[x := \sigma] = \tau[x := \sigma]\underline{r}(A[x := \sigma])$, *if* x *is not bound in* A *or in* $\tau\underline{r}A$;

iv) $\mathbf{APP}^E \vdash A \;\Rightarrow\; \mathbf{APP}^E \vdash \tau\underline{r}A$ *for some* τ;

v) $\mathbf{APP}^E \vdash A \;\leftrightarrow\; \exists x\, x\underline{r}A \;\leftrightarrow\; \tau_A\underline{r}A$ *for negative* A;

vi) *for every instance* $EAC(A,B)$ *of* EAC *there is a term* σ_A *(depending on* A*, not on* B*) with*

$$\mathbf{APP}^E \vdash \sigma_A\, \underline{r}\, EAC(A,B);$$

vii) $\mathbf{APP}^E + EAC \vdash A \;\leftrightarrow\; \exists x\, x\underline{r}A$;

viii) $\mathbf{APP}^E \vdash \exists x\, x\underline{r}A \;\Longleftrightarrow\; \mathbf{APP}^E + EAC \vdash A$;

ix) *let* Γ *be defined as in* 3.9, *but now for the language of* $\mathbf{APP}^E$, *then*

$$A \in \Gamma \;\Rightarrow\; \mathbf{APP}^E \vdash (\exists x\, x\underline{r}A) \to A;$$

x) $\mathbf{APP}^E + EAC$ *is conservative over* $\mathbf{APP}^E$ *with respect to* Γ.

PROOF. i), ii), iii) Formula induction.
iv) As 3.3. The new axioms of $\mathbf{APP}^E$ are dealt with as follows:
$\lambda y.p\tau 0\underline{r}(E\tau \to \exists x(x = \tau))$;
$\lambda y.0\underline{r}(\exists x(x = \tau) \to E\tau)$;
the components of STR and SUB are realized by
$\lambda y.0$, $\lambda y.p00$;
$\lambda x.xy\underline{r}(\forall x A \to A[x := y])$;
$\lambda x.pyx\underline{r}(A[x := y] \to \exists x A)$;

$p0(\lambda x.0)\underline{r}SAX^E$.

v) Formula induction.

vi) Take $\sigma_A := \lambda z.p(\lambda x.p_1(zx\tau_A))(\lambda xv.p_2(zx\tau_A))$ and proceed as in the proof of 3.5, using $\tau_A\underline{r}A(x) \leftrightarrow \exists v\ v\underline{r}A(x)$.

vii) Formula induction, as 3.6.

viii) As 3.7.

ix) Analogous to 3.10.

x) As 3.11. □

Before we transfer theorem 3.13 to $\underset{\sim}{\mathrm{HA}}$ we define a slight modification of $\underset{\sim}{\mathrm{APP}}^E$.

3.14. DEFINITION. $\underset{\sim}{\mathrm{APP}}^E_1$ is $\underset{\sim}{\mathrm{APP}}^E$ plus quantifiers $\forall x \in N$, $\exists y \in N$ (they are not abbreviations, but part of the language). Of course, the axioms

$$\forall\mathrm{NAX} \qquad \forall x \in N\ A \ \leftrightarrow\ \forall x(x \in N \rightarrow A)$$

$$\exists\mathrm{NAX} \qquad \exists x \in N\ A \ \leftrightarrow\ \exists x(x \in N \wedge A)$$

are added, and we extend the definition of $\underline{r}$ with

$$\tau\underline{r}\forall x \in N\ A := \forall x \in N\ \tau x\underline{r}A,$$

$$\tau\underline{r}\exists x \in N\ A := p_1\tau \in N \wedge p_2\tau\underline{r}A[x := p_1\tau].$$

Theorem 3.13 holds for $\underset{\sim}{\mathrm{APP}}^E_1$ as well: one only has to observe that $\forall$NAX, $\exists$NAX are realized by $p(\lambda xyz.x)(\lambda x.x00)$ resp. $p(\lambda x.p(p_1x)(p0(p_2x)))(\lambda x.p(p_1x)(p_2(p_2x)))$.

From here till the end of this section, $\underline{r}$ denotes realizability in $\underset{\sim}{\mathrm{APP}}^E_1$.

3.15. DEFINITION. Realizability in $\underset{\sim}{\mathrm{HA}}^*$ is denoted by $t\underline{r}_1A$ and defined by

$$t\underline{r}_1A := (t^\circ\underline{r}A^\circ)'.$$

3.16. REMARK. Kleene's original realizability may be defined for $\underset{\sim}{\mathrm{HA}}^*$ as follows:

$$t\underline{r}_k(s_1 = s_2) := Et \wedge s_1 = s_2$$

$$\begin{array}{ll}
t\underline{r}_k(A\wedge B) & := j_1(t)\underline{r}_kA \wedge j_2(t)\underline{r}_kB\\
t\underline{r}_k(A\to B) & := Et \wedge \forall x(x\underline{r}_kA \to \{t\}(x)\underline{r}_kB)\\
t\underline{r}_k\forall xA & := \forall x\ \{t\}(x)\underline{r}_kA\\
t\underline{r}_k\exists xA & := Ej_1(t) \wedge j_2(t)\underline{r}_k(A[x := j_1(t)])
\end{array}$$

$\underline{r}_k$ is virtually the same as $\underline{r}_1$, i.e. we have

$$\underset{\sim}{\mathrm{HA}}^* \vdash t\underline{r}_kA \leftrightarrow t\underline{r}_1A.$$

This can be verified with formula induction, using $\underset{\sim}{\mathrm{HA}}^* \vdash t^{\circ\,\prime} = t$, $A^{\circ\,\prime} \leftrightarrow A$ (Ch.II, 4.5) and (for $A = \forall xB$) the definition of $\tau\underline{r}\forall x\in N\ B$ in $\underset{\sim}{\mathrm{APP}}{}_1^E$.

3.17. LEMMA. i) $\underset{\sim}{\mathrm{HA}}^* \vdash \tau'\underline{r}_1A \leftrightarrow (\tau\underline{r}A^\circ)'$;

ii) $\underset{\sim}{\mathrm{HA}}^* \vdash \tau'\underline{r}_1A' \leftrightarrow (\tau\underline{r}A)'$

PROOF. i)

$$\begin{array}{lll}
\tau'\underline{r}_1A & = (\tau^{\prime\circ}\underline{r}A^\circ)' & \text{(def. of } \underline{r}_1\text{)}\\
 & = ((x\underline{r}A^\circ)[x := \tau^{\prime\circ}])' & (3.13.(iii))\\
 & = (x\underline{r}A^\circ)'[x := \tau^{\prime\circ\prime}] & \text{(by def. of } {}'\text{)}\\
 & \equiv_{\mathrm{HA}^*} (x\underline{r}A^\circ)'[x := \tau'] & \text{(Ch.II, 4.5.(i))}\\
 & = ((x\underline{r}A^\circ)[x := \tau])' & \text{(by def. of } {}'\text{)}\\
 & = (\tau\underline{r}A^\circ)' & (3.13.(iii))
\end{array}$$

ii) With formula induction (using $\underset{\sim}{\mathrm{HA}}^* \vdash t^{\circ\,\prime} = t$, $A^{\circ\,\prime} \leftrightarrow A$) we prove $\underset{\sim}{\mathrm{HA}}^* \vdash (\tau\underline{r}A^{\prime\circ})' \leftrightarrow (\tau\underline{r}A)'$; from this (ii) follows, for by (i) we have $\underset{\sim}{\mathrm{HA}}^* \vdash (\tau\underline{r}A^{\prime\circ})' \leftrightarrow \tau'\underline{r}_1A'$. □

Now we have the following pendant of 3.13:

3.18. THEOREM.

i) $\underset{\sim}{\mathrm{HA}}^* \vdash A \Rightarrow \underset{\sim}{\mathrm{HA}}^* \vdash t\underline{r}_1A$ *for some* t *;*

ii) $\underset{\sim}{\mathrm{HA}}^* \vdash A \leftrightarrow \exists x\ x\underline{r}_1A$ *for negative* A *;*

iii) $\underset{\sim}{\mathrm{HA}}^*$ *realizes* ECT_0 *;*

iv) $\underset{\sim}{\mathrm{HA}}^* + \mathrm{ECT}_0 \vdash A \leftrightarrow \exists x\ x\underline{r}_1A$ *;*

v) $\underset{\sim}{\mathrm{HA}}^* \vdash \exists x\ x\underline{r}_1A \Leftrightarrow \underset{\sim}{\mathrm{HA}}^* + \mathrm{ECT}_0 \vdash A$ *;*

vi) *let* Γ *be defined as in* 3.9.(ii) *but now for the language of* $\underset{\sim}{\mathrm{HA}}^*$, *then* $A \in \Gamma_1 \Rightarrow \underset{\sim}{\mathrm{HA}}^* \vdash (\exists x\ x\underline{r}_1 A) \to A$ *;*

vii) $\underset{\sim}{\mathrm{HA}}^* + \mathrm{ECT}_0$ *is conservative over* $\underset{\sim}{\mathrm{HA}}^*$ *with respect to* Γ.

PROOF. i) Let $\vec{x}$ be the free variables of A. Then

$$\begin{aligned}
\underset{\sim}{\mathrm{HA}}^* \vdash A \ &\Rightarrow\ \underset{\sim}{\mathrm{APP}}_1^E \vdash \vec{x} \in N \to A^\circ && \text{(Ch.II, 4.2.(ii))}\\
&\Rightarrow\ \underset{\sim}{\mathrm{APP}}_1^E \vdash \vec{x} \in N \to \tau\underline{r}A^\circ && \text{(3.13.(iv))}\\
&\Rightarrow\ \underset{\sim}{\mathrm{HA}}^* \vdash (\tau\underline{r}A^\circ)' && \text{(Ch.II, 4.4)}\\
&\Rightarrow\ \underset{\sim}{\mathrm{HA}}^* \vdash \tau'\underline{r}_1 A && \text{(3.17.(i))}
\end{aligned}$$

ii) If A negative, then so is A°, and by 3.13.(v)

$$\underset{\sim}{\mathrm{APP}}_1^E \vdash \vec{x} \in N \to (A^\circ \leftrightarrow \exists x\ x\underline{r}A^\circ).$$

By Ch.II, 4.4, 4.5:

$$\underset{\sim}{\mathrm{HA}}^* \vdash A \leftrightarrow \exists x(x\underline{r}A^\circ)'$$

and by the definition of $\underline{r}_1$, this is just (ii).

iii) Let $\mathrm{ECT}_0(A,B)$ be an instance of ECT_0, then $\underset{\sim}{\mathrm{HA}}^* \vdash \mathrm{ECT}_0(A,B) \leftrightarrow$ $\leftrightarrow (\mathrm{EAC}(A^\circ,B^\circ))'$, so by (i) there is a term t with

$$\text{(1)} \qquad \underset{\sim}{\mathrm{HA}}^* \vdash t\underline{r}_1((\mathrm{EAC}(A^\circ,B^\circ))' \to \mathrm{ECT}_0(A,B)).$$

By 3.13.(vi) and Ch.II, 4.4:

$$\underset{\sim}{\mathrm{HA}}^* \vdash (\sigma_A \circ \underline{r}\mathrm{EAC}(A^\circ,B^\circ))'.$$

With 3.17.(ii):

$$\text{(2)} \qquad \underset{\sim}{\mathrm{HA}}^* \vdash \sigma_A \circ' \underline{r}_1(\mathrm{EAC}(A^\circ,B^\circ))'.$$

We combine (1) and (2):

$$\underset{\sim}{\mathrm{HA}}^* \vdash \{t\}(\sigma_A \circ')\underline{r}_1 \mathrm{ECT}_0(A,B).$$

iv) By 3.13.(vii) and 2.18.(i):

$$\underset{\sim\sim}{\mathrm{HA}}^* + \mathrm{ECT}_0 \vdash A^{\circ\,\prime} \leftrightarrow \exists x(x\underline{r}A^\circ)';$$

now apply 3.17.(i).

v) Follows from (iii) and (iv), as in 3.7.

vi) If $A \in \Gamma$ then $A^\circ \in \Gamma$; now the result follows from 3.13.(ix), as in (ii).

vii) Follows from (v) and (vi). □

3.19. REMARKS.

i) Using $'' : \underset{\sim\sim}{\mathrm{APP}}^E \to \underset{\sim\sim}{\mathrm{EL}}^*$ (defined in Ch.II, 4.9) and $\underset{\sim\sim}{\mathrm{APP}}_2^E$ (= $\underset{\sim\sim}{\mathrm{APP}}_1^E$ + quantifiers $\forall x \in (N \Rightarrow N)$, $\exists y \in (N \Rightarrow N)$), we can define $\underline{r}_2$ for $\underset{\sim\sim}{\mathrm{EL}}^*$ by $t\underline{r}_2A := (t^\circ\underline{r}A^\circ)''$. $\underline{r}_2$ is equivalent to the realizability for functions first formulated in Kleene & Vesley's [KV65]; see also [T73]. With GC instead of ECT_0, a theorem like 3.18 can be given for $\underline{r}_2$.

ii) We sketch how to show that $\underset{\sim\sim}{\mathrm{APP}}$, $\underset{\sim\sim}{\mathrm{APP}}^E$ have the disjunction property (DP), the existence property (EP) and the numerical existence property (EP(N)):

DP	$\vdash A \vee B$	$\Rightarrow$	$\vdash A$ or $\vdash B$,
EP	$\vdash \exists x A(x)$	$\Rightarrow$	$\vdash A(\tau)$ for some term τ ,
EP(N)	$\vdash \exists x \in N\ A(x)$	$\Rightarrow$	$\vdash A(\bar{n})$ for some numeral $\bar{n}$.

To prove these properties for a theory, one often uses the so-called $\underline{q}$-realizability, a modification of $\underline{r}$ (see e.g. Troelstra [T73]). Following an idea by Grayson [Gr81], we define another variant $\underline{g}$ of $\underline{r}$:

$$\tau\underline{g}(A \to B) := \forall x(x\underline{g}A \to \tau x\underline{g}B) \wedge (A \to B),$$

the other clauses are like those for $\underline{r}$.

$\underline{g}$ has the characteristic property

(1) $\vdash (\exists x\ x\underline{g}A) \to A$ for all A .

The soundness proof for $\underline{g}$ runs parallel to that for $\underline{r}$: the 'realizing terms' are the same. So if $\vdash \exists x A(x)$ then $\vdash \tau\underline{g}\exists x A(x)$ for some τ , i.e. $\vdash p_2\tau\underline{q}A(p_1\tau)$, hence (by (1)) $\vdash A(p_1\tau)$, and we have EP. For EP(N) we use the term model of Ch.II, 5.13 by which we have $\vdash \tau \in N \Rightarrow$ $\Rightarrow \vdash \tau = \bar{n}$ for some $\bar{n}$; DP follows from EP(N).

iii) Feferman gives in [Fe75], [Fe79] a definition of $\underline{r}$ for his applicative systems on which our theories APP, APP^E are inspired. He proves soundness without formulating an axiomatization result. The results we derived for $\underline{r}_1$ and HA^*, and for $\underline{r}_2$ and EL^* are not new: they can all be found in [T73].

§4. Skolem functions and forcing.

We are going to prove that APP + EAC is conservative over HA in this section. This is done by the introduction and elimination of Skolem functions for arithmetical formulae $\exists nA(n)$, denoted by ε_A (the choice of notation is inspired by Hilbert's ε-symbol; see 4.22). We start with defining APP(ε) by adding the ε_A to APP.

4.1. DEFINITION. i) A formula $A = A(\vec{x})$ is called *arithmetical* if:

a) all its quantifiers range over N, i.e. occur in contexts $\forall y \in N$, $\exists z \in N$;

b) all its free variables are restricted to N, so $A \equiv A \wedge \vec{x} \in N$.

ii) APP(ε) is APP plus constants ε_A for every arithmetical formula $A = A(m,n)$ of APP(ε), and the schema εAX: this is

$$\varepsilon\mathrm{AX}(A) \qquad \forall\vec{m}(\exists nA(\vec{m},n) \to \exists n(A(\vec{m},n) \wedge n = \varepsilon_A\vec{m}))$$

for all arithmetical A.

4.2. LEMMA.

i) APP(ε) $\vdash A \leftrightarrow \exists x\ x\underline{r}A$ *for negative* A;

ii) APP + EAC $\vdash A \Rightarrow$ APP(ε) $\vdash A$ *for arithmetical* A.

PROOF. i) As for APP (3.4).

ii) Let A be arithmetical, APP + EAC $\vdash$ A. Then APP $\vdash \exists x\ x\underline{r}A$ (by 3.7), hence

(1) APP(ε) $\vdash \exists x\ x\underline{r}A$.

By εAX, we have $\exists n B(\vec{m},n) \leftrightarrow B(\vec{m},\varepsilon_{\bar{B}} m)$ for all subformulae $\exists nB$ of A, so we find a negative formula A^- of $APP(\varepsilon)$ with

(2) $\qquad \underset{\sim}{APP}(\varepsilon) \vdash A \leftrightarrow A^-.$

By (i), (2) implies

(3) $\qquad \underset{\sim}{APP}(\varepsilon) \vdash \forall x(x\underline{r}A \rightarrow \tau x\underline{r}A^-)$ for some τ .

As A^- is negative we have, by (i)

(4) $\qquad \underset{\sim}{APP}(\varepsilon) \vdash \exists x\ x\underline{r}A^- \rightarrow A^-.$

Now (1), (2), (3), (4) yield

$$\underset{\sim}{APP}(\varepsilon) \vdash A.$$

□

With 4.2.(ii) we are one step away from the desired conservation result: only

(5) $\qquad \underset{\sim}{APP}(\varepsilon) \vdash A \Rightarrow \underset{\sim}{APP} \vdash A$ for arithmetical A

is required. We prove (5) as follows. If $\underset{\sim}{APP}(\varepsilon) \vdash A$, then $\underset{\sim}{APP} + \varepsilon AX(A_0) + \ldots + \varepsilon AX(A_n) \vdash A$ for some $A_0,\ldots,A_n$. The instances $\varepsilon AX(A_i)$ are eliminated one by one by forcing. To show this, we start with $\underset{\sim}{APP}(\varepsilon,A_0)$: this is $\underset{\sim}{APP}$ + the constant ε + $(\varepsilon AX(A_0)$ with ε instead of ε_{A_0}). For $\underset{\sim}{APP}(\varepsilon,A_0)$ we define *forcing*, an interpretation in $\underset{\sim}{APP}$.

4.3. <u>CONVENTION</u>. We use the set-and-element notation $\tau \in A$ (τ a term, A a formula), with the meaning $A[x := \tau]$.

4.4. <u>DEFINITION</u>. i) Let $M = M(x)$ be a formula of $\underset{\sim}{APP}$. We say that M is a *monoid* if:

$$\underset{\sim}{APP} \vdash \lambda x.x \in M;$$

$$\underset{\sim}{APP} \vdash f,\ g \in M \rightarrow \lambda x.f(gx) \in M.$$

$f, g, h, \ldots$ are used for elements of a monoid M.

ii) Let M be a monoid. $\Vdash_M A$ (A *is forced by* M) is defined by

$$\begin{aligned}
&\Vdash_M P && := \forall f \in M\ \exists g \in M\ \forall h \in M\ P[\varepsilon := f(g(h0))] \qquad (P \text{ prime})\\
&\Vdash_M (A \wedge B) && := \ \Vdash_M A \wedge \Vdash_M B\\
&\Vdash_M (A \to B) && := \forall f \in M(\Vdash_M (A[\varepsilon := f\varepsilon]) \to \Vdash_M (B[\varepsilon := f\varepsilon]))\\
&\Vdash_M \forall xA && := \forall x \Vdash_M A\\
&\Vdash_M \exists xA && := \forall f \in M\ \exists g \in M\ \exists x \Vdash_M (A[\varepsilon := f(g\varepsilon)])
\end{aligned}$$

If it is not important which monoid M is meant, or if this is clear from the context, we write $\Vdash$ for $\Vdash_M$ and $\forall f$, $\exists g$ for $\forall f \in M$, $\exists g \in M$.

The thing to do now is to prove the soundness of $\Vdash$ as interpretation of $\utilde{\mathrm{APP}}(\varepsilon,A_0)$ in $\utilde{\mathrm{APP}}$. Unfortunately this is not possible: the special monoid M_0 we need to get $\varepsilon\mathrm{AX}(A_0)$ forced (see 4.15) does not yield e.g. $\Vdash_{M_0} \exists x\ x = \varepsilon$. The problem lies in quantification over terms containing ε, and forces us to the following detour: we define a weakening $\utilde{\mathrm{APP}}(\varepsilon,A_0)^-$ of $\utilde{\mathrm{APP}}(\varepsilon,A_0)$ for which we can prove that $\Vdash_{M_0}$ is sound, and we show that $\utilde{\mathrm{APP}}(\varepsilon,A_0)$ can be interpreted in $\utilde{\mathrm{APP}}(\varepsilon,A_0)^-$.

4.5. DEFINITION. i) $\utilde{\mathrm{APP}}(\varepsilon,A_0)^-$ is $\utilde{\mathrm{APP}}(\varepsilon,A_0)$ with $\forall\mathrm{AX}$ $(\forall xA \to A[x := \tau])$, $\exists\mathrm{AX}$ $(A[x := \tau] \to \exists xA)$ restricted to τ not containing ε and with $=\!\mathrm{AX}$, SUB and the axioms for the constants (except ε) written with terms (possibly containing ε) instead of variables.

ii) The mapping $^{\varepsilon}$: $\utilde{\mathrm{APP}}(\varepsilon,A_0) \to \utilde{\mathrm{APP}}(\varepsilon,A_0)^-$ is defined by

$$\begin{aligned}
x^{\varepsilon} &:= x\varepsilon && (x \text{ a variable})\\
c^{\varepsilon} &:= c && (c \text{ a constant})\\
(\sigma\tau)^{\varepsilon} &:= \sigma^{\varepsilon}\tau^{\varepsilon}\\
(\sigma = \tau)^{\varepsilon} &:= (\sigma^{\varepsilon} = \tau^{\varepsilon})\\
(\tau \in N)^{\varepsilon} &:= \exists x \in N(x = \tau^{\varepsilon})
\end{aligned}$$

$^{\varepsilon}$ commutes with the logical operators.

4.6. LEMMA.

i) $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- \vdash \sigma = \tau \to (A[x := \sigma] \leftrightarrow A[x := \tau])$;

ii) $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- \vdash (\forall x \in N\ A)^\varepsilon \leftrightarrow \forall x \in N(A^\varepsilon[x\varepsilon := x])$,

$(\exists x \in N\ A)^\varepsilon \leftrightarrow \exists x \in N(A^\varepsilon[x\varepsilon := x])$;

iii) $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- \vdash A^\varepsilon \leftrightarrow A$ *for* A *arithmetical and closed;*

iv) $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0) \vdash A \Rightarrow \underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- \vdash A^\varepsilon$.

PROOF. i) As Ch.II, lemma 3.4.(i). We need the term variant of =AX, SUB here, since we no longer have quantification over all terms.

ii) $(\forall x \in N\ A)^\varepsilon = \forall x(\exists y \in N(y = x\varepsilon) \to A^\varepsilon) \equiv \forall x \forall y \in N(y = x\varepsilon \to A^\varepsilon) \equiv$
$\equiv \forall x \in N(A^\varepsilon[x\varepsilon := x])$;

the last equivalence follows from (i), the fact that x occurs only in the context $x\varepsilon$ in A^ε, and from $\forall y \in N\ \exists x\ y = x\varepsilon$ (put $x := ky$). Similarly for the second half.

iii) Follows from (ii), by formula induction.

iv) Induction over the length of a proof of A.

Propositional axioms and rules: trivial.

$\forall$AX: by (i) and the definition of $^\varepsilon$ we have $(A[x := \tau])^\varepsilon \equiv A^\varepsilon[x := \lambda\varepsilon.\tau^\varepsilon]$; now $\lambda\varepsilon.\tau^\varepsilon$ is ε - free, so we have $\forall x A^\varepsilon \to (A[x := \tau])^\varepsilon$ in $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^-$, i.e. $(\mathrm{AX2})^\varepsilon$.

$\exists$AX: analogously.

$\forall$-R, $\exists$-R: easy.

=AX, SUB: follow from the corresponding axioms formulated with terms in $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^-$.

Axioms for the constants (except ε): idem.

IND: follows from (ii).

$\varepsilon\mathrm{AX}(A_0)$: follows from (iii). □

4.7. Now we set out to show

(1) for all monoids M,
$\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- - \varepsilon\mathrm{AX}(A_0) \vdash A \Rightarrow \underset{\sim}{\mathrm{APP}} \vdash (\Vdash_M A)$;

(2) $\underset{\sim}{\mathrm{APP}} \vdash (\Vdash_M \varepsilon\mathrm{AX}(A_0))$ for some monoid M.

Before proving (1) we rewrite the definition of $\Vdash$. We use the abbreviations

$$\Box A := \forall f\ A[\varepsilon := f\varepsilon],$$

$$\nabla A := \forall f \exists g\ A[\varepsilon := f(g\varepsilon)];$$

the symbol $\Box$ is borrowed from modal logic, ∇ can be compared with $\Box\Diamond$ in modal logic (especially S4).
We adopt the (natural) convention to work out $\Box$, ∇ from the outside: so $\nabla\Box A = \forall f\exists g\Box(A[\varepsilon := f(g\varepsilon)]) = \forall f\exists g\forall h\ A[\varepsilon := f(g(h\varepsilon))]$. If we now define ${}^{\Box}$: $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^{-} \to \underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^{-}$ by

$$\begin{array}{lll} P^{\Box} & := \nabla\Box P & (\text{ P prime}) \\ (A\wedge B)^{\Box} & := A^{\Box}\wedge B^{\Box} & \\ (A\to B)^{\Box} & := \Box(A^{\Box}\to B^{\Box}) & \\ (\forall xA)^{\Box} & := \forall xA^{\Box} & \\ (\exists xA)^{\Box} & := \nabla\exists xA^{\Box} & \end{array}$$

then $\Vdash A = A^{\Box}[\varepsilon := 0]$.

We list some properties of $\Box$, ∇ :

4.8. <u>LEMMA</u>. *In* $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^{-} - \varepsilon\mathrm{AX}(A_0)$ *we have*

(3) $\vdash A \Rightarrow\ \vdash\Box A$,

(4) $\Box A\to A$,

(5) $\Box A\to\Box\Box A$,

(6) $\nabla\nabla A\to\nabla A$,

(7) $\nabla A\to\Box\nabla A$,

(8) $\Box A\to\nabla A$,

(9) $\Box(A\to B)\to(\Box A\to\Box B)$,

(10) $\Box(A\to B)\to(\nabla A\to\nabla B)$,

(11) $\forall x\Box A\to\Box\forall xA$.

<u>PROOF</u>. (3): if A is an axiom of $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^{-} - \varepsilon\mathrm{AX}(A_0)$, then so is $A[\varepsilon := f\varepsilon]$; analogously for rules. So if $\vdash A$, then $\vdash A[\varepsilon := f\varepsilon]$, hence $\vdash\forall f\ A[\varepsilon := f\varepsilon]$, i.e. $\vdash\Box A$.

(4)-(11): follow from the definition of $\Box$, ∇ and from the fact that M is a monoid. $\Box$

4.9. LEMMA. *In* $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- - \varepsilon\mathrm{AX}(A_0)$ *we have*

(12) $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$,

(13) $\nabla(A \wedge B) \rightarrow (\nabla A \wedge \nabla B)$,

(14) $\nabla\Box(A \wedge B) \leftrightarrow (\nabla\Box A \wedge \nabla\Box B)$,

(15) $\Box(\Box A \rightarrow B) \leftrightarrow \Box(\Box A \rightarrow \Box B)$,

(16) $\nabla(A \rightarrow B) \rightarrow (\Box A \rightarrow \nabla B)$,

(17) $\Box\forall xA \leftrightarrow \forall x\Box A$,

(18) $\nabla\forall xA \rightarrow \forall x\nabla A$,

(19) $\exists x\Box A \rightarrow \Box\exists xA$,

(20) $\exists x\nabla A \rightarrow \nabla\exists xA$.

PROOF. (12)-(20) can all be derived from (3)-(11) but sometimes a simpler proof is found by writing out the definitions of $\Box$, ∇ . We only give the proof of (14), which is rather involved.

$\rightarrow$: $\nabla\Box(A \wedge B) \overset{12}{\rightarrow} \nabla(\Box A \wedge \Box B) \overset{13}{\rightarrow} (\nabla\Box A \wedge \nabla\Box B)$.

$\leftarrow$: $\vdash A \rightarrow (B \rightarrow (A \wedge B))$

$\Rightarrow \vdash \Box\Box\Box(A \rightarrow (B \rightarrow (A \wedge B)))$ (by (3))

$\Rightarrow \vdash \Box\nabla\Box A \rightarrow \Box\nabla\Box(B \rightarrow (A \wedge B))$ (by (9), (10))

$\Rightarrow \vdash \Box\nabla\Box A \rightarrow (\nabla\Box\Box B \rightarrow \nabla\nabla\Box(A \wedge B))$ (by (9), (16), (10))

$\Rightarrow \vdash \nabla\Box A \rightarrow (\nabla\Box B \rightarrow \nabla\Box(A \wedge B))$ (by (7), (5), (6))

$\Rightarrow \vdash (\nabla\Box A \wedge \nabla\Box B) \rightarrow \nabla\Box(A \wedge B)$.

$\Box$

4.10. LEMMA. *In* $\underset{\sim}{\mathrm{APP}}(\varepsilon,A_0)^- - \varepsilon\mathrm{AX}(A_0)$ *we have*

(21) $A^\Box \leftrightarrow \Box A^\Box \leftrightarrow \nabla A^\Box$.

PROOF. By (4), (8), it suffices to show $A^\Box \rightarrow \Box A^\Box$, $\nabla A^\Box \rightarrow A^\Box$. This is done with formula induction: we treat the case $A = B \rightarrow C$, which is least trivial.

A = B→C: $A^\square = \square(B^\square \to C^\square) \overset{5}{\to} \square\square(B^\square \to C^\square) = \square A^\square$;

$\nabla A^\square = \nabla\square(B^\square \to C^\square) \overset{4,7}{\to} \square\nabla(B^\square \to C^\square) \overset{16}{\to} \square(\square B^\square \to \nabla C^\square) \overset{IH}{\to} \square(B^\square \to C^\square) =$

$= A^\square$.

□

4.11. LEMMA. $\mathbf{APP}(\varepsilon,A_0)^- - \varepsilon AX(A_0) \vdash A \Rightarrow \mathbf{APP}(\varepsilon,A_0)^- - \varepsilon AX(A_0) \vdash A^\square$.

PROOF. Induction over the length of a proof of A.

→AX: $(A \to A)^\square = \square(A^\square \to A^\square)$ and this is derivable, by (3).

∀AX: $(\forall xA \to A[x := \tau])^\square = \square(\forall xA^\square \to A[x := \tau]^\square)$: now $A[x := \tau]^\square = A^\square[x := \tau]$ (for τ is ε - free), so $(AX2)^\square$ is derivable, using (3).

∃AX: $(A[x := \tau] \to \exists xA)^\square = \square(A[x := \tau]^\square \to \nabla\exists xA^\square) \equiv \square(\nabla A^\square[x := \tau] \to \nabla\exists xA^\square)$: for the last step we used (21) and the fact that τ is ε - free. The last formula is derivable using (3) and (10).

PR1-4: straightforward.

PR5: $((A \wedge B) \to C)^\square = \square((A^\square \wedge B^\square) \to C^\square) \Longleftrightarrow \square(A^\square \to (B^\square \to C^\square)) \Longleftrightarrow$

$\overset{21}{\Longleftrightarrow} \square(\square A^\square \to (B^\square \to C^\square)) \overset{15}{\Longleftrightarrow} \square(\square A^\square \to \square(B^\square \to C^\square)) \overset{21}{\Longleftrightarrow} \square(A^\square \to \square(B^\square \to C^\square)) =$

$= (A \to (B \to C))^\square$.

∀-R: easy.

∃-R: $(A \to B)^\square = \square(A^\square \to B^\square) \overset{4}{\Rightarrow} (A^\square \to B^\square) \Rightarrow (\exists xA^\square \to B^\square) \overset{3}{\Rightarrow} \square\square(\exists xA^\square \to B^\square) \Rightarrow$

$\overset{10}{\Rightarrow} \square(\nabla\exists xA^\square \to \nabla B^\square) \overset{21}{\Rightarrow} \square(\nabla\exists xA^\square \to B^\square) = (\exists xA \to B)^\square$.

All non-logical axioms except IND can be written in the form $P \wedge Q \to R$ with P, Q, R prime. Now $\square(P \wedge Q \to R) \overset{5}{\to} \square\square\square(P \wedge Q \to R) \overset{9,10}{\to}$

$\overset{9,10}{\to} \square(\nabla\square(P \wedge Q) \to \nabla\square R) \overset{14}{\to} \square(\nabla\square P \wedge \nabla\square Q \to \nabla\square R) = (P \wedge Q \to R)^\square$, so this last formula is derivable, since $\square(P \wedge Q \to R)$ is (by (3)).

IND: $IND^\square = \square(A(0)^\square \wedge \forall x\, \square(x \in N \wedge A(x)^\square \to A(Sx)^\square) \to \forall x\, \square(x \in N \to A(x)^\square))$ and this formula follows from $\square IND(A^\square)$ (using (5), (9), (11), (21)), which is derivable (by (3)).

□

4.12. LEMMA. $\mathbf{APP}(\varepsilon,A_0)^- - \varepsilon AX(A_0) \vdash A \Rightarrow \mathbf{APP} \vdash A[\varepsilon := 0]$.

PROOF. Evident. □

4.13. LEMMA. $\underset{\sim}{\mathrm{APP}}(\varepsilon, A_0)^- - \varepsilon\mathrm{AX}(A_0) \vdash A \Rightarrow \underset{\sim}{\mathrm{APP}} \vdash (\Vdash A)$.

PROOF. Recall $\Vdash A = A^{\square}[\varepsilon := 0]$ and combine 4.11 and 4.12. □

4.14. LEMMA. $A \equiv (\Vdash A)$ *if* A *does not contain* ε.

PROOF. Easy: we only need that a monoid is inhabited (it is, by $\lambda x.x$). □

With lemma 4.13 we proved (1) of 4.7. We now define the monoid M_0 needed for (2) of 4.7:

4.15. DEFINITION.

$$M_0 := \{f \mid \forall\vec{m}(\forall x(fx\vec{m} = x\vec{m}) \vee \exists n(A_0(\vec{m},n) \wedge \forall x(fx\vec{m} = n)))\}.$$

4.16. LEMMA. M_0 *is a monoid.*

PROOF. $\lambda x.x \in M_0$ is obvious. To prove closure under $\circ$ (composition of functions), we argue as follows. Assume $f, g \in M_0$; we want $f \circ g \in M_0$, i.e. for all $\vec{m}$

$$\forall x(f(gx)\vec{m} = x\vec{m}) \vee \exists n(A_0(\vec{m},n) \wedge \forall x(f(gx)\vec{m} = n)). \tag{22}$$

$f \in M_0$, so $\forall x(fx\vec{m} = x\vec{m})$ (I) or $\exists n(A_0(\vec{m},n) \wedge \forall x(fx\vec{m} = n))$ (II).

I): $g \in M_0$, so $\forall x(gx\vec{m} = x\vec{m})$ (IA) or $\exists n(A_0(\vec{m},n) \wedge \forall x(gx\vec{m} = n))$ (IB).

IA): let x be arbitrary. Now $f(gx)\vec{m} = gx\vec{m} = x\vec{m}$, hence $\forall x(f(gx)\vec{m} = x\vec{m})$, which implies (22).

IB): now $A_0(\vec{m},n) \wedge \forall x(gx\vec{m} = n)$ for some n. Let x be arbitrary, then $f(gx)\vec{m} = gx\vec{m} = n$, hence $\exists n(A_0(\vec{m},n) \wedge \forall x(f(gx)\vec{m} = n))$ and this implies (22).

II): now $A_0(\vec{m},n) \wedge \forall x(fx\vec{m} = n)$ for some n. Let x be arbitrary, then $f(gx)\vec{m} = n$, hence $\exists n(A_0(\vec{m},n) \wedge \forall x(f(gx)\vec{m} = n))$ and this implies (22).

□

4.17. LEMMA. $\underset{\sim}{\mathrm{APP}} \vdash (\Vdash_{M_0} \varepsilon\mathrm{AX}(A_0))$.

PROOF. Without loss of generality we assume $A_0 = A_0(m,n)$, so $\vec{m} = m$.

We let f, g, h, f', g', h' range over M_0. Now $\Vdash_{M_0} \varepsilon\mathrm{AX}(A_0) =$ $\Vdash_{M_0} (\forall m(m \in N \wedge \exists n(n \in N \wedge A_0(m,n) \to \exists n(n \in N \wedge A_0(m,n) \wedge n = \varepsilon m)))$; $A_0(m,n)$ is ε-free, so by 4.14 this is equivalent to

$$(23) \qquad \forall m \forall f(\exists n A_0(m,n) \to \forall g \exists h \exists n(A_0(m,n) \wedge \\ \wedge \forall f' \exists g' \forall h'(n = (f \circ g \circ h \circ f' \circ g' \circ h')0m))).$$

(23) follows from

$$(24) \qquad A_0(m,n_0) \wedge g \in M_0 \to \exists h \exists n(A_0(m,n) \wedge \forall x(n = g(hx)m)).$$

We prove (24). Assume $A_0(m,n_0)$, $g \in M_0$. By the definition of M_0, we can distinguish two cases:

i) $\forall x(gxm = xm)$. Define $h := \lambda xy.\Delta n_0(xy)my$, so $hxm = n_0$ and $hxm' =$ $= xm'$ if $m' \neq m$; hence $h \in M_0$ and $\forall x\ g(hx)m = n_0$, so $\exists h \exists n(A_0(m,n) \wedge \forall x(n = g(hx)m))$.

ii) $\exists n(A_0(m,n) \wedge \forall x(gxm = n))$. Now put $h := \lambda x.x$ and we have $\exists h \exists n(A_0(m,n) \wedge \forall x(n = g(hx)m))$.

Now (24) is proved, and we conclude $\Vdash_{M_0} \varepsilon\mathrm{AX}(A_0)$. □

4.18. <u>LEMMA</u>. $\mathrm{APP}(\varepsilon,A_0)^- \vdash A \Rightarrow \mathrm{APP} \vdash (\Vdash_{M_0} A)$.

<u>PROOF</u>. Combine 4.13 and 4.17. □

4.19. <u>LEMMA</u>. $\mathrm{APP}(\varepsilon,A_0) \vdash A \Rightarrow \mathrm{APP} \vdash A$ *for arithmetical* A.

<u>PROOF</u>. If $\mathrm{APP}(\varepsilon,A_0) \vdash A$, then (by 4.6.(iii), (ii)) $\mathrm{APP}(\varepsilon,A_0) \vdash A$, so (with 4.18) $\mathrm{APP} \vdash \Vdash_{M_0} A$; now apply 4.14 to obtain $\mathrm{APP} \vdash A$. □

4.20. <u>THEOREM</u>. $\mathrm{APP} + \mathrm{EAC} \vdash A \Rightarrow \mathrm{APP} \vdash A$ *for arithmetical* A.

<u>PROOF</u>. Let A be arithmetical, and assume $\mathrm{APP} + \mathrm{EAC} \vdash A$. Then $\mathrm{APP}(\varepsilon) \vdash A$ by 4.2.(iii), so $\mathrm{APP} + \varepsilon\mathrm{AX}(A_0) + \ldots + \varepsilon\mathrm{AX}(A_n) \vdash A$. By applying 4.19 n+1 times (for $A_0,\ldots,A_n$) we get $\mathrm{APP} \vdash A$. □

4.21. <u>COROLLARY</u>. $\mathrm{APP} + \mathrm{EAC}$ *is conservative over* HA.

<u>PROOF</u>. Combine theorem 5.20 of Ch.II with 4.20. □

4.22. REMARKS.

i) The idea of Skolem functions first appeared in Skolem [Sk20]. In Hilbert's [H23] we find the logical function $\tau(A)$ or $\tau_a(A(a))$ with the axiom $A(\tau(A)) \to A(a)$; he also mentions the relation with the axiom of choice. In classical logic, $\tau(A)$ can be thought of as the Skolem function of $\neg A$; moreover, quantification can be defined with τ by $\forall a\, A(a) := A(\tau_a(A(a)))$, $\exists a\, A(a) := \neg A(\tau_a(\neg A(a)))$. In [H26], Hilbert uses for the first time the symbol ε named after him, in the axiom $A(a) \to A(\varepsilon A)$.

ii) In [Go76], Goodman proves that $\mathbf{HA}^{\omega} + AC$ is conservative over $\mathbf{HA}$. His proof is based on the interpretation (akin to realizability) of $\mathbf{HA}^{\omega}$ into his arithmetic theory of constructions $\mathbf{ATC}$; in [Go73] he showed that $\mathbf{ATC}$ is conservative over $\mathbf{HA}$ via an argument resembling both forcing and the elimination of choice sequences. He presents a more direct proof in [Go78] using what he calls relativised realizability, a combination of realizability and forcing. Beeson gives in [Be79] another proof in which realizability and forcing are used separately. Our proof of $\mathbf{APP} + EAC$ conservative over $\mathbf{APP}$ is based on a study of Beeson's argument.

§5. Inductive definitions.

In this section we introduce inductive definitions and investigate to what extent they are preserved under realizability and forcing. In either case the monotonicity of the predicate operator associated with the inductive definition plays a decisive role.

5.1. CONVENTIONS. We extend the set-and-membership notation as follows:

$$A \subset B := \forall x(x \in A \to x \in B)$$

$$A \equiv B := A \subset B \wedge B \subset A$$

$$A \cap B := A \wedge B$$

$$A \Rightarrow B := A \to B$$

We use P, Q as free unary predicate variables, and the rule

$$\vdash A(P) \;\Rightarrow\; \vdash A(B) \quad \text{for all formulae } B\,.$$

Predicate operators are written Γ_A, with the meaning given by

$$\tau \in \Gamma_A(B) \ := \ \tau \in A[P := B].$$

5.2. DEFINITION. Inductive definitions are considered as first-order definitions of the least fixed point of predicate operators: for such an operator $\Gamma = \Gamma_A$, we introduce the predicate constant I_Γ and the axioms $ID(\Gamma, I_\Gamma)$:

$ID1(\Gamma, I_\Gamma)$ $\quad \Gamma(I_\Gamma) \subset I_\Gamma$,

$ID2(\Gamma, I_\Gamma)$ $\quad \Gamma(P) \subset P \ \rightarrow \ I_\Gamma \subset P$.

5.3. DEFINITION. A predicate operator $\Gamma = \Gamma_A$ is called *monotone* if

$$\vdash P \subset Q \rightarrow \Gamma(P) \subset \Gamma(Q).$$

5.4. LEMMA. *If* P *occurs only positively in* A, *then* Γ_A *is monotone.*

PROOF. Easy, with formula induction. □

Now let $\underset{\sim}{T}$ be some theory, e.g. an extension of $\underset{\sim\sim}{\mathrm{APP}}$, for which $\underline{r}$ is sound, i.e.

$$\underset{\sim}{T} \vdash A \ \Rightarrow \ \underset{\sim}{T} \vdash \tau \underline{r} A \quad \text{for some term } \tau .$$

5.5. DEFINITION. i) The mapping ^{r} is defined by

$$A^r := (x)_1 r(A[x := (x)_2]) \quad \text{if } A \text{ is a formula},$$

$$\Gamma_A(B)^r := \Gamma_{A^r}(B^r),$$

$$P^r := P, \quad P \text{ a predicate variable}.$$

ii)

$$\underline{\tau} \ := \ \ulcorner_{<(x)_1, \tau(x)_1(x)_2> \in P}\urcorner$$

$$\sigma \cdot \tau \ := \lambda xy.\tau x(\sigma xy).$$

iii) We extend the definition of $\underline{r}$ (3.1) by

$$\tau\underline{r}(\sigma \in P) := \langle\sigma,\tau\rangle \in P \quad (P \text{ a predicate variable}).$$

5.6. LEMMA. i) $\tau\underline{r}(\sigma \in A) = \langle\sigma,\tau\rangle \in A^r$;

ii) $\tau\underline{r}(A \subset B) \leftrightarrow A^r \subset \underline{\tau}(B^r)$;

iii) $\underline{\tau}$ *is monotone;*

iv) $\underline{\sigma}(\underline{\tau}(A)) \equiv \underline{\sigma\bullet\tau}(A)$.

PROOF. i) A is a predicate variable P : immediate, by 5.5.(iii) and $P^r = P$.

A a formula: $\langle\sigma,\tau\rangle \in A^r = ((x)_2\underline{r}(A[x := (x)_1]))[x := \langle\sigma,\tau\rangle] = \tau\underline{r}(A[x := \sigma]) = \tau\underline{r}(\sigma \in A)$.

A is of the form $\Gamma_B(C)$: then

$$\begin{aligned}\langle\sigma,\tau\rangle \in A^r &= \langle\sigma,\tau\rangle \in \Gamma_{B^r}(C^r) = \langle\sigma,\tau\rangle \in B^r[P := C^r] = \\ &= \langle\sigma,\tau\rangle \in B^r[P^r := C^r] = \langle\sigma,\tau\rangle \in (B[P := C])^r = \\ &= \tau\underline{r}(\sigma \in B[P := C]) \qquad (\text{for } B[P := C] \text{ is a formula}) \\ &= \tau\underline{r}(\sigma \in \Gamma_B(C)).\end{aligned}$$

ii)

$$\begin{aligned}\tau\underline{r}(A \subset B) &= \tau\underline{r}(\forall x(x \in A \to x \in B)) \\ &= \forall x\forall u(u\underline{r}(x \in A) \to \tau xu\underline{r}(x \in B)) \\ &\leftrightarrow \forall xu(\langle x,u\rangle \in A^r \to \langle x,\tau xu\rangle \in B^r) \\ &\leftrightarrow \forall x(x \in A^r \to \langle (x)_1,\tau(x)_1(x)_2\rangle \in B^r) \\ &= A^r \subset \underline{\tau}(B^r).\end{aligned}$$

iii) By 5.4.

iv)

$$\begin{aligned}\underline{\sigma}(\underline{\tau}(A)) &= \underline{\sigma}(A[x := \langle (x)_1,\tau(x)_1(x)_2\rangle]) \\ &= A[x := \langle (x)_1,\tau(x)_1(x)_2\rangle][x := \langle (x)_1,\sigma(x)_1(x)_2\rangle] \\ &= A[x := \langle (x)_1,\tau(x)_1(\sigma(x)_1(x)_2)\rangle] \\ &= \underline{\lambda xy.\tau x(\sigma xy)}(A) \\ &= \underline{\sigma\bullet\tau}(A).\end{aligned}$$

□

In the sequel, we write Γ^r for Γ_{A^r} if $\Gamma = \Gamma_A$, and I^r for I_{Γ^r} if $I = I_\Gamma$.

5.7. LEMMA. *If* Γ *is monotone, then*

$$\underset{\sim}{T} + \mathrm{ID}(\Gamma^{r}, I^{r}) \vdash (\mathrm{ID}(\Gamma, I) \text{ is realized}).$$

PROOF. ID1: by ID1(Γ^{r}, I^{r}), we have $\Gamma^{r}(I^{r}) \subset I^{r}$; using $\underline{\lambda xy.x}P \equiv P$, we get $\Gamma^{r}(I^{r}) \subset \underline{\lambda xy.x}(I^{r})$, i.e.

$$\underline{\lambda xy.x}\ \underline{r}\ (\Gamma(I) \subset I).$$

ID2: since Γ is monotone and $\underline{r}$ is sound for $\underset{\sim}{T}$, we have, for some term σ:

$$\forall u(P \subset \underline{u}(Q) \to \Gamma^{r}(P) \subset \underline{\sigma u}(\Gamma^{r}(Q))). \tag{1}$$

We want $\tau\underline{r}(\Gamma(P) \subset P \to I \subset P)$ for some τ, i.e.

$$\forall v(\Gamma^{r}(P) \subset \underline{v}(P) \to I^{r} \subset \underline{\tau v}(P)). \tag{2}$$

Assume

$$\Gamma^{r}(P) \subset \underline{v}(P). \tag{3}$$

By (1) $(u := \tau v,\ \ P := \underline{\tau v}(P),\ \ Q := P)$:

$$\underline{\tau v}(P) \subset \underline{\tau v}(P) \to \Gamma^{r}(\underline{\tau v}(P)) \subset \underline{\sigma(\tau v)}(\Gamma^{r}(P)),$$

so

$$\Gamma^{r}(\tau v(P)) \subset \underline{\sigma(\tau v)}(\Gamma^{r}(P)). \tag{4}$$

(3) implies (using 5.6.(iii) and (iv)):

$$\underline{\sigma(\tau v)}(\Gamma^{r}(P)) \subset \underline{\sigma(\tau v)\cdot v}(P). \tag{5}$$

Combining (4) and (5):

$$\Gamma^{r}(\underline{\tau v}(P)) \subset \underline{\sigma(\tau v)\cdot v}(P). \tag{6}$$

Now if $\tau v = \sigma(\tau v)\cdot v$ then

$$(7) \qquad \Gamma^{r}(\underline{\tau v}(P)) \subset \underline{\tau v}(P),$$

so by $ID2(\Gamma^{r}, I^{r})$ we get

$$I^{r} \subset \underline{\tau v}(P),$$

the conclusion of (2). So we are ready if $\tau v = \sigma(\tau v)\cdot v$ holds. Here we use the fixed point operator ϕ of Ch.II, 3.7: put

$$\tau := \phi(\lambda xv.\sigma(xv)\cdot v),$$

then

$$\tau v = \phi(\lambda xv.\sigma(xv)\cdot v)v = (\lambda xv.\sigma(xv)\cdot v)\tau v = \sigma(\tau v)\cdot v$$

and we are done. □

Now we turn to forcing. Let $\underset{\sim}{T}(\varepsilon)$ be an extension of $\underset{\sim}{T}$ with the constant ε and axioms for ε. We assume that the combination of ${}^{\varepsilon}$ and $\Vdash_{M}$ (M a monoid in $\underset{\sim}{T}$) is sound for $\underset{\sim}{T}(\varepsilon)$, i.e.

$$\underset{\sim}{T}(\varepsilon) \vdash A \Rightarrow \underset{\sim}{T} \vdash (\Vdash_{M}(A^{\varepsilon})).$$

5.8. <u>DEFINITION</u>. i) For convenience, we put

$$f \Vdash A := \Vdash_{M}(A^{\varepsilon}[\varepsilon := f\varepsilon]),$$

$$g \geq f := f, g \in M \wedge \exists h \in M(g = f\circ h).$$

ii) The mapping ^{F} is defined by

$$A^{F} := (x)_{2} \Vdash (A[x := (x)_{1}]) \qquad \text{if } A \text{ is a formula,}$$

$$(\Gamma_{A}(B))^{F} := \Gamma_{A^{F}}(B^{F}),$$

$$P^{F} := P.$$

iii) $[f) := \{x \mid (x)_2 \geq f\}$.

iv) We extend the definition of $\Vdash$ by

$$f \Vdash (\tau \in P) := \langle \tau, f \rangle \in P.$$

5.9. LEMMA. i) $f \Vdash (\tau \in A) = \langle \tau, f \rangle \in A^F$;

ii) $f \Vdash (A \to B) \leftrightarrow \forall g \geq f(g \Vdash A \to g \Vdash B)$;

iii) $f \Vdash \forall x A \leftrightarrow \forall x(f \Vdash A)$;

iv) $f \Vdash (A \subset B) \leftrightarrow A^F \subset ([f) \Rightarrow B^F)$.

PROOF. i) A is a predicate variable P: by $P^F = P$ and 5.8.(iv).
A a formula: $\langle \tau, f \rangle \in A^F = f \Vdash (A[x := \tau]) = f \Vdash (\tau \in A)$.
A is of the form $\Gamma_B(C)$: then

$$\begin{aligned} \langle \tau, f \rangle \in A^F &= \langle \tau, f \rangle \in \Gamma_{B^F}(C^F) = \langle \tau, f \rangle \in B^F[P := C^F] \\ &= \langle \tau, f \rangle \in B^F[P^F := C^F] = \langle \tau, f \rangle \in (B[P := C])^F \\ &= f \Vdash (\tau \in (B[P := C])) \qquad \text{(for } B[P := C] \text{ is a formula)} \\ &= f \Vdash (\tau \in \Gamma_B(C)). \end{aligned}$$

ii)
$$\begin{aligned} f \Vdash (A \to B) &= \Vdash (A^\varepsilon[\varepsilon := f\varepsilon] \to B^\varepsilon[\varepsilon := f\varepsilon]) \\ &= \forall g(\Vdash (A^\varepsilon[\varepsilon := f\varepsilon][\varepsilon := g\varepsilon]) \to \Vdash (B^\varepsilon[\varepsilon := f\varepsilon][\varepsilon := g\varepsilon])) \\ &= \forall g(\Vdash (A^\varepsilon[\varepsilon := f(g\varepsilon)]) \to \Vdash (B^\varepsilon[\varepsilon := f(g\varepsilon)])) \\ &= \forall g(f \circ g \Vdash A \to f \circ g \Vdash B) \\ &\leftrightarrow \forall g \geq f(g \Vdash A \to g \Vdash B). \end{aligned}$$

iii) Trivial.

iv)
$$\begin{aligned} f \Vdash (A \subset B) &= f \Vdash \forall x(x \in A \to x \in B) \\ &= \forall x \forall g \geq f(g \Vdash (x \in A) \to g \Vdash (x \in B)) \\ &\leftrightarrow \forall x(x \in A^F \to ((x)_2 \geq f \to x \in B^F)) \\ &= A^F \subset ([f) \Rightarrow B^F). \end{aligned}$$

□

From now on, we write Γ^F for Γ_{A^F} if $\Gamma = \Gamma_A$, and I^F for I_{Γ^F} if $I = I_\Gamma$.

5.10. LEMMA. *If* Γ *is monotone, then*

$$\underset{\sim}{T} + \mathrm{ID}(\Gamma^F, I^F) \vdash \lambda x.x \Vdash \mathrm{ID}(\Gamma, I).$$

PROOF. ID1: by $\mathrm{ID1}(\Gamma^F, I^F)$, we have $\Gamma^F(I^F) \subset I^F$; as $I^F \subset ([\lambda x.x) \Rightarrow I^F)$, we get $\Gamma^F(I^F) \subset ([\lambda x.x) \Rightarrow I^F)$, i.e. $\lambda x.x \Vdash (\Gamma(I) \subset I)$.
ID2: we want $\lambda x.x \Vdash (\Gamma(P) \subset P \to I \subset P)$, i.e. for all $f \in M$

$$(8) \qquad \Gamma^F(P) \subset ([f) \Rightarrow P) \to I^F \subset ([f) \Rightarrow P).$$

So assume

$$\Gamma^F(P) \subset ([f) \Rightarrow P).$$

This implies

$$(9) \qquad ([f) \Rightarrow \Gamma^F(P)) \subset ([f) \Rightarrow P).$$

Γ is monotone and $\Vdash$ is sound, so we have

$$Q \subset ([f) \Rightarrow P) \to \Gamma^F(Q) \subset ([f) \Rightarrow \Gamma^F(P)).$$

Now put $Q := ([f) \Rightarrow P)$, then we get

$$\Gamma^F([f) \Rightarrow P) \subset ([f) \Rightarrow \Gamma^F(P)).$$

Together with (9):

$$\Gamma^F([f) \Rightarrow P) \subset ([f) \Rightarrow P);$$

with $\mathrm{ID2}(\Gamma^F, I^F)$, this yields

$$I^F \subset ([f) \Rightarrow P),$$

the conclusion of (8). □

5.11. DEFINITION. i) ID_1 is the axiom scheme of non-iterated inductive definitions in $\mathbf{APP}$, i.e. the instances of ID_1 are $ID(\Gamma,I)$ with $\Gamma = \Gamma_A$ where A is a formula in the language of $\mathbf{APP}$, containing P only positively. Such predicate operators Γ are called positive.

ii) $\mathbf{ID}_1 := \mathbf{HA} + (ID_1$ for the language of $\mathbf{HA}$).

Now we are able to prove some conservation results.

5.12. THEOREM. $\mathbf{APP} + EAC + ID_1$ *is conservative over* $\mathbf{APP} + ID_1$.

PROOF. One easily verifies: if Γ positive, then so are Γ^r, Γ^F. By 5.4, all Γ occurring in ID_1 are monotone, so by 5.7, 5.10 $\underline{r}$ and $\Vdash_M$ (M any monoid) are sound for $\mathbf{APP} + ID_1$. Now we can extend 3.7, 4.2.(iii) and 4.20 to $\mathbf{APP} + ID_1$ and the result follows. □

5.13. LEMMA. $\mathbf{APP} + ID_1$ *is conservative over* $\mathbf{ID}_1$.

PROOF. As Ch.II, 5.11 - 5.20. For the analogue of 5.19 we must show

$$\mathbf{ID}_1 \vdash (\underline{n} \in I_\Gamma)^{*T} \leftrightarrow n \in I_\Gamma,$$

where $\Gamma = \Gamma_A$, $(t \in I_\Gamma)^* = (t^* \in I_{\Gamma^*})$ with $\Gamma^* = \Gamma_{A^*}$, $(\tau \in I_\Gamma)^T =$ $= (\tau^T \in I_{\Gamma^T})$ with $\Gamma_A^T = \Gamma_{A^T}$. This is proved using

$$A \equiv B \rightarrow I_{\Gamma_A} \equiv I_{\Gamma_B} .$$

□

5.14. THEOREM. $\mathbf{APP} + EAC + ID_1$ *is conservative over* $\mathbf{ID}_1$.

PROOF. Combine 5.12, 5.13. □

§6. Martin-Löf's theory $\underset{\sim}{\mathrm{ML}}_0$.

In this final section we turn to the basic theory $\underset{\sim}{\mathrm{ML}}_0$ of extensional types by Martin-Löf. We do not give an extensive description, but refer the reader to [Ma75] and [Ma82] by Martin-Löf and to [DT84] by Diller & Troelstra, which contains a survey of $\underset{\sim}{\mathrm{ML}}_0$ on which our treatment is based.

We concentrate on the relation between $\underset{\sim}{\mathrm{ML}}_0$ and $\underset{\sim}{\mathrm{HA}}$. In [DT84] one finds the interpretations $^\wedge$ of $\underset{\sim}{\mathrm{HA}}$ into $\underset{\sim}{\mathrm{ML}}_0$ and *, mapping $\underset{\sim}{\mathrm{ML}}_0$ into $\underset{\sim}{\mathrm{APP}}^E$ (which is called $\underset{\sim}{\mathrm{APP}}$ there): dropping formulae $E\tau$ in the definition of * results in a mapping of $\underset{\sim}{\mathrm{ML}}_0$ in $\underset{\sim}{\mathrm{APP}}$ as defined in Ch.II (i.e. with total application). We prove here that $\underset{\sim}{\mathrm{ML}}_0$ is conservative over $\underset{\sim}{\mathrm{HA}}$. This is done by defining *extensional realizability* $\underline{e}$ for $\underset{\sim}{\mathrm{APP}}$, which can be considered as the composition of $^\wedge$ and *; the rest of the argument closely follows the proof of the conservation theorem for $\underset{\sim}{\mathrm{APP}}$ + EAC (see §4). Finally, we discuss the problem of axiomatizing $\underline{e}$.

6.1. The mapping $^\wedge$: $\underset{\sim}{\mathrm{HA}} \to \underset{\sim}{\mathrm{ML}}_0$.
See [DT84], 5.5. We assume that the primitive recursive functions of $\underset{\sim}{\mathrm{HA}}$ are defined using 0, S, k, s, r and that these constants also occur in $\underset{\sim}{\mathrm{ML}}_0$. Then:

$$(s = t)^\wedge = I(N,s,t)$$

$$(A \wedge B)^\wedge = \Sigma x \in A^\wedge . B^\wedge$$

$$(A \to B)^\wedge = \Pi x \in A^\wedge . B^\wedge$$

$$\forall n A(n)^\wedge = \Pi n \in N . A^\wedge(n)$$

$$\exists n A(n)^\wedge = \Sigma n \in N . A^\wedge(n)$$

Without proof we state:

6.2. LEMMA. *If the free variables of the* $\underset{\sim}{\mathrm{HA}}$*-formula* A *are among* $\vec{m}$, *then*

$$\underset{\sim}{\mathrm{HA}} \vdash A \Rightarrow \underset{\sim}{\mathrm{ML}}_0 \vdash (\vec{m} \in N \Rightarrow t \in A^\wedge) \quad \text{for some term } t \,;$$

here $\vec{m} \in N$ *abbreviates the context* $m_1 \in N, \ldots, m_k \in N$. □

6.3. The mapping * : $\underset{\sim}{\mathrm{ML}}_0 \to \underset{\sim}{\mathrm{APP}}$.

See [DT84], 6.3. * associates with every formula A of $\underset{\sim}{\mathrm{ML}}_0$ a formula $A^* = A^*(x,y)$ which we suggestively write $\{(x,y)|A^*\}$. We identify r of $\underset{\sim}{\mathrm{ML}}_0$ with the recursor term R of $\underset{\sim}{\mathrm{APP}}$ (see Ch.II, 3.8) and e of $\underset{\sim}{\mathrm{ML}}_0$ with 0 . Then

$$\begin{aligned} N^* &= \{(x,y)|x,y \in N \wedge x=y\} \\ I(A,s,t) &= \{(0,0)|(s,t) \in A^*\} \\ \Pi x \in A.B(x)^* &= \{(f,g)|\forall xy((x,y) \in A^* \to (fx,gy) \in B^*(x))\} \\ \Sigma x \in A.B(x)^* &= \{(x,y)|((x)_1,(y)_1) \in A^* \wedge ((x)_2,(y)_2) \in B^*((x)_1)\} \end{aligned}$$

6.4. LEMMA. $\underset{\sim}{\mathrm{ML}}_0 \vdash s=t \in A \Rightarrow \underset{\sim}{\mathrm{APP}} \vdash (s,t) \in A^*$.

PROOF. See 6.3.1 in [DT84]. □

We now combine $^\wedge$ and * in the following definition of *extensional realizability* $\underline{e}$ for $\underset{\sim}{\mathrm{APP}}$:

6.5. DEFINITION. $(\sigma,\tau)\underline{e}A$ is defined by

$$\begin{aligned} &(\sigma,\tau)\underline{e}(\rho_1=\rho_2) &&:= \sigma=\tau=0 \wedge \rho_1=\rho_2 \\ &(\sigma,\tau)\underline{e}(\rho \in N) &&:= \rho=\sigma=\tau \in N \\ &(\sigma,\tau)\underline{e}(A \wedge B) &&:= ((\sigma)_1,(\tau)_1)\underline{e}A \wedge ((\sigma)_2,(\tau)_2)\underline{e}B \\ &(\sigma,\tau)\underline{e}(A \to B) &&:= \forall xy((x,y)\underline{e}A \to (\sigma x,\tau y)\underline{e}B) \\ &(\sigma,\tau)\underline{e}\forall xA(x) &&:= \forall x((\sigma,\tau)\underline{e}A(x)) \\ &(\sigma,\tau)\underline{e}\exists xA(x) &&:= \exists x((\sigma,\tau)\underline{e}A(x)) \end{aligned}$$

$\tau\underline{e}A$ abbreviates $(\tau,\tau)\underline{e}A$.

6.6. LEMMA. i) $(\sigma,\tau)\underline{e}A \to (\tau,\sigma)\underline{e}A$;

ii) $(\sigma,\tau)\underline{e}\forall nA(n) \leftrightarrow \forall n(\sigma n,\tau n)\underline{e}A(n)$;

iii) $(\sigma,\tau)\underline{e}\exists nA(n) \leftrightarrow (\sigma)_1=(\tau)_1 \in N \wedge ((\sigma)_2,(\tau)_2)\underline{e}A((\sigma)_1)$.

PROOF. Straightforward. □

Before we prove that $\underline{e}$ is sound, we establish our claim that its restriction to HA is the composition of $^\wedge$ and *. For simplicity, we assume that HA is a subtheory of APP.

6.7. LEMMA. APP $\vdash (\sigma,\tau) \in A^{\wedge *} \leftrightarrow (\sigma,\tau)\underline{e}A$ *for* A *in* HA.

PROOF. Induction over the logical complexity of A:

A prime: then $A = (s = t)$. Now

$$A^\wedge = I(N,s,t),$$
$$A^{\wedge *} = \{(0,0) \mid (s,t) \in N \wedge s = t\}, \quad \text{so}$$
$$(\sigma,\tau) \in A^{\wedge *} = (\sigma = \tau = 0 \wedge s = t \in N);$$

as we have APP $\vdash s \in N,\ t \in N$ (for s, t are terms of HA), this is equivalent to $s = t \wedge \sigma = \tau = 0$, i.e. $(\sigma,\tau)\underline{e}A$.

A = B ∧ C:

$$\begin{aligned}(\sigma,\tau) \in A^{\wedge *} &= (\sigma,\tau) \in (\Sigma x \in B^\wedge . C^\wedge)^* \\ &= ((\sigma)_1,(\tau)_1) \in B^{\wedge *} \wedge ((\sigma)_2,(\tau)_2) \in C^{\wedge *} \\ &\equiv (\sigma,\tau)\underline{e}(B \wedge C), \quad \text{by ind. hyp.}\end{aligned}$$

A = B → C:

$$\begin{aligned}(\sigma,\tau) \in A^{\wedge *} &= (\sigma,\tau) \in (\Pi x \in B^\wedge . C^\wedge)^* \\ &= \forall xy((x,y) \in B^{\wedge *} \to (\sigma x,\tau y) \in C^{\wedge *}) \\ &\equiv (\sigma,\tau)\underline{e}(B \to C), \quad \text{by ind. hyp.}\end{aligned}$$

A = ∀nB(n):

$$\begin{aligned}(\sigma,\tau) \in A^{\wedge *} &= (\sigma,\tau) \in (\Pi x \in N . B(n)^\wedge)^* \\ &= \forall xy((x,y) \in N \to (\sigma x,\tau y) \in B(x)^{\wedge *}) \\ &\equiv \forall x \in N (\sigma x,\tau y) \in B(x)^{\wedge *} \\ &\equiv (\sigma,\tau)\underline{e}\forall x \in N\ B(x), \quad \text{by ind. hyp. and 6.6.(ii).}\end{aligned}$$

A = ∃nB(n):

$$\begin{aligned}(\sigma,\tau) \in A^{\wedge *} &= (\sigma,\tau) \in (\Sigma n \in N . B(n)^\wedge)^* \\ &= ((\sigma)_1,(\tau)_1) \in N \wedge ((\sigma)_2,(\tau)_2) \in B((\sigma)_1)^{\wedge *} \\ &\equiv (\sigma,\tau)\underline{e}\exists x(x \in N \wedge B(x)), \quad \text{by ind. hyp. and 6.6.(iii).}\end{aligned}$$

□

6.8. <u>LEMMA</u>. *(Soundness of* $\underline{e}$ *.)* $\mathbf{APP} \vdash A \Rightarrow \mathbf{APP} \vdash \tau\underline{e}A$ *for some* closed *term* τ .

<u>PROOF</u>. For the propositional axiom and rules we can copy the corresponding parts of the proof in 3.3. $\forall AX$ and $\exists AX$ are $\underline{e}$ - realized by $\lambda x.x$; for $\forall$-R, $\exists$-R we have that the conclusion is $\underline{e}$ - realized by the same term as the premiss; here we use that the term realizing the premiss is closed. The realizing terms for the non-logical axioms are different form those of the soundness proof for $\underline{r}$, but are not hard to find. We give some examples:

$$\langle 0,\lambda xy.0\rangle\underline{e} = AX,$$

$$\lambda x.\langle\langle 0,0\rangle,\lambda y.y\rangle\underline{e}SUB,$$

$$\lambda yx.R(y)_1(\lambda uv.(y)_2\langle Pdu,v\rangle)\underline{e}IND.$$

□

As in §4, we use the extension $\mathbf{APP}(\varepsilon)$ of $\mathbf{APP}$ to prove $A \leftrightarrow \exists x\ x\underline{e}A$ for arithmetical A .

6.9. <u>DEFINITION</u> of τ_A for arithmetical A .

$$\begin{aligned}
\tau_{\sigma_1=\sigma_2} &:= 0\\
\tau_{\rho\in N} &:= \rho\\
\tau_{A\wedge B} &:= \langle\tau_A,\tau_B\rangle\\
\tau_{A\to B} &:= k\tau_B\\
\tau_{\forall nA} &:= \lambda n.\tau_A\\
\tau_{\exists nB} &:= \langle\varepsilon_A\vec{m},\tau_A[n:=\varepsilon_A\vec{m}]\rangle \quad \text{if} \quad A = A(\vec{m},n).
\end{aligned}$$

6.10. <u>LEMMA</u>. *For arithmetical* A *:*

i) $\mathbf{APP}(\varepsilon) \vdash A \to \tau_A\underline{e}A;$

ii) $\mathbf{APP}(\varepsilon) \vdash \exists xy((x,y)\underline{e}A) \to A.$

PROOF. Simultaneous induction over A.

A prime, $A = B \wedge C$: easy.

$A = B \rightarrow C$:

i) Assume $B \rightarrow C$. By the induction hypothesis, we have $\exists xy((x,y)\underline{e}B) \rightarrow B$ and $C \rightarrow \tau_C\underline{e}C$, so $\exists xy((x,y)\underline{e}B) \rightarrow \tau_C\underline{e}B$. By logic and 6.6.(iii) this implies $\forall xy((x,y)\underline{e}B \rightarrow (\tau_C,\tau_C)\underline{e}C)$, i.e. $\tau_A\underline{e}A$.

ii) Assume $\exists xy((x,y)\underline{e}(B \rightarrow C))$, i.e. $\exists xy\forall zu((z,u)\underline{e}B \rightarrow (xz,yu)\underline{e}C)$. Together with $B \rightarrow \tau_B\underline{e}B$ and $\exists vw((v,w)\underline{e}C \rightarrow C)$ (by the induction hypothesis) this yields $B \rightarrow C$.

$A = \forall n(n \in N \rightarrow B(n))$: as above, using 6.6.(ii).

$A = \exists n(n \in N \wedge B(n))$:

i) Assume $\exists n \in N(B(n))$, then (by εAX) $B(\vec{m},\varepsilon\vec{m})$. The induction hypothesis gives us $B(\vec{m},n) \rightarrow \tau_B\underline{e}B(\vec{m},n)$, so with substitution we get $\tau_B[n := \varepsilon\vec{m}]\underline{e}B(\vec{m},\varepsilon\vec{n})$ i.e. $\tau_A\underline{e}A$.

ii) Assume $\exists xy((x,y)\underline{e}(\exists n \in N\ B(n)))$, i.e. $\exists xy((x)_1 = (y)_1 \in N \wedge ((x)_2,(y)_2)\underline{e}B((x)_1))$, so by induction hypothesis $\exists x((x)_1 \in N \wedge B((x)_1))$, i.e. $\exists n \in N\ B(n)$.

□

6.11. COROLLARY. $\mathbf{APP}(\varepsilon) \vdash A \leftrightarrow \exists x\ x\underline{e}A$ *for arithmetical* A. □

6.12. THEOREM. $\mathbf{ML}_0$ *is conservative over* $\mathbf{HA}$.

PROOF. Assume $\mathbf{ML}_0 \vdash t \in A^\wedge$, t some term of $\mathbf{ML}_0$, A a formula of $\mathbf{HA}$. By 6.4:

$$\mathbf{APP} \vdash t \in A^{\wedge *}.$$

With 6.7:

$$\mathbf{APP} \vdash t\underline{e}A.$$

By 6.11, and the fact that $\mathbf{APP}(\varepsilon)$ extends $\mathbf{APP}$:

$$\mathbf{APP}(\varepsilon) \vdash A,$$

so with 4.19:

$$\mathbf{APP} \vdash A$$

and hence (for $\underset{\sim}{\text{APP}}$ is conservative over $\underset{\sim}{\text{HA}}$)

$$\underset{\sim}{\text{HA}} \vdash A.$$

□

6.13. <u>REMARKS</u>.

i) It is tempting to think that $(x,y)\underline{e}A$ is a transitive relation in x and y, i.e.

$$(\rho,\sigma)\underline{e}A \wedge (\sigma,\tau)\underline{e}A \to (\rho,\tau)\underline{e}A.$$

However, the proof by formula induction breaks down at $A = \exists zB(z)$, for we do not have, in general

$$\exists z((\rho,\sigma)\underline{e}B(z)) \wedge \exists z((\sigma,\tau)\underline{e}B(z)) \to \exists z((\rho,\tau)\underline{e}B(z)).$$

Neither are we able to derive $(\sigma,\tau)\underline{e}A \to \sigma\underline{e}A$. As a consequence, we have no proof of the *projectiveness* of $\underline{e}$: this is the property

$$\exists xy((x,y)\underline{e}A) \leftrightarrow \exists uv((u,v)\underline{e}(\exists xy((x,y)\underline{e}A)))$$

This last fact blocks the (obvious) way to an axiomatization result for $\underline{e}$, viz. the way we followed in §3 when treating $\underline{r}$.

ii) Other versions of extensional realizability have been defined and studied in [Be82] by Beeson and [Gr82] by Grayson. Our definition differs from those in that it is based on the fact that $\underset{\sim}{\text{APP}}$ allows quantification over *all* objects.

CHAPTER IV. EXTENDED BAR INDUCTION

§1. Introduction.

1.1. In this chapter, we study the principle of extended bar induction (EBI). Our main result is that $\mathbf{APP}+\mathrm{EBI}$ proves the same arithmetical theorems as $\mathbf{ID}_1$ (theorem 5.8; see Ch.III, 5.11 for a definition of $\mathbf{ID}_1$). As a corollary, we obtain

$$\mathbf{EL}^* + \mathrm{EBI} \text{ is conservative over } \mathbf{ID}_1 \cap L(\mathbf{HA}).$$

1.2. To formulate EBI, we extend $\mathbf{APP}$ to $\mathbf{APP}^*$ by adding new variables α, β, ... for *sequences* of objects; they may occur without restriction in terms and formulae. We add the following quantifier rules and axioms:

$$\forall R_{SEQ} \qquad \frac{A \to B}{A \to \forall\alpha B} \qquad (\alpha \text{ not free in } A)$$

$$\exists R_{SEQ} \qquad \frac{A \to B}{\exists\alpha A \to B} \qquad (\alpha \text{ not free in } B)$$

$$\forall AX_{SEQ} \qquad \forall\alpha A\alpha \to A\beta$$

$$\exists AX_{SEQ} \qquad A\beta \to \exists\alpha A\alpha$$

The other new axioms are:

$$\mathrm{SEQAX1} \qquad \forall\alpha\forall n\exists x(\alpha n = x)$$

SEQAX2 $\forall x\exists\alpha\forall n(xn = \alpha n)$

SEQAX3 $\forall\alpha\beta\exists\gamma\forall n(\gamma n = \langle\alpha n, \beta n\rangle)$

SEQAX4 $\forall\alpha x\exists\beta(\beta 0 = x \wedge \forall n(\beta(n+1) = \alpha n))$

N.B. The axioms $\forall x Ax \to A\tau$, $A\tau \to \exists x Ax$ remain restricted to $\tau \in L(\underset{\sim}{APP})$; as a consequence, we do not have e.g. $\exists x(x = \alpha)$.

1.3. REMARK.
$\underset{\sim}{APP}^*$ is the first part of extending $\underset{\sim}{APP}$ to $\underset{\sim}{T}_1^*$, a theory with *choice* sequences (see §2). In this sense, $\underset{\sim}{APP}^*$ is comparable with $\underset{\sim}{EL}^*$ (see [T77], 5.2).
It is consistent to assume $\alpha, \beta, \ldots$ in $\underset{\sim}{APP}^*$ to be *lawlike* (if we consider the objects of $\underset{\sim}{APP}$ to be lawlike). This follows from

(1) $$\underset{\sim}{APP}^* \nvdash \neg\forall\alpha\exists x\forall n(\alpha n = xn),$$

a consequence of 1.5. So the sequences $\alpha, \beta, \ldots$ in $\underset{\sim}{APP}^*$ are not really choice sequences yet - that requires $\underset{\sim}{CS}$ - like axioms, viz. ECS1-4 in 2.1. See also 2.6.

1.4. The interpretation A^- of a formula $A = A(\alpha,\beta,\ldots)$ of $\underset{\sim}{APP}^*$ in $\underset{\sim}{APP}$ is straightforward: replace the sequence variables $\alpha, \beta, \ldots$ by object variables $a, b, \ldots$.

1.5. LEMMA. $\underset{\sim}{APP}^* \vdash A \Rightarrow \underset{\sim}{APP} \vdash A^-$.

PROOF. Straightforward. □

1.6. COROLLARY. $\underset{\sim}{APP}^*$ *is conservative over* $\underset{\sim}{APP}$. □

1.7. The sequences $\alpha, \beta, \ldots$ we introduced above can be looked at from two points of view:
 i) as objects (not in the range of the variables $x, y, \ldots$ of $\underset{\sim}{APP}$) with some special properties as stated in the axioms: the corresponding equality is $\alpha = \beta$, equality between objects;
 ii) as sequences of objects $\alpha 0, \alpha 1, \ldots$: here the appropriate equality is $\alpha \equiv \beta$, where $\equiv$ is defined by

$$(\sigma \equiv \tau) := \forall n(\sigma n = \tau n).$$

Warning: the rôle of $=$, $\equiv$ is not the same as in other publications on choice sequences.

Now it is the second point of view which concerns us here, and we would like to have the following substitution property:

$$(2) \qquad \alpha \equiv \beta \rightarrow (A\alpha \leftrightarrow A\beta).$$

(2) is derivable in $\underset{\sim}{\mathrm{APP}}^*$ in case α occurs *regularly* in $A\alpha$, i.e. only in contexts $\alpha\tau$ where τ is a natural number.

1.8. DEFINITION. i) A formula A of $\underset{\sim}{\mathrm{APP}}^*$ is called *regular* if all its *free* sequence variables occur regularly in A.
ii) A formula A is called *totally regular* if *all* its (free and bound) sequence variables occur regularly in A.

We do not want to restrict our formal language to regular formulae in order to obtain (2): that would require a complicated definition of different sorts of terms, conflicting with the type-free and flexible character of $\underset{\sim}{\mathrm{APP}}$. To be able to formulate a weaker but valid version of (2), we use a well-known method for making predicates extensional: define

$$A(\alpha_1,\ldots,\alpha_n)^e := \exists\beta_1\ldots\beta_n(\alpha_1 \equiv \beta_1 \wedge \ldots \wedge \alpha_n \equiv \beta_n \wedge A(\beta_1,\ldots,\beta_n));$$

here $\alpha_1,\ldots,\alpha_n$ are the sequence variables occurring free in A. Now A^e is always regular and we have, in $\underset{\sim}{\mathrm{APP}}^*$:

$$A \rightarrow A^e$$

$$A \leftrightarrow A^e \text{ for regular } A$$

$$\alpha \equiv \beta \rightarrow ((A\alpha)^e \leftrightarrow (A\beta)^e).$$

1.9. Some notation and conventions.

A finite sequence $x_0,\ldots,x_{n-1}$ is coded by an object f iff:

$$(3) \qquad \begin{cases} f0 = n \\ f(i+1) = x_i \quad (0 \le i < n). \end{cases}$$

This coding is not unique, of course: one easily constructs f, g with $f0 = g0 = n$, $\forall i < n(f(i+1) = g(i+1))$ and $f(n+1) \neq g(n+1)$. However, we shall write $<x_0,\dots,x_{n-1}>$ for f satisfying (3), but only in cases where no ambiguity can occur.

It is not hard to define in APP the functions $(\cdot)_{\bullet}$, lth, $*$, $\hat{}$, $\bar{}$, $< >$ satisfying

$$(<x_0,\dots,x_{n-1}>)_i = x_i \qquad (0 \leq i < n)$$

$$\mathrm{lth}(<x_0,\dots,x_{n-1}>) = n$$

$$<x_0,\dots,x_{n-1}>*<y_0,\dots,y_{m-1}> = <x_0,\dots,x_{n-1},y_0,\dots,y_{m-1}>$$

$$\hat{x} = <x>$$

$$\mathrm{lth}(< >) = 0$$

$$\bar{a}n = <a0,\dots,a(n-1)>.$$

The equivalence relation $\sim$ between finite sequences is defined by

$$x \sim y := (\mathrm{lth}\ x = \mathrm{lth}\ y \in N \ \wedge \ \forall i < \mathrm{lth}\ x\ ((x)_i = (y)_i).$$

$*$ is also used to denote concatenation of a finite sequence with an infinite one: if a is (thought of as) an infinite sequence $a0, a1,\dots,$ then

$$(<x_0,\dots,x_{n-1}>*a)m = \begin{cases} x_m & \text{if } m < n, \\ a(n-m) & \text{if } m \geq n. \end{cases}$$

In the sequel, we shall often use the notation ϕ_x, defined by

$$\phi_x := \lambda a.\phi(x*a).$$

1.10. We adopt a set-and-membership notation, defined by

$$\tau \in A := A[x := \tau]$$
$$A \subset B := \forall x(x \in A \to x \in B)$$
$$A \equiv B := A \subset B \ \wedge \ B \subset A$$

$$A \cap B := A \wedge B$$

$$A \Rightarrow B := A \to B.$$

We also put

$$\tau \in x := \mathrm{lth}\ x \in N \wedge \forall i < \mathrm{lth}\, x\, (\tau i = (x)_i)$$

$$\tau \in \overline{A} := \forall n (\tau n \in A)$$

$$\tau \in A_x := x * \tau \in A$$

$$N^{<\omega} := \mathrm{lth}\ x \in N \wedge \forall i < \mathrm{lth}\, x\, ((x)_i \in N)$$

$$N^{\omega} := \forall n (xn \in N)$$

$$\begin{aligned} \mathrm{Tree}(A) := {} & \forall x \in A (\mathrm{lth}\ x \in N) \wedge \\ & \forall xy (x \sim y \wedge x \in A \to y \in A) \wedge \\ & \langle\rangle \in A \wedge \\ & \forall xy (x * y \in A \to x \in A) \wedge \\ & \forall x \in A \exists y (x * \hat{y} \in A). \end{aligned}$$

$$\forall \hat{x} \in A \ldots := \forall x (\hat{x} \in A \to \ldots).$$

1.11. To the equivalence relations $\sim$ (for finite sequences) and $\equiv$ (for sets), we add:

$$\sigma \equiv \tau := \forall n (\sigma n = \tau n),$$

$$\phi =_A \psi := \forall \alpha \in \overline{A} (\phi\alpha = \psi\alpha)$$

$$f \equiv_A g := \forall \alpha \in \overline{A} (f\alpha \equiv g\alpha).$$

These relations satisfy the following properties.

1.12. LEMMA.

i) $x \sim y \wedge \alpha \in x \to \alpha \in y$;

ii) $x \sim y \wedge x \in A \wedge \mathrm{Tree}(A) \to y \in A$;

iii) $x \sim y \;\rightarrow\; x{*}\alpha \equiv y{*}\alpha$;

iv) $x \sim y \wedge \mathrm{Tree}(A) \;\rightarrow\; A_x \equiv A_y$;

v) $\alpha \equiv \beta \;\rightarrow\; \bar{\alpha}n \sim \bar{\beta}n$;

vi) $\alpha \equiv \beta \wedge \alpha \in x \;\rightarrow\; \beta \in x$;

vii) $\phi =_A \psi \wedge x \in A \wedge \mathrm{Tree}(A) \;\rightarrow\; \phi_x =_{A_x} \psi_x$;

viii) $A \equiv B \;\rightarrow\; \bar{A} \equiv \bar{B}$;

ix) $A \equiv B \wedge \mathrm{Tree}(A) \;\rightarrow\; \mathrm{Tree}(B)$;

x) $\mathrm{Tree}(A) \;\rightarrow\; A \equiv A_{<\,>}$.

PROOF. Easy. □

1.13. Definition of EBI.

We define

$$\mathrm{Bar}(A,P) := \forall \alpha \in \bar{A} \exists n P(\bar{\alpha}n),$$

$$\mathrm{Mon}(A,P) := \forall xy(x{*}y \in A \wedge Px \rightarrow P(x{*}y)),$$

$$\mathrm{Ind}(A,P) := \forall x \in A(\forall y(x{*}\hat{y} \in A \rightarrow P(x{*}\hat{y})) \rightarrow Px).$$

Now EBI(A,P) reads

$$\mathrm{Tree}(A) \wedge \mathrm{Bar}(A,P) \wedge \mathrm{Mon}(A,P) \wedge \mathrm{Ind}(A,P) \;\rightarrow\; P{<\,>}.$$

EBI(A) is EBI(A,P) for all *regular* $P \in L(\utilde{\mathrm{APP}}^*)$, and EBI is EBI(A) for all $A \in L(\utilde{\mathrm{APP}})$ (hence *not* containing sequence variables). BI is defined as $\mathrm{EBI}(N^{<\omega})$.

For more information on EBI see [T80], §1.
N.B. Our EBI corresponds with EBI" in [T80]; moreover, our restriction on A in the definition of EBI does not play a role there.

The main result of this chapter is:

1.14. THEOREM. $\mathbf{APP}^* + \mathrm{EBI}$ *and* $\mathbf{ID}_1$ *prove the same arithmetical theorems.*

Here $\mathbf{ID}_1$ is $\mathbf{HA} + \mathrm{ID}_1$, i.e. intuitionistic arithmetic + non-iterated inductive definitions with positive operator form (see Ch.III, §5).

As a corollary, we have

$$\mathbf{EL}^* + \mathrm{EBI}_{\mathrm{ar}} \text{ is conservative over } \mathbf{ID}_1 \cap L(\mathbf{HA}),$$

where $\mathrm{EBI}_{\mathrm{ar}}$ is $\mathrm{EBI}(A)$ for all arithmetical A.

The steps of the proof of (3) are:

i) we formulate a theory $\mathbf{T}_1^*$, an extension of $\mathbf{APP}^*$ with tree variables, inductively defined sets and choice-sequence-like axioms for $\alpha, \beta, \ldots$; EBI is derivable in T_1^*;

ii) $\mathbf{T}_1^*$ is interpreted in $\mathbf{T}_2$ (a theory without sequence variables) by forcing, which can also be formulated as an elimination translation in the sense of [KT70] and [T80];

iii) $\mathbf{T}_2$ is reduced to $\mathbf{T}_3$, a theory without tree variables;

iv) $\mathbf{T}_3$ is shown to be contained in $\mathbf{APP} + \mathrm{EAC} + \mathrm{ID}_1$;

v) as was proved in Ch.3, §5, $\mathbf{APP} + \mathrm{EAC} + \mathrm{ID}_1$ proves the same arithmetical theorems as $\mathbf{ID}_1$;

vi) finally we observe, using a result by Sieg [BFPS81], that $\mathbf{ID}_1$ and $\mathbf{ID}_1(\mathcal{O})$ prove the same arithmetical theorems, and we show that $\mathrm{ID}_1(\mathcal{O})$ is contained in $\mathrm{APP}^* + \mathrm{EBI}$, which closes the circle.

§2. The theory $\mathbf{T}_1^*$.

In this section we define the theory $\mathbf{T}_1^*$ and show, among other things, that $\mathbf{T}_1^* \vdash \mathrm{EBI}$.

2.1. The language of $\mathbf{T}_1^*$ consists of that of $\mathbf{APP}^*$ plus variables S, T, ... for trees and the constants U (the universal tree) and I_0 (for inductively defined sets of functions). $\mathbf{T}_1^*$ has tree terms, defined as follows:

i) U and all tree variables are tree terms;

ii) if V, W are tree terms, then so is $V \times W$;

iii) if V is a tree term and τ a term, then V_τ is a tree term.

New prime formulae are $\tau \in V$ and $\tau \in I_0(V)$, V a tree term. We assume $\times$, $[\ ]_0$, $[\ ]_1$ to be defined satisfying

$$\langle x_0,\dots,x_{n-1}\rangle \times \langle y_0,\dots,y_{n-1}\rangle = \langle\langle x_0,y_0\rangle,\dots,\langle x_{n-1},y_{n-1}\rangle\rangle;$$

$$[\langle x_0,\dots,x_{n-1}\rangle]_i = \langle (x_0)_i,\dots,(x_{n-1})_i\rangle \qquad (i = 0,1).$$

Now we can give the new rules and axioms of $\underset{\sim}{T}_1^*$:

$\forall R_{TR}$ $\quad \dfrac{A \to B}{A \to \forall TB}$ $\quad$ (T not free in A)

$\exists R_{TR}$ $\quad \dfrac{A \to B}{\exists TA \to B}$ $\quad$ (T not free in B)

$\forall AX_{TR}$ $\quad \forall TA(T) \to A(S)$

$\exists AX_{TR}$ $\quad A(S) \to \exists TA(T)$

TRAX1 $\quad$ Tree(T) for all tree variables T

TRAX2 $\quad \tau \in U \leftrightarrow \mathrm{lth}\ \tau \in N$

TRAX3 $\quad \sigma \in V_\tau \leftrightarrow \tau * \sigma \in V$

TRAX4 $\quad \tau \in V \times W \leftrightarrow [\tau]_0 \in V \wedge [\tau]_1 \in W$

TRAX5 $\quad \mathrm{Tree}(A) \to \exists T(T \equiv A)$ $\quad$ for $A \in L^-(\underset{\sim}{APP})$
(i.e. $A \in L(\underset{\sim}{APP})$, A $\vee$-, $\exists$-free)

TRAX6 $\quad \forall T \forall x(x \in T \to \exists S(S \equiv T_x))$

TRAX7 $\quad \forall TT' \exists S(S \equiv T \times T')$

TRAX8 $\quad I_0(T) \equiv I_0(T_{\langle\,\rangle})$

In I_0AX1-3, I_1AX we use ϕ, f as variables of $\underset{\sim}{APP}$ (i.e. ranging over objects). In the rest of this chapter, we shall often use ϕ and ψ for elements of some $I_0(T)$, and f, g, h, ... for elements of some $I_1(S,T)$.

I_0AX1 $\quad \forall \alpha \in \overline{T}(\phi\alpha = x) \to \phi \in I_0(T)$

I_0AX2 $\quad \forall \hat{x} \in T(\phi_{\hat{x}} \in I_0(T_{\hat{x}})) \to \phi \in I_0(T)$

I_0AX3 $\forall x \in T \forall \phi [\exists y \forall \alpha \in \overline{T}_x (\phi\alpha = y) \vee \forall \hat{y} \in T_x (\phi_{\hat{y}} \in P(x * \hat{y})) \rightarrow \phi \in Px]$
$\rightarrow \forall x \in T(I_0(T_x) \subset P(x))$

I_1AX $\forall \alpha \in \overline{S} \forall f \in I_1(S,T) \exists \beta \in \overline{T}(\beta = f\alpha)$

here I_1 is defined by

$$f \in I_1(S,T) \leftrightarrow \forall n(\lambda x.fxn \in I_0(S)) \wedge \forall a \in \overline{S}(fa \in \overline{T}).$$

In the next five axioms, A and B contain no free sequence variables besides those shown.

ECS1 $\forall a \in \overline{T}\ Aa \rightarrow \forall \alpha \in \overline{T}\ A\alpha$ for prime A

ECS2 $\forall T \forall f \in I_1(T,U)(\forall \alpha \in \overline{T}\ A(f\alpha) \rightarrow \forall \alpha \in \overline{T}\ B(f\alpha)) \rightarrow \forall \alpha (A\alpha \rightarrow B\alpha)$

ECS3 $\forall \alpha \in \overline{T} \exists x A(\alpha, x) \rightarrow \exists \phi \in I_0(T) \forall \alpha \in \overline{T}\ A(\alpha, \phi\alpha)$

ECS4 $\forall \alpha \in \overline{T} \exists \beta A(\alpha, \beta) \rightarrow \exists f \in I_1(T,U) \forall \alpha \in \overline{T}\ A(\alpha, f\alpha)$

EAC $\forall x(Ax \rightarrow \exists y B(x,y)) \rightarrow \exists f \forall x (Ax \rightarrow B(x, fx))$
A $\vee$-, $\exists$-free.

2.2. REMARKS.

A) Not all tree terms V satisfy Tree(V): e.g. for $V = T_\tau$ this is only the case if $\tau \in T$.

B) By I_0AX1-3, $I_0(T)$ is an inductively defined set of functions ϕ defined on sequences α with $\forall n(\overline{\alpha}n \in T)$ (so α is an 'infinite branch' of T). I_0AX1 states that all constant functions ϕ are in $I_0(T)$, by I_0AX2 one can prove e.g. that $\lambda\alpha.\alpha 0$, $\lambda\alpha.\alpha 1$, ... are in $I_0(T)$; the schema I_0AX3 expresses that $I_0(T)$ is the smallest set satisfying I_0AX1 and I_0AX2.
$I_1(S,T)$ is a set of functions from $\overline{S}$ to $\overline{T}$, and consists by definition of those functions the projection of which are elements of $I_0(S)$.
$I_0(T)$, $I_1(S,T)$ are sometimes called I_0-resp. I_1-sets. They are investigated in §3.

C) Comparing $\underset{\sim}{T}_1^*$ with $\underset{\approx}{CS}^*$ in [T80], we observe the following differences (besides the choice of $\underset{\approx}{APP}$ resp. $\underset{\approx}{EL}$ as basic system):

i) $\underset{\sim}{T}_1^*$ has tree variables, whereas $\underset{\approx}{CS}^*$ has type constants (types are subsets of N).
It is shown in [T80] (2.5, 2.6) that EBI(A) (EBI"(A) in the notation used there), A a subtree of $N^{<\omega}$, can be reduced to EBI($B^{<\omega}$), $B \subset N$; however, this method of reduction is based on decidable equality on N, and can therefore not be applied in our context (unless we would restrict EBI to subtrees of $N^{<\omega}$).
Tree *variables* in $\underset{\sim}{T}_1^*$ are needed to formulate the axiom ECS2; it is weaker than its counterpart in $\underset{\approx}{CS}^*$.

ii) the functionals in $I_0(T)$, $I_1(S,T)$ are not coded by neighbourhood functions as in $\underset{\approx}{CS}^*$ (using K_σ, $K_{\sigma,\tau}$), but are directly present in $\underset{\sim}{T}_1^*$; this allows a more direct treatment (cf. §3).

iii) the trees in $\underset{\sim}{T}_1^*$ for which $I_0(S)$ is defined can be seen as trees definable in $\underset{\approx}{APP}$; hence I_0AX1-3 may be thought of as a schema of *non-iterated* inductive definitions. In $\underset{\approx}{CS}^*$, however, the defining formula of a type σ may contain inductively defined sets K_σ, which makes the defining axioms of the K_τ equivalent to *finitely iterated* inductive definitions.

2.3. We now give some properties of $\underset{\sim}{T}_1^*$. In some proofs, we use facts about I_0, I_1 which are proved afterwards in §3.

2.4. <u>LEMMA</u>. $\forall T \exists a (a \in \overline{T})$.

<u>PROOF</u>. Tree(T), so $\forall x \in T \exists y (x * \hat{y} \in T)$. With EAC: $\exists f \forall x \in T(x * \langle fx \rangle \in T)$. Now define

$$a0 := f\langle\ \rangle,$$

$$a(n+1) := f(\bar{a}n),$$

then $\forall n(\bar{a}n \in T)$. □

2.5. <u>COROLLARY</u>. $\forall T \exists \alpha (\alpha \in \overline{T})$ *(by* SEQAX2*)*.

We show that $\underset{\sim}{T}_1^*$ is a proper extension of $\underset{\sim}{\mathrm{APP}}^*$.

2.6. <u>LEMMA</u>. $\underset{\sim}{T}_1^* \vdash \neg\forall\alpha\exists x\forall n(xn = \alpha n)$.

<u>PROOF</u>. Assume $\forall\alpha\exists x\forall n(xn = \alpha n)$, then (by ECS3) $\forall\alpha\forall n(\phi\alpha n = \alpha n)$ for some $\phi \in I_0(U)$. But by 3.10.(i) such a ϕ is continuous, so the value of $\phi\alpha$ is determined by an initial segment of α : contradiction. □

2.7. <u>COROLLARY</u>. $\underset{\sim}{T}_1^*$ *properly extends* $\underset{\sim}{\mathrm{APP}}^*$.

<u>PROOF</u>. Combine 2.6 with (1) in 1.3. □

2.8. <u>DEFINITION</u>. We define four schemata: EAD, ECS2', ECS3' and ECS4'. EAD is a weakening of the axiom of analytic data AD in [T80]; ECS2' is a relativized version of ECS2; ECS3' and ECS4' are extensions of ECS3 and ECS4 to arbitrary regular formulae A .

EAD $\quad A\alpha \to \exists T\exists f \in I_1(T,U)(\exists\beta \in \overline{T}(f\beta = \alpha) \wedge \forall\beta \in \overline{T}\ A(f\beta))$

ECS2' $\quad \forall S\forall f \in I_1(S,T)(\forall\alpha \in \overline{S}\ A(f\alpha) \to \forall\alpha \in \overline{S}\ B(f\alpha)) \to \forall\alpha \in \overline{T}(A\alpha \to B\alpha)$

in EAD, ECS2', A contains no free sequence variables besides α .

ECS3' $\quad \forall\alpha \in \overline{T}\exists x A(\alpha,x) \to \exists S\exists\gamma \in \overline{S}\exists\phi \in I_0(S\times T)\forall\alpha \in \overline{T}\ A(\alpha,\phi(\gamma\times\alpha))$

ECS4' $\quad \forall\alpha \in \overline{T}\exists\beta A(\alpha,\beta) \to \exists S\exists\gamma \in \overline{S}\exists f \in I_1(S\times T,U)\forall\alpha \in \overline{T}\ A(\alpha,f(\gamma\times\alpha))$

in ECS3', ECS4', A is regular and may contain free sequence variables besides α .

2.9. <u>LEMMA</u>. i) EAD *and* ECS2 *are equivalent, i.e.*

$$\underset{\sim}{T}_1^* - \mathrm{ECS2} \vdash \mathrm{ECS2} \leftrightarrow \mathrm{EAD}.$$

ii) $\underset{\sim}{T}_1^* \vdash$ ECS2'.

<u>PROOF</u>. i) : by logic, we have

$$\forall S\forall g \in I_1(S,U)(\forall\alpha \in \overline{S}\ A(g\alpha)) \to$$
$$\to \forall\alpha \in \overline{S}\exists T\exists f \in I_1(T,U)(\exists\beta \in \overline{T}(f\beta = g\alpha) \wedge \forall\beta \in \overline{T}\ A(f\beta));$$

to see this, take $T := S$, $f := g$. So, by ECS2 we have

$$\forall\alpha(A\alpha \to \exists T\exists f \in I_1(T,U)(\exists\beta \in \overline{T}(f\beta = \alpha) \wedge \forall\beta \in T\ A(f\beta)))$$

i.e. EAD.

$\leftarrow$: assume

$$(1) \qquad \forall T\forall f \in I_1(T,U)(\forall\alpha \in \overline{T}\ A(f\alpha) \to \forall\alpha \in \overline{T}\ B(f\alpha)),$$

take any α and assume $A\alpha$. By EAD:

$$\exists S\exists g \in I_1(S,U)(\exists\beta \in \overline{S}(g\beta = \alpha) \wedge \forall\beta \in \overline{S}\ A(g\beta))$$

so, by (1)

$$\exists S\exists g \in I_1(S,U)(\exists\beta \in \overline{S}(g\beta = \alpha) \wedge \forall\beta \in \overline{S}\ A(g\beta))$$

and hence $B\alpha$, by the substitution property of $=$.

ii) Easy, take $\alpha \in \overline{T} \wedge A(\alpha)$ for A. □

2.10. LEMMA. i) *For regular* A, *we have in* $\underset{\sim}{T}^*_1$

$$(2) \qquad \forall\alpha \in \overline{S}\forall\beta \in \overline{T}\ A(\alpha,\beta) \leftrightarrow \forall\alpha \in \overline{S\times T}\ A(\pi_0\alpha,\pi_1\alpha)$$

(*see* 3.6 *for a definition of* π_0, π_1).

ii) $\underset{\sim}{T}^*_1 \vdash$ ECS3', ECS4'.

PROOF. i) By SEQAX3, we have $\forall\alpha \in \overline{S}\forall\beta \in \overline{T}\exists\gamma \in \overline{S\times T}\ (\gamma \equiv \alpha\times\beta)$ ($\alpha\times\beta$ is defined in 3.6) and, by 3.8.(v) and I_1AX, we also have $\forall\gamma \in \overline{S\times T}\exists\alpha \in \overline{S}\exists\beta \in \overline{T}(\pi_0\gamma = \alpha \wedge \pi_1\gamma = \beta)$. Together with the substitution property for $\equiv$ w.r.t. regular formulae (1.8) this yields (2).

ii) We first prove ECS3'. Assume

$$\forall\alpha \in \overline{T}\exists x A(\alpha,x)$$

where A is regular. Without loss of generality we assume that A contains as free sequence variables besides α only β_0 and β_1, so $A = A(\alpha,x,\beta_0,\beta_1)$. Let $\beta := \beta_0\times\beta_1$ then, by (i)

$$\forall\alpha \in \overline{T}\exists x A(\alpha,x,\pi_0\beta,\pi_1\beta).$$

By EAD (which is derivable in $\underset{\sim}{T}{}_1^*$, by 2.9.(i)), there are S, $f \in I_1(S,U)$, $\gamma_0 \in \overline{S}$ with $f\gamma_0 = \beta$ and

$$\forall\gamma \in \overline{S}\forall\alpha \in \overline{T}\exists x A(\alpha,x,\pi_0(f\gamma),\pi_1(f\gamma)).$$

Apply (i):

$$\forall\alpha \in \overline{S\times T}\ \exists x A(\pi_0\alpha,x,\pi_0(f(\pi_1\alpha)),\pi_1(f(\pi_1\alpha))).$$

Now with ECS3:

$$\exists\phi \in I_0(S\times T)\forall\alpha \in \overline{S\times T}\ A(\pi_0\alpha,\phi\alpha,\pi_0(f(\pi_1\alpha)),\pi_1(f(\pi_1\alpha)))$$

which is equivalent to

$$\exists\phi \in I_0(S\times T)\forall\gamma \in \overline{S}\ \forall\alpha \in \overline{T}\ A(\alpha,\phi(\gamma\times\alpha),\pi_0(f\gamma),\pi_1(f\gamma))$$

hence (take $\gamma := \gamma_0$, and use $f\gamma_0 = \beta$, $\beta = \beta_0\times\beta_1$ and (i))

$$\exists\phi \in I_0(S\times T)\forall\alpha \in \overline{T}\ A(\alpha,\phi(\gamma_0\times\alpha),\beta_0,\beta_1)$$

so

$$\exists S\exists\gamma \in \overline{S}\exists\phi \in I_0(S\times T)\forall\alpha \in T\ A(\alpha,\phi(\gamma\times\alpha),\beta_0,\beta_1).$$

ECS4' is derived analogously. □

2.11. DEFINITION. EIUS, extended induction over unsecured sequences, is defined by

EIUS $\quad \forall S\ \forall\gamma \in \overline{S}\ \forall\phi \in I_0(S\times T) \cap (\overline{S\times T} \Rightarrow N)$
$$(\forall\alpha \in \overline{T}\ Q(\bar{\alpha}(\phi(\gamma\times\alpha))) \wedge \mathrm{Mon}(T,Q) \wedge \mathrm{Ind}(T,Q) \to Q\langle\ \rangle).$$

2.12. LEMMA. $\underset{\sim}{T}{}_1^* \vdash$ EIUS.

PROOF. Use I_0AX3 with

$$\phi \in P(x) \;:=\; \phi \in (\overline{(S\times T)_x} \Rightarrow N) \wedge \forall\alpha \in \overline{T}_x\ Q(x*\alpha(\phi(\gamma\times\alpha))) \wedge$$
$$\wedge\, \mathrm{Mon}(T,Q) \wedge \mathrm{Ind}(T,Q)$$

to prove

$$\forall x \in T\ \forall\gamma \in \overline{S}\ \forall\phi \in I_0((S\times T)_x) \cap (\overline{(S\times T)_x} \Rightarrow N)$$
$$(\forall\alpha \in T_x\ Q(x*\bar{\alpha}(\phi(\gamma\times\alpha))) \wedge \mathrm{Mon}(T,Q) \wedge \mathrm{Ind}(T,Q) \to Q(x));$$

then take $x := \langle\,\rangle$. For details, see 3.2.1 and 5.7.4 in [KT70]. □

2.13. LEMMA. $\underset{\sim}{T}{}_1^* \vdash \mathrm{EBI}(A)$ *for* $A \in L^-(\underset{\sim}{APP})$.

PROOF. Assume Tree(A), then $A \equiv T$ for some T (by TRAX5); and by ECS3'

$$\forall\alpha \in \overline{T}\ \exists n\ P(\bar{\alpha}n) \to$$
$$\to \exists S\ \exists\gamma \in \overline{S}\ \exists\phi \in I_0(S\times T) \cap (\overline{S\times T} \Rightarrow N)\forall\alpha \in \overline{T}\ P(\bar{\alpha}(\phi(\gamma\times\alpha)))$$

for regular P. Now apply EIUS. □

2.14. THEOREM. $\underset{\sim}{T}{}_1^* \vdash \mathrm{EBI}(A)$ *for all* $A \in L(\underset{\sim}{APP})$.

PROOF. Let $A \in L(\underset{\sim}{APP})$. By Ch.III, 2.5 we have $x \in A \leftrightarrow \exists y A^-(x,y)$ for some $A^- \in L^-(\underset{\sim}{APP})$. Assume Tree(A), Bar(A,P), Mon(A,P), Ind(A,P), and define

$$x^k := \overline{\lambda n.((x)_n)_0}\ k,$$

so $\langle x_0,\ldots,x_{n-1}\rangle^k = \langle (x_0)_0,\ldots,(x_{k-1})_0\rangle$, and put

$$x \in B \;:=\; \mathrm{lth}\,x \in N \wedge \forall n < \mathrm{lth}\,x\ \ A^-(x^n,((x)_n)_1),$$
$$Q(x) \;:=\; P(x^{\mathrm{lth}\,x}).$$

$x \in B$ means: $x^{\mathrm{lth}\,x} \in A$ and, for every $n < \mathrm{lth}\,x$, $((x)_n)_1$ is the 'witnessing information' that $x^n \in A$.
One easily derives Tree(B), Bar(B,Q), Mon(B,Q), Ind(B,Q); hence by 2.13 (observe that $B \in L^-(\underset{\sim}{APP})$) $Q\langle\,\rangle$, so $P\langle\,\rangle$ (for $\langle\,\rangle^0 = \langle\,\rangle$). □

§3. Inductively defined functionals.

Here we establish the properties of I_0, I_1 that are needed in §2 and §4. For this, we define the theories $\underset{\sim}{T}_2^*$ and $\underset{\sim}{T}_2$.

3.1. DEFINITION. $\underset{\sim}{T}_2^*$ is obtained from $\underset{\sim}{T}_1^*$ by omitting the axioms ECS1-4; if we also drop the sequence variables $\alpha, \beta, \ldots$, their axioms and rules, and replace α, β in I_0AX1, 3 and I_1AX by the object variables a, b, we get the theory $\underset{\sim}{T}_2$. So $\underset{\sim}{T}_2$ is an extension of $\underset{\sim}{\text{APP}}$ + EAC with tree variables and inductively defined sets of functionals.

3.2. LEMMA. $\underset{\sim}{T}_2^* \vdash A \Rightarrow \underset{\sim}{T}_2 \vdash A^-$,

where $^-$: $\underset{\sim}{T}_2^* \to \underset{\sim}{T}_2$ *is the extension of the mapping* $^-$ *of* 1.3 *to* $\underset{\sim}{T}_2^*$.

PROOF. As in 1.4. □

3.3. COROLLARY. $\underset{\sim}{T}_2^*$ *is conservative over* $\underset{\sim}{T}_2$.

3.4. LEMMA. *In* $\underset{\sim}{T}_2^*$ *we have*

i) $\alpha \equiv \beta \wedge \alpha \in \overline{T} \wedge \phi \in I_0(T) \to \phi\alpha = \phi\beta$;

ii) $\alpha \equiv \beta \wedge \alpha \in \overline{T} \wedge f \in I_1(T,S) \to f\alpha \equiv f\beta$;

iii) $\phi =_T \psi \wedge \phi \in I_0(T) \to \psi \in I_0(T)$;

iv) $f \equiv_T g \wedge f \in I_1(T,S) \to g \in I_1(T,S)$;

v) $S \subset T \to I_0(T) \subset I_0(S)$;

vi) $S_1 \subset T_1 \wedge T_2 \subset S_2 \to I_1(T_1,T_2) \subset I_1(S_1,S_2)$;

vii) $\phi \in I_0(T) \to \phi =_T \phi_{<>}$.

PROOF. (i), (iii) and (v) are proved using I_0AX3, taking for $\phi \in P(x)$ respectively

$$\forall\alpha\beta(\alpha \in \overline{T_x} \wedge \alpha \equiv \beta \to \phi\alpha = \phi\beta),$$

$$\forall\psi(\psi \equiv_{T_x} \phi \to \psi \in I_0(T_x))$$

and

$$\phi \in I_0(S_x);$$

also TRAX8 is used.

(ii), (iv), (vi) follow from (i), (iii), (v) and the definition of I_1.
(vii) follows from (i) (for $\alpha \equiv \langle\,\rangle * \alpha$). □

3.5. <u>LEMMA</u>. i) $\underset{\sim}{T}{}_2^* \vdash \phi \in I_0(T) \leftrightarrow \forall y \in T(\mathrm{lth}\, y = n \rightarrow \phi_y \in I_0(T_y))$;

ii) *let* $\phi \in I_0(T)$, $\phi \in \overline{T} \Rightarrow N$, *then*

$$\forall\psi(\forall\alpha \in \overline{T}(\psi_{\bar{\alpha}(\phi\alpha)} \in I_0(T_{\bar{\alpha}(\phi\alpha)})) \leftrightarrow \psi \in I_0(T)).$$

<u>PROOF</u>. i) The case $n = 0$ follows from $\mathrm{lth}\, y = 0 \leftrightarrow y \sim \langle\,\rangle$, $T \equiv T_{\langle\,\rangle}$ and $\phi =_T \phi_{\langle\,\rangle}$. For $n = 1$, ← follows with I_0AX2, → with I_0AX3 where $\phi \in P(x) := \forall\hat{y} \in T_x(\phi_{\hat{y}} \in I_0(T_{x*\hat{y}}))$. For $n > 1$, use induction over N.
ii) Use (i) and I_0AX3 with

$$\phi \in P(x) :=$$

$$\phi \in (\overline{T}_x \Rightarrow N) \rightarrow \forall\psi(\forall\alpha \in \overline{T}_x(\psi_{\bar{\alpha}(\phi\alpha)} \in I_0(T_{x*\bar{\alpha}(\phi\alpha)})) \rightarrow \psi \in I_0(T_x)).$$

□

3.6. <u>DEFINITIONS</u>. We define

$$\alpha\times\beta := \lambda n.\langle\alpha n,\beta n\rangle$$

$$\pi_i := \lambda\alpha n.(\alpha n)_i \qquad (i = 0,1)$$

$$x* := \lambda\alpha.x*\alpha$$

$$f\otimes g := \lambda\alpha n.\langle f\alpha n, g\alpha n\rangle$$

$$f\circ g := \lambda\alpha.f(g\alpha)$$

3.7. <u>LEMMA</u>. i) $\forall n(\lambda\alpha.\alpha n \in I_0(T))$;

ii) $\forall\phi \in I_0(T)\forall x(\lambda\alpha.x(\phi\alpha) \in I_0(T))$;

iii) $\forall\phi,\psi \in I_0(T)(\lambda\alpha.\langle\phi\alpha,\psi\alpha\rangle \in I_0(T))$.

PROOF. i) induction over N, using I_0AX1,2.

ii) induction over I_0.

iii) double induction over I_0. □

3.8. LEMMA.

i) $\lambda\alpha.\phi\alpha+1 \in I_0(T)$;

ii) $\lambda\alpha.\alpha \in I_1(T,T)$;

iii) $\phi,\psi \in I_0(T) \to \lambda\alpha.\max(\phi\alpha,\psi\alpha) \in I_0(T)$;

iv) $x \in T \to x^* \in I_1(T_x,T)$;

v) $\pi_i \in I_1(T_0 \times T_1, T_i)$ $(i = 0,1)$;

vi) $\forall f \in I_1(S,T_1)\forall g \in I_1(S,T_2)(f \otimes g \in I_1(S,T_1 \times T_2))$.

PROOF.

i) by 3.7.(ii).

ii) by 3.7.(i) and the definition of I_1.

iii) combine 3.7.(iii), (ii).

iv) use I_0AX1, 3.7.(i) and the definition of I_1.

v) by 3.7.(i) and the definition of I_1.

vi) by 3.7.(iii). □

For the important lemma 3.11 we need not only to know that all $\phi \in I_0(T)$ are continuous, but also that any such ϕ has a *modulus* $\delta \in I_0(T) \cap (\overline{T} \Rightarrow N)$ which is also its own modulus; analogous for $I_1(S,T)$.

3.9. DEFINITION. Let $\phi \in I_0(T)$, $f \in I_1(S,T)$.

i) $\delta\, mod\, \phi := \delta \in (\overline{T} \Rightarrow N) \wedge \forall\alpha\beta \in \overline{T}(\bar{\alpha}(\delta\alpha) = \bar{\beta}(\delta\alpha) \to \phi\alpha = \phi\beta)$;

ii) $\delta \in M_0(T) := \delta \in I_0(T) \wedge \delta\, mod\, \delta$;

iii) $d\, Mod\, f := d \in (N \Rightarrow (\overline{S} \Rightarrow N)) \wedge \forall n\forall\alpha\beta \in \overline{S}(\bar{\alpha}(dn\alpha) = \bar{\beta}(dn\alpha) \to \overline{f\alpha n} = \overline{f\beta n})$.

iv) $d \in M_1(S) := \forall n(dn \in M_0(S))$.

3.10. LEMMA. i) $\forall\phi \in I_0(T)\exists\delta \in M_0(T)(\delta\, mod\, \phi)$;

ii) $\forall f \in I_1(S,T)\exists d \in M_1(S)(d\, Mod\, f)$.

PROOF. i) Use I_0AX3 with $\phi \in P(x) := \exists\delta \in M_0(T_x)(\delta\ mod\ \phi)$.

- $\forall\alpha \in T_x(\phi\alpha = y)$: take $\delta := \lambda\alpha.0$.
- Assume $\forall\hat{y} \in T_x \exists\delta \in M_0(T_{x*\hat{y}})(\delta\ mod\ \phi_{\hat{y}})$.

By EAC:

$$\exists D \forall\hat{y} \in T_x(Dy \in M_0(T_{x*\hat{y}}) \wedge Dy\ mod\ \phi_{\hat{y}}).$$

Define

$$\delta' := \lambda\alpha.D(\alpha 0)(\lambda n.\alpha(n+1))+1,$$

then $\delta' \in I_0(T_x)$ (by I_0AX2 and 3.8.(i)), $\delta'\ mod\ \phi$ and $\delta'\ mod\ \delta'$.

ii) Assume $f \in I_1(S,T)$, so by the definition of I_1 we have $\forall n(\lambda\alpha.f\alpha n \in I_0(S))$. With (i):

$$\forall n \exists\delta \in M_0(S)(\delta\ mod\ \lambda\alpha.f\alpha n);$$

using EAC, we find some D with

$$\forall n(Dn \in M_0(S) \wedge Dn\ mod\ \lambda\alpha.f\alpha n).$$

Now define d by

$$\begin{cases} d0 := D0 \\ d(n+1) := \lambda\alpha.\max(dn\alpha, D(n+1)\alpha), \end{cases}$$

then $d \in M_1(S)$ (by 3.8.(iii), induction over n) and $d\ Mod\ f$. □

3.11. LEMMA. *(Closure of I_0-and I_1-sets under composition.)*

i) $\forall\phi \in I_0(S)\forall f \in I_1(S,T)(\phi\circ f \in I_0(T))$;

ii) $\forall f \in I_1(S,T)\forall g \in I_1(S',S)(f\circ g \in I_1(S',T))$.

PROOF. i) We use I_0AX3 with $\phi \in P(x) := \forall T \forall f \in I_1(T,S_x)(\phi\circ f \in I_0(T))$.

- $\forall\alpha \in \overline{S_x}(\phi\alpha = y)$: then $\phi\circ f$ is also constant on $\overline{T}$, and (by I_0AX1) in $I_0(T)$.

- Assume

(1) $$\forall \hat{y} \in S_x \forall T \forall f \in I_1(T, S_{x*\hat{y}})(\phi_{\hat{y}} \circ f \in I_0(T)),$$

and let $g \in I_1(T, S_x)$. Then $\lambda\alpha.g\alpha 0 \in I_0(T)$, so by 3.10.(i) δ *mod* $\lambda\alpha.g\alpha 0$ for some $\delta \in M_0(T)$. Now let $\alpha \in \overline{T}$ be arbitrary and define $z := g\alpha 0$. Then

$$\forall \beta \in T_{\bar{\alpha}(\delta\alpha)}(g(\bar{\alpha}(\delta\alpha)*\beta)0 = z).$$

Define

$$h := \lambda\beta n.g(\bar{\alpha}(\delta\alpha)*\beta)(n+1),$$

then, by 3.7.(i), for all n

$$\lambda\beta.h\beta n = \lambda\beta.g(\bar{\alpha}(\delta\alpha)*\beta)(n+1) \in I_0(T_{\bar{\alpha}(\delta\alpha)}),$$

so $h \in I_1(T_{\bar{\alpha}(\delta\alpha)}, S_{\hat{z}})$ by the definition of I_1. Now

$$\begin{aligned}\phi_{\hat{y}} \circ h &= \lambda\beta.\phi(\langle g(\bar{\alpha}(\delta\alpha)*\beta)0\rangle * \lambda n.g(\bar{\alpha}(\delta\alpha)*\beta)(n+1)) \\ &= \lambda\beta.\phi(g(\bar{\alpha}(\delta\alpha)*\beta)) \\ &= (\phi \circ g)_{\bar{\alpha}(\delta\alpha)}\end{aligned}$$

By (1), $\phi_{\hat{y}} \circ h \in I_0(T_{\bar{\alpha}(\delta\alpha)})$, so with 3.5.(ii) we have $\phi \circ g \in I_0(T)$.
ii) Easy, use $\lambda\alpha.((f \circ g)\alpha)n = (\lambda\alpha.(f\alpha)n) \circ g$, (i) and the definition of I_1. □

3.12. <u>LEMMA</u>. *Let* $\delta \in M_0(T)$, *and let* A *satisfy*

(2) $$\forall x \in T \forall pq(\forall \alpha \in \overline{T}_x(p\alpha = q\alpha) \to (A(x,p) \leftrightarrow A(x,q))).$$

Then:

i) $$\forall \alpha \in \overline{T} \exists \phi \in I_0(T_{\bar{\alpha}(\delta\alpha)})A(\bar{\alpha}(\delta\alpha), \phi) \to \exists \psi \in I_0(T) \forall \alpha \in \overline{T}\ A(\bar{\alpha}(\delta\alpha), \psi_{\bar{\alpha}(\delta\alpha)});$$

ii) $$\forall \alpha \in \overline{T} \exists f \in I_1(T_{\bar{\alpha}(\delta\alpha)}, S)A(\bar{\alpha}(\delta\alpha), f) \to \exists g \in I_1(T,S) \forall \alpha \in \overline{T}\ A(\bar{\alpha}(\delta\alpha), g_{\bar{\alpha}(\delta\alpha)}).$$

PROOF. i) Assume $\forall\alpha \in \overline{T}\exists\phi \in I_0(T_{\bar{\alpha}(\delta\alpha)})A(\bar{\alpha}(\delta\alpha),\phi)$. Using EAC, we find a Φ with

$$(3) \qquad \forall\alpha \in \overline{T}(\Phi\alpha \in I_0(T_{\bar{\alpha}(\delta\alpha)}) \wedge A(\bar{\alpha}(\delta\alpha),\Phi\alpha)).$$

We also have, by 2.5, $\forall x \in T\exists\beta(\beta \in \overline{T}_x)$; EAC gives us an F with $\forall x \in T(Fx \in \overline{T}_x)$, i.e. $\forall x \in T(x{*}Fx \in \overline{T})$.
Now define, for $\alpha \in \overline{T}$:

$$\alpha_\delta := \bar{\alpha}(\delta\alpha){*}F(\bar{\alpha}(\delta\alpha)),$$

then $\alpha_\delta \in \overline{T}$ and $\delta(\alpha_\delta) = \delta\alpha$ (for δ *mod* δ) so $\bar{\alpha}_\delta(\delta(\alpha_\delta)) = \bar{\alpha}(\delta\alpha)$; also, by δ *mod* δ

$$(4) \qquad \forall\beta \in \overline{T}_{\bar{\alpha}(\delta\alpha)}((\bar{\alpha}(\delta\alpha){*}\beta)_\delta \equiv \alpha\delta).$$

Define

$$\psi := \lambda\alpha.(\Phi\alpha_\delta)(\lambda n.\alpha(n+\delta\alpha))$$

then, by (4)

$$(5) \qquad \forall\beta \in \overline{T}_{\bar{\alpha}(\delta\alpha)}(\psi_{\bar{\alpha}(\delta\alpha)}\beta = \Phi\alpha_\delta\beta).$$

Now (3) gives

$$\forall\alpha \in \overline{T}(\Phi\alpha_\delta \in I_0(T_{\bar{\alpha}(\delta\alpha)}) \wedge A(\bar{\alpha}(\delta\alpha),\Phi\alpha_\delta))$$

so, with (2) and (5)

$$\forall\alpha \in \overline{T}(\psi_{\bar{\alpha}(\delta\alpha)} \in I_0(T_{\bar{\alpha}(\delta\alpha)}) \wedge A(\bar{\alpha}(\delta\alpha),\psi_{\bar{\alpha}(\delta\alpha)}))$$

With 3.5.(ii):

$$\exists\psi \in I_0(T)\forall\alpha \in \overline{T}\ A(\bar{\alpha}(\delta\alpha),\psi_{\bar{\alpha}(\delta\alpha)}).$$

ii) Analogously. □

3.13. LEMMA. *Let* $\varepsilon \in M_0(T)$, $f \in I_1(S,T)$, $\alpha \in \overline{S}$. *Then*

$$\exists\delta \in M_0(S)\forall\beta \in \overline{S}_{\bar{\alpha}(\delta\alpha)}(f(\bar{\alpha}(\delta\alpha)*\beta) \in \overline{f\alpha}(\varepsilon(f\alpha))).$$

Remark. The existence of δ follows from the continuity of ε and f ; $\delta \in M_0(S)$ requires a more subtle argument.

PROOF. $f \in I_1(S,T)$ implies (by 3.10.(ii)) $d\,\mathit{Mod}\,f$ for some $d \in M_1(S)$, so

$$\forall n\,\forall\alpha \in \overline{S}\,\forall\beta \in \overline{S}_{\bar{\alpha}(dn\alpha)}(f(\bar{\alpha}(dn\alpha)*\beta) \in \overline{f\alpha}n).$$

Define

$$\delta := \lambda\alpha.d(\varepsilon(f\alpha))\alpha,$$

then

$$\forall\alpha \in \overline{S}\ \forall\beta \in \overline{S}_{\bar{\alpha}(\delta\alpha)}(f(\bar{\alpha}(\delta\alpha)*\beta) \in \overline{f\alpha}(\varepsilon(f\alpha))).$$

It remains to be shown that $\delta \in I_0(S)$ and $\delta\,\mathit{mod}\,\delta$. Now $\varepsilon \in M_0(T)$, $f \in I_1(S,T)$, so $\varepsilon\circ f \in I_0(S)$; let $\eta \in M_0(S)$, $\eta\,\mathit{mod}\,\varepsilon\circ f$ (using 3.10.(i)), then

$$\forall\alpha \in \overline{S}\,\exists n\,\forall\beta \in \overline{S}_{\bar{\alpha}(\eta\alpha)}(\varepsilon\circ f)(\bar{\alpha}(\eta\alpha)*\beta) = n. \tag{6}$$

Now, by definition of δ

$$\forall\alpha \in \overline{S}[\delta_{\bar{\alpha}(\eta\alpha)} = \lambda\beta.d((\varepsilon\circ f)(\bar{\alpha}(\eta\alpha)*\beta))(\bar{\alpha}(\eta\alpha)*\beta)]$$

so, by (6)

$$\forall\alpha \in \overline{S}\,\exists n\,[\delta_{\bar{\alpha}(\eta\alpha)} = \lambda\beta.dn(\bar{\alpha}(\eta\alpha)*\beta) = (dn)_{\bar{\alpha}(\eta\alpha)}].$$

By 3.5.(i) we get $\forall\alpha \in \overline{S}(\delta_{\bar{\alpha}(\eta\alpha)} \in I_0(S_{\bar{\alpha}(\eta\alpha)}))$ and with 3.5.(ii) this gives $\delta \in I_0(S)$.

To see that $\delta\,\mathit{mod}\,\delta$, assume $\bar{\alpha}(\delta\alpha) = \bar{\beta}(\delta\alpha)$, i.e.

$$\bar{\alpha}(d(\varepsilon(f\alpha))\alpha) = \bar{\beta}(d(\varepsilon(f\alpha))\alpha). \tag{7}$$

$d \in M_1(S)$, so $\forall n(dn \in M_0(S))$, hence (7) implies

(8) $$d(\varepsilon(f\alpha))\alpha = d(\varepsilon(f\alpha))\beta.$$

Also $d\,\mathit{Mod}\,f$, so with (8)

$$\overline{f\alpha}(\varepsilon(f\alpha)) = \overline{f\beta}(\varepsilon(f\alpha));$$

with $\varepsilon\,\mathit{mod}\,\varepsilon$ this yields $\varepsilon(f\alpha) = \varepsilon(f\beta)$. Combining this with (8), we conclude $d(\varepsilon(f\alpha))\alpha = d(\varepsilon(f\beta))\beta$, i.e. $\delta\alpha = \delta\beta$. □

§4. Forcing.

4.1. In this section we interpret $\underset{\sim}{T}_1^*$ in $\underset{\sim}{T}_2$. This interpretation is presented in two ways: first as an elimination translation (in the sense of [KT70] and [T80]), which is somewhat easier to understand, then as a definition of forcing, which has a more semantic flavour.

4.2. To describe the elimination translation, we consider $\forall\alpha \in \overline{S}$, $\exists\beta \in \overline{T}$ as *quantifiers*, not as abbreviations of $\forall\alpha(\alpha \in \overline{S} \to \ldots)$ etc; $\forall\alpha$, $\exists\beta$ are read as $\forall\alpha \in \overline{U}$, $\exists\beta \in \overline{U}$. Also $\forall m$, $\exists n$ are considered as quantifiers ranging over N. Now the elimination translation for formulae without free sequence variables reads

$$\ulcorner P \urcorner = P \qquad (P \text{ prime})$$

$$\ulcorner A \wedge B \urcorner = \ulcorner A \urcorner \wedge \ulcorner B \urcorner$$

$$\ulcorner A \to B \urcorner = \ulcorner A \urcorner \to \ulcorner B \urcorner$$

$$\ulcorner \forall x A \urcorner = \forall x \ulcorner A \urcorner$$

$$\ulcorner \exists x A \urcorner = \exists x \ulcorner A \urcorner$$

$$\ulcorner \forall n A \urcorner = \forall n \ulcorner A \urcorner$$

$$\ulcorner \exists n A \urcorner = \exists n \ulcorner A \urcorner$$

$$\ulcorner \forall T A \urcorner = \forall T \ulcorner A \urcorner$$

$$\ulcorner \exists T A \urcorner = \exists T \ulcorner A \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ P\alpha \urcorner = \forall a \in \bar{T}\ Pa \qquad (\text{P prime})$$

$$\ulcorner \forall \alpha \in \bar{T}(A \wedge B) \urcorner = \ulcorner \forall \alpha \in \bar{T}\ A \urcorner \wedge \ulcorner \forall \alpha \in \bar{T}\ B \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}(A\alpha \rightarrow B\alpha) \urcorner = \forall S \forall f \in I_1(S,T)(\ulcorner \forall \alpha \in \bar{S}\ A(f\alpha) \urcorner \rightarrow \ulcorner \forall \alpha \in \bar{S}\ B(f\alpha) \urcorner)$$

$$\ulcorner \forall \alpha \in \bar{T}\ \forall x A \urcorner = \forall x \ulcorner \forall \alpha \in \bar{T}\ A \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \exists x A x \urcorner = \exists \phi \in I_0(T) \ulcorner \forall \alpha \in \bar{T}\ A(\phi\alpha) \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \forall n A \urcorner = \forall n \ulcorner \forall \alpha \in \bar{T}\ A \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \exists n A \urcorner = \exists \phi \in I_0(T) \cap (\bar{T} \Rightarrow N) \ulcorner \forall \alpha \in \bar{T}\ A(\phi\alpha) \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \forall \beta \in \bar{S}\ A(\alpha,\beta) \urcorner = \forall f \in I_1(T\times S,T) \forall g \in I_1(T\times S,S) \ulcorner \forall \alpha \in \overline{T\times S}\ A(f\alpha,g\alpha) \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \exists \beta \in \bar{S}\ A(\alpha,\beta) \urcorner = \exists g \in I_1(T,S) \ulcorner \forall \alpha \in \bar{T}\ A(\alpha,g\alpha) \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \forall S\ A \urcorner = \forall S \ulcorner \forall \alpha \in \bar{T}\ A \urcorner$$

$$\ulcorner \forall \alpha \in \bar{T}\ \exists S\ A \urcorner = \exists S \ulcorner \forall \alpha \in \bar{T}\ A \urcorner$$

$$\ulcorner \exists \alpha \in \bar{T}\ A\alpha \urcorner = \exists a \in \bar{T} \ulcorner Aa \urcorner$$

A few examples:

i)

$$\begin{aligned} \ulcorner \mathrm{SEQAX1} \urcorner &= \ulcorner \forall \alpha \forall n \exists x(\alpha n = x) \urcorner \\ &= \forall n \ulcorner \forall \alpha \exists x(\alpha n = x) \urcorner \\ &= \forall n\ \exists \phi \in I_0(U) \ulcorner \forall \alpha(\alpha n = \phi\alpha) \urcorner \\ &= \forall n\ \exists \phi \in I_0(U) \forall a(an = \phi a); \end{aligned}$$

using 3.7.(i) and 3.2, we see that this interpretation of SEQAX1 is true in $\underset{\sim}{T}_2$.

ii) $$\ulcorner \mathrm{SEQAX2} \urcorner = \ulcorner \forall x\, \exists \alpha\, \forall n (xn = \alpha n) \urcorner$$
$$= \forall x\, \exists a\, \forall n (xn = an),$$

which is also true in $\underset{\sim}{T}_2$.

iii) $$\ulcorner \forall \alpha\, \exists x\, \forall n (\alpha n = xn) \urcorner = \exists \phi \in I_0(U)\, \ulcorner \forall \alpha\ \forall n (\alpha n = \phi \alpha n) \urcorner$$
$$= \exists \phi \in I_0(U)\, \forall n \ulcorner \forall \alpha (\alpha n = \phi \alpha n) \urcorner$$
$$= \exists \phi \in I_0(U)\, \forall n\, \forall a (an = \phi an),$$

and this is definitely *not* true in $\underset{\sim}{T}_2$, for by 3.2 and 3.10.(i) the value of ϕa is completely determined by an initial segment of a.

4.3. Now we turn to forcing. First we introduce the concept of *distinguished terms* of some formula A: these are certain term occurrences in A, usually indicated by $\vec{p}$ ($= p_1, \ldots, p_n$). Sometimes they are underlined to distinguish them, and we write $A = A(\vec{p})$ or $A = A(\underline{\vec{p}})$. This concept is needed for the following important definition.

4.4. DEFINITION. Let A be a formula with distinguished terms $\vec{p}$, and let f be some term. The *restriction* of A *along* f is defined by

$$A{\upharpoonright}f := A[\vec{p} := \vec{p}{\upharpoonright}f],$$

where $\vec{p}{\upharpoonright}f$ stands for $p_1 \circ f, \ldots, p_n \circ f$; they are exactly the distinguished terms of $A{\upharpoonright}f$.

4.5. EXAMPLES.

i) $A{\upharpoonright}f = A$ if A contains no distinguished terms.

ii) $(\underline{\phi}a = \underline{\psi}b){\upharpoonright}f = ((\underline{\phi \circ f})a = (\underline{\psi \circ f})b)$.

4.6. In the definition of forcing we shall give in a moment, we associate to every formula A of $\underset{\sim}{T}_1^*$ and tree variable T a formula $T \Vdash A$ (T *forces* A) of $\underset{\sim}{T}_2$. If A contains the choice variables $\alpha_1, \ldots, \alpha_n$ free, then we associate the free $\underset{\sim}{\mathrm{APP}}$-variables $f_1, \ldots, f_n$ to $\alpha_1, \ldots, \alpha_n$ and put

$$T \Vdash A(\alpha_1,\ldots,\alpha_n) := \forall S(f_1,\ldots,f_n \in I_1(S,T) \to$$
$$\to S \Vdash A(f_1,\ldots,f_n)).$$

For formulae without free sequence variables and with distinguished terms $\vec{p}$, we define

$$
\begin{array}{lll}
T \Vdash P & := \forall a \in \overline{T}(P[\vec{p} := \vec{p}a]) & \text{for prime } P \\
T \Vdash A \wedge B & := T \Vdash A \wedge T \Vdash B & \\
T \Vdash A \to B & := \forall S \forall f \in I_1(S,T)(S \Vdash (A \upharpoonleft f) \to S \Vdash (B \upharpoonleft f)) & \\
T \Vdash \forall x A & := \forall x(T \Vdash A) & \\
T \Vdash \exists x A(x) & := \exists \phi \in I_0(T)(T \Vdash A(\underline{\phi})) & \\
T \Vdash \forall n A & := \forall n(T \Vdash A) & \\
T \Vdash \exists n A & := \exists \phi \in I_0(T) \cap (\overline{T} \Rightarrow N)(T \Vdash A(\underline{\phi})) & \\
T \Vdash \forall S A & := \forall S(T \Vdash A) & \\
T \Vdash \exists S A & := \exists S(T \Vdash A) & \\
T \Vdash \forall \alpha A(\alpha) & := \forall S \forall f \in I_1(S,T) \forall g \in I_1(S,U)(S \Vdash (A \upharpoonleft f)(\underline{g})) & \\
 & \text{N.B. } (A \upharpoonleft f)(\underline{g}) \text{ is to be read as } (A \upharpoonleft f)[\alpha := \underline{g}] & \\
T \Vdash \exists \alpha A(\alpha) & := \exists g \in I_1(T,U)(T \Vdash A(\underline{g})) &
\end{array}
$$

4.7. EXAMPLES.

i) $T \Vdash \text{SEQAX1} \; = T \Vdash \forall \alpha \forall n \exists x(\alpha n = x)$

$= \forall S \forall f \in I_1(S,T) \forall g \in I(S,U)(S \Vdash (\forall n \exists x(\alpha n = x)) \upharpoonleft f[\alpha := \underline{g}])$

$\equiv \forall S \forall g \in I_1(S,U)(S \Vdash (\forall n \exists x(\alpha n = x))[\alpha := \underline{g}])$

$= \forall S \forall g \in I_1(S,U)(S \Vdash \forall n \exists x(\underline{g}n = x))$

$= \forall S \forall g \in I_1(S,U) \forall n \exists \phi \in I_0(S) \; S \Vdash (\underline{g}n = \underline{\phi})$

$= \forall S \forall g \in I_1(S,U) \forall n \exists \phi \in I_0(S) \forall a \in \overline{S}(gan = \phi a)$

ii) $T \Vdash \text{SEQAX2} \; = \; T \Vdash \forall x \exists \alpha \forall n(xn = \alpha n)$

$= \forall x \; T \Vdash \exists \alpha \forall n(xn = \alpha n)$

$$= \forall x \exists g \in I_1(T,U)\ T \Vdash \forall n(xn = \underline{g}n)$$

$$= \forall x \exists g \in I_1(T,U) \forall n \forall a \in \overline{T}(xn = gan)$$

iii) $$T \Vdash \forall \alpha \exists x \forall n(\alpha n = xn) =$$

$$= \forall S \forall f \in I_1(S,T) \forall g \in I_1(S,U)\ S \Vdash (\exists x \forall n(\alpha n = xn)) \upharpoonright f)[\alpha := \underline{g}]$$

$$\equiv \forall S \forall g \in I_1(S,U)\ S \Vdash \exists x \forall n(\underline{g}n = xn)$$

$$= \forall S \forall g \in I_1(S,U) \exists \phi \in I_0(S)\ S \Vdash \forall n(\underline{g}n = \underline{\phi}n)$$

$$= \forall S \forall g \in I_1(S,U) \exists \phi \in I_0(S) \forall n \forall a \in \overline{S}(gan = \phi an)$$

To show that forcing and the elimination translation are equivalent interpretations, we need the so-called monotonicity property of $\Vdash$ (proved in 4.10), and 4.12.(iii).

4.8. LEMMA. *For totally regular formulae* A *we have*

$$\underset{\sim}{T}_2 \vdash T \Vdash A(\vec{\underline{p}}) \leftrightarrow \ulcorner \forall \alpha \in \overline{T}\ A(\vec{p}\alpha) \urcorner .$$

PROOF. Formula induction. Most cases are trivial or easy, except $A = \forall \beta \in \overline{T'}B(\vec{p},\beta)$. By 4.12.(iii), $T \Vdash \forall \beta \in \overline{T'}B(\vec{p},\beta)$ is equivalent to

(1) $$\forall S \forall f \in I_1(S,T) \forall g \in I_1(S,T')(S \Vdash B(\underline{\vec{p}\circ f},\underline{g}));$$

also

$$\ulcorner \forall \alpha \in \overline{T}\ \forall \beta \in \overline{T'}B(\vec{p}\alpha,\beta) \urcorner =$$

$$= \forall f \in I_1(T\times T',T) \forall g \in I_1(T\times T',T') \ulcorner \forall \alpha \in \overline{T\times T'}B(\vec{p}(f\alpha),g\alpha) \urcorner ,$$

which is equivalent to

(2) $$\forall f' \in I_1(T\times T',T) \forall g' \in I_1(T\times T',T')(T\times T' \Vdash B(\underline{g'},\underline{\vec{p}\circ f'})).$$

(1) → (2) is evident: take $S := T\times T'$. For (2) → (1) we argue as follows. By 4.10, (2) implies

(3) $$\forall f' \in I_1(T{\times}T',T)\forall g' \in I_1(T{\times}T',T')\forall S\forall h \in I_1(S,T{\times}T')$$
$$(S \Vdash B(\underline{g'\circ h},\underline{\vec{p}\circ f'\circ h})).$$

Now take $h := f\otimes g$, $f' := \pi_0$, $g' := \pi_1$, use $\pi_0\circ(f\otimes g) \equiv_S f$, $\pi_1\circ(f\otimes g) \equiv_S g$, and we get (1). □

We shall now prove some lemmata needed for the soundness theorem for $\Vdash$.

4.9. LEMMA. *(substitution).*

i) $$p =_T q \to (T \Vdash A(\underline{p}) \leftrightarrow T \Vdash A(\underline{q}));$$

ii) $$T \Vdash A(\tau) \leftrightarrow T \Vdash A(\underline{\lambda\alpha.\tau}),\ \tau \textit{ a term of } L(\mathrm{APP}).$$

PROOF. Straightforward, with formula induction. □

4.10. LEMMA. *(monotonicity).*

$$T \Vdash A \leftrightarrow \forall S\forall f \in I_1(S,T)(S \Vdash (A{\upharpoonleft}f)).$$

PROOF. ← follows from $\lambda\alpha.\alpha \in I_1(T,T)$ (3.8.(ii)).
→ is proved with formula induction: as an example, we treat the cases $A = \exists xB$ and $A = \forall\alpha B$.

$\underline{A = \exists xB(x)}$: assume $T \Vdash \exists xB(x)$, i.e.

$$\exists\phi \in I_0(T)\ T \Vdash B(\underline{\phi}).$$

By induction hypothesis:

$$\exists\phi \in I_0(T)\ \forall S\,\forall f \in I_1(S,T)\ S \Vdash (B{\upharpoonleft}f)(\underline{\phi\circ f})$$

so, with lemma 3.11.(i):

$$\forall S\forall f \in I_1(S,T)\exists\psi \in I_0(S)\ S \Vdash (B{\upharpoonleft}f)(\underline{\psi})$$

i.e. $\forall S\forall f \in I_1(S,T)\ S \Vdash \exists xB(x)$.

$\underline{A = \forall\alpha B(\alpha)}$: assume $T \Vdash \forall\alpha B(\alpha)$, i.e.

$$\forall S\forall f \in I_1(S,T)\forall g \in I_1(S,U)\ S \Vdash (B{\upharpoonleft}f)(\underline{g});$$

with lemma 3.11.(ii):

$$\forall S'\forall f' \in I_1(S',T)\ \forall S\ \forall f \in I_1(S,S')\forall g \in I_1(S,U)\ S \Vdash (B1f'1f)\underline{g}$$

i.e. $\forall S'\forall f' \in I_1(S',T)\ S' \Vdash (\forall\alpha B(\alpha)1f')$. □

4.11. LEMMA. *(bar-property).*

$$\forall\delta \in M_0(T)(\forall a \in \bar{T}(T_{\bar{a}(\delta a)} \Vdash (A1\bar{a}(\delta a)*)) \leftrightarrow T \Vdash A).$$

PROOF. ← follows from the previous lemma and lemma 3.8.(iv). → requires formula induction: we consider the key cases $A = B \to C$, $A = \exists xB$.

$\underline{A = B \to C}$: assume $\delta \in M_0(T)$ and $\forall a \in \bar{T}(T_{\bar{a}(\delta a)} \Vdash ((B \to C)1\bar{a}(\delta a)*))$, i.e.

(1) $\forall a \in \bar{T}\ \forall S\ \forall f \in I_1(S,T_{\bar{a}(\delta a)})(S \Vdash (B1(\bar{a}(\delta a)*)\circ f) \to S \Vdash (C1(\bar{a}(\delta a)*)\circ f))$

and let $g \in I_1(S,T)$, $b \in \bar{S}$. By lemma 3.13:

(2) $\exists\eta \in M_0(S)\forall a \in \bar{S}_{\bar{b}(\eta b)}(g(\bar{b}(\eta b)*a) \in \overline{gb}(\delta(gb)))$.

Define h by

$$h := \lambda an.g(\bar{b}(\eta b)*a)(n+\delta(gb)),$$

then $(\overline{gb}(\delta(gb))*)\circ h = g\circ(\bar{b}(\eta b)*)$ (by (2)) and $h \in I_1(S_{\bar{b}(\eta b)},T_{\overline{gb}(\delta(gb))})$ (by 3.7.(i), 3.8.(iv), 3.11.(ii)). So, by (1) ($a := gb$, $S := S_{\bar{b}(\eta b)}$, $f := h$):

(3) $S_{\bar{b}(\eta b)} \Vdash (B1g\circ(\bar{b}(\eta b)*)) \to S_{\bar{b}(\eta b)} \Vdash (C1g\circ(\bar{b}(\eta b)*))$.

Since we also have (lemma 4.10 with $f := (\bar{b}(\eta b)*)$)

(4) $S \Vdash B1g \to S_{\bar{b}(\eta b)} \Vdash (B1g\circ(\bar{b}(\eta b)*))$

and, by induction hypothesis

(5) $\forall b \in \bar{S}(S_{\bar{b}(\eta b)} \Vdash (C1g\circ(\bar{b}(\eta b)*)) \to S \Vdash C1g)$

we get (combining (3), (4), (5))

$$\forall S \forall g \in I_1(S,T)(S \Vdash B1g \to S \Vdash C1g)$$

i.e. $S \Vdash B \to C$.

$\underline{A = \exists x B(x)}$: assume $\delta \in M_0(T)$ and $\forall a \in \overline{T}(T_{\bar{a}(\delta a)} \Vdash (\exists x B(x)1(\bar{a}(\delta a)*)))$, i.e.

$$\forall a \in \overline{T} \exists \phi \in I_0(T_{\bar{a}(\delta a)})(T_{\bar{a}(\delta a)} \Vdash (B1(\bar{a}(\delta a)*))(\underline{\phi})).$$

By 3.12.(i) and 4.9.(i):

$$\exists \psi \in I_0(T) \forall a \in \overline{T}(T_{\bar{a}(\delta a)} \Vdash (B1(\bar{a}(\delta a)*))(\underline{\psi \circ (\bar{a}(\delta a)*)})).$$

With the induction hypothesis:

$$\exists \psi \in I_0(T)(T \Vdash B(\underline{\phi})),$$

i.e. $T \Vdash \exists x B(x)$. □

4.12. LEMMA.

i) $T \Vdash \forall n\ An \leftrightarrow T \Vdash \forall x(x \in N \to Ax)$;

ii) $T \Vdash \exists n\ An \leftrightarrow T \Vdash \exists x(x \in N \wedge Ax)$;

iii) $T \Vdash \forall \alpha \in \overline{S}\ A\alpha \leftrightarrow \forall T' \forall f \in I_1(T',T) \forall g \in I_1(T',S)(T' \Vdash (A1f)(\underline{g}))$;

iv) $T \Vdash \exists \alpha \in \overline{S}\ A\alpha \leftrightarrow \exists g \in I_1(T,S)(T \Vdash A(\underline{g}))$;

v) $T \Vdash A(f\underline{g}) \leftrightarrow T \Vdash A(\underline{f \circ g})$;

vi) $\vdash (T \Vdash A1f \to T \Vdash B1f) \Rightarrow \vdash \forall T(T \Vdash (A \to B))$;

vii) *if* A *contains no free sequence variables and no distinguished terms, then:*

a) $S \Vdash A \leftrightarrow T \Vdash A$;

b) $T \Vdash \exists x A \leftrightarrow \exists x(T \Vdash A)$;

viii) *if* $A \in L(\underset{\sim}{T}_2)$, *then* $(T \Vdash A) \leftrightarrow A$.

PROOF. i), ii) Easy, write out the definition of $T \Vdash \forall x \ldots$, $T \Vdash \exists x \ldots$ and use 4.10.

iii) $\forall\alpha \in \overline{S}\ A\alpha$ abbreviates $\forall\alpha(\forall n(\alpha n \in S) \rightarrow A\alpha)$, so writing out $T \Vdash \forall\alpha \in \overline{S}\ A\alpha$ yields

$$\forall T'\forall f \in I_1(T',T)\forall g \in I_1(T',U)\forall T''\forall h \in I_1(T'',T')$$
$$(\forall n\forall a \in \overline{T''}(g(ha)n \in S) \rightarrow T'' \Vdash ((A\upharpoonright f)(\underline{g})\upharpoonright h));$$

this is equivalent to (use 3.4.(vi), 3.11.(ii))

$$\forall T'\forall f \in I_1(T',T)\forall g \in I_1(T',U)\forall T''\forall h \in I_1(T'',T')$$
$$(g\circ h \in I_1(T'',S) \rightarrow T'' \Vdash (A\upharpoonright f\circ h)(\underline{g\circ h})),$$

and it is not hard to see that this is equivalent to the second formula of (iii).

iv) Easy.

v) Formula induction.

vi) Easy.

vii) a): by 4.5 and the fact $S \Vdash A = S \Vdash (A\upharpoonright f)$.

b): $T \Vdash \exists x Ax = \exists\phi \in I_0(T)(T \Vdash A\phi)$; as ϕ is continuous, we have $\phi\circ(y*)$ is constant, for some $y \in T$, so by 4.10 and 4.9.(i) $\exists x(Ty \Vdash A(\underline{\lambda a.x}))$; hence $\exists x(T \Vdash Ax)$, by (a) and 4.9.(ii).

viii) Formula induction: use (vi). □

4.13. THEOREM. *(Soundness.)*

$$\underset{\sim}{T}{}_1^* \vdash A \Rightarrow \underset{\sim}{T}{}_2 \vdash \forall T(T \Vdash A).$$

PROOF. Induction over the length of a proof of A .

Logical axioms and rules of $\underset{\sim}{\mathrm{APP}}$:

$A \rightarrow A$, $\forall x\ Ax \rightarrow A\tau$: trivial, for τ contains no choice variables.

$A\tau \rightarrow \exists x\ Ax$: use 4.9.(ii) and I_0AX1.

$\dfrac{A}{B \rightarrow A}$: trivial, by 4.5.

$\dfrac{A \to B \quad B \to C}{A \to C}$: easy, by 3.11.(ii).

$\dfrac{A \qquad A \to B}{B}$: easy, by 3.8.(ii).

$\dfrac{A \to B \quad A \to C}{A \to (B \wedge C)}$: trivial.

$\dfrac{(A \wedge B) \to C}{A \to (B \to C)}$: assume $T \Vdash A \wedge B \to C$, i.e.

$$(6) \qquad \forall S \forall f \in I_1(S,T)(S \Vdash A\upharpoonright f \wedge S \Vdash B\upharpoonright f \to S \Vdash C\upharpoonright f).$$

This implies

$$\forall S \forall f \in I_1(S,T) \forall S' \forall g \in I_1(S',S)(S \Vdash A\upharpoonright f\circ g \wedge S \Vdash B\upharpoonright f\circ g \to S \Vdash C\upharpoonright f\circ g).$$

Distribute $\forall S$, $\forall g \in I_1(S',S)$:

$$\forall S \forall f \in I_1(S,T)(\forall S' \forall g \in I_1(S',S)\ S' \Vdash A\upharpoonright f\circ g \to$$
$$\to \forall S' \forall g \in I_1(S',S)(S' \Vdash B\upharpoonright f\circ g \to S' \Vdash C\upharpoonright f\circ g)).$$

With 4.10:

$$(7) \qquad \forall S \forall f \in I_1(S,T)(S \Vdash A\upharpoonright f \to$$
$$\forall S' \forall g \in I_1(S',S)(S' \Vdash B\upharpoonright f\circ g \to S' \Vdash C\upharpoonright f\circ g))$$

i.e. $T \Vdash A \to (B \to C)$.
The other way round is easier: take $S' := S$, $g := \lambda x.x$ in (7) and we get (6).

$\dfrac{A \to B}{A \to \forall x B}$: trivial.

$\dfrac{A \to B}{\exists x A \to B}$: assume $T \Vdash A \to B$, i.e.

$$(8) \qquad \forall S \forall f \in I_1(S,T)(S \Vdash (A\upharpoonright f)(x) \to S \Vdash B\upharpoonright f).$$

Let $f \in I_1(S,T)$ and assume

$$S \Vdash (A1f)(\underline{\phi}) \quad \text{for some} \quad \delta \in I_0(S).$$

By 3.10.(i) $\delta\ \mathit{mod}\ \phi$ for some $\delta \in M_0(S)$. Now, by 4.11:

$$\forall a \in \bar{S}\ S_{\bar{a}(\delta a)} \Vdash (A1f\circ(\bar{a}(\delta a)*))(\underline{\phi\circ(\bar{a}(\delta a)*)}).$$

Since $\delta\ \mathit{mod}\ \phi$, we have

$$\forall a \in \bar{S}\ \exists x\, \forall b \in \bar{S}_{\bar{a}(\delta a)}\ \phi(\bar{a}(\delta a)*b) = x,$$

so, with 4.9.(ii)

$$\forall a \in \bar{S}\ \exists x\ S_{\bar{a}(\delta a)} \Vdash (A1f\circ(\bar{a}(\delta a)*))(x).$$

With (8) this gives

$$\forall a \in \bar{S}(S_{\bar{a}(\delta a)} \Vdash B1(f\circ(\bar{a}(\delta a)*)))$$

which implies (by 4.11) $S \Vdash B1f$.
So we have shown

$$\forall S \forall f \in I_1(S,T)(\exists\phi(S \Vdash (A1f)(\underline{\phi}) \to S \Vdash B1f)$$

i.e. $T \Vdash (\exists xA \to B)$.

Non-logical axioms of $\underset{\sim}{\mathrm{APP}}$: most of them present no problems. We only consider IND:

assume $T \Vdash A01f$ and $T \Vdash \forall n(An \to A(n+1))1f$, i.e.
$\forall n\, \forall S\, \forall g \in I_1(S,T)(S \Vdash An1f\circ g \to S \Vdash A(n+1)1f\circ g)$; then $\forall n(T \Vdash An1f \to$
$\to T \Vdash A(n+1)1f)$, so with $T \Vdash A01f$ we get $\forall n(T \Vdash An)$.

Axioms and rules of $\underset{\sim}{\mathrm{APP}}^*$ for sequence variables:

$\forall R_{SEQ}$: let $A = A(\vec{\beta})$, $B = B(\alpha,\vec{\beta})$. Now $\underset{\sim}{T}_2 \vdash \forall T(T \Vdash (A \to B))$ reads

$$\underset{\sim}{T}_2 \vdash \forall T(f,\vec{g} \in I_1(S,T) \to$$
$$\to \forall S'\forall h \in I_1(S',S)(S' \Vdash A(\vec{f}1h) \to S' \Vdash B(g\circ h,\vec{f}1h)));$$

we quantify over S, $\vec{f}$, g, take $S' := S$, $T := U$, $h := \lambda x.x$ and get

$$\underset{\sim}{T}_2 \vdash \forall S \forall f, \vec{g} \in I_1(S,U)(S \Vdash A(\vec{f}) \to S \Vdash B(g,\vec{f}));$$

now take $\vec{f} := \vec{f} \circ h \circ k$ and use 3.11.(ii) and $I_1(S,T) \subset I_1(S,U)$ (by 3.4.(vi)):

$$\underset{\sim}{T}_2 \vdash \forall S''T(\vec{f} \in I_1(S'',T) \to \forall S' \forall h \in I_1(S',S'') \forall S \forall k \in I_1(S,S')$$
$$(S \Vdash A(\vec{f} \circ h \circ k) \to \forall g \in I_1(S,U)(S \Vdash B(g,\vec{f} \circ h \circ k)))).$$

Distribute $\forall S$, $\forall k \in I_1(S,S')$ and apply 4.5:

$$\underset{\sim}{T}_2 \vdash \forall T(\vec{f} \in I_1(S'',T) \to \forall S' \forall h \in I_1(S',S'')(S' \Vdash A(\vec{f} \circ h) \to$$
$$\to \forall S' \forall k \in I_1(S,S') \forall g \in I_1(S,U)(S \Vdash B(g, \vec{f} \circ h \circ k))))$$

i.e. $\underset{\sim}{T}_2 \vdash \forall T(T \Vdash (A \to \forall \alpha B))$.

$\underline{\exists R_{SEQ}}$: as above, but simpler: write out $U \Vdash (A \to B)$ and use $I_1(S,T) \subset I_1(S,U)$.

$\underline{\forall \alpha A \alpha \to A \beta}$: let $A = A(\alpha, \vec{\gamma})$. Now $T \Vdash (\forall \alpha A \alpha \to A \beta)$ reads

$$g, \vec{h} \in I_1(S,T) \to \forall S' \forall k \in I_1(S',S)(\forall S'' \forall l \in I_1(S'',S')$$
$$\forall f \in I_1(S'',U)(S'' \Vdash A(f, \vec{h} \circ k \circ l)) \to S' \Vdash A(g \circ k, \vec{h} \circ k))$$

and this holds in $\underset{\sim}{T}_2$: to see this, take $S'' := S$, $l := \lambda x.x$, $f := g \circ k$ and use $I_1(S,T) \subset I_1(S,U)$.

$\underline{A \beta \to \exists \alpha A \alpha}$: let $A = A(\alpha, \vec{\gamma})$. Now $T \Vdash (A \beta \to \exists \alpha A \alpha)$ reads

$$g, \vec{h} \in I_1(S,T) \to \forall S' \forall k \in I_1(S',S)(S' \Vdash A(g \circ k, \vec{h} \circ k) \to$$
$$\to \exists f \in I_1(S',U)(S' \Vdash A(f, \vec{h} \circ k)))$$

which evidently holds (take $f := g \circ k$).

SEQAX1: $T \Vdash \forall\alpha\, \forall n\, \exists x(\alpha n = x)$ reads (see 4.7.(i))

$$\forall S \forall g \in I_1(S,U)\, \forall n\, \exists \phi \in I_0(S) \forall a \in \overline{S}(gan = \phi a)$$

and this holds by the definition of $I_1(S,U)$.

SEQAX2: $T \Vdash \forall x\ \exists\alpha\ \forall n(xn = \alpha n)$ reads (4.7.(ii))

$$\forall x \exists g \in I_1(T,U)\, \forall n\, \forall a \in \overline{T}(xn = gan)$$

and this is a consequence of I_0AX1 and the definition of I_1.

SEQAX3: $T \Vdash \forall\alpha\beta\ \exists\gamma\ \forall n(\gamma n = \langle\alpha n, \beta n\rangle)$ reads

$$\forall S \forall f \in I_1(S,U) \forall S' \forall g \in I_1(S',S) \forall h \in I_1(S',U)$$
$$\exists k \in I_1(S,U) \forall n \forall \alpha \in \overline{S}'(kan = \langle f(ga)n, han\rangle)$$

and this follows from 3.8.(vi) and 3.11.(ii).

SEQAX4: $T \Vdash \forall\alpha x\ \exists\beta(\beta 0 = x \wedge \forall n(\beta(n+1) = \alpha n))$ reads

$$\forall S \forall f \in I_1(S,U)\ \forall x\, \exists g \in I_1(S,U)(\forall a \in \overline{S}(ga0 = x)\ \wedge$$
$$\wedge\ \forall n \forall a \in \overline{S}(ga(n+1) = fan))$$

and this follows from the definition of $I_1(S,U)$.

Tree axioms and rules of $\underset{\sim}{T}_1^*$:

$\forall R_{TR}$, $\exists R_{TR}$, $\forall AX_{TR}$, $\exists AX_{TR}$: easy, since $\forall T$, $\exists T$ commute with $\Vdash$.

TRAX1-8: also easy, for they do not contain sequence variables.

I_0*AX1*: $S \Vdash (\forall\alpha \in \overline{T}(\phi\alpha = x) \to \phi \in I_0(\tau))$ reads

$$\forall S'(\forall S'' \forall g \in I_1(S'',T) \forall a \in \overline{S}''(\phi(ga) = x) \to \phi \in I_0(\tau))$$

and this holds in $\underset{\sim}{T}_2$ (take $S'' := T$, $g := \lambda x.x$).

I_0*AX2*: easy, as it contains no sequence variables.

$\underline{I_0AX3}$: using $(\exists x\ Ax \to B) \leftrightarrow \forall x(Ax \to B)$ and $(A \vee B \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$, we can rewrite I_0AX3 without $\vee$ and $\exists$. Now the proof of $T \Vdash I_0AX3$ is analogous to that for $T \Vdash IND$.

$\underline{I_1AX}$: $T \Vdash \forall\alpha \in \overline{S}_1 \forall f \in I_1(S_1,S_2)\exists\beta \in \overline{S}_2\forall n(\beta n = f\alpha n)$ reads

$$\forall T'\forall g \in I_1(T',S_1)\forall f \in I_1(S_1,S_2)\exists h \in I_1(T',S_2)$$
$$\forall a \in \overline{T}'\ \forall n\ (han = f(ga)n)$$

and this follows from 3.11.(ii).

$\underline{ECS1}$: $S \Vdash (\forall a \in \overline{T}\ Aa \to \forall\alpha \in \overline{T}\ A\alpha)$ reads (remember that A is prime)

$$\forall S'(\forall a \in \overline{T}\ Aa \to \forall S''\forall f \in I_1(S'',T)\forall b \in \overline{S''}A(fb))$$

and this follows from the definition of I_1.

$\underline{ECS2}$: both $S \Vdash (\forall\alpha(A\alpha \to B\alpha))$ and $S \Vdash (\forall T\forall f \in I_1(T,U)(\forall\alpha \in \overline{T}\ A(f\alpha) \to$ $\to \forall\alpha \in \overline{T}\ B(f\alpha)))$ are equivalent to

$$\forall T\forall f \in I_1(T,U)(T \Vdash Af \to T \Vdash Bf);$$

use 4.7.(iv) for the second equivalence.

$\underline{ECS3}$: $S \Vdash \forall\alpha \in \overline{T}\exists x\, A(\alpha,x)$ and $S \Vdash \exists\phi \in I_0(T)\forall\alpha \in \overline{T}\ A(\alpha,\phi\alpha)$ are equivalent to

$$\exists\phi \in I_0(T)(T \Vdash A(\lambda x.x,\phi)).$$

$\underline{ECS4}$: analogous to ECS3.

$\underline{EAC}$: easy, by 4.12.(vii) (recall that EAC does not contain free sequence variables).

□

We complete the picture of $\underset{\sim}{T}_1^*$, $\underset{\sim}{T}_2$ and $\Vdash$ as follows.

.14. THEOREM. i) *Let* A *be a completely regular formula of* $\underset{\sim}{T}_1^*$. *Then*

$$\underset{\sim}{T}_1^* \vdash A \leftrightarrow (T \Vdash A).$$

ii) *Let* A *be a formula of* $\underset{\sim}{T}_2$. *Then*

$$\underset{\sim}{T}_2 \vdash A \leftrightarrow (T \Vdash A).$$

PROOF. i) With formula induction we show, for completely regular A:

$$\underset{\sim}{T}_1^* \vdash \forall\alpha \in \overline{T}\ A(\vec{p}\alpha) \leftrightarrow T \Vdash A(\underline{\vec{p}});$$

from this (i) follows.

A prime: by ECS1.

$A = B \wedge C$, $A = \forall x Bx$: easy.

$A = B \rightarrow C$: simple, use ECS2.

$A = \forall\beta B\beta$: by the definition of $\Vdash$ and the induction hypothesis we see that $T \Vdash \forall\beta B(\vec{p},\beta)$ is equivalent to

$$(1) \qquad \forall S \forall f \in L_1(S,T) \forall g \in I_1(S,U) \forall\alpha \in \overline{S}\ B(\vec{p}(f\alpha),g\alpha);$$

now (1) $\leftrightarrow \forall\alpha \in \overline{T}\ \forall\beta B(\vec{p}\alpha,\beta)$: $\leftarrow$ is evident; for $\rightarrow$, take $S := T{\times}U$, $f := \pi_0$, $g := \pi_1$ and use substitution for $\equiv$ (A is regular, hence B).

$A = \exists\beta B\beta$: use ECS4 and the induction hypothesis.

$A = \exists x Bx$: analogous.

ii) We prove with formula induction:

$$\underset{\sim}{T}_2 \vdash A(\vec{x}) \leftrightarrow T \Vdash A(\vec{p}),$$

here the $\vec{p}$ are constant parameters with value $\vec{x}$, i.e. $\forall a\ \vec{p}a = \vec{x}$. From this (ii) follows.

A prime, $A = B \wedge C$, $A = B \rightarrow C$, $A = \forall y B$: easy.

$A = \exists y By$: now $T \Vdash \exists y B(y,\vec{p}) = \exists\phi \in I_0(T)(T \Vdash B(\phi,\vec{p}))$; by the induction hypothesis and 3.10.(i) this is equivalent to

$$\exists\delta \in M_0(T)\exists\phi \in I_0(T)(\delta\ \mathit{mod}\ \phi\ \wedge$$
$$\wedge\ \forall a \in \overline{T}(T_{\bar{a}(\delta a)} \Vdash B(p1(\bar{a}(\delta a)*),\phi\circ(\bar{a}(\delta a)*))))$$

i.e. (by 4.4.(i))

$$\exists\delta \in M_0(T)\exists\phi \in I_0(T)(\delta\ \mathit{mod}\ \phi \wedge$$
$$\wedge\ \forall a \in \overline{T}\exists y(T_{\bar{a}(\delta a)} \Vdash B(\vec{p}1(\bar{a}(\delta a)*),\lambda z.y))).$$

With the induction hypothesis:

$$\exists\delta \in M_0(T)\exists\phi \in I_0(T)(\delta\ \mathit{mod}\ \phi \wedge \forall a \in \overline{T}\ \exists y\ B(\vec{x},y))$$

i.e. $\exists y B(\vec{x},y)$. □

§5. Reduction to $\mathbf{ID}_1$.

In this section the proof of our main theorem is completed.

5.1. First we define a new theory $\mathbf{T}_3$ which looks like $\mathbf{T}_2$, but without tree variables. Let I_0AX1'-$3'$ be the following axiom schemata (A an arbitrary negative formula of $\mathbf{APP}$):

I_0AX1' $\quad \mathrm{Tree}(A) \wedge \forall a \in \overline{A}(\phi a = x) \to \phi \in I_0(A)$

I_0AX2' $\quad \mathrm{Tree}(A) \wedge \forall\hat{x} \in A(\phi_{\hat{x}} \in I_0(A_{\hat{x}})) \to \phi \in I_0(A)$

I_0AX3' $\quad \mathrm{Tree}(A) \wedge \forall x \in A\forall\phi[\exists y\forall a \in \overline{A}_x(\phi a = y) \vee (\forall\hat{y} \in A(\phi_{\hat{y}} \in P(x*\hat{y})) \to \phi \in P(x))] \to$
$\qquad\qquad \to \forall x \in A(I_0(A_x) \subset P(x))$

Now $\mathbf{T}_3 := \mathbf{APP} + I_0AX1'$-$3' + \mathrm{EAC}$.

5.2. THEOREM. $\mathbf{T}_2 \vdash A \Rightarrow \mathbf{T}_3 \vdash A$ *for* $A \in L(\mathbf{T}_3)$.

PROOF. A detailed proof would be long and tedious, so we confine ourselves to a sketch. Let $\mathbf{T}_2^f$ be an arbitrary subtheory of $\mathbf{T}_2$ with only finitely many instances of TRAX5, say for the formulae $A_1,\ldots,A_n$. We assume $FV(A_i) \subset \{x,z_i\}$, $i = 1,\ldots,n$ (the variable x is used to define the set A_i; see 1.9). For technical reasons, we add $A_0 := (1\mathrm{th}\ x \in N)$ to the list $A_1,\ldots,A_n$. We shall define an interpretation $^f : \mathbf{T}_2^f \to \mathbf{T}_3$ satisfying

$$\underset{\sim}{T}_2^f \vdash A \Rightarrow \underset{\sim}{T}_3 \vdash A^f;$$

from this the theorem follows.
The naive idea for ^{f} is: replace formulae $\forall T\ A[\tau_j \in T]_j$ by

$$\bigwedge_{i=0}^{n} (\mathrm{Tree}(A_i) \rightarrow A[A_i(\tau_j)]_j).$$

But this is not enough, for the A_i may contain parameters, and we also have to deal with the closure conditions $\forall T \forall x(x \in T \rightarrow \exists S(S \equiv T_x))$ (TRAX6) and $\forall TT'\exists S(S \equiv T\times T')$ (TRAX7). This leads us to considering the 'universe of trees' of $\underset{\sim}{T}_2^f$, which consists of the trees defined by $A_0,\dots,A_n$, closed off under taking subtrees and products.

We recall the notation $\times$, $[\ ]_0$, $[\ ]_1$ from 2.1 and define the following notation:

$$x^{< >} := x,$$

$$x^{y*\hat{0}} := [x^y]_0, \qquad x^{y*\hat{1}} := [x^y]_1;$$

here $y = \langle y_0,\dots,y_n\rangle$, $y_i = 0$ or 1 $(i = 0,\dots,n)$. Such a sequence y is called a 0-1-*sequence* and we call x^y the y-*projection* of x.
An example:

$$x^{<1,0,0>} = [[[x]_1]_0]_0.$$

We now have e.g.

$$x \in (T_1\times T_2)\times T_3 \leftrightarrow x^{<0,0>} \in T_1 \wedge x^{<0,1>} \in T_2 \wedge x^{<1>} \in T_3.$$

The idea now is to code the trees of the 'universe of trees' of $\underset{\sim}{T}_2^f$ by quintuples y, z, u, v, m which satisfy

i) z, u, v are finite sequences with length m;

ii) z is a sequence of parameters;

iii) u is a finite sequence of different finite 0-1-sequences;

iv) v is a sequence of natural numbers $\leq n$;

v) $\langle\,\rangle$, z, u, v, m code a tree which contains y.

(i) - (v) are collected in $Adm(y,z,u,v,m)$:

$$\begin{aligned} Adm(y,z,u,v,m) := {} & lth\, z = lth\, u = lth\, v = m \wedge m \in N \wedge \\ & \forall i < m(lth(u)_i \in N \wedge (v)_i \in N) \wedge \\ & \forall ij < m((u)_i = (u)_j \to i = j) \wedge \\ & \forall i < m \forall k < lth(u)_i(((u)_i)_k \in \{0,1\}) \wedge \\ & Tree(T(\langle\,\rangle,x,z,u,v,m)) \wedge \\ & T(\langle\,\rangle,y,z,u,v,m), \end{aligned}$$

where

$$T(y,x,z,u,v,m) := \forall i < m(\bigwedge_{j=0}^{n} (j = (v)_i \to (y*x)^{(u)_i} \in A_j[z := (z)_i])).$$

We call $\{x | T(y,x,z,u,v,m)\}$ the tree coded by y,z,u,v,m; it consists of those x for which holds:

> for any $i < m$, the $(u)_i$ - projection of $y*x$ is in the tree defined by the formula $A_{(v)_i}$ with parameters $(z)_i$.

Now the definition of ^{f} is as follows.

$$(\forall TB)^f = \forall yzuvm(Adm(y,z,u,v,m) \to (B[T := T(y,x,z,u,v,m)])^f)$$

$$(\exists TB)^f = \exists yzuvm(Adm(y,z,u,v,m) \wedge (B[T := T(y,x,z,u,v,m)])^f)$$

> ^{f} commutes with $\forall x$, $\exists y$, $\wedge$, $\vee$, $\to$ and leaves prime formulae unchanged.

By this definition of ^{f}, we get formulae like $\tau \in (T(y,x,z,u,v,m))_\sigma$ and $\tau \in (T(y_1, x, z_1, u_1, v_1, m_1) \times T(y_2, x, z_2, u_2, v_2, m_2))$; to interpret these we recall the conventions

$$\tau \in A := A[x := \tau]$$

$$\tau \in A_\sigma := \sigma * \tau \in A$$

from 1.9, and adopt the following:

$$\tau \in U := \mathrm{lth}\,\tau \in N,$$

$$\tau \in A \times B := [\tau]_0 \in A \wedge [\tau]_1 \in B.$$

We check the soundness of ^{f} in the version

$$\underset{\sim}{T}_2^f \vdash A \Rightarrow \underset{\sim}{T}_3 \vdash (\forall \vec{T}\, A)^f,$$

where T are the free tree variables of A. By the definition of ^{f}, we only have to inspect the rules and axioms concerning trees, and EAC.

$\forall R_{TR}$, $\exists R_{TR}$, $\forall AX_{TR}$, $\exists AX_{TR}$: easy.

TRAX1: $(\forall T(\mathrm{Tree}(T)))^f$ follows from the definition of Adm.

TRAX2-4: trivial, by the conventions mentioned above.

TRAX5: we only have instances with A_i, $1 \leq i \leq n$. Now

$$A_i(x,z) \leftrightarrow T(<>, x, \hat{z}, <>, \hat{i}, 1)$$

and, by $\mathrm{Tree}(A_i(x,z))$, we have $\mathrm{Adm}(<>, \hat{z}, <>, \hat{i}, 1)$.

TRAX6: if T is coded by y, z, u, v, m, then take $y*x$, z, u, v, m as code for S $(\equiv T_x)$.

TRAX7: if T is coded by y, z, u, v, m and T' by y', z', u', v', m', then take $y \times y'$, $z*z'$, $<(u)_0*\hat{0},\ldots,(u)_{m-1}*\hat{0},(u')_0*\hat{1},\ldots,(u')_{m'-1}*\hat{1}>$, $v*v'$, $m+m'$ as a code for S $(\equiv T \times T')$.

TRAX8: easy, for in $\underset{\sim}{T}_3$ we have

$$\mathrm{Tree}(A) \wedge \mathrm{Tree}(B) \wedge A \equiv B \rightarrow I_0(A) \equiv I_0(B)$$

for $A \in \mathcal{L}^-(\mathrm{APP})$ (to be proved with induction over $I_0(A)$, $I_0(B)$), and also $\mathrm{Tree}(A) \rightarrow A \equiv A_{<>}$.

EAC: it is enough to show that A^f is negative if A is. By the definition of ^{f} we only have to check that $T(y,x,z,u,v,m)$ is negative, and this follows from the fact that the A_i are negative (by the restriction on TRAX5).
This ends the proof. □

Now we compare $\underset{\sim}{T}_3$ with $\underset{\sim}{APP} + EAC + ID_1$ (see Ch.III, §5 for inductive definitions).

5.3. LEMMA. $\underset{\sim}{T}_3 \vdash A \Rightarrow \underset{\sim}{APP} + EAC + ID_1 \vdash A$ *for* $A \in L(\underset{\sim}{APP})$.

PROOF. We shall show that $I_0AX1\text{-}3'$ follow from ID_1. Let $B_A = B_A(P,z)$ be defined by (we write $\langle x,\phi\rangle$ for z):

$$B_A(P,\langle x,\phi\rangle) := [\exists y \forall a(\forall n(x*\bar{a}n \in A) \rightarrow \phi a = y) \vee$$

$$\vee \forall y(x*\hat{y} \in A \rightarrow \langle x*\hat{y},\phi_{\hat{y}}\rangle \in P)] \rightarrow \langle x,\phi\rangle \in P.$$

Γ_{B_A} is the predicate operator with

$$z \in \Gamma_{B_A}(P) \leftrightarrow B_A(P,z).$$

We write I_A for I_Γ (Γ abbreviates Γ_{B_A}), the least fixed point of Γ_{B_A}; by ID_1 we have

(1) $\qquad \Gamma(I_A) \subset I_A,$

(2) $\qquad \Gamma(P) \subset P \rightarrow I_A \subset P.$

Now we define $I_0(A_x)$ explicitly by

$$\phi \in I_0(A_x) := \langle x,\phi\rangle \in I_A;$$

writing out (1), (2) and substituting $\phi \in I_0(A_x)$ for $\langle x,\phi\rangle \in I_A$ and $\phi \in P(x)$ for $\langle x,\phi\rangle \in P$ yields $I_0AX1'\text{-}3'$, even without the condition Tree(A). □

5.4. <u>THEOREM</u>. $\mathbf{APP}^* + \mathrm{EBI} \vdash A \;\Rightarrow\; \mathbf{ID}_1 \vdash A$ *for* $A \in L(\mathbf{HA})$.

<u>PROOF</u>. Let $\mathbf{APP}^* + \mathrm{EBI} \vdash A$, $A \in L(\mathrm{HA})$. Then, by 2.11:

$$\mathbf{T}_1^* \vdash A.$$

By 4.13 and 4.14.(ii) (A is a fortiori in the language of $\mathbf{T}_2$):

$$\mathbf{T}_2^* \vdash A.$$

By 5.2 and 5.3:

$$\mathbf{APP} + \mathrm{EAC} + \mathrm{ID}_1 \vdash A.$$

Finally, by Ch.III, 5.13:

$$\mathbf{ID}_1 \vdash A.$$

□

To establish that $\mathbf{ID}_1$ *axiomatizes* the arithmetical fragment of $\mathbf{APP} + \mathrm{EBI}$, we prove the converse of the previous theorem. We shall use a result by Sieg, for which we first need a definition.

5.5. <u>DEFINITION</u>. Let $\{\cdot\}(\cdot)$ be the Kleene-bracket-notation as introduced in Ch.II, 4.3. Without loss of generality we may assume that $\forall n\, \{0\}(n) = 0$. We define the axioms $\mathcal{O}$AX1-3:

$\mathcal{O}$AX1 $\quad 0 \in \mathcal{O}$

$\mathcal{O}$AX2 $\quad \forall n(\{x\}(n){\downarrow} \wedge \{x\}(n) \in \mathcal{O}) \to x \in \mathcal{O}$

$\mathcal{O}$AX3 $\quad A(0) \wedge \forall x[\forall n(\{x\}(n){\downarrow} \wedge A(\{x\}(n))) \to Ax] \to \forall x \in \mathcal{O}\; Ax$

$\mathcal{O}$ is called the inductively defined tree class of the first order. We also put

$$\mathrm{ID}_1(\mathcal{O}) := \mathcal{O}\mathrm{AX1} + \mathcal{O}\mathrm{AX2} + \mathcal{O}\mathrm{AX3},$$

$$\mathbf{ID}_1(\mathcal{O}) := \mathbf{HA} + \mathrm{ID}_1(\mathcal{O}).$$

5.6. THEOREM. *(Sieg).* $\mathbf{ID}_1$ *and* $\mathbf{ID}_1(\mathcal{O})$ *prove the same arithmetical theorems.*

PROOF. Follows from [BFPS81], Ch.III, Theorem 3.2.3. □

5.7. LEMMA. $\mathbf{ID}_1(\mathcal{O}) \vdash A \Rightarrow \mathbf{APP}^* + BI \vdash A$.

PROOF. We interpret $x \in \mathcal{O}$ by

$$(1) \qquad \forall \alpha \in N^{\omega} \exists n(fx(\bar{\alpha}n) = 0 \wedge \forall m < n(fx(\bar{\alpha}m) > 0))$$

where f is the function satisfying

$$fx\langle\rangle = x,$$

$$fx(y * \hat{z}) = \{fxy\}(z).$$

We verify that $\mathcal{O}$AX1-3 become derivable in $\mathbf{APP}^* + BI$ under this interpretation. $\mathcal{O}$AX1 and $\mathcal{O}$AX2 follow, without using BI, by writing out their interpretation and using the definition of f; for $\mathcal{O}$AX3 we do need BI. Assume

i) $A0$,

ii) $\forall x(\forall n\, A(\{x\}(n)) \to A(x))$,

iii) $x \in \mathcal{O}$, i.e. $\forall \alpha \in N\, \exists n(fx(\bar{\alpha}n) = 0 \wedge \forall m < n(fx(\bar{\alpha}m) > 0))$.

Put

$$By := \forall z \in N^{<\omega} A(fx(y * z)).$$

Then

a) $Bar(N^{<\omega}, B)$, by (i), (ii) and $\forall n(\{0\}(n) = 0)$;

b) $Mon(N^{<\omega}, B)$, by the definition of B;

c) $Ind(N^{<\omega}, B)$, by (ii) and the definition of f;

d) $Tree(N^{<\omega})$.

So with BI we get $B\langle\rangle$, hence $A(fx\langle\rangle)$, i.e. Ax. We conclude $\forall x \in \mathcal{O}\, Ax$, so $\mathcal{O}$AX3 is derived. □

5.8. THEOREM. $\mathbf{APP}^* + EBI \vdash A \Leftrightarrow \mathbf{ID}_1 \vdash A$ *for* $A \in \mathcal{L}(HA)$.

PROOF. Combine 5.4 and 5.7. □

We now formulate the principal corollary. Let $\mathbf{EL}^*$ be the theory $\mathbf{EL}$ (see Ch.II, 4.7), but with α, β, ... as sequence variables. In $\mathbf{EL}^*$ we can write down Tree(A), Bar(A,P), Mon(A,P), Ind(A,P) and EBI(A,P) just as in $\mathbf{APP}^*$ (now x, y range over natural numbers); EBI for $\mathbf{EL}^*$ is defined as EBI(A,P) for all $P \in L(\mathbf{EL}^*)$ and all $A \in L(\mathbf{HA})$.

5.9. THEOREM. $\mathbf{EL}^* + \mathrm{EBI}$ *and* $\mathbf{ID}_1$ *prove the same arithmetical theorems.*

PROOF. We interpret $\mathbf{EL}^*$ in $\mathbf{APP}^*$ by extending the interpretation $^\circ$ of $\mathbf{HA}$ into $\mathbf{APP}^E$ (Ch.II, 4.1) with the identity for $\forall\alpha$, $\exists\alpha$. It is not difficult to show that A° always is a regular formula; this is used to obtain

$$\mathbf{EL}^* + \mathrm{EBI} \vdash A \;\Rightarrow\; \mathbf{T}_1 \vdash A^\circ .$$

Combining this with 5.4 and Ch.II, 4.5.(ii) we get

$$\mathrm{EL}^* + \mathrm{EBI} \vdash A \;\Rightarrow\; \mathbf{ID}_1 \vdash A,$$

the first half of the theorem. The proof for the inverse implication runs parallel to 5.7. □

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Variables, metavariables.

P, f	I.3.1
α, β,...	I.5.1
a, b, c,..., x, y, z	II.2.1
ρ, σ, τ,...	II.2.1
m, n,...	II.2.1
a, b, c, d,...	II.4.7
f, g, h,...	III.4.4
P, Q	III.5.1
α, β,...	IV.1.2
ϕ, ψ,...	IV.2.1
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S, T, ...	IV.2.1
V, W, ...	IV.2.1

Other notation and symbols.

SAMENVATTING

In dit proefschrift worden formele theorieën van intuïtionistische logica en intuïtionistische wiskunde bestudeerd. Hoofddoel van het onderzoek is het karakteriseren van het rekenkundig fragment van de theorie $\underset{\sim}{\mathrm{EL}}$ + EBI, elementaire analyse plus het axiomaschema van 'uitgebreide versperringsinductie' (Extended Bar Induction). De weg naar dit doel voert ons in hoofdstuk I langs intuïtionistische logica met descriptoren: dit zijn operatoren die, toegepast op een formule $A(x)$, het unieke object x met de eigenschap A opleveren (indien zo'n object bestaat). In hoofdstuk II bestuderen we twee theorieën $\underset{\sim}{\mathrm{APP}}$ en $\underset{\sim}{\mathrm{APP}}^{E}$, beide gebaseerd op type-vrije applicatie; in $\underset{\sim}{\mathrm{APP}}$ is deze applicatie totaal, in $\underset{\sim}{\mathrm{APP}}^{E}$ partieel. Aangetoond wordt dat $\underset{\sim}{\mathrm{APP}}$ een conservatieve uitbreiding is van $\underset{\sim}{\mathrm{HA}}$, intuïtionistische rekenkunde. Hoofdstuk III is gewijd aan $\underset{\sim}{\mathrm{APP}}$ plus EAC, een 'uitgebreid keuze-axioma' (Extended Axiom of Choice). Ook $\underset{\sim}{\mathrm{APP}}$ + EAC blijkt conservatief over $\underset{\sim}{\mathrm{HA}}$ te zijn. Een zijpad voert over aan het eind van dit hoofdstuk naar $\underset{\sim}{\mathrm{ML}}_0$, de basis van P. Martin-Löf's extensionele typen-theorieën. In het vierde en laatste hoofdstuk betrekken we het axioma EBI in het onderzoek. Via een aantal uitbreidingen van $\underset{\sim}{\mathrm{APP}}$ met o.a. keuzerijen en boomvariabelen reduceren we $\underset{\sim}{\mathrm{EL}}$ + EBI tot de theorie $\underset{\sim}{\mathrm{ID}}_1$, intuïtionistische rekenkunde uitgebreid met (niet-geïtereerde) inductieve definities.

STELLINGEN

bij het proefschrift

Theories of Type-free Application and Extended Bar Induction

van Gerard R. Renardel de Lavalette

1. Aan de theorie APP, gedefinieerd in hoofdstuk II, §2 van dit proefschrift kan een bewijsbaarheidspredikaat $p \,\square\, A$ worden toegevoegd, met de betekenis p is een bewijs van A. Een natuurlijke axiomatisering is:

$$A \leftrightarrow \exists p(p \,\square\, A)$$

$$\begin{aligned}
p \,\square\, (A \wedge B) &\leftrightarrow (p)_1 \,\square\, A \wedge (p)_2 \,\square\, B \\
p \,\square\, (A \to B) &\leftrightarrow (p)_1 \,\square\, (\forall q(q \,\square\, A \to (p)_2 q \,\square\, B)) \\
p \,\square\, \forall x A x &\leftrightarrow (p)_1 \,\square\, (\forall x((p)_2 x \,\square\, Ax)) \\
p \,\square\, \exists x A x &\leftrightarrow (p)_1 \,\square\, A(p)_2
\end{aligned}$$

Zij $\mathrm{APP}^{\square}$ de aldus gedefinieerde theorie. Dan geldt, voor formules A van APP:

i) $\mathrm{APP} + \mathrm{AC} \vdash A \Rightarrow \mathrm{APP}^{\square} \vdash A$,

ii) $\mathrm{APP}^{\square} \vdash A \Rightarrow \mathrm{APP} + \mathrm{EAC} \vdash A$.

Uit (ii) en uit stelling 4.21 van hoofdstuk III van dit proefschrift volgt:

iii) $\mathrm{APP}^{\square}$ conservatief over HA.

2. Zij $A \to B$ een afleidbare formule in de intuïtionistische predikatenlogica, en zij I de interpolant van $A \to B$, verkregen uit het bewijs van K. Schütte van de interpolatiestelling voor de intuïtionistische predikaten logica in diens artikel "Der Interpolationssatz der Intuitionistischen Prädikatenlogik". Dan geldt:

> elke predikaatletter die een strikt positief voorkomen heeft in I, komt strikt positief voor in A en positief in B.

Hierbij is het begrip *strikt positief voorkomen* gedefinieerd door: p komt strikt positief voor in A als p uitsluitend in positieve subformules van A voorkomt.

K. SCHÜTTE, Der Interpolationssatz der
intuitionistischen Prädikaten-
logik, Mathematische Annalen 148,
p. 192-200 (1962).

3. Zij $\{t(n)\}_{n=0}^{\infty}$ een rij reële getallen waarvoor geldt

$$\begin{aligned} t(0) &= 0 \\ t(1) &= 1 \\ t(n+2) &= a\cdot t(n) + 2b\cdot t(n+1) \qquad (n \geq 0) \end{aligned}$$

waarbij $ab \neq 0$, $b^2 + a > 0$. Dan geldt

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{a^{(2^i)}}{t(2^{i+1})} &= \frac{a}{b} + b + \sqrt{b^2 + a} \qquad \text{als} \quad b < 0 \\ &= \frac{a}{b} + b - \sqrt{b^2 + a} \qquad \text{als} \quad b > 0 \end{aligned}$$

Problem E2922 (proposed by Roger
Cuculière, Paris, France),
American Mathematical Monthly 89
(1), p. 63 (1982).

4. Zij

$$V_n^k = \{(a_1,\ldots,a_k) \,|\, a_i \in \mathbb{Z}/n,\ i = 1,\ldots k\}$$

de verzameling van rijtjes met lengte k van niet-negatieve gehele getallen kleiner dan n. Twee elementen $\vec{a} = (a_1,\ldots,a_k)$ en $\vec{b} = (b_1,\ldots,b_k)$ van V_n^k worden *equivalent* genoemd als er een getal d is met $0 \leq d < k$, zodanig dat

$$a_1 = b_{d+1},\ a_2 = b_{d+2},\ \ldots\ ,\ a_{k-d} = b_k,$$
$$a_{k-d+1} = b_1,\ \ldots\ ,\ a_k = b_0\ .$$

Notatie: $\vec{a} \sim \vec{b}$.

Er geldt: het aantal equivalentieklassen $\|V_n^k/{\sim}\|$ van V_n^k wordt gegeven door

$$\|V_n^k/\sim\| = \frac{1}{k} \sum_{dd'=k} \varphi(d)\cdot n^{d'} .$$

Hierbij is φ de indicator-functie van Euler.

5. Dankzij de mogelijkheid van kunstlens-implantatie is de grijze staar de bes behandelbare ouderdomskwaal.

6. De toevoeging van een correspondentierubriek, waarin op korte termijn bekno te reacties op verschenen artikelen geplaatst kunnen worden, zal de waarde van menig wetenschappelijk tijdschrift ten goede komen.

7. In hoofdstuk 6 van zijn proefschrift *Judging* geeft H.J.M. Boukema een logische analyse van een arrest van het Europese Hof van Justitie. Hiertoe ge bruikt hij de propositielogica in de zgn. Poolse notatie. Hij concludeert:

> 'The above analysis of the Van Duijn Case by means of the propositional calculus of modern logic and the examination of the arguments of this case by means of counter-formula method do not let the Court's reasoning appear as logically sound in all respects.'
>
> (H.J.M. Boukema, Judging, Tjeenk Willink, 1980, p. 128).

Zowel zijn keuze van het logisch systeem als de onzorgvuldige wijze waarop d auteur de betreffende gedeelten van het arrest in logische formules vertaal ondermijnen deze conclusie.

8. 'Hardy va lui rendre visite à l'hôpital, et lui dit qu'il a pris un taxi. Ramanujan demande le numéro de la voiture: 1729. "Quel beau nombre! s'écrie-t-il; c'est le plus petit qui soit deux fois une somme de deux cubes!" En effet, 1729 est égal à 10 au cube plus 9 au cube, et aussi à 12 au cube plus 1 au cube. Il fallut six mois à Hardy pour le démontrer, et le même problème n'est pas encore résolu pour la quatrième puissance.'

> (L. Pauwels & J. Bergier, Le matin des magiciens, Editions Gallimard, 1960, p. 555-556.)

De auteurs van dit citaat getuigen van een ernstige onderschatting van Hardys rekenkundige vermogens, òf van een gebrek aan eigen vaardigheid op dit gebied.