Hybrid Sabotage Modal Logic

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Abstract. We introduce a new hybrid modal logic HSML for reasoning about sabotage-style graph games with edge deletions, and provide a complete Hilbert-style axiomatization. We extend the completeness analysis to protocol models with restrictions on available edge deletions, and clarify the connections between HSML-style logics of edge deletions and recent modal logics for stepwise point deletion from graphs.

Keywords: Modal logic, graph change, sabotage game, link deletion.

1 Introduction

Sabotage games were introduced in [17] as a model for algorithmic behavior under disturbance, a topic of increasing interest when analyzing abuses of and threats to computational systems such as the Internet. The idea is that in a task involving stepwise traversal of a graph by a player called 'Traveler', the disturbing influence becomes a counter-player called 'Demon' who starts each round by cutting some available link in the graph. The resulting sabotage game is determined, and winning conditions and winning invariants can be defined in a natural associated modal logic SML which has a standard modality for accessible nodes from the current point as well as a new 'deletion modality' describing what is true at the current point after some link has been deleted from the graph.

There is a strand of literature exploring applications and technical properties of sabotage games and their modal logic. [13] proved that model checking for SML is Pspace-complete, while satisfiability is undecidable. [4] gave a bisimulation-style characterization of SML under translation as a fragment of first-order logic, as well as a complete tableau system for validity, and similar results were obtained independently in [2] in a more general study of modal logics of graph change. More recent results include [12] on sabotage modal logics with definable link deletions, and a Zero-One Law for SML, [14], showing that in the long run as finite graph size increases, the sabotage game is massively in favor of Traveler, who wins at any position with probability 1. In terms of applications of the game, one interesting proposal using sabotage games for learning scenarios is found in [8]. The background to these publications is a more general investigation of the connections between modal logics and existing or newly designed graph games, advocated in the programmatic paper [21], with concrete case studies in [26,9] on 'poison games', and [15] on modal logics of fact change.

A natural and straightforward issue left open in this literature is a *Hilbert-style axiomatization* of SML, which would be useful for actual standard reasoning about sabotage games or related dynamic scenarios. Such an axiomatization must exist by the known effective translation of SML into first-order logic, but finding a concrete workable proof system has turned out surprisingly difficult. The present paper fills this gap, at least for a mild hybrid modal extension of the original SML language called HSML, and explores some broader implications of this result. The technique used for our completeness theorem stems from a recent axiomatization of a basic modal logic MLSR for stepwise object deletion (or alternatively, of 'quantification without replacement') in [22], that we adapt to the sabotage setting, and simplify considerably.

Once we have the connection between the standard semantics of SML and the proof system in our completeness proof, a natural follow-up question arises. Can one modulate this relationship between semantics and proof system so as to get completeness for other natural semantics for modal logics of graph change? We show how this can be done for a new *'protocol version'*, [11], of SML that restricts the available deletions for the Demon. Next, we turn to the general issue of relating modal logics for deleting edges and for deleting vertices from graphs. We embed the sabotage logic HSML faithfully into MLSR by encoding edge deletion as vertex deletion, and also provide a partial converse. We end with identifying a few further topics that seem amenable to our style of analysis, including interpolation for HSML and axiomatizing its schematic validities.

Relation to DEL For readers familiar with dynamic-epistemic logic(DEL), [23], [5], [18], an analogy may be helpful. A system like 'public announcement logic' (PAL) has modalities for actions $!\varphi$ of deleting all points that satisfy $\neg\varphi$ from a given graph model. PAL is decidable thanks to 'recursion axioms' that push dynamic modalities through complex postconditions. However, if we perform deletions step by step, we get the above logic MLSR which is undecidable, [22], since arbitrary sequences of deletions require storage in an unbounded memory, a device allowing for encoding of undecidable computational problems. The situation is similar with link deletions. There are complete and decidable dynamic-epistemic logics for uniform definable link cutting (an example of such a system occurs in the Appendices to this paper), but in contrast, SML and HSML maintain sequences of arbitrary stepwise link deletions that require memory, and thus incur higher complexity. Even so, research questions about SML show many similarities with those for PAL and MLSR. One might even think that the link deletion case is essentially the same subject as the point deletion case, but more precise information on the true connections will be found in Section 5 below.

Relation to hybrid modal logic In this paper, we employ devices from hybrid logic, [3], to boost the expressive power of the original sabotage modal logic just enough to allow for a Hilbert-style axiomatization. However, this choice of a surplus is not unique. We focus on nominals plus the @-operator as a convenient syntax, but a version of SML extended with nominals and global existential and universal modalities would also be worth investigating. Moreover, we just determine the most general logic of the above games. Sabotage logics for specific classes of graphs may well be axiomatizable using further proof-theoretic techniques from hybrid logic, such as those presented in [7]. Finally, one could also turn the tables, and in the spirit of [2], view our results from a hybrid perspective as exploring fragments of the full first-order language that arise as hybrid languages are enriched with modalities for various forms of graph change.

2 Hybrid Sabotage Modal Logic

2.1 Language and semantics

We start by introducing the basic notions of the system HSML. For details of modal logic that we do not explain, we refer to [6].

Definition 1 (Language). *Let* $\mathsf{Prop} = \{p, q, r, \ldots\}$ *be a nonempty countable set of propositional variables disjoint from a nonempty countable set of nominals* $\mathsf{Nom} = \{a, b, c, d, \ldots\}$. *The* hybrid modal sabotage language HSML *is defined over the set of atoms* $\mathsf{Prop} \cup \mathsf{Nom}$ *by the following grammar:*

$$\varphi ::= a \mid p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \blacklozenge\varphi \mid @_a\varphi$$

Definition 2 (Model). *A* model $\mathfrak{M} = (W, R, V)$ *for HSML is a standard modal relational model with worlds* W*, accessibility relation* R *and valuation function* V*, subject to the condition that* V *assigns singleton subsets of* W *to nominals.*

Definition 3 (Truth conditions). *The semantics of HSML is as follows:*

$\mathfrak{M}, w \models a$	iff	$w \in V(a)$
$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models @_a\varphi$	iff	$\mathfrak{M}, v \models \varphi$ *where* $V(a) = \{v\}$
$\mathfrak{M}, w \models \neg\varphi$	iff	*not* $\mathfrak{M}, w \models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \phi$	iff	$\mathfrak{M}, w \models \varphi$ *and* $\mathfrak{M}, w \models \phi$
$\mathfrak{M}, w \models \Diamond\varphi$	iff	$\mathfrak{M}, v \models \varphi$ *for some* v *with* Rwv
$\mathfrak{M} = (W, R, V), w \models \blacklozenge\varphi$	iff	*there is a pair* $(u, v) \in R$ *such that* $(W, R \setminus (u, v), V), w \models \varphi$.

The deletion diamond modality of SML and its universal dual $\blacksquare = \neg\blacklozenge\neg$ describes effects of cutting arbitrary links, one at a time, allowing one to express, e.g., winning patterns for Traveler in sabotage games by modal combinations $\blacksquare\Diamond$. For more on the expressive power of this device, cf. [4].

However, using nominals, we can define still more, in particular, the following useful operator describing the effect of cutting a specific named link:

$$\langle a|b\rangle\varphi := (@_a\Diamond b \wedge \blacklozenge(@_a\neg\Diamond b \wedge \varphi)) \vee (@_a\neg\Diamond b \wedge \varphi)$$

Informally, this formula says that after cutting a possibly existent link between the world named "a" and the world named "b", φ will hold. The first disjunct describes the effects of actually cutting such a link, the second disjunct takes care of the case that no link connected a and b. Formally, let $\mathfrak{M} = (W, R, V), \mathfrak{M}^{(a|b)} = (W, R^{(a|b)}, V)$, where $R^{(a|b)} = R \setminus \{(u, v) \mid \mathfrak{M}, u \models a \text{ and } \mathfrak{M}, v \models b\}$. Unpacking the above truth conditions, it is easy to see that the following holds:

Fact 1 $\mathfrak{M}, w \models \langle a|b\rangle\varphi$ *iff* $\mathfrak{M}^{(a|b)}, w \models \varphi$.

In what follows, we will often need finite sequences of link cuts, and accordingly, we will use the notation $\mathfrak{M}^{\overline{(a|b)}_n}$ for the model $(((\mathfrak{M}^{(a_1|b_1)})^{(a_2|b_2)})^{\cdots})^{(a_n|b_n)}$ and $\overline{\langle a|b\rangle}_n\varphi$ for the formula $\langle a_1|b_1\rangle \dots \langle a_n|b_n\rangle\varphi$ when $n \geq 1$. Moreover, in the special case of $n = 0$ we let $\mathfrak{M}^{\overline{(a|b)}_n}$ denote $\mathfrak{M}$ while $\overline{\langle a|b\rangle}_n\varphi$ denotes φ.

2.2 A proof system for HSML

Using our named link-cutting device, we now present the proof system **HSML** in Table 1. Its first module consists of standard axioms and derivation rules from the minimal modal logic with hybrid additions, [3], the second module is the usual minimal modal logic for the sabotage modality, the third module contains dynamic-epistemic style recursion axioms for definable link cutting, because of which the logic is not closed under uniform substitution, and the fourth module contains the crucial derivation rule connecting the deletion modality and the named link cutting modality.[5]

The first two modules drive standard modal completeness arguments, the third and fourth capture the arbitrary deletion modality $\blacklozenge$. In particular, the finite prefixes of deletions in the rule schema (B-Mix) allow for reasoning about models arising from an initial one after finite histories of link cutting.[6]

For an illustration of how one can work with this calculus, we derive a few inference rules and theorems in the above proof system. Some of these principles will be useful in our proof for the strong completeness of **HSML** in Section 3.

Fact 2 *Replacement of Equivalents:* $\dfrac{\varphi \leftrightarrow \psi}{\langle a|b\rangle\varphi \leftrightarrow \langle a|b\rangle\psi}$ *can be derived in HSML.*

Proof. First note that the monotonicity rule for $\blacklozenge$: $\dfrac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi}$ is a derivable rule since $\blacksquare$ is a K operator. Next, we derive $\langle a|b\rangle\varphi \to \langle a|b\rangle\psi$ from $\vdash \varphi \to \psi$.

1. $\vdash \varphi \to \psi$ (assumption)
2. $\vdash (@_a\neg\Diamond b \land \varphi) \to (@_a\neg\Diamond b \land \psi)$ (from 1 by the propositional logic CPL)
3. $\vdash \blacklozenge(@_a\neg\Diamond b \land \varphi) \to \blacklozenge(@_a\neg\Diamond b \land \psi)$ (from 2 and the distribution rule for $\blacklozenge$)
4. $\vdash (@_a\Diamond b \land \blacklozenge(@_a\neg\Diamond b \land \varphi)) \to (@_a\Diamond b \land \blacklozenge(@_a\neg\Diamond b \land \psi))$ (from 3 and CPL)

[5] We will often make tacit appeals to a proof rule of Replacement of Equivalents in what follows, but this is derivable in the system **HSML** as presented here.

[6] In the logic PAL, finite sequences of announcements can be compressed to one by the Composition Axiom. However, it is easy to show that no such compression is possible in HSML, unless we define complex modalities for simultaneous link cuts.

Axioms and rules for basic hybrid modal logic

All tautologies of classical propositional logic, plus Modus Ponens (CPL)
All axioms of the minimal modal logic for $\Box$, plus the Necessitation Rule

Axioms and rules of hybrid logic for $@_a$:

$@_a(\varphi \to \psi) \to (@_a\varphi \to @_a\psi), \quad a \wedge @_a\varphi \to \varphi$

$@_a\varphi \leftrightarrow \neg @_a \neg\varphi, \quad a \wedge \varphi \to @_a\varphi, \quad @_a a, \quad @_a b \leftrightarrow @_b a$

$@_a b \wedge @_b\varphi \to @_a\varphi, \quad @_b @_a\varphi \leftrightarrow @_a\varphi, \quad \Diamond @_a\varphi \to @_a\varphi$

$(Name): \dfrac{c \to \varphi}{\varphi} (c \notin \varphi), \quad (Nec): \dfrac{\varphi}{@_a\varphi}$

$(Paste): \dfrac{@_a\Diamond b \wedge @_b\varphi \to \delta}{@_a\Diamond\varphi \to \delta}$ ($b \notin \varphi$, δ and a are distinct from b)

Distribution Axiom for $\blacksquare$ $\quad \blacksquare(\phi \to \psi) \to (\blacksquare\phi \to \blacksquare\psi)$

Necessitation Rule for $\blacksquare$ $\quad \dfrac{\varphi}{\blacksquare\varphi}$

Recursion axioms for $\langle a|b\rangle$

$\langle a|b\rangle c \leftrightarrow c \quad \langle a|b\rangle p \leftrightarrow p \quad \langle a|b\rangle\neg\varphi \leftrightarrow \neg\langle a|b\rangle\varphi \quad \langle a|b\rangle(\varphi \wedge \psi) \leftrightarrow (\langle a|b\rangle\varphi \wedge \langle a|b\rangle\psi)$

$\langle a|b\rangle @_c\varphi \leftrightarrow @_c\langle a|b\rangle\varphi \quad \langle a|b\rangle\Diamond\varphi \leftrightarrow ((a \wedge \Diamond(\neg b \wedge \langle a|b\rangle\varphi)) \vee (\neg a \wedge \Diamond\langle a|b\rangle\varphi))$

Inference rule for $\blacklozenge$, $\langle a|b\rangle$

(B-Mix) : $\dfrac{@_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \to \theta}{@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi \to \theta}$

where $n \geq 0$; the new nominals a_{n+1}, b_{n+1} are distinct from c and other nominals in $\overline{\langle a|b\rangle}_n$ and do not occur in φ or θ.

Table 1. The Hilbert-style proof system **HSML**

5. $\vdash (@_a\Diamond b \wedge \blacklozenge(@_a\neg\Diamond b \wedge \varphi)) \vee (@_a\neg\Diamond b \wedge \varphi) \to (@_a\Diamond b \wedge \blacklozenge(@_a\neg\Diamond b \wedge \varphi)) \vee (@_a\neg\Diamond b \wedge \psi)$ (from 4 and CPL)
6. $\vdash \langle a|b\rangle\varphi \to \langle a|b\rangle\psi$ (from 5 and the definitions of $\langle a|b\rangle\varphi$ and $\langle a|b\rangle\psi$)

The derivation of the other direction of the equivalence, namely $\vdash \langle a|b\rangle\psi \to \langle a|b\rangle\varphi$ from $\vdash \psi \to \varphi$, proceeds analogously. Putting all this together, it follows that $\dfrac{\varphi \leftrightarrow \psi}{\langle a|b\rangle\varphi \leftrightarrow \langle a|b\rangle\psi}$ is an admissible inference rule in HSML.

Fact 3 *The formula:* $\overline{\langle a|b\rangle}_n(\varphi \wedge \psi) \leftrightarrow (\overline{\langle a|b\rangle}_n\varphi \wedge \overline{\langle a|b\rangle}_n\psi)$ *is provable in* **HSML**.

Proof. This formula generalizes the distribution of $\langle a|b\rangle$ over conjunction, which is a recursion axiom for $\langle a|b\rangle$ reflecting the fact that link cutting between named points is an operation that is a partial function on models. The proof of the Fact involves an iterated appeal to the recursion axiom for the conjunction, with successive substitutions licensed by Replacement of Equivalents.

Another simple useful fact is this.

Fact 4 *The formula:* $@_a\Diamond b \wedge \langle a|b\rangle\psi \to \blacklozenge\psi$ *is provable in* **HSML**.

Proof. This formula specifies the effect of cutting the link between a and b in terms of $\blacklozenge$. It follows easily from the above definition of the link-cutting modality $\langle a|b\rangle\varphi$ plus an appeal to CPL and the minimal modal logic for $\blacklozenge$.

Next come two facts whose proofs are more complex than the preceding ones.

Fact 5 $\overline{\langle a|b\rangle}_n\Diamond\psi \leftrightarrow \bigvee_{S\subseteq[n]}(\bigwedge_{m\in S} a_m \wedge \bigwedge_{m\in[n]-S}\neg a_m \wedge \Diamond(\bigwedge_{m\in S}\neg b_m \wedge \overline{\langle a|b\rangle}_n\psi))$ *is provable in* **HSML** *for any natural nunber* $n \in \mathbb{N}$*, where* $[n]$ *with* $n \geq 1$ *denotes the set* $\{1, \ldots, n\}$ *while* $[0]$ *denotes the empty set* $\emptyset$.

Proof. For the case that $n = 0$, the formula reduces to $\Diamond\psi \leftrightarrow \Diamond\psi$, which is a tautology. For the case that $n = 1$, the formula reduces to $\langle a_1|b_1\rangle\Diamond\varphi \leftrightarrow ((a_1 \wedge \Diamond(\neg b_1 \wedge \langle a_1|b_1\rangle\varphi)) \vee (\neg a_1 \wedge \Diamond\langle a_1|b_1\rangle\varphi))$, which is a recursion axiom for $\langle a_1|b_1\rangle$.

Next, we prove the general case, where each subset S of $[n]$ specifies a possible case. In each possible case, the left side specifies what happens to those worlds to which the current world has access to after the sequence of link cuttings.

Suppose that for all $0 \leq n \leq k$ and for all formulas ψ, we have already shown:

$$\vdash \overline{\langle a|b\rangle}_n\Diamond\psi \leftrightarrow \bigvee_{S\subseteq\ [n]}(\bigwedge_{m\in S} a_m \wedge \bigwedge_{m\in[n]-S}\neg a_m \wedge \Diamond(\bigwedge_{m\in S}\neg b_m \wedge \overline{\langle a|b\rangle}_n\psi)) \quad (I.H.)$$

We are going to to prove the assertion for $n = k + 1$.

For the sake of simplifying notation, let $\langle c_k\rangle$ denote $\langle a_k|b_k\rangle$, $\overline{\langle c\rangle}_k$ denote $\overline{\langle a|b\rangle}_k$ for $k \in \mathbb{N}$ and $\Theta^S_n\psi$ denote $\bigvee_{S\subseteq\ [n]}(\bigwedge_{m\in S} a_m \wedge \bigwedge_{m\in[n]-S}\neg a_m \wedge \Diamond(\bigwedge_{m\in S}\neg b_m \wedge \overline{\langle a|b\rangle}_n\psi))$.

By the definition of $\overline{\langle c\rangle}_{k+1}$, we have

$$\vdash \overline{\langle c\rangle}_{k+1}\Diamond\psi \leftrightarrow \overline{\langle c\rangle}_k\langle c_{k+1}\rangle\Diamond\psi$$

Applying the Replacement of Equivalents rule k times to the recursion axiom $\vdash \langle c_{k+1}\rangle\Diamond\psi \leftrightarrow (a_{k+1} \wedge \Diamond(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \vee (\neg a_{k+1} \wedge \Diamond\langle c_{k+1}\rangle\psi)$, we obtain

$$\vdash \overline{\langle c\rangle}_k\langle c_{k+1}\rangle\Diamond\psi \leftrightarrow \overline{\langle c\rangle}_k((a_{k+1} \wedge \Diamond(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \vee (\neg a_{k+1} \wedge \Diamond\langle c_{k+1}\rangle\psi))$$

It follows that

$$\vdash \overline{\langle c\rangle}_{k+1}\Diamond\psi \leftrightarrow \overline{\langle c\rangle}_k((a_{k+1} \wedge \Diamond(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \vee (\neg a_{k+1} \wedge \Diamond\langle c_{k+1}\rangle\psi))$$

Next, after applying the recursion axioms several times to the latter part of the above formula, it follows that

$$\vdash \overline{\langle c\rangle}_{k+1}\Diamond\psi \leftrightarrow ((a_{k+1} \wedge \overline{\langle c\rangle}_k\Diamond(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \vee (\neg a_{k+1} \wedge \overline{\langle c\rangle}_k\Diamond\langle c_{k+1}\rangle\psi)) \quad (*)$$

Let α and β denote $\overline{\langle c\rangle}_k\Diamond(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)$ and $\overline{\langle c\rangle}_k\Diamond\langle c_{k+1}\rangle\psi$ respectively. Then, by applying the inductive hypothesis to α, β, we obtain the two facts

$$\vdash \alpha \leftrightarrow \Theta_k^S(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)$$
$$\vdash \beta \leftrightarrow \Theta_k^S(\langle c_{k+1}\rangle\psi)$$

Now replacing α, β by equivalent formulas in the formula $(*)$, we get

$$\vdash \overline{\langle c\rangle}_{k+1}\Diamond\psi \leftrightarrow ((a_{k+1} \wedge \Theta_k^S(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \vee (\neg a_{k+1} \wedge \Theta_k^S(\langle c_{k+1}\rangle\psi)))$$

Focusing on the right part of this formula, we get the following equivalence:

$$\vdash a_{k+1} \wedge \Theta_k^S(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi) \leftrightarrow$$
$$\bigvee_{S\subseteq\ [k]} (\bigwedge_{m\in S\cup\{k+1\}} a_m \wedge \bigwedge_{m\in[k]-S} \neg a_m \wedge \Diamond(\bigwedge_{m\in S\cup\{k+1\}} \neg b_m \wedge \overline{\langle c\rangle}_{k+1}\psi))$$
$$\vdash \neg a_{k+1} \wedge \Theta_k^S\Diamond\langle c_{k+1}\rangle\psi \leftrightarrow \bigvee_{S\subseteq\ [k]} (\bigwedge_{m\in S} a_m \wedge \bigwedge_{m\in[k+1]-S} \neg a_m \wedge \Diamond(\bigwedge_{m\in S} \neg b_m \wedge \overline{\langle c\rangle}_{k+1}\psi))$$

Notice how a_{k+1} and $\neg a_{k+1}$ distribute over the big disjunctions and how the $\neg b_{k+1}$ gets out of $\overline{\langle c\rangle}_k$ by the recursion axiom for norminals and merged into the big conjunction. Furthermore, by some combinatoric inference, we have $2^{[k+1]} = 2^{[k]} \cup \{S \cup \{k+1\} \mid S \in 2^{[k]}\}$. It thus follows that

$$\vdash (a_{k+1} \wedge \Theta_k^S(\neg b_{k+1} \wedge \langle c_{k+1}\rangle\psi)) \wedge (\neg a_{k+1} \wedge \Theta_k^S(\langle c_{k+1}\rangle\psi)) \leftrightarrow \Theta_{k+1}^S\psi$$

That is,

$$\vdash \overline{\langle c\rangle}_{k+1}\Diamond\psi \leftrightarrow \bigvee_{S\subseteq[k+1]} (\bigwedge_{m\in S} a_m \wedge \bigwedge_{m\in[k+1]-S} \neg a_m \wedge \Diamond(\bigwedge_{m\in S} \neg b_m \wedge \overline{\langle c\rangle}_{k+1}\psi))$$

which is what we needed to prove.

Finally, we show how the B-Mix rule can be used to prove a basic principle about the interaction between $\Diamond$ and $\blacklozenge$.

Fact 6 $\vdash_{\text{HSML}} \blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi$.

Proof. 1. $\vdash \langle a|b\rangle\varphi \leftrightarrow (@_a\Diamond b \wedge \blacklozenge(@_a\neg\Diamond b \wedge \varphi)) \vee (@_a\neg\Diamond b \wedge \varphi)$ (by definition)

2. $\vdash @_a\Diamond b \wedge \langle a|b\rangle\varphi \to \blacklozenge\varphi$ (from 1)
3. $\vdash \Diamond(@_a\Diamond b \wedge \langle a|b\rangle\varphi) \to \Diamond\blacklozenge\varphi$ (from 2 in the minimal modal logic K)
4. $\vdash \Box @_a\Diamond b \wedge \Diamond\langle a|b\rangle\varphi \to \Diamond(@_a\Diamond b \wedge \langle a|b\rangle\varphi)$ (theorem of the logic K)
5. $\vdash \Box @_a\Diamond b \wedge \Diamond\langle a|b\rangle\varphi \to \Diamond\blacklozenge\varphi$ (from 3 and 4)
6. $\vdash \langle a|b\rangle\Diamond\varphi \leftrightarrow ((a \wedge \Diamond(\neg b \wedge \langle a|b\rangle\varphi)) \vee (\neg a \wedge \Diamond\langle a|b\rangle\varphi))$ (axiom for $\langle a|b\rangle$)
7. $\vdash \langle a|b\rangle\Diamond\varphi \to \Diamond\langle a|b\rangle\varphi$ (from 6)
8. $\vdash \Box @_a\Diamond b \wedge \langle a|b\rangle\Diamond\varphi \to \Diamond\blacklozenge\varphi$ (from 5 and 7)
9. $\vdash @_a\Diamond b \to \Box @_a\Diamond b$ (theorem of hybrid logic)

10. $\vdash @_a\Diamond b \wedge \langle a|b\rangle\Diamond\varphi \to \Diamond\blacklozenge\varphi$ (from 8 and 9)
11. $\vdash @_c(@_a\Diamond b \wedge \langle a|b\rangle\Diamond\varphi \to \Diamond\blacklozenge\varphi)$, for c not occurring in φ (**Nec** rule for $@_c$)
12. $\vdash @_c(@_a\Diamond b \wedge \langle a|b\rangle\Diamond\varphi) \to @_c\Diamond\blacklozenge\varphi$ (from 11)
13. $\vdash @_c\blacklozenge\Diamond\varphi \to @_c\Diamond\blacklozenge\varphi$ (from 12 using the B-Mix rule)
14. $\vdash @_c(\blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi)$ (from 13 in hybrid logic)
15. $\vdash c \to (\blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi)$ (from 14 in hybrid logic)
16. $\vdash \blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi$ (from 15 by the Name rule).

It may be of interest to note that the converse implication $\Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi$ is not valid in HSML, as can be seen by giving a simple countermodel.

Readers who want to get still more familiar with the proof system **HSML** may find the implication $\langle a|b\rangle\langle c|d\rangle\varphi \to \langle c|d\rangle\langle a|b\rangle\varphi$ a useful further exercise.

3 Soundness and Strong Completeness for HSML

We now turn to the meta-properties of the proof system **HSML**.

Theorem 1 (Soundness). *All provable formulas* **HSML** *are valid.*

The soundness of most principles in the above proof system is immediate, we only concentrate on those that deserve special attention.

Fact 7 *The axiom* $\langle a|b\rangle\Diamond\varphi \leftrightarrow ((a \wedge \Diamond(\neg b \wedge \langle a|b\rangle\varphi)) \vee (\neg a \wedge \Diamond\langle a|b\rangle\varphi))$ *is valid.*

Proof. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W, R', V)$, where $R' = R \setminus \{(u, v) \,|\, \mathfrak{M}, u \models a \text{ and } \mathfrak{M}, v \models b\}$, i.e., the pair named by (a, b) has been deleted.

From left to right, if $\mathfrak{M}, w \models \langle a|b\rangle\Diamond\varphi$, then $\mathfrak{M}', w \models \Diamond\varphi$, so $\mathfrak{M}', v \models \varphi$ for some v with $R'wv$, and $\mathfrak{M}, v \models \langle a|b\rangle\varphi$. Case 1: $\mathfrak{M}, w \models a$. Then $\mathfrak{M}', w \models a$, and so $\mathfrak{M}', v \models \neg b$, whence $\mathfrak{M}, v \models \neg b$, and taking together, $\mathfrak{M}, v \models \neg b \wedge \langle a|b\rangle\varphi$ and $\mathfrak{M}, w \models (a \wedge \Diamond(\neg b \wedge \langle a|b\rangle\varphi)$: the first disjunct on the right. Case 2: $\mathfrak{M}, w \models \neg a$. Then, since $\mathfrak{M}, v \models \langle a|b\rangle\varphi$, we get the second disjunct: $\mathfrak{M}, w \models \neg a \wedge \Diamond\langle a|b\rangle\varphi$.

From right to left, a similar semantic argument will work, essentially reversing the preceding steps, including the case distinction.

Fact 8 *The B-Mix rule is sound.*

Proof. Assume that $@_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_n \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \to \theta$ is valid, where the nominals a_{n+1} and b_{n+1} are different from c and any nominals in the sequence $(a|b)_n$ and do not occur in φ and θ. Consider any HSML model $\mathcal{M}$ and point w such that $\mathcal{M}, w \models @_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi$. According to the truth conditions for the link deletion modalities, there must be a still available link deletion $(d|d')$ after the links defined in the sequence $\overline{\langle a|b\rangle}_n$ have been cut such that φ is true after the deletion. Now take two fresh nominals a_{n+1} and b_{n+1} not occurring in the formulas so far, such that $V(a_{n+1}) = V(d)$ and $V(b_{n+1}) = V(d')$. Then the antecedent of the assumed validity is satisfied, and we get $\mathcal{M}, w \models \theta$.

We have seen how the B-Mix rule is used in the proof system to prove significant theorems. In the following completeness proof, we will see it is also essential for constructing a special type of maximally consistent sets.

Theorem 2. *The proof system* **HSML** *is strongly complete.*

The proof to follow uses the technique introduced for the modal logic MLSR in [22]. In the present setting, this involves combining basic modal logic, hybrid logic, the key HSML modality $\blacklozenge$ for arbitrary link deletion in a graph, and its interaction with the above defined modality for deletion of named links. A noteworthy difference with the cited reference is our simplification in defining the latter modality, cf. Fact 1 in Section 2.1, so we can do without DEL-style link cutting modalities as additional primitives.

As is standard in completeness proofs, it suffices to show that any **HSML**-consistent set of formulas is satisfiable in a HSML model.

The first step is to prove that any **HSML**-consistent set can be extended to a maximally consistent set ('**HSML**-MCS') satisfying the following properties.

Definition 4 (Named, pasted, mixed, B-mixed). *A set of formulas Γ is (a)* named *if it contains a nominal, (b)* pasted *if $@_a\Diamond\varphi \in \Gamma$ implies that there is some nominal b such that the formula $@_a\Diamond b \wedge @_b\varphi \in \Gamma$, (c)* mixed *if $\overline{\langle a|b\rangle_n}\blacklozenge\varphi \in \Gamma$ implies that $\overline{\langle a|b\rangle_n}(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \in \Gamma$ for some nominals a_{n+1}, b_{n+1}, and finally, (d) Γ is* B-mixed *if $@_c\overline{\langle a|b\rangle_n}\blacklozenge\varphi \in \Gamma$ implies that $@_c\overline{\langle a|b\rangle_n}(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \in \Gamma$ for some nominals a_{n+1}, b_{n+1}.*

The properties *named* and *pasted* are needed to deal with the hybrid component of the logic while *mixed* and *B-mixed* are for the link-cutting part. The property *mixed* will become relevant later in Lemma 2.

Lemma 1 (Lindenbaum Lemma). *Let Nom' be a countably infinite set of nominals disjoint from Nom, and let $\mathcal{L}'$ be the language obtained by adding these new nominals to $\mathcal{L}$. Every* **HSML***-consistent set of formulas in language $\mathcal{L}$ can be extended to a named, pasted and B-mixed* **HSML***-MCS in the language $\mathcal{L}'$.*

Proof. Given a consistent set of $\mathcal{L}$-formulas Σ, let Σ_d to be $\Sigma \cup \{d\}$, where d is an arbitrary new nominal in Nom'. Σ_d is consistent. For suppose not. Then for some conjunction of formulas θ from Σ, $\vdash d \to \neg\theta$. But the new nominal d does not occur in θ, and so, by the Name rule, $\vdash \neg\theta$. This contradicts the consistency of Σ: so Σ_d must be consistent.

Next, enumerate all the formulas of $\mathcal{L}'$ (this includes the nominals in Nom'). We define a sequence of consistent sets as follows. Let Σ^0 be the set Σ_d just constructed. Now, working inductively, suppose we have defined Σ^m, where $m \geq 0$. Let φ_{m+1} be the $m+1$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows. If $\Sigma^{m+1} \cup \{\varphi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

- $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_a\Diamond b \wedge @_b\varphi\}$, if φ_{m+1} is of the form $@_a\Diamond\varphi$.
  Here b is the first nominal in the enumeration not occurring in Σ^m or $@_a\Diamond\varphi$.

– $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi)\}$, if φ_{m+1} is of the form $@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi \in \Gamma$. Here a_{n+1}, b_{n+1} are the first two nominals in the enumeration that do not occur in Σ^m or $@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi$.
– $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\}$ if φ_{m+1} is not of the form $@_a\Diamond\varphi$ or $@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi$.

Let $\Sigma^+ = \bigcup_{n\geq 0}\Sigma^n$. Clearly this set is named, maximal, pasted and B-mixed. It is also consistent. For expansions of the first kind, consistency preservation is guaranteed by the Paste rule. For expansions of the second kind, if the set obtained is not consistent, then for some conjunction of formulas θ from the set $\Sigma^m \cup \{\varphi_{m+1}\}$,

$$\vdash @_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi \to \neg\theta.$$

By the B-mix rule, $\vdash \varphi_{m+1} \to \neg\theta$, contradicting the consistency of $\Sigma^m \cup \{\varphi_{m+1}\}$.

Next, each **HSML**-MCS Γ induces a family of maximally consistent sets.

Definition 5. *The* named set Δ_a yielded by Γ *is* $\{\varphi \mid @_a\varphi \in \Gamma\}$.

Now we can define the modal model that will satisfy our given consistent set.

Definition 6 (Named model). *The* named model generated by Γ *is the tuple* $\mathfrak{M}^\Gamma = (W^\Gamma, R^\Gamma, V^\Gamma)$ *where (a)* W^Γ *consists of all named sets yielded by* Γ*, (b)* $R^\Gamma uv$ *iff for all formulas* φ *with* $\varphi \in v$*, we have* $\Diamond\varphi \in u$*, and finally (c)* $V^\Gamma(o) = \{w \in W^\Gamma \mid o \in w, o \in \mathsf{Prop} \cup \mathsf{Nom}\}$.

This model has the following basic properties that can be shown just as in standard completeness proofs for hybrid logic, [3].

Lemma 2 (Existence Lemma). *Let* Γ *be a named, pasted and B-mixed* **HSML**-*MCS and let* $\mathfrak{M}^\Gamma = (W^\Gamma, R^\Gamma, V^\Gamma)$ *be the named model yielded by* Γ.

(a) All named sets Δ_a *yielded by* Γ *are* **HSML**-*MCSs.*

(b) If $u \in W^\Gamma$ *and* $\Diamond\varphi \in u$*, then there is some* $v \in W^\Gamma$ *with* $R^\Gamma uv$ *and* $\varphi \in v$.

(c) All named sets Δ_a *yielded by* Γ *are mixed.*

Proof. We only prove the least standard third item. Assume that $\overline{\langle a|b\rangle}_n\blacklozenge\varphi \in \Delta_c$, i.e., $@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi \in \Gamma$. Since the set Γ is B-mixed, we have $@_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \in \Gamma$ for some nominals a_{n+1}, b_{n+1}, and so we have immediately that $\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \in \Delta_c$. This means that Δ_c is mixed.

Now comes the part of the proof where we need to consider models arising after link deletions, in order to deal with the modality $\blacklozenge$. In addition to the preceding named model, we introduce the following new models.

Definition 7 (Derived Henkin model). *Let* $\overline{\langle a|b\rangle}_n = \langle a_1|b_1\rangle \dots \langle a_n|b_n\rangle$*. The* derived Henkin model *from a named model* $\mathfrak{M}^\Gamma$ *generated by* Γ *is the tuple*

$$\mathfrak{M}^\Gamma : \overline{\langle a|b\rangle}_n = (W^{\overline{\langle a|b\rangle}_n}, R^{\overline{\langle a|b\rangle}_n}, V^{\overline{\langle a|b\rangle}_n})$$

with worlds, accessibilty and valuations defined as follows:

– $W^{\overline{\langle a|b\rangle}_n} = \{(w, \overline{\langle a|b\rangle}_n) \mid w \in W^\Gamma\}$
– $R^{\overline{\langle a|b\rangle}_n}((u, \overline{\langle a|b\rangle}_n), (v, \overline{\langle a|b\rangle}_n))$ *if*
  (a) $R^\Gamma uv$ *and (b)* $a_i \notin u$ *or* $b_i \notin v$ *for all* $i \leq n$
– $V^{\overline{\langle a|b\rangle}_n}(o) = \{(w, \overline{\langle a|b\rangle}_n) \mid w \in V^\Gamma(o), o \in \mathsf{Prop} \cup \mathsf{Nom}\}$.

We stipulate that $\mathfrak{M}^\Gamma : \overline{\langle a|b\rangle}_0 = \mathfrak{M}^\Gamma$.

Each point in the derived Henkin Model induces the following set of formulas:

$$\Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w) = \{\varphi \mid \overline{\langle a|b\rangle}_n \varphi \in w\}$$

We now prove the crucial Truth Lemma: for derived Henkin models, membership in these induced sets and truth in the corresponding worlds coincide.

Lemma 3 (Truth Lemma). *For all formulas* φ*, finite sequences* $\overline{\langle a|b\rangle}_n$ *and points* w *in a named model* $\mathfrak{M}$ *yielded by* Γ*, we have that, for any* $n \geq 0$*:*

$$\mathfrak{M} : \overline{\langle a|b\rangle}_n, (w, \overline{\langle a|b\rangle}_n) \models \varphi \;\; \textit{iff} \;\; \varphi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$$

Proof. The proof is by induction on the formulas φ. For brevity, we will write $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \varphi$, leaving out the sequence notation $\overline{\langle a|b\rangle}_n$.

(a) *Atomic formulas.* We only prove the case for p, the one for nominals is similar. $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models p$ iff $w \in V(p)$ iff $p \in w$ (by the definition of V in derived Henkin models) iff $\overline{\langle a|b\rangle}_n p \in w$ (by the recursion axiom for p) iff $p \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$.

(b) *Negations.* $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \neg\psi$ if $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \not\models \psi$ iff (by the inductive hypothesis) $\psi \notin \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$ iff $\overline{\langle a|b\rangle}_n \psi \notin w$ iff $\neg\overline{\langle a|b\rangle}_n \psi \in w$ iff (by the recursion axiom for $\neg\psi$) $\overline{\langle a|b\rangle}_n \neg\psi \in w$ iff $\neg\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$.

(c) *Conjunction.* The proof is like the preceding one, using the inductive hypothesis and the recursion axiom for conjunctions under link cutting modalities.

(d) *@ Operators.* $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models @_c\psi$ iff $\mathfrak{M} : \overline{\langle a|b\rangle}_n, \Delta_c \models \psi$ iff (by the inductive hypothesis) $\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, \Delta_c)$ iff (by definition) $\overline{\langle a|b\rangle}_n \psi \in \Delta_c$ iff (noting that $\alpha \in \Delta_c$ iff $@_c\alpha \in \Delta_a$ for any nominal a) $@_c\overline{\langle a|b\rangle}_n \psi \in w$ iff (by the recursion axiom for $@_c$) $\overline{\langle a|b\rangle}_n @_c\psi \in w$ iff $@_c\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$.

(e) $\Diamond$ *modality.* In the case of $n = 0$, the assertion reduces to the standard modal case, whose proof is well-known, [6]. So let us focus on the case $n \neq 0$. From left to right, let $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \Diamond\psi$. Then there is a v with $R^{\overline{\langle a|b\rangle}_n}((w, \overline{\langle a|b\rangle}_n), (v, \overline{\langle a|b\rangle}_n))$ and $\mathfrak{M} : \overline{\langle a|b\rangle}_n, v \models \psi$. By the inductive hypothesis, $\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, v)$, i.e., $\overline{\langle a|b\rangle}_n \psi \in v$. Since $R^{\overline{\langle a|b\rangle}_n}(w, \overline{\langle a|b\rangle}_n)(v, \overline{\langle a|b\rangle}_n)$, it follows that Rwv. Thus by the definition of R in a named model, $\Diamond\overline{\langle a|b\rangle}_n \psi \in w$. Now, by the definition of the relations $R^{\overline{\langle a|b\rangle}_n}$, $a_x \notin w$ or $b_x \notin v$ for any $x \in [1, ..., n]$. In particular, for any $x \in [1, ..., n]$, if $a_x \in w$, $b_x \notin v$. Starting from a_1, either $a_1 \wedge \Diamond(\neg b_1 \wedge \overline{\langle a|b\rangle}_n \psi) \in w$ or $\neg a_1 \wedge \Diamond \overline{\langle a|b\rangle}_n \psi \in w$. By the recursion axiom for $\Diamond$, we then get that $\langle a_1|b_1\rangle \Diamond \langle a_2|b_2\rangle \ldots \langle a_n|b_n\rangle \psi \in w$. Repeating this argument, we can push $\Diamond$ to the innermost position, which gives us the desired result $\overline{\langle a|b\rangle}_n \Diamond \psi \in w$. That is, $\Diamond \psi \in \varphi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$.

From right to left: let $\Diamond \psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$, i.e., $\overline{\langle a|b\rangle}_n \Diamond \psi \in w$. By Fact 5, we have the set $S = \{x \in [1, ..., n] \mid a_x \in w\}$ such that $\Diamond(\bigwedge_{x \in S} \neg b_x \wedge \overline{\langle a|b\rangle}_n \psi) \in w$. By the Existence Lemma for $\Diamond$, there is a v with $\bigwedge_{x \in S} \neg b_x \wedge \overline{\langle a|b\rangle}_n \psi \in v$, which implies that $\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, v)$. By the inductive hypothesis, $\mathfrak{M} : \overline{\langle a|b\rangle}_n, v \models \psi$. Also, by the definition of S and $\bigwedge_{x \in S} \neg b_x \in v$, we have for any $x \in [1, ..., n]$, $a_x \notin w$ or $b_x \notin v$. By the definition of $R^{\overline{\langle a|b\rangle}_n}$ and Rwv, then $R^{\overline{\langle a|b\rangle}_n}((w, \overline{\langle a|b\rangle}_n), (v, \overline{\langle a|b\rangle}_n))$. Therefore, $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \Diamond \psi$.

(f) *The deletion modality* $\blacklozenge$. From left to right, let $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \blacklozenge \psi$. Then there is a link in $\mathfrak{M} : \overline{\langle a|b\rangle}_n$, say $((\Delta_{a_{n+1}}, \overline{\langle a|b\rangle}_n), (\Delta_{b_{n+1}}, \overline{\langle a|b\rangle}_n))$ (the naming of the link is guaranteed by our model construction) such that $\mathfrak{M} : \overline{\langle a|b\rangle}_{n+1}, w \models \psi$. Then by the inductive hypothesis $\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_{n+1}, w)$, i.e., $\overline{\langle a|b\rangle}_{n+1} \psi \in w$. Moreover, our model construction even yields that $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models @_{a_{n+1}} \Diamond b_{n+1}$. But then, by cases already proved, it follows that $@_{a_{n+1}} \Diamond b_{n+1} \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$, i.e., $\overline{\langle a|b\rangle}_n @_{a_{n+1}} \Diamond b_{n+1} \in w$. Now recall the definition of named link cutting $\langle a_{n+1}|b_{n+1}\rangle \psi$ in the language of HSML. We noted earlier that $@_{a_{n+1}} \Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle \psi \rightarrow \blacklozenge \psi$ is a theorem of **HSML**, and using the principles of the minimal logic K for $\langle a|b\rangle$, we get $\overline{\langle a|b\rangle}_n \blacklozenge \psi \in w$. Thus $\blacklozenge \psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$.

Finally, from right to left, let $\blacklozenge \psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$. By the Existence Lemma, w is mixed, and so $\psi \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_{n+1}, w)$ and $@_{a_{n+1}} \Diamond b_{n+1} \in \Phi(\mathfrak{M}, \overline{\langle a|b\rangle}_n, w)$ for new nominals a_{n+1} and b_{n+1} that do not occur in ψ. Thus, by the inductive hypothesis, $\mathfrak{M} : \overline{\langle a|b\rangle}_{n+1}, w \models \psi$. Also, by inductive cases already proved, $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models @_{a_{n+1}} \Diamond b_{n+1}$, and hence $((\Delta_{a_{n+1}}, \overline{\langle a|b\rangle}_n), (\Delta_{b_{n+1}}, \overline{\langle a|b\rangle}_n)) \in R^{\overline{\langle a|b\rangle}_n}$. Now $R^{\overline{\langle a|b\rangle}_{n+1}}$ equals the relation $R^{\overline{\langle a|b\rangle}_n} \setminus ((\Delta_{a_{n+1}}, \overline{\langle a|b\rangle}_n), (\Delta_{b_{n+1}}, \overline{\langle a|b\rangle}_n))$ while the valuation functions in all derived Henkin models are the same modulo the indexical sequences, we have $\mathfrak{M} : \overline{\langle a|b\rangle}_n, w \models \blacklozenge \psi$.

As usual, this finalizes the proof of the completeness theorem, since all formulas in the initially given set Γ will be true at the initial world of the named model induced by some arbitrary maximally consistent extension of Γ.

4 Protocol HSML

Having analyzed HSML on standard models, we now consider a natural generalization, also known from dynamic-epistemic logic, [11]. Suppose that not all link deletions are available, for instance, to the Demon in a sabotage game. This gives more general 'protocol models' for scenarios where agents operate under various constraints. There are several types of protocols, less or more general, cf. [19], but we will only analyze one particular case here.

Definition 8 (Protocols). *Let* $\Sigma = \{(a|b) \mid a, b \in \mathsf{NOM}\}$*. Members of the set* Σ^* *of all finite sequences of elements in* Σ *are called* histories*. A subset* S *of* Σ^* *is* closed under initial segments *if for any* $h \in S$*, its initial segments* $h' \sqsubseteq h$ *are also in* S*. A* protocol *is a set of histories closed under taking initial segments. Any HSML model* $\mathfrak{M} = (W, R, V)$ *has an associated set* $\mathsf{Prtcl}(\mathfrak{M})$ *of* feasible protocols f *satisfying the following condition: if* $(a_1|b_1)\ldots(a_i|b_i) \in f$*, then (a)* $(V(a_i), V(b_i)) \in R$*, and (b) for each* $j < i, V(a_i) \neq V(a_j)$ *or* $V(b_i) \neq V(b_j)$*.*

Here a history represents a sequence of successive link deletions in the given model, and a protocol defines which such sequences are allowed, for various reasons that may depend on the precise application. Condition (a) on protocols states that all links to be deleted actually exist, condition (b) states that no link is deleted twice, clearly minimal conditions on executable protocols.

Definition 9 (Protocol model). *Given a HSML model* $\mathfrak{M} = (W, R_0, V_0)$ *and one of its feasible protocols* f*, the* protocol model $\mathfrak{F} = \mathsf{Forest}(\mathfrak{M}, f) = (H, R, V)$ *is defined from initial worlds and link cut histories as follows:*

(a) $H = \{w\sigma \mid w \in W, \sigma \in f\}$*.*

(b) Rhh' *iff* $h = w\sigma$ *and* $h' = v\sigma$ *for some* $\sigma \in f$ *and* $w, v \in W$ *satisfying* R_0wv *while* $V_0(a) \neq w$ *or* $V_0(b) \neq v$ *for any* $(a|b) \in \sigma$*.*

(c) $V(o) = \{w\sigma \in H \mid V_0(a) = w\}$ *where* $o \in \mathsf{Prop} \cup \mathsf{Nom}$*.*

The semantics of HSML is easily lifted to protocol models:

Definition 10 (Truth conditions). *Given a protocol model* $\mathfrak{F} = \langle H, R, U\rangle$ *and a world* $h = w\sigma \in H$*, truth is defined by the following conditions:*

$\mathfrak{F}, w\sigma \models o$	*iff*	$w\sigma \in V(o)$*, where* $o \in \mathsf{Prop} \cup \mathsf{Nom}$
$\mathfrak{F}, h \models \neg\varphi$	*iff*	*not* $\mathfrak{F}, h \models \varphi$
$\mathfrak{F}, h \models \varphi_1 \wedge \varphi_2$	*iff*	$\mathfrak{F}, h \models \varphi_1$ *and* $\mathfrak{F}, h \models \varphi_2$
$\mathfrak{F}, w\sigma \models @_a\varphi$	*iff*	*there is* $v\sigma \in V(a)$ *such that* $\mathfrak{F}, v\sigma \models \varphi$
$\mathfrak{F}, h \models \Diamond\varphi$	*iff*	*there is* $h' \in H$ *such that* Rhh' *and* $\mathfrak{F}, h \models \varphi$
$\mathfrak{F}, w\sigma \models \blacklozenge\varphi$	*iff*	*there is* $\sigma' = \sigma(a|b) \in f$ *s.t.* $\mathfrak{F}, w\sigma' \models \varphi$*.*

The syntactic definition of $\langle a|b\rangle$ is the same as that in HSML:

$$\langle a|b\rangle\varphi := (@_a\Diamond b \wedge \blacklozenge(@_a\neg\Diamond b \wedge \varphi)) \vee (@_a\neg\Diamond b \wedge \varphi)$$

The following proposition describing its effect can easily be verified.

Fact 9 $\mathfrak{F}, w\sigma \models \langle a|b\rangle\varphi$ *iff*

(a) $\sigma(a|b) \in f$ *and* $\mathfrak{F}, w\sigma(a|b) \models \varphi$*, or (b)* $\mathfrak{F}, w\sigma \models @_a\neg\Diamond b \wedge \varphi$

A Hilbert-style proof system for Protocol HSML is presented in Table 2. The difference with the axiom system **HSML** is that deletions are no longer freely available, so we need to modify some of the recursion axioms for named link cuts. For instance, the earlier axiom $\langle a|b\rangle p \leftrightarrow p$ now becomes

$$\langle a|b\rangle p \leftrightarrow \langle a|b\rangle\top \wedge p$$

Axioms and rules for basic hybrid modal logic

See Table 1

K axiom for $\blacksquare$ & Necessitation Rule for $\blacksquare$

See Table 1

Invariance axiom for $\langle a|b\rangle$

$@_c\langle a|b\rangle\top \leftrightarrow \langle a|b\rangle\top$

Recursion axioms for $\langle a|b\rangle$

$\langle a|b\rangle c \leftrightarrow \langle a|b\rangle\top \wedge c \quad \langle a|b\rangle p \leftrightarrow \langle a|b\rangle\top \wedge p \quad \langle a|b\rangle\neg\varphi \leftrightarrow \neg\langle a|b\rangle\varphi$

$\langle a|b\rangle\neg\phi \leftrightarrow \langle a|b\rangle\top \wedge \neg\langle a|b\rangle\phi \quad \langle a|b\rangle @_c\varphi \leftrightarrow @_c\langle a|b\rangle\varphi$

$\langle a|b\rangle\Diamond\varphi \leftrightarrow ((a \wedge \Diamond(\neg b \wedge \langle a|b\rangle\varphi)) \vee (\neg a \wedge \Diamond\langle a|b\rangle\varphi))$

Inference rule for $\blacklozenge$

$$(\text{B-}Mix): \ \frac{@_c\overline{\langle a|b\rangle}_n(@_{a_{n+1}}\Diamond b_{n+1} \wedge \langle a_{n+1}|b_{n+1}\rangle\varphi) \to \theta}{@_c\overline{\langle a|b\rangle}_n\blacklozenge\varphi \to \theta}$$

where $n \geq 0$; the new nominals a_{n+1}, b_{n+1} are distinct from c and other nominals in $\overline{\langle a|b\rangle}_n$ and do not occur in φ or θ.

Table 2. The Hilbert-style proof system for protocol HSML

which also contains irreducible protocol information about available deletions.[7] In addition, the system contains a new principle expressing that the protocol is 'uniform': the available deletions are the same at each world:

$$@_c\langle a|b\rangle\top \leftrightarrow \langle a|b\rangle\top$$

Theorem 3. *Protocol HSML is strongly complete.*

Proof. The completeness proof follows the same pattern as our earlier one for **HSML**. We merely sketch some salient steps that require attention.

For a start, the Lindenbaum Lemma can be proved just as before. With a little more care, we can also still have the earlier named models:

Definition 11. *We say that $\Delta_a = \{\varphi \mid @_a\varphi \in \Gamma\}$ is the* protocol named set yielded by Γ. *A named model is a tuple $(\mathfrak{M}^\Gamma, f^\Gamma) = ((W^\Gamma, R^\Gamma, V^\Gamma), f^\Gamma)$ where*

(a) W^Γ is the set of all named set yielded by Γ
(b) $R^\Gamma uv$ iff for all formulas φ, $\varphi \in v$ implies $\Diamond\varphi \in u$

[7] The modified recursion axioms allow new situations. E.g., $\neg\langle a|b\rangle p \wedge \neg\langle a|b\rangle\neg p$ is not satisfiable in HSML, but in Protocol HSML it is true in a model where $(a, b) \notin f$.

(c) $V^\Gamma(o) = \{w \in W^\Gamma \mid o \in w, o \in \mathsf{Prop} \cup \mathsf{Nom}\}$
(d) $f^\Gamma = \{\overline{(a|b)}_n : \overline{\langle a|b\rangle}_n \top \wedge \bigwedge_{i=0}^{n-1} \overline{\langle a|b\rangle}_i @_{a_{i+1}} \Diamond b_{i+1} \in \Gamma\}$.

Here the condition $\bigwedge_{i=0}^{n-1} \overline{\langle a|b\rangle}_i @_{a_{i+1}} \Diamond b_{i+1} \in \Delta_c$ picks out all those sequences of link cuts admissible according to Γ which do not include any vacuous cuts.

Lemma 4. *If* $\overline{\langle a|b\rangle}_n \top \in \Gamma$*, then there is* $\sigma = \overline{(c|d)}_m \in f^\Gamma$ *s.t. for all* φ, $m \leq n$*:*
(a) $\overline{\langle c|d\rangle}_m \varphi \in \Gamma$ *iff* $\overline{\langle a|b\rangle}_n \varphi \in \Gamma$*,* *(b)* $\mathfrak{F}, w \models \overline{\langle c|d\rangle}_m \varphi$ *iff* $\mathfrak{F}, w \models \overline{\langle a|b\rangle}_n \varphi$

Proof. We can get the sequence $\overline{(c|d)}_m$ from $\overline{(a|b)}_n$ by deleting all those pairs $(a_i|b_i)$ for which $\overline{\langle a|b\rangle}_i @_{a_{i+1}} \Diamond b_{i+1} \notin \Gamma$.

Next comes a slightly different route from the completeness proof for **HSML**.

Lemma 5. *For all formulas* φ*, finite sequences* $\sigma = \overline{(a|b)}_n \in f^\Gamma$ *and points* w *in the generated protocol named model* $\mathfrak{F} = \mathsf{Forest}(\mathfrak{M}, f)$ *yielded by* Γ*,*

$$\mathfrak{F}, w\sigma \models \varphi \text{ iff } \overline{\langle a|b\rangle}_n \varphi \in w$$

Proof. The proof is by induction on the formulas φ.

(a) *Atomic propositions and nominals.* Given that $\sigma \in f$, which implies that $\overline{\langle a|b\rangle}_n \top \in w$, we have $\mathfrak{F}, w\sigma \models p$ iff $w\sigma \in V(p)$ iff $p \in w$ iff $\overline{\langle a|b\rangle}_n p \in w$.
The case of nominals is similar.

(b) *Negations.* Since $\overline{\langle a|b\rangle}_n \top \in w$, we can use the modified recursion axiom to get $\mathfrak{F}, w\sigma \models \neg\psi$ iff $\mathfrak{F}, w \not\models \psi$ iff $\overline{\langle a|b\rangle}_n \psi \notin w$ iff $\neg\overline{\langle a|b\rangle}_n \psi \in w$ iff $\overline{\langle a|b\rangle}_n \neg\psi \in w$.

As for further inductive steps, the cases for the operators $\wedge$, $@_c$, $\Diamond$ and $\blacklozenge$ are similar to those in the proof of the Truth Lemma for HSML in Section 3, using the derived Henkin Model, since the recursion axioms for these operators have not changed. Here we treat @ and $\blacklozenge$ as examples.

(c) We have the following equivalences: $\mathfrak{F}, w\sigma \models @_c\psi$ iff $\mathfrak{F}, \Delta_c\sigma \models \psi$ iff $\overline{\langle a|b\rangle}_n \psi \in \Delta_c$ iff $@_c\overline{\langle a|b\rangle}_n \psi \in w$ iff $\overline{\langle a|b\rangle}_n @_c\psi \in w$.

(d) First, assume that $\mathfrak{F}, w\sigma \models \blacklozenge\psi$. Then there is $\sigma' = \sigma(a|b)_{n+1} \in f$ such that $\mathfrak{F}, w\sigma' \models \psi$. Thus by the inductive hypothesis $\overline{\langle a|b\rangle}_{n+1}\psi \in w$. Since $\mathfrak{F}, w\sigma \models @_{a_{n+1}} \Diamond b_{n+1}$, by the cases we have proved, it follows that $\overline{\langle a|b\rangle}_n @_{a_{n+1}} \Diamond b_{n+1} \in w$. Therefore, $\overline{\langle a|b\rangle}_n \blacklozenge\psi \in w$.

Next, assume that $\overline{\langle a|b\rangle}_n \blacklozenge\psi \in w$. By the Existence Lemma, w is mixed, and so we have $\overline{\langle a|b\rangle}_{n+1}\psi \in w$ for some a_{n+1} and b_{n+1}. By definition of f, $\sigma' = \sigma(a|b)_{n+1} \in f$. By the inductive hypothesis, $\mathfrak{F}, w\sigma' \models \psi$. Therefore $\mathfrak{F}, w\sigma \models \blacklozenge\psi$.

The key Truth Lemma follows immediately from Lemma 4 and Lemma 5.

Lemma 6 (Truth Lemma). *For all formulas* φ*, finite sequences* $\sigma = \overline{(a|b)}_n$ *and points* w *in the named protocol model* $\mathfrak{F} = \mathsf{Forest}(\mathfrak{M}, f)$ *yielded by* Γ*,*

$$\mathfrak{F}, w \models \overline{\langle a|b\rangle}_n \varphi \text{ iff } \overline{\langle a|b\rangle}_n \varphi \in w$$

This finalizes the proof of the completeness theorem for Protocol HSML.

Remark: Comparing the two completeness proofs. The 'full protocol' $full(\mathfrak{M})$ for a model $\mathfrak{M}$ consists of all possible histories of link cuts. The derived Henkin model of Definition 7 in the completeness proof for HSML is in essence the full protocol model of the named model of Definition 6. The difference is only notational: a history $w\sigma$ in the full protocol model of $\mathcal{M}$ is attached to the model, becoming one of its pointed derived Henkin models $\mathcal{M} : \sigma, w$. Also, the truth conditions of $\Diamond$, $\blacklozenge$ in the full protocol model $\mathsf{Forest}(\mathfrak{M}, full(\mathfrak{M})), w\sigma$ are as in the derived Henkin model $\mathcal{M} : \sigma, w$. Thus one could also start with a completeness proof in the format that we have given here for protocol models, and then derive one for standard models as a special case.

Discussion: Reducing HSML and Protocol HSML. Given the analogy in completeness proofs, it is a natural question how HSML and Protocol HSML are related. For instance, can one find a formula φ' for every formula φ such that φ is satisfiable in HSML iff φ' is satisfiable in protocol HSML? One might think of such a formula φ' as a conjunction of the form

$$\bigwedge_{a,b \in Q} \overline{\langle a|b \rangle}_n \top \wedge \varphi$$

The first conjunct says that all link cuts explicitly involved in φ are admissible, where Q is the set of all nominals that occur in φ, possibly plus some new ones. The problem, however, is that not all relevant link cuts need be explicitly stated in a given formula φ, as illustrated in the following example.

$$\varphi := \Diamond \top \ \wedge \ \neg \blacklozenge p \ \wedge \ \neg \blacklozenge \neg p$$

is not satisfiable in HSML. Since there are no nominals in the formula φ, φ' would equal φ by the above method. However, φ' is satisfiable in protocol HSML.

Are there better reductions? And what about the opposite direction, from Protocol HSML to HSML? While we believe that mutual reductions indeed exist for dynamic-epistemic PAL and Protocol PAL, we are not sure that they extend to sabotage logics, and hence leave these matters as open problems.

5 Comparing link deletion and point deletion

A natural companion to link or edge deletion in graphs is deletion of vertices or points. The modal logic MLSR for stepwise point deletion of [22], mentioned in the introduction as the inspiration for our completeness proof, adds a modality $\langle -\varphi \rangle \psi$ for stepwise world removal to the basic hybrid modal logic:

$$\varphi ::= a \mid p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid @_a\varphi \mid \langle -\varphi \rangle \varphi$$

Formulas $\langle -\varphi \rangle \psi$ have the following truth condition in models $\mathfrak{M} = (W, R, V)$:

$$\mathfrak{M}, s \models \langle -\varphi \rangle \psi \ \text{ iff } \text{ there is a } t \neq s \text{ with } \mathfrak{M}, t \models \varphi \text{ and } \mathfrak{M} - \{t\}, s \models \psi$$

In MLSR, the universal modality is definable as follows: $U\varphi := \varphi \wedge \neg\langle - \neg\varphi\rangle\top$, so it is freely available in our later proofs. It follows that the hybrid notion $@_a\varphi$ is also definable, although this notation is used as primitive in MLSR for greater perspicuity of its proof system.

5.1 From link deletion to point deletion

Intuitively, deleting links can be simulated by deleting points in models of the right kind. We will make this precise by embedding the logic HMSL into MLSR.

Consider the following fragment of the language of MLSR, with atomic propositions from $\mathsf{Prop} \cup \{i\}$ including a distinguished proposition letter i, and Nom[8]:

$$\varphi ::= a \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond(i \wedge \Diamond\varphi) \mid @_a\varphi \mid \langle -(i \wedge \Diamond\neg i)\rangle\varphi$$

We can translate the language of HSML into this fragment of MLSR.

Definition 12 (Translation I). *Here is the* HSML-to-MLSR translation*:*

(a) $\mathsf{Tr}(a) = a,\quad \mathsf{Tr}(p) = p,\quad \mathsf{Tr}(\neg\varphi) = \neg\mathsf{Tr}(\varphi),\quad \mathsf{Tr}(\varphi \wedge \psi) = \mathsf{Tr}(\varphi) \wedge \mathsf{Tr}(\psi)$

(b) $\mathsf{Tr}(@_a\varphi) = @_a\mathsf{Tr}(\varphi),\quad \mathsf{Tr}(\Diamond\varphi) = \Diamond(i \wedge \Diamond\mathsf{Tr}(\varphi)),\quad \mathsf{Tr}(\blacklozenge\varphi) = \langle -(i \wedge \Diamond\neg i)\rangle\mathsf{Tr}(\varphi).$

Next, we define models for MLSR where this translation makes sense.

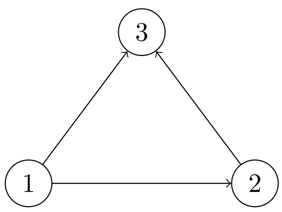
Definition 13 (Transformed models I). *Given a model* $\mathfrak{M}_0 = (W_0, R_0, V_0)$ *for HSML, the model* $\mathcal{F}(\mathfrak{M}_0) = (W, R, V)$ *for MLSR is defined as follows:*

(a) $W = W_0 \cup W_i$ *where* $W_i = \{(w, v, i) \mid (w, v) \in R_0$ *and* $w, v \in W_0\}$

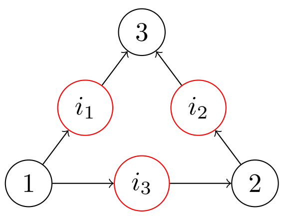
(b) $R = \{(w, (w, v, i)), ((w, v, i), v) \mid (w, v) \in R_0\}$

(c) $V : \mathsf{Nom} \cup \mathsf{Prop} \cup \{i\} \to W$ *is a valuation function such that* $V(o) = V_0(o)$ *for* $o \in \mathsf{Prop} \cup \mathsf{Nom}$ *and* $V(i) = W_i$*.*

Example 1. In the graph depicted below, $\mathcal{F}(\mathfrak{M}_0)$ is the transformed model of $\mathfrak{M}_0$. The link $(1,3)$ is represented by i_1 in the transformed model. The sentence 'I can travel from 1 to 3' can be faithfully translated as 'I can first travel from 1 to i_1, and then to 3', while deleting the link $(1,3)$ can be faithfully represented as deleting a point in the model $\mathcal{F}(\mathfrak{M}_0)$, namely, the node i_1.



$\mathfrak{M}_0$



$\mathcal{F}(\mathfrak{M}_0)$

Now we have the following result connecting the two languages.

[8] For the purpose of this section, we can use $\langle -i\rangle\varphi$ rather than $\langle -(i \wedge \Diamond\neg i)\rangle\varphi$ in the language. But the result in Appendix B needs the latter form which can specify more information about the models.

Fact 10 *For any formula φ in the language of HSML and any model $\mathfrak{M}_0$,*

$$\mathfrak{M}_0, w \models \varphi \text{ iff } \mathcal{F}(\mathfrak{M}_0), w \models \mathsf{Tr}(\varphi),$$

where $w \in W_0$ and $\mathcal{F}(\mathfrak{M}_0)$ is constructed from $\mathfrak{M}_0$ as in the preceding definition.

A result on equivalence of validities in the two logics follows immediately.

Corollary 1. *We have the following equivalence:*

$$\models_{\mathcal{C}} \varphi \text{ iff } \models_{\mathcal{F}(\mathcal{C})} \neg i \to \mathsf{Tr}(\varphi)$$

where $\mathcal{C}$ denotes the class of all models for HMSL, while $\mathcal{F}(\mathcal{C})$ denotes the class of all MLSR models constructed from these.

However, this result does not tell us that we can embed HMSL into the logic MLSR as it stands over arbitrary models: for that, we need to define the special class $\mathcal{F}(\mathcal{C})$ in the language of MLSR.

This can be done, with the caveat that we need extend the language of MLSR by adding the reverse operator $\Diamond^{-1}$ as in temporal logic:

$\mathfrak{M}, s \models \Diamond^{-1}\psi$ iff $\mathfrak{M}, t \models \psi$ for some t with Rts.

In this extended language for MLSR, that we will denote by MLSR$^+$, we can define the special class $\mathcal{F}(\mathcal{C})$ using the formulas listed in Table 3.

Table 3. Defining $\mathcal{F}(\mathcal{C})$ in MLSR$^+$

a. $i \to (\Diamond\neg i \wedge \langle - \top\rangle\neg\Diamond\top)$ 1. an i point has exactly one successor, which is a $\neg i$ point
b. $i \to (\Diamond^{-1}\neg i \wedge \langle - \top\rangle\neg\Diamond^{-1}\top)$ 2. an i point has exactly one predecessor, which is a $\neg i$ point
c. $\neg i \to \Box\, i$ 3. if a $\neg i$ point has successors, then they are i points
d. $\neg i \wedge \Diamond\top \to [-\neg i]\langle - \top\rangle\Box\Diamond\top$ 4. if a $\neg i$ point has two or more different i successors, then these cannot have the same successor.

Proposition 1.

$$\mathfrak{M} \in \mathcal{F}(\mathcal{C}) \text{ iff } \mathfrak{M} \models_{\mathrm{MLSR}^+} A$$

where A is the conjunction of the four MLSR formulas in Table 3.

Proof. For a start, note that all the listed properties a,b,c,d in Table 3 hold for all models in $\mathcal{F}(\mathcal{C})$. Next, it is easy to see that, for any model $\mathfrak{M}$, $\mathfrak{M} \models$ a iff it satisfies property 1, and likewise for b and 2, and c and 3. Given this, it suffices to focus on establishing the following claim:

If $\mathfrak{M}$ satisfies properties 1, 2 and 3, $\mathfrak{M} \in \mathcal{F}(\mathcal{C})$ iff $\mathfrak{M} \models$ d.

From left to right, assume that $\mathfrak{M} \in \mathcal{F}(\mathcal{C})$. Given any point w in $\mathfrak{M}$ with $\mathfrak{M}, w \models \neg i \wedge \Diamond\top$, we prove that $\mathfrak{M}, w \models [-\neg i]\langle -\top\rangle\Box\Diamond\top$. When deleting any $\neg i$ point $v \neq w$, there are two cases. Case 1: The deleted $\neg i$ point $v \neq w$ is the successor of some i successor of w. In this case, by deleting this i-predecessor of v, since $\mathfrak{M} \in \mathcal{F}(\mathcal{C})$, all other i-successors of w must have a $\neg i$ successor. Case 2: Otherwise, deleting any i point suffices to keep $\Box\Diamond\top$ true at w.

From right to left, given a model $\mathfrak{M}$ which satisfies properties 1, 2 and 3, but lacks 4, assume that some $\neg i$ world w has two i successors s and t which share the same successor v. By the assumption then $\mathfrak{M}, w \models \neg i \wedge \Diamond\top$. Moreover, v is a $\neg i$ world and the unique successor of both s and t because $\mathfrak{M}$ has properties 1, 2. Next we prove that $\mathfrak{M}, w \models \langle -\neg i\rangle[-\top]\Diamond\Box\bot$. After deleting the world v, w has two successors s and t without a successor. Thus no matter which world we choose to delete($[-\top]$), $\Diamond\Box\bot$ is satisfied at w. It follows that $\mathfrak{M}, w \not\models d$.

Now we have the following result connecting the two languages.

Corollary 2. *For every formula φ in the language of HSML,*

$$\models_{\mathrm{HSML}} \varphi \;\; \textit{iff} \;\; \models_{\mathrm{MLSR}^+} UA \to (\neg i \to \mathsf{Tr}(\varphi))$$

Note that here we add the universal operator U in front of A, because UA rather than A can make sure the model of MLSR$^+$ which refutes $UA \to (\neg i \to \mathsf{Tr}(\varphi))$ *at a certain world*, according to Proposition 1, is a model in $\mathcal{F}(\mathcal{C})$.

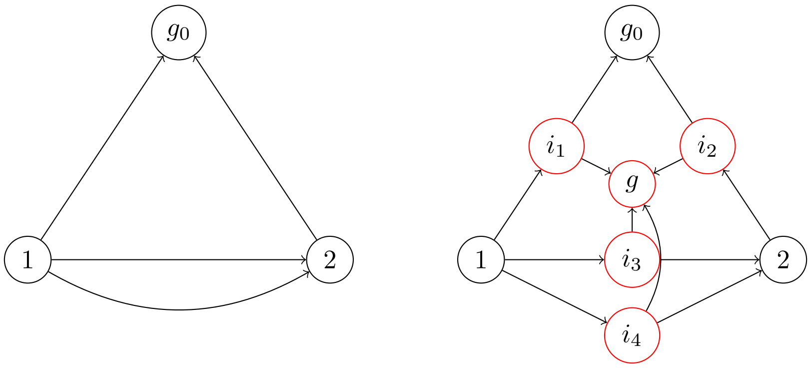
In Appendix B, we present a slightly more complex method for obtaining an analogue to Corollary 2 which needs no extension of the language of MLSR.

Digression: sabotage games. The above model transformation also implies the equivalence of the multi-link version of the sabotage game with single-point destinations, [17], and the single-link version with multiple destinations. This result first appeared in Lemma 1 of [13]. We flesh out the details of its proof to show how it relates to the above embedding result.

Fact 11 *Let* Ind *be an arbitrary set of individuals, and let the map of a multi-link version sabotage game be* $\mathfrak{M}_0 = (W_0, R_0^i, V_0)$*. Then the Traveler has a winning strategy starting at* $w \in W_0$ *in* $\mathfrak{M}_0$ *iff the Traveler has a winning strategy on* w *in* $\mathcal{F}(\mathfrak{M}_0) = (W, R, V)$ *where* $W = \{g\} \cup W_0 \cup \bigcup_{i\in\mathsf{Ind}}\{(w, v, i) \mid (w, v) \in R_0^i \text{ and } w, v \in W_0\}$*,* $R = \{(w, (w, v, i)), ((w, v, i), v), ((w, v, i), g) \mid (w, v) \in R_0^i\}$ *and the valuation function* V *is the same as the old* V_0 *except that it makes the new node* g *one of the goals.*

Before giving the proof, we first illustrate the definition of $\mathcal{F}(\mathfrak{M}_0)$ in the fact and how it works in the proof by the following example.

Example 2. In the graph below, the Traveler has a winning strategy on a node in $\mathfrak{M}_0$ iff the Traveler has a winning strategy on the corresponding node in $\mathcal{F}(\mathfrak{M}_0)$. The added goal point g_1 plays a crucial role here. The upper $(1,2)$ link is represented by the three links $(1,i_3)$, $(i_3,2)$, (i_3,g) in the transformed model in the sense that the Traveler can still move from 1 to 2 as long as the three links all remain untouched, while such a travel is no longer possible once at least one of the three links has been deleted.



$\mathfrak{M}_0$ $\qquad$ $\mathcal{F}(\mathfrak{M}_0)$

Proof. To start our proof, for $(a,b)\in R$, we define this mapping f into $\bigcup_i R_0^i$:

$$f((a,b))=\begin{cases}(u,v)_i \text{ if } & (a,b)=(u,(u,v,i))\\ (u,v)_i \text{ if } & (a,b)=((u,v,i),v)\\ (u,v)_i \text{ if } & (a,b)=((u,v,i),g)\end{cases}$$

We show that deleting a link (a,b) in $\mathcal{F}(\mathfrak{M}_0)$ is equivalent to deleting $f((a,b))$ in $\mathfrak{M}$, while deleting $(u,v)_i$ in $\mathfrak{M}$ is equivalent to deleting any (a,b) such that $f((a,b))=(u,v)_i$ in $\mathcal{F}(\mathfrak{M}_0)$. More precisely, Traveler can go from u to v through R_i in $\mathfrak{M}$ iff Traveler can go from u to v via (u,v,i) in $\mathcal{F}(\mathfrak{M}_0)$. Based on the mapping f defined above, both Traveler and Demon can derive winning strategies in one model from winning strategies in the other. We have two cases.

Case 1. If Traveler can move from u to v by R_i in $\mathfrak{M}_0$, then $(u,v)_i$ was not deleted. So in $\mathcal{F}(\mathfrak{M}_0)$, $(u,(u,v,i))$, $((u,v,i),v)$, $((u,v,i),g)$ are all not deleted. On u, Traveler first goes to (u,v,i) through the link $(u,(u,v,i))$. Then Demon has to delete the link $((u,v,i),g)$, or Traveler will win immediately. Traveller now moves to v through the link $((u,v,i),v)$.

Case 2. If Traveler cannot go from u to v by R_i in $\mathfrak{M}$, then $(u,v)_i$ was deleted. So in $\mathcal{F}(\mathfrak{M}_0)$, at least one of $(u,(u,v,i))$, $((u,v,i),v)$, $((u,v,i),g)$ was deleted. If $(u,(u,v,i))$ or $((u,v,i),v)$ was deleted, then there is no path from u to (u,v,i) to v. If $((u,v,i),g)$ was deleted, then Demon can cut $((u,v,i),v)$ when Traveler reaches (u,v,i). Therefore, Traveler can no longer go to v through (u,v,i).

5.2 From point deletion to link deletion

At this point, it is natural to seek a converse system embedding, adding a converse modality $\Diamond^{-1}$ to HSML to match the extension we made for MLSR. However, there is a mismatch here, since point removal in MLSR refers to a formula, while link deletion is arbitrary in HSML. Still, an embedding may be obtained by generalizing the link deletion operator $\blacklozenge$ of HSML to a conditional version.

More precisely, we embed MLSR$^+$ into an extended logic HSML$^+$, whose language is as follows with $a \in \mathsf{Nom}$, $p \in \mathsf{Prop}$ and e a special nominal:

$$\varphi ::= a \mid e \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Diamond^{-1}\varphi \mid \blacklozenge^{\varphi}_{\varphi}\varphi$$

The truth condition for $\blacklozenge^{\varphi}_{\psi}\chi$ reads: [9]

$\mathfrak{M} = (W, R, V), w \models \blacklozenge^{\varphi}_{\psi}\chi$ iff there is a pair $(s,t) \in R$ such that $\mathfrak{M}, s \models \varphi$, $\mathfrak{M}, t \models \psi$ and $(W, R \setminus (s,t), V), w \models \chi$.

Note that this covers the original sabotage modality as the special case $\blacklozenge^{\top}_{\top}\chi$. However, even using its named deletions, HSML$^+$ cannot define the universal modality $U\varphi$. But we do have the following restricted version

$$\forall\varphi := \neg\blacklozenge^{\neg\varphi}_{\top}\top \wedge \neg\blacklozenge^{\top}_{\neg\varphi}\top$$

This says that no links can be cut to or from $\neg\varphi$-worlds (such worlds will be called *isolated*), or in other words, all worlds which are in a R-relation with some world satisfy φ. This notion will suffice for our later purposes.

Next, the special nominal e is key to making point deletions in an MLSR$^+$ model become link deletions in a matching HSML$^+$ model.

Definition 14 (Transformed models II). *Given a model $\mathfrak{M}_0 = (W_0, R_0, V_0)$ for* MLSR$^+$*, the model $\mathcal{G}(\mathfrak{M}_0) = (W, R, V)$ for* HSML$^+$ *is defined as follows:*

(a) $W = W_0 \cup \{w_e\}$

(b) $R = R_0 \cup \{(w, w_e)) \mid w \in W_0\}$

(c) $V : \mathsf{Nom} \cup \mathsf{Prop} \cup \{e\} \to W$ *is a valuation function such that* $V(o) = V_0(o)$ *for* $o \in \mathsf{Prop} \cup \mathsf{Nom}$ *and* $V(e) = \{w_e\}$.

The following translation makes the effect of deleting a point v in $\mathfrak{M}_0$ equivalent to the effect of deleting the corresponding link (v, w_e) in $\mathcal{F}(\mathfrak{M}_0)$.

Definition 15 (Translation II). *The* MLSR$^+$-to-HSML$^+$ translation *is this:*

(a) $\mathbf{Tr}(o) = o$, $\quad \mathbf{Tr}(\neg\varphi) = \neg\mathbf{Tr}(\varphi)$, $\quad \mathbf{Tr}(\varphi \wedge \psi) = \mathbf{Tr}(\varphi) \wedge \mathbf{Tr}(\psi)$,

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[9] $\blacklozenge^{\varphi}_{\phi}\psi$ cannot be defined in the language of HSML. Let $M_1 = (W_1, R_1, V_1)$ with $W_1 = \{w_1, w_2\}, R_1 = \{(w_1, w_1), (w_2, w_2)\}, V(p) = \{w_2\}, V(a) = \{w_1\}$ for any $a \in \mathsf{Nom}$. $M_2 = (W_2, R_2, V_2)$ with $W_2 = \{v_1, v_2\}, R_2 = \{(v_1, v_1), (v_2, v_2)\}, V(p) = \emptyset, V(a) = \{v_1\}$ for any $a \in \mathsf{Nom}$. It is easy to see that there is an HSML-style bisimulation (cf. Appendix A) between (M_1, w_1) and (M_2, v_1), and so each formula α is true at w_1 iff α is true at v_1. However, $M_1, w_1 \vDash \blacklozenge^{a}_{\top}\blacklozenge^{p}_{\top}a$, $M_2, v_1 \nvDash \blacklozenge^{a}_{\top}\blacklozenge^{p}_{\top}a$.

(b) $\mathbf{Tr}(\Diamond\varphi) = \Diamond(\Diamond e \wedge \mathbf{Tr}(\varphi))$, $\quad \mathbf{Tr}(\Diamond^{-1}\varphi) = \Diamond^{-1}(\Diamond e \wedge \mathbf{Tr}(\varphi))$,

(c) $\mathbf{Tr}(@_a\varphi) = @_a(\Diamond e \wedge \mathbf{Tr}(\varphi))$, $\quad \mathbf{Tr}(\langle -\varphi\rangle\psi) = \blacklozenge_e^{\mathbf{Tr}(\varphi)\wedge\Diamond e}(\Diamond e \wedge \mathbf{Tr}(\psi))$

This translation *relativizes* the operators in MLSR$^+$-formulas φ syntactically to refer only to those worlds in $\mathcal{G}(\mathfrak{M}_0)$ that satisfy either $\Diamond e$ or e. An easy induction on formulas implies the following semantic invariance property:

Proposition 2. *For all HSML-models $\mathfrak{N}$, and all worlds w satisfying $\Diamond e$, we have that $\mathfrak{N}, w \models \mathbf{Tr}(\varphi)$ iff $\mathfrak{N} \mid (\Diamond e \vee e), w \models \mathbf{Tr}(\varphi)$, where $\mathfrak{N} \mid (\Diamond e \vee e)$ is the submodel of $\mathfrak{N}$ consisting of all worlds that satisfy $\Diamond e \vee e$.*

In particular, once a link R from a world v to w_e has been deleted in a model $\mathcal{G}(\mathfrak{M}_0)$, v falls outside of the relativized model $\mathcal{G}(\mathfrak{M}_0) \mid (\Diamond e \vee e)$ and plays no role any more in the evaluation of translated formulas. Thus, the effect on such formulas is the same as if the world v had been deleted.

These observations are the key to the following result.

Fact 12 *For any formula φ of* MLSR$^+$*, any model $\mathfrak{M}_0$ and world $w \in W_0$,*

$$\mathfrak{M}_0, w \models \varphi \text{ iff } \mathcal{G}(\mathfrak{M}_0), w \models \mathbf{Tr}(\varphi),$$

where $\mathcal{G}(\mathfrak{M}_0)$ is constructed from $\mathfrak{M}_0$ as in Definition 14.

Proof. The proof is by induction, and we only sketch the crucial case of the point-deletion modality. Recall that $\mathfrak{M}_0, s \models \langle -\varphi\rangle\psi$ iff there exists a world $t \neq s$ such that $\mathfrak{M}_0, t \models \varphi$ and $\mathfrak{M}_0 - \{t\}, s \models \psi$, where $\mathfrak{M}_0 - \{t\}$ is the submodel of $\mathfrak{M}_0$ in which the world t has been deleted. By the inductve hypothesis, we have that (a) $\mathcal{G}(\mathfrak{M}_0), t \models \mathbf{Tr}(\varphi)$, and (b) $\mathcal{G}(\mathfrak{M}_0 - \{t\}), s \models \mathbf{Tr}(\psi)$. To see that the formula $\mathbf{Tr}(\langle -\varphi\rangle\psi)$ as defined above is true at s in $\mathcal{G}(\mathfrak{M}_0)$, we cut the link from t to e, and need to have $\mathbf{Tr}(\psi)$ true at s. However, this follows from (b) above plus Proposition 2, since $\mathcal{G}(\mathfrak{M}_0 - \{t\})$ equals the relativization of the model $\mathcal{G}(\mathfrak{M}_0)$ after the link cut from t to e to only those worlds that satisfy $\Diamond e \vee e$.

Our remaining task is to suitably define the class of models $\mathcal{G}(\mathcal{C}) = \{\mathcal{G}(\mathfrak{M}_0) \mid \mathfrak{M}_0 \in \mathcal{C}\}$, where $\mathcal{C}$ is the class of all models for MLSR$^+$. We start with two simple auxiliary observations about the defining HSML$^+$-formula (where we recall that our special universal modality $\forall$ ranges only over non-isolated points).

Proposition 3. *For any* HMSL$^+$ *model $\mathfrak{N}$,*

(a) If $\mathfrak{N} \in \mathcal{G}(\mathcal{C})$, then $\mathfrak{N}, w \models \forall((\neg e \to \Diamond e) \wedge (e \to \Box\bot))$

(b) Let $\mathfrak{N} - \mathsf{ISO}$ be the model obtained by removing all isolated worlds in $\mathfrak{N}$. If, for some world w, $\mathfrak{N}, w \models \forall((\neg e \to \Diamond e) \wedge (e \to \Box\bot))$, then $\mathfrak{N} - \mathsf{ISO} \in \mathcal{G}(\mathcal{C})$.

We now obtain the following reduction from MLSR$^+$ to HMSL$^+$.

Fact 13 *For each formula φ in the language of* MLSR$^+$*,*

$$\models_{\text{MLSR}^+} \varphi \text{ iff } \models_{\text{HSML}^+} \forall((\neg e \to \Diamond e) \wedge (e \to \Box\bot)) \to (\neg e \wedge \Diamond e \to \mathsf{Tr}(\varphi))$$

Proof. From right to left, this is straightforward. Suppose that the stated formula is valid in HSML$^+$, and let $\mathfrak{M}, s$ be any pointed model for MLSR$^+$. By Proposition 3.(a), we have that the HSML$^+$-model $\mathcal{G}(\mathfrak{M}) \models \forall((\neg e \to \Diamond e) \wedge (e \to \Box\bot))$. By the definition of the mapping $\mathcal{G}$, we have that s satisfies $\neg e \wedge \Diamond e$. It follows from the assumption that $\mathcal{G}(\mathfrak{M}) \models \mathbf{Tr}(\varphi)$, and so by Fact 12, $\mathcal{M}, s \models \varphi$.

From left to right, we argue by contraposition. Suppose that some HSML$^+$-model $\mathfrak{N}$ and world s make the following formulas true: (a) $\forall((\neg e \to \Diamond e) \wedge (e \to \Box\bot))$, (b) $\neg e$, (c) $\Diamond e$, and (d) $\neg\mathbf{Tr}(\varphi)$. Now remove all isolated points from $\mathfrak{N}$ to obtain the model $\mathfrak{N}$-ISO. It is easy to verify that, in this model, (a), (b) and (c) above still hold at s. [In particular, given the assumptions, neither s nor e are isolated points, so they stay in.] But (d) remains true as well, by an appeal to Proposition 2. The reason is that, given the truth of (a), $\mathfrak{N}$-ISO equals the relativized model $\mathfrak{N} \mid (\Diamond e \vee e)$. But then, finally, Proposition 3.(b) gives us an MLSR$^+$-model $\mathfrak{M}$ with $\mathcal{G}(\mathfrak{M}) = \mathfrak{N}$-ISO where φ is false at s.

Finally, we close the circle of our two system embeddings so far by showing that the extended language HSML$^+$ can also be embedded into MLSR$^+$. One just extends the translation function in Definition 12 by adding the two clauses

$$\mathsf{Tr}(\Diamond^{-1}\varphi) = \Diamond^{-1}(i \wedge \Diamond^{-1}\mathsf{Tr}(\varphi))$$
$$\mathsf{Tr}(\blacklozenge^{\varphi}_{\psi}\chi) = \langle -(i \wedge \Diamond^{-1}(\neg i \wedge \mathsf{Tr}(\varphi)) \wedge \Diamond(\neg i \wedge \mathsf{Tr}(\psi))\rangle \mathsf{Tr}(\chi)$$

It is not hard to verify that we have the following new result:

Proposition 4. *For any formula φ in the language of* HSML$^+$*,*

$$\models_{\mathrm{HSML}^+} \varphi \;\; \textit{iff} \;\; \models_{\mathrm{MLSR}^+} UA \to (\neg i \to \mathsf{Tr}(\varphi))$$

Thus, we can embed slight extensions of HSML into matching extensions of MLSR and vice versa. This gives substance to the intuition that point deletion and link deletion are closely related in a logical perspective.[10] We leave obtaining sharper and more parsimonious reductions as an open problem.[11]

6 Conclusion

We have axiomatized the logic HSML of arbitrary link deletion in a Hilbert-style format using modest hybrid additions to the original language of sabotage modal logic which allow for defining an auxiliary companion modality of named link

[10] Point deletion and arrow deletion are also close in *Arrow Logic*, [16], [24], which treats arrows as objects representing transitions, but we have not been able to establish a precise connection between this research line and our logics of graph change.

[11] We have encountered quite a few forms of deletion by now. PAL deletes all points satisfying a certain property, and the counterpart for this is the DEL-style logic of uniform definable link cutting in Appendix A. One can also delete definable objects or definable links stepwise, as we have analyzed here. As a specialization of this, there is deletion of arbitrary points or links, or just individual named objects or links. We leave a comparison of the latter variants to further study.

deletion simplifying the proof system. In addition, we have used our setting to provide mutual reductions between existing modal logics of point deletion and link deletion that suggest more unity to logics of graph-changing games than might have been apparent at first sight.

We believe that the technique of axiomatization via a companion modality definable in the logic, which simplifies the one in [22] to which our general treatment remains indebted, can be applied to many further logics of graph change in the literature, and also to further kinds of semantics beyond the protocol models whose logic we have axiomatized.

In our view, two major open problems remain for judging the virtues of working with HSML. A first concern are the *schematically valid* formulas of HSML that remain valid under substitution of arbitrary formulas for atomic formulas. Most, but not all of the principles in our axiomatizations were schematically valid: in particular, the recursion axiom for proposition letters was not. [10] axiomatizes the schematically valid formulas of public announcement logic using an abstract poly-modal semantics with modal and dynamic accessibility relations which can be seen as a generalization of our protocol models, cf. also [25]. We believe that our approach lifts to such a setting, but this needs to be verified.

Another major question concerning HSML (also open for its parent logic SML) is an *interpolation theorem*. It is not hard to see that the proof techniques for hybrid logic in [1] extend to HSML, but they require adding downarrow binders to our language which lack motivation in the setting of graph games. Whether we can do without them is an open problem at the present stage.

Finally, as already suggested in the Introduction, our results for HSML can also be seen as a case study for broader issues in dynamic-epistemic and hybrid logics. We leave these to further investigation.

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A Appendix: HSML with general link cutting

The nominal link cutting operator $\langle a|b\rangle$ in HSML sufficed for proving completeness. But dynamic-epistemic logic has complete systems for general link cutting modalities $\langle\varphi|\psi\rangle$ that describe the new model after cutting all the links $\{(w,v)\,|\,\mathfrak{M},w\models\varphi \text{ and } \mathfrak{M},v\models\psi\}$ simultaneously from a current model, [20].

Fact 14 *General definable link cutting is not definable in HSML.*

Proof. We first extend the SML-bisimulations of [4] as follows.

Definition 16 (HS-bisimulation). *Let $\mathcal{M}_1=(W_1,R_1,V_1)$, $\mathcal{M}_2=(W_2,R_2,V_2)$ be models for the language $\mathcal{L}$. A relation $Z\subseteq W_1\times W_2$ is a* HS-bisimulation *between M_1 and M_2 if the following conditions are satisfied:*

(a) for atoms: if w_1Zw_2, then $w_1\in V_1(a)$ iff $w_2\in V_2(a)$ for $a\in\mathsf{Prop}\cup\mathsf{Nom}$.
(b) forth and back conditions for $\Diamond$ are as usual.
(c) forth condition for $\blacklozenge$: if w_1Zw_2, $\mathcal{M}_1'$ is a new model obtained from $\mathcal{M}_1$ by cutting a link, then there exists a new model $\mathcal{M}_2'$ obtained from $\mathcal{M}_2$ by cutting a link such that w_1Zw_2, where Z is an HS-bisimulation between $\mathcal{M}_1'$ and $\mathcal{M}_2'$. The back condition for $\blacklozenge$ is the obvious converse.
(d) All points named by the same nominal are related by Z.

The following is easy to prove by induction on formulas.

Fact 15 *HSML-formulas are invariant for HS-bisimulations.*

Now we can give a concrete example to prove that $\langle\varphi|\psi\rangle$ cannot be defined in HSML. Let $M_1=(W_1,R_1,V_1)$ with $W_1=\{w_1,w_2,w_3\}, R_1=\{(w_2,w_3)\}, V(p)=\{w_2\}, V(a)=\{w_1\}$ for any $a\in\mathsf{Nom}$. $M_2=(W_2,R_2,V_2)$ with $W_2=\{v_1,v_2,v_3\}$, $R_2=\{(v_2,v_3)\}, V(p)=\emptyset, V(a)=\{v_1\}$ for any $a\in\mathsf{Nom}$. It is easy to see that $(M_1,w_1)Z(M_2,v_1)$, which means that any formula α is true at w_1 iff α is true at v_1. However, $M_1,w_1\nvDash\langle p|\top\rangle\blacklozenge\top$, $M_2,v_1\vDash\langle p|\top\rangle\blacklozenge\top$, which leads to a contradiction.

Adding a general link cutting operator $\langle\varphi|\psi\rangle$ to HSML yields a logic GHSML whose syntax is given by

$$\varphi ::= a \mid p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \Diamond\varphi \mid \blacklozenge\varphi \mid @_i\varphi \mid \langle\varphi|\psi\rangle\alpha$$

with $p\in\mathsf{Prop}$, $a\in\mathsf{Nom}$. Dual modal operators $\Box,\blacksquare,[\varphi|\psi]$ are defined as usual. The truth condition for $\langle\varphi|\psi\rangle$ is as follows:

$(W,R,V),w\vDash\langle\varphi_1|\varphi_2\rangle\psi$ iff $(W,R',V),w\vDash\psi$, where
$R'=R\backslash\{(w_1,w_2)\mid(W,R,V),w_i\vDash\varphi_i \text{ for } i=1,2\}$

The logic GHSML can be axiomatized in the same style as HSML, with the one difference that the link cutting modality is now a primitive of the system, for which we have recursion axioms in standard dynamic-epistemic style:

(a) $\langle\varphi|\psi\rangle a \leftrightarrow a$

(b) $\langle\varphi|\psi\rangle p \leftrightarrow p$

(c) $\langle\varphi|\psi\rangle\neg\alpha \leftrightarrow \neg\langle\varphi|\psi\rangle\alpha$

(d) $\langle\varphi|\psi\rangle(\alpha \wedge \beta) \leftrightarrow \langle\varphi|\psi\rangle\alpha \wedge \langle\varphi|\psi\rangle\beta$

(e) $\langle\varphi|\psi\rangle@_a\alpha \leftrightarrow @_a\langle\varphi|\psi\rangle\alpha$

(f) $\langle\varphi|\psi\rangle\Diamond\alpha \leftrightarrow ((\varphi \wedge \Diamond(\neg\psi \wedge \langle\varphi|\psi\rangle\alpha)) \vee (\neg\varphi \wedge \Diamond\langle\varphi|\psi\rangle\alpha))$

A completeness proof can be given for this extended proof system in the same style as the one we gave for HSML, though its details will now be closer to the completeness proof for MLSR in [22].

B Another approach to embedding HSML into MLSR

We assume the setting of Section 5, but now introduce the following notions in order to tighten up the translation provided there.

Definition 17 (Named Pseudo Transformed models). *A* named pseudo transformed model $\mathfrak{M}_P$ *is a named MLSR model in which for any* $a, b \in \mathsf{Nom}$*, the following formulas are true globally:* $i \to (\Diamond a \to \Box(a \wedge \neg i))$, $\neg i \to \Box i$, $\Diamond(b \wedge i \wedge \Diamond a) \to \neg\Diamond(\neg b \wedge i \wedge \Diamond a)$ *and* $b \wedge \Diamond(i \wedge a) \to @_b\neg i \wedge (\neg b \to \neg\Diamond a)$.

A named pseudo transformed model ('nptm', for short) is close to a transformed model, but there are some differences. In an nptm, i-worlds may have neither successors nor predecessors. But in a transformed model, an i-world must have both a $\neg i$-successor and a $\neg i$-predecessor. However, in an nptm, if an i-world has a successor or a predecessor, it must be a $\neg i$-world and unique.

Let $\mathcal{S}$ denote the class of all named pseudo transformed models.

Proposition 5. $\neg i \wedge \mathsf{Tr}(\varphi)$ *is satisfiable in a transformed model iff* $\neg i \wedge \mathsf{Tr}(\varphi)$ *is satisfiable in a named pseudo transformed model.*

Proof. First, given a transformed model $\mathcal{F}(\mathfrak{M}_0)$ and a translated formula $\mathsf{Tr}(\varphi)$ true at a $\neg i$-point w in it, by adding norminals to the language and naming all points in $\mathcal{F}(\mathfrak{M}_0)$, we get an nptm with $\mathsf{Tr}(\varphi)$ still true at w.

Next, given an nptm $\mathfrak{M}_P$, we delete all i-points without successors and then add for each i-point without predecessors a $\neg i$-point linking to it (with no restriction on how atomic propositions except for i are assigned to these new points). Let $\mathfrak{M}'_P$ be the resulting transformed model. Then we prove by formula induction that, for any formula $\neg i \wedge \mathsf{Tr}(\varphi)$ and $\neg i$-point in both $\mathfrak{M}_P$ and $\mathfrak{M}'_P$, the formula is true at w in $\mathfrak{M}_P - B$ iff it is true at w in $\mathfrak{M}'_P - B$, where B is a finite subset of i-points which are in both $\mathfrak{M}_P$ and $\mathfrak{M}'_P$.[12]

[12] Here are the key cases. (a) For $\Diamond(i \wedge \Diamond\varphi)$, note that neither adding $\neg i$-points to initial i-points nor deleting dead end i-points affects the links that make $\Diamond(i \wedge \Diamond\varphi)$ true at w in $\mathfrak{M}_P$. Therefore, by the inductive hypothesis, $\Diamond(i \wedge \Diamond\varphi)$ is also true at w in $\mathfrak{M}'_P$. From $\mathfrak{M}'_P$ to $\mathfrak{M}_P$, the same argument applies. (b) For $\langle -i \wedge \Diamond\neg i\rangle\varphi$, note that neither the deleted points nor the added points satisfy $i \wedge \Diamond\neg i$, so when evaluating the formula $\langle -i \wedge \Diamond\neg i\rangle\varphi$, we can always delete the same points in $\mathfrak{M}_P$ and $\mathfrak{M}'_P$ that satisfy $i \wedge \Diamond\neg i$. The equivalence follows by the inductive hypothesis.

Corollary 3.

$$\models_{\mathcal{C}} \varphi \ \textit{ iff } \ \models_{\mathcal{S}} \neg i \to \mathsf{Tr}(\varphi)$$

Adding the four formulas in Definition 17 as axioms to obtain a proof system $\mathbf{MLSR}(\mathcal{C})$, soundness and completeness go through – noting that deleting points from an nptm still yields a named pseudo transformed model. Putting things together, one then obtains the desired

Corollary 4.

$$\vdash_{\mathbf{HSML}} \varphi \ \textit{ iff } \ \vdash_{\mathbf{MLSR}(\mathcal{C})} \neg i \to \mathsf{Tr}(\varphi)$$