SET THEORY OF INFINITE IMPERFECT INFORMATION GAMES

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Abstract. We survey the recent developments in the investigation of Blackwell determinacy axioms. It is generally believed that ordinary (Gale-Stewart) determinacy and Blackwell determinacy for infinite games are equivalent in a strong sense, but this conjecture (of Tony Martin's) has not yet been proved. The paper is a snapshot of the current state-of-the-art knowledge in this area.

1. Introduction

For most mathematicians, the term "game theory" evokes associations of applications in Economics and Computer Science: it is often associated with the Prisoner's Dilemma or other applications in the social sciences. Game theory is not perceived as an area of mathematics but rather as an area in which mathematics is applicable.

In addition to this famous area of game theory that has been repeatedly honoured by the Nobel Prize for Economics, games also play an important rôle in mathematics and computer science, often connected with logic. We use the name *Logic & Games* for the research field that connects techniques from logic in order to investigate games, or game theoretic methods in order to prove theorems of logic. This field itself is broad and has many subfields that are as diverse as logic itself. One of its subfields is the set theoretic study of infinite games.

It was the Polish school of topologists and measure theorists that connected game theory to set theory. According to Steinhaus [$\mathrm{St}_1$65, p. 464], Banach and Mazur knew in the 1930s that there is a non-determined infinite game (constructed by a use of the axiom of choice) and that there is a connection between games and the Baire property.

Gale and Stewart in their seminal [$\mathrm{GaSt}_2$53] presented the general theory of infinite games, and Mycielski and Steinhaus proposed a set theoretic analysis of game-related axioms in [$\mathrm{MySt}_1$62].

In the 1960s, the early set theoretic investigation of the game theoretic axioms proposed by Mycielski and Steinhaus was mainly done by Mycielski and a growing group of Californian logicians, among them John Addison, Tony Martin, Yiannis Moschovakis and Bob Solovay at the University of California at Berkeley and Los Angeles. In the 1970s, the Los Angeles area

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set theory seminar with researchers from UCLA and the California Institute of Technology (including prominently the researchers mentioned earlier and Alekos Kechris, John Steel, and later Hugh Woodin) became known as "the Cabal" and its regular meetings together with a conference series called the *Very Informal Gathering* produced a theory that is now known as *Cabal-style Descriptive Set Theory* and was published in the four proceedings volumes of the Cabal seminar [KeMo78, $\mathrm{KeMa_1Mo81}$, $\mathrm{KeMa_1Mo83}$, $\mathrm{KeMa_1St_088}$] and codified in Moschovakis' text book [Mo80].

The Cabal has investigated the consequences of game theoretic axioms in set theory, and they have unveiled deep connections between these axioms and the foundations of mathematics and metamathematics. On their way, they have also developed a lot of set theoretic techniques for dealing with infinite games. This theory has been developed in the setting of two-player perfect information games. Although the Cabal has been looking at variants of determinacy with different sets of possible moves ($\mathsf{AD}_{\mathbb{R}}$) and variants of determinacy for games of different transfinite lengths [$\mathrm{St_088}$, Ne04], they didn't give up the general perfect information structure.

In this survey paper, we shall extend results from Cabal-style set theory to a broader class of games, namely to a type of imperfect information games used in statistics and called *Blackwell games.* The imperfection of the information in Blackwell games is very minor: instead of playing infinitely many times in turn, the players play infinitely many times simultaneously and then reveal their moves. Many game-theoretic phenomena cannot be modelled with Blackwell games. However, Blackwell games are so far the only type of imperfect information games that have been investigated from the point of view of set theory. It will turn out in § 5 (Theorem 12) that if we restrict our attention to set-theoretic strength, even this puny amount of imperfect information can be avoided, and we end up doing an analysis of perfect information games played with probabilistic strategies.

In §§ 2 and 3, we introduce the basic notions of the theory of determinacy of perfect and imperfect information games, including the protagonist of this paper, the *axiom of Blackwell determinacy* $\mathsf{Bl\text{-}AD}$. We relate the development of the 1990s on Blackwell determinacy in §§ 4 and 5, in particular, we discuss Martin's equivalence conjecture (Conjecture 8) and the Martin-Vervoort Zero-One Laws (Theorem 10 and Corollary 11). These developments led to a considerable simplification of the formulation of axioms of Blackwell determinacy that opened up the possibility of generalization in several directions. In §§ 6 and 7, we discuss results pointing towards the truth of Martin's conjecture: the computation of the consistency strength of $\mathsf{Bl\text{-}AD}$ due to Martin, Neeman and Vervoort (Theorem 14) and some partial results concerning infinitary combinatorics and hierarchies of sets and functions due to the present author.[1] In §§ 8 and 9, we finally discuss weaker and

[1]A few years ago, the present author has published another survey on Blackwell games entitled "*Consequences of the Axiom of Blackwell Determinacy*" [Lö02b]. That survey

stronger versions of BI-AD. The paper closes with a list of open questions in the appendix.

2. Games in Set Theory

Our paper is written for readers with some background in set theory. We are using standard notation from set theory throughout the paper. As usual in set theory, we shall be working on Baire space ω^ω with the standard topology generated by the basic open sets $[s] := \{x \in \omega^\omega \ ; \ s \subseteq x\}$ for finite sequences $s \in \omega^{<\omega}$. It is well-known that this topological space is homeomorphic to the irrational numbers, so it is safe to call its elements **real numbers**.

Throughout we shall work in the theory $\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$. This small fragment of the axiom of choice is necessary for the definition of axioms of Blackwell determinacy. In §§ 7.2 and 7.3, we shall extend our basic theory to $\mathsf{ZF}+\mathsf{DC}$ and $\mathsf{ZF}+\mathsf{DC}_\mathbb{R}$, respectively, but not without explicitly mentioning it in the results.

We look at infinite games of the following type: we have two players, called player I and player II, and a fixed payoff set $A \subseteq \omega^\omega$. The players play infinitely often natural numbers, player I begins with x_0, player II answers with x_1, then player I plays x_2 and so on. After infinitely many rounds of the game, they have produced a sequence $x := \langle x_i \ ; \ i \in \omega\rangle \in \omega^\omega$ which they compare to the payoff set A. If $x \in A$, then player I wins, otherwise, player II wins. We shall be using the standard notation for infinite games: If $x \in \omega^\omega$ is the sequence of moves for player I and $y \in \omega^\omega$ is the sequence of moves for player II, we let $x * y$ be the sequence constructed by playing x against y, *i.e.*,

$$(x*y)(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1. \end{cases}$$

Conversely, if $x \in \omega^\omega$ is a run of a game, then we let x_I be the part played by player I and x_II be the part played by player II, *i.e.*, $x_\mathrm{I}(n) = x(2n)$ and $x_\mathrm{II}(n) = x(2n+1)$. The game just described will be denoted by $\mathrm{G}(A)$.

Let us denote by ω^{Even} the set of finite sequences of natural numbers of even length, by ω^{Odd} the set of such sequences of odd length; then we call a function $\sigma : \omega^{\mathrm{Even}} \to \omega$ a **(pure) strategy for player I** and a function $\sigma : \omega^{\mathrm{Odd}} \to \omega$ a **(pure) strategy for player II**. Clearly, you can let strategies play against each other, recursively producing an infinite sequence of natural numbers. If σ is a strategy for player I and τ is a strategy for player II, then we denote the recursively define outcome of playing σ against τ by $\sigma * \tau$.

A strategy σ for player I (τ for player II) is called **winning in** $\mathrm{G}(A)$ if for every strategy τ for player II (σ for player I), we have $\sigma * \tau \in A$ ($\sigma * \tau \notin A$). We call a set A **determined** if one of the two players has a winning strategy.

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focusses on set-theoretic properties that provably hold under the assumption of BI-AD; the material corresponds roughly to the content of § 7 of this paper.

As an aside, let us mention that for pure strategies it is not even necessary to win against all strategic opponents; it is enough to win against opponents that do not react to our moves ("blindfolded opponents"): A real $x \in \omega^\omega$ defines **blindfolded strategies** σ_x and τ_x by

$$\sigma_x(s) := \tau_x(s) := x\left(\left\lceil \frac{\mathrm{lh}(s) - 1}{2} \right\rceil\right).$$

We write $\sigma * x := \sigma * \tau_x$ and $x * \tau = \sigma_x * \tau$.

Proposition 1. If σ is a pure strategy for player I and A is an arbitrary payoff set, then the following are equivalent for the game on A:

(**i**) For all $x \in \omega^\omega$, we have $\sigma * x \in A$, and
(**ii**) σ is a winning strategy.

The analogous statement holds for strategies for player II.

Proof. We only have to show (**i**)⇒(**ii**). Without loss of generality, assume that σ is a strategy for player I. Let τ be a pure counterstrategy such that $\sigma * \tau \notin A$. Let $x := (\sigma * \tau)_{\mathrm{II}}$. Then $\sigma * x = \sigma * \tau_x = \sigma * \tau \notin A$. □

Games of the type $\mathrm{G}(A)$ have been investigated by Zermelo in [Ze13] as a formalization of the game of chess. The notion of determinacy was introduced by Gale and Stewart in [GaSt$_2$53] where they also proved two fundamental theorems about this notion: they proved that every open set $A \subseteq \omega^\omega$ is determined and they constructed (using the axiom of choice) a non-determined set. Generalizing the open determinacy theorem, Wolfe [Wo55] proved the determinacy of all $\mathbf{\Sigma}^0_2$ sets, then Morton Davis [Da63] extended this to the determinacy of all $\mathbf{\Sigma}^0_3$ sets. At the very next level lurked the first metamathematical surprise: while the determinacy of all $\mathbf{\Sigma}^0_4$ sets is provable in ZFC [Pa72], any proof of it must essentially use set theoretic techniques (due to a theorem of Harvey Friedman and Tony Martin).[2] This work culminated in Tony Martin's 1975 proof of *Borel determinacy* [Ma$_1$75].

In fact, Borel determinacy is the best possible answer in ZFC: in Gödel's constructible universe, there is a non-determined Π^1_1 set. More precisely, the determinacy of all Π^1_1 sets is equivalent to the existence of $0^\#$ by a theorem of Harrington [Ha78]. Again, this equivalence theorem was the start of a new development in which many researchers connected the theory of determinacy to large cardinal axioms.

In a very natural way, this investigation converged with other developments in seemingly unrelated areas of set theory to give a tremendously

[2]The exact result is: Let ZC^- be Zermelo set theory without power set, let $\mathsf{ZFC}^- := \mathsf{ZC}^- + $*Ersetzung*, let $\mathsf{Ers}(\Sigma_1)$ be *Ersetzung* restricted to Σ_1 formulas, and let β_0 be the least ordinal such that $\mathbf{L}_{\beta_0} \models \mathsf{ZFC}^-$ (if it exists). Then:

$$\mathsf{ZC}^- + \mathsf{Ers}(\Sigma_1) + \mathsf{Det}(\Sigma^0_4) \vdash \text{“}\beta_0 \text{ exists”}.$$

deep understanding of the structure of the set-theoretic continuum combining descriptive set theory, the theory of large cardinals and core model theory.

In light of the fact that Gale and Stewart had constructed a non-determined set (using the axiom of choice), it is surprising that one of the most central axioms of this theory was the full **axiom of determinacy** AD stating that all sets are determined. Despite the fact that it contradicts AC, the axiom AD allows a very detailed structure theory in its models (in particular, AD implies that all sets of reals have so-called *regularity properties* like Lebesgue measurability, the perfect set property and the Baire property; *cf.* § 7.1) and models of AD show up in prominent places as soon as the set theoretic universe is rich enough.[3]

A discussion of the rich theory of determinacy and its interesting history is far beyond the scope of this survey, and we refer the reader to [Ka94, §§ 27-32]. Some of the consequences of AD are discussed in §§ 7.1, 7.2 and 7.3.

3. When is an imperfect information game determined?

If you show the definitions and results of § 2 to a typical game theorist, he or she will note that all we have said so far concerns a very special class of games. Our games discussed so far have the following properties:

- They are *games of perfect information*: At each stage, both players know the exact situation of the game. This excludes that moves are made simultaneously ("scissors-stone-paper"), that cards are hidden or private knowledge of one player, and similar phenomena.
- They are games of *perfect recall*: Both players have unlimited storage for the moves that have been made.
- In our games we have *absolute noncooperation*; not only are our games zero-sum, but we always have winner and loser and no other relevant parameter. This excludes phenomena of compromise and trade-off.
- We consider only *two-player games*. Coalitions only play a role for games with more than two players.

It would be very interesting to develop set theoretic infinite versions of games of full imperfect information, of imperfect recall, of cooperation and with more than two players. However, the only area that has been investigated so far is a rather small class of imperfect information games. It is those games that we shall now look into.

Games with imperfect information were investigated by Johann von Neumann whose famous *minimax theorem* (1928) is the imperfect information

[3]The precise statement of this is the following theorem of Woodin: "Assume that there are ω Woodin cardinals and a measurable cardinal above them. Then $\mathbf{L}(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$."

analogue of Zermelo's theorem of the determinacy of finite perfect information games. Infinite versions of von Neumann's games were introduced by David Blackwell in [$\mathrm{Bl}_0$69] who proved that if the payoff set of a certain game modelled with probability measures is a countable unions of closed sets, then one of the players can approximate an optimal strategy.[4]

Blackwell's games are very close to the games of § 2: we still have the two players and a fixed payoff set $A \subseteq \omega^\omega$. The players play infinitely often natural numbers x_i, but this time they don't do this in turn, but always play their numbers x_{2i} and x_{2i+1} simultaneously. As before, after infinitely many rounds of the game, they have produced a sequence $x := \langle x_i \ ; \ i \in \omega \rangle \in \omega^\omega$ which they compare to the payoff set A. If $x \in A$, then player I wins, otherwise, player II wins. We denote this game by $\mathrm{B}(A)$.

For player I, there is almost no change: in $\mathrm{G}(A)$, being the one who has to move first, he was already at a disadvantage. The change for player II, however, is profound: in $\mathrm{G}(A)$, he was allowed to base his move x_{2i+1} on his knowledge of x_{2i}. As these two moves are now played simultaneously, this is not possible anymore. This also changes the notion of strategy: in the game $\mathrm{B}(A)$, both players I and II have to use functions from ω^{Even} to ω.

In such a situation, the notion of winning strategy (and hence the notion of determinacy) is not helpful anymore, as the following example shows.

We consider the Gale-Stewart and Blackwell variants of the game of "playing different reals" in which player I wins if for all i, players I and II have played different numbers in round i. More formally, let $M := \{x \in \omega^\omega \ ; \forall i(x(2i) \neq x(2i+1)\}$, a closed subset of ω^ω. This game will serve as an illustrative example throughout this paper. Clearly, player II has a winning strategy in the game $\mathrm{G}(M)$.

Observation 2. None of the players has a winning strategy in $\mathrm{B}(M)$.

Proof. A winning strategy of player I in $\mathrm{B}(M)$ would be one in $\mathrm{G}(M)$ contradicting the fact that player II has a winning strategy there.

Now let σ be a strategy for player II. In the Blackwell game, this means that $\sigma : \omega^{\mathrm{Even}} \to \omega$. Recursively define a real x as follows: Assume that $x{\restriction}n$ is already defined, then let $x(n) := \sigma(x{\restriction}n * \sigma) + 1$. Then clearly, if player I plays x, he will never play the same as σ in the same round, and thus win. □

Despite the fact that the notion of winning strategy doesn't yield an analysis of the game $\mathrm{B}(M)$, there is a clear and easy intuition that the game is easier for player I: after all, the players make infinitely many moves independently, and if only one of them produces different results, player I wins. So, just randomizing at each step will eventually pay off for player I.

We shall now mould this intuition into a precise definition:

[4] Blackwell calls these games "Games with Slightly Imperfect Information" in his [$\mathrm{Bl}_0$97].

Let $\mathrm{Prob}(\omega)$ be the set of probability measures on ω. We call a function $\sigma : \omega^{\mathrm{Even}} \to \mathrm{Prob}(\omega)$ a **Blackwell strategy for player I** and a function $\tau : \omega^{\mathrm{Odd}} \to \mathrm{Prob}(\omega)$ a **Blackwell strategy for player II** if for all $n, m \in \omega$ and $s \in \omega^{\mathrm{Even}}$, we have $\tau(s^\frown\langle n\rangle) = \tau(s^\frown\langle m\rangle)$.

If σ and τ are Blackwell strategies for players I and II, respectively , then they allow to describe the (randomized) behaviour of the two players: Player I chooses his first move x_0 according to the probability measure $\sigma(\langle\,\rangle)$, then player II consults his strategy about the measure $\tau(\langle x_0\rangle)$ (which may not depend on x_0 if τ is a Blackwell strategy) and plays according to that probability measure, and so on.

Let

$$\nu(\sigma,\tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \omega^{<\omega}$, we can define

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\restriction}i)(\{s(i)\}).$$

Using $\mathsf{AC}_\omega(\mathbb{R})$ to pick Borel codes, this generates a Borel probability measure on ω^ω which can be seen as a measure of how well the strategies σ and τ performs against each other. If B is a Borel set, $\mu_{\sigma,\tau}(B)$ is interpreted as the probability that the result of the game ends up in the set B when player I randomizes according to σ and player II according to τ.

For any Blackwell strategy σ for player I or τ for player II we can define a measure for its quality (the **value of the strategy**) by

$$\mathrm{val}^A_{\mathrm{I}}(\sigma) := \inf\{\mu^-_{\sigma,\tau}(A)\,;\, \tau \text{ is Blackwell strategy for player II}\}, \text{ and}$$

$$\mathrm{val}^A_{\mathrm{II}}(\tau) := \sup\{\mu^+_{\sigma,\tau}(A)\,;\, \sigma \text{ is Blackwell strategy for player I}\}.^5$$

Now we define the **value sets** for player I and player II by

$$V_{\mathrm{I}}(A) := \{\mathrm{val}^A_{\mathrm{I}}(\sigma)\,;\, \sigma \text{ is a Blackwell strategy for player I}\}, \text{ and}$$

$$V_{\mathrm{II}}(A) := \{\mathrm{val}^A_{\mathrm{II}}(\tau)\,;\, \tau \text{ is a Blackwell strategy for player II}\}.$$

Then $V_{\mathrm{II}}(A)$ lies entirely above $V_{\mathrm{I}}(A)$ in the sense that for all $v \in V_{\mathrm{II}}(A)$ and $v^* \in V_{\mathrm{I}}(A)$ we have $v \geq v^*$. If now these two sets $V_{\mathrm{I}}(A)$ and $V_{\mathrm{II}}(A)$ touch each other in a point p (depicted in Figure 1), then the outcome of the game is determined in the following sense: if both players play rationally, then the probability that player I wins is arbitrarily close to p. In this case, we say that the payoff set **has a value**.

In the other case, when the sets $V_{\mathrm{I}}(A)$ and $V_{\mathrm{II}}(A)$ don't touch each other, then the interval between the supremum v^- of $V_{\mathrm{I}}(A)$ (the **lower value**) and the infimum v^+ of $V_{\mathrm{II}}(A)$ (the **upper value**) is an area of indeterminacy:

[5] Here, μ^+ denotes outer measure and μ^- denotes inner measure with respect to μ in the usual sense of measure theory. If A is Borel, then $\mu^+(A) = \mu^-(A) = \mu(A)$ for Borel measures μ.

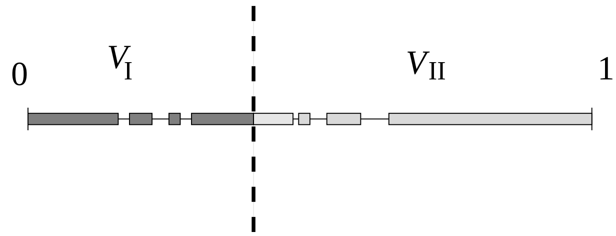


FIGURE 1. Value sets V_{I} and V_{II} touch each other: The outcome is stochastically determined if both players approximate optimal play

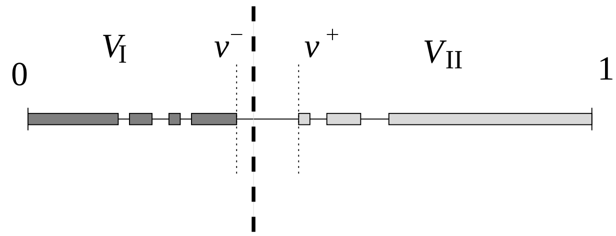


FIGURE 2. Value sets V_{I} and V_{II} don't touch: The payoff set is not Blackwell determined

Player I can bound his chance of winning from below by v^- and player II can bound player I's chance of winning from above by v^+, but since these are not the same number, the real outcome can be somewhere in the interval (as depicted in Figure 2).

What sets have a value? A first restriction is given by the following very simple example: Let $H := \{x \in \omega^\omega\,;\, x(0) \geq x(1)\}$. This is a clopen payoff set that corresponds to the game "who knows the larger natural number?".

Observation 3. The game $\mathrm{B}(H)$ does not have a value.

Proof. We shall show that the lower value of this game is 0 and the upper value is 1: Let σ be any strategy for player I. We shall show that it has value $< \varepsilon$ for every $\varepsilon > 0$. Since all moves after the first two are irrelevant for the payoff, a strategy for player I is just a probability measure on ω. We know that $\sum_{i\in\omega} \sigma(\{i\}) = 1$, so find a natural number n such that $\sum_{i<n} \sigma(\{i\}) > 1 - \varepsilon$. Then take the trivial strategy τ that assigns the total measure to the number n. The probability that player I wins against τ using the strategy σ is $< \varepsilon$.

The proof that the upper value is 1 is exactly the same. $\square$

This simple game shows that we should restrict our games to finitely many possible moves, for instance, binary choices. Therefore, for a pointclass Γ, we shall write $\mathsf{Val}(\Gamma)$ for "Every set $A \subseteq 2^\omega$ such that $A \in \Gamma$ is Blackwell determined." It is easy to see [Lö02a, Corollary 4.4] that 2 can be replaced by any finite set with more than one element (if Γ is reasonably closed).

Proofs of Blackwell determinacy had a development not unsimilar to the development of determinacy proofs described in § 2. As mentioned, von Neumann's Minimax theorem takes the role of Zermelo's theorem. In [$\text{Bl}_0$69], Blackwell proved the Blackwell determinacy of all $\mathbf{\Sigma}^0_2$ sets. There were results extending Blackwell's result by Orkin [Or72] and Maitra and Sudderth [Ma_0Su92], before Blackwell in an extended abstract [$\text{Bl}_0$97] revived the interest in the foundational questions. In particular, he asked whether all Borel sets are Blackwell determined. Instigated by this, Marco Vervoort [Ve95, Ve96] proved $\mathbf{\Sigma}^0_3$ Blackwell determinacy, before Martin solved Blackwell's question (*cf.* § 4).

Before continuing with describing the developments of the 1990s (§ 4), let us stop for a moment and ask ourselves what we can do with these new games.

Theorem 4 (Vervoort, 1995)**.** Let $\mathbf{\Gamma}$ be a boldface pointclass. Assume $\mathsf{Val}(\mathbf{\Gamma})$. Then all sets in $\mathbf{\Gamma}$ are Lebesgue measurable.

Proof. Let $A \subseteq 2^\omega$ be a set in $\mathbf{\Gamma}$. Let us define a continuous function as follows: for $s \in 2^{2n}$, we let

$$\widehat{s}(k) := |s(2k) - s(2k+1)|, \text{ and}$$

let $\pi : 2^\omega \to 2^\omega$ be the function induced by $s \mapsto \widehat{s}$. For given strategies σ and τ, let $\widehat{\mu_{\sigma,\tau}}$ be the pull-back measure with respect to π, *i.e.*,

$$\widehat{\mu_{\sigma,\tau}}(A) := \mu_{\sigma,\tau}(\pi^{-1}[A]).$$

If either of σ or τ is the "randomize" strategy assigning probability $\frac{1}{2}$ to both 0 and 1, then $\widehat{\mu_{\sigma,\tau}}$ is Lebesgue measure on 2^ω.

Now let $A^* := \pi^{-1}[A] \in \mathbf{\Gamma}$. We get that

$$\sup V_{\mathrm{I}}(A^*) \leq \widehat{\mu_{\sigma,\tau}}^-(A) \leq \widehat{\mu_{\sigma,\tau}}^+(A) \leq \inf V_{\mathrm{II}}(A^*),$$

and since $\sup V_{\mathrm{I}}(A^*)$ and $\inf V_{\mathrm{II}}(A^*)$ coincide by assumption, A is Lebesgue measurable. □

Vervoort's proof of Theorem 4 gave rise to a new proof of Lebesgue measurability from determinacy due to Tony Martin which was published in [$\text{Ma}_1$03].

The proof of Theorem 4 is specific for Lebesgue measurability and is not easily adapted to other regularity properties like the Baire property or the perfect set property. We shall discuss this in more detail in § 7.1.

4. Martin's Theorem and Conjecture

A priori, there is no immediate connection between the axioms $\mathsf{Det}(\mathbf{\Gamma})$ and $\mathsf{Val}(\mathbf{\Gamma})$. If player II has a winning strategy in $\mathrm{G}(A)$, then he may use the extra information that player II has not at his disposal in the game $\mathrm{B}(A)$. If the game $\mathrm{B}(A)$ has a value, especially if the value is neither 0 or 1, then this doesn't give a hint on how to construct a winning strategy in $\mathrm{G}(A)$.

This lack of connection is illustrated by our game "playing different reals" from § 3:

Proposition 5. Let $M := \{x \in 2^{\mathbb{N}}\,;\, \forall n(x(2n) \neq x(2n+1)\}$. Player II has a winning strategy in $\mathrm{G}(M)$ while the value of $\mathrm{B}(M)$ is 1.

Proof. We already indicated in the discussion after Observation 2 that $\mathrm{B}(M)$ is easier for player I than for player II. Using the definitions from § 3, we can this make more precise now:

We shall show that $\sup V_{\mathrm{I}} = 1$. For any strategy τ for player II, the "randomize" strategy $\sigma(s)(\{0\}) = \sigma(s)(\{1\}) = \frac{1}{2}$ (which we already used in the proof of Theorem 4) will give $\mu_{\sigma,\tau}(M) = 1$. □

As a consequence, we cannot expect a simple translation of good strategies in $\mathrm{G}(A)$ into good strategies in $\mathrm{B}(A)$. However, Martin showed that you can still mimick $\mathrm{B}(A)$ with a number of perfect information games. In particular, fixing a set $A \subseteq 2^{\omega}$, Martin defined for each $v \in (0, 1]$ a set A_v that intuitively express bounds for the lower and upper values:

Lemma 6 (Martin, 1998)**.** If player I has a winning strategy for $\mathrm{G}(A_v)$, then $\sup V_{\mathrm{I}}(A) \geq v$. If player II has a winning strategy for $\mathrm{G}(A_v)$, then $\inf V_{\mathrm{II}} \leq v$.

Proof. For the definition of A_v, *cf.* [Ma$_1$98, p. 1569]. The two statements of the lemma are [Ma$_1$98, Lemma 1.3] and [Ma$_1$98, Lemma 1.6]. □

Martin used Lemma 6 to derive the following result:

Theorem 7 (Martin, 1998)**.** If $\mathbf{\Gamma}$ is a boldface pointclass, then $\mathsf{Det}(\mathbf{\Gamma})$ implies $\mathsf{Val}(\mathbf{\Gamma})$.

Of the two possible implications between $\mathsf{Det}(\mathbf{\Gamma})$ and $\mathsf{Val}(\mathbf{\Gamma})$, this was probably the one less expected to hold. Since having a value talks about a much more general class of strategies, it was natural to assume that $\mathsf{Val}(\mathbf{\Gamma})$ should be stronger than the axiom that only talks about pure strategies, a rather restricted class of strategies.

The uniform nature of Martin's result and the fact that having a value was seen as the intuitively stronger of the two properties led to the natural conjecture:

Conjecture 8 (Martin)**.** Let $\mathbf{\Gamma}$ be a boldface pointclass. Then $\mathsf{Det}(\mathbf{\Gamma})$ and $\mathsf{Val}(\mathbf{\Gamma})$ are equivalent.

Martin's Conjecture 8 has proved to be more difficult than people thought at first and it is still open in its general form. The most intriguing open instance of Conjecture 8 is the question whether "Every set $A \subseteq 2^\omega$ has a value" implies AD.

5. Zero-One Laws and Optimal Strategies

In a surprising development of the years 1999 and 2000, it became clear that the issue of imperfect information does not play any rôle for the question raised by Martin as Conjecture 8.

For this, let us look at a class of games that lie between $\mathrm{G}(A)$ and $\mathrm{B}(A)$: they are played like $\mathrm{G}(A)$, *i.e.*, the players move in turn and there is no information hidden for either of the players, but they are played with probabilistic strategies. Formally, we call functions $\sigma : \omega^{\mathrm{Even}} \to \mathrm{Prob}(\omega)$ **mixed strategies for player I** and functions $\tau : \omega^{\mathrm{Odd}} \to \mathrm{Prob}(\omega)$ **mixed strategies for player II**. Note that every mixed strategy for player I is also a Blackwell strategy for player I, but not so for player II. Mixed strategies σ and τ give rise to a Borel probability measure on ω^ω as did Blackwell strategies, and we define **mixed values** for mixed strategies in analogy to the values of § 3. The game $\mathrm{M}(A)$ is the game with payoff A played with mixed strategies. We say that A is **Blackwell determined** if the upper and lower mixed value of $\mathrm{M}(A)$ coincide. We also say that A **has a mixed value**. For a pointclass Γ, we write $\mathsf{Bl\text{-}Det}(\Gamma)$ for "all sets in Γ are Blackwell determined".

Proposition 9. For every boldface pointclass $\boldsymbol{\Gamma}$, we have that $\mathsf{Val}(\boldsymbol{\Gamma})$ implies $\mathsf{Bl\text{-}Det}(\boldsymbol{\Gamma})$.

Proof. You can easily see $\mathrm{M}(A)$ as a Blackwell game by defining $\pi : x \mapsto x^*$ via

$$\begin{aligned} x^*(2n) &:= x(4n), \text{ and} \\ x^*(2n+1) &:= x(4n+3). \end{aligned}$$

Then $\mathrm{M}(A)$ and $\mathrm{B}(\pi^{-1}[A])$ are the same game. □

The mixed game $\mathrm{M}(A)$ is essentially a perfect information game, and this has important consequences for the possible mixed values. As a first step, Martin and Vervoort could prove a first **Zero-One Law** for mixed games in the year 2000:

Theorem 10 (Martin-Vervoort Zero-One Law, 2000)**.** Suppose A has a mixed value. Then the mixed value is either 0 or 1.

With the Martin Vervoort Zero-One Law in mind, we can define a very strong property for mixed strategies:

A mixed strategy for player I is now called **optimal for** A if for every mixed strategy τ for player II, there is a Borel subset $B \subseteq A$ such that $\mu_{\sigma,\tau}(B) = 1$. Similarly, a mixed strategy τ for player II is called **optimal**

for A if for every mixed strategy σ for player I, there is a Borel superset $B \supseteq A$ such that $\mu_{\sigma,\tau}(B) = 0$.

Vervoort was able to use the Martin-Vervoort Zero-One-Law to prove the existence of optimal strategies:

Corollary 11 (Vervoort Strong Zero-One Law, 2000)**.** Let $\mathbf{\Gamma}$ be a boldface pointclass. Suppose that $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ holds and $A \in \mathbf{\Gamma}$. Then there is either an optimal strategy for player I or an optimal strategy for player II in $\mathrm{M}(A)$.

Proof. Proofs of both Theorem 10 and Theorem 11 can be found as [Ve00, Theorems 5.3.3 & 5.3.4] or [$\mathrm{Ma_1}$NeVe03, Lemmas 3.7 & 3.10]. □

Using the Zero-One Laws, we can now prove the converse of Proposition 9 and rephrase our definition of Blackwell determinacy:

Theorem 12 (Martin)**.** For boldface pointclasses $\mathbf{\Gamma}$, the statements $\mathsf{Val}(\mathbf{\Gamma})$ and $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ are equivalent.

Proof. This result has been mentioned in passing in [$\mathrm{Ma_1}$NeVe03, p. 619]. One just has to check that the proof of Lemma 6 still goes through with optimal strategies instead of winning strategies. □

Synthetizing what we have said about Blackwell determinacy so far, we can give a new but equivalent definition of the notion as follows: We call a set A is **Blackwell determined** if either player I or player II has an optimal strategy in the game $\mathrm{M}(A)$. By the Strong Zero-One Law, this will give the same notion restricted to boldface pointclasses. In §§ 8 & 9, we shall use this definition as the basis for generalizations.

6. Consistency Strength of Blackwell Determinacy

Martin's proof of Theorem 7 immediately yields upper bounds for the consistency strength of axioms of Blackwell determinacy: for instance, if there is a measurable cardinal, then $\mathsf{Bl\text{-}Det}(\mathbf{\Pi}^1_1)$ must hold.

If Martin's Conjecture 8 is true, then all of these upper bounds should be sharp and the corresponding lower bound should be provable as well. As a first result, Martin proved the existence of sharps from $\mathsf{Bl\text{-}Det}(\mathbf{\Pi}^1_1)$ thereby establishing the equivalence of determinacy and Blackwell determinacy at the $\mathbf{\Pi}^1_1$ level. (This result is published with a full proof in [Lö04, Theorem 3.8].)

It was open in the years 1998 and 1999 whether you could get any stronger large cardinal assumptions out of Blackwell determinacy. Then, the Zero-One Law (Theorem 10) allowed Martin, Neeman and Vervoort to simulate the proof of the Third Periodicity Theorem [Mo80, § 6E] with mixed strategies. In particular, they showed

Lemma 13 (Martin-Neeman-Vervoort)**.** Call a pointclass Γ **weakly scaled** if every set in Γ admits a scale $\{\varphi_n\,;\, n \in \omega\}$ such that every φ_n is a Γ-norm. If Γ is a weakly scaled pointclass and $\Delta := \Gamma \cap \check{\Gamma}$, then $\mathsf{Bl\text{-}Det}(\Delta)$ implies $\mathsf{Det}(\Delta)$.

From this, you can get the equivalence between Blackwell determinacy and determinacy for many pointclasses (thus proving many instances of Martin's Conjecture 8):

Theorem 14 (Martin-Neeman-Vervoort)**.** Let $\mathbf{\Gamma}$ be either $\mathbf{\Delta}^1_{2n}$, $\mathbf{\Sigma}^1_{2n}$, $\mathbf{\Delta}^1_{2n+1}$, $\Game^n(<\omega^2\text{-}\mathbf{\Pi}^1_1)$, or $\wp(\omega^\omega)\cap\mathbf{L}(\mathbb{R})$. Then $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Det}(\mathbf{\Gamma})$.

Proof. This is Theorem 5.1, Corollary 5.3, Theorem 5.4, Theorem 5.6, and Theorem 5.7 in [Ma$_1$NeVe03]. □

Theorem 14 settles the question of the consistency strength of the axiom of Blackwell determinacy $\mathsf{Bl\text{-}AD}$ completely: If $\mathbf{V}\models\mathsf{Bl\text{-}AD}$, then this is still true in $\mathbf{L}(\mathbb{R})$, but there we get $\mathbf{L}(\mathbb{R})\models\mathsf{AD}$ by Theorem 14. Consequently, the consistency strength of $\mathsf{Bl\text{-}AD}$ is the same as that of AD.

7. Approaching Equivalence

While Theorem 14 settles the question about the consistency strength of Blackwell determinacy, it leaves open the most vexing question that we mentioned before: does $\mathsf{Bl\text{-}AD}$ imply AD?

Looking at the type of AD-arguments that still work under the assumption of $\mathsf{Bl\text{-}AD}$, we see that the most serious problem with $\mathsf{Bl\text{-}AD}$ is that pure strategies generate continuous choice-like functions whereas even optimal mixed strategies don't generate a function at all. In other words: We are missing a principle that given an optimal strategy for player I in $\mathrm{M}(A)$ allows a parametrised choice of an element of the payoff set A (and similarly for player II and the complement of A). Such a principle called the **parametrised choice principle** (PCP) has been defined in [Lö06] where we show that (in the base theory $\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})+\mathsf{Bl\text{-}AD}$), AD and PCP are equivalent.

Thus, PCP can be seen as the difference between AD and $\mathsf{Bl\text{-}AD}$. If we can show PCP from $\mathsf{Bl\text{-}AD}$, Martin's Conjecture 8 would be proved.

The direct approaches of proving determinacy from Blackwell determinacy have not been successful so far. A different approach is to take some of the very characteristic consequences of the axiom of determinacy and prove that they hold under $\mathsf{Bl\text{-}AD}$.

This has been successful in some case. We focus here on three very characteristic features of AD-set theory: regularity properties of all sets of reals (§ 7.1), the very concrete combinatorial theory of small cardinals (up to $\aleph_{\varepsilon_0}$ and slightly higher; § 7.2), and the existence of global hierarchies of sets of reals (§ 7.3).

7.1. **Regularity Properties.** One of the most basic consequences of AD is the fact that sets of reals have all desirable properties. These properties are typically called **regularity properties**, and examples are Lebesgue measurability, the Baire property, and the perfect set property.[6]

Working in $\mathsf{ZF} + \mathsf{AD}$, Mycielski and Swierczkowski showed in [MySw64] that every set is Lebesgue measurable. A well-known topological game argument due to Banach and Mazur [Ke94, 8H] establishes that every set has the Baire property. Finally, Davis proved in [Da63] that every set has the perfect set property. Most of these proofs use a modest amount of coding, but typically link the regularity property of a set A directly to the determinacy of some set A^* simply definable from A.

Let us describe this in more detail for the perfect set property: Fix some bijection $\pi : \omega \to 2^{<\omega}$. We now recursively define a function $\pi : \omega^\omega \to \omega^\omega$: Given some $x \in \omega^\omega$, let $x^{(0)} := \pi(x(0))$, $x^{(2i+1)} := x^{(2i)\frown}\langle x(2i+1)\rangle$, and $x^{(2i+2)} := x^{(2i+1)\frown}\pi(x(2i+2))$. We write $N_{x,i}$ for the length of the sequence $x^{(i)}$. Then the function π defined by $\pi(x) := \bigcup_{i\in\omega} x^{(i)}$ is continuous.

For $A \subseteq 2^\omega$, let $A^* := \{x\,;\,\pi(x) \in A\}$ be the π-preimage of A. There is a well-known connection between A^* and the perfect set property of A:

Fact 15. For every set A, A has the perfect set property if and only if A^* is determined.

Proof. Cf. [Ke94, Theorem (21.1)]. □

Thus, AD implies that every set of reals has the perfect set property. What if we replace AD by $\mathsf{Bl\text{-}AD}$? From Theorem 4 we know that every set is Lebesgue measurable, but as we mentioned earlier, its proof was very heavily using the affinity of the definition of Blackwell determinacy to measures and doesn't seem to have analogues for other regularity properties.

One could ask whether analogues of Fact 15 could hold for Blackwell determinacy. Unfortunately, Hjorth showed that the answer is no. This argument was sketched in in [$\mathrm{Ma}_1$98, p. 1580]. We give a more detailed proof here: A measure μ is called **atomless** if every singleton has μ-measure 0. A set is called **universally zero** if it has μ-measure 0 for every atomless measure μ.

Theorem 16 (Hjorth)**.** Assume AC. Then there is a set A with the following properties:

1. A^* is Blackwell determined, but
2. A does not have the perfect set property.

Proof. Using the axiom of choice, pick one element from each WO_α and call the resulting set A. This is clearly an uncountable set which (by the boundedness lemma) cannot have a perfect subset. Another boundedness argument shows easily that A is universally null [Lö01, Proposition 3.4].

[6] A set A has the **Baire property** if there is a Borel set B such that $B \triangle A$ is a countable union of nowhere dense sets. It has the **perfect set property** if it is either countable or contains a nonempty perfect subset.

We shall now show that player II has an optimal strategy in $\mathrm{M}(A^*)$. This strategy is again the "randomize strategy" τ assigning $\frac{1}{2}$ to both 0 and 1. We shall show that for any strategy σ for player I, $\mu_{\sigma,\tau}(A^*) = 0$. Look at the pullback measure ν by the function π, *i.e.*, the measure defined by $\nu(X) = \mu_{\sigma,\tau}(\pi^{-1}[X])$.

For any fixed $x \in 2^\omega$, we have

$$\pi^{-1}[\{x\}] \subseteq \{\, y\,;\, \forall n(\, y(2n+1) = x(N_{y,2n})\,)\,\}\,,$$

and consequently, $\mu_{\sigma,\tau}(\pi^{-1}[\{x\}]) = 0$. Therefore, ν is an atomless measure, and thus $\nu(A) = \mu_{\sigma,\tau}(A^*) = 0$. □

As a consequence of Theorem 16, any proof of the perfect set property from Blackwell determinacy cannot follow the simple path outlined by Fact 15. In fact, it is still open whether the perfect set property can be proved from Blackwell determinacy alone.

Question 17. Does Bl-AD imply that all sets of reals have the perfect set property?

Similarly, nothing is known about other regularity properties. For the regularity property sometimes called "Martin measurability"[7], there is a construction similar to Theorem 16 in [Lö01, Theorem 6.3]: Under the assumption of AC + CH, there is a set which is closed under Turing equivalence and Blackwell determined, but not Martin measurable.

Question 18. Does Bl-AD imply that all sets of reals have the Baire property?

Question 19. Does Bl-AD imply that all Turing closed sets of reals are Martin measurable ("Turing determinacy")?

7.2. **Infinitary Combinatorics.** While the consequences of AD for regularity properties of sets of real are the most well-known properties of the playful universe, the combinatorial structure theory for small cardinals like $\aleph_1$, $\aleph_2$, $\aleph_{\omega+1}$ and many others is definitely the most striking property. It starkly contrasts with the fact that ZFC cannot even determine the continuum function for these cardinals: under AD many if not all cardinals have some strong combinatorial properties, and there is a very clear pattern which cardinals have what properties.

If we let

$$\Theta := \sup\{\xi\ ;\ \xi \text{ is the length of a prewellordering of } \omega^\omega\},$$

then this ordinal measures the range of the effect that the reals can have on the combinatorics of ordinals. It is known that AD implies that Θ is a reasonably large cardinal: for instance, $\aleph_\Theta = \Theta$ by a result of Solovay's [Ka94,

[7]Let X be a set closed under Turing equivalence, *i.e.*, if $x \equiv_{\mathrm{T}} y$ and $x \in X$, then $y \in X$. Then we say that X is **Martin measurable** if it contains a Turing cone or is disjoint from a Turing cone.

Exercise 28.17], and all of the cardinals that we called "small cardinals" in the last paragraph are well below Θ.

Moreover, all of the small cardinals can be explicitly identified as ultrapowers with concretely given measures. The identification of cardinals as concrete ultrapowers together with the determination of their combinatorial properties is called a **measure analysis**. It is rather peculiar that measure analyses have little to do with actual games: typically, we need the strong partition property (see below) of one cardinal that allows us to define measures on it. The proof of the strong partition property uses games, but the subsequent analysis doesn't. For an historical account and introduction of the basics, we refer the reader to [Ka94, Chapter 28]. If the reader is interested in the full structure theory of cardinals, we refer him or her to [Ja∞] or [JaLö06]. In this section, we shall use DC in order to guarantee that ultrapowers by measures are well-founded. We can therefore identify an ultrapower of an ordinal by a measure with the ordinal isomorphic to it via the Mostowski collapse.

Let κ be a cardinal. We say that κ has the **strong partition property** if $\kappa \to (\kappa)^\kappa$ holds, and that κ has the **weak partition property**, if for all $\lambda < \kappa$, $\kappa \to (\kappa)^\lambda$ holds. Both the weak and the strong partition property of any uncountable cardinal severely violate the axiom of choice, and they are extremely strong combinatorial properties.

Theorem 20 (Martin (1971), Kleinberg (1977))**.** Assume AD. Then $\aleph_1$ has the strong partition property, and the club filter $\mathcal{C}$ on $\aleph_1$ is a normal measure (*i.e.*, a σ-complete normal ultrafilter). Assuming DC, the recursively defined ultrapowers

$$\begin{aligned} \kappa_1 &:= \aleph_1, \\ \kappa_{n+1} &:= {\kappa_n}^{\aleph_1}/\mathcal{C} \end{aligned}$$

can be computed as $\kappa_n = \aleph_n$ and all of them are Jónsson cardinals.

Proof. Cf. [Ka94, Theorem 28.12]. A full proof of the Kleinberg part of this theorem can be found in [Kl77]. □

Theorem 20 gives an indication of how the strong partition property generates measures that are used to concretely describe small cardinals. We define the projective ordinals by

$$\boldsymbol{\delta}^1_n := \sup\{\xi \ ; \ \xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \boldsymbol{\Delta}^1_n\}.$$

The ordinal $\boldsymbol{\delta}^1_n$ can be seen as the limit of influence that the $\boldsymbol{\Delta}^1_n$ sets and functions have on combinatorics on the ordinals. Kunen and Martin proved in 1971 [Ke78, Theorem 3.12] that under AD, we have $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, so it is enough to compute the odd projective ordinals. This computation was done by Steve Jackson:

Theorem 21 (Jackson)**.** Define $\mathbf{e}_1 := 0$ and $\mathbf{e}_{n+1} = \omega^{\mathbf{e}_n}$ and assume AD and DC . Then for every $n \in \omega$,

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathbf{e}_{2n+1}+1},$$

and all odd projective ordinals have the strong partition property.

Proof. Cf. [Ja88] and [Ja99]. □

Using the combinatorial properties for the projective ordinals, Jackson was able to give a measure analysis for the cardinals below the supremum of the projective ordinals, $\aleph_{\varepsilon_0}$. This analysis uses Jackson's notion of a **description**; there is a very simple and concrete measure analysis for cardinals below $\boldsymbol{\delta}^1_5$ due to Jackson and his student Khafizov [JaKh06]. The underlying methodology of **canonical measure assignments** is described in detail in [JaLö06].

Jackson's analysis can be pushed even further and works up to $\aleph_{\omega_1}$ (...). Even beyond that, we find many strong partition cardinals below Θ by work of Kechris, Kleinberg, Moschovakis and Woodin [KeKlMoWo81]:

Theorem 22. Assume AD and DC. Then there are cofinally many strong partition cardinals below Θ.

All in all, we get a picture as in the left column of Figure 3: we have the canonical measure assignment analysis below $\boldsymbol{\delta}^1_5$, then Jackson's description analysis up to $\boldsymbol{\delta}^1_\omega = \aleph_{\varepsilon_0+1}$, and even beyond to $\aleph_{\omega_1}$, and finally much less information (but still cofinally many strong partition cardinals) between $\aleph_{\omega_1}$ and Θ.

It is this picture that we should like to recreate under the assumption of $\mathsf{BI\text{-}AD}$. However, progress has been modest. One of the very first steps on the way to computing the projective ordinals is to ascertain that they are actually cardinals. The proof of this under AD uses the Moschovakis Coding Lemma. We write CL for the following statement:

> Let $\leq$ be a $\boldsymbol{\Delta}^1_n$ prewellordering of $X \subseteq \omega^\omega$ with length ξ and associated norm φ. Then for every function $f : \xi \to \wp(\omega^\omega)$ there is a $g : \xi \to \wp(\omega^\omega)$ with
> (1) For all $\eta < \xi$, $g(\eta) \subseteq f(\eta)$,
> (2) for all $\eta < \xi$, if $f(\eta)$ is nonempty, then so is $g(\eta)$, and
> (3) $\{\langle x, y\rangle \; ; \; x \in X \;\&\; y \in g(\varphi(x))\}$ is a $\boldsymbol{\Sigma}^1_n$ set.

Theorem 23 (Moschovakis)**.** Assume AD. Then CL holds.

It is unknown whether CL follows from $\mathsf{BI\text{-}AD}$, and it is in general unknown whether the $\boldsymbol{\delta}^1_n$ are cardinals.

Question 24. Does $\mathsf{BI\text{-}AD}$ imply CL?

We can however, still show some results for cardinals at the bottom of the hierarchy depicted in Figure 3 by following the ideas of the proof of Theorem 20:

Theorem 25. Assume BI-AD. Then $\aleph_1$ has the strong partition property, and for all natural numbers n, the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ have the countable partition property, *i.e.*, for all $\alpha < \omega_1$, the partition relation $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^\alpha$ holds.

We get a canonical measure assignment for all cardinals $\aleph_n$ which also determines their cofinalities as

$$\mathrm{cf}(\aleph_n) := \begin{cases} \aleph_0 & \text{if } n = 0, \\ \aleph_1 & \text{if } n = 1, \\ \aleph_2 & \text{otherwise} \end{cases}$$

and allows us to compute $\boldsymbol{\delta}^1_3$:

Corollary 26. Assume BI-AD. Then $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$.

Proof. By Theorem 25, we know that $\boldsymbol{\delta}^1_3$ is a regular cardinal. By Theorem 14, we get PD and that every real has a sharp. This allows us to apply results of Martin that say that $\boldsymbol{\delta}^1_2 = \aleph_2 < \boldsymbol{\delta}^1_3 \leq \aleph_{\omega+1}$. But with the cofinality function as given avove, and since $\boldsymbol{\delta}^1_3$ is regular, $\aleph_{\omega+1}$ is the only possible choice left for it. □

Since we even know that $\boldsymbol{\delta}^1_4$ is a cardinal, we cannot compute it under the assumption of BI-AD.

Question 27. Does BI-AD imply that $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$?

As mentioned, the measure analysis due to Jackson does not really use games to a large extent. A lot of Jackson's analysis can be derived abstractly from the strong partition properties and some additional combinatorial properties of cardinals. This raises the question whether we can prove these strong partition results for higher projective ordinals:

Question 28. Does BI-AD imply that $\boldsymbol{\delta}^1_3$ (or in general, $\boldsymbol{\delta}^1_{2n+1}$) has the strong partition property?

7.3. **Hierarchies of sets of reals.** Almost as striking as the detailed combinatorial analysis of cardinals is the fact that AD gives us a wellordered hierarchy of sets of reals. From a topological point of view, there is no reducibility relation on sets of reals more natural than **Wadge reducibility** and **Lipschitz reducibility** defined by

$$A \leq_{\mathrm{W}} B :\Longleftrightarrow \text{ there is a continuous function } f \text{ with } A = f^{-1}[B], \text{ and}$$

$$A \leq_{\mathrm{L}} B :\Longleftrightarrow \text{ there is a Lipschitz function } f \text{ with } A = f^{-1}[B].$$

These relations are partial preorders inducing equivalence relations $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{L}}$ which in turn give rise to a degree structure (with a partial ordering). Under the assumption of the axiom of choice, these partial preorders can be quite wild:

Proposition 29. If A is an uncountable set of reals without a nonempty perfect subset, then there is some Π^0_2 set P such that A and P are incomparable in the $\leq_{\mathrm{W}}$ relation.

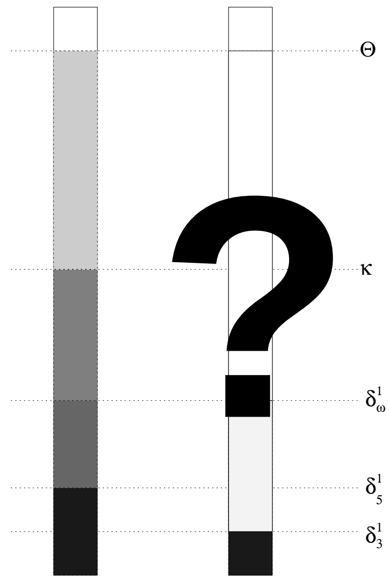


FIGURE 3. *Left.* The extent of measure analyses under AD. The darkest area corresponds to the analysis via canonical measure assignments up to $\boldsymbol{\delta}^1_5$, the dark area to the analysis via description theory up to $\boldsymbol{\delta}^1_\omega = \aleph_{\varepsilon_0+1}$ and $\kappa := \aleph_{\omega_1}$, the light area corresponds to a cofinal sequence of strong partition cardinals. *Right.* The extent of measure analyses under Bl-AD. Almost nothing is known beyond $\boldsymbol{\delta}^1_3$.

Proof. Consider $P := \{x\,;\,\forall n\exists k > n(x(k) = 0)\}$. It is easy to see that if $P \leq_{\mathrm{W}} X$, then X contains a nonempty perfect set. So, by our assumption, we have that $P \not\leq_{\mathrm{W}} A$. But we cannot have $A \leq_{\mathrm{W}} P$, as then A would be $\boldsymbol{\Pi}^0_2$ as a continuous preimage of a Π^0_2 set. This would contradict the fact that all Borel sets have the perfect set property. Therefore A and P are $\leq_{\mathrm{W}}$-incomparable. □

Both $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ have an equivalent definition in terms of games: For an element $x \in \omega^\omega$ we say that it is **finite** if $D_x := \{n\,;\,x(n) > 0\}$ is finite, and for a nonfinite x, we let $d_x : \omega \to \omega$ be the increasing enumeration of D_x. Then we can define $\widehat{x}(n) := x(d_x(n)) - 1$.[8] For given A and B, we can define $X^{\mathrm{L}}_{A,B} := \{x\,;\,x_{\mathrm{I}} \in A \not\leftrightarrow x_{\mathrm{II}} \in B\}$ and $X^{\mathrm{W}}_{A,B} := \{x\,;\,x_{\mathrm{II}}$ is finite or $x_{\mathrm{I}} \in A \not\leftrightarrow \widehat{x_{\mathrm{II}}} \in B\}$.

Proposition 30. For any two sets of reals A and B, we have:

(1) $A \leq_{\mathrm{L}} B$ if and only if player II has a winning strategy in the game $\mathrm{G}(X^{\mathrm{L}}_{A,B})$.

[8]This operation corresponds to considering the move 0 as a passing move. We think of x as a sequence of real moves (where $x(n) > 0$ represents the number $x(n)-1$) and passes (coded by $x(n) = 0$). The real $\widehat{x}$ corresponds to deleting the passes from the real x.

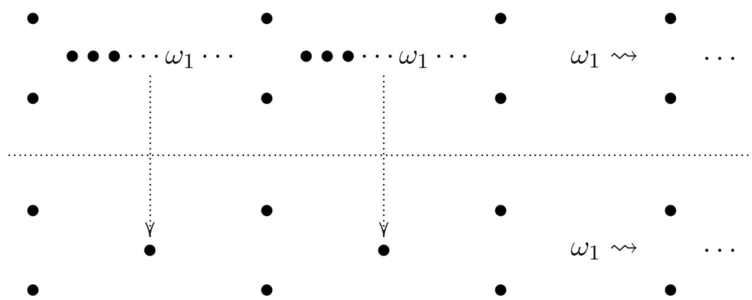


FIGURE 4. The Lipschitz hierarchy (above the dotted line) with blocks of ω_1 classes between non-selfdual pairs, and the Wadge hierarchy (below the dotted line) where these blocks collapse to a single class. At levels of uncountable cofinality, both hierarchies have a non-selfdual pair.

(2) $A \leq_{\mathrm{W}} B$ if and only if player II has a winning strategy in the game $\mathrm{G}(X^{\mathrm{W}}_{A,B})$.

Now that we reduced the existence of continuous and Lipschitz functions to gane-theoretic terms, we get a structure result under AD.

Lemma 31 (Wadge's Lemma)**.** For any two Borel sets A and B, either $A \leq_{\mathrm{W}} B$ or $\omega^\omega \backslash B \leq_{\mathrm{W}} A$. Moreover, assuming AD, this is true for any two sets of reals.

Much more is true: Using a technique called the **Martin-Monk technique**, Martin was able to prove (in $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$) the wellfoundedness of the relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ and Steel and van Wesep completely determined the structure of the Wadge hierarchy: Both hierarchies are semi-linear (*i.e.*, antichains have length at most 2) and wellfounded. The antichains of length 2 are called a **non-selfdual pair**, and these pairs occur in the Lipschitz hierarchy exactly at level α where $\mathrm{cf}(\alpha) > \omega$ and $\alpha = 0$. In the Wadge hierarchy, the blocks of selfdual classes of length ω_1 collapse to a single class, and hence the Wadge hierarchy has non-selfdual pairs at every second successor levels and limit levels of uncountable cofinality.

This pattern is depicted in Figure 4 and is highly characteristic of AD. For more details, we refer the reader to the survey paper [An∞].

Of course, the game-theoretic description in Proposition 30 immediately suggests a Blackwell version of these hierarchies: we define

$$A \leq_{\mathrm{BlL}} B :\Longleftrightarrow \text{ player II has an optimal strategy in } \mathrm{G}(X^{\mathrm{L}}_{A,B}), \text{ and}$$

$$A \leq_{\mathrm{BlW}} B :\Longleftrightarrow \text{ player II has an optimal strategy in } \mathrm{G}(X^{\mathrm{W}}_{A,B}).$$

Can we get a similar structure theory for the Blackwell Wadge and Lipschitz hierarchy under the assumption of $\mathsf{Bl\text{-}AD}$? The basic theory of Blackwell Lipschitz degrees works smoothly:

Theorem 32. Assume $\mathsf{Bl\text{-}AD}$. Say that a Blackwell Lipschitz degree $\mathbf{d}$ is called a **successor degree** if there is a degree $\mathbf{p} <_{\mathrm{BlL}} \mathbf{d}$ such that there is no $\mathbf{e}$ with $\mathbf{p} <_{\mathrm{BlL}} \mathbf{e} <_{\mathrm{BlL}} \mathbf{d}$. We say that $\mathbf{d}$ is **of countable cofinality** if there is a sequence $\langle [A_n]_{\mathrm{BlL}}\,;\, n \in \omega\rangle$ without a greatest element such that

$$A \equiv_{\mathrm{BlL}} \bigoplus_{n\in\omega} A_n.$$

Then the Blackwell Lipschitz degrees are semi-linearly ordered, every successor degree is a selfdual degree, and the non-selfdual degrees are exactly the first two degrees and the non-successor degrees which are not of countable cofinality.

However, the Martin-Monk technique does not seem to work with $\mathsf{Bl\text{-}AD}$. As a consequence, we do not know whether our new hierarchy is wellfounded.

Question 33. Does $\mathsf{Bl\text{-}AD} + \mathsf{DC}_{\mathbb{R}}$ imply that the Blackwell Lipschitz hierarchy is wellfounded?

Since our analysis of $\leq_{\mathrm{BlW}}$ and $\leq_{\mathrm{BlL}}$ did not yield the wished results, we could look for other global hierarchies that we know to exist under the assumption of AD. One of them is the hierarchy generated by the First Periodicity Theorem, investigated as "Steel hierarchy" in [Ch00] and [Du03].

Instead of sets of reals, we look at surjective functions $\varphi : \omega^\omega \to \alpha$ (for some ordinal α) called **norms**. Each norm φ defines an order

$$\leq_\varphi := \{\langle x, y\rangle\,;\, \varphi(x) \leq \varphi(y)\}.$$

For two norms φ and ψ, we say that φ is **FPT-reducible to** ψ (for "**F**irst **P**eriodicity **T**heorem"; in symbols: $\varphi \leq_{\mathrm{FPT}} \psi$) if there is a continuous function $F : \omega^\omega \to \omega^\omega$ such that for all $x \in \omega^\omega$, we have

$$\varphi(F(x)) \leq \psi(x).$$

As with $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$, FPT-reducibility can be expressed in game terms. Given φ and ψ, we look at the set $X_{\varphi,\psi} := \{x\,;\, x_{\mathrm{II}} \text{ is finite or } \varphi(x_{\mathrm{I}}) > \psi(\widehat{x_{\mathrm{II}}})\}$.

Proposition 34. For any two norms φ and ψ, we have $\varphi \leq_{\mathrm{FPT}} \psi$ if and only if player II has a winning strategy in $\mathrm{G}(X_{\varphi,\psi})$.

Theorem 35 (Moschovakis)**.** Assume AD and $\mathsf{DC}_{\mathbb{R}}$. Then $\leq_{\mathrm{FPT}}$ is a prewellordering.

Proof. This is essentially the proof of the First Periodicity Theorem of Moschovakis [AdMo68]. For more details, *cf.* [Mo80, 6B]. □

As the Martin-Monk technique, the proof of Theorem 35 involves filling an infinite diagram of games using strategies. However, this proof is much simpler than the Martin-Monk proof and can be adapted to the Blackwell determinacy situation. In analogy with $\leq_{\mathrm{FPT}}$ and Proposition 34, we define

$$\varphi \leq_{\mathrm{BlFPT}} \psi :\Longleftrightarrow \text{ player II has an optimal strategy in } \mathrm{G}(X_{\varphi,\psi}).$$

Theorem 36. Assume $\mathsf{Bl\text{-}AD}$ and $\mathsf{DC}_{\mathbb{R}}$. Then $\leq_{\mathrm{BlFPT}}$ is a prewellordering.

Proof. Cf. [Lö05a, Theorem 3.2]. □

Theorem 36, along with Theorem 4 is one of the few consequences of $\mathsf{Bl\text{-}AD}$ that holds for all sets of reals and not just for a definable initial segment of the Wadge hierarchy.

8. Weaker Axioms of Blackwell Determinacy

So far, we have looked at three types of games on a given payoff set A: the perfect information game with pure strategies $\mathrm{G}(A)$ ("Gale-Stewart game"), the game with slightly imperfect information with Blackwell strategies $\mathrm{B}(A)$ ("Blackwell game"), and the perfect information game with mixed strategies $\mathrm{M}(A)$. All of these games were symmetric in the sense that both players were using strategies of the same type.

Proposition 1 told us that Gale-Stewart games allow some flexibility concerning the status of the two players: if player I has a strategy winning against all blindfolded strategies, then he has a strategy winning against all pure strategies. This is not true for optimal strategies as was indicated by Proposition 5:

Observation 37. Consider once again the game of playing different reals with payoff set $M := \{x \in 2^{\mathbb{N}} ; \forall n(x(2n) \neq x(2n+1)\}$ as in § 3. We have already seen (Proposition 5) that player II has a winning strategy in $\mathrm{G}(M)$ and player I has an optimal strategy in $\mathrm{B}(M)$. Clearly, in $\mathrm{M}(M)$, player II has an optimal strategy (use the winning strategy from $\mathrm{G}(M)$).

But at the same time, in $\mathrm{M}(M)$, player I has a strategy with the following property: for every blindfolded strategy τ for player II, $\mu_{\sigma,\tau}(M) = 1$.

So, player I has a strategy which is optimal against blindfolded opponents, but cannot be optimal against all opponents.

Proof. Essentially, this is the proof of Proposition 5: we showed that the "randomize" strategy $\sigma(s)(\{0\}) = \sigma(s)(\{1\}) = \frac{1}{2}$ is optimal in $\mathrm{B}(M)$, but every blindfolded strategy is a Blackwell strategy. □

Observation 37 suggests the definition of weaker optimality properties. If we denote the classes of mixed, pure and blindfolded strategies with $\mathcal{S}_{\mathrm{mixed}}$, $\mathcal{S}_{\mathrm{pure}}$, and $\mathcal{S}_{\mathrm{blindfolded}}$, respectively, and let $\mathcal{S}$ be any of these classes, then we can say that a mixed strategy σ is **$\mathcal{S}$-optimal** for the payoff set $A \subseteq \omega^\omega$ if for all $\tau_* \in \mathcal{S}$ for player II, $\mu^-_{\sigma,\tau_*}(A) = 1$ (and similarly for mixed strategies of player II). Clearly, $\mathcal{S}_{\mathrm{mixed}}$-optimality is the same as optimality in the sense of § 5. Proposition 1 can now be rephrased as "for a pure strategy, being a winning strategy and being $\mathcal{S}_{\text{blindfolded}}$-optimal are equivalent".

We can now call a set $A \subseteq \omega^\omega$ **purely** or **blindfoldedly Blackwell determined** if either player I or player II has an $\mathcal{S}_{\text{pure}}$-, or $\mathcal{S}_{\text{blindfolded}}$-optimal strategy, respectively, and we write pBl-AD, and blBl-AD for the full axioms claiming pure or blindfolded Blackwell determinacy for all sets.

Note that Observation 37 raises the question whether blBl-AD behaves like a determinacy statement at all: blindfolded Blackwell determinacy lacks the basic dichotomy property of determinacy theory, as in $\mathrm{M}(M)$, both players have an $\mathcal{S}_{\text{blindfolded}}$-optimal strategy.

It turns out that despite of this behaviour, the axiom blBl-AD and its definable fragments still give us logical strength:

Theorem 38 (Martin)**.** The statements blBl-Det($\mathbf{\Pi}^1_1$), pBl-Det($\mathbf{\Pi}^1_1$), and Det($\mathbf{\Pi}^1_1$) are equivalent.

Proof. Cf. [Lö04, Corollary 3.9] □

Theorem 39. The axioms blBl-AD and pBl-AD imply the existence of an inner model with a strong cardinal.

Proof. Cf. [Lö04, Corollary 4.9]. □

There is no general method to deal with blBl-AD: it is still unknown whether blBl-AD implies that all sets are Lebesgue measurable, and since this is used in the proof of Theorem 25, we cannot get the results on infinitary combinatorics from § 7.2. We were able to derive a consequence violating the axiom of choice in [Lö04, Lemma 3.12], though:

Corollary 40. The axiom blBl-AD violates the axiom of choice.

Moving from blindfolded Blackwell determinacy to pure Blackwell determinacy, we shall be able to show a conditional equivalence theorem. We identify the set of pure strategies with the set $\mathbb{N}^{(\omega^{<\omega})}$. For any mixed strategy τ, we shall define a probability measure V_τ on $\mathbb{N}^{(\omega^{<\omega})}$ which we shall call the **Vervoort code of** τ. If p_0, ..., p_n are elements of $\omega^{<\omega}$ and N_0, ..., N_n are natural numbers, define

$$\mathrm{V}_\tau(\{\tau^*\,;\,\tau^*(p_0) = N_0\,\&\,...\,\&\,\tau^*(p_n) := N_n\}) = \prod_{i=0}^{n} \tau(p_i)(\{N_i\}),$$

and let V_τ be the unique extension of this function to the Borel σ-algebra on $\mathbb{N}^{(\omega^{<\omega})}$.

Theorem 41 (Vervoort)**.** If σ and τ are mixed strategy and B is a Borel subset of ω^ω, then

$$\begin{aligned}\mu_{\sigma,\tau}(B) &= \int \mu_{\sigma,x}(B)\,d\mathrm{V}_\tau(x)\\ &= \int \mu_{x,\tau}(B)\,d\mathrm{V}_\sigma(x)\end{aligned}$$

Proof. This is essentially [Ve95, Theorem 6.6]. □

Theorem 42. Let $\mathbf{\Gamma}$ be a boldface pointclass such that all sets in $\mathbf{\Gamma}$ are Lebesgue measurable. Let $A \in \mathbf{\Gamma}$ and assume that A is purely Blackwell determined. Then A is Blackwell determined. Moreover, every $\mathcal{S}_{\mathrm{pure}}$-optimal strategy is actually $\mathcal{S}_{\mathrm{mixed}}$-optimal.

Proof. Note that for any σ and τ, the $\mu_{\sigma,\tau}$-measurability of a set A is equivalent to the Lebesgue measurability of some continuous preimage of A, so in our situation, all sets in $\mathbf{\Gamma}$ are $\mu_{\sigma,\tau}$-measurable.

Now since A is purely Blackwell determined, let σ be (without loss of generality) an $\mathcal{S}_{\mathrm{pure}}$-optimal strategy for player I. Let τ be an arbitrary mixed strategy for player II. We shall show that for every Borel superset $B \supseteq A$, we have $\mu_{\sigma,\tau}(B) = 1$. Because A is $\mu_{\sigma,\tau}$-measurable, this proves the claim.

By our assumption, we know that for every pure strategy τ^* and every Borel $B \supseteq A$, we have $\mu_{\sigma,\tau^*}(B) = 1$. But this means that the function

$$f : \mathbb{N}^{(\omega^{<\omega})} \to \mathbb{R} : x \mapsto \mu_{\sigma,x}(B)$$

is the constant function with value 1.

Using Theorem 41, we get

$$\mu_{\sigma,\tau}(B) = \int \mu_{\sigma,x}(B)\, d\mathrm{V}_\tau(x) = \int f(x)\, d\mathrm{V}_\tau(x) = 1.$$

□

Corollary 43. Let $\mathbf{\Gamma}$ be a boldface pointclass such that all sets in $\mathbf{\Gamma}$ are Lebesgue measurable. Then $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$.

Using the Martin-Neeman-Vervoort Equivalence Theorem 14, we can bootstrap some equivalence for pure Blackwell determinacy from Corollary 43:

Corollary 44. Let $\mathbf{\Gamma} \in \{\mathbf{\Delta}^1_2, \mathbf{\Sigma}^1_2, \mathbf{\Delta}^1_3\}$. Then $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Det}(\mathbf{\Gamma})$.

Proof. "$\mathbf{\Delta}^1_2$ & $\mathbf{\Sigma}^1_2$": By Theorem 38, we have $\mathsf{Det}(\mathbf{\Pi}^1_1)$ and thus by the usual Solovay unfolding argument, every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable. Now Corollary 43 and Theorem 14 finish the proof.

"$\mathbf{\Delta}^1_3$": By the first argument, we get $\mathsf{Det}(\mathbf{\Sigma}^1_2)$. Again, we use Solovay unfolding to get the Lebesgue measurability of all $\mathbf{\Sigma}^1_3$ sets and finish the proof with Corollary 43 and Theorem 14. □

The methods of Theorem 14 do not extend to prove the equivalence of $\mathsf{Bl\text{-}Det}(\mathbf{\Sigma}^1_3)$ and $\mathsf{Det}(\mathbf{\Sigma}^1_3)$ (*cf.* [Ma$_1$NeVe03, p. 633]). This halts our progress on the third level of the projective hierarchy. Of course, $\mathsf{pBl\text{-}Det}(\mathbf{\Sigma}^1_3)$ gives $\mathsf{Det}(\mathbf{\Delta}^1_3)$ by Corollary 44, hence the Lebesgue measurability of all $\mathbf{\Sigma}^1_3$ sets and thus $\mathsf{Bl\text{-}Det}(\mathbf{\Sigma}^1_3)$ by Corollary 43.

Both for blindfolded Blackwell determinacy and pure Blackwell determinacy, it is open whether they are equivalent to Blackwell determinacy (and then, if Martin's Conjecture 8 is true) to determinacy.

Question 45. Are pBl-AD or blBl-AD equivalent to AD?

Building on the results from [Lö04], Greg Hjorth (2002, personal communication) found a proof of "blBl-AD implies the existence of an inner model with a Woodin cardinal" using Cabal-style descriptive set theory and inner model theory.

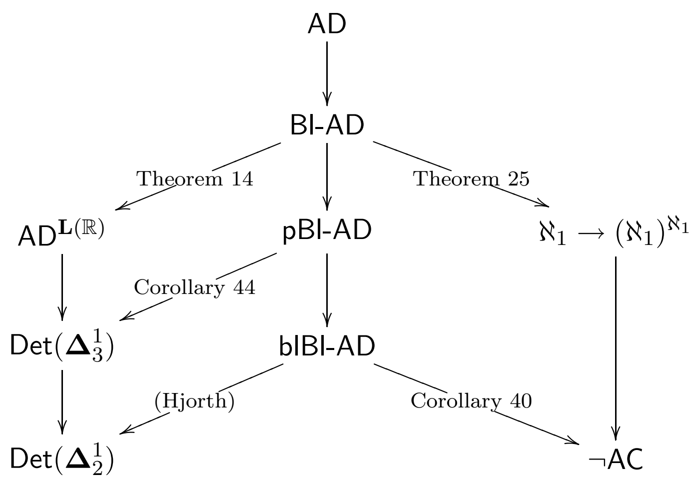


FIGURE 5. Diagram of axioms of determinacy and consequences

9. Stronger Axioms of Blackwell Determinacy

Axioms of determinacy have been extended into several directions (*cf.* [$\mathrm{St}_0$88, p. 96]: researchers have investigated long games (*i.e.*, transfinite games of length $\alpha > \omega$) and games with more than countably many moves (*e.g.*, real number moves). We write AD^α for the axiom of determinacy for games of length α and AD_X for the axiom of determinacy for games in which the players play elements of X in each move.

For perfect information games, there is a limit to both α and X:

Theorem 46. The axioms AD^{ω_1}, AD_{ω_1} and $\mathsf{AD}_{\wp(\mathbb{R})}$ are inconsistent.

Proof. For AD^{ω_1} take the following game: Player I plays 0s for some countable time α. After he plays the first 1, player II has to react by using the next ω rounds to produce an element of WO_α. If the game is determined, player II must have a winning strategy, but this would be a choice function for the family $\{\mathrm{WO}_\alpha\,; \alpha < \omega_1\}$ and thus an uncountable wellordered set of reals which would contradict AD.

The games for AD_{ω_1} and $\mathsf{AD}_{\wp(\mathbb{R})}$ are similar: let player I play α or WO_α itself instead of α many 0s. □

Also, Blass has proved that there is a link between long games and games with uncountably many moves (*cf.* Theorem 54). In this section, we shall investigate what is known about the Blackwell analogues of these extensions.

9.1. **Uncountable sets of moves.** When we want to extend the notion of Blackwell determinacy to subsets of X^ω (in place of ω^ω), almost all of the definitions are extended in the obvious way: mixed strategies become functions from $X^{<\omega}$ to Prob, and the rest of the definitions stay the same using this new extended definition of mixed strategy.

We have to be careful, however, as our definition of Blackwell determinacy involves extending the function $\mu_{\sigma\tau}$ defined on the basic open sets to the entire σ-algebra of the Borel sets. In the case of ω^ω, we needed $\mathsf{AC}_\omega(\mathbb{R})$ in order to show that the set of interpretations of Borel codes is the Borel σ-algebra.

When moving to X^ω, there are two potential problems:

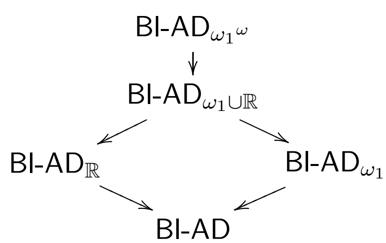
(1) Our definition of the measure $\mu_{\sigma,\tau}$ depends on the values of σ and τ on singleton sets. For uncountable sets X, this might not be enough to properly define a measure. We can easily deal with this problem by requiring the measures to have countable support (*cf.* [Ma$_1$NeVe03, Remark 3.1]): Let Prob(X) the set of probability measures on X with countable support, *i.e.*, those measures μ on X such that there is a countable subset $Y \subseteq X$ such that $\mu(Y) = 1$.

(2) If X is large, then the extension of $\mu_{\sigma,\tau}$ from a function defined on the basic open sets to a function defined on the σ-algebra of Borel sets may require more choice than we have at our disposal in our base theory $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$. This is not the case if there is a surjection from $\mathbb{R}$ onto $\mathbb{R} \times X^\omega$ [Lö05b, Proposition 2].

Proposition 47. For $X \in \{\mathbb{R}, \omega_1, \omega_1 \cup \mathbb{R}, {\omega_1}^\omega\}$, any function $\mu_{\sigma,\tau}$ defined on the basic open sets is extendible to the Borel σ-algebra.

Using Proposition 47, we can now use the obvious analogues of the definitions in § 3 to define the notion of **Blackwell determinacy** for sets in X^ω if $X \in \{\mathbb{R}, \omega_1, \omega_1 \cup \mathbb{R}, {\omega_1}^\omega\}$, and define $\mathsf{Bl\text{-}AD}_X$ to mean "Every $A \subseteq X^\omega$ is Blackwell determined".

These new axioms of Blackwell determinacy obviously give the following implicational structure:



In the following, we shall show that the top two of the strong axioms of Blackwell determinacy are inconsistent. Note that since there is a surjection from $\wp(\mathbb{R})$ onto $\omega_1 \cup \mathbb{R}$ this means that there cannot be a reasonable consistent version of $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$.

Lemma 48. If μ is a probability measure on X with countable support then there is a unique $\mathbf{m}_\mu > 0$ such that

$$\{x\,;\,\mu(\{x\}) = \mathbf{m}_\mu\} \text{ is nonempty and finite, and}$$

$$\{x\,;\,\mu(\{x\}) > \mathbf{m}_\mu\} = \varnothing.$$

Proof. For $\varepsilon > 0$, let $X_\varepsilon := \{x\,;\,\mu(\{x\}) \geq \varepsilon\}$. By σ-additivity, it is impossible that any set X_ε is infinite. Now pick $z \in \mathrm{supp}(\mu)$ and let $\varepsilon := \mu(\{z\})$, so that X_ε is finite and nonempty. Define

$$\mathbf{m}_\mu := \max\{\mu(\{x\})\,;\,x \in X_\varepsilon\}.$$

□

Theorem 49. Let X be linearly ordered by $\preceq$, $n \in \mathbb{N}$, $A \subseteq X^n$ and τ be a optimal strategy for player II in the game on $X^n \backslash A$. Then there is a definable function $\hat{\tau}$ defined on the mixed strategies such that $\mathrm{ran}(\hat{\tau}) \subseteq A$.

Proof. Given a mixed strategy σ, we define recursively

$$x_{2i} := \min_{\preceq}\{x\,;\,\sigma(\langle x_0, \ldots, x_{2i-1}\rangle)(\{x\}) = \mathbf{m}_{\sigma(\langle x_0,\ldots,x_{2i-1}\rangle)}\}, \text{ and}$$

$$x_{2i+1} := \min_{\preceq}\{x\,;\,\tau(\langle x_0, \ldots, x_{2i}\rangle)(\{x\}) = \mathbf{m}_{\tau(\langle x_0,\ldots,x_{2i}\rangle)}\}.$$

By Lemma 48, the minimum is taken over a nonempty finite linearly ordered set, so this sequence is definable in ZF. We let $\hat{\tau}(\sigma)$ be the sequence $\langle x_0, \ldots, x_{n-1}\rangle$ and claim that for all σ, $\hat{\tau}(\sigma) \in A$:

Clearly, $\mathbf{m} := \mu_{\sigma,\tau}(\{\langle x_0, \ldots, x_{n-1}\rangle\})$ is the finite product of the positive numbers $\mathbf{m}_{\sigma(\langle x_0,\ldots,x_{2i-1}\rangle)}$ and $\mathbf{m}_{\tau(\langle x_0,\ldots,x_{2i}\rangle)}$, and therefore strictly positive. If $\langle x_0, \ldots, x_{n-1}\rangle \notin A$, then $\mu_{\sigma,\tau}(X^n \backslash A) \geq \mathbf{m}$, contradicting the optimality of τ. □

For sets Y and Z, we define the class of Y-Z-choice games $\mathrm{CG}_{Y,Z}(A)$ as follows: If $A : Y \to \wp(Z)$ is a family of nonempty subsets of Z indexed by elements of Y, then the game $\mathrm{CG}_{Y,Z}(A)$ is the two-round game in which player I plays an element $y \in Y$, player II follows up with playing an element $z \in Z$, and player II wins if $z \in A(y)$.

Theorem 50. If $X := Y \cup Z$ is linearly ordered and $\mathsf{Bl\text{-}AD}_X$ (is defined and) holds, then $\mathsf{AC}_Y(Z)$ holds.[9]

Proof. Let $A : Y \to \wp(Z)$ be a family of nonempty sets. Because the sets are nonempty, player I cannot have an optimal strategy in $\mathrm{GC}_{Y,Z}(A)$. Let τ be an optimal strategy for player II. Let σ_y be defined by $\sigma_y(\varnothing)(\{y\}) := 1$. Then

$$f : y \mapsto \hat{\tau}(\sigma_y)$$

is a definable choice function by Theorem 49. □

[9]The requirement that $\mathsf{Bl\text{-}AD}_X$ be defined is just a reminder that we need to use Proposition 47 or a similar result.

Corollary 51. The following implications between axioms of Blackwell determinacy and choice principles hold:

(1) $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ implies $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$,
(2) $\mathsf{Bl\text{-}AD}_{\omega_1\cup\mathbb{R}}$ implies $\mathsf{AC}_{\omega_1}(\mathbb{R})$.

Consequently, $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is strictly stronger than $\mathsf{Bl\text{-}AD}$, and $\mathsf{Bl\text{-}AD}_{\omega_1\cup\mathbb{R}}$ and $\mathsf{Bl\text{-}AD}_{\omega_1{}^\omega}$ are inconsistent.

Proof. It is well-known that if $\mathbb{R}$ is not wellordered, then $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ is false in $\mathbf{L}(\mathbb{R})$. But if $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is true in $\mathbf{V}$, then $\mathbf{L}(\mathbb{R}) \models \mathsf{Bl\text{-}AD}$. As discussed in the proof of Theorem 46, the choice principle $\mathsf{AC}_{\omega_1}(\mathbb{R})$ contradicts AD, as it allows to construct an uncountable sequence of reals. Using Theorem 25, we can show that it contradicts $\mathsf{Bl\text{-}AD}$ as well. □

Of course, the gap left by Theorem 46 and Corollary 51 suggests the obvious question:

Question 52. Is $\mathsf{Bl\text{-}AD}_{\omega_1}$ inconsistent?

Another question that is raised by Corollary 51 is the one about the strength of $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$: we know that if $\mathsf{Bl\text{-}AD}$ is consistent, then so is $\mathsf{Bl\text{-}AD} + \neg\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. In the case of $\mathsf{AD}_{\mathbb{R}}$, much more is known: we know that the consistency strength of $\mathsf{AD}_{\mathbb{R}}$ is considerably higher than that of AD. Work of de Kloet, Kieftenbeld and the present author [dKKiLö∞] suggests that we can follow the lines of Solovay's analysis of $\mathsf{AD}_{\mathbb{R}}$ from [So78] and prove analogous theorems for $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. But:

Question 53. Does $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ imply $\mathsf{AD}_{\mathbb{R}}$?

9.2. **Long Blackwell games.** The underlying idea used here to define long Blackwell games goes back to David de Kloet's undergraduate research thesis [dK05]. De Kloet introduces two different versions of the *axiom of real Blackwell determinacy* in this paper, the one that we discussed in § 9.1 and another one that he calls the *Euclidean variant*. De Kloet's Euclidean variant is closely connected to what we shall call $\mathsf{Bl\text{-}AD}^{\omega^2}$.

We shall be defining Blackwell determinacy for games of length $\omega \cdot n$ and ω^2. We say that a **transfinite mixed strategy** is a function from sequences of natural numbers of length $< \omega^2$ into Prob. Given two transfinite mixed strategies σ and τ, we shall now define a Borel measure $\mu^n_{\sigma,\tau}$ on $\omega^{\omega\cdot n}$. As before, we shall be defining the measure on the basic open subsets and then (using $\mathsf{AC}_\omega(\mathbb{R})$) extend this to all Borel sets. The basic open sets of the topological space $\omega^{\omega\cdot n}$ are of the type

$$[s_0, ..., s_{n-1}] := \{x\,;\, \forall i < n\,(\, s_i \subseteq (x)_i\,)\},$$

where $(x)_i(k) := x(\omega \cdot i + k)$ for $i < n$.

The definition will be by recursion on n. If $n = 1$, we can just use the ordinary definition of $\mu_{\sigma,\tau}$ for ordinary mixed strategies. Let us assume that we have defined a measure $\mu^n_{\sigma,\tau}$ on $\omega^{\omega\cdot n}$ which we now want to extend to a measure $\mu^{n+1}_{\sigma,\tau}$ on $\omega^{\omega\cdot(n+1)}$.

Fix $x \in \omega^{\omega\cdot n}$ and $s \in \omega^{<\omega}$ and follow the general idea of the definition of $\mu_{\sigma,\tau}$ in the length ω games. Almost as in § 3, we let

$$\nu_x(\sigma,\tau)(t) := \begin{cases} \sigma(x^\frown t) & \text{if } \mathrm{lh}(t) \text{ is even, and} \\ \tau(x^\frown t) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \omega^{<\omega}$, we can define

$$f_{\sigma,\tau}(x,s) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu_x(\sigma,\tau)(s{\upharpoonright}i)(\{s(i)\}).$$

Finally, we set

$$\mu^{n+1}_{\sigma,\tau}([s_0,...,s_n]) := \int_{[s_0,\dots,s_{n-1}]} f_{\sigma,\tau}(x,s)\; d\mu^n_{\sigma,\tau}(x).$$

Clearly, the measures $\mu^{n+1}_{\sigma,\tau}$ and $\mu^n_{\sigma,\tau}$ **cohere**, *i.e.*,

$$\mu^{n+1}_{\sigma,\tau}([s_0,...,s_{n-1},\varnothing]) = \mu^n_{\sigma,\tau}([s_0,...,s_{n-1}]),$$

and thus generate a Borel measure on ω^{ω^2} by the Kolmogorov consistency theorem.

We have now defined a Borel measure on ω^{ω^2} for each pair of mixed strategies σ and τ. The notion of **optimality** can now be literally copied from the context of length ω games in § 3, and for any $\alpha \leq \omega^2$, the axiom $\mathsf{Bl\text{-}AD}^\alpha$ can be defined as "For every set $A \subseteq \omega^\alpha$, one of the two players has an optimal strategy in $\mathrm{M}(A)$".

We now have a definition of $\mathsf{Bl\text{-}AD}_\mathbb{R}$ from § 9.1 and a definition of $\mathsf{Bl\text{-}AD}^{\omega^2}$. Blass [$\mathrm{Bl}_1$75] proved that the perfect information analogues of these two axioms are equivalent:[10]

Theorem 54 (Blass)**.** The following are equivalent:

(1) $\mathsf{AD}_\mathbb{R}$, and
(2) AD^{ω^2}.

Can we do the same for the Blackwell axioms?

Question 55. Are $\mathsf{Bl\text{-}AD}_\mathbb{R}$ and $\mathsf{Bl\text{-}AD}^{\omega^2}$ equivalent?

Appendix: List of Open Questions

The central open question in this area is of course Martin's Conjecture 8. Its positive solution solves most of the open questions listed below (an asterisk marks those that would be solved by a proof of Martin's Conjecture). Some of the open questions have definable versions: for instance, "Does $\mathsf{Bl\text{-}AD}$ imply that all sets of reals have the perfect set property?" has the definable versions "If $\mathbf{\Gamma}$ is a boldface pointclass, does $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ imply that

[10] *Cf.* [LöRo02] for an improvement to $\mathsf{AD}^{\omega\cdot 2}$ on the basis of a strong result due to Woodin.

all sets in $\mathbf{\Gamma}$ (or $\exists^{\mathbb{R}}\mathbf{\Gamma}$) have the perfect set property?". Those questions with definable versions are marked with a dagger.

Conjecture 8	$\mathsf{Bl\text{-}AD}$ implies AD.
Question 17	Does $\mathsf{Bl\text{-}AD}$ imply that all sets of reals have the perfect set property?*†
Question 18	Does $\mathsf{Bl\text{-}AD}$ imply that all sets of reals have the Baire property?*†
Question 19	Does $\mathsf{Bl\text{-}AD}$ imply that all Turing closed sets of reals are Martin measurable?*†
Question 24	Does $\mathsf{Bl\text{-}AD}$ imply the Moschovakis Covering Lemma?*
Question 27	Does $\mathsf{Bl\text{-}AD}$ imply $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$?*
Question 28	Does $\mathsf{Bl\text{-}AD}$ imply that $\boldsymbol{\delta}^1_3$ has the strong partition property?*
Question 33	Does $\mathsf{Bl\text{-}AD} + \mathsf{DC}_{\mathbb{R}}$ imply that $\leq_{\mathrm{BlL}}$ is wellfounded?*
Question 45	Are $\mathsf{pBl\text{-}AD}$ or $\mathsf{blBl\text{-}AD}$ equivalent to AD?
Question 52	Is $\mathsf{Bl\text{-}AD}_{\omega_1}$ inconsistent?
Question 53	Does $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ imply $\mathsf{AD}_{\mathbb{R}}$?
Question 55	Are $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ and $\mathsf{Bl\text{-}AD}^{\omega^2}$ equivalent?

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