Metric and Layered Temporal Logic for Time Granularity

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To the memory of my father.

Contents

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Udine — Angelo Montanari
August, 1996.

Chapter 1

The general picture

1.1 Introduction

This dissertation is about the design of temporal logics that deal with changing time granularities. *Time granularity* can be defined as the resolution power of the temporal qualification of a statement. Providing a formalism with the concept of time granularity makes it possible to specify time information with respect to differently-grained temporal domains. This does not merely mean that one can use different time units–say, months and days–to represent time quantities in a *unique flat temporal model*, but it involves more difficult semantic issues related to the problem of assigning a proper meaning to the association of statements with the different temporal domains of a *layered temporal model* and of switching from one domain to a coarser/finer one.

The original motivation of the work described in this dissertation was the design of a temporal logic suitable for the specification of real-time systems whose components evolve according to different time scales. Nevertheless, there are significant similarities between the problems it addresses and those dealt with by the current research on logics that deal with changing contexts and perspectives. Indeed, even if it has been developed in a temporal framework, our proposal actually outlines the basic features of a general *logic of granularity*. In this respect, it can be seen as a generalization of Rescher and Garson's topological logic [107] to layered structures. Moreover, it presents interesting connections with the logics of contexts discussed in [8, 22], where modalities are used to shift variables, domains, and interpretation functions from one context to another. More generally, the design of these types of logics is emerging as a relevant research topic in the broader area of combination of logics, theories, and structures, at the intersection of logic with artificial intelligence, computer science, and computational linguistics, e.g., [15, 16, 46, 47]. In this dissertation, we will devise suitable combination techniques both to define temporal logics and to prove logical properties of these logics, such as completeness and decidability.

As noticed in [9], being able to provide and relate temporal representations at different 'grain levels' of the same reality is an important research theme for temporal logic and a major requirement for many applications. With regard to *logical specifications*, there exists a large class of real-time systems whose components have dynamic behavior regulated by very different time constants (granular real-time systems). A good specification language must enable one to specify and verify the components of a granular system and their interactions in a simple and intuitively clear way, see e.g., [32, 34, 45]. With regard to *temporal databases*, when information is collected from different sources which are not under the same control, differently-grained time-stamps are associated with different data. To guarantee consistency either the data must be converted into a uniform representation that is independent of time-granularity, or temporal operations must be generalized to cope with data associated with different temporal domains. In both cases, a precise semantics for time granularity is needed; see e.g., [13, 14, 31, 27, 41, 85, 122, 125, 123]. With regard to *problem solving*, intelligent temporal reasoning systems should be able to switch among time granularities in order to provide either quick coarse-grain answers, or slower fine-grain ones, depending on the requirements for responsiveness and quality of the answer, see e.g., [35, 40, 43, 62, 75, 78, 97, 112]. Finally, shifts in the temporal perspective occur very often in *natural language communication*, and thus the ability of supporting and relating a variety of temporal models, at different grain sizes, is a relevant feature for the task of natural language understanding, see e.g., [49]. For all these application domains (and many others), the flatness of the temporal model underlying most logics of time proposed in the literature is a major drawback.

Despite the widespread recognition of its relevance in the fields of formal specifications, knowledge representation and reasoning and temporal databases, there is a lack of a systematic framework for time granularity. To the best of our knowledge, besides in the above mentioned papers, time granularity or related concepts have been discussed in [30, 44, 55, 67, 76, 83, 111, 113, 126]. In particular, Lamport introduces different temporal views in order to find a convenient temporal representation of computational processes [76]. Hobbs proposes a formal characterization of the general notion of granularity, but gives no special attention to time granularity [67]. He only sketches out a rather restrictive mapping of continuous time into discrete times using the situation calculus formalism. Clifford and Rao provide a set-theoretic formalization of time granularity, but they do not attempt to relate the truth value of statements to time granularity [30]. Galton and Shoham give significant categorizations of statements based on their temporal properties that are strictly related to the concept of time granularity even if it is not explicitly considered [55, 113]. Finally, extensions to existing languages for formal specifications, knowledge representation, and temporal databases to support a limited concept of time granularity have been proposed by Roman [111], Evans and Montanari et al. [44, 83], and Wiederhold et al. [126], respectively.

In this dissertation, we propose a metric and layered temporal logic for time granularity, and we show how to use it to specify granular real-time systems. We start by considering the purely metric fragment in isolation. We define a general two-sorted framework where a number of metric temporal logics, having a different expressive power, can be defined as suitable combinations of a temporal component and an algebraic one. Then, we exploit the proposed framework to study completeness issues for the various systems of metric temporal logic. Despite their relevance, these issues have been ignored or only partially addressed in the literature. The next step is the definition of a many-layer metric temporal logic, embedding the notion of time granularity. We identify the main functionalities a logic for time granularity must support and the constraints it must satisfy. In particular, we identify the set of properties constraining the relations between time instants belonging to different layers. Then, we axiomatically define a metric and layered temporal logic, viewed as the combination of a number of differently-grained (single-layer) metric temporal logics, and we study its logical properties. We devote special attention to the decidability problem. We identify relevant classes of metric and layered temporal structures, and show that the corresponding theories are decidable. More precisely, we prove the decidability of the validity and satisfiability problems for the theory of finitely-layered metric temporal structures, and for two relevant theories of ω-layered metric temporal structures. These decidability results provide useful insights about the relations between many-layer and flat metric temporal systems, e.g., they answer the natural question whether, and under which conditions, many-layer temporal systems can be reduced to flat ones. In the last part of the dissertation, we concentrate on the problem of executing metric and layered temporal logics. However, instead of proposing any specific-system oriented solution, we devise a general computational strategy which has a value of its own, and whose range of applicability is not restricted to temporal logics.

In the following section, we discuss the application of metric and layered temporal logic to the specification of granular real-time systems. The last section briefly outline the general structure of the dissertation.

1.2 The specification of granular real-time systems

1.2.1 The addition of a metric of time

Logic-based methods for representing and reasoning about temporal information have proved to be highly beneficial in the area of formal specifications [58, 115]. Timing properties play a major role in the specification of reactive and concurrent software systems that operate in real-time, which are among the most critical software systems. They constrain the interactions between different components of the system as well as between the system and its environment, and minor changes in the precise timing of interactions may lead to radically different behaviors. Plants or weapon control devices, "fly by wire" aircraft, time critical information systems and embedded applications are only some examples of the important family of real-time systems.

Temporal logic has been successfully used for modeling and analyzing the behavior of reactive and concurrent systems (cf. Pnueli [103] and Manna and Pnueli [79]). It supports semantic model checking, which can be used to verify consistency of specifications, and to check positive and negative examples of system behavior against specifications; it also supports pure syntactic deduction, which may be used to prove properties of systems.

Unfortunately, most common specification languages [17, 50, 64, 63, 71] are inadequate for real-time applications: they cannot deal with temporal properties in a simple and satisfactory way, because they lack an explicit and quantitative representation of time. A few remarkable exceptions, however, do exist. They are extensions of Petri Nets [81, 106, 59] or versions of Temporal Logic [2, 57, 72, 101], which support direct and quantitative specifications of temporal properties and relevant validation activities.

In this dissertation, we will present (a suitable extension of) *metric temporal logics* which provide a uniform framework in which both qualitative and quantitative timing properties of real-time systems can be expressed by means of a parametrized operator of (relative) temporal realization. The main issues to be confronted when formalizing a metric temporal logic for executable specifications are:

Expressiveness (definability). Is the metric temporal logic powerful enough to express both the properties of the underlying temporal structure and the timing requirements of the specified real-time systems?

Soundness and completeness. Is the metric temporal logic equipped with a sound and complete axiomatization?

Decidability. Which properties of the specified real-time system can be automatically verified? Most temporal logics for real-time systems proposed in the literature cannot be decided (cf. Henzinger [65]). Some of them recover decidability sacrificing completeness with respect to the original model class.

Executability. How can we prove the consistency and adequacy of specifications? In principle, decidability proof methods (e.g. via Büchi automata) outline an effective procedure to prove the satisfiability and/or validity of a formula. But as soon as certain assumptions about the nature of the temporal domain and the available set of primitive operations are relaxed, the satisfiability/validity problem becomes undecidable (Alur and Henzinger [2]). An alternative approach consists in looking at metric temporal logics as particular polymodal logics and supporting derivability by means of proof procedures for nonclassical logics or via translations in first-order theories (cf. Ohlbach [100]). In this case, providing the logic with a sound and complete axiomatization becomes a central issue.

All these issues will be addressed in this dissertation.

1.3 The addition of time granularity

There are, however, systems whose temporal specification is far from being simple even with timed Petri Nets or metric temporal logic. Consider the wide-ranging class of real-time systems whose components have dynamic behaviours regulated by very different—even by orders of magnitude—time constants (hereafter *granular real-time systems*). For instance, a pondage power station consists of a reservoir, with filling and emptying times of days or weeks, generator units, possibly changing state in a few seconds, and electronic control devices, evolving in milliseconds or even less. A complete specification of the power station must include the description of these components and of their interactions.

A natural description of the temporal evolution of the reservoir state will probably use days: "During rainy weeks, the level of the reservoir increases 1 meter a day". The description of the control devices behaviour may use microseconds: "When an alarm comes from the level sensors, send an acknowledge signal in 50 microseconds". We say that systems of such a type have *different time granularities*. It is somewhat unnatural to compel the specifier of these systems to use a unique time granularity, microseconds in the previous example, to describe the behaviour of all the components. For instance, the specifier of the requirements for a pondage power plant should not be compelled to write sentences like "the filling of the reservoir must be completed within n microseconds". A good language must allow the specifier to easily describe all simple and intuitively clear facts. A major issue of specification languages is in fact the naturalness of the notation. Then, *different time granularities* must be a feature of a specification language for granular real-time systems.

In particular, the addition of time granularity to a specification language for granular real-time systems enhances its modularity, abstraction, and flexibility.

Modular specifications. It allows one to maintain the representations of the dynamics of different components of the specified system, that evolve according to different time constants, as separate as possible. For instance, the temporal evolution of the basin level of a hydroelectric plant depends on at least three different processes that evolve according to very different time constants: the flow of water, whose time constant is *day*, the opening and the closing of the radial gates, whose time constant is *minute*, and the electronic control, whose time constant is *microsecond*.

Incremental specifications. Often, assigning a meaning to a statement in a domain whose time granularity is finer than the original one requires some *extra knowledge*. In some case, as in the previous examples, such an extra knowledge is *implicit* in the use of natural language. In other cases, the change of time granularity is paired with a *refinement process*. As an example, such a process is often applied when complex specifications are written in an incremental way. Indeed, the refinement of a given specification level generally requires the definition of higher-level predicates in terms of more detailed ones, and such a refinement often involves a change of time granularity. For instance, the radial gate opening can be described as a whole with respect to *minutes*, while it can be decomposed into a number of component subprocesses with respect to *seconds*.

Flexible specifications. Finally, time granularity increases both the temporal distinctions that a language can make and the distinctions that it can leave unspecified. This means that considering two events as simultaneous or temporally distinct, or two time-varying relations as temporally overlapped or disjoint, depends on the granularity one refers to. Typically, the standard components of a granular real-time system, whose behavior is well-known, can be specified at coarse time granularities, while the most innovative ones require a detailed specification at fine time granularities. Time granularity makes it possible to differentiate the level of refinement of the specifications of different component of a (granular) real-time system.

The red thread connecting the chapters of this dissertation is the definition of a temporal logic for time granularity, the proof of its fundamental logical properties, and the illustration of some examples of its application to the specification of granular real-time systems.

1.4 Outline of the dissertation

In the following, we briefly outline the structure of the dissertation, and summarize the main contributions included in each chapter.

In Chapters 2 and 3, we define the basic systems of two-sorted metric temporal logic and extend them with time granularity. The logical properties of the proposed systems are proved by means of *modal and temporal logic* techniques. In Chapter 4, we concentrate on decidability issues for metric and layered temporal logics. In order to prove the decidability of different classes of layered structures, we exploit tools and techniques borrowed from *formal languages and automata theory.* Finally, in Chapter 5, we deal with the problem of supporting derivability in metric and layered temporal logics. The proposed solution is based on a *set-theoretic translation method* for polymodal logics, paired with *automated theorem proving* in first-order set theories.

The main contributions of each chapter are the following ones.

In Chapter 2, we explore *completeness issues* of metric temporal logic (*MTL* for short). We do this by starting with a very basic system, and we build on it either by adding axioms or by enriching the underlying structures. We view *MTLs* as two-sorted logics having both formulae and parameters; formulae are evaluated at time instants, while parameters take values in an (ordered) abelian group of temporal displacements. We first define a minimal *MTL* that can be seen as the metric counterpart of minimal tense logic, and we provide it with a sound and complete axiomatization. Next, we characterize the class of two-sorted frames with a linearly ordered temporal domain. Then, we extend our systems with the ability to mix temporal and displacement formulae to make their logical machinery sufficiently powerful. In the last part of the chapter, we show how *MTL* can be viewed as a Propositional Dynamic Logic where programs have been replaced by displacements. Such a rewriting will be fully exploited in Chapter 5 to support *MTL* executability. In the conclusions, we provide an assessment of the work and outline further directions of research, including the possibility of using the proposed two-sorted framework for characterizing a variety of *MTLs* simply by changing the requirements on the algebraic and/or temporal components. We also briefly discuss decidability issues in *MTLs*.

In Chapter 3, we develop a *metric and layered temporal logic* (*MLTL* for short), extending *MTL* with time granularity, and show how it can be used for specifying granular real-time systems. *MLTL* replaces the flat temporal domain of *MTLs/* with a temporal universe consisting of a set of differently-grained temporal domains. Such a temporal universe identifies the temporal domains relevant to a granular system specification and defines the relations between instants belonging to different domains. To qualify formulae with respect to the temporal universe, *MLTL* is provided with an operator of *contextualization* that identifies the domains a given formula refers to. Within each temporal domain, it is then possible to talk about truth and falsehood of formulae at different time instants by means of a *displacement* operator. Finally, a *projection* operator is added to constrain the relationships between formulae associated with differently-grained domains.

The combined use of these operators allows one to represent a granular system by properly connecting a set of differently-grained formulae. In the simplest case, such a representation consists of the logical composition of a number of formulae referring to different temporal domains. In more complex cases, the projection operator is used to deal with nested quantifications of differently-grained temporal displacements (e.g., to specify the condition: "there exist some days during which the plant remains inactive for some hours"), or to specify the composition of differently-grained temporal displacements (e.g., to specify the condition: "in twenty seconds, five minutes will have passed from the occurrence of the fault"). Moreover, *MLTL* can be provided with *consistency rules* that, given the truth value of a formula with respect to the domains it explicitly refers to, constrain its truth value with respect to other domains. To this end, we define two consistency rules that respectively allow one to project temporal formulae from coarser to finer domains (*downward temporal projection*) and from finer to coarser ones (*upward temporal projection*), and then show that they are interdeducible.

In order to guarantee the usefulness of *MLTLs* as formal tools, it is necessary to show some basic decidability properties. In Chapter 4, we present some *decidable theories of metric and layered temporal structures.* The decidability problem for the pure metric (non-granular) fragment has been addressed by Alur and Henzinger in [2] which was, in fact, our starting point. They showed that, under suitable assumptions about the temporal domain and the associated operations, the validity and satisfiability problems for real-time logics extending propositional temporal logics with metric features are decidable. These problems can indeed be reduced to the corresponding problem for a decidable theory: the well-known theory $S1S$.

In the first part of Chapter 4, we present a first extension of their results, aiming at dealing with time granularity. Such an extension allows one to treat situations in which a finite number of coarsenings/refinements of the temporal domain is sufficient. The key idea to deal with the resulting finitely-layered metric temporal structures is to reformulate the decidability problem into an equivalent one relative to the finest metric component (layer). We first formally defined the theory T_{ML} of finitely-layered metric temporal structures, and the associated second-order language $\mathcal{L}^2_{ML}$. Then, we defined a computable function τ which translates each sentence ϕ of the language $\mathcal{L}^2_{ML}$ for T_{ML} into a sentence $\tau(\phi)$ of the language $\mathcal{L}^2$ underlying the theory $S1S$ so that $\tau(\phi)$ is valid (satisfiable) in $S1S$ if and only if ϕ is valid (satisfiable) in T_{ML}. The translation is actually performed in two steps: we first embed finitely-layered metric temporal structures into (flat) metric temporal structures; then, we reduce metric temporal structures to *S1S* structures. Hence, in both the original work by Alur and Henzinger and the above mentioned extension to finitely layered temporal structures, the basic tool for proving decidability properties is the theory $S1S$, and the basic engine is Büchi theorem on the decidability of regular ω-languages.

In the second part of Chapter 4, we deal with the more general case in which the underlying temporal structure consists of infinitely many temporal layers (ω-layered, k-refinable, metric temporal structures). We introduce the second-order language $\mathcal{L}^2_{\omega ML^k}$ for ω-layered (k-refinable) metric temporal structures, and show how to interpret it over different classes of structures. We first consider the case of temporal structures in which there is a finest temporal domain together with an infinite number of coarser and coarser domains (*upward unbounded layered structures*). We define a proper decidable extension of $S1S$, called $S1S^k$, and prove that the decidability of the satisfiability (resp. validity) problem for the theory of upward unbounded layered structures can be reduced to the corresponding problem for $S1S^k$. Next, we deal with the problem of deciding infinitely refinable structures (*downward unbounded layered structures*), and we prove that the decidability of the satisfiability (resp. validity) problem for the theory of such structures can be reduced to the decidability of the satisfiability (resp. validity) problem for SkS, the well-known monadic second-order decidable theory of k successors [118].

In Chapter 5, we show how to support MTL executability via a set-theoretic translation of its PDL rewriting described in Chapter 2. We first present a novel *set-theoretic translation method* for polymodal logics that reduces derivability in a large class of propositional polymodal logics to derivability in a very weak first-order set theory Ω. Unlike most existing translation methods, the one we propose applies to any normal (complete) finitely axiomatizable polymodal logic, regardless of whether it is first-order complete or an explicit semantics is available. In the first part of the chapter, we introduce the set-theoretic translation method and prove its soundness and completeness with respect to (complete) monomodal logics. Then, we generalize the translation to modal logics involving many accessibility relations. We preliminary consider the case of polymodal logic, and then modify the resulting translation to deal with MTL. In the last part of the chapter, we briefly summarize some related works devoted to (i) a systematic analysis of extensions of the proposed set-theoretic translation method, (ii) a comparison between the proposed translation and the standard one, and (iii) an analysis of the computational properties of Ω.

Chapter 2 is based on the paper "Completeness results for two-sorted metric temporal logics" (with Maarten de Rijke), published in the Proceedings of the *4th International Conference on Algebraic Methodology and Software Technology* (an extended and revised version has been submitted to a special issue of *Theoretical Computer Science* including a selection of the papers presented at the Conference). Chapter 3 is based on the papers: "Dealing with Different Time Granularities in Formal Specifications of Real-Time Systems" (with Edoardo Corsetti and Elena Ratto), published in the *Journal of Real-Time Systems*, "Embedding Time Granularity in a Logical Specification Language for Synchronous Real-Time Systems" (with Emanuele Ciapessoni, Edoardo Corsetti, and Pierluigi San Pietro), published in *Science of Computer Programming*, and "A Metric and Layered Temporal Logic for Time Granularity, Synchrony and Asynchrony" published in the Proceedings of the *ICTL'94 Workshop*. The first part of Chapter 4 is based on the paper "Decidability results for metric and layered temporal logics" (with Alberto Policriti), published in the *Notre Dame Journal of Formal Logic*. Chapter 5 is based on the papers: "A set-theoretic translation method for polymodal logics" (with Giovanna D'Agostino and Alberto Policriti), published in the *Journal of Automated Reasoning*, "Set-theoretic decidability results for modal theorem proving" (with Giovanna D'Agostino and Alberto Policriti), published in the Proceedings of the *5th Italian Conference on Theoretical Computer Science*, and "Modal deduction in Second-Order Logic and Set Theory - I" (with Johan van Benthem, Giovanna D'Agostino, and Alberto Policriti), to be published in the *Journal of Logic and Computation.*

Chapter 2

Two-sorted metric temporal logic

2.1 Introduction

In this chapter, we define basic systems of two-sorted Metric Temporal Logic (*MTL* for short) which offer a uniform logical framework for specifying qualitative and quantitative timing properties of (non-granular) real-time systems. In the next chapter, we will show how to extend *MTL* with time granularity.

The idea of a logic of positions (topological, or metric, logic) has originally been formulated by Rescher and Garson [107]. They defined the basic features of the logic, and showed how to give it a temporal interpretation. The logic of positions extends propositional logic with a parametrized operator P_α of positional realization. Such an operator allows one to constrain the truth value of a proposition at position α. The parameter α denotes either (i) an absolute position or (ii) a displacement with respect to the current position which is left implicit. According to interpretation (ii), $P_\alpha q$ is true at the position i if and only if q is true at a position j at distance α from i. In [107], Rescher and Garson introduced two axiomatizations of the logic of positions that differ from each other in the interpretation of parameters. Later, Rescher and Urquhart [108] proved the soundness and completeness of the axiomatization based on an absolute interpretation of parameters through a reduction to monadic quantification theory. Independently, a metric temporal logic has been developed by Koymans [72] to support the specification and verification of real-time systems. He extended the standard model for temporal logic based on point structures with a distance function that measures, for any pair of time points, how far they are apart in time. He provided the logic with a sound axiomatization, but no proof of completeness was given.

The chapter is mainly devoted to explore (soundness and) completeness issues for basic systems of *MTL*. We do this by starting with a very basic system, and build on it either by adding axioms or by enriching the underlying structures. We view *MTLs* as two-sorted logics having both formulae and parameters; formulae are evaluated at time instants while parameters take values in an (ordered) abelian group of temporal displacements. In Section 2.2, we define a minimal metric logic that can be seen as the metric counterpart of minimal tense logic, and we provide it with a sound and complete axiomatization. Then, we show how to obtain the metric temporal logic of linear orders by adding an ordering over displacements. In Section 2.4, we consider general *MTLs* allowing quantification over algebraic variables and free mixing of algebraic formulae and temporal propositional symbols. Finally, in Section 2.5, we propose a reformulation of *MTL* in the style of Propositional Dynamic Logic. This alternative characterization of *MTL* will be at the basis of its execution via translation into a suitable set-theory, as shown in Chapter 5. At the end, we provide an assessment of the work and outline further directions of research, including the possibility of using the proposed two-sorted framework for characterizing a variety of metric (temporal) logics simply by changing the requirements on its algebraic and/or temporal components. We also consider the decidability problem of *MTL*, and briefly discuss how standard methods from modal logic, such as filtration, can be used to deal with this problem.

2.2 The basic metric logic

In this section we define the minimal metric temporal logic MTL_0, and consider some of its natural extensions.

Language. We define a two-sorted temporal language for our basic calculus MTL_0. First, its algebraic part is built up from a non-empty set A of *constants* denoting the group elements. The set of terms over A, $T(A)$, is the smallest set such that (1) $A \subseteq T(A)$, and (2) if α, $\beta \in T(A)$ then $(\alpha + \beta)$, $(-\alpha), 0 \in T(A)$. Next, the temporal part of the language is built up from a non-empty set Φ of *proposition letters*. The set of MTL_0-formulae over Φ and A, $F(\Phi, A)$, is the smallest set such that (1) $\Phi \subseteq F(\Phi, A)$, and (2) if ϕ, $\psi \in F(\Phi, A)$ and $\alpha \in T(A)$, then $\neg\phi$, $\phi \wedge \psi$, $\Delta_\alpha \phi$ (and its dual $\nabla_\alpha \phi := \neg\Delta_\alpha\neg\phi$), $\perp \in F(\Phi, A)$. We will adopt the following notational conventions: p, q, ... denote proposition letters; ϕ, ψ, ... denote MTL_0-formulae; Σ, Γ, ... denote sets of MTL_0-formulae; α, β, ... denote algebraic terms.

Structures. We define a *two-sorted frame* to be a triple $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$, where T is the set of (time) points over which temporal formulae are evaluated, $\mathfrak{D}$ is the algebra of metric displacements in whose domain D terms take their values, and $\mathrm{DIS} \subseteq T \times D \times T$ is an accessibility relation relating pairs of points and displacements.

We require the following properties to hold for the components of two-sorted frames. First, $\mathfrak{D}$ should be an abelian group, that is, a 4-tuple $(D, +, -, 0)$ where $+$ is a binary function of *displacement composition*, $-$ is a unary function of *inverse displacement*, and 0 is the *zero displacement* constant, such that:

(i)	$\alpha + \beta = \beta + \alpha$	(commutativity of $+$)
(ii)	$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$	(associativity of $+$)
(iii)	$\alpha + 0 = \alpha$	(zero element of $+$)

(*iv*) $\alpha + (-\alpha) = 0$ (inverse)

Second, we require the displacement relation DIS to respect the converse operation of the abelian group in the following sense: if $\mathrm{DIS}(i, \alpha, j)$ then $\mathrm{DIS}(j, -\alpha, i)$.

We turn a two-sorted frame $\mathfrak{F}$ into a *two-sorted model* by adding an interpretation for our algebraic terms, and a valuation for atomic temporal formulae. An *interpretation* for algebraic terms is given by a function $g : A \to D$ that is automatically extended to all terms from $T(A)$. A *valuation* is simply a function $V : \Phi \to 2^T$. Then, we say that an equation $\alpha = \beta$ is *true* in a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ whenever $g(\alpha) = g(\beta)$. Next, *truth* of temporal formulae is defined by

$$
\begin{array}{rll}
\mathfrak{M}, i \Vdash p & \text{iff} & i \in V(p) \\
\mathfrak{M}, i \Vdash \bot & & \text{never} \\
\mathfrak{M}, i \Vdash \neg\phi & \text{iff} & \mathfrak{M}, i \nVdash \phi \\
\mathfrak{M}, i \Vdash \phi \wedge \psi & \text{iff} & \mathfrak{M}, i \Vdash \phi \text{ and } \mathfrak{M}, i \Vdash \psi \\
\mathfrak{M}, i \Vdash \Delta_\alpha \phi & \text{iff} & \text{there exists } j \text{ such that } \mathrm{DIS}(i, g(\alpha), j) \text{ and } \mathfrak{M}, j \Vdash \phi.
\end{array}
$$

To avoid messy complications we only consider one-sorted consequences $\Gamma \models \phi$; for algebraic formulae '$\Gamma \models \phi$' means 'for all two-sorted models $\mathfrak{M}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$'; for temporal formulae it means 'for all models $\mathfrak{M}$, and times instants i, if $\mathfrak{M}, i \Vdash \Gamma$, then $\mathfrak{M}, i \Vdash \phi$'.

A simple example. Even though the language of MTL_0 is very poor, it already allows us to express conditions on real-time systems. As a first example, consider a communication channel C that outputs each message with a delay δ with respect to its input time, and that neither generates nor loses messages (cf. Montanari et al. [29, 84]). C can be specified as follows:

$$out \leftrightarrow \Delta_{-\delta} in.$$

This example can easily be generalized to the case of a channel C that collects messages from n different sources $S_1, \ldots, S_n$ and outputs them with delay δ. To exclude that two input events can occur simultaneously, we add the constraint:

$$\forall i, j\, \neg(in(i) \wedge in(j) \wedge i \neq j),$$

which is shorthand for

$$\neg(in(1) \wedge in(2)) \wedge \ldots \wedge \neg(in(n-1) \wedge in(n)).$$

Then the behavior of C is specified by the formula

$$\forall i\, (out(i) \leftrightarrow \Delta_{-\delta} in(i)),$$

which is shorthand for a finite conjunction.

Notice that preventing input events from occurring simultaneously also guarantees that output events do not occur simultaneously.

Suppose now that C outputs the messages it receives from $S_1, \ldots, S_n$ with delays $\delta_1, \ldots, \delta_n$, respectively. Constraining input events not to occur simultaneously no longer guarantees that there are no conflicts at output time. A simple strategy of conflict resolution consists in assigning a different priority to messages coming from different knowledge sources, so that, when a conflict occurs, C only outputs the message with highest priority. Accordingly, the specification of C is modified, preserving the requirement that it does not generate messages, but relaxing the requirement that it does not lose messages.

Assume that $S_1, \ldots, S_n$ are listed in decreasing order of priority. The behavior of C can be specified as follows:

$$\forall i\,(out(i) \leftrightarrow (\Delta_{-\delta_i} in(i) \wedge \neg\exists j\,(\Delta_{-\delta_j} in(j) \wedge j < i)))),$$

which is a shorthand for

$$(out(1) \leftrightarrow \Delta_{-\delta_1} in(1)) \wedge (out(2) \leftrightarrow (\Delta_{-\delta_2} in(2) \wedge \neg\Delta_{-\delta_1} in(1))) \wedge \ldots \wedge$$

$$\wedge(out(n) \leftrightarrow (\Delta_{-\delta_n} in(n) \wedge (\neg\Delta_{-\delta_1} in(1) \wedge \ldots \wedge \neg\Delta_{-\delta_{n-1}} in(n-1))).$$

More complex examples are given in later sections.

Axioms. Our basic calculus MTL_0 has two components. On the one hand it has the usual laws of algebraic logic to deal with the displacements:

(Ref)	$\vdash \alpha = \alpha$	for all terms α (reflexivity)
(Sym)	$\vdash \alpha = \beta \implies \vdash \beta = \alpha$	(symmetry)
(Tra)	$\vdash \delta = \alpha,\ \alpha = \beta \implies \vdash \delta = \beta$	(transitivity)
(Rep)	$\vdash \alpha = \beta \implies \vdash \delta(\alpha/x) = \delta(\beta/x)$	(replacement)
(Sub)	$\vdash \alpha = \beta \implies \vdash \alpha(\delta/x) = \beta(\delta/x)$	(substitution),

as well as the above axioms (*i*)–(*iv*) for abelian groups. Here, $\beta(\alpha/x)$ denotes the result of substituting α for all occurrences of x in β.

The second component of MTL_0 governs the temporal aspect of our structures; its axioms are the usual axioms of propositional logic plus

(Ax1)	$\nabla_\alpha(p \to q) \to (\nabla_\alpha p \to \nabla_\alpha q)$	(normality)
(Ax2)	$p \to \nabla_\alpha \Delta_{-\alpha} p,$	(symmetry)

and its rules are modus ponens and

(NEC)	$\vdash \phi \implies \vdash \nabla_\alpha \phi$	(necessitation rule for ∇_α)
(REP)	$\vdash \phi \leftrightarrow \psi \implies \vdash \chi(\phi/p) \leftrightarrow \chi(\psi/p)$	(replacement)
	where (ϕ/p) denotes substitution of ϕ for the variable p	
(LIFT)	$\vdash \alpha = \beta \implies \vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi$	(transfer of identities).

Axiom (Ax1) is the usual distribution axiom; axiom (Ax2) expresses that a displacement α is the converse of a displacement $-\alpha$. The rules (NEC) and (REP) are familiar from modal logic, and the rule (LIFT) allows us to transfer provable algebraic identities from the displacement domain to the temporal domain.

A *derivation in MTL_0* is a sequence of terms and/or formulae $\sigma_1, \ldots, \sigma_n$ such that each σ_i ($1 \leq i \leq n$) is either an axiom, or obtained from $\sigma_1, \ldots, \sigma_{n-1}$ by applying one of the derivation rules of MTL_0. We write $\vdash_{MTL_0} \sigma$ to denote that there is a derivation in MTL_0 that ends in σ. It is an immediate consequence of this definition that $\vdash_{MTL_0} \alpha = \beta$ iff $\alpha = \beta$ is provable (in algebraic logic) from the axioms of abelian groups only: whereas we can lift algebraic information from the displacement domain to the temporal domain using the (LIFT) rule, there is no way in which we can import temporal information into the displacement domain. As with consequences, we only consider one-sorted inferences '$\Gamma \vdash \phi$'.

Completeness. In this subsection we prove completeness for the basic calculus MTL_0. Our strategy will be to construct a canonical-like model by taking the free abelian group over our algebraic elements as the displacement component, by taking the familiar canonical model as the temporal component, and by linking the two in a suitable way.

The displacement domain. Recall that $T(A)$ is the collection of all algebraic terms built up from the elements of A. Define a congruence relation θ on $T(A)$ by taking

$$(\alpha, \beta) \in \theta \ \text{ iff } \ \vdash_{MTL_0} \alpha = \beta.$$

Then the *canonical displacement domain* $\mathfrak{D}^0$ is constructed by taking

$$\begin{aligned} D^0 &= T(A)/\theta \\ \alpha/\theta + \beta/\theta &= (\alpha + \beta)/\theta \\ -\alpha/\theta &= (-\alpha)/\theta \\ 0 &= 0/\theta. \end{aligned}$$

That $\mathfrak{D}^0$ is indeed an abelian group is easily shown using the defining axioms and rules of MTL_0. The group $\mathfrak{D}^0$ is known as the *free abelian group over A* (cf. Burris and Sankappanavar [21]).

We interpret our terms using the *canonical mapping* $g : T(A) \to \mathfrak{D}^0$ defined by $\alpha \mapsto \alpha/\theta$.

The temporal domain. A set of MTL_0-formulae is *maximal MTL_0-consistent* (or: an MCS) if it is MTL_0-consistent and it does not have proper MTL_0-consistent extensions. The *canonical temporal domain* T^0 is constructed by taking

$$T^0 = \{\Sigma \mid \Sigma \text{ is maximal } MTL_0\text{-consistent } \}.$$

Define a *canonical valuation* V^0 by putting $V^0(p) = \{\Sigma \mid p \in \Sigma\}$.

The canonical model for MTL_0. We almost have all the ingredients to define a canonical model for MTL_0; we only need to define a displacement relation $\mathrm{DIS}^0 \subseteq T^0 \times D^0 \times T^0$. This is done as follows:

$$\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma) \quad \text{iff} \quad \text{for every formula } \gamma,\ \gamma \in \Gamma \text{ implies } \Delta_\alpha\gamma \in \Sigma$$
$$\text{(equivalently: for all formulae } \sigma, \text{ if } \nabla_\alpha\sigma \in \Sigma \text{ then } \sigma \in \Gamma).$$

Note that if $(\alpha, \beta) \in \theta$, then $\vdash \alpha = \beta$, hence $\vdash \nabla_\alpha\phi \leftrightarrow \nabla_\beta\phi$ by the (LIFT) rule, for all formulae ϕ. From this one easily derives that the definition of DIS^0 does not depend on the representative we take for α/θ.

Also, $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ implies $\mathrm{DIS}^0(\Gamma, -\alpha/\theta, \Sigma)$: if $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ and $\sigma \in \Sigma$, then $\nabla_\alpha\Delta_{-\alpha}\sigma \in \Sigma$ by axiom (Ax2), hence $\Delta_{-\alpha}\sigma \in \Gamma$.

Then, the *canonical model* for MTL_0 is the model $\mathfrak{M}^0 = (T^0, \mathfrak{D}^0; \mathrm{DIS}^0; V^0, g)$.

2.2.1. THEOREM. (Completeness) *MTL_0 is sound and complete for the class of all MTL_0-frames.*

Proof. Proving soundness is left to the reader. To prove completeness we show that every consistent set of MTL_0-formulae is satisfiable in a model based on a two-sorted frame.

Let Σ be a MTL_0-consistent set of formulae; by standard techniques we can extend it to a maximal MTL_0-consistent set Σ^+ that lives somewhere in the canonical model $\mathfrak{M}^0$ for MTL_0. To complete the proof of the theorem it suffices to establish the following Truth Lemma. For all MTL_0-formulae ϕ and all $\Sigma \in T^0$:

$$\phi \in \Sigma \quad \text{iff} \quad \mathfrak{M}^0, \Sigma \Vdash \phi.$$

The proof of the lemma is by induction on ϕ. The atomic case is immediate from the definition of V^0, and the boolean cases are immediate from the induction hypothesis. So consider a modal formula $\Delta_\alpha\phi$.

($\Leftarrow$) Assume $\Sigma \Vdash \Delta_\alpha\phi$. Then there exists Γ such that $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ and $\Gamma \Vdash \phi$. By induction hypothesis $\phi \in \Gamma$, so $\Delta_\alpha\phi \in \Sigma$.

($\Rightarrow$) If $\Delta_\alpha\phi \in \Sigma$, then, to prove $\Sigma \Vdash \Delta_\alpha\phi$, we need to find a Γ with $\phi \in \Gamma$ and $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$. Such a Γ exists if we can show that $\{\phi\} \cup \{\psi \mid \nabla_\alpha\psi \in \Sigma\}$ is consistent — but this can be done by standard modal means.

This completes the proof of the Truth Lemma, and hence the proof of the completeness theorem. $\dashv$

Imposing additional constraints. For many purposes two-sorted frames as we have studied them so far are too simple. In particular, they don't satisfy all the natural conditions one may want to impose on the displacement relation. Examples of such properties that arise in application areas such as real-time system specification include

Transitivity:	$\forall i, j, k, \alpha, \beta\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \beta, k) \rightarrow \mathrm{DIS}(i, \alpha + \beta, k))$
Quasi-functionality:	$\forall i, j, j', \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \alpha, j') \rightarrow j = j')$
Reflexivity:	$\forall i\, \mathrm{DIS}(i, 0, i)$
Antisymmetry:	$\forall i, j, \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \alpha, i) \rightarrow i = j \wedge \alpha = 0)$.

As in standard modal and temporal logic only some of the natural properties we want to impose on structures are expressible. In particular, the first three of the above properties are expressible in metric temporal logic, as follows (cf. Montanari et al. [29]):

(Ax3)	$\nabla_{\alpha+\beta}p \rightarrow \nabla_\alpha\nabla_\beta p$	(transitivity)
(Ax4)	$\Delta_\alpha p \rightarrow \nabla_\alpha p$	(quasi-functionality w.r.t. the 3rd argument)
(Ax5)	$\nabla_0 p \rightarrow p$	(reflexivity)

In the case of Transitivity, Quasi-functionality, and Reflexivity we are able to extend the basic completeness result fairly effortlessly because the corresponding temporal formulae are so-called Sahlqvist formulae. And the important feature of Sahlqvist formulae is that they are *canonical* in the sense that they are validated by the frame underlying the canonical model defined in the proof of Theorem 2.2.1 (cf. Goldblatt [61] for analogous arguments in standard modal and temporal logic, or De Rijke and Venema [110] for the general picture). As a consequence we have the following:

2.2.2. THEOREM. (Completeness) *Let $X \subseteq \{\mathrm{Ax3}, \mathrm{Ax4}, \mathrm{Ax5}\}$. Then MTL_0X is complete with respect to the class of frames satisfying the properties expressed by the axioms in X.*

Further natural properties like

Euclidicity: $\forall i, j, k, \alpha, \beta\,((\mathrm{DIS}(i,\alpha,j) \wedge \mathrm{DIS}(i,\alpha+\beta,k)) \rightarrow \mathrm{DIS}(j,\beta,k))$,

which is represented in metric temporal logic by the formula (Montanari et al. [29]):

$$\Delta_\alpha\nabla_\beta p \rightarrow \nabla_{\alpha+\beta}p \qquad \text{(Euclidicity)},$$

can already be derived from MTL_0Ax3.

In the case of Antisymmetry, we have to do more work. First of all, Antisymmetry is not expressible in the basic metric language. One can use a standard unfolding argument to prove this claim (as in ordinary modal logic). Despite the undefinability of Antisymmetry, we can prove a completeness result for the class of antisymmetric two-sorted frames: we will now show that MTL_0 is complete with respect to such frames; we use a technique based on Burgess' chronicle construction (cf. Burgess [20]).

2.2.3. DEFINITION. Below we write $\rightsquigarrow_\alpha$ for the canonical displacement relation defined in the proof of Theorem 2.2.1: $\Sigma \rightsquigarrow_\alpha \Gamma$ if for all $\gamma \in \Gamma$, $\Delta_\alpha\gamma \in \Sigma$.

Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame, and g an interpretation of the algebraic terms on $\mathfrak{F}$. A *chronicle* τ on $\mathfrak{F}$ and g is a function τ such that τ assigns to each $i \in T$ an MCS $\tau(i)$.

A chronicle τ is *coherent* if for all α, $\mathrm{DIS}(i,\alpha,j)$ implies $\tau(i) \rightsquigarrow_\alpha \tau(j)$. Moreover, τ is *prophetic* (resp. *historic*) if it is coherent and satisfies condition 1 (resp. 2):

1. if $\Delta_\alpha\phi \in \tau(i)$, then there exists j such that $\mathrm{DIS}(i, g(\alpha), j)$, and $\phi \in \tau(j)$;
2. if $\Delta_{-\alpha}\phi \in \tau(i)$, then there exists j such that $\mathrm{DIS}(j, g(\alpha), i)$, and $\phi \in \tau(j)$.

Finally, τ is *perfect* if it is both prophetic and historic.

Let V be a valuation in $(T, \mathfrak{D}; \mathrm{DIS}; g)$. The *induced chronicle* is a function τ_V such that $\tau_V(i) = \{\phi \mid i \in V(\phi)\}$, for each $i \in T$. It is easy to see that τ_V is always perfect.

Conversely, if τ is a perfect chronicle, then it naturally induces a valuation V_τ defined by $V_\tau(p) = \{i \mid p \in \tau(i)\}$.

2.2.4. LEMMA. *Let τ be a perfect chronicle on a two-sorted frame $(T, \mathfrak{D}; \mathrm{DIS})$. If $V = V_\tau$ is the valuation induced by τ, then $\tau = \tau_V$ is the chronicle induced by V, that is, $V(\phi) = \{i \mid \phi \in \tau(i)\}$. Any member of any $\tau(i)$ is thus satisfiable in $(T, \mathfrak{D}; \mathrm{DIS}; g)$.*

By definition, MTL_0 is complete for the class of all antisymmetric two-sorted frames iff every consistent formula ϕ is satisfiable on a model based on an antisymmetric two-sorted frame. By Lemma 2.2.4, this is equivalent to the existence of a perfect chronicle τ on some anti-symmetric two-sorted frame $(T, \mathfrak{D}; \mathrm{DIS})$ and an interpretation g such that $\phi \in \tau(i)$ for some $i \in T$. We now construct such T, $\mathfrak{D}$, DIS, g and τ.

Let T_∞ be a countably infinite set of time instants, and M the set of tuples $(T_n$, $\mathfrak{D}$, DIS_n, g, $\tau_n)$ such that:

(a) T_n is a non-empty finite subset of T_∞;
(b) $\mathfrak{D}$ is the free abelian group over the set A, and g is the canonical interpretation, as in the proof of Theorem 2.2.1;
(c) $\mathrm{DIS}_n \subseteq T_n \times D^0 \times T_n$ is antisymmetric;
(d) τ_n is a coherent chronicle on $(T_n, \mathfrak{D}; \mathrm{DIS}_n)$ and g.

2.2.5. DEFINITION. We say that a 5-tuple $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g)$ in M *is extended by* a 5-tuple $\mu_m = (T_m, \mathfrak{D}, \mathrm{DIS}_m, \tau_m, g)$ in M if: (1) $T_n \subseteq T_m$; (2) $\mathrm{DIS}_n = \mathrm{DIS}_m \cap (T_n \times D^0 \times T_n)$; and (3) $\tau_n \subseteq \tau_m$.

A conditional requirement of the form specified in Definition 2.2.3 (1) (resp. (2)) is called *unborn* for $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g) \in M$, if its antecedent is not fulfilled. This is the case when $i \notin T_n$, or $i \in T_n$, but $\Delta_\alpha\phi \notin \tau_n(i)$ (resp. $\Delta_{-\alpha}\phi \notin \tau_n(i)$). It is called *alive* for μ_n if its antecedent is fulfilled, but its consequent is not. This is the case when $i \in T_n$ and $\Delta_\alpha\phi \in \tau_n(i)$ (resp. $\Delta_{-\alpha}\phi \in \tau_n(i)$), but there is no $j \in T_n$ such that $\mathrm{DIS}_n(i, g(\alpha), j)$ (resp. $\mathrm{DIS}_n(j, g(\alpha), i)$), and $\phi \in \tau_n(j)$. Finally, it is called *dead* if its consequent is fulfilled.

2.2.6. LEMMA. *Consider $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g) \in M$. For any requirement as in Definition 2.2.3 (1) (resp. (2)) which is alive for μ_n, there exists an extension $\mu_m \in M$ for which it is dead.*

Proof. Consider a requirement as in Definition 2.2.3 (1). Assume $i \in T_n$ and $\Delta_\alpha\phi \in \tau_n(i)$. By the proof of Theorem 2.2.1 there exists an MCS Γ such that $\tau_n(i) \rightsquigarrow_\alpha \Gamma$ and $\phi \in \Gamma$. Define μ_m as follows:

- $T_m = T_n \cup \{j\}$;
- $\mathrm{DIS}_m = \mathrm{DIS}_n \cup \{(i, \beta, j)\}$;
- $\tau_m = \tau_n \cup \{(j, \Gamma)\}$. $\dashv$

2.2.7. THEOREM. (Completeness) *MTL_0 is complete with respect to the class of all antisymmetric two-sorted frames.*

Proof. Let ϕ_0 be a consistent formula. We construct an antisymmetric two-sorted frame $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$, an interpretation g, and a perfect chronicle τ on $\mathfrak{F}$ and g such that $\phi_0 \in \tau(i_0)$ for some $i_0 \in T$.

First, let $\mathfrak{D}$ be the free algebra over the set A, and g the canonical interpretation. Second, take a countably infinite set S, and fix an enumeration $i_0, i_1, \ldots$ of S, and an enumeration ϕ_0, ϕ_1, ... of all formulae. Then, to each conditional requirement of the form specified in Definition 2.2.3 (1) (resp. (2)), with $i = i_n$ and $\phi = \phi_m$, we assign the number $2 \cdot 5^n \cdot 7^m$ (resp. $3 \cdot 5^n \cdot 7^m$). Moreover, we take an MCS Γ with $\phi_0 \in \Gamma$, and define $\mu_0 = (T_0, \mathfrak{D}, \mathrm{DIS}_0, \tau_0, g)$, where $T_0 = \{i_0\}$, $\mathrm{DIS}_0 = \emptyset$, and $\tau_0 = \{(i_0, \Gamma)\}$. If μ_n is defined, we consider the requirement with the least code number among all requirements which are alive for μ_n. By Lemma 2.2.6 we can choose an extension μ_{n+1} of μ_n for which that requirement is dead.

Let T, DIS and τ be respectively defined as follows: $T = \bigcup_n T_n$, $\mathrm{DIS} = \bigcup_n \mathrm{DIS}_n$, and $\tau = \bigcup_n \tau_n$. $(T, \mathfrak{D}; \mathrm{DIS})$ is an antisymmetric two-sorted frame and τ is a perfect chronicle on this frame and g. $\dashv$

When metric temporal logic is employed for specifying real-time systems, one further condition is usually imposed on the displacement relation. Since the behavior of real-time systems is essentially modeled in terms of infinite sequences of states/events, it is natural to require the closure of the temporal domain under displacements. Such a requirement is captured by imposing seriality of the displacement relation:

Seriality: $\forall i, \alpha \exists j\, \mathrm{DIS}(i, \alpha, j)$,

which can be axiomatized as

(Ax6) $\nabla_\alpha p \to \Delta_\alpha p$ (seriality)
(or, equivalently, $\Delta_\alpha \top$).

Again, the basic completeness result can be extended without effort because the corresponding temporal formula is a Sahlqvist formula. Moreover, it is interesting to study the interplay between Seriality and the properties of Transitivity, Quasi-functionality and Reflexivity.

The addition of Seriality turns Quasi-functionality into Functionality:

$$\Delta_\alpha p \leftrightarrow \nabla_\alpha p.$$

Therefore, each occurrence of Δ_α can be replaced by ∇_α. This allows us, for instance, to merge Transitivity and Euclidicity

$$\nabla_{\alpha+\beta} p \leftrightarrow \nabla_\alpha \nabla_\beta p.$$

Moreover, it is easy to see that the addition of Seriality forces ∇_α and $\neg$ to commute

$$\nabla_\alpha \neg p \leftrightarrow \neg \nabla_\alpha p.$$

Given the Distributivity of ∇_α over $\wedge$, we conclude that ∇_α distributes over all truth functional connectives.

2.3 Two-sorted frames based on ordered groups

For a variety of application purposes, our basic calculus and its semantics need to be extended with orderings. In particular, a linear order on the temporal domain is needed in many application areas; for instance, in real-time specifications we want to guarantee that between any two time instants there is a unique displacement. In the following, we achieve this by adding a total ordering on the displacement domain D.

In the definition of a two-sorted frame we replace the abelian component by an *ordered* abelian group. That is, by a structure $\mathfrak{D} = (D, +, -, 0, <)$, where $(D, +, -, 0)$ is an abelian group, and $<$ is an irreflexive, asymmetric, transitive and linear relation that satisfies the comparability property $(viii)$ below:

(v)	$\neg(\alpha < \alpha)$
(vi)	$\neg(\alpha < \beta \wedge \beta < \alpha)$
(vii)	$\alpha < \beta \wedge \beta < \gamma \to \alpha < \gamma$
$(viii)$	$\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha.$

Next, there are two axioms expressing the relation between $+$ and $-$, and $<$:

(ix)	$\alpha < \beta \to \alpha + \gamma < \beta + \gamma$
(x)	$\alpha < \beta \to -\beta < -\alpha.$

One can use various languages to talk about ordered abelian groups. We do not have any clear preference, as long as the language used can be equipped with a complete axiomatization. We will simply use full first-order logic over $=$, $<$ to reason about the ordered abelian component of our two-sorted frames.

To be precise, our metric temporal language for talking about two-sorted frames based on an ordered abelian group, has a first-order component built up from terms in $T(A)$ and predicate symbols $=$ and $<$; its temporal component is as before.

The interpretation of this language on two-sorted frames based on an ordered abelian group is fairly straightforward: the first-order component is interpreted on the group, and the temporal component on the temporal domain. Validity in this language is easily axiomatized; for the displacement component we take the axioms and rules of identity, ordered abelian groups, strict linear order together with any complete calculus for first-order logic; and for the temporal component we take the same axioms as in the case of MTL_0: axioms (Ax1), (Ax2) and the rules modus ponens, (NEC), (REP) and (LIFT). Let MTL_1 denote the resulting two-sorted calculus.

2.3.1. THEOREM. (Completeness) *MTL_1 is complete with respect to the class of two-sorted frames based on ordered abelian groups.*

Proof. We can simply repeat the proof of Theorem 2.2.1 here, and replace the free algebra construction of the displacement domain by a Henkin construction for first-order logic. $\dashv$

2.3.1 Deriving a temporal ordering

Given that we have an ordering $<$ on the algebraic component of our frames, a natural definition for an ordering $\ll$ on the temporal frame suggests itself:

$$i \ll j \text{ iff for some } \alpha > 0, \mathrm{DIS}(i, \alpha, j). \tag{2.1}$$

So i and j are $\ll$-related if there exists a positive displacement between them. Using the relation $\ll$, we can define the qualitative operators F, P of non-metric temporal logic as follows:

$$\mathfrak{M}, i \Vdash F\phi := \exists j \, (i \ll j \wedge j \Vdash \phi) \text{ and } \mathfrak{M}, i \Vdash P\phi := \exists j \, (j \ll i \wedge j \Vdash \phi).$$

We will not consider this extension.

Additional properties. The definition of $\ll$ given in (2.1) does not produce a temporal ordering with all the natural properties that we usually expect it to have. In particular, unless we put further restrictions on the relation of temporal displacement, $\ll$ will not be a strict linear order, and there may be time instants without a unique temporal distance between them.

To repair this situation, we assume that the displacement relation DIS satisfies the following properties: transitivity, quasi-functionality, reflexivity (as defined in Section 2), and total connectedness and quasi-functionality w.r.t. the second argument:

(xi) $\quad \forall i, j \exists \alpha \, \mathrm{DIS}(i, \alpha, j)$ (total connectedness)
(xii) $\quad \forall i, j, \alpha, \beta \, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \beta, j) \rightarrow \alpha = \beta)$
(quasi-functionality w.r.t. the 2nd argument).

Given these assumptions on the displacement relation, we can show that the temporal relation $\ll$ as defined in (2.1) is a strict linear order. To see that $\ll$ is transitive, assume that $i \ll j \ll k$. Then there exist α, β with $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, k)$. Hence $\mathrm{DIS}(i, \alpha + \beta, k)$ and $i \ll k$.

For irreflexivity, assume $i \ll i$. Then $\mathrm{DIS}(i, \alpha, i)$ for some $\alpha > 0$. By reflexivity of DIS, $\mathrm{DIS}(i, 0, i)$, hence, by quasi-functionality of the second argument, $\alpha = 0$ — a contradiction.

For asymmetry, assume $i \ll j \ll i$. Then $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, i)$ for some $\alpha, \beta > 0$. Then $\mathrm{DIS}(j, -\alpha, i)$ and so $\beta = -\alpha$, by quasi-functionality of the second argument again, which yields a contradiction.

Finally, to prove totality, take any two i, j. By total connectedness there exists α such that $\mathrm{DIS}(i, \alpha, j)$. By axiom $(viii)$, $\alpha > 0 \vee \alpha = 0 \vee 0 > \alpha$. If $\alpha > 0$, then $i \ll j$. If $\alpha = 0$, then by quasi-functionality and reflexivity of DIS, $i = j$. And if $\alpha < 0$, then $-\alpha > 0$ and $\mathrm{DIS}(j, -\alpha, i)$, so $j \ll i$.

Let us call a two-sorted frame *nice* if it is transitive, reflexive, totally-connected, and quasi-functional in both the 2nd and 3rd argument of its displacement relation; a model is *nice* if it is based on an nice frame.

The next obvious question is: can we characterize the nice frames in the language of MTL_1? The answer is 'no'. To see this, we adapt two truth preserving constructions from standard modal logic to the present setting. For the sake of simplicity, we confine ourselves to frames that share the same displacement domain; however, the definitions are easily generalized to the general case.

2.3.2. DEFINITION. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. The *disjoint union* of $\mathfrak{F}$ and $\mathfrak{F}'$ is the two-sorted frame $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}, \mathrm{DIS}'')$. Here, T'' is the disjoint union of T and T', while the displacement relation DIS'' is just the disjoint union of DIS and DIS'.

2.3.3. THEOREM. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and $\mathfrak{F} \uplus \mathfrak{F}'$ their disjoint union. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$ and $\mathfrak{F}' \models \alpha = \beta$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \alpha = \beta$, and, for all formulae ϕ, if $\mathfrak{F} \models \phi$ and $\mathfrak{F}' \models \phi$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \phi$.*

2.3.4. THEOREM. *There is no modal formula ϕ that expresses total connectedness of two-sorted frames.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under disjoint union. An intuitive account of this negative conclusion can be given noticing that disjoint unions are not totally connected frames "by definition".

Suppose that there exists a formula ϕ expressing total connectedness. By Theorem 2.3.3, it follows that ϕ is valid in the disjoint union $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}; \mathrm{DIS}'')$ of any two frames $\mathfrak{F}$ and $\mathfrak{F}'$ validating ϕ. Take $i \in \mathfrak{F}$ and $j \in \mathfrak{F}'$; by definition of $\mathfrak{F} \uplus \mathfrak{F}'$, it follows that there exists no $\alpha \in \mathfrak{D}$ such that $\mathrm{DIS}''(i, \alpha, j)$. $\dashv$

2.3.5. DEFINITION. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. A *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ is a mapping $f : T \to T'$ such that:

1. if $\mathrm{DIS}(i, \alpha, j)$, then $\mathrm{DIS}'(f(i), \alpha, f(j))$;
2. if $\mathrm{DIS}'(f(i), \alpha, j')$, then for some $j \in T$ both $f(j) = j'$ and $\mathrm{DIS}(i, \alpha, j)$ hold.

2.3.6. THEOREM. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and f a surjective bounded morphism from $\mathfrak{F}$ to $\mathfrak{F}'$. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$, then $\mathfrak{F}' \models \alpha = \beta$. And, for all formulae ϕ, if $\mathfrak{F} \models \phi$, then $\mathfrak{F}' \models \phi$.*

2.3.7. THEOREM. *There is no modal formula ϕ that expresses quasi-functionality w.r.t. the second argument of the displacement relation.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under bounded morphisms. Suppose that there exists a formula ϕ expressing quasi-functionality with respect to the second argument of the accessibility relation.

Consider the two-sorted frames $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ such that $T = \{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4\}$, $T' = \{i', j'\}$, DIS contains $(i_1, 1, j_1)$, $(i_1, 2, j_3)$, $(i_2, 2, j_1)$, $(i_2, 1, j_3)$, $(i_3, 1, j_2)$, $(i_3, 2, j_4)$, $(i_4, 1, j_4)$, and $(i_4, 2, j_2)$, together with the converse triplets $(j_1, -1, i_1)$,

$(j_3, -2, i_1)$, and so on, while $\mathrm{DIS}' = \{(i', 1, j'), (i', 2, j'), (j', -2, i'), (j', -1, i')\}$. Clearly, $\mathfrak{F}$ satisfies the requirement of quasi-functionality, while $\mathfrak{F}'$ does not.

Now, consider the mapping $f : T \to T'$ defined by $f(i_1) = f(i_2) = f(i_3) = f(i_4) = i'$, $f(j_1) = f(j_2) = f(j_3) = f(j_4) = j'$. It is easy to verify that f is a surjective bounded morphism. Then, from $\mathfrak{F} \models \phi$ Theorem 2.3.6 allows us to infer that $\mathfrak{F}' \models \phi$, and we have a contradiction. $\dashv$

Enriching the language. Given that nice frames cannot be characterized in the language of MTL_1, a possible way out consists in enriching the language to enable us to express the properties of total connectedness and quasi-functionality of the displacement relation in its 2nd argument. We briefly show that those properties can actually be expressed by adding to the language the future and past operators F, P, the difference operator $\mathcal{D}$, and by allowing that information from the temporal domain is lifted to the displacement domain by permitting the two languages to be mixed.

First, the *difference operator* (de Rijke [109]) is a unary modal operator $\mathcal{D}$ that allows us to model unbounded jumps. Its semantic interpretation is defined as follows:

$$(\mathfrak{F}, V), i \Vdash \mathcal{D}\phi \text{ iff } \exists j\, (j \neq i \wedge (\mathfrak{F}, V), j \Vdash \phi),$$

with dual $\overline{\mathcal{D}}$:

$$(\mathfrak{F}, V), i \Vdash \overline{\mathcal{D}}\phi \text{ iff } \forall j\, (j \neq i \to (\mathfrak{F}, V), j \Vdash \phi).$$

The difference operator and its dual allow us to define three derived unary operators $\mathcal{E}$, its dual $\mathcal{A}$, and $\mathcal{U}$ that respectively model truth in at least one world, truth in all worlds, and truth in a unique world:

$$\mathcal{E}\phi \equiv \mathcal{D}\phi \vee \phi,\ \ \mathcal{A}\phi \equiv \overline{\mathcal{D}}\phi \wedge \phi,\ \text{ and } \mathcal{U}\phi \equiv \mathcal{E}(\phi \wedge \neg\mathcal{D}\phi).$$

In a language in which the algebraic and temporal formulae may be mixed, properties (*xi*) and (*xii*) can be expressed by means of the qualitative operators F, P and $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{U}$ as follows:

(Ax7) $\mathcal{D}p \to Fp \vee Pp$ (total connectedness of DIS)

(Ax8) $\mathcal{U}p \wedge \mathcal{U}q \to (\mathcal{E}(p \wedge \Delta_\alpha q) \wedge \mathcal{E}(p \wedge \Delta_\beta q) \to \alpha = \beta)$
(quasi-functionality of DIS w.r.t. the 2nd argument).

However, we prefer to remain within the original language of MTL_1 and reason about nice frames there, mainly because adding the axioms Ax7 and Ax8 forces us to give up the simplicity of the basic calculus and to include non-standard derivation rules to govern the difference operator. As we will show below, the logic of nice frames can be captured in the original language.

Completeness for nice frames. Instead of increasing the expressive power of metric temporal logic, we leave it as it stands, and prove a completeness result for nice frames in the old language. We will do this in two steps. We first prove completeness with respect to totally connected frames via some sort of generated submodel construction, and then we prove the full result.

Here's the idea for the case of total connectedness. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame. The *master relation* on $\mathfrak{F}$ is defined by

$$(i, j) \in \mathit{Master} \;\text{ iff }\; (i, j) \in (\ll \cup \gg)^*.$$

Thus i, j are in the master relation iff there exists a zigzag path along the displacement relation from i to j in the following sense:

$$\mathrm{DIS}(i, \alpha_1, j_1), \mathrm{DIS}(j_1, \alpha_2, j_2), \ldots, \mathrm{DIS}(j_n, \alpha_{n+1}, j),$$

where $\alpha_1, \ldots, \alpha_n \in D$, and $j_1, \ldots, j_n \in T$.

A *point-generated component* of a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ is a model $(T', \mathfrak{D}; \mathrm{DIS}'; g, V')$ such that for some $i \in T$,

- $T' = \{j \in T \mid (i, j) \in \mathit{Master}\}$
- $\mathrm{DIS}' = \mathrm{DIS} \cap (T' \times D \times T')$
- $V'(p) = V(p) \cap T'$, for all p.

2.3.8. PROPOSITION. *Let $\mathfrak{M}'$ be a point-generated component of a model $\mathfrak{M}$ based on a two-sorted frame with ordered abelian group. If $\mathfrak{M}$ has a transitive displacement relation, then $\mathfrak{M}'$ has a transitive and totally connected displacement relation.*

2.3.9. LEMMA. *Let $\mathfrak{M}'$ be a point-generated component of a two-sorted model $\mathfrak{M}$. Then $\mathfrak{M}'$ satisfies exactly the same algebraic formulae as $\mathfrak{M}$. Moreover, for all $i \in T'$ and for all temporal formulae ϕ we have $\mathfrak{M}, i \Vdash \phi$ iff $\mathfrak{M}', i \Vdash \phi$.*

MTL_1Ax3 extends MTL_1 with the transitivity axiom $\nabla_{\alpha+\beta} p \to \nabla_\alpha \nabla_\beta p$.

2.3.10. THEOREM. (Completeness) *MTL_1Ax3 is sound and complete with respect to the class of two-sorted frames based on ordered abelian groups whose displacement relation is transitive and totally connected.*

Proof. We only prove completeness, and to establish this it suffices to show that every MTL_1Ax3-consistent set of formulae is satisfiable in a model based on a frame of the right kind.

Let Γ be a MTL_1Ax3-consistent set of formulae. By a Sahlqvist style argument (cf. Theorem 2.2.2) it is easily seen that Γ is satisfiable in a model $\mathfrak{M}$ based on a two-sorted frame with a transitive displacement relation, say at a time instant i. Let $\mathfrak{M}'$ be a point-generated component of $\mathfrak{M}$ that contains i. By Proposition 2.3.8 $\mathfrak{M}'$ has a transitive and totally connected displacement relation, and by Lemma 2.3.9 we have $\mathfrak{M}', i \Vdash \Gamma$, as required. $\dashv$

To prove completeness w.r.t. the class of nice frames, we need to carry out a second construction. First, call a two-sorted frame *almost nice* if it is transitive, reflexive, totally-connected, and quasi-functional in the 3rd argument of its displacement relation; a model is *almost nice* if it is based on an almost nice frame. So a frame is nice if it is almost nice and quasi-functional in the 2nd argument of its displacement relation.

Now, to build a nice model we will take an almost nice model and carefully unfold it. To be precise, let $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ be an almost nice model, and let $i \in T$. The *i-stratification of* $\mathfrak{M}$ is the model $\mathfrak{M}' = (T', \mathfrak{D}; \mathrm{DIS}'; V', g)$ which is defined as follows:

$$\begin{aligned}
T' &= \{(0,i)\} \cup \{(\alpha, j) \mid \mathrm{DIS}(i, \alpha, j) \text{ in } \mathfrak{M}\} \\
\mathrm{DIS}_0 &= \{((0,i), \alpha, (\alpha, j)) \mid (\alpha, j) \in T'\} \cup \{((\alpha, j), -\alpha, (0, i)) \mid (\alpha, j) \in T'\} \\
\mathrm{DIS}_1 &= \{((\alpha, j), \beta - \alpha, (\beta, k)) \mid (\alpha, j), (\beta, k) \in T'\} \\
\mathrm{DIS}' &= \mathrm{DIS}_0 \cup \mathrm{DIS}_1 \\
V'(p) &= \{(\alpha, j) \in T' \mid j \in V(p)\}.
\end{aligned}$$

Observe that $\mathrm{DIS}_0 \subseteq \mathrm{DIS}_1$.

2.3.11. Proposition. *Let $\mathfrak{M}$ be an almost nice model, and let $i \in \mathfrak{M}$. The i-stratification of $\mathfrak{M}$ is nice.*

Proof. We first observe that for any pairs (α, j), $(\gamma, k) \in T'$, and $\beta \in \mathfrak{D}$, if it holds that $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ then $\beta = \gamma - \alpha$.

Now, to prove the proposition, we have to check the nice-ness properties. First of all, we show that $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ implies $\mathrm{DIS}'((\gamma, k), -\beta, (\alpha, j))$. By the observation $\beta = \gamma - \alpha$. Also, (α, j), $(\gamma, k) \in T'$ implies $\mathrm{DIS}'((\gamma, k), \alpha - \gamma, (\alpha, j))$, that is, $\mathrm{DIS}'((\gamma, k), -\beta, (\alpha, j))$.

Next, we show that DIS' is reflexive. As $\mathfrak{M}$ is assumed to be reflexive, we have $\mathrm{DIS}(i, 0, i)$, hence $\mathrm{DIS}((0, i), 0, (0, i))$. As to other points $(\alpha, j) \in T'$, $\mathrm{DIS}_1((\alpha, j), \alpha - \alpha, (\alpha, j))$, by definition of DIS_1, and thus $\mathrm{DIS}'((\alpha, j), 0, (\alpha, j))$.

To see that DIS' is quasi-functional with respect to its 3rd argument, assume that both $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\alpha, j), \beta, (\gamma', k'))$ hold. We need to show that $\gamma = \gamma'$ and $k = k'$. First of all, $\beta = \gamma - \alpha = \gamma' - \alpha$, hence $\gamma = \gamma'$. Therefore, $\mathrm{DIS}(i, \gamma, k)$ and $\mathrm{DIS}(i, \gamma, k')$. So by the assumption that DIS is quasi-functional in its 3rd argument, $k = k'$.

Given that $\mathfrak{M}$ is total, the totality of its i-stratifications is immediate.

Transitivity of $\mathfrak{M}'$ may be established as follows: assume that both $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\gamma, k), \beta', (\delta, l))$ hold. Then $\mathrm{DIS}'((\alpha, j), \delta - \alpha, (\delta, l))$. As $\beta + \beta' = (\gamma - \alpha) + (\delta - \gamma)$, we are done.

Finally, to prove quasi-functionality of DIS' in its 2nd argument, assume that we have both $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\alpha, j), \beta', (\gamma, k))$. It follows that $\beta = \gamma - \alpha = \beta'$. $\dashv$

2.3.12. Proposition. *Let $\mathfrak{M}$ be an almost nice model, and let $\mathfrak{M}'$ be an i-stratification of $\mathfrak{M}$. For all formulae ϕ, j in $\mathfrak{M}$, and (α, j) in $\mathfrak{M}'$, we have $\mathfrak{M}, j \Vdash \phi$ iff $\mathfrak{M}', (\alpha, j) \Vdash \phi$.*

Proof. This is by induction on ϕ. The base case and the boolean cases are trivial. So consider a temporal formula $\Delta_\gamma \psi$. Assume first that $j \Vdash \Delta_\gamma \psi$. Then there exists k with $\mathrm{DIS}(j, \gamma, k)$. Now, let α be such that $(\alpha, j) \in T'$. Then $\mathrm{DIS}(i, \alpha, j)$, and hence $\mathrm{DIS}(i, \alpha + \gamma, k)$ and $(\alpha + \gamma, k) \in T'$. By definition, $\mathrm{DIS}_0((0, i), \alpha, (\alpha, j))$ and $\mathrm{DIS}_0((0, i), \alpha + \gamma, (\alpha + \gamma, k))$. But then $\mathrm{DIS}'((\alpha, j), \gamma, (\alpha + \gamma, k))$. By induction hypothesis, $(\alpha + \gamma, k) \Vdash \psi$, hence $(\alpha, j) \Vdash \Delta_\gamma \psi$.

Conversely, assume that $(\alpha, j) \Vdash \Delta_\gamma \psi$. Then there exists $(\beta, k) \in T'$ such that both $\mathrm{DIS}'((\alpha, j), \gamma, (\beta, k))$ and $(\beta, k) \Vdash \psi$ hold. Hence $\gamma = \beta - \alpha$. By construction we must have $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(i, \beta, k)$ and hence $\mathrm{DIS}(j, \beta - \alpha, k)$. As $k \Vdash \psi$ (by induction hypothesis) and $\gamma = \beta - \alpha$, this implies $j \Vdash \Delta_\gamma \psi$, as required. ⊣

We are ready now for a completeness result for the class of nice frames. Let MTL_2 denote the extension of MTL_1 with axioms Ax3, Ax4 and Ax5 (expressing transitivity, quasi-functionality of DIS in its 3rd argument, and reflexivity, respectively). By an easy adaptation of the proof of Theorem 2.3.10, MTL_2 is sound and complete w.r.t. the class of almost nice frames.

2.3.13. THEOREM. (Completeness) *MTL_2 is sound and complete with respect to the class of nice frames.*

Proof. We only show that every MTL_2-consistent set of temporal formulae is satisfiable on a nice model. Let Γ be such a set. By earlier remarks Γ is satisfiable on an almost nice model at some time instant i. Let $\mathfrak{M}'$ be the i-stratification of $\mathfrak{M}$. By Propositions 2.3.11 and 2.3.12 $\mathfrak{M}'$ is a nice model that satisfies Γ at i. ⊣

2.3.2 Adding discreteness

One natural specialization of the metric temporal logic of linear orders consists in the addition of discreteness. As with the earlier addition of an ordering, we will constrain the domain of temporal displacements to be discrete and show that the discreteness of the temporal domain necessarily follows.

The discreteness of the domain of displacements is expressed by the following axiom:

$$(xiii) \qquad \forall\alpha\exists\beta,\beta'(\alpha < \beta \wedge \forall\gamma(\alpha < \gamma \rightarrow (\beta = \gamma \vee \beta < \gamma)) \wedge \beta' < \alpha \wedge \\ \forall\delta(\delta < \alpha \rightarrow (\beta' = \delta \vee \beta' < \delta)))$$

The discreteness of the temporal domain follows as shown by the following proposition.

2.3.14. PROPOSITION. *Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame based on a discrete ordered abelian group $\mathfrak{D}$. For all $i, j \in T$, there exist only finitely many k such that $i \ll k \ll j$.*

Proof. Left to the reader. ⊣

An interesting consequence of restricting ourselves to discrete temporal domains is that bounded response and invariance properties like

$$p \rightarrow \exists x(0 \leq x < \delta \wedge \Delta_x q),$$

and

$$p \rightarrow \forall x(0 \leq x < \delta \rightarrow \nabla_x q)$$

become expressible in the basic systems of metric temporal logics (devoid of quantification and mixed formulae).

The restricted quantification involved in bounded response properties can indeed be replaced by a finite disjunction of formulae of the form $\Delta_\alpha q$ (one disjunct for each displacement α—there exists a finite number of such displacements—such that $0 \leq \alpha < \delta$). Analogously, the restricted quantification involved in bounded invariance properties can be replaced by a finite conjunction of formulae of the form $\Delta_\alpha q$.

On the other hand, unrestricted quantification involved in unbounded versions of response and invariance properties like

$$p \to \exists x (0 < x \wedge \Delta_x q),$$

and

$$p \to \forall x (0 < x \to \nabla_x q),$$

as well as nested quantification in the formula

$$\exists x (0 < x \wedge \Delta_x p \wedge \forall y (0 \leq y < x \to \nabla_y q))$$

cannot be captured by basic metric temporal logics. This deficiency can be overcome by using the qualitative operators F, P and/or the operators *Since* and *Until*. The above introduced properties can indeed be represented as $p \to Fp$, $p \to Gp$, and q *Until* p, respectively. However, this solution requires the addition of the axioms for the qualitative operators and of the axioms constraining the relationships between the qualitative operators and the operator of temporal realization, as well as a completeness proof for the resulting logical system. We do not consider such extensions.

2.4 Increased interaction

So far we have only considered simple languages that allow us to lift information from the algebraic domain to the temporal domain but not vice versa. For application purposes they have to be extended. As an example, consider an automatic reply system that, whenever it receives a message, sends an acknowledgment with a delay less than δ. Such a bounded response property can be represented by the following formula:

$$p \to \exists x \, (0 \leq x < \delta \wedge \Delta_x q),$$

where p and q denote the receipt of the message and its acknowledgment, respectively. However, the languages considered so far cannot express such conditions as they lack quantification and constrain displacements to occur as parameters of the operator of temporal realization only.

In this section, we will show how the ability of freely mixing temporal and displacement formulae enables us to exploit more complex ways of interaction between the two domains. Our first goal is to define the logic $Q\text{-}MTL_0$ and its language.

Language. Let A denote a set of algebraic constants, and X a collection of algebraic variables; a denotes a typical element of A, x a typical element of X. The set of algebraic terms $T(X \cup A)$ is built up as follows:

$$\alpha ::= 0 \mid a \mid x \mid \alpha + \alpha \mid -\alpha.$$

Using this, we define the formulae of $Q\text{-}MTL_0$:

$$\phi ::= p \mid \neg\phi \mid \alpha = \beta \mid \alpha < \beta \mid \phi \wedge \phi \mid \Delta_\alpha \phi \mid \forall x\, \phi.$$

Thus, we allow quantification over algebraic variables and free mixing of algebraic formulae and temporal propositional symbols.

Structures. Starting from an ordered two-sorted frame $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ we arrive at a $Q\text{-}MTL_0$-model by adding a valuation V and an interpretation function g for the algebraic terms, as in Section 2.3. What remains to be defined is the way we evaluate our new mixed formulae at time instances. For the atomic case we stipulate the obvious definition:

$$\begin{aligned} \mathfrak{M}, i \Vdash \alpha = \beta \quad &\text{iff} \quad g(\alpha) = g(\beta) \\ \mathfrak{M}, i \Vdash \alpha < \beta \quad &\text{iff} \quad g(\alpha) < g(\beta). \end{aligned}$$

Thus, the truth value of formulae of the form $\alpha = \beta$ and $\alpha < \beta$ is determined by referring only to the algebraic component.

Next, to evaluate quantified formulae $\forall x\, \phi$ at a point in time, we write $g =_x g'$ to denote that the assignments g and g' agree on all algebraic variables except maybe x. Then

$$(\mathfrak{F}; V, g), i \Vdash \forall x\, \phi \quad \text{iff} \quad (\mathfrak{F}; V, g'), i \Vdash \phi,$$

for all assignments g' such that $g =_x g'$.

2.4.1. Remark. Note that in the traditional terminology from quantified modal logic, our semantic structures implement a *fixed-domain* approach with a *rigid* (objectual) interpretation of terms (cf. Garson [56]. Indeed, we assume that there exists a single domain of quantification for all time points which contains all the possible values for displacements.

An example. Consider a traffic light controller C. When the request button is pushed, the controller makes a pedestrian light turn green within a given time bound after which the light remains green for a certain amount of time (cf. Henzinger et al. [66]). Moreover, assume that C takes a unit of time to switch the light and that the time needed for its internal operations is negligible.

We require that C satisfies the following conditions:

(i) whenever a pedestrian pushes the request button ('request is true'), then the light is green within 5 time units and remains green for at least 10 time units (this condition guarantees that no pedestrian waits for more than 5 time units, and that he or she is given at least 10 time units to cross the road);

(ii) whenever request is true, then it is false within 20 time units (this condition ensures that the request button is reset);

(iii) whenever request has been false for 20 time units, the light is red (this condition should prevent the light from always being green).

The behavior of C can be formally specified in $Q\text{-}MTL_0$ as the conjunction of the following formulae:

$$\begin{aligned} &\mathit{request} \;\rightarrow\; \exists x(0 < x \leq 5 \wedge \forall y(x \leq y < x+10 \rightarrow \nabla_y \mathit{lightIsGreen})),\\ &\mathit{request} \;\rightarrow\; \exists z(0 \leq z \leq 20 \wedge \Delta_z \neg \mathit{request}),\\ &\forall x(0 \leq x < 20 \rightarrow \nabla_x \neg \mathit{request}) \rightarrow \nabla_{20} \mathit{lightIsRed}, \end{aligned}$$

together with a formula stating that at each time instant the traffic light is either red or green:

$$\mathit{lightIsGreen} \leftrightarrow \neg \mathit{lightIsRed}.$$

Different implementations of C, all satisfying the given specification, can be obtained by making different assumptions about the value of temporal parameters, e.g., by varying the delay between requests and resets.

It is worth noting that, even if there are no restrictions on the frequency of requests, the above specification is appropriate only if that frequency is low; otherwise, it may happen that switching the light to red is delayed indefinitely.

To overcome this problem, we can constrain the duration of the periods during which the traffic light is green and those during which it is red. As an example, we can replace conditions (i)–(iii) by the following ones:

(iv) whenever a pedestrian pushes the request button and the light has been red for at least 20 time units, then the light is green within 5 time units and for at least 20 time units;

(v) whenever a pedestrian pushes the request button and the light has been red for x time units, with x less than 20 time units, then the light is green within $(20-x)+5$ time units and for at least 20 time units;

(vi) the light cannot be green for more than 20 time units;

(vii) the light must be red for at least 20 time units.

Conditions (iv)–(vii) can be specified in $Q\text{-}MTL_0$ as follows, using the event *pushButton* instead of the property *request*:

- $\mathit{pushButton} \wedge \forall x\,(-20 < x < 0 \rightarrow \nabla_x \mathit{lightIsRed}) \rightarrow$
  $\exists y\,(0 < y \leq 5 \wedge \forall z\,(y \leq z < y+20 \rightarrow \nabla_z \mathit{lightIsGreen})),$
- $\forall x\,(\mathit{pushButton} \wedge -20 < x < 0 \wedge \Delta_x \mathit{lightIsGreen} \wedge \forall y(x < y \leq 0 \rightarrow \nabla_y \mathit{lightIsRed}) \rightarrow$
  $\exists z\,(0 < z \leq (20-x)+5 \wedge \forall w(z \leq w < z+20 \rightarrow \nabla_w \mathit{lightIsGreen}))),$
- $\forall x\,(\forall y(0 \leq y < x \rightarrow \nabla_y \mathit{lightIsGreen}) \rightarrow x \leq 20),$
- $\mathit{lightIsGreen} \wedge \Delta_1 \mathit{lightIsRed} \rightarrow \forall x\,(0 < x \leq 20 \rightarrow \nabla_x \mathit{lightIsRed}).$

Axioms. Our next goal is to arrive at a complete axiomatization of validity in the language of $Q\text{-}MTL_0$. To the axioms of MTL_2 we will add a number of axiom schemata governing the behavior of quantifiers and substitutions. First of all, we have

(Ax9) $\forall x\,(\phi \to \psi) \leftrightarrow (\forall x\,\phi \to \forall x\,\psi)$ (functionality)
(Ax10) $\phi \to \forall x\,\phi$, for x not in ϕ (elimination of vacuous quantifier)
(Ax11) $\forall x\,\phi \to \phi(\alpha/x)$, with α free for x in ϕ (universal instantiation)

and the rule:

(UG) $\vdash \phi \implies \vdash \forall x\,\phi$ (universal generalization).

We also add the Barcan formula:

(Ax12) $\forall x\,\nabla_\alpha \phi \to \nabla_\alpha \forall x\,\phi$, with $x \notin \alpha$,

where $x \notin \alpha$ denotes that $x \neq \alpha$ and x does not occur in α. Furthermore, we have the following axioms relating algebraic terms and temporal operators:

(Ax13) $\alpha = \beta \to \forall x \nabla_x \alpha = \beta$
(Ax14) $\alpha \neq \beta \to \forall x \nabla_x \alpha \neq \beta$
(Ax15) $\alpha < \beta \to \forall x \nabla_x \alpha < \beta$
(Ax16) $\alpha \not< \beta \to \forall x \nabla_x \alpha \not< \beta$.

2.4.2. Remark. It is worth noting that the requirement that neither $x = \alpha$ nor $x \in \alpha$ in (Ax11) is essential to guarantee the soundness of the Barcan formula, as is shown by the following example. Suppose that the Barcan formula holds without restrictions. Let x be a variable (over displacements). From axiom (Ax2), by (UG), (Ax11) and Modus Ponens, we obtain $p \to \nabla_x \Delta_{-x} p$. Then, by (UG), (Ax9), (Ax10) and Modus Ponens, it follows that $p \to \forall x\, \nabla_x \Delta_{-x} p$. Now, since the Barcan formula holds without restrictions, we obtain by Modus Ponens that $p \to \nabla_x \forall x\, \Delta_{-x} p$, which clearly is not a valid formula.

Also, axiom (Ax16) can actually be derived from the other axioms.

2.4.3. Remark. Note that we also have converses to (Ax13)–(Ax16):

$$\begin{aligned} \forall x\,\nabla_x(\alpha = \beta) &\to \nabla_0(\alpha = \beta), \text{ by (Ax11)} \\ &\to \alpha = \beta, \text{ by (Ax5)}, \end{aligned}$$

and similarly for the other cases. As a consequence we have that for purely algebraic formulae ϕ the following equivalence is provable: $\phi \leftrightarrow \forall x\, \nabla_x \phi$.

2.4.4. Lemma. *Q-MTL$_0$ derives the following formula:*

(*T*1) $\nabla_\alpha \forall x\,\phi \to \forall x\,\nabla_\alpha \phi$, *with* $x \notin \alpha$ *(converse of the Barcan formula)*

Proof. We have

$$\begin{aligned} &\vdash \nabla_\alpha \forall x\,\phi \to \nabla_\alpha \phi, \text{ by (Ax10), (NEC) and (Ax1)} \\ &\Rightarrow \vdash \forall x\,\nabla_\alpha \forall x\,\phi \to \forall x\,\nabla_\alpha \phi, \text{ by (UG) and (Ax8)} \\ &\Rightarrow \vdash \nabla_\alpha \forall x \phi \to \forall x \nabla_\alpha \phi, \text{again by (Ax10)}. \end{aligned}$$

Observe that (T1) together with the Barcan formula allows us to conclude that the domain of temporal displacements does not change when we move from one time point to another. ⊣

Completeness. To prove a completeness result for $Q\text{-}MTL_0$ we can follow the general pattern of the completeness proofs given in Sections 2.2 and 2.3, but the presence of mixed formulae complicates some of the details. We use a variant of Hughes and Cresswell's [68] method for proving axiomatic completeness in the presence of the Barcan formula.

First, a *Henkin-formula* with respect to a variable y is defined as follows

1. Any formula of the form $\exists x\, \phi \to \phi(y/x)$ is a Henkin formula with respect to y.
2. If ψ is a Henkin-formula with respect to y, χ is any formula not containing y free, and α is an algebraic term not containing y, then $\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to y.

Henkin-formulae that differ only in that each is a Henkin-formula with respect to a different variable will be said to have the same *Henkin-form*. A set of formulae has the *Henkin-property* if it contains at least one Henkin formula of every Henkin-form.

2.4.5. LEMMA. *If ψ is a Henkin-formula with respect to y, then $\vdash \exists y\, \psi$.*

Proof. We argue by induction on Henkin-formulae. If ψ is of the form $\exists x\, \phi \to \phi(y/x)$, then, using the validity of $\exists y\, (\exists x\, \phi \to \phi(y/x))$ for y not free in ϕ, we get $\vdash \exists y\, \psi$.

Suppose that ψ is a Henkin-formula with respect to y, and that $\vdash \exists y\, \psi$. Assume that y is not free in the formula χ and doesn't occur in the term α; we have to show that $\vdash \exists y\, (\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi))$. Observe

$$
\begin{aligned}
&\vdash \exists y\, \psi \\
&\quad \Rightarrow\ \vdash \Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \exists y\, \psi),\ \text{by standard modal reasoning,} \\
&\quad \Rightarrow\ \vdash \Delta_\alpha \chi \to \Delta_\alpha \exists y\, (\chi \wedge \psi),\ \text{as } y \text{ is not free in } \chi, \\
&\quad \Rightarrow\ \vdash \Delta_\alpha \chi \to \exists y\, \Delta_\alpha(\chi \wedge \psi),\ \text{by the Barcan formula,} \\
&\quad \Rightarrow\ \vdash \exists y\, (\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)),\ \text{as } y \text{ does not occur in } \alpha \text{ and is not free in } \chi. \quad \dashv
\end{aligned}
$$

2.4.6. LEMMA. *Assume Σ is a consistent set of formulae, none of which contains any occurrence of y, and let ψ be a Henkin formula with respect to y. Then $\Sigma \cup \{\psi\}$ is consistent.*

Proof. Let $\Sigma' \subseteq \Sigma$ be finite. It suffices to show that $\Sigma' \cup \{\psi\}$ is consistent. Suppose otherwise. Then

$$
\begin{aligned}
\vdash \bigwedge \Sigma' \to \neg\psi \quad &\Rightarrow\ \vdash \bigwedge \Sigma' \to \forall y\, \neg\psi \\
&\Rightarrow\ \vdash \exists y\, \psi \to \neg \bigwedge \Sigma' \\
&\Rightarrow\ \vdash \neg \bigwedge \Sigma',\ \text{by Lemma 2.4.5,}
\end{aligned}
$$

which contradicts the consistency of Σ. $\dashv$

2.4.7. LEMMA. *Every consistent formula is contained in a maximal consistent set with the Henkin-property.*

Proof. This is standard; use Lemma 2.4.6. $\dashv$

2.4.8. LEMMA. *Let Σ be a maximal consistent set of formulae with the Henkin-property. Let $\Delta_\alpha\psi \in \Sigma$. Then there exists a maximal consistent set of formulae Γ with the Henkin-property such that*

$$\{\psi\} \cup \{\chi \mid \nabla_\alpha\chi \in \Sigma\} \subseteq \Gamma.$$

Proof. Define $\Gamma_0 := \{\psi\}$. Take the first Henkin-form in some fixed enumeration of all Henkin-forms, and enumerate the Henkin-formulae of the first form as $\delta_{11}, \ldots, \delta_{1n}, \ldots$. By assumption Σ has the Henkin-property, hence it contains a formula of the form $\Delta_\alpha\psi \to \Delta_\alpha(\psi \wedge \delta_{1i_1})$. Put $\Gamma_1 := \Gamma_0 \cup \{\delta_{1i_1}\}$.

In general, given that for the first m Henkin-forms we have added Henkin-formulae δ_{1i_1}, $\ldots$, δ_{mi_m}, we consider a formula $\delta_{(m+1)i_{(m+1)}}$ of the $(m+1)$-th form, which is such that

$$\Delta_\alpha(\psi \wedge \delta_{1i_1} \wedge \ldots \wedge \delta_{mi_m}) \to \Delta_\alpha(\psi \wedge \delta_{1i_1} \wedge \ldots \wedge \delta_{mi_m} \wedge \delta_{(m+1)i_{(m+1)}})$$

is in Σ, and obtain Γ_{m+1} as $\Gamma_m \cup \{\delta_{(m+1)i_{(m+1)}}\}$. Let $\Gamma' = \bigcup_m \Gamma_m$. Then Γ' has the Henkin-property.

Next, add $\{\chi \mid \nabla_\alpha\chi \in \Sigma\}$ to Γ' to obtain Γ''; this can be done without destroying consistency. Finally, increase Γ'' to a maximal consistent set Γ in the usual way. $\dashv$

We can now embark on the completeness proof for *Q-MTL*$_0$. Let Σ be a maximal *Q-MTL*$_0$-consistent set of formulae that has the Henkin property. Using Σ we will define a canonical model $\mathfrak{M}^0 = (T^0, \mathfrak{D}^0; \mathrm{DIS}^0; V^0, g)$ as follows.

The displacement domain. Using a Henkin construction, we build a displacement domain $\mathfrak{D}^0$ from Σ. In this domain the (displacement) objects are equivalences classes of terms modulo the congruence relation θ, where θ is 'provable equality according to Σ': $(\alpha, \beta) \in \theta$ iff $\Sigma \vdash \alpha = \beta$. The interpretation function $g : T(X \cup A) \to \mathfrak{D}^0$ is defined in the obvious way by putting $g(\alpha) = \alpha/\theta$.

The displacement relation. Define the relation DIS^0 as in the unquantified case: for maximal consistent sets Γ_1, Γ_2, and for every term $\gamma \in T(X \cup A)$, define

$$\begin{array}{lll}\mathrm{DIS}^0(\Gamma_1, g(\gamma), \Gamma_2) & \text{iff} & \text{for every formula } \sigma,\ \sigma \in \Gamma_2 \text{ implies } \Delta_\gamma\sigma \in \Gamma_1 \\ & & \text{(equivalently: for all } \sigma\text{, if } \nabla_\gamma\sigma \in \Gamma_1 \text{ then } \sigma \in \Gamma_2).\end{array}$$

The temporal domain. The *canonical temporal domain* T^0 consists of all maximal consistent sets Γ with the Henkin-property such that for some α, $\mathrm{DIS}^0(\Sigma, g(\alpha), \Gamma)$. Define the *canonical valuation* V^0 by putting $V^0(p) = \{\Gamma \mid p \in \Gamma\}$, for all proposition letters p.

2.4.9. LEMMA. *For all $\Gamma \in T^0$, and all algebraic terms α, β we have that $(\alpha = \beta) \in \Gamma$ iff $(\alpha = \beta) \in \Sigma$, and similarly for formulae of the form $\alpha < \beta$.*

Proof. As $\Gamma \in T^0$, we have $\mathrm{DIS}^0(\Sigma, g(\gamma), \Gamma)$ for some γ. Then $(\alpha = \beta) \in \Gamma$ implies $\Delta_\gamma(\alpha = \beta) \in \Sigma$, and so $(\alpha = \beta) \in \Sigma$ by axiom (Ax14) and universal instantiation. Conversely, $(\alpha = \beta) \in \Sigma$ implies $\nabla_\gamma(\alpha = \beta) \in \Sigma$, by axiom (Ax13), implies $(\alpha = \beta) \in \Gamma$. $\dashv$

2.4.10. Theorem. (Completeness) *Q-MTL_0 is sound and complete for the class of all Q-MTL_0-frames.*

Proof. Take a consistent formula ϕ, and let Σ be a maximal consistent extension of $\{\phi\}$ with the Henkin-property. Construct the canonical model $\mathfrak{M}^0$ for Σ as defined above. To establish the completeness of Q-MTL_0 we need to check that $\mathfrak{M}^0$ validates the axioms of Q-MTL_0, but this is clear. On top of that we need a truth lemma for Q-MTL_0.

We first treat the case of atomic algebraic formulae. Let $\Gamma \in T^0$; then $\mathrm{DIS}^0(\Sigma, g(\gamma), \Gamma)$ for some γ. Then $(\alpha = \beta) \in \Gamma$ iff $(\alpha = \beta) \in \Sigma$ (by Lemma 2.4.9) iff $g(\alpha) = g(\beta)$ iff $\Gamma \Vdash \alpha = \beta$, as required.

The remaining atomic cases and the boolean cases are straightforward. The case of the universal quantifier is the same as in standard completeness proofs for first-order logic. So let us consider the case of ∇_α. We have to show that

$$\text{if } \neg\nabla_\alpha\phi \in \Gamma_1, \text{ then } \exists\Gamma_2 \left(\mathrm{DIS}^0(\Gamma_1, g(\alpha), \Gamma_2) \text{ and } \neg\phi \in \Gamma_2\right),$$

where Γ_1, $\Gamma_2 \in T^0$.

By Lemma 2.4.8 the set $\{\phi \mid \nabla_\alpha\phi \in \Gamma_1\} \cup \{\neg\psi\}$ can be extended to a maximal consistent set Γ_2 with the Henkin-property. Clearly, $\Gamma_1 \in T^0$ and $\mathrm{DIS}^0(\Gamma_1, g(\alpha), \Gamma_2)$ implies $\Gamma_2 \in T^0$, by axiom (Ax3). Finally, $\{\phi \mid \nabla_\gamma\phi \in \Gamma_1\}$ is a subset of Γ_2, so $\mathrm{DIS}^0(\Gamma_1, g(\gamma), \Gamma_2)$ holds, as required. $\dashv$

Enriching the temporal component. For most application purposes the language of Q-MTL_0 (or a minor extension thereof) suffices. However, if full quantificational force of the temporal domain is required, the above techniques can easily be extended, as we will demonstrate now.

We consider a rich language in which the temporal component is based on a full first-order language instead of a propositional one. We consider the system Q-MTL_1.

The *language* Q-MTL_1 is built up using algebraic terms specified by

$$\alpha ::= 0 \mid a \mid x \mid \alpha + \alpha \mid -\alpha,$$

as before, and using a disjoint collection of 'temporal' variables S, typically denoted with s, t, ...; these are the variables that we will quantify over in the quantified temporal part of our language. Next, we define the formulae of Q-MTL_1:

$$\phi ::= Rt_1 \dots t_n \mid \neg\phi \mid \alpha = \beta \mid \alpha < \beta \mid \phi \wedge \phi \mid \Delta_\alpha\phi \mid \forall x\, \phi \mid \forall s\, \phi.$$

Thus, we can quantify using displacement variables x, or using 'temporal' variables s.

The *models* of Q-MTL_1 are structures of the form

$$\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; O, V, g).$$

O is the domain of individual objects; the function V assigns a member of O to each individual temporal variable. For every n-ary (temporal) predicate letter R, $V(R)$ is a collection of $(n+1)$-tuples $(u_1, \dots, u_n, w)$, where u_1, ..., $u_n \in O$ and $w \in T$.

Given this set-up, we can calculate the truth value for all formulae ϕ in the following manner:

$$\begin{array}{rll}
(\mathfrak{F}; O, V, g), i \Vdash R(s_1, \ldots, s_n) & \text{iff} & (V(s_1), \ldots, V(s_n), i) \in V(R) \\
(\mathfrak{F}; O, V, g), i \Vdash \alpha = \beta & \text{iff} & g(\alpha) = g(\beta) \\
(\mathfrak{F}; O, V, g), i \Vdash \alpha < \beta & \text{iff} & g(\alpha) < g(\beta) \\
(\mathfrak{F}; O, V, g), i \Vdash \forall x\, \phi & \text{iff} & (\mathfrak{F}; O, V, g'), i \Vdash \phi \text{ for all assignments } g' \\
& & \text{such that } g =_x g' \\
(\mathfrak{F}; O, V, g), i \Vdash \forall s\, \phi & \text{iff} & (\mathfrak{F}; O, V', g), i \Vdash \phi \text{ for all valuations } V' \\
& & \text{such that } V =_s V' \\
(\mathfrak{F}; O, V, g), i \Vdash \Delta_\alpha \phi & \text{iff} & (\mathfrak{F}; O, V, g), j \Vdash \phi \text{ for some time instant } j \\
& & \text{with DIS}(i, g(\alpha), j).
\end{array}$$

2.4.11. Remark. Observe that, just as with *Q-MTL*$_0$ models, in *Q-MTL*$_1$ models, the displacement domain is constant over all time instants, as are the truth values of the purely algebraic formulae. And the newly added individual objects domain is constant across all time instants, but, of course, (purely) temporal formulae may differ in truth value from one time instance to another.

Next, we specify the *axioms* of *Q-MTL*$_1$. To the axioms of *Q-MTL*$_0$ we add quantificational axioms for the temporal quantifiers, as well as the rule of universal generalization and the Barcan formula for the temporal quantifiers:

(Ax9′) $\forall s\,(\phi \to \psi) \leftrightarrow (\forall s\,\phi \to \forall s\,\psi)$ (functionality)
(Ax10′) $\phi \to \forall s\,\phi$, for s not in ϕ (elimination of vacuous quantifier)
(Ax11′) $\forall s\,\phi \to \phi(t/s)$, with t free for s in ϕ (universal instantiation)

and the rule:

(UG′) $\vdash \phi \implies \vdash \forall s\,\phi$ (universal generalization).

We also add the Barcan formula:

(Ax12′) $\forall s\, \nabla_\alpha \phi \to \nabla_\alpha \forall s\, \phi$.

2.4.12. Theorem. (Completeness) *Q-MTL*$_1$ *is sound and complete.*

Sketch of the proof. To establish the completeness of *Q-MTL*$_1$ using the proof technique of Theorem 2.4.10 we need to adopt the notions of a Henkin-formula and a Henkin-form (page 31) as follows. Let r be either a displacement variable or a temporal variable.

1. Any formula of the form $\exists x\,\phi \to \phi(y/x)$ is a Henkin formula with respect to y.
2. Any formula of the form $\exists s\,\phi \to \phi(t/s)$ is a Henkin formula with respect to t.
3. If ψ is a Henkin-formula with respect to y, χ is any formula not containing y free, and α is an algebraic term not containing y, then $\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to y.

4. If ψ is a Henkin-formula with respect to t, χ is any formula not containing t free, then $\Delta_\alpha\chi \rightarrow \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to t.

As before, Henkin-formulae that differ only in that each is a Henkin-formula with respect to a different variable of the same sort (i.e., either they are all displacement variables, or all temporal variables) will be said to have the same Henkin-form. A set of formula has the Henkin-property if it contains at least one Henkin formula of every Henkin-form.

We leave it to the reader to verify that given the above notions of Henkin-formula, Henkin-form, and Henkin-property, Lemma's 2.4.5–2.4.8 remain valid.

The canonical model for Q-MTL_1 is built up in the same way as for Q-MTL_0, except for the fact that we need to specify a domain of individual objects O and a valuation V; the former will simply be the collection of all temporal variables, and the latter is defined by $V(R) = \{(u_1, \ldots, u_n, \Gamma) \mid R(u_1, \ldots, u_n) \in \Gamma\}$, where R is an n-ary predicate symbol. With this modification a truth lemma can be established as in the proof of Theorem 2.4.10. $\dashv$

2.5 A PDL-like reformulation of metric temporal logic

In this section, we show how to reinterpret (propositional) MTL as a Propositional Dynamic Logic. In the concluding remarks, we will briefly sketch a possible exploitation of this correspondence to prove decidability results for basic systems of (propositional) MTL. In Chapter 5, we will show how such a reinterpretation of (propositional) MTL can be used to support its execution.

The idea is to interpret basic MTL systems as polymodal logics with a set of modal operators ∇_α indexed by temporal displacements. Unlike PDL, MTL does not encompass any operation corresponding to the PDL program $?\phi$, which is mapped into an accessibility relation $R_{?\phi}$ whose definition depends on the considered model. This allows us to express the semantics of MTL in terms of standard frames instead of standard models, a standard model $\mathfrak{M}$ simply being a model based on a standard frame. Moreover, MTL has no infinitary operations (like the operation $(.)^*$ of PDL), but its finitary structure is richer (e.g., the operation $+$ of MTL satisfies the properties of inverse and commutativity).

Each modal operator ∇_α is interpreted as a distinct accessibility relation R_α. Two-sorted frames $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and models $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ are replaced by pairs $\mathfrak{F} = (T, \{R_\alpha : \alpha \in T(A)\})$ and triplets $\mathfrak{M} = (T, \{R_\alpha : \alpha \in T(A)\}, V)$, respectively, where $\mathrm{DIS}(i, g(\alpha), j)$ if and only if $R_\alpha(i, j)$, and

$$\mathfrak{M}, i \Vdash \nabla_\alpha\phi \;\;\Leftrightarrow\;\; \forall j(R_\alpha(i,j) \Rightarrow \mathfrak{M}, j \Vdash \phi).$$

Standard frames. Let $\circ$ and $^{-1}$ be respectively a binary and a unary operation over the set $\{R_\alpha : \alpha \in T(A)\}$ defined as follows:

$$\begin{aligned} R_\alpha \circ R_\beta &= \{(i,k) : \exists j(R_\alpha(i,j) \wedge R_\beta(j,k))\} \\ (R_\alpha)^{-1} &= \{(j,i) : R_\alpha(i,j)\}. \end{aligned}$$

Moreover, let I be the identity relation over T, that is, $I = \{(i,i) : i \in T\}$.

We define a frame $\mathfrak{F}$ to be *standard* if it satifies the following conditions: the structure $\langle\{R_\alpha : \alpha \in T(A)\}, \circ\rangle$ is a commutative group and $R_{\alpha+\beta} = R_\alpha \circ R_\beta, R_{-\alpha} = (R_\alpha)^{-1}$, and $R_0 = I$.

A *standard model* $\mathfrak{M}$ is simply a model based on a standard frame.

It is possible to show that, in any standard frame $\mathfrak{F}$, transitivity directly follows from the definition of $\circ$. Moreover, it is worth noting that, in order to define a standard frame $\mathfrak{M}$, it suffices to provide each modal operator ∇_α, indexed by an *atomic* displacement $\alpha \in A$, with an interpretation R_α, that is, to specify the structure: $(T, \{R_\alpha : \alpha \in A\}, V)$. The relations R_α, for all non-atomic displacements α, can then be inductively defined according to the above conditions giving the intended meaning of $+, -$, and 0.

Axiomatization. Let us consider the PDL-like reformulation of MTL (MTL-as-PDL hereafter) whose axioms are

(i)	$\nabla_{\alpha+\beta} p \leftrightarrow \nabla_{\beta+\alpha} p$
(ii)	$\nabla_{\alpha+(\beta+\gamma)} p \leftrightarrow \nabla_{(\alpha+\beta)+\gamma} p$
(iii)	$\nabla_{\alpha+0} p \leftrightarrow \nabla_\alpha p$
(iv)	$\nabla_{\alpha+(-\alpha)} p \leftrightarrow \nabla_0 p$
(Ax1)	$\nabla_\alpha (p \rightarrow q) \rightarrow (\nabla_\alpha p \rightarrow \nabla_\alpha q)$
(Ax2)	$p \rightarrow \nabla_\alpha \Delta_{-\alpha} p$
(Ax3)	$\nabla_{\alpha+\beta} p \rightarrow \nabla_\alpha \nabla_\beta p$
(Ax5')	$\nabla_0 p \leftrightarrow p$
(Ax6)	$\nabla_\alpha p \rightarrow \Delta_\alpha p$

and whose rules are

(Rep)	$\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi \implies \vdash \nabla_{[\alpha/x]\delta} \phi \leftrightarrow \nabla_{[\beta/x]\delta} \phi$	(replacement)
	where $[\alpha/x]$ denotes substitution of α for the variable x	
(Sub)	$\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi \implies \vdash \nabla_{[\delta/x]\alpha} \phi \leftrightarrow \nabla_{[\delta/x]\beta} \phi$	(substitution)
(NEC)	$\vdash \phi \implies \vdash \nabla_\alpha \phi$	(necessitation rule for ∇_α),

plus modus ponens and replacement (REP) and uniform substitution (SUB) of propositional variables

(REP)	$\vdash \phi \leftrightarrow \psi \implies \vdash \chi(\phi/p) \leftrightarrow \chi(\psi/p)$	(replacement)
	where (ϕ/p) denotes substitution of ϕ for the variable p	
(SUB)	$\vdash \phi \leftrightarrow \psi \implies \vdash \phi(\chi/p) \leftrightarrow \psi(\chi/p)$	(uniform substitution).

It is worth noting that from *Euclidicity* ($\Delta_\alpha \nabla_\beta p \rightarrow \nabla_{\alpha+\beta} p$) and *Seriality* ($Ax6$), it easily follows that

$$\nabla_\alpha \nabla_\beta p \rightarrow \nabla_{\alpha+\beta} p. \tag{2.2}$$

(Sketch of the proof.) From $\nabla_\alpha p \rightarrow \Delta_\alpha p$, we obtain $\nabla_\alpha \nabla_\beta p \rightarrow \Delta_\alpha \nabla_\beta p$ by uniform substitution; the conclusion follows from *Euclidicity* by Syllogism.

Pairing 2.2 with *Transitivity*, we obtain $\nabla_\alpha \nabla_\beta p \leftrightarrow \nabla_{\alpha+\beta} p$. Notice also that $Ax5'$ guarantees that for all $i \in T$ there exists one ($\nabla_0 p \rightarrow p$ corresponds to the condition

$\forall i\ R_0(i,i)$) and only one ($p \to \nabla_0 p$ corresponds to the condition $\forall i,j(R_0(i,j) \to i=j)$) j, that is, i itself, such that $R_0(i,j)$.

Completeness. The completeness proof for MTL-as-PDL is standard: there are no operators like $(.)^*$ to complicate matters, and we can just use a canonical model method.

Formally, let $\mathfrak{M}^0 = (T^0, \{R^0_\alpha : \alpha \in T(A)\}, V^0)$ be the canonical model for MTL-as-PDL, whose components are defined as follows:

T^0 is the set of maximal MTL-consistent sets,
$R^0_\alpha(i,j)$ if and only if $\{\phi : \nabla_\alpha \phi \in i\} \subseteq j$, and
$V^0 = \{i \in T^0 : p \in i\}$.

We first show that $\mathfrak{M}^0$ is a standard model. To this end, we must verify that it satisfies all the conditions for standard models. In the following, for each condition, we will explicitly prove that this is the case. However, since all of the axioms are Sahqlvist forms, and therefore they are canonical, the corresponding frame conditions can actually be *derived from them.* As an example, axiom Ax3 immediately says that $R_{\alpha\circ\beta} \subseteq R_{\alpha+\beta}$, and this will hold in the canonical model by canonicity. From the converse of the above axiom (i.e. Theorem 2.2), we get the converse of the condition.

Condition 1: $R^0_{\alpha+\beta} = R^0_\alpha \circ R^0_\beta$.

We separately prove that (i) $R^0_\alpha \circ R^0_\beta \subseteq R^0_{\alpha+\beta}$ and (ii) $R^0_{\alpha+\beta} \subseteq R^0_\alpha \circ R^0_\beta$. From now on, we will use $(i,j) \in R_\alpha$ as an alternative (equivalent) notation for $R_\alpha(i,j)$.

Proof of (i). Suppose that $(i,k) \in R^0_\alpha \circ R^0_\beta$. By definition, this means that there exists $j \in T^0$ such that $\{\phi : \nabla_\alpha \phi \in i\} \subseteq j$ and $\{\phi : \nabla_\beta \phi \in j\} \subseteq k$. We want to prove that $(i,k) \in R^0_{\alpha+\beta}$, that is, $\{\phi : \nabla_{\alpha+\beta}\phi \in i\} \subseteq k$.

Let us assume that $\nabla_{\alpha+\beta}\phi \in i$. Since $\nabla_{\alpha+\beta}\phi \to \nabla_\alpha\nabla_\beta\phi \in i$, it follows that $\nabla_\alpha\nabla_\beta\phi \in i$. Hence $\nabla_\beta\phi \in j$ (by definition of j) and $\phi \in k$ (by definition of k).

Proof of (ii). Suppose that $(i,k) \in R^0_{\alpha+\beta}$. By definition, this means that $\{\phi : \nabla_{\alpha+\beta}\phi \in i\} \subseteq k$. We want to prove that $(i,k) \in R^0_\alpha \circ R^0_\beta$, that is, there exists $j \in T^0$ such that $\{\phi : \nabla_\alpha\phi \in i\} \subseteq j$ and $\{\phi : \nabla_\beta\phi \in j\} \subseteq k$. By Lindenbaum Lemma, it suffices to show that the set

$$j_0 = \{\phi : \nabla_\alpha\phi \in i\} \cup \{\Delta_\beta\psi : \psi \in k\}$$

is consistent.

The proof is by contradiction. Suppose that j_0 is inconsistent. This means that there exist $n, m > 0$ such that $\vdash \phi_1 \wedge \ldots \wedge \phi_n \wedge \Delta\psi_1 \wedge \ldots \wedge \Delta\psi_m \to \bot$, with $\phi_i \in i$, for $i = 1, \ldots, n$, and $\Delta\psi_j \in k$, for $j = 1, \ldots, m$. Let ψ be equal to $\psi_1 \wedge \ldots \wedge \psi_m$. It is straightforward to prove $\vdash \Delta\psi \to \Delta\psi_1 \wedge \ldots \wedge \Delta\psi_m$. From this, it follows that $\vdash \phi_1 \wedge \ldots \wedge \phi_n \to \neg\Delta_\beta\psi$ and then $\vdash \nabla_\alpha\phi_1 \wedge \ldots \wedge \nabla_\alpha\phi_n \to \nabla_\alpha\neg\Delta_\beta\psi$. Since $\nabla_\alpha\phi_j \in i$, for $j = 1, \ldots, n$, it follows that $\nabla_\alpha\neg\Delta_\beta\psi$, which is equivalent to $\nabla_\alpha\nabla_\beta\neg\psi$, belongs to i. From $\nabla_\alpha\nabla_\beta\neg\phi \to \nabla_{\alpha+\beta}\neg\phi \in \phi$, it follows that $\nabla_{\alpha+\beta}\neg\phi \in \phi$ and therefore $\neg\phi \in k$, which is a contradiction because $\psi_j \in k$, for $j = 1, \ldots, m$.

The structure of the proofs for conditions 2, 4, and 5 is essentially the same.

Condition 2: $R^0_\alpha \circ R^0_\beta = R^0_\beta \circ R^0_\alpha$.

We prove that $R^0_\alpha \circ R^0_\beta \subseteq R^0_\beta \circ R^0_\alpha$ (the proof of the opposite inclusion is similar). Suppose that $(i,k) \in R^0_\alpha \circ R^0_\beta$. Since, by condition 1, $R^0_\alpha \circ R^0_\beta \subseteq R^0_{\alpha+\beta}$, it follows that $(i,k) \in R^0_{\alpha+\beta}$, that is, $\{\phi : \nabla_{\alpha+\beta}\phi \in i\} \subseteq k$. Assume now that $\nabla_{\beta+\alpha}\phi \in i$. Since $\nabla_{\beta+\alpha}\phi \rightarrow \nabla_{\alpha+\beta}\phi \in i$, it follows that $\nabla_{\alpha+\beta}\phi \in i$ and thus $\phi \in k$. Hence, $\{\phi : \nabla_{\beta+\alpha}\phi \in i\} \subseteq k$ and therefore $(i,k) \in R^0_{\beta+\alpha}$. Since, by condition 1, $R^0_{\beta+\alpha} \subseteq R^0_\beta \circ R^0_\alpha$, it follows that $(i,k) \in R^0_\beta \circ R^0_\alpha$.

Condition 4: $R^0_\alpha \circ R^0_0 = R^0_\alpha$.

Since, by condition 1, $R^0_\alpha \circ R^0_0 = R^0_{\alpha+0}$, $R^0_\alpha \circ R^0_0 = R^0_\alpha$ if and only if $R^0_{\alpha+0} = R^0_\alpha$. We prove that $R^0_{\alpha+0} \subseteq R^0_\alpha$ (the proof of the opposite inclusion is similar). Let $(i,j) \in R^0_{\alpha+0}$. By definition, this means that $\{\phi : \nabla_{\alpha+0}\phi \in i\} \subseteq j$. Assume now that $\nabla_\alpha\phi \in i$. Since $\nabla_\alpha\phi \rightarrow \nabla_{\alpha+0}\phi \in i$, it follows that $\nabla_{\alpha+0}\phi \in i$ and thus $\phi \in j$. Hence, $\{\phi : \nabla_\alpha\phi \in i\} \subseteq j$ and therefore $(i,j) \in R^0_\alpha$.

Condition 5: $R^0_\alpha \circ R^0_{-\alpha} = R^0_0$.

Since, by condition 1, $R^0_\alpha \circ R^0_{-\alpha} = R^0_{\alpha+(-\alpha)}$, $R^0_\alpha \circ R^0_{-\alpha} = R^0_0$ if and only if $R^0_{\alpha+(-\alpha)} = R^0_0$. We prove that $R^0_{\alpha+(-\alpha)} \subseteq R^0_0$ (the proof of the opposite inclusion is similar). Let $(i,j) \in R^0_{\alpha+(-\alpha)}$. By definition, this means that $\{\phi : \nabla_{\alpha+(-\alpha)}\phi \in i\} \subseteq j$. Assume now that $\nabla_0\phi \in i$. Since $\nabla_0\phi \rightarrow \nabla_{\alpha+(-\alpha)}\phi \in i$, it follows that $\nabla_{\alpha+(-\alpha)}\phi \in i$ and thus $\phi \in j$. Hence, $\{\phi : \nabla_0\phi \in i\} \subseteq j$ and therefore $(i,j) \in R^0_0$.

Condition 6: $R^0_{-\alpha} = (R^0_\alpha)^{-1}$.

Let us show that (i) $(R^0_\alpha)^{-1} \subseteq R^0_{-\alpha}$ and (ii) $R^0_{-\alpha} \subseteq (R^0_\alpha)^{-1}$.

Proof of (i). Assume that $(j,i) \in (R^0_\alpha)^{-1}$. By definition of $(.)^{-1}$, this means that $(i,j) \in R^0_\alpha$, that is, $\{\phi : \nabla_\alpha\phi \in i\} \subseteq j$. In particular, it holds that $\{\Delta_{-\alpha}\phi : \nabla_\alpha\Delta_{-\alpha}\phi \in i\} \subseteq j$. This implies that $\{\Delta_{-\alpha}\phi : \phi \in i\} \subseteq j$. On the contrary, suppose that there exists ϕ such that $\phi \in i$ and $\Delta_{-\alpha}\phi \notin j$. Since $\phi \rightarrow \nabla_\alpha\Delta_{-\alpha}\phi \in i$, it follows that $\nabla_\alpha\Delta_{-\alpha}\phi \in i$ and thus $\Delta_{-\alpha}\phi \in j$ (contradiction). Finally, since $\{\Delta_{-\alpha}\phi : \phi \in i\} \subseteq j$ if and only if $\{\phi : \nabla_{-\alpha}\phi \in j\} \subseteq i$, it follows that $(j,i) \in R^0_{-\alpha}$.

Proof of (ii). Assume that $(j,i) \in R^0_{-\alpha}$. By definition, this means that $\{\phi : \nabla_{-\alpha}\phi \in j\} \subseteq i$. In particular, it holds that $\{\Delta_\alpha\phi : \nabla_{-\alpha}\Delta_\alpha\phi \in j\} \subseteq i$. This implies that $\{\Delta_\alpha\phi : \phi \in j\} \subseteq i$. As before, suppose that there exists ϕ such that $\phi \in j$ and $\Delta_\alpha\phi \notin i$. Since $\phi \rightarrow \nabla_{-\alpha}\Delta_\alpha\phi \in j$, it follows that $\nabla_{-\alpha}\Delta_\alpha\phi \in j$ and thus $\Delta_\alpha\phi \in i$ (contradiction). Finally, since $\{\Delta_\alpha\phi : \phi \in j\} \subseteq i$ if and only if $\{\phi : \nabla_\alpha\phi \in i\} \subseteq j$, it follows that $(i,j) \in R^0_\alpha$ and therefore, by definition, $(j,i) \in (R^0_\alpha)^{-1}$.

Notice that in the proof of (ii) we have assumed that $\phi \rightarrow \nabla_{-\alpha}\Delta_\alpha\phi$ is a theorem of *MTL*-as-*PDL*. To prove this fact, we must first show that $\phi \rightarrow \nabla_{-\alpha}\Delta_{-(-\alpha)}\phi$ is a theorem of*MTL*-as-*PDL*, and then show that $\Delta_{-(-\alpha)}\phi \leftrightarrow \Delta_\alpha\phi$ holds. To obtain the first result, we must interpret *MTL*-as-*PDL* axioms as having universal import, and therefore we need some instantiation mechanism. The easiest way of achieving this is viewing axioms as axiom schemes: we take as axioms all formulae of a given form; so by Ax2 ($p \rightarrow \nabla_\alpha\Delta_{-\alpha}$), we have got $p \rightarrow \nabla_{-\beta}\Delta_{--\beta}$. As an alternative, we can consider all the displacement variables occurring in *MTL*-as-*PDL* axioms as (implicitly) universally quantified, and add an axiom of universal instatiation[1]. The second result can be proved as follows:

[1]As already noticed, what really distinguishes extended MTL from basic MTL is the ability of mixing

$$\Delta_\alpha\phi \leftrightarrow \Delta_{\alpha+0}\phi \leftrightarrow \Delta_{\alpha+(-\alpha+(-(-\alpha)))}\phi \leftrightarrow \Delta_{(\alpha+(-\alpha))+(-(-\alpha))}\phi \leftrightarrow$$
$$\leftrightarrow \Delta_{0+(-(-\alpha))}\phi \leftrightarrow \Delta_{-(-\alpha)+0}\phi \leftrightarrow \Delta_{-(-\alpha)}\phi.$$

Condition 7: $R_0^0 = I$.

Let us show that (i) $R_0^0 \subseteq I$ and (ii) $I \subseteq R_0^0$.

Proof of (i). Assume that $(i,j) \in R_0^0$ and let $\phi \in i$. Since $\phi \to \nabla_0\phi \in i$, we have that $\nabla_0\phi \in i$. By definition of R_0^0 ($(i,j) \in R_0^0$ if and only if $\{\phi : \nabla_0\phi \in i\} \subseteq j$), it follows that $\phi \in j$ and thus $i \subseteq j$. Since i and j are maximal consistent sets, we can conclude that $i = j$.

Proof of (ii). We must show that for all $i \in T$, $(i,i) \in R_0^0$. By definition, $(i,i) \in R_0^0$ if and only if $\{\phi : \nabla_0\phi \in i\} \subseteq i$. Assume that $\nabla_0\phi \in i$. Since $\nabla_0\phi \to \phi \in i$, from $\nabla_0\phi \in i$ it follows that $\phi \in i$ and thus $\{\phi : \nabla_0\phi \in i\} \subseteq i$.

Notice that if we replace $Ax5'$ by $\nabla_0 p \to p$, we can only prove that $I \subseteq R_0^0$.

To conclude the completeness proof we only need the truth lemma, whose proof is straightforward, and thus omitted.

2.5.1. THEOREM. *MTL-as-PDL is (sound and) complete with respect to the class of standard frames.*

It is possible to show that MTL-as-PDL is equivalent to MTL as stated by the followig Proposition.

2.5.2. PROPOSITION. *MTL-as-PDL derives all and only the (temporal) theorems of MTL.*

(Sketch of the proof.) We have shown (cf. Theorem 2.5.1) that MTL-as-PDL is (sound and) complete with respect to the class of standard frames. Now, we can turn every model of MTL into a MTL-as-PDL one, and conversely. So: if ϕ is not provable in MTL, then, by completeness there exists an MTL model refuting ϕ; turn this into a MTL-as-PDL model. The MTL-as-PDL model should refute ϕ as well; so then ϕ is not provable in MTL-as-PDL. Likewise, any non-theorem of MTL-as-PDL can be shown to be a non-theorem of MTL, by turning MTL-as-PDL models into MTL models in a truth preserving way.

Concluding remarks

In this chapter we have proved completeness results for basic systems of metric temporal logic. We started with the minimal calculus and showed how to extend it to obtain the logic of two-sorted frames with a linear temporal order in which there exists a unique temporal distance between any two time instants. After that we considered general metric temporal logics allowing quantification over algebraic and temporal variables and free mixing of algebraic and temporal formulae.

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displacement and temporal formulae. Quantifiers are already present in basic MTL, but only the possibility of mixing the two types of formulae makes their use effective.

We traced a sort of preferred path from the minimal metric temporal logic MTL_0 to the quantified metric temporal logic $Q\text{-}MTL_0$, passing through the (unquantified) metric temporal logic of linear orders MTL_2. In fact, the proposed two-sorted framework allows one to characterize a variety of metric temporal logics simply by weakening or strengthening the requirements on the algebraic and temporal components and their interaction. For example, in certain application areas it seems natural to abandon the requirement that the displacement relation is quasi-functional with respect to its third argument; one situation where this comes up is in the use of our metric (temporal) logics for specifying the spatial behavior of read and write heads of a hard disk. Developing this more liberal approach to interpreting metric (temporal) languages is part of our ongoing research. We are currently considering also the problem of establishing what kinds of richer PDL correspond to quantified systems of MTL.

In this chapter, we have not discussed decidability issues. It is known that a negative result holds for $Q\text{-}MTL_0$. Burgess [20] shows that the decision problem for quantified metric temporal logic is equivalent to that for the set of all universal monadic second-order formulae true in all ordered abelian groups, and he proves that the decision problem for the validity of first-order formulae involving a single binary predicate, which is known to be undecidable, can be reduced to this equivalent problem.

As to the decidability question for propositional metric temporal logics, we are currently trying to exploit the link between (propositional) metric temporal logics and versions of *propositional dynamic logic*, defined in the previous section, with a view to importing results and techniques on decidability from the latter. Roughly, our strategy is the following. In the previous section, we have proposed a reinterpretation of the propositional metric language on multi-modal models of the form $(W, \{R_\alpha \mid \alpha \text{ is an algebraic term } \}, V)$, and defined the semantics of a modal operator Δ_α in terms of the relation R_α. To prove the decidability of a metric temporal logic, one should then show that it has the finite model property with respect to the above multi-modal models, and the key tool in doing so will be (an adaptation of) the filtration method familiar from modal and dynamic logic (cf. Goldblatt [61]).

Chapter 3

Metric and layered temporal logic

3.1 Introduction

In this chapter, we extend metric temporal logic(s) with the notion of time granularity. The resulting metric and layered temporal logic allows one to build granular real-time system specifications by referring to the natural time granularity in any component of the specification, and by properly constraining the interactions between differently-grained components.

A first attempt at extending logical specification languages for incorporating time granularity is reported in [33, 34, 32, 82]. It basically consists of translation mechanisms that maps a formula associated with a given time granularity into a corresponding formula associated with a finer one. In such a way, a model of a specification involving different granularities can be built by translating everything to the finest granularity. However, in general, there is no standard way to define the meaning of a formula when moving from a time granularity to another one. Thus, more information is needed from the specifier to drive the translation of the formulae. The main idea is that when we state that a predicate p holds at a given time t_1, where t_1 is a term in temporal domain T^1, we mean that p holds in a subset of the interval corresponding to t_1 in the finer domain T^2. Such a subset could be the whole interval, or even a single instant, or a scattered sequence of smaller intervals, and so on. For instance, saying that "the light has been switched on at time t_{min} (t_{min} is measured in minutes)" may correspond to a predicate *switching_on* that is true just at the minute t_{min} and just at one second within t_{min}. Instead, saying that an employee works at the day t_d generally means that there are several minutes, during the day t_d, where the predicate *work* holds for the employee. These are not necessarily contiguous. Thus, the specifier will be provided with suitable tools to qualify the subset of the intervals of time that, in the finer time granularity, correspond to the given instant in the coarse granularity.

In this chapter, we substantially revise such an approach. We extend the basic logical language with contextual and projection operators that make it possible to logically compose formulae associated with different time granularity and to explicitly switch from one granularity to another. Moreover, we provide the resulting language with a model-theoretic semantics and a sound axiomatization. The proposed semantics expresses more general and complete properties of time granularity than the transformational semantics given before. Besides, the axiomatic system provides a better clarification of the meaning of time granularity and gives the possibility of doing inferences from a granular specification.

The chapter is organized as follows. In Section 3.2, we discuss the main issues to be confronted when formalizing the concept of time granularity. Then, in Section 3.3, we describe the basic steps required to extend the temporal structure for two-sorted metric temporal logic with time granularity. In Section 3.4, we formally define syntax and semantics of a metric and layered temporal logic, together with its axiomatization. In Section 3.5, we give some examples of temporally layered specifications. Finally, in Section 3.6 we consider the so-called *alignment problem* of temporal domains, and then, in Section 3.7, we show how to deal with it in the proposed logic. Conclusions provide an assessment of the approach, discuss open issues and outline possible extensions. Routine proofs can be found in [84].

3.2 Time granularity issues

The main problems we have to solve to give a formal meaning to the use of different time granularities in a formal language are the qualification of statements with respect to time granularity and the definition of the links between statements associated with a given time granularity, like *days*, and statements associated with another granularity, like *microseconds*.

Sometimes, this problem has an obvious solution that consists in using *different time units* - say, months and minutes - to measure time quantities in a *unique dynamic model*. For instance, the problem of specifying a pondage power plant through a set of states and transitions requires the definition of the temporal constraints of the system. A description of the plant could include states such as *empty_reservoir, full_reservoir, open_sluice_gate, closed_sluice_gate*, together with the transitions between these states. A numeric value is associated with each transition, which is the time needed for its completion. We can easily state that moving from *empty_reservoir* to *full_reservoir* by applying a given input of water per second takes 2 months, whereas moving from *open_sluice_gate* to *closed_sluice_gate*, when applying the command *close_sluice_gate*, takes 2 minutes. All that is needed is that, syntactically, the user may attach a suitable label to temporal terms specifying the unit for them. Semantically, a possible interpreter for such a language could easily build a global state of the system bound to a time instant that is measured in the *finest time unit*. Simple multiplications would be needed when executing transitions measured in a coarser scale. At most, some level of nondeterminism could arise from the fact that, generally, when we say that "a reservoir is filled within 2 months" we do not mean that it is filled in exactly $2 * 30 * 24 * 60 * 60$ seconds (assuming that every month has exactly 30 days), but in an approximation of such a number whose bounds could be either explicitly stated by the user - say, 5 days - or stated a priori on the basis of the adopted time unit - more than 1 month and less than 3 months. In this case, therefore, a model of the system using different time granularities is just an abbreviation for a model on the finest time unit.

In most granular systems, however, the treatment of different time granularities involves more difficult semantic problems. Consider, for instance, the sentence "every month, if an employee works, then he gets his salary". It could be formalized, in a first order language, by a formula such as:

$$\forall t_m, emp(work(emp, t_m) \rightarrow get\_salary(emp, t_m)),$$

with an obvious meaning of the used symbols, once it is stated that the subscript m denotes the fact that t is measured by the time unit of *months*.
Another requirement can be expressed by the sentence "an employee must complete every received job within 3 days". It is formalized by the formula:

$$\forall t_d, emp, job(get\_job(emp, job, t_d) \rightarrow job\_done(emp, job, t_d + 3)),$$

where the subscript d denotes that t is measured by the time unit of *days*.
Assume now that the two formulae are part of the specification of the same office system. We need a *common model* for both formulae. As done before, we could choose the finest temporal domain, i.e., the set of (times measured by) *days*, as the common domain. Then, a term labeled by m would be translated into a term labeled d by multiplying its value by 30. However, clearly the statement "every month, if an employee works, then he gets his salary" is different from the statement "every day, if an employee works, then he gets his salary". In fact, working for a month means that one works for 22 days in the month, whereas getting a monthly salary means that there is one day when one gets the salary for the month. Similarly, stating that "every day of a given month it rains" does not mean, in general, that "it rains for all seconds of all days of the month". On the contrary, if one states that "a car has been moving for three hours at a speed greater than 30 km per hour", he usually means that for all seconds included in the considered three hours the car has been moving at the specified speed. The above examples show that the models associated with temporal statements are likely to change interpretation when switching from one time granularity to another one. The addition of a concept ot time granularity is thus necessary to allow one to build granular temporal models by referring to the 'natural scale' in any component of the model and by properly constraining the interactions between differently-grained components.

Further difficulties arise from the so-called *alignment problem* of temporal domains [32]. It can be illustrated by the following examples. Consider the sentence "tomorrow I will eat". If one interprets it in the domain of hours, its meaning is that there will be several hours, starting from the next midnight until the following one, when it will be true that I eat, *no matter in which hour of the present day this sentence is claimed*. Thus, if the sentence is claimed at 1 a.m., it will be true that "I eat" in times t whose distance d from the current instant is such that $23 \leq d < 47$. Instead, if the same sentence is claimed at 10 p.m. of the same day, d will be such that $2 \leq d < 26$. Consider now the sentence "dinner will be ready in one hour". If it is interpreted in the domain of minutes, its meaning is that dinner will be ready in 60 minutes starting from the minute when it is claimed. Therefore, if the sentence is claimed at minute, say, 10, or 55, of a given hour, always it will be true that “dinner is ready” at time t whose distance d from such a minute is *exactly* 60 minutes. Clearly, the two examples require two different semantics. We call cases of the first and second type *synchronous* and *asynchronous*, respectively. We first provide a comprehensive treatment of the synchronous case; then, we generalize the proposed solution to deal also with the asynchronous one.

3.3 Embedding time granularity in the temporal structure

3.3.1 The notion of temporal universe

The temporal universe $\mathcal{T}$ is the union of a set of disjoint *temporal domains*, that is, $\mathcal{T} = \bigcup_{i \in M} T^i$, where M is an initial segment of $\mathbb{N}$ (possibly equal to $\mathbb{N}$)[1]. The set of domains $\mathcal{C} = \{T^i : i \in M\}$ is totally ordered on the basis of the degree of fineness (coarseness) of its elements. Let $\prec$ be such a *granularity* relation. For each $i \in M$, $T^i \prec T_{i+1}$ and the granularity of T_{i+1} is said finer than the granularity of T^i. As an example, consider the temporal universe including *years, months, weeks* and *days*. The domains are ordered by granularity as follows: $years \prec months \prec weeks \prec days$. We also introduce a finer relation $\supset$ on the set of domains of a temporal universe, called *disjointedness* relation. It is a partial ordering relation modeling a natural notion of inclusion between domains. It allows us to rule out pairs of domains like *months* and *weeks* for which an instant of a finer domain (*weeks*) can be astride two instants of the coarser one (*months*). With respect to the previous example, the domains are ordered by disjointedness as follows: $years \supset months, months \supset days, weeks \supset days$. This means that *years* are pairwise disjoint when viewed as sets of *months*; the same holds for *months* when viewed as sets of *days*[2].

Each domain is *discrete* with the possible exception of the finest domain(s) that may be dense. The reason is that, at least as long as one is interested in denumerable domains as we are, each dense domain is already at the finest level of granularity, and it allows any degree of precision in measuring time displacements. As a consequence, for dense domains we must distinguish granularity from metric, while for discrete domains we can define granularity in terms of set cardinality and assimilates it to a natural notion of metric. Mapping, say, a set of rational numbers into another set of rational numbers would only mean changing

[1] In the area of logical specifications as well as in the area of temporal databases, most applications are concerned with a *finite* set of temporal domains. The relevance of the infinite case will be pointed out in Chapter 4.

[2] To understand why the disjointedness relation is notated as strict inclusion, consider the case of the three domains of *years, months*, and *hours*. For any *year*, and any *month* belonging to it, the set of *days* of the *month* is a proper subset of the set of *days* of the *year*. The generalization to the case of n domains ordered by disjointedness is straightforward.

the unit of measure with no semantic effect. Just in the same way one could decide to describe geometric facts by using, say, Kmeters and centimetres. However, if Kmeters are measured by rational numbers, the same level of precision as with centimetres can be achieved. Instead, the key point in time granularity is that saying that something holds for all days in a given interval does not imply that it holds every second within the same interval [32]. For the sake of simplicity, we assume that each domain is discrete.

For each ordered pair T^i, T^j, with $T^i \prec T^j$, a mapping is defined that maps each element t^i of T^i into an interval of contiguous elements of T^j, whose width is called the *conversion factor* between T^i and T^j with respect to i. In general, the value of the conversion factors of elements belonging to the same domain may be different. This dependency on time instants is introduced to deal with pair of domains like *real months* and *days* for which a different number of instants of the finer domains (28 or 29, 30 and 31 *days*) corresponds to different instants of the coarser one (*months*). Furthermore, such a decomposition function maps contiguous instants into contiguous intervals and preserves the ordering of domains. If $T^i \supset T^j$ then the intervals are disjoint, e.g., in the case of the mapping from *minutes* to *seconds*, otherwise the intervals can meet at their endpoints, e.g., in the case of the mapping from *months* to *weeks*. We also require that the set union of the intervals of T^j belonging to the range of the decomposition function is equal to T^j. Finally, for each i, j, k, we require that if $T^i \supset T^k \supset T^j$, then the decomposition function from T^i to T^j is equal to the composition of the decomposition functions from T^i to T^k and from T^k to T^j.

For certain classes of temporal universes, we assume that for each pair of temporal domains T^i, T_j the conversion factor is constant (homogeneity assumption). In such a case, conversion factors provide a relative measurement of the granularity of each ordered pair of domains T^i and T^j. This assumption is useful, for instance, to deal with *legal months*.

In general, there are several ways to define these mappings, each one satisfying the required properties. According to the intended meaning of the mappings as *decomposition functions*, each element of T^i is mapped into the set of elements of T^j that compose it.

For each pair T^i, T^j, with $T^i \prec T^j$, we also define an *abstraction* function that maps each element j of T^j into an interval I_i of contiguous elements of T^i such that j belongs to the intersection of the intervals of T^j resulting from the application of the decomposition function to the elements of I_i. The uniqueness of such intervals can be easily deduced from the definition of the decomposition functions. If $T^i \supset T^j$, each interval I_i is a singleton (we call its element the *coarse grain equivalent* of j with respect to T^i), and therefore the abstraction function can be redefined as a mapping from T^j on T^i.

3.3.2 Temporal universe formalization

In this section, the concept of temporal universe is formally defined by means of the relations of *displacement*, *contextualization*, and *projection*.

The relation of displacement. For each temporal domain T^i, a ternary relation $\mathrm{DIS}_i \subseteq T^i \times D \times T^i$ is defined that relates pairs of time instants of T^i and displacements. For the sake of simplicity, we confine ourselves to relations DIS_i that share the same displacement domain. The *displacement relation* DIS is equal to $\bigcup_{i \in M} \mathrm{DIS}_i$. As shown in Chapter 2, specific constraints can be added to displacement relations to force them to satisfy properties such as reflexivity, quasi-functionality, transitivity. We assume that all the domains satisfy the same constraints.

The relation of contextualization. The relation of contextualization $\mathrm{CONT} \subseteq \mathcal{T} \times \mathcal{C}$ associates each instant of the temporal universe with the temporal domain it belongs to. In its full generality, such a relation allows one to deal with possibly overlapping domains. For the sake of simplicity, however, we restrict ourselves to the case of non-overlapping domains. Moreover, we require that the set of domains constitutes a partition of the temporal universe, that is, we require that each time instant belongs to one (domains cover the temporal universe) and only one (domains are disjoint) domain, and that for each domain there exists at least one instant belonging to it (domains are not empty). To capture this requirement, it suffices to constrain the relation of contextualization to be a total function $\mathrm{CONT} : \mathcal{T} \to \mathcal{C}$, with range $\mathrm{CONT}(\mathcal{T})$ equal to $\mathcal{C}$.

The relation of projection. The relation of *projection* $\updownarrow \subseteq \mathcal{T} \times \mathcal{T}$ $(= \bigcup_{i,j \in M} T^i \times T^j)$ embeds the decomposition and abstraction functions in the temporal structure. In order to constrain its semantics, we preliminary require that, for each ordered pair of domains T^i, T^j and each i in T^i, there exists a *conversion factor* that expresses the numerical relationship between the granularities of T^i and T^j with respect to i. Let C_F be the function that, for each ordered pair T^i, T^j and each i in T^i, returns the relevant conversion factor. Formally, for each domain T^i, with $i \in M$, we define a function $C_F : T^i \times \mathcal{C} \times \mathcal{C} \to Q$ which satisfies the following properties:

(a) conversion factors from each domain into itself are equal to 1:

$$\forall T^i, i(i \in T^i \to C_F(i, T^i, T^i) = 1);$$

(b) conversion factors from coarser to (strictly) finer domains are greater than 1:

$$\forall T^i, T^j, i((i \in T^i \wedge T^i \prec T^j) \to C_F(i, T^i, T^j) > 1);$$

(c) conversion factors of symmetrical and disjoint pairs of domains are reciprocal:

$$\forall T^i, T^j, i, j((i \in T^i \wedge j \in T^j \wedge T^i \supset T^j \wedge \updownarrow(i,j)) \to C_F(i, T^i, T^j) * C_F(j, T^j, T^i) = 1);$$

(d) conversion factors of disjoint domains are compositional:

$$\forall T^i, T^j, T^k, i((T^i \supset T^j \supset T^k \wedge i \in T^i) \to C_F(i, T^i, T^k) = \sum_{t \in \{j : j \in T^j \wedge \updownarrow(i,j)\}} C_F(t, T^j, T^k)).$$

Let us assume T^k to be equal to T^j in (d). From (a), it follows that:

(e) the conversion factor between T^i and T^j, with $T^i \supset T^j$, with respect to $i \in T^i$ is equal to the cardinality of the set of $j \in T^j$ such that $\updownarrow(i,j)$:

$$\forall T^i, T^j, i((T^i \supset T^j \wedge i \in T^i) \rightarrow C_F(i, T^i, T^j) = \sharp\{j : j \in T^j \wedge \updownarrow(i,j)\}).$$

We also require that the relation of projection satisfies the following properties:

- *reflexivity*
  every time instant projects on itself
  $$\forall t \ \updownarrow(t,t)$$
- *symmetry*
  if i downward (upward) projects on j, then j upward (downward) projects on i
  $$\forall i, j(\updownarrow(i,j) \rightarrow \updownarrow(j,i))$$
- *downward transitivity*
  if $T^i \supset T^j \supset T^k$ and i of T^i projects on j of T^j and j projects on k of T^k, then i projects on k
  $$\forall T^i, T^j, T^k, i, j, k((T^i \supset T^j \supset T^k \wedge \\ i \in T^i \wedge j \in T^j \wedge k \in T^k \wedge \updownarrow(i,j) \wedge \updownarrow(j,k)) \rightarrow \updownarrow(i,k))$$
- *downward/upward transitivity (case 1)*
  if $T^i \supset T^k \supset T^j$ and i of T^i projects on j of T^j and j projects on k of T^k, then i projects on k
  $$\forall T^i, T^j, T^k, i, j, k((T^i \supset T^k \supset T^j \wedge \\ i \in T^i \wedge j \in T^j \wedge k \in T^k \wedge \updownarrow(i,j) \wedge \updownarrow(j,k)) \rightarrow \updownarrow(i,k))$$
- *order preservation*
  the linear order of domains is preserved by the projection relation. For each T^i and T^j we require that the projection intervals are ordered but possibly meet
  $$\forall T^i, T^j, i, i', j, j'((i \in T^i \wedge i' \in T^i \wedge j \in T^j \wedge j' \in T^j \wedge \\ \updownarrow(i,j) \wedge \updownarrow(i',j') \wedge \exists \alpha(\alpha > 0 \wedge \mathrm{DIS}(i, \alpha, i'))) \rightarrow \exists \beta(\beta \geq 0 \wedge \mathrm{DIS}(j, \beta, j')))$$
  For pairs of domains ordered by disjointedness, we require the stronger property that projection intervals are disjoint
  $$\forall T^i, T^j, i, i', j, j'((T^i \supset T^j \wedge i \in T^i \wedge i' \in T^i \wedge j \in T^j \wedge j' \in T^j \wedge \\ \updownarrow(i,j) \wedge \updownarrow(i',j') \wedge \exists \alpha(\alpha > 0 \wedge \mathrm{DIS}(i, \alpha, i'))) \rightarrow \exists \beta(\beta > 0 \wedge \mathrm{DIS}(j, \beta, j')))$$
  Strong order preservation and symmetry properties allow us to prove the uniqueness of coarse grain equivalents for disjoint domains
  $$\forall T^j, T^i, j, i, i'((T^i \supset T^j \wedge j \in T^j \wedge i \in T^i \wedge i' \in T^i \wedge \updownarrow(i,j) \wedge \updownarrow(i',j)) \rightarrow i = i')$$

Together with properties (b) and (c) of convertion factors, it allows us to generalize property (e) to the property:

$$\forall T^i, T^j, i(i \in T^i \rightarrow \lceil C_F(i, T^i, T^j) \rceil = \sharp\{j : j \in T^j \wedge \updownarrow(i, j)\})$$

stating that, for each pair of disjoint domains T^i, T^j, and each $i \in T^i$, the $\lceil\ \rceil$ of the value of the relevant conversion factor is exactly the number of $j \in T^j$ such that $i \rightarrow j$.

- *contiguity*
  the projection relation maps an instant into an interval of contiguous instants on a given domain, i.e.
  there exist at least $\lceil C_F(i, T^i, T^j) \rceil$ contiguous instants of T^j related to each instant i of T^i:

  $$\forall T^i, T^j, i(i \in T^i \rightarrow \exists j(j \in T^j \wedge \forall \alpha, j'((0 \leq \alpha < \lceil C_F(i, T^i, T^j) \rceil \wedge j' \in T^j \wedge \mathrm{DIS}(j, \alpha, j')) \rightarrow \updownarrow(i, j'))))$$

  and there exist at most $\lceil C_F(i, T^i, T^j) \rceil$ contiguous instants of T^j related to i:

  $$\forall T^i, T^j, i(i \in T^i \rightarrow \exists j(j \in T^j \wedge \forall j'((\updownarrow(i, j') \wedge j' \in T^j) \rightarrow \exists \alpha(0 \leq \alpha < \lceil C_F(i, T^i, T^j) \rceil \wedge \mathrm{DIS}(j, \alpha, j'))))$$

In all the previous formulae, t, i, j, and k are quantified over the temporal domain $\mathcal{T}$ (if not further constrained), while α and β are quantified over the algebraic domain $\mathcal{D}$ (if not further constrained).

For particular kinds of temporal universe, we can also require that the projection satisfies the property of homogeneity.

- *homogeneity*
  for each pair of disjoint domains of the temporal universe, the homogeneity property requires that there exists a *constant* conversion factor expressing the numerical relationship between their granularities

  $$\forall T^i, T^j((T^i \supset T^j \vee T^j \supset T^i) \rightarrow \exists C_{i,j} \forall i(i \in T^i \rightarrow C_F(i, T^i, T^j) = C_{i,j}))$$

  Hereinafter, to avoid messy complications, we will assume the homogeneity property. Clearly, such an assumption precludes us to deal with domains like real months. However, the proposed formalization can be easily generalized to the non-homogeneous case.
  Notice that, under the assumption of homogeneity, the function C_F that computes the conversion factors can be redefined as a binary function $C_F : \mathcal{C} \times \mathcal{C} \rightarrow Q$. Moreover, pairing the contiguity and the homogeneity properties we obtain that, for each pair T^i and T^j, there exist exactly $C_{i,j}$ contiguous instants of T^j related to each instant of T^i.

Many other relevant properties can be derived from the given ones including:

- *totality (seriality)*
  the projection relation is defined for each instant of every domain of the temporal universe
  $$\forall i, T^j \exists j (j \in T^j \wedge \updownarrow(i,j))$$
- *coverage*
  for each instant j and each domain T^i, there exist an instant j', a displacement α, and an instant i belonging to T^i such that $\mathrm{DIS}(j,\alpha,j')$, j' projects on i, and i projects on j
  $$\forall j, T^i \exists \alpha, j', i(\mathrm{DIS}(j,\alpha,j') \wedge \updownarrow(j',i) \wedge i \in T^i \wedge \updownarrow(i,j))$$
- *upward transitivity*
  if $T^k \supset T^j \supset T^i$ and i of T^i projects on j of T^j and j projects on k of T^k, then i projects on k
  $$\forall T^i, T^j, T^k, i, j, k((T^k \supset T^j \supset T^i \wedge i \in T^i \wedge j \in T^j \wedge k \in T^k \wedge \updownarrow(i,j) \wedge \updownarrow(j,k)) \rightarrow \updownarrow(i,k))$$
- *downward/upward transitivity (case 2)*
  if $T^k \supset T^i \supset T^j$ and i of T^i projects on j of T^j and j projects on k of T^k, then i projects on k
  $$\forall T^i, T^j, T^k, i, j, k((T^k \supset T^i \supset T^j \wedge i \in T^i \wedge j \in T^j \wedge k \in T^k \wedge \updownarrow(i,j) \wedge \updownarrow(j,k)) \rightarrow \updownarrow(i,k))$$

3.4 Metric and layered temporal logic (MLTL)

In this section we define syntax, semantics, and axiomatization of metric and layered temporal logic (MLTL). We also provide a number of examples to which the reader can refer while considering formal definitions.

Language. The language for MLTL is a three-sorted temporal language extending the (two-sorted) language for *MTL* with a *context sort*. Formally, let $T(X \cup A)$ and Φ be respectively the sets of algebraic terms and proposition letters, let C be a non-empty set of context constants denoting the domains into which the temporal universe is partitioned, and let Y be a collection of context variables. The set of context terms $T(C \cup Y)$ is the set union $C \cup Y$. The definition of $T(A \cup X)$ given in Chapter 2 is extended with the following rule: if $c_1, c_2 \in T(C \cup Y)$, then $c_f(c_1, c_2)$ is an algebraic term, where $c_f : T(C \cup Y) \times T(C \cup Y) \rightarrow T(A \cup X)$ is a binary function that maps each pair of contexts (temporal domains) into an algebraic term (displacement). Furthermore, we add two binary relations $\prec$ and $\supset$ over $T(C \cup Y) \times T(C \cup Y)$.

Let us extend the conventional notations of *MTL* with the following one: $c, c_1, c_2, \ldots$ denote context terms. The formulae of *MLTL* are defined as follows:

$$\phi ::= p \mid \neg\phi \mid \alpha = \beta \mid \alpha < \beta \mid c_1 \prec c_2 \mid c_1 \supset c_2 \mid \phi \wedge \phi \mid \Delta_\alpha \phi \mid \Delta^c \phi \mid \Diamond\phi \mid \forall x\, \phi \mid \forall y\, \phi,$$

where $x \in X$, $y \in Y$, $\alpha, \beta \in T(X \cup A)$, $c, c_1, c_2 \in T(C \cup Y)$, and $p \in \Phi$.

The operator Δ^c is called the *contextual operator*. When applied to a formula ϕ, it restricts the evaluation of ϕ to the time instants belonging to the context (denoted by) c. Moreover, $\Delta^c\phi$ conventionally evaluates to false outside the context c. The dual operator ∇^c is defined as follows:

$$\nabla^c\phi := \neg\Delta^c\neg\phi$$

In constrast to Δ^c, ∇^c conventionally evaluates to true outside the context c.

The operator $\Diamond$ is called the *projection operator*. When applied to a formula ϕ, it allows one to evaluate ϕ at the time instants related to the current one by the projection relation. The dual operator $\Box$ is defined as follows:

$$\Box\phi := \neg\Diamond\neg\phi.$$

It is worth noting that the combined use of the displacement (Δ_α) and contextual (Δ^c) operators makes it possible to define a notion of *contextualized* (or *local*) *displacement*. Formally, we introduce a derived operator Δ^c_α defined as follows:

$$\Delta^c_\alpha\phi := \Delta^c\Delta_\alpha\phi,$$

together with the dual one ∇^c_α

$$\nabla^c_\alpha\phi := \nabla^c\nabla_\alpha\phi.$$

Such an operator allows one to view the context term c as the sort of the algebraic term α (*multisorted algebraic terms*). In such a way, the composition of contextual and displacement operators can be seen as a new typed operator, the *contextual displacement* operator Δ^c_α.

Structures. We define a *three-sorted frame* to be a tuple $\mathfrak{F} = (\mathcal{T}, \mathcal{C}, \mathfrak{D}; \mathrm{DIS}, \mathrm{CONT}, \updownarrow)$, where $\mathcal{T}$ is the temporal universe over which *MLTL*-formulae are evaluated, $\mathcal{C}$ is the set of temporal domains over which context terms are interpreted, $\mathfrak{D} = (D, +, -, 0)$ is the algebra of metric displacements in whose domain D algebraic terms take their values, $\mathrm{DIS} = \bigcup_{i \in M} \mathrm{DIS}_i$ is the displacement relation, $\mathrm{CONT} \subseteq \mathcal{T} \times \mathcal{C}$ is the relation of contextualization, and $\updownarrow \subseteq \mathcal{T} \times \mathcal{T}$ is the projection relation.

To turn a three-sorted frame $\mathfrak{F}$ into a *three-sorted model* $\mathfrak{M}$, let us first add the interpretations for the context and algebraic terms, and the valuation for atomic temporal formulae. An interpretation for context terms is given by a function $h : C \cup Y \to \mathcal{C}$. The interpretation g for algebraic terms given in Chapter 2 is extended by requiring that $g(c_f(c_1, c_2)) = \lceil C_F(h(c_1), h(c_2)) \rceil$. The valuation V for propositional variables as well as the valuation of atomic formulae of the forms $\alpha = \beta$ and $\alpha < \beta$ are defined as in Chapter 2. Then, we say that an atomic formula of the form $c_1 \prec c_2$ (resp. $c_1 \supset c_2$) is *true* in a model $\mathfrak{M} = (\mathfrak{F}; V, g, h)$ whenever $h(c_1) \prec h(c_2)$ (resp. $h(c_1) \supset h(c_2)$).

Next, the *truth* of the temporal formulae $\Delta_\alpha\phi$, $\Delta^c\phi$, and $\Diamond\phi$ is defined by

$$\mathfrak{M}, i \Vdash \Delta_\alpha\phi \quad \text{iff} \quad \text{there exists } j \text{ such that } \mathrm{DIS}(i, g(\alpha), j) \text{ and } \mathfrak{M}, j \Vdash \phi$$
$$\mathfrak{M}, i \Vdash \Delta^c\phi \quad \text{iff} \quad \mathrm{CONT}(i, h(c)) \text{ and } \mathfrak{M}, i \Vdash \phi$$

$$\mathfrak{M}, i \Vdash \Diamond\phi \quad \text{iff} \quad \text{there exists } j \text{ such that } \updownarrow(i,j) \text{ and } \mathfrak{M}, j \Vdash \phi.$$

Notice that the semantic clause for $\Delta_\alpha\phi$ is the same clause given for such a formula in MTL, except for the replacement of the old definition of DIS by the new one.

The semantic clauses for the dual operators $\nabla_\alpha\phi, \nabla^c\phi$, and $\Diamond\phi$, and for the derived operator $\Delta_\alpha^c\phi$ can be easily derived from the previous ones:

$$\begin{array}{rcl} \mathfrak{M}, i \Vdash \nabla_\alpha\phi & \text{iff} & \text{for all } j \text{ such that } \mathrm{DIS}(i, g(\alpha), j), \mathfrak{M}, j \Vdash \phi \\ \mathfrak{M}, i \Vdash \nabla^c\phi & \text{iff} & \text{if } \mathrm{CONT}(i, h(c)), \text{ then } \mathfrak{M}, i \Vdash \phi \\ \mathfrak{M}, i \Vdash \Box\phi & \text{iff} & \text{for all } j \text{ such that } \updownarrow(i,j), \mathfrak{M}, j \Vdash \phi \\ \mathfrak{M}, i \Vdash \Delta_\alpha^c\phi & \text{iff} & \mathrm{CONT}(i, h(c)) \text{ and there exists } j \text{ such that} \\ & & \mathrm{DIS}(i, g(\alpha), j) \text{ and } \mathfrak{M}, j \Vdash \phi. \end{array}$$

Finally, to evaluate quantified formulae $\forall y\,\phi$, with $y \in Y$, at a time point i of the temporal universe, we write $h =_y h'$ to denote that the assignments h and h' agree on all context variables except maybe y. Then

$$(\mathfrak{F}; V, g, h), i \Vdash \forall y\,\phi \quad \text{iff} \quad (\mathfrak{F}; V, g, h'), i \Vdash \phi,$$

for all assignments h' such that $h =_x h'$.

The notions of satisfiability, validity, and logical consequence given for *MTL* can be easily generalized to *MLTL*. Furthermore, the stratified structure of *MLTL*-frames makes it possible to define the notions of *local* satisfiability, *local* validity, and *local* logical consequence, restricting the general notions of satisfiability, validity, and logical consequence to a specific domain of the temporal universe.

Examples. The following examples illustrate the main kinds of relations between different components of specification that can be expressed in *MLTL*. More substantial examples are given in the next section.

In the simplest cases, *MLTL* specifications are obtained by contextualizing formulae and composing them by means of logical connectives.

3.4.1. Example. For instance, the sentence:

"Men work every month and eat every day"

is specified by the formula:

$$\forall x_{man}(\forall\alpha\nabla_\alpha^{month} work(x_{man}) \wedge \forall\beta\nabla_\beta^{day} eat(x_{man})).$$

The projection operator is needed when displacements over different temporal domains have to be composed.

3.4.2. Example. For instance, the sentence:

"In twenty seconds five minutes will have passed from the occurrence of the fault" is specified by the formula:

$$\Delta_{20}^{second} \Diamond \Delta_{-5}^{minute} fault.$$

It is possible to give a stronger interpretation of the sentence, which is expressed by the formula:

$$\Delta_{20}^{second} \Diamond \Delta_{-5}^{minute} fault \wedge \forall \alpha (0 \leq \alpha < 20 \rightarrow \neg \Delta_{\alpha}^{second} \Diamond \Delta_{-5}^{minute} fault).$$

It is worth noting that this example may raise a question about the usefulness of the $\Diamond$ operator that could seem redundant. This is because the specified condition involves a shift from the coarser to the finer domain, and to model these kinds of shift we can equivalently use the $\Diamond$ and the $\Box$ operators. It is, however, immediate to see that this is not the case anymore when the shift occurs in the opposite direction.

Contextual and projection operators can also be paired to specify nested quantifications. Some typical situations, together with their formalization, are captured by the following examples.

3.4.3. EXAMPLE. The sentence:

"There exist some days during which the plant works every hour"

is specified by the formula:

$$\exists \alpha \Delta_{\alpha}^{day} \Box \nabla^{hour} work(plant).$$

The sentence:

"There exist some days during which the plant remains inactive for several hours"

is specified by the formula:

$$\exists \alpha \Delta_{\alpha}^{day} \Diamond \Delta^{hour} inactive(plant).$$

The sentence:

"Every day there exist some hours during which the plant is in production"

is specified by the formula:

$$\forall \alpha \nabla_{\alpha}^{day} \Diamond \Delta^{hour} in\_production(plant).$$

The sentence:

"The plant is monitored by the remote system each minute of every hour"

is specified by the formula:

$$\forall \alpha \nabla_{\alpha}^{hour} \Box \nabla^{minute} monitor(remote - system, plant).$$

Axioms. An axiomatization of validity in the language of *MLTL* can be obtained by adding to the axioms and rules of *MTL* a number of axiom schemata and rules governing the behavior of the contextual and projection operators as well as the relations between these operators and the displacement one. We partition the additional axiom schemata and rules in two sets. The first set includes axiom schemata and rules expressing basic logical properties of the contextual and projection operators; the second one collects axiom schemata that codify specific properties of layered temporal structures.

The basic logical properties of *MLTL* operators are expressed by the following axiom schemata:

(Ax17)	$\nabla^c(\phi \to \psi) \to (\nabla^c\phi \to \nabla^c\psi)$	(normality of ∇^c)
(Ax18)	$\Delta^c\phi \to \phi$	("necessity" for Δ^c)
(Ax19)	$\nabla^c\nabla^c\phi \equiv \nabla^c\phi$	(idempotency of ∇^c)
(Ax20)	$\nabla^c\nabla_\alpha\phi \equiv \nabla_\alpha\nabla^c\phi$	(commutativity of ∇^c and ∇_α)
(Ax21)	$\Box(\phi \to \psi) \to (\ \Box\phi \to \Box\psi)$	(normality of $\Box$),

where c is a context term, and by the rules:

(∇^c-NEC)	$\vdash \phi \longrightarrow \vdash \nabla^c\phi$	(necessitation rule for ∇_c)
($\Box$-NEC)	$\vdash \phi \longrightarrow \vdash \Box\phi$	(necessitation rule for $\Box$).

We also add the Barcan formula for both the contextual and the projection operator:

(Ax22)	$\forall x\nabla^c\phi \to \nabla^c\forall x\phi$, with $x \neq c$	(Barcan formula for ∇^c)
(Ax23)	$\forall x\Box\phi \to \Box\forall x\phi$	(Barcan formula for $\Box$).

Furthermore, we have the following axioms relating algebraic terms and contextual operators:

(Ax24)	$\alpha = \beta \to \forall x\nabla^x\alpha = \beta$
(Ax25)	$\alpha \neq \beta \to \forall x\nabla^x\alpha \neq \beta$
(Ax26)	$\alpha < \beta \to \forall x\nabla^x\alpha < \beta$
(Ax27)	$\alpha \not< \beta \to \forall x\nabla^x\alpha \not< \beta$.

Axioms (Ax28)-(Ax31) relating context terms ordered by granularity ($\prec$) or by disjointedness ($\supset$) and displacement operators as well as axioms (Ax32)-(Ax35) relating context terms and contextual operators can be easily obtained from axioms (Ax13)-(Ax16) and (Ax23)-(Ax26), respectively, by substituting $\prec$ for $=$ and $\supset$ for $<$.

On the basis of the above axiom schemata and rules, we can observe that the projection operator $\Diamond$ and the dual operator $\Box$ behave as the usual modal operators of possibility and necessity. On the contrary, the behavior of the contextual operator Δ^c (and of the dual operator ∇^c) is less standard, and deserves further consideration. In the following, we report a number of theorems that contribute to a better clarification of the behavior of contextual operators (the proofs are given in [84]).

As usual, given the definition of Δ^c, it is immediate to prove that $\Delta^c\phi \leftrightarrow \neg\nabla^c\neg\phi$, together with its corollaries $\nabla^c\neg\phi \leftrightarrow \neg\Delta^c\phi$ and $\neg\nabla^c\phi \leftrightarrow \Delta^c\neg\phi$. Such theorems, together with the usual substitution rule of equivalents, allow us to replace ∇^c with $\neg\Delta^c\neg$, and vice versa, in any formula. Moreover, axiom (Ax17), together with the rule (∇^c-NEC), allows us to deduce the distributivity of ∇^c with respect to $\wedge$ and then, by duality, the distributivity of Δ^c with respect to $\vee$. Then, from (Ax17), (∇^c-NEC), the duality of ∇^c and Δ^c, and the distributivity of Δ^c with respect to $\vee$, it follows that:

$$\nabla^c\phi \rightarrow (\Delta^c\phi \leftrightarrow \Delta^c\top).$$

Pairing the duality of ∇^c and Δ^c, and axiom (Ax18), we obtain that $\phi \rightarrow \nabla^c\phi$, and then $\Delta^c\phi \rightarrow \nabla^c\phi$. This last result, together with the distributivity of Δ^c with respect to $\vee$ and the duality of ∇^c and Δ^c, allows us to derive that:

$$\Delta^c\phi \leftrightarrow (\Delta^c\top \wedge \nabla^c\phi)$$

From this theorem, it is easy to prove the distributivity of Δ^c with respect to $\wedge$ and then, by duality, the distributivity of ∇^c with respect to $\vee$. Moreover, together with axiom (Ax18) and the distributivity of Δ^c with respect to $\wedge$, such a theorem allows us to deduce that:

$$\Delta^c(\phi \wedge \psi) \leftrightarrow (\Delta^c\phi \wedge \psi),$$

together with the dual one:

$$\nabla^c(\phi \vee \psi) \leftrightarrow (\nabla^c\phi \vee \psi)$$

These formulae can be easily generalized to whatever conjunction and disjunction of formulae. They show that whenever contextualization is applied to a single conjunct (resp. disjunct), it is automatically lifted to the whole conjunction (resp. disjunction).

From the same theorem and axiom (Ax19), it is possible to derive that:

$$\Delta^c\nabla^c\phi \leftrightarrow \Delta^c\phi.$$

This formula (and the dual one) as well as (Ax19) (and the dual formula) can be viewed as *reduction rules* that can be used to simplified nested occurrences of contextual operators.

Besides the fundamental logical properties of contextual and projection operators, we can axiomatize properties of temporal structures. The following axiom schemata capture the properties specified in Section 3.3.2:

(Ax36) $\phi \rightarrow \exists c \Delta^c\phi$ (domain covering)

(Ax37) $\forall c_1, c_2(\Delta^{c_1}\Delta^{c_2}\top \rightarrow c_1 = c_2)$ (domain disjointedness)

(Ax38) $\Box\phi \rightarrow \phi$ (reflexivity)

(Ax39) $\phi \rightarrow \Box\Diamond\phi$ (symmetry)

(Ax40) $\forall c_1, c_2, c_3((c_1 \supset c_2 \supset c_3 \wedge \nabla^{c_1}\Box\nabla^{c_3}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\Box\nabla^{c_3}\phi)$
(downward transitivity)

(Ax41) $\forall c_1, c_2, c_3((c_1 \supset c_3 \supset c_2 \wedge \nabla^{c_1}\Box\nabla^{c_3}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\Box\nabla^{c_3}\phi)$
(downward/upward transitivity - case 1)

(Ax42) $\forall c_1, c_2(\exists\alpha(\alpha > 0 \wedge \Delta^{c_1}_{\alpha}\Diamond\Delta^{c_2}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\exists\beta(\beta \geq 0 \wedge \nabla_{\beta}\phi))$
(weak order preserving)

(Ax42') $\forall c_1, c_2(\exists\alpha(\alpha > 0 \wedge \Delta^{c_1}_{\alpha}\Diamond\Delta^{c_2}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\exists\beta(\beta > 0 \wedge \nabla_{\beta}\phi))$
(strong order preserving)

(Ax43) $\forall c_1, c_2\exists\alpha(\alpha = c_f(c_1, c_2) \wedge \nabla^{c_1}(\Diamond\Delta^{c_2}\forall\beta(0 \leq \beta < \alpha \rightarrow \nabla_{\beta}\phi) \leftrightarrow$

$\exists\gamma(0 \leq \gamma < \alpha \wedge \Box\nabla^{c_2}\nabla_\gamma\phi)))$

(contiguity)

(Ax44) $\quad \forall c_1, c_2 \exists \alpha \; c_f(c_1, c_2) = \alpha$ (homogeneity)

where α, β, and γ are algebraic variables and c, c_1, c_2, and c_3 are context variables.

Putting together the Barcan formula for $\Box$ and axiom (Ax39) (symmetry), we obtain $\Box\forall x\phi \rightarrow \forall x\Box\phi$, and thus $\forall x\Box\phi \leftrightarrow \Box\forall x\phi$. Moreover, from the given axioms it is also possible to prove the following theorems expressing derived properties of temporal structures:

$\forall c_2(\Box\nabla^{c_2}\phi \rightarrow \Diamond\Delta^{c_2}\phi)$ (totality)

$\forall c_1, c_2, c_3((c_3 \supset c_2 \supset c_1 \wedge \nabla^{c_1}\Box\nabla^{c_3}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\Box\nabla^{c_3}\phi)$

(upward transitivity)

$\forall c_1, c_2, c_3((c_3 \supset c_1 \supset c_2 \wedge \nabla^{c_1}\Box\nabla^{c_3}\phi) \rightarrow \nabla^{c_1}\Box\nabla^{c_2}\Box\nabla^{c_3}\phi)$

(downward/upward transitivity - case 2)

$\forall c_1(\phi \rightarrow \exists x\Delta_x\Diamond\Delta^{c_1}\Diamond\phi)$ (coverage)

It is worth noting that, as long as we restrict ourselves to temporal structures provided with a (specific) *finite* number of temporal domains, all quantifications over context variables occuring in MLTL axioms can be viewed as shorthands for finite conjunctions (universal quantifications) and disjunctions (existential quantifications).

Preservation. Let us consider now the following problem: given the truth value of a formula with respect to a certain domain, can we constrain (and how) its truth value with respect to the other domains? Notice that the language of *MLTL* makes it possible to write formulae involving switching across domains, but the proposed axiomatization does not impose any general constraint on the relations among the truth values of a formula with respect to different domains. In Chapter 4, we will give an example of a proposition which is true at each instant of a given domain, and false with respect to each instant of another one. Therefore, in principle, we can only record the links explicitly provided by the specifier, and cannot impose any other constraint about the truth value of a formula with respect a domain different from the given one. Nevertheless, from a practical point of view, it makes sense to look for general rules that capture *typical* relations.

In the following, we define two consistency rules that allow one to project temporal formulae from coarser to finer domains (*downward temporal projection*) and from finer to coarser ones (*upward temporal projection*), respectively. For each pair of domains T^i, T^j, with T^i coarser than T^j, *downward temporal projection* states that if a fact p is true at a time instant $i \in T^i$, then there exists at least one time instant $j \in T^j$, belonging to its decomposition, such that p is true at j. For each pair of domains T^i, T^j, with T^i finer than T^j, *upward temporal projection* states that if p is true at each time instant $i \in T^i$ such that $j \rightarrow i$, then p is true at time j. Formally, *downward temporal projection* is defined by the formula:

$$\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1}(\phi \rightarrow \Diamond\Delta^{c_2}\phi)),$$

while *upward temporal projection* is defined by the formula:

$$\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1}(\Box\nabla^{c_2}\phi \rightarrow \phi)),$$

where (in both cases) ϕ is a formula devoid of any occurrence of the displacement, contextual and projection operators.

The following proposition shows that the formulae defining downward and upward temporal projection are actually equivalent.

3.4.4. PROPOSITION. *The formulae defining downward and upward temporal projection are* interdeducible.

Proof. Let us show that, assuming $\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1}(\phi \rightarrow \Diamond\Delta^{c_2}\phi))$, we can derive $\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1}(\Box\nabla^{c_2}\phi \rightarrow \phi))$. The proof of the opposite direction is similar, and thus omitted. First of all, by exploiting simple properties of the projection operators, we can prove, using the standard logical machinery, that $\nabla^{c_1}(\phi \rightarrow \Diamond\Delta^{c_2}\phi)$ is logically equivalent to $\nabla^{c_1}(\Box\nabla^{c_2}\neg\phi \rightarrow \neg\phi)$. By sillogism, we obtain that $c_1 \supset c_2 \rightarrow \nabla^{c_1}(\Box\nabla^{c_2}\neg\phi \rightarrow \neg\phi)$. Finally, after simple logical manipulations, we can applied universal generalization to conclude that $\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1}(\Box\nabla^{c_2}\phi \rightarrow \phi))$ (a detailed proof can be found in [84]). $\dashv$

Downward temporal projection provides the *weakest semantics* that can be attached to an assertion in a domain finer than the original one, provided that such an assertion is not wholistic. Wholistic assertions indeed relate to the structure of the interval over which they hold as a whole, and they do not hold over any proper subinterval of it. Thus, such assertions cannot be projected across domains. Most often, however, the semantics interpretation underlying downward temporal projection is too weak so that user qualifications are needed. In general, it is possible to provide domain-specific categorizations of assertions according to their behaviour under downward temporal projection. Such categorizations allow one to introduce and characterize primitive ontological concepts, such as event, property, fact, and process, in terms of their temporal projection[3]. In Section 3.7, we will introduce some specializations of the basic projection operator $\Diamond$ that allow one to define different types of downward temporal projection, distinguishing among assertions that hold at one and only one instant of the finer domain (*punctual*), assertions that hold at each instant j of the finer domain such that $\updownarrow (i, j)$ (*continuous and pervasive*), assertions that hold over scattered sequence of intervals of the finer domain whose element j all satisfy the condition $\updownarrow (i, j)$ (*bounded sequence*), and so on [82].

Soundness and completeness. We conclude this section discussing the soundness and completeness of the proposed *MLTL* axiomatization with respect to the class of metric and layered temporal structures. The soundness result for *MLTL* can be proved as usual by checking that each *MLTL* axiom is a valid formula and that each rule preserves validity. The verification that tautologies are valid, and that basic logical rules preserve validity is completely standard. The proof of the soundness of axioms expressing basic properties of temporal operators can be obtained from the semantic definition of the language. Similarly,

[3]Similar categorizations of temporal assertions have been proposed by Roman in [111] and by Shoham in [113]. Other, more sophisticated, categorizations can be found in the literature on natural language semantics.

the proof of soundness of the rules governing the behavior of the displacement, contextual and projection operators can be easily built up by exploiting the notion of validity. Finally, the soundness of the axioms expressing specific properties of the temporal universe is proved by showing that for all relevant three-sorted frames and valuations, each axiom evaluates to true at each time instant of the temporal universe.

3.4.5. Theorem. (Soundness) *MLTL is sound for the class of (metric and layered) three-sorted frames.*

The soundness proof is given in [84].

We did not directly address here the problem of proving the completeness of the *MLTL* axiomatization. Nevertheless, in the next chapter we will prove that the theories of significant classes of metric and layered temporal structures are decidable. Axiomatic completeness follows as a by-product of decidability, even though the axioms are not produced explicitly. As for the completeness of the proposed *MLTL* axiomatization, there are at least two possible approaches to such a problem. On the one hand, one can adopt the direct approach of building a canonical model for *MLTL*. Even though there seem to be no specific technical problems to solve, the process of canonical model construction is undoubtedly very demanding in view of the size and complexity of the *MLTL* axiom system. On the other hand, one can follow the approach outlined by Finger and Gabbay in [47], viewing *MLTL* as the combination of a number of differently-grained metric temporal logics, and determining what constraints such a combination must satisfy to guarantee the transference of the completeness results for *MTL* given in Chapter 2 from the component metric temporal logics to the combined one. This second approach seems the most promising one with respect to the problem of mastering the complexity of the *MLTL* axiomatization.

3.5 Examples of layered specifications

In the previous section, we proposed some simple examples of properties involving time granularity to demonstrate the expressive power of *MLTL*. In this section, we will show how *MLTL* can be exploited to specify the behaviour of granular real-time systems such as a monitoring system and a high voltage station [29, 32].

3.5.1. Example. (**Monitoring system**) Let S be a monitoring system consisting of a monitor M and a remote system R. The purpose of M is to monitor the state of R. The monitoring is performed through a procedure which requires R to periodically send to M a message containing information about its state. In order for the remote system to be considered in a correct state, the temporal distance between two consecutive messages must not exceed one *hour*: if one message is sent within one *hour* from the preceding one, M will wait for the next one. Otherwise, M starts a procedure to verify whether R is in a correct state. In the verification procedure, M sends to R a control signal that lasts one *second*. If R replies within 5 *seconds* to this control signal with an answer signal, followed, within the successive 5 *seconds*, by required information message, then M concludes that

R is still in a correct state and waits for the next message. Otherwise, M concludes that R state is incorrect, delivering an idle message 10 *seconds* after the emission of control signal. There is no restoration from the idle condition.

It can be noted that the above reported informal specification of the expected system behavior admits at least two distinct interpretations: these correspond to the possibility that the whole system, that is, all the processes that concur to the execution of the system task, is synchronized with the switching from one *hour* to the next one, or else that its evolution is essentially independent with respect to the synchronization events. If we adopt the first interpretation, since the system is synchronized at the first *second* in every *hour*, the control message is issued at a fixed *second* (e.g., the first *second*) of the second *hour* following the last synchronous *hour* containing, in any position, the occurrence of an event of message sending. On the contrary, if we adopt the second interpretation ignoring any synchronization with the domain of *hours*, the monitor M emits a control signal when exactly 3600 *seconds* have elapsed from the precise *second* when the last message was sent by the remote unit R (we assume that there are no delays due to trasmission, that is, messages are received at the same *second* at which they are sent). This is an instance of the alignment problem which, as discussed in Section 3.2, arises when one relates the descriptions of the evolution of a system at different time granularities.

In the following we confine ourselves to the first interpretation, and show how *MLTL* allows us to specify the intended behavior of the monitoring system. In the next section, we will discuss the alignment problem in detail and we will show how to support the other interpretation.

The monitoring system clearly operates in two modes, the normal "read and wait" mode and the verification mode. The normal behavior takes place over the domain of *hours*, while the fast verification procedure operates over the domain of *seconds*. The formal specification of the monitoring system will thus be based on a temporal universe composed of the two domains of *hours* and *seconds*. It will consist of the logical conjunction of three different components C1, C2, and C3.

(C1) The verification procedure starts if the last *hour* M does not receive any message from R, and R has never been declared idle:

$$\forall \alpha \nabla_{\alpha}^{minute}(control \leftrightarrow (\nabla_{-1} \Box \nabla^{seconds} \neg message \wedge \neg SomPast(idle))),$$

where $SomPast(\phi)$ is a shorthand for $\neg \forall \alpha(\alpha < 0 \rightarrow \nabla_{\alpha}(\neg \phi))$.
(C2) The idle declaration:

$$\forall \alpha \nabla_{\alpha}^{seconds}(\nabla_{10} idle \leftrightarrow (control \wedge (Lasts_{e,e}(\neg answer, -6) \vee$$
$$\exists \beta(1 \leq \beta \leq 5 \wedge \nabla_{\beta}(answer \wedge Lasts_{e,e}(\neg message, 6)))))),$$

where $Lasts_{e,e}(\phi, \alpha)$ is a shorthand for $\forall \beta(0 < \beta < \alpha \rightarrow \nabla^{\beta} \phi)$.
(C3) An answer from R can only be received within 5 *seconds* from the issue of a control message:

$$\forall \alpha \nabla_{\alpha}^{seconds}(answer \rightarrow \exists \beta(-5 \leq \beta \leq -1 \wedge \nabla_{\beta} control)).$$

3.5.2. Example. (**High voltage station**) Let us discuss now an excerpt of the specification of a supervisor that automates the activities of a High Voltage (HV) station, devoted to the end user distribution of the energy generated by power plants[4]. Each station is composed of bays, connecting generation units and distribution line. A bay consists of circuit breakers and insulators. They are both switches, but an expensive circuit breaker can interrupt current in a very short time (50 *milliseconds* or even less), while a cheap insulator is not able to interrupt a flowing current and has switching time of a few *seconds*.

Let us consider a simple HV station consisting of two bars $b1$ and $b2$ connected to different power units, a distribution line l and two bays, pb (parallel bay) and lb (line bay). The parallel bay shorts circuit between the two bars $b1$ and $b2$; it is composed of two insulators, $ip1$ and $ip2$, and one circuit breaker *cbp*. It is in the state *closed* if all its switches are closed, it is *open* otherwise. The line bay connects the distribution line either with the first or the second bar. It is composed of three insulators $ilb1$, $ilb2$, $il1$ and one circuit breaker *cbp*. It is in the state $closed\_on\_b1$ if $ilb1, cbl$ and $il1$ are closed, while it is in the state $closed\_on\_b2$ if $ilb2, cbl$ and $il1$ are closed.

We report here the specification of the change from $b1$ to $b2$ of the bar connected to the line. The supervisor must close the parallel bay pb first, this action taking 10 *seconds*, then it closes the insulator $ilb2$ and opens the insulator $ilb1$ in 5 *seconds*. Lastly, it opens the parallel bay, taking other 10 *seconds*.

In order to formally specify the functioning of the supervisor, for every action we identify the time granularity with respect to which where it can be considered as an instantaneous event. The change of the bar takes about *30 seconds*, opening and closing the parallel bay *10 seconds*, switching the insulators *5 seconds*, switching of circuit breakers *50 milliseconds*. The predicates *change_bar_from_b1_to_b2*, *closed_pb*, *open_pb*, *close_ilb1*, *close_ilb2*, *open_ilb1*, etc., denote the corresponding commands sent to the various devices by the supervisor. The existential projection operator $\Diamond$ is used to connect formulae on different domains.

The change of bar is described by the formula below, specifying the sequence of actions taken by the supervisor:

$$\forall\alpha\nabla_{\alpha}^{30sec}(change\_bar\_from\_b1\_to\_b2 \rightarrow \Diamond(\Delta^{10sec}close\_pb\wedge$$
$$\Delta_{3}^{5sec}close\_ilb2 \wedge \Delta_{4}^{5sec}\Diamond\Delta^{10sec}open\_par\_bay)).$$

The effect of closing the parallel bay is specified by the following formula:

$$\forall\alpha\nabla_{\alpha}^{10sec}(close\_pb \rightarrow \Diamond(\Delta^{5sec}close\_ip1\wedge$$
$$\Delta_{1}^{5sec}close\_ip2 \wedge \Delta_{1}^{5sec}\Diamond\Delta^{50milli}close\_cb)).$$

The opening of the parallel bay is perfectly symmetrical to its closing.

[4]This example has been provided by the Centro Ricerche in Automatica (CRA) of the Ente Nazionale per l'Energia Elettrica (ENEL).

3.6 Dealing with the alignment problem

In Section 3.2, we briefly discussed the so-called *alignment problem* of temporal domains. We analyzed the sentences "tomorrow I will eat" and "dinner will be ready in one hour", and showed that the way in which we interpret them with respect to a domain finer than the domain they explicitly refer to is different. Then, in Section 3.5 (Example 3.5.1), we showed that even the same sentence may admit different interpretations with respect to a finer domain. The informal description of the normal "read and wait" operating mode of the monitoring system indeed includes the requirement that, as long as the remote system is in a correct state, it sends a message to the monitor every hour, and we already pointed out that such a requirement admits at least two different interpretations with respect to the domain of *second*. The same situation actually arises whenever we try to assign a meaning to this requirement with respect to any domain finer than the domain of *hour*.

The alignment problem can be formally characterized as follows. First of all, observe that such a problem arises with statements, associated with a given *domain* T^i, that assert the truth of some fact F at an instant $i' \in T^i$ located at *distance* α_i (either in the future or in the past) from the current instant $i \in T^i$, when we try to interpret them with respect to a finer domain T^j. Such statements indeed admit a univocal interpretation with respect to the domain T^i they refer to, that is, when evaluated at $i \in T^i$, they state that there exists an instant $i' \in T^i$ such that $\mathrm{DIS}_i(i, \alpha_i, i')$ and F holds at i'. However, there is no way of preserving such a univocity of interpretation with respect to any domain T^j finer than T^i. There are at least two alternative (limit) interpretations. According to the first one, when evaluated at an instant $j \in T^j$, with $\updownarrow(i,j)$, the above statements assert that there exist an instant $j' \in T^j$ and a displacement α_j such that $\updownarrow(i',j')$, $(\alpha_i - 1) * C_F(T^i,T^j) < \alpha_j < (\alpha_i + 1) * C_F(T^i,T^j)$, $\mathrm{DIS}_j(j,\alpha_j,j')$, and F holds at j'. According to second one, when evaluated at an instant $j \in T^j$, with $\updownarrow(i,j)$, they assert that there exist an instant $j' \in T^j$ and a displacement α_j such that $\updownarrow(i',j')$, $\alpha_j = \alpha_i * C_F(T^i,T^j)$, and F holds at j'.

The first interpretation assumes that there exists a unique global clock, and that all domain are synchronized with respect to it; the second interpretation assumes that each domain is provided with its own clock, and thus ignores any synchronization of domains. We call interpretations of the first and second type *synchronous* and *asynchronous*, respectively. They are indistinguishable with respect to the domain to which formulae (statements) explicitly refer (and to all coarser domains), but differ from each other when formulae (statements) are projected on any finer domain. It is worth noting that, in the synchronous case, if the projected formula is true with respect to a given instant belonging to the projection interval of the current instant, then it is true with respect to any other instant of such an interval, while, in the asynchronous case, the projected formula can be true with respect to a given instant of the projection interval and false with respect to all the others. This can be formally expressed saying that asynchronous models are a *proper subset* of synchronous ones.

The projection operators of *MLTL* $\Diamond$ and $\Box$ are based on the synchronous interpretation. Supporting the asynchronous interpretation mainly requires two extensions:

(i) replacing the notion of current instant with the notion of *vector of current instants*;
(ii) defining an *asynchronous projection operator* that, for each ordered pair of domains T^i, T^j, maps the current instant of T^i into the current instant of T^j.

The notion of vector of current instants deserves a detailed analysis. Let $\mathcal{T} = \bigcup_{i \in M} T^i$ be the temporal universe and $\vec{v}$ be a M-dimensional vector of current instants. For any $i \in M$, the i-th component of $\vec{v}$, denoted by $[\vec{v}]_i$, is a time instant of T^i that belongs to the projection interval of $[\vec{v}]_j$ on T^i, for all $1 \leq j \leq i-1$. For any ordered pair $i, j \in M$, it is possible to define a notion of *phase displacement* between the domains T^i and T^j, with respect to the current instant $[\vec{v}]_i$ of T^i, as the temporal distance between the first element of the projection interval of $[\vec{v}]_i$ on T^j and $[\vec{v}]_j$. Such a notion allows us to *replace* the vector of current instants by as many vectors of phase displacements as the domains of the temporal universe are (one for any possible choice of a specific current instant from the vector of current instants). More formally, for each vector of current instants $\vec{v}$ and each index $i \in M$, we define a vector of phase displacements $\vec{\alpha_i}$ such that, for each $j \in M$, $[\vec{\alpha_i}]_j$ is equal to the phase displacement between T^i and T^j with respect to $[\vec{v}]_i$. Under the assumption that $\updownarrow$ is homogeneous, however, the phase displacement between T^i and T^j does not depend on $[\vec{v}]_i$. Therefore, we can define a *phase displacement* function $P_D : \mathcal{C} \times \mathcal{C} \rightarrow D$ that, for each ordered pair of domains T^i, T^j, returns their phase displacement $P_D(T^i, T^j) = [\vec{\alpha_i}]_j$. The resulting M vectors of phase displacements are clearly not independent. In the case of a temporal universe totally ordered with respect to the disjointedness relationship $\supset$, for each ordered pair of domains T^i, T^j, with $T^i \supset T^j$, and each domain T^k, the relations between $\vec{\alpha_i}$ and $\vec{\alpha_j}$ are expressed by the following conditions:

(a) if $T^k \supseteq T^i$ then $[\vec{\alpha_i}]_k = [\vec{\alpha_j}]_k = 0$;
(b) if $T^i \supset T^k$ and $T^k \supseteq T^j$ then $[\vec{\alpha_i}]_k \geq 0$ and $[\vec{\alpha_j}]_k = 0$;
(c) if $T^j \supset T^k$ then $[\vec{\alpha_i}]_k = [\vec{\alpha_i}]_j \cdot C_F(T^j, T^k) + [\vec{\alpha_j}]_k$,

where $T^i \supseteq T^j$ stands for $T^i \supset T^j \vee T^i = T^j$.

In the next section, we will provide *MLTL* with the capability of supporting both synchronous and asynchronous interpretations. In particular, we will show that the replacement of the vector of current instants with M vectors of phase displacements will allow us to keep the semantics of the extended language simpler and closer to the original one. Indeed, instead of interpreting formulae with respect to a vector of current instants, we will continue to interpret them with respect to a single current instant taken from such a vector (given the corresponding M vectors of phase displacements), and define the way in which such an instant must be updated when (non-zero) displacements or projections are performed. More precisely, let $\vec{v}$ be the vector of current instants, $[\vec{v}]_i$ be the current instant with respect to which the considered formula is interpreted, and $\vec{\alpha_i}$, with $i \in M$, be the associated vector of phase displacements. The execution of a *(non-zero) displacement* within the current domain modifies all the current instants of the finer domains and possibly the current instants of the coarser ones, but leaves the M vectors of phase displacements unchanged. Then, the new vector of current instants can be easily determined on the basis of the new current instant (still belonging to T^i) and the given phase displacement function[5]. The execution of an *asynchronous projection* from T^i on T^j changes the temporal domain and forces $[\vec{v}]_j$ to become the new current instant with respect to which the projected formula must be interpreted, and to replace $\vec{\alpha_i}$ by $\vec{\alpha_j}$.

3.7 Supporting synchronous and asynchronous interpretations

In this section, we extend basic *MLTL* to support synchronous and asynchronous interpretations of the relations betweeen differently-grained components of a granular real-time system in a uniform framework. We will first define the asynchronous projection operator in terms of the basic operators of displacement, contextualization, and (synchronous) projection. Then, we will show how both synchronous and asynchronous projection operators can actually be defined in terms of a third (simpler) projection operator, called the *aligned projection operator*.

The extended language. First, we revise the definition of the set of algebraic terms $T(A \cup X)$ given for basic *MLTL*, adding a constant $\alpha_{i,j}$, for each ordered pair of indices $i, j \in M$, and the rule: if $c_1, c_2 \in T(C \cup Y)$, then $p_d(c_1, c_2) \in T(A \cup X)$, where $p_d : T(C \cup Y) \times T(C \cup Y) \to D$ is an interpreted function symbol mapping each ordered pair of contexts (temporal domains) into an algebraic term (phase displacement). Moreover, we add a new projection operator $\Diamond_\alpha$, called the *asynchronous projection operator*. When applied to a formula ϕ, it allows one to evaluate ϕ with respect to the time instants belonging to the vector of current instants $\vec{v}$. The dual operator $\Box_\alpha$ is defined as usual as $\neg\Diamond_\alpha\neg$.

The asynchronous projection operator $\Box_\alpha$ can be defined in terms of the displacement, contextual, and projection operators as follows:

$$\begin{aligned}\Box_\alpha\phi := \forall c_1, c_2 \exists \alpha, \beta(\alpha = p_d(c_1, c_2) \wedge \beta = c_f(c_1, c_2) \wedge 0 \leq \alpha < \beta \wedge \\ \nabla^{c_1}\Diamond\Delta^{c_2}(\Delta_\alpha\phi \wedge \forall\gamma(0 \leq \gamma < \beta \to \Delta_\gamma p)) \wedge \nabla_1\Box\nabla^{c_2}\neg p),\end{aligned}$$

where p is a syntactically univocal propositional letter.

Let us consider now the behavior of $\Box_\alpha$ in the particular case in which $P_D(T^i, T^j)$ is equal to 0, for each $i, j \in M$ (*synchronization of current instants*). In such a case, whatever is the current instant $[\vec{v}]_i$ with respect to which the formula $\Box_\alpha\phi$ is evaluated, $\Box_\alpha$ acts as an aligned projection operator mapping the current instant $[\vec{v}]_i$ on the first instant of its projection intervals on T^j, for each $j \in M$. Obviously, as soon as the vector of current instants changes and the elements of the new vector are no more synchronized, the behaviors of the asynchronous and aligned projection operators become different. However, we can introduce a new projection operator $\Box_0$, called the *aligned projection operator*, that maps the current instant i on the first instant of its projection interval on T^j, for each

[5]It is worth noting that if the assumption that $\updownarrow$ is homogeneous is relaxed, the invariance under displacement of the vectors of phase displacements is no more guaranteed.

$j \in M$, and redefine the *asynchronous projection operator* by means of this new operator, the displacement operator and the contextual one. Furthermore, it is possible to show that also the original (synchronous) projection operator $\Box$ can be redefined in terms of the displacement, contextual, and aligned projection operators. This means that we could take the displacement, contextual, and aligned projection operators as primitive and deriving both the synchronous and asynchronous projection operators from them.
The aligned projection operator can be defined in terms of the displacement, contextual, and projection operators as follows:

$$\Box_0\phi := \forall c_1, c_2 \exists \beta(\beta = c_f(c_1, c_2) \wedge \nabla^{c_1}\Diamond\Delta^{c_2}(\phi \wedge \forall\gamma(0 \leq \gamma < \beta \rightarrow \Delta_\gamma p)) \wedge \nabla_1\Box\nabla^{c_2}\neg p).$$

Notice that, unlike the definition of $\Box_\alpha$, the definition of $\Box_0$ does not involve the function p_d. The dual operator $\Diamond_0\phi$ is defined as usual.
The asynchronous projection operator can be easily defined in terms of the aligned one as follows:

$$\Box_\alpha\phi := \forall c_1, c_2 \exists \alpha, \beta(\alpha = p_d(c_1, c_2) \wedge \beta = c_f(c_1, c_2) \wedge 0 \leq \alpha < \beta \wedge \nabla^{c_1}\Box_0\nabla^{c_2}\nabla_\alpha\phi).$$

Furthermore, if we assume the aligned projection operator $\Box_0$ as a primitive one, we can also define the synchronous projection operator as follows:

$$\Box\phi := \forall c_1, c_2 \exists \beta(\beta = c_f(c_1, c_2) \wedge \forall\alpha(0 \leq \alpha < \beta \rightarrow \nabla^{c_1}\Box_0\nabla^{c_2}\nabla_\alpha\phi)).$$

Both the proposed definitions of $\Box_\alpha$ make use of the function p_d, which is needed to determine the value of the phase displacement between the domain the current instant belongs to and each projection domain. However, we do not need to use such a function in case of contextualized projections, where both the original and the projection domain are fixed once and for all. In such a case, we only need to introduce a suitable algebraic constant denoting the phase displacement between the two specific domains. In general, we need a distinct constant $\alpha_{i,j}$ for each phase displacement $[\vec{\alpha_i}]_j$, with $i, j \in M$. For each ordered pair of contexts c_i, c_j, we can define a contextualized asynchronous projection operator $\Box_\alpha^{c_i,c_j}$ in the following way:

$$\Box_\alpha^{c_i,c_j}\phi := \alpha_{i,j} < c_f(c_i, c_j) \wedge \nabla^{c_i}\Box_0\nabla^{c_2}\nabla_{\alpha_{i,j}}\phi.$$

The dual operator $\Diamond_\alpha^{c_1,c_2}$ is defined as usual. The idea of contextualized projections can be immediately generalized to support the projection on a specific instant of the projection interval different from the first one. For instance, to map the current hour into its sixteenth minute, we can use the projection operator $\Box_{15}^{hour,minute}$ defined as follows[6]:

$$\Box_{15}^{hour,minute}\phi := \nabla^{hour}\Box_0\nabla^{minute}\nabla_{15}\phi$$

[6]It is worth noting that specifying a numerical value for the displacement from the beginning of the projection interval makes sense only if the projection domain is univocally determined, given that the same numerical value denotes a different displacement with respect to different domains [86].

A detailed example of the use of the extended language for specifying a granular real-time system is given in [86].

A further extension to the language is actually needed to support asynchronous upward and downward temporal projection. In basic *MLTL*, upward and downward temporal projections are based on the $\Box$ and $\Diamond$ operators, respectively, and thus they behave according to a synchronous interpretation. To allow the user to specify the intended behavior of formulae under projection, we extend the alphabet of the basic *MLTL* language with a *mode sort*. As long as we only need to distinguish between synchronous and asynchronous interpretations, two constants s and a of mode sort, that respectively denote synchronous and asynchronous interpretations, are sufficient. Mode constants come into play as a second parameter of the contextual operator. When applied to a formula ϕ, the resulting contextual operator $\Delta^{c,m}$ determine both the domain of ϕ and the way in which upward and downward temporal projections act on ϕ. More precisely, the first parameter c of $\Delta^{c,m}$ is still a term of context sort denoting the temporal domain the formula ϕ refers to; the second parameter m of mode sort constrains the temporal projection of ϕ across domains. In case this second parameter is missing, the synchronous interpretation is taken as default.

The extended structures. Three-sorted frames can be easily generalized to incorporate the mode sort. The definition of model remains essentially unchanged, except for the fact that (temporal) formulae are evaluated with respect to a vector of time instants rather than at a time instant. Nevertheless, given the correspondence between vectors of time instants and vectors of phase displacements, the valuation of a formula ϕ with respect to a vector of current instants $\vec{v}$ can be defined in terms of the valuation of ϕ at a single current instant of such a vector, provided that the interpretation g for algebraic terms given in Section 3.4 is extended by requiring that $g(p_d(c_1, c_2)) = [\vec{\alpha_i}]_j$, with $h(c_1) = T^i$, $h(c_2) = T^j$, and $g(\alpha_{i,j}) = [\vec{\alpha_i}]_j$ (for all $i, j \in M$). Notice that this allows us to keep the definition of satisfiability, validity and logical consequence unchanged.

With regard to the operators, the mode characterization of contextual operators does not affect their semantics, but only constrains the applicability of temporal projection (see below). As for the asynchronous projection operator $\Diamond_\alpha$, the truth of the formula $\Diamond_\alpha\phi$ can be defined as follows. Given a vector of current instants $\vec{v}$, $\Diamond_\alpha\phi$ evaluates to true at the (specific) current instant $[\vec{v}]_i$ if and only if there exists $j \in M$ such that ϕ evaluates to true at $[\vec{v}]_j$. Unlike the case of the asynchronous projection operator, the notion of vector of current instants is not involved in the definition of the semantics of the aligned projection operator: the formula $\Diamond_0\phi$ evaluates to true at a time instant i if and only if there exists $j \in M$ such that ϕ evaluates to true at the first instant of the projection interval of i on T^j.

The truth of the temporal formulae $\Diamond_\alpha\phi$ and $\Diamond_0\phi$ can be formally defined by means of two functions, $first : \mathcal{T} \times \mathcal{D} \to \mathcal{T}$ and $last : \mathcal{T} \times \mathcal{D} \to \mathcal{T}$. For each time instant $i \in \mathcal{T}$ and each temporal domain $T^j \in \mathcal{C}$, the function *first* (resp. *last*) determines the minimum (resp. maximum) time instant belonging to the projection interval of i on T^j, with respect to the ordering over T^j. The function *first* and *last* can be formally defined as follows:

$first(i,T^j) = j$ iff $j \in T^j \wedge \updownarrow(i,j) \wedge \forall j'((j' \in T^j \wedge \updownarrow(i,j')) \rightarrow \exists \alpha_j(\alpha_j \geq 0 \wedge \mathrm{DIS}_j(j,\alpha_j,j')))$

and

$last(i,T^j) = j$ iff $j \in T^j \wedge \updownarrow(i,j) \wedge \forall j'((j' \in T^j \wedge \updownarrow(i,j')) \rightarrow \exists \alpha_j(\alpha_j \leq 0 \wedge \mathrm{DIS}_j(j,\alpha_j,j')))$.

The following proposition expresses basic relations between the functions *first* and *last* (for the sake of simplicity, we assume the functionality of each DIS_i, with $i \in M$):

3.7.1. PROPOSITION. *For any pair of domains T^i and T^j, ordered by disjointedness ($T^i \supset T^j$), and any instant $i \in T^i$:*

(a) $\mathrm{DIS}_j(first(i,T^j), \lceil C_F(T^i,T^j)\rceil + (-1), last(i,T^j))$;

(b) $\mathrm{DIS}_j(last(i,T^j), 1, first(succ(i),T^j))$, *where $succ(i)$ denotes the (unique) time instant $i' \in T^i$ such that* $\mathrm{DIS}_i(i,1,i')$.

The proof of (a) directly follows from the properties of contiguity and homogeneity of the relation of projection. It is worth noting that (a) also holds for pairs of domains T^i and T^j such that either $T^j \supset T^i$ or $T^i = T^j$. In such a case, $first(i,T^j)$ is indeed equal to $last(i,T^i,T^j)$, and thus $\mathrm{DIS}_j(first(i,T^j),0,last(i,T^j))$ holds. Such an relation can actually be taken as a definition of the function *first* in terms of the function *last*, or vice versa.

The proof of (b) involves the properties of symmetry, strong order preservation and coverage of the projection relation. The proofs of both relations are given in [86].

The *truth* of the temporal formulae $\Diamond_0\phi$ and $\Diamond_\alpha\phi$ is defined by

$$\mathfrak{M}, i \Vdash \Diamond_0\phi \quad \text{iff} \quad \text{there exist } T^j \text{ and } j \text{ such that } j \in T^j, j = first(i,T^j), \text{ and } \mathfrak{M}, j \Vdash \phi;$$

$$\mathfrak{M}, i \Vdash \Diamond_\alpha\phi \quad \text{iff} \quad \text{there exist } T^i, T^j, \text{ and } j \text{ such that } i \in T^i, j \in T^j, \mathrm{DIS}_j(first(i,T^j), P_D(T^i,T^j), j), \text{ and } \mathfrak{M}, j \Vdash \phi.$$

On the basis of the relations expressed by Proposition 3.7.1, it is possible to prove that the given definition of $\Box_0$ in terms of the displacement, contextual, and projection operators captures the intended meaning of $\Box_\alpha$ as expressed by the above truth definition.

3.7.2. LEMMA. *For any instant $i \in \mathcal{T}$ and any formula ϕ, the formula defining $\Box_0\phi$ in terms of the displacement, contextual, and projection operators evaluates to true at i if and only if for any temporal domain T^j, ϕ evaluates to true at the initial point of the interval of T^j on which i is projected.*

The proof is given in [86]. On the basis of Lemma 3.7.2, we can easily prove that the definition of $\Box_\alpha$ in terms of the displacement, contextual, and aligned projection operators captures the intended meaning of $\Box_0$ as expressed by the above truth definition.

3.7.3. THEOREM. *For any T^i, $i \in T^i$ and ϕ, the formula defining the $\Box_\alpha\phi$ operator in terms of the displacement, contextual, and aligned projection operators evaluates to true at i if and only if, for each T^j,*

$$\phi \textit{ evaluates to true at } j \in T^j \textit{ such that } \mathrm{DIS}_j(first(i,T^j), P_D(T^i,T^j), j).$$

Axioms. The properties of the aligned and asynchronous projection operators can be derived from the basic axioms for displacement, contextual and (synchronous) projection operators. We only need to add the axioms constraining the relations between phase displacements. They essentially codify conditions (a)-(c) given in Section 3.6, and are not reported here.

We already observed that mode qualifications of contextual operators come into play in upward and downward temporal projections. In the following, we will show how to generalize the definition of temporal projection given in Section 3.4 to support both synchronous and asynchronous interpretations.

The definition of downward (synchronous) temporal projection remains unchanged; the formula is slightly modified by adding the mode qualification:

$$\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1,s}(\phi \rightarrow \Diamond\Delta^{c_2}\phi)),$$

where s is the mode constant denoting the synchronous interpretation.

Besides, we define two new consisteny rules of downward and upward *asynchronous* temporal projection. For each pair of domains T^i, T^j, with T^i coarser than T^j, downward asynchronous temporal projection states that if a fact p is true at the current time instant i of T^i, then p is true at the current time instant j of T^j. Moreover, for each pair of domains T^i, T^j, with T^i finer than T^j, upward asynchronous temporal projection states that if p is true at the current time instant i of T^i, then p is true at the current time instant j of T^j. Notice that, in the asynchronous case, upward and downward temporal projections become perfectly symmetric.
Formally, *downward asynchronous projection* is defined by the following formula:

$$\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1,a}(\phi \rightarrow \Diamond_\alpha\Delta^{c_2}\phi)),$$

where ϕ is a formula devoid of any occurrence of the displacement, contextual and projection operators, $\Diamond_\phi$ is the asynchronous projection operator, and a is the mode constant denoting the synchronous interpretation, while *upward asynchronous projection* is defined by the formula:

$$\forall c_1, c_2(c_1 \supset c_2 \rightarrow \nabla^{c_1,a}(\Box_\alpha\nabla^{c_2}\phi \rightarrow \phi)).$$

As in the synchronous case, it is easy to show that upward and downward asynchronous projections are interdeducible. Moreover, pairing the formulae for the synchronous and asynchronous cases, it is straightforward to prove that each asynchronous model is also a synchronous one, but not vice versa. In [86], we also show that under the assumption that $\updownarrow$ is homogeneous, displacement and projection operators *commute*, modulo the conversion factor, that is, formulae can be first translated of a given displacement over the original domain and then synchronously (resp. asynchronously) projected, or they can be first synchronously (resp. asynchronously) projected and then translated of a distance equal to the fine grain equivalent of the originally specified displacement.

Finally, even if the asynchronous projection operator has been introduced with the specific goal of constraining projected formulae to be evaluated at the current time instant of the projection domain, it is immediate to consider possible extensions of this operator supporting ontological characterizations of assertions. As shown in [113], primitive ontological concepts as event, property, fact and process, can be defined in terms of the behaviour under temporal projection of the corresponding assertions by replacing the point domain over which phase displacement are interpreted with an interval domain. We do not consider here these (straightforward) extensions.

Concluding remarks

When building specifications for time-dependent systems–whether plant control systems, office systems, or whatever–it may happen that different components of such systems have quite different dynamic behaviours, bound to different time granularities. Common formal languages impose the use of a unique time granularity, a restriction that can make formal specifications of such systems quite cumbersome and unnatural. In this chapter, we defined a metric and layered temporal logic that makes it possible to deal with different time granularities in real-time, granular system specifications.

The effectiveness of MLTL as specification language can be improved in several directions. First of all, in order to cope with the complexity of real-world application domains, MLTL can be provided with high-level linguistic primitives that express common ontological concepts, such as event, property, fact, and process, and with further abstraction and modularization mechanisms,e.g, mechanisms provided by the object-oriented programming paradigm. In this dissertation, we do not consider these extensions. The interested reader can consult [34]. Second, suitable fragments of MLTL can be identified both to tailor the proposed general framework to specific applications and to make it possible to prove interesting logical properties such as decidability and completeness. We will address decidability issues in the next chapter. Third, in order to prove that MLTL specifications are consistent (*verification* task) and that they actually satisfy the required properties (*validation* task), we need to make them executable. The problem of executing metric (and layered) temporal logic is dealt with in Chapter 5.

Chapter 4

Decidable theories of layered temporal structures

4.1 Introduction

In this chapter, we study the decidability of the validity and satisfiability problems for *MLTL*. Decidable theories of metric and layered temporal structures are obtained by imposing suitable constraints on the temporal framework for time granularity defined in Chapter 3. More precisely, monadic second-order languages for time granularity supporting the displacement, contextualization, and projection functionalities are considered, and the theories of finitely-layered temporal structures, upward unbounded layered structures, and downward unbounded layered structures are shown to be decidable.

In [2], Alur and Henzinger showed that, under suitable assumptions about the temporal domain and the associated operations, the validity and satisfiability problems for real-time logics are decidable. These problems can be reduced, through coding into the theory *S1S*, to the decidable problem of determining whether or not the language recognized by a given Büchi automaton is empty [118]. More precisely, the problem of checking the validity of a formula $\mathcal{F}$ can be reduced to the decidable problem of checking whether or not the language recognized by the Büchi automaton corresponding to $\neg\mathcal{F}$ is empty, while the problem of checking the satisfiability of a formula $\mathcal{F}$ can be reduced to the decidable problem of checking whether or not the language recognized by the Büchi automaton corresponding to $\mathcal{F}$ is not empty. Our goal is to generalize this result to temporal logics combining metric and layered features.

When faced with a combined logic, there are at least two possible approaches to the problem of establishing its logical properties such as decidability, soundness and completeness. The first identifies what constraints the combination method must satisfy to guarantee the transfer of logical properties from the component logics to the combined one; examples can be found in [47, 92]. Otherwise, instead of lifting logical properties from the components to the combined logic, one can try to obtain a reduction to one of components and solve the problem for that one component. In the first part of this chapter, we follow the latter strategy by embedding *finitely-layered metric temporal structures* into their finest metric component, and then reducing the decidability of the theory of the finest component to a theory that is known to be decidable, namely $S1S$; cf. [42, 105].

In the second part of the chapter, we consider the more general case in which the underlying temporal structure consists of infinitely many temporal layers (ω-layered, k-refinable, metric temporal structures), and show that more powerful engines are necessary to deal with such structures. We introduce the second-order language $\mathcal{L}^2_{\omega LM^k}$ for ω-layered (k-refinable) metric temporal structures, and show how to interpret it over different classes of structures. We first consider the case of temporal structures in which there is a finest temporal domain together with a infinite number of coarser and coarser domains (*upward unbounded layered structures*). To deal with such structures we use a more expressive theory, that we called $S1S^k$, which is a proper extension of $S1S$. The decidability of $S1S^k$ is shown using a more powerful basic engine, namely the decidability of ω-languages recognized by k-ary Systolic Tree Automata. Such class of ω-languages has been recently proved to properly extend the class of regular ω-languages and to have the same closure and decidability properties [95, 96]. Since all the basic closure properties of regular ω-languages hold also for systolic tree ω- languages, a direct correspondence with the above mentioned second-order theory $S1S^k$ (properly extending $S1S$) can be established. From the one hand, upward unbounded layered structures provide an interesting example of application for the decidability of systolic tree ω-languages. On the other hand, we believe that the second-order theory $S1S$ is too weak to deal with infinitely coarsening domains, and that $S1S^k$ is a somehow "minimal" theory able to deal with such a case. Next, we deal with the problem of deciding infinitely refinable structures (*downward unbounded layered structures*), and we prove that the decidability of the satisfiability (resp. validity) problem for the theory of such structures can be reduced to the decidability of the satisfiability (resp. validity) problem for SkS, the well-known monadic second-order decidable theory of k successors [118].

The chapter is organized as follows. In Section 4.2, we introduce the theory of finitely-layered metric temporal structures. In Section 4.3, we show how to reduce the decidability problem for this theory to the decidability problem for $S1S$. In Section 4.4, we provide some background knowledge about systolic and Rabin tree automata. Then, in Section 4.5, we define the theory $S1S^k$. In Section 4.6, we formally define the theories of upward (resp. downward) unbounded layered structures, and prove that they are decidable. In Section 4.7, we show how the basic functionalities of metric and layered temporal logic can be expressed in $\mathcal{L}^2_{\omega LM^k}$. The concluding remarks point out possible further developments of the work done.

4.2 The theory of finitely-layered temporal structures

Let $\mathcal{L}^2_{nLM}$ be the second-order language for the theory of finitely-layered metric temporal structures T_{nLM}. It includes individual variables $\vec{\mathtt{x}}, \vec{\mathtt{y}}, \ldots$ and uninterpreted unary predicate symbols, the constant symbol $\vec{\mathtt{0}}$, the unary function symbols $\vec{+}_{\mathtt{1}}\mathtt{1}, \ldots, \vec{+}_{\mathtt{n}}\mathtt{1}$ (local successors), the unary (interpreted) predicate symbols $\vec{\mathtt{T}}^{\mathtt{1}}, \ldots, \vec{\mathtt{T}}^{\mathtt{n}}$ (contextualizations), the binary relational symbols $\vec{\leq}_{\mathtt{1}}, \ldots, \vec{\leq}_{\mathtt{n}}$ (local orderings), $\uparrow$ (upward projection) and $\downarrow$ (downward projection), $\vec{\equiv}_{\mathtt{1,2}}, \vec{\equiv}_{\mathtt{1,3}}, \ldots, \vec{\equiv}_{n,2}, \vec{\equiv}_{n,3}, \ldots$ (local congruences), and quantification of individual variables and (uninterpreted) unary predicate symbols. The first-order fragment of $\mathcal{L}^2_{nLM}$ is denoted by $\mathcal{L}_{nLM}$. We restrict ourselves to formulae that contain no free individual variables. Setting up the structures in which $\mathcal{L}^2_{nLM}$ can be interpreted is our next task; it takes quite a bit of work.

In order to simplify the generalization of known decidability results for real-time logics to metric and finitely-layered ones, we first reformulate the definition of three-sorted frame given in Chapter 3 in a vectorial fashion. We define a *finitely-layered metric temporal structure* as a tuple

$$(\vec{\mathcal{T}}, \vec{T}^1, \ldots, \vec{T}^n, \vec{\leq}_1, \ldots, \vec{\leq}_n, \uparrow, \downarrow, \vec{\equiv}_{1,2}, \vec{\equiv}_{1,3}, \ldots, \vec{\equiv}_{n,2}, \vec{\equiv}_{n,3}, \ldots, \vec{+}_1 1, \ldots, \vec{+}_n 1, \vec{0}).$$

$\vec{\mathcal{T}}$ is the carrier set of the structure, and it is called the *temporal universe*. The n components $\vec{T}^1, \ldots, \vec{T}^n$ are sets of temporal vectors corresponding to the interpretation of the n unary predicates $\vec{\mathtt{T}}^{\mathtt{1}}, \ldots, \vec{\mathtt{T}}^{\mathtt{n}}$, respectively. The temporal universe $\vec{\mathcal{T}}$ is equal to $\bigcup_{i=1}^n \vec{T}^i$. The set of domains is totally ordered by inclusion: $\vec{T}^1 \supset \vec{T}^2 \supset \ldots \supset \vec{T}^n$, and thus $\vec{\mathcal{T}} = \vec{T}^1$. Let us call $\supset$ the *disjointedness* relation (even if, in Chapter 3, we have seen that the disjointedness relation usually defines a partial ordering over domains, we restrict ourselves to the case in which domains are totally ordered by disjointedness). For each pair of domains $\vec{T}^i, \vec{T}^j$, we say that the granularity of $\vec{T}^i$ is coarser (resp. finer) than the granularity of $\vec{T}^j$ if and only if $\vec{T}^i \supset \vec{T}^j$ (resp. $\vec{T}^j \supset \vec{T}^i$). Formally, the disjointedness relation on $\{\vec{T}^1, \ldots, \vec{T}^n\}$ is a total ordering $\supset$ such that $\vec{T}^i \supset \vec{T}^j$, for $1 \leq i < n-1$ and $i < j \leq n$. Each vector $\vec{x}$ such that $\vec{T}^i$ is the finest domain to which it belongs is called a *time instant* of $\vec{T}^i$. A *fine membership* relation $\in'$ is defined such that $\vec{x} \in' \vec{T}^i$ if and only if $\vec{x} \in \vec{T}^i \wedge \vec{x} \notin \vec{T}^{i+1}$. Since n is finite, for each $\vec{x} \in \vec{\mathcal{T}}$, there exists one and only one $\vec{T}^i$ such that $\vec{x} \in' \vec{T}^i$. Moreover, for each pair of consecutive domains $\vec{T}^i, \vec{T}^{i+1}$, with $1 \leq i < n$, we assume that there exists a natural number $cf_{i,i+1}$, called the *conversion factor* between $\vec{T}^i$ and $\vec{T}^{i+1}$, that expresses the ratio between the granularities of time instants finely belonging to the two domains (*homogeneity* assumption).

Furthermore, $\vec{\leq}_1, \ldots, \vec{\leq}_n$ are binary relations of local temporal ordering over $\vec{T}^1, \ldots, \vec{T}^n$, respectively; $\uparrow$ and $\downarrow$ are binary relations of upward and downward projection over $\vec{\mathcal{T}}$; $\vec{\equiv}_{i,2}$, $\vec{\equiv}_{i,3}, \ldots$ are binary relations of local time congruence over $\vec{T}^i$, for $1 \leq i \leq n$; $\vec{+}_1 1, \ldots, \vec{+}_n 1$ are unary successor functions of temporal displacement over $\vec{T}^1, \ldots, \vec{T}^n$, respectively; and $\vec{0}$ is the zero vector (see below).

To specify the components of finitely-layered metric temporal structures, we introduce a representation for temporal vectors. For $1 \leq i \leq n$, we represent the set $\{\vec{x} \mid \vec{x} \in' \vec{T}^i\}$ as the generalized cartesian product $\mathbb{N} \times \prod_{k=1}^{i-1}[0, cf_{k,k+1})$, where each pair $[0, cf_{k,l})$ denotes an interval of natural numbers. The representation of the set $\{\vec{x} \mid \vec{x} \in \vec{T}^i\}$ is thus simply $\bigcup_{j=i}^{n} \mathbb{N} \times \prod_{k=1}^{j-1}[0, cf_{k,k+1})$. Furthermore, for $k = 1, \ldots, n$, a function $[\cdot]_k : \vec{\mathcal{T}} \mapsto \mathbb{N} \cup \{\bot\}$ can be defined such that, for each $\vec{x} (\in' T^i)$, $[\vec{x}]_k$ is equal to the k-th component of $\vec{x}$ if $k \leq i$, and to $\bot$ otherwise.

The above representation of temporal vectors can be interpreted as follows. Time instants finely belonging to $\vec{T}^1$ take value over (a temporal domain isomorphic to) $\mathbb{N}$. Let us call their values *absolute* temporal positions. The representation of an instant $\vec{x}$ finely belonging to $\vec{T}^i$, with $1 < i \leq n$, consists of two different parts: the specification of its (absolute) position $[\vec{x}]_1$ with respect to $\vec{T}^1 \backslash \vec{T}^2$, where $\backslash$ denotes set-theoretic difference, plus the specification of $i-1$ nested displacements $[\vec{x}]_2, \ldots, [\vec{x}]_i$ with respect to $\vec{T}^2 \backslash \vec{T}^3, \ldots, \vec{T}^i \backslash \vec{T}^{i+1}$, respectively.

4.2.1. Example. Consider a temporal universe consisting of hours, minutes, and seconds. An hour is specified by its absolute value, e.g. hour 4011, a minute is specified by the hour it belongs to plus a displacement with respect to the first minute of such an hour, e.g. the sixteenth minute of hour 4011 is represented by the pair $(4011, 15)$, a second is specified by the hour it belongs to plus a displacement with respect to the first second of the minute it belongs to, which in its turn is specified in the same way with respect to the hour, e.g. the third second of the sixteenth minute of hour 4011 is represented by the triplet $(4011, 15, 2)$.

We now define local orderings, congruences, successors, and upward and downward projections. For $i = 1, \ldots, n$, the local ordering $\vec{\leq}_i$ between any pair of vectors $\vec{x}, \vec{y} \in \vec{T}^i$ is defined in terms of ordering of their components.

4.2.2. Definition. (Local ordering) For each domain $\vec{T}^i$, a *local ordering* $\vec{\leq}_i$ (lexicographical ordering) is defined such that, for each pair of vectors $\vec{x}, \vec{y} \in \vec{T}^i$,

$$\vec{x} \vec{\leq}_i \vec{y}$$

$$\text{iff}$$

$$\forall j (1 \leq j \leq i \rightarrow [\vec{x}]_j = [\vec{y}]_j) \vee \exists j (1 \leq j \leq i \wedge \forall k (1 \leq k < j \rightarrow [\vec{x}]_k = [\vec{y}]_k) \wedge [\vec{x}]_j < [\vec{y}]_j).$$

A notion of local equality $\vec{=}_i$ of two instants $\vec{x}, \vec{y} \in \vec{T}^i$ can be derived immediately.

The relations of upward projection $\uparrow \subseteq \vec{\mathcal{T}} \times \vec{\mathcal{T}}$ and downward projection $\downarrow \subseteq \vec{\mathcal{T}} \times \vec{\mathcal{T}}$ are defined in terms of the notions of prefix and extension, respectively.

4.2.3. Definition. (Prefix and extension) For all $\vec{x} \in' \vec{T}^i$, a (non empty) *prefix* of $\vec{x}$ is a time instant $\vec{y} \in' \vec{T}^j$, with $1 \leq j \leq i$, such that $[\vec{x}]_k = [\vec{y}]_k$, for $k = 1, \ldots, j$. For all $\vec{x} \in' \vec{T}^i$, an *extension* of $\vec{x}$ is a time instant $\vec{y} \in' \vec{T}^j$, with $i \leq j \leq n$, such that $[\vec{x}]_k = [\vec{y}]_k$, for $k = 1, \ldots, i$.

4.2.4. Definition. (Upward and downward projections) For each pair of vectors $\vec{x}, \vec{y} \in \vec{\mathcal{T}}$, $\uparrow(\vec{x}, \vec{y})$ holds if and only if $\vec{y}$ is a prefix of $\vec{x}$, while $\downarrow(\vec{x}, \vec{y})$ holds if and only if $\vec{y}$ is an extension of $\vec{x}$.

It is immediate to see that the projection relation $\updownarrow$, defined in Chapter 3, is equal to the set union of upward and downward projections, that is $\updownarrow = \uparrow \cup \downarrow$.

4.2.5. PROPOSITION. *For any temporal domain $\vec{T}^i$ and any pair of vectors $\vec{x}, \vec{y} \in' \vec{T}^i$, if $\vec{x}$ is not equal to $\vec{y}$, then there exists no vector $\vec{z}$ such that $\downarrow(\vec{x}, \vec{z})$ and $\downarrow(\vec{y}, \vec{z})$.*

Local congruence relations $\vec{\equiv}_{i,2}, \vec{\equiv}_{i,3}, \ldots$ between pairs of vectors $\vec{x}, \vec{y}$ belonging to the same domain $\vec{T}^i$ are defined in terms of (standard) congruence relations between their i-th components x_i, y_i.

4.2.6. DEFINITION. (Local congruence) For each domain $\vec{T}^i$, each pair of vectors $\vec{x}, \vec{y} \in \vec{T}^i$, and each natural number d, a *local congruence relation* $\equiv_{i,d}$ is defined as follows:

$$\vec{x} \vec{\equiv}_{i,d} \vec{y} \quad \text{iff} \quad [\vec{x}]_i \equiv_d [\vec{y}]_i.$$

The apparently stronger notion of local congruence $\vec{\equiv}'_{i,d}$ between $\vec{x}, \vec{y} \in' \vec{T}^i$ that holds whenever all the components are congruent modulo-d, can be defined as follows:

$$\vec{x} \vec{\equiv}'_{i,d} \vec{y} \ \text{ iff } \ \forall j (1 \leq j \leq i \rightarrow \vec{x} \vec{\equiv}_{j,d} \vec{y}),$$

where $\forall j (1 \leq j \leq i \rightarrow \vec{x} \vec{\equiv}_{j,d} \vec{y})$ is a shorthand for $\vec{x} \vec{\equiv}_{1,d} \vec{y} \wedge \ldots \wedge \vec{x} \vec{\equiv}_{i,d} \vec{y}$.

Finally, for each $\vec{T}^i$, a unary successor function $\vec{+}_i 1$ is defined.

4.2.7. DEFINITION. (Local successor) Let $\vec{T}^i$ be a temporal domain, and $\vec{x} = \langle x_1, \ldots, x_j \rangle$, with $j \geq i$, be an element of $\vec{T}^i$. The application of $\vec{+}_i 1$ to $\vec{x}$ is defined as follows:

$$\vec{x} \vec{+}_i 1 = \begin{cases} \langle x_1, \ldots, x_i + 1, \ldots, x_j \rangle & \text{if } \ i = 1 \vee x_i + 1 < cf_{i-1,i}, \\ \langle x_1, \ldots, x_{i-1}, 0, \ldots, x_j \rangle \vec{+}_{i-1} 1 & \text{otherwise,} \end{cases}$$

where 0 and +1 are the constant 0 and the successor function of natural numbers, respectively.

Notice that even if local successors are specified within a given domain, they can actually propagate to different domains.

In order to define an *interpretation* for the language $\mathcal{L}^2_{nLM}$, it is useful to introduce an alternative (pseudo) vectorial representation according to which the i-th component of a vector denotes an absolute position with respect to $\vec{T}^i \setminus \vec{T}^{i+1}$. Such a representation can be automatically derived from the above given one. For each domain $\vec{T}^i$ and each vector $\vec{x} \in' \vec{T}^i$, let us transform $\vec{x}$ into a (pseudo)vector $\vec{y}$ such that $[\vec{y}]_1 = [\vec{x}]_1$, and, for each $j = 2, \ldots, i$, $[\vec{y}]_j = (\ldots(([\vec{x}]_1 \cdot cf_{1,2} + [\vec{x}]_2) \cdot cf_{2,3} + [\vec{x}]_3) \ldots) \cdot cf_{j-1,j} + [\vec{x}]_j = [\vec{y}]_{j-1} + [\vec{x}]_j$.

4.2.8. EXAMPLE. Assume the temporal universe of Example 4.2.1. The representation of the third second of the sixteenth minute of hour 4011 becomes $\langle 4011, 240675, 14440502 \rangle$.

According to this alternative representation, for each $i = 1, \ldots, n$, the set $\{\vec{x} \mid \vec{x} \in' \vec{T}^i\}$ becomes a suitable subset of the product $\mathbb{N} \times \cdots \times \mathbb{N}$ (i times). This representation is redundant, because each component $[\vec{y}]_j$ of a vector $\vec{y} \in' \vec{T}^i$, with $1 < j \leq i$, codifies complete information about all the components of lower index. It is indeed easy to prove that $[\vec{y}]_{j-1}$ is equal to the (unique) natural number such that $[\vec{y}]_{j-1} \cdot cf_{i-1,i} \leq [\vec{y}]_j < ([\vec{y}]_{j-1}+1) \cdot cf_{i-1,i}$. Moreover, once $[\vec{y}]_{j-1}$ has been determined, it can be used to determine $[\vec{y}]_{j-2}$, and so on, until the first component is reached. Thus, the finest component of a (pseudo)vector implicitly provides complete information about all the other components. Even if it is less elegant than the original representation, we will use this (pseudo) vectorial representation to make the definition of the semantic interpretation of $\mathcal{L}^2_{nLM}$-formulae simpler.

Let ϕ be a formula of $\mathcal{L}^2_{nLM}$, with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_m}$. Unary and binary relational symbols, and constants and function symbols are mapped onto the corresponding components of the temporal structure. Thus, an interpretation $\mathcal{I}$ for ϕ is given by $m \cdot n$ sets $p^{\mathcal{I}}_{1,1}, \ldots, p^{\mathcal{I}}_{m,n} \subseteq \mathbb{N}$, where, for each set $p^{\mathcal{I}}_{k,i}$, k and i indicate the indices of the predicate $\mathtt{p_k}$ and of the set of time instants of $\vec{T}^i$, respectively. For $k = 1, \ldots, m$, $p^{\mathcal{I}}_{k,1}, \ldots, p^{\mathcal{I}}_{k,n-1}, p^{\mathcal{I}}_{k,n}$ we define the interpretation $p^{\mathcal{I}}_k$ of $\mathtt{p_k}$ with respect to the sets $\vec{T}^1 \setminus \vec{T}^2, \ldots, \vec{T}^{n-1} \setminus \vec{T}^n, \vec{T}^n$ by stipulating that $\mathtt{p_k}$ holds on a vector $\vec{x}$ if and only if for some i, $\mathtt{p_k}$ holds for the i-th component of $\vec{x}$ (i.e., $[\vec{x}]_i \in p^{\mathcal{I}}_{k,i}$).

4.2.9. Example. Assume the temporal universe of the previous examples. Moreover, let $\vec{x}$ be the vector of Example 4.2.8, $\mathtt{p_k}$ be a predicate symbol of $\mathcal{L}^2_{nLM}$, and $\mathcal{I}$ be an interpretation for $\mathcal{L}^2_{nLM}$. According to the given definition, $\mathtt{p_k}$ holds at $\vec{x}$ if, for instance, $240675 \in p^{\mathcal{I}}_{k,2}$.

4.2.1 Supporting basic MLTL functionalities

Now that we have defined our language for talking about layered and metric temporal structures, we show how it can express the three key features of metric and layered temporal logics defined in Section 1: contextualization, and granular and metric displacement, thus showing the expressiveness of the language $\mathcal{L}^2_{nLM}$, and its usefulness as a framework for studying metric and layered temporal logics. We will also introduce the notions of (global) temporal ordering and congruence.

Contextualization restricts the range of possible values of a given vector variable $\vec{\mathtt{x}}$, by constraining (the value of) $\vec{\mathtt{x}}$ to belong to a given domain $\vec{T}^i$. In $\mathcal{L}^2_{nLM}$, contextualization is expressed by means of the n unary predicates $\vec{\mathtt{T}}^1(\vec{\mathtt{x}}), \ldots, \mathtt{T}^n(\vec{\mathtt{x}})$. Contextualization is formally defined as follows. For the sake of readability, we will use a set notation $\vec{\mathtt{x}} \in \vec{\mathtt{T}}^i$ (resp. $\vec{\mathtt{x}} \notin \vec{\mathtt{T}}^i$) instead of $\vec{\mathtt{T}}^i(\vec{\mathtt{x}})$ (resp. $\neg\vec{\mathtt{T}}^i(\vec{\mathtt{x}})$).

4.2.10. Definition. (Contextualization) For each vector variable $\vec{\mathtt{x}}$ and each unary predicate $\vec{\mathtt{T}}^i$, with $1 \leq i \leq n$, $\vec{\mathtt{x}} \in \vec{\mathtt{T}}^i$ holds if and only if (the value of) $\vec{\mathtt{x}}$ belongs to the domain $\vec{T}^i$.

The total ordering of temporal domains defined by granularity allows us to easily prove that the formula:

$$\forall\vec{x}(\vec{x}\in\vec{T}^{i}\rightarrow\forall j(1\leq j\leq i\rightarrow\vec{x}\in\vec{T}^{j})),$$

is valid in all finitely-layered metric temporal structures (as usual, the consequent of the outermost implication stands for "$\vec{x}\in\vec{T}^{1}\wedge\ldots\wedge\vec{x}\in\vec{T}^{i}$"). Contextualization also allows us to prove the following proposition.

4.2.11. PROPOSITION. *For each pair of vector variables* $\vec{x},\vec{y}$, $\vec{x}=\vec{y}\leftrightarrow\exists i(\vec{x}\in'\vec{T}^{i}\wedge\vec{y}\in'\vec{T}^{i}\wedge\vec{x}\overset{\rightarrow}{=}_{i}\vec{y}))$, *where the right-hand side formula stands for* "$(\vec{x}\in'\vec{T}^{1}\wedge\vec{y}\in'\vec{T}^{1}\wedge\vec{x}=_{1}\vec{y})\vee\ldots\vee(\vec{x}\in'\vec{T}^{n}\wedge\vec{y}\in'\vec{T}^{n}\wedge\vec{x}=_{n}\vec{y})$" *and* "$\vec{x}\in'\vec{T}^{i}\leftrightarrow(\vec{x}\in\vec{T}^{i}\wedge\vec{x}\notin T^{i+1})$".

It follows that two vectors *finely* belonging to different domains are distinct.

Contextualization can occur in different types of formulae. As an example, it is involved in formulae stating that there exists a time instant belonging to a given domain $\vec{T}^{i}$ at which a formula ϕ is true, and in formulae stating that ϕ is true at each instant of a given domain $\vec{T}^{i}$ (restricted quantification). These formulae take the forms $\exists\vec{x}(\vec{x}\in'\vec{T}^{i}\wedge\phi(\vec{x}))$ and $\forall\vec{x}(\vec{x}\in'\vec{T}^{i}\rightarrow\phi(\vec{x}))$, respectively.

4.2.12. DEFINITION. (Granular ordering and equivalence) We define a partial ordering $\gg$ over $\mathcal{T}$ based on the 'grain-size' of vectors ($\vec{x}\gg\vec{y}$ if and only if $\vec{x}$ is *coarser than* $\vec{y}$) as follows:

$$\vec{x}\gg\vec{y}\text{ iff }\exists i,j(\vec{x}\in'\vec{T}^{i}\wedge\vec{y}\in'\vec{T}^{j}\wedge i<j),$$

where the right-hand side formula stands for "$(\vec{x}\in'\vec{T}^{1}\wedge\vec{y}\in'\vec{T}^{2})\vee(\vec{x}\in'\vec{T}^{1}\wedge\vec{y}\in'\vec{T}^{3})\vee\ldots\vee(\vec{x}\in'\vec{T}^{n-1}\wedge\vec{y}\in'\vec{T}^{n})$". Moreover, an equivalence relation $\sim$ over $\mathcal{T}$, such that $\vec{x}\sim\vec{y}$ if and only if $\vec{x}$ is *as coarse as* $\vec{y}$, can be defined as follows:

$$\vec{x}\sim\vec{y}\text{ iff }\exists i(\vec{x}\in'\vec{T}^{i}\wedge\vec{y}\in'\vec{T}^{i}),$$

where the right-hand side formula stands for "$(\vec{x}\in'\vec{T}^{1}\wedge\vec{y}\in'\vec{T}^{1})\vee\ldots\vee(\vec{x}\in'\vec{T}^{n}\wedge\vec{y}\in'\vec{T}^{n})$".

Granular displacement is directly supported by upward and downward projections. As in the case of contextualizations, we adopt a set notation $\vec{y}\in\uparrow(\vec{x})$ (resp. $\vec{y}\notin\uparrow(\vec{x})$) instead of $\uparrow(\vec{x},\vec{y})$ (resp. $\neg\uparrow(\vec{x},\vec{y})$).

4.2.13. DEFINITION. (Granular displacement) For each pair of vector variables $\vec{x},\vec{y}$, $\vec{y}\in\uparrow(\vec{x})$ holds if and only if (the value of) $\vec{y}$ belongs to the *upward projection* of (the value of) $\vec{x}$, while $\vec{y}\in\downarrow(\vec{x})$ holds if and only if (the value of) $\vec{y}$ belongs to the *downward projection* of (the value of) $\vec{x}$.

Granular displacements allow one to express conditions on the belonging of an instant to the projection of another one. For instance, the constraint that $\vec{y}$ must belong to the downward projection of $\vec{x}$ is expressed by the atomic formula $\vec{y}\in\downarrow(\vec{x})$. Moreover, existential and universal quantifications under projection can be used to state that there exists $\vec{y}$ belonging to the downward projection of $\vec{x}$ such that a formula ϕ is true at $\vec{y}$, as well as to state that ϕ is true at each $\vec{y}$ belonging to the downward projection of

$\vec{x}$ (restricted quantification). These formulae take the forms $\exists \vec{\mathtt{y}}(\vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}}) \wedge \phi(\vec{\mathtt{y}}))$ and $\forall \vec{\mathtt{y}}(\vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}}) \rightarrow \phi(\vec{\mathtt{y}}))$, respectively.

The relations $\uparrow$ and $\downarrow$ can be specialized to restrict upward and downward projections to a specific domain. For each domain $\vec{T}^j$, the restriction of upward (resp. downward) projection to $\vec{T}^j$ denoted by $\uparrow^j \subseteq \vec{\mathcal{T}} \times \vec{T}^j$ (resp. $\downarrow_j \subseteq \vec{\mathcal{T}} \times \vec{T}^j$) is defined as follows (according to the set notation):

$$\vec{\mathtt{y}} \in \uparrow^{\mathtt{j}}(\vec{\mathtt{x}}) \text{ iff } \vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}}) \wedge \vec{\mathtt{y}} \in' \vec{\mathtt{T}}^{\mathtt{j}} \text{ (resp. } \vec{\mathtt{y}} \in \downarrow_{\mathtt{j}}(\vec{\mathtt{x}}) \text{ iff } \vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}}) \wedge \vec{\mathtt{y}} \in' \vec{\mathtt{T}}^{\mathtt{j}}).$$

4.2.14. PROPOSITION. *For each $\vec{x} \in \vec{\mathcal{T}}$ and $1 \leq j \leq n$, there exists at most one vector $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in \uparrow^j(\vec{x})$. More precisely, if $\vec{x} \in' \vec{T}^i$ and $j \leq i$, then there exists one and only one vector $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in \uparrow^j(\vec{x})$, while, if $\vec{x} \in' \vec{T}^i$ and $i < j$, then there are no vectors $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in \uparrow^j(\vec{x})$. Moreover, for each $\vec{x} \in \vec{\mathcal{T}}$ and $1 \leq j \leq n$, if $\vec{x} \in' \vec{T}^i$ and $i \leq j$, then there exist $cf_{i,j}$ vectors $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in \downarrow_j(\vec{x})$, where $cf_{i,i} = 1$ and, for $j > i$, $cf_{i,j} = cf_{i,i+1} \cdot \ldots \cdot cf_{j-1,j}$, while if $\vec{x} \in' \vec{T}^i$ and $j < i$, then there exist no vectors $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in \downarrow_j(\vec{x})$.*

The following example gives a natural explanation of the proposed structure for time granularity in terms of specialized upward and downward projections.

4.2.15. EXAMPLE. Consider a temporal universe consisting of hours, minutes, and seconds, and let $\vec{x}$ be the second represented by the triplet $(4011, 15, 2)$. The prefix of $\vec{x}$ with respect to $\vec{T}^1$, $\uparrow^1(\vec{x})$, is 4011, its prefix with respect to $\vec{T}^2$, $\uparrow^2(\vec{x})$, is $(4011, 15)$, the minimal extension of $\uparrow^1(\vec{x})$ on $\vec{T}^2$, $\downarrow_2(\uparrow^1(\vec{x}))$, is $(4011, 0)$, the minimal extension of $\uparrow^2(\vec{x})$ on $\vec{T}^3$, $\downarrow_3(\uparrow^2(\vec{x}))$, is $(4011, 15, 0)$. It is immediate to see that 15 and 2 can actually be interpreted as nested displacements.

Specializations of upward and downward projections also allow us to define a relation of *temporal ordering* $\vec{\lesssim}$ over the temporal universe $\vec{\mathcal{T}}$ based on the local orderings $\vec{\lesssim}_1, \ldots, \vec{\lesssim}_n$ over $\vec{T}^1, \ldots, \vec{T}^n$.

4.2.16. DEFINITION. (Temporal ordering and equivalence) A *temporal ordering* over $\vec{\mathcal{T}}$ is binary relation $\vec{\lesssim}$ such that, for each pair of vector variables $\vec{\mathtt{x}}, \vec{\mathtt{y}}$,

$$\vec{\mathtt{x}} \vec{\lesssim} \vec{\mathtt{y}} \text{ iff } \exists \mathtt{i}, \mathtt{j}(\vec{\mathtt{x}} \in' \vec{\mathtt{T}}^{\mathtt{i}} \wedge \vec{\mathtt{y}} \in' \vec{\mathtt{T}}^{\mathtt{j}} \wedge (\vec{\mathtt{x}} \vec{\lesssim}_{\mathtt{i}} \vec{\mathtt{y}} \vee \vec{\mathtt{x}} \vec{\lesssim}_{\mathtt{j}} \vec{\mathtt{y}})),$$

where the right-hand side formula is the usual shorthand. On the basis of $\vec{\lesssim}$, it is immediate to define a binary relation $\vec{\simeq}$ of *temporal equivalence* such that, for each pair of vector variables $\vec{\mathtt{x}}, \vec{\mathtt{y}}$, $\vec{\mathtt{x}} \vec{\simeq} \vec{\mathtt{y}}$ holds if and only if $\vec{\mathtt{x}} \vec{\lesssim} \vec{\mathtt{y}} \wedge \vec{\mathtt{y}} \vec{\lesssim} \vec{\mathtt{x}}$ holds.

Remark. The relation of temporal equivalence induces as many classes of equivalent vectors (*clusters*) as the vectors belonging to the coarsest domain are. In particular, it puts in the same class a vector and all its extensions. As a consequence, it may happen that vectors that are locally ordered become members of the same class, that is, vectors that are temporally distinguishable with respect to the temporal domain they finely belong to become temporally indistinguishable with respect to coarser domains.

Just like local ordering relations, local congruence relations can be lifted to the temporal universe $\vec{\mathcal{T}}$.

4.2.17. Definition. (Temporal congruence) For each natural number d, a *temporal congruence* over $\vec{\mathcal{T}}$ is binary relation $\vec{\equiv}_d$ such that, for each pair of vector variables $\vec{\mathtt{x}}, \vec{\mathtt{y}}$,

$$\vec{\mathtt{x}}\vec{\equiv}_{\mathtt{d}}\vec{\mathtt{y}} \text{ iff } \exists \mathtt{i},\mathtt{j}(\vec{\mathtt{x}} \in' \vec{\mathtt{T}}^{\mathtt{i}} \wedge \vec{\mathtt{y}} \in' \vec{\mathtt{T}}^{\mathtt{j}} \wedge (\vec{\mathtt{x}}\vec{\equiv}_{\mathtt{i},\mathtt{d}}\vec{\mathtt{y}} \vee \vec{\mathtt{x}}\vec{\equiv}_{\mathtt{j},\mathtt{d}}\vec{\mathtt{y}})).$$

Finally, let us consider the operation of *metric displacement*. It can be defined in terms of the n local functions $\vec{+}_{\mathtt{1}}\mathtt{1}, \dots, \vec{+}_{\mathtt{n}}\mathtt{1}$ as follows.

4.2.18. Definition. (Metric displacement) A temporal successor function $\vec{+}1$ is defined over $\vec{\mathcal{T}}$ such that, for each $\vec{\mathtt{x}}$, whose value belongs to $\vec{\mathcal{T}}$,

$$\vec{\mathtt{y}} = \vec{\mathtt{x}}\vec{+}\mathtt{1} \text{ iff } \exists \mathtt{i}(\vec{\mathtt{x}} \in' \vec{\mathtt{T}}^{\mathtt{i}} \wedge \vec{\mathtt{y}}\vec{=}_{\mathtt{i}}\vec{\mathtt{x}}\vec{+}_{\mathtt{i}}\mathtt{1}),$$

where the formula on the right-hand side is the usual shorthand.

It is clear that, as long as we are interested in supporting the basic functionalities of MLTL, a proper fragment of $\mathcal{L}^2_{nLM}$ is sufficient, including (uninterpreted) unary predicate symbols, the constant symbol $\vec{0}$, the unary function symbol $\vec{+}\mathtt{1}$ (metric displacement), the unary predicate symbols $\vec{\mathtt{T}}^{\mathtt{1}}, \dots, \vec{\mathtt{T}}^{\mathtt{n}}$ (contextualizations), the binary relational symbols $\vec{\leq}$ (temporal ordering), $\uparrow$ and $\downarrow$ (granular displacements), $\vec{\equiv}_2, \vec{\equiv}_3, \dots$ (temporal congruences), and quantification over individual variables and uninterpreted unary predicate symbols.

In conclusion, we point out that the theory of finitely-layered metric temporal structures does not impose any constraint on the relationships among the truth values of free predicate symbols with respect to the different domains. As an example, it may happen that a given predicate $\mathtt{p}$ is true with respect to some (all) instants of $\vec{T}^i \setminus T^{i+1}$ and false with respect to all instants of $T^j \setminus T^{j+1}$, with $1 \leq i < n$, $1 < j < n$, and $i < j$. This situation is described by the following example.

4.2.19. Example. Consider a temporal universe consisting of three temporal domains $\vec{T}^1, \vec{T}^2$, and $\vec{T}^3$. Assume $cf_{1,2} = 6$ and $cf_{2,3} = 3$. The proposition: "at the current instant an even number of atomic ($\vec{T}^3$) instants have passed" always holds in $\vec{T}^1 \setminus \vec{T}^2$, while it is true at every odd instant in $\vec{T}^2 \setminus \vec{T}^3$. A proposition that it is always true in $\vec{T}^1 \setminus \vec{T}^2$ and always false in $\vec{T}^2 \setminus \vec{T}^3$ can now easily be built: "at the current and next instant an even number of atomic instants have passed".

However, projection relations can be used to codify specific *consistency rules* that, given the truth value of a formula with respect to a certain domain, allow us to constrain its truth value with respect to other domains.

In Chapter 3, we introduced a downward consistency rule capturing the weakest semantics that can be attached to a formula (devoid of temporal operators) in a domain finer than the original one. We can reformulate it as follows.

4.2.20. Definition. *For any domain $\vec{T}^i$, with $1 \leq i < n$, and any vector $\vec{x} \in \vec{T}^i$, if a formula ϕ with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_m}$ holds at $\vec{x}$, then there exists a vector $\vec{y}$ such that $\vec{y} \in\downarrow_{i+1}(\vec{x})$ and ϕ holds at $\vec{y}$:*

$$\forall\vec{x}\exists\vec{y}(\vec{x} \in \vec{T}^i \rightarrow (\vec{y} \in\downarrow_{i+1}(\vec{x}) \wedge \phi(\vec{x}, p_1, \ldots, p_m) \rightarrow \phi(\vec{y}, p_1, \ldots, p_m)).$$

The iterated application of the downward consistency rule allows us to constrain the truth value of a formula ϕ with respect to any domain coarser than the domain of $\vec{x}$. As shown in Chapter 3, it is possible to prove that the downward consistency rule is interdeducible with an *upward consistency rule* stating that, for any domain $\vec{T}^i$, with $1 \leq i < n$, and any vector $\vec{x} \in \vec{T}^i$, if a formula ϕ with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_m}$ holds at all vectors $\vec{y}$ such that $\vec{y} \in\downarrow_{i+1}(\vec{x})$, then ϕ holds at $\vec{x}$. The addition of the *consistency* requirement restricts the set of interpretations to those satisfying the following conditions:

(a) if $x \in p^{\mathcal{I}}_{i,j}$, then, for all $k = j+1, \ldots n$, there exists y such that $cf_{j,k} \cdot x \leq y < cf_{j,k} \cdot (x+1)$ and $y \in p^{\mathcal{I}}_{i,k}$ (*downward consistency*);

(b) if, for all y, $cf_{j,k} \cdot x \leq y < cf_{j,k} \cdot (x+1)$ implies $y \in p^{\mathcal{I}}_{i,k}$, then $x \in p^{\mathcal{I}}_{i,j}$ (*upward consistency*).

4.3 Decidability of finitely-layered temporal structures

To prove the decidability of the theory of finitely-layered metric temporal structures T_{nLM}, we will show how to define a computable function τ which translates each sentence ϕ of the language $\mathcal{L}^2_{nLM}$ for T_{nLM} into a sentence $\tau(\phi)$ of $\mathcal{L}^2$ so that $\tau(\phi)$ is valid (satisfiable) in *S1S* if and only if ϕ is valid (satisfiable) in T_{nLM}. The translation is actually performed in two steps: we first embed finitely-layered metric temporal structures into (flat) metric temporal structures; then, we reduce metric temporal structures to *S1S* structures.

The language $\mathcal{L}^2_M$ for the theory of (flat) metric temporal structures T_M is the second-order language with uninterpreted unary predicate symbols, the constant symbol $\mathtt{0}$, the unary function symbol $\mathtt{+1}$, the binary relational symbols $\leq$ and $\equiv_2, \equiv_3, \ldots$, and quantification over individual variables and unary predicate symbols. As before, $\mathcal{L}_M$ denotes the first-order fragment of $\mathcal{L}^2_M$. We interpret $\mathcal{L}^2_M$ over the natural numbers $\mathbb{N}$, with $\leq$ being interpreted as the usual linear order, and only consider formulae without free individual variables. Let ϕ be a formula of $\mathcal{L}^2_M$ with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_m}$. As in the case of $\mathcal{L}^2$, an interpretation $\mathcal{I}$ for ϕ is given by n sets $p^{\mathcal{I}}_1, \ldots, p^{\mathcal{I}}_m \subseteq \mathbf{N}$. In such a case, $\mathbb{N}$ plays the role of the discrete temporal domain over which the predicates $\mathtt{p_1}, \ldots, \mathtt{p_n}$ take value.

In Section 4.3.1, we briefly summarize existing decidability results for real-time logics. In Section 4.3.2 we translate each sentence ϕ of $\mathcal{L}^2_{nLM}$ into a sentence $\tau_1(\phi)$ of $\mathcal{L}^2_M$; then, in Section 4.3.3 we translate each sentence ψ of $\mathcal{L}^2_M$ into a sentence $\tau_2(\psi)$ of $\mathcal{L}^2$. The function τ is obtained composing τ_1 and τ_2.

4.3.1 Decidability results for real-time logics

Real-time logics extend linear propositional temporal logic (PTL) with an explicit notion of time. PTL is provided with a notion of state (of computation), and it is interpreted over infinite sequences of (computation) states. It is widely used to specify and verify reactive and concurrent programs/systems, e.g. [79]. Qualitative timing constraints expressing safety and liveness properties of programs/systems can indeed be easily coded in PTL. As an example, a response property of the form "each p-state is followed by a q-state" is specified in PTL by the formula $\Box(p \rightarrow \Diamond q)$.

Let $\mathcal{L}^2$ be the second-order language with uninterpreted unary predicate symbols, the binary relational symbol $\leq$, and quantification over individual variables and unary predicate symbols, and $\mathcal{L}$ denote the first-order fragment of $\mathcal{L}^2$. The response property can be expressed in $\mathcal{L}$ by the formula "$\forall i(p(i) \rightarrow \exists j(i \leq j \wedge q(j)))$". PTL corresponds to a proper subset of $\mathcal{L}$, but it has the same expressive power of $\mathcal{L}$ (see [51]). $\mathcal{L}^2$ can be interpreted over the natural numbers $\mathbb{N}$, with $\leq$ interpreted as the usual linear order[1]. Let ϕ be a formula of $\mathcal{L}^2$ with free predicate symbols $p_1, \ldots, p_m$, and without free individual variables. An interpretation $\mathcal{I}$ for ϕ is given by m sets $p_1^{\mathcal{I}}, \ldots, p_m^{\mathcal{I}} \subseteq \mathbf{N}$. Alternatively, $\mathcal{I}$ can be described as an infinite sequence of states $\sigma = \sigma_0, \sigma_1, \ldots$, with $\sigma_i \subseteq \{p_1, \ldots, p_m\}$ for $i \geq 0$, such that $p_j \in \sigma_i$ if and only if $i \in p_j^{\mathcal{I}}$. The set of models of ϕ, i.e., the set of interpretations that satisfy ϕ, is denoted by $\mathcal{M}(\phi)$. PTL-formulae can be translated into $\mathcal{L}$-formulae without changing their set of models. $\mathcal{L}^2$ is essentially the language underlying the second-order theory of one successor *S1S*, because $\leq$ is definable in terms of the successor and hence inessential. Büchi connected *S1S* with finite automata over infinite words [18], and used this relationship to prove the decidability of *S1S* [19].

PTL cannot be used to specify real-time systems, because it cannot express quantitative timing constraints, such as deadlines and timing delays. To overcome this shortcoming PTL has been extended with explicit time references (Timed PTL [2]). The resulting real-time logics have explicit notions of state and time, and are interpreted over infinite sequences of *timed states*.

Real-time logics are characterized by three main 'parameters': the temporal domain, the primitive operations defined over it, and the time function that maps each state into its time. Different choices of the parameter values make the validity/satisfiability problems for real-time logics decidable or undecidable. Most real-time logics proposed in the literature cannot be decided, thus failing in establishing the proper balancing between expressiveness and decidability. Some of them recover decidability sacrificing completeness. In [2], Alur and Henzinger showed that the choice of taking $\mathbb{N}$ with linear order and congruence relations as the time theory and constraining the time function to be (at least weakly) monotonic makes real-time logics decidable. Formally, let $\mathcal{L}_T^2$ be the temporal extension of $\mathcal{L}^2$ (and $\mathcal{L}_T$ be its first-order fragment). Besides the state sort, $\mathcal{L}_T^2$ has a time sort, over which the constant 0, the successor function $+1$, the order relation $\leq$, and the

[1]Remember that over natural numbers the constant 0 and the successor function $+1$ can be derived from $\leq$ using first-order quantification as follows: $x = 0$ if and only if $\forall y(x \leq y)$ and $y = +1(x)$ if and only if $x < y \wedge \forall z(x < z \rightarrow y \leq z)$.

congruence relations $\equiv_2, \equiv_3, \ldots$ are defined. Moreover, a mapping f from states to times is given. Each interpretation $\mathcal{I}$ for $\phi \in \mathcal{L}_T^2$ can be viewed as a pair (σ, ρ) (*sequence of timed states*), where σ is an infinite sequence of states and $\rho = f^{\mathcal{I}}$. The set of models of ϕ is denoted by $\mathcal{M}_T(\phi)$. $\mathcal{L}_T^2$-formulae can be used to express properties of sequences of timed states. As an example, a bounded response time property of the form "each p-state is followed by a q-state within 1 time unit" can be expressed by the $\mathcal{L}_T$-formula "$\forall i(p(i) \rightarrow \exists j(i \leq j \wedge q(j) \wedge f(j) \leq f(i)+1))$", where $f(j) \leq f(i)+1$ holds if and only if either $f(j) = f(i)$ or $f(j) = f(i)+1$.

A formula $\phi \in \mathcal{L}_T^2$ is satisfiable (valid) if and only if ϕ is satisfied by at least one (all) sequence of timed states. The second-order theory of timed state sequences is the set of all valid $\mathcal{L}_T^2$-formulae. Timed PTL is an elementary, yet expressively complete, fragment of such a theory. Alur and Henzinger proved that this theory is decidable, by showing the finite-state character of temporal information needed to determine the truth value of a $\mathcal{L}_T^2$-formula ϕ with respect to a given interpretation $\mathcal{I}$ (information contained in $f^{\mathcal{I}}$) [2].

As an example, consider the formula expressing the bounded response time property. A sequence of timed states for this formula specifies the truth values of p and q, and the value of f, at each state $i \geq 0$. For each state i, let us denote the time difference $f^{\mathcal{I}}(i) - f^{\mathcal{I}}(i-1)$, with $f^{\mathcal{I}}(-1) = 0$, by $df^{\mathcal{I}}(i)$. Even if $df^{\mathcal{I}}$ takes value over $\mathbb{N}$, to determine the truth value of the considered formula with respect to the given interpretation $\mathcal{I}$, it suffices to know, for each state i, if $df^{\mathcal{I}}(i)$ is equal to 0, or it is equal to 1, or it is greater than or equal to 2. This allows us to model $df^{\mathcal{I}}$ by means of three monadic predicates over the state sort $Tdiff_0$, $Tdiff_1$, and $Tdiff_{\geq 2}$ only (*time-difference predicates*). A notion of *extended state sequence* for the given formula can thus be defined as a state sequence in the propositions $p, q, Tdiff_0, Tdiff_1$, and $Tdiff_{\geq 2}$ such that (i) it agrees with the original timed state sequence on p and q, and (ii) codifies constraints on the time distances between states in terms of time-difference predicates. The same technique can be used to model time-congruence relations in terms of a finite number of monadic *time-congruence predicates* $Tcong_{i,j}$ over the state sort. As a general rule, it is possible to prove that, given a formula $\phi \in \mathcal{L}_T^2$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$ for ϕ with the same underlying extended state sequence, $\mathcal{I} \in \mathcal{M}_T(\phi)$ if and only if $\mathcal{J} \in \mathcal{M}_T(\phi)$[2]. This means that the extended state sequence underlying a given interpretation $\mathcal{I}$ contains enough information to decide whether or not ϕ is true with respect to $\mathcal{I}$. Therefore, each formula ϕ can be characterized in terms of the set $\mathcal{M}_T^*(\phi)$ of the extended state sequences underlying its interpretations rather than in terms of the set $\mathcal{M}_T(\phi)$.

The main outcome of Alur and Henzinger's decidability results is the method they outline. They have proved that metric temporal information (differences and congruences over the time sort) can be modeled by means of a finite set of monadic predicates over the state sort. Their proof relies on the finite-state character of (metric) temporal information, which can be expressed as follows: each temporal property that partitions an infinite set of states (instants) into a finite set of classes can be *finitely* modeled and it is thus decidable. In the following, we generalize Alur and Henzinger's decidability results to finitely-layered

[2] The original proof is given in [2]. Corrections and remarks on this proof can be found in [89].

metric temporal structures proving that temporal contextualization and projection can be finitely modeled.

4.3.2 Flattening the finitely-layered structure

Let us define the translation function τ_1 that maps each formula $\phi \in \mathcal{L}^2_{nLM}$ into a formula $\tau_1(\phi) \in \mathcal{L}^2_M$. We preliminary replace the relations $\vec{\lesssim}, \vec{\equiv}_2, \vec{\equiv}_3, \ldots$ (as well as $\sim, \gg, \simeq, \updownarrow, \uparrow^i, \downarrow_i$) and the function $\vec{+}1$ by their definitions in terms of $\vec{\mathtt{T}}^{\mathtt{1}}, \ldots, \vec{\mathtt{T}}^{\mathtt{n}}, \vec{\lesssim}_{\mathtt{1}}, \ldots, \vec{\lesssim}_{\mathtt{n}}, \uparrow, \downarrow, \vec{\equiv}_{\mathtt{1},\mathtt{d_1}}, \ldots, \vec{\equiv}_{\mathtt{n},\mathtt{d_n}}$, and $\vec{+}_{\mathtt{1}}\mathtt{1}, \ldots, \vec{+}_{\mathtt{n}}\mathtt{1}$. Moreover, we assume without loss of generality that terms appearing in atomic formulae which are not equalities are variables (notice that this is correct in view of Proposition 4.2.11). We first define the behavior of τ_1 on terms; then, we specify its application to atomic formulae; finally, we show how to deal with quantifiers and logical connectives.

Terms of $\mathcal{L}^2_{nLM}$ are defined as follows: (i) the zero vector $\vec{\mathtt{0}}$ is a term; (ii) each vector variable $\vec{\mathtt{x}}$ is a term; (iii) if $\vec{\mathtt{t}}$ is a term, then $\vec{\mathtt{t}}\vec{+}_{\mathtt{i}}\mathtt{1}$ is a term; (iv) nothing else is a term. The translation of terms is performed by means of the following rules:

(r_1) $\tau_1(\vec{\mathtt{0}}) = \mathtt{0}$; (r_2) $\tau_1(\vec{\mathtt{x}}) = \mathtt{x}$; (r_3) $\tau_1(\vec{\mathtt{t}}\vec{+}_{\mathtt{i}}\mathtt{m}) = \tau_1(\vec{\mathtt{t}}) + \mathtt{m} \cdot cf_{\mathtt{i,n}}$,

where $\vec{+}_{\mathtt{i}}\mathtt{m}$ denotes m superpositions of $\vec{+}_{\mathtt{i}}\mathtt{1}$.

Once the preliminary replacements have been performed, atomic formulae of $\mathcal{L}^2_{nLM}$ can only take one of the following forms:

1. $\vec{\mathtt{t}}_1 = \vec{\mathtt{t}}_2$ (*term equality*);
2. $\vec{\mathtt{x}} \in \vec{\mathtt{T}}^{\mathtt{i}}$ (*$\vec{x}$ belongs to $\vec{T}^i$*);
3. $\vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})$ (*$\vec{y}$ belongs to the upward projection of $\vec{x}$*);
4. $\vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}})$ (*$\vec{y}$ belongs to the downward projection of $\vec{x}$*);
5. $\vec{\mathtt{x}}\vec{\lesssim}_{\mathtt{i}}\vec{\mathtt{y}}$ (*$\vec{x} \in \vec{T}^i$ does not follow $\vec{y} \in \vec{T}^i$*);
6. $\vec{\mathtt{x}} \equiv_{\mathtt{i,d}} \vec{\mathtt{y}}$ (*$\vec{x} \in \vec{T}^i$ is congruent modulo-d with $\vec{y} \in \vec{T}^i$*);
7. $\mathtt{p_k}(\vec{\mathtt{x}})$ (*p_k, with $1 \le k \le m$, holds in $\vec{x}$*).

In the translation τ_1 of a sentence ϕ, each vector variable $\vec{\mathtt{x}}$, occurring in ϕ, will be replaced by $n+1$ variables $\mathtt{x_1}, \ldots, \mathtt{x_n}, \mathtt{x}$, where $\mathtt{x_1}, \ldots, \mathtt{x_n}$ represent the starting points of the time intervals of $\mathbf{N}$ to which $\mathtt{x}$ may belong. Intuitively, $\mathtt{x_1}, \ldots, \mathtt{x_n}$ represent the projections of the (absolute positions of the) components of $\vec{\mathtt{x}}$. Whatever the formula ϕ is, $\mathtt{x_1}, \ldots, \mathtt{x_n}, \mathtt{x}$ must satisfy the following constraints:

(a) for $i = 1, \ldots, n-1$, $\mathtt{x_i} \equiv_{cf_{\mathtt{i,n}}} \mathtt{0}$;
(b) for $i = 1, \ldots, n-1$, $\mathtt{x_i} \le \mathtt{x_{i+1}} < \mathtt{x_i} + cf_{\mathtt{i,n}}$;
(c) for $i = 1, \ldots, n$, $\mathtt{x_i} \le \mathtt{x} \rightarrow \mathtt{x} < \mathtt{x_i} + cf_{\mathtt{i,n}}$;
(d) $\mathtt{x_1} \le \mathtt{x}$,

where $cf_{i,n} = cf_{i,i+1} \cdot \ldots \cdot cf_{n-1,n}$ is the conversion factor between $\vec{T}^i$ and $\vec{T}^n$.

The first two conditions codify basic properties of temporal structures: (a) says that the time instants of $\vec{T}^i$ are encoded by intervals starting at $k \cdot cf_{i,n}$ and ending at $(k+1) \cdot cf_{i,n}$, for $k = 0, 1, \ldots$; (b) guarantees that the intervals starting at $x_1, \ldots, x_n$, are ordered by inclusion according to granularity. For $i \geq 1$, (c) will enforce $\vec{x} \in \vec{T}^i$ to be equivalent to $x_i \leq x$. Accordingly, (d) expresses the fact that, for every $\vec{x}$, $\vec{x} \in \vec{T}^1$.

For every vector variable $\vec{\mathtt{x}}$, the formula $\xi(\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{x})$ defined as:

$$\bigwedge_{\mathtt{i}=1}^{\mathtt{n}-1} (\mathtt{x}_\mathtt{i} \equiv_{cf_{\mathtt{i},\mathtt{n}}} 0 \wedge \mathtt{x}_\mathtt{i} \leq \mathtt{x}_{\mathtt{i}+1} < \mathtt{x}_\mathtt{i} + cf_{\mathtt{i},\mathtt{n}}) \wedge \bigwedge_{\mathtt{i}=1}^{\mathtt{n}} (\mathtt{x}_\mathtt{i} \leq \mathtt{x} \rightarrow \mathtt{x} < \mathtt{x}_\mathtt{i} + cf_{\mathtt{i},\mathtt{n}}) \wedge \mathtt{x}_1 \leq \mathtt{x},$$

will be introduced by the translation in order to guarantee (a)–(d) to hold. Since each individual variable $\vec{\mathtt{x}}$ occurring in ϕ is quantified, $\xi(\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{x})$ will be introduced during the translation of $\forall \vec{\mathtt{x}}$ or of $\exists \vec{\mathtt{x}}$ to constrain the relationships among $\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{x}$.

Term equality. In view of Proposition 4.2.11, atomic formulae of the form (1) expressing term equality are translated as follows:

$$\tau_1(\vec{\mathtt{t}}_1 = \vec{\mathtt{t}}_2) \;=\; \tau_1(\vec{\mathtt{t}}_1) = \tau_1(\vec{\mathtt{t}}_2)$$

Contextualizations. Atomic formulae of the form (2) constrain (the value of) $\vec{\mathtt{x}}$ to belong to a specific domain $\vec{T}^i$ of the temporal universe. The application of τ_1, together with condition (c) in $\xi(\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{x})$, restricts the set of admissible values for $\mathtt{x}$ to the interval $[x_i, x_i + cf_{i,n})$.

Accordingly, the translation of $\vec{\mathtt{x}} \in \vec{\mathtt{T}}^\mathtt{i}$, with $1 \leq i \leq n$, is defined as follows:

$$\tau_1(\vec{\mathtt{x}} \in \vec{\mathtt{T}}^\mathtt{i}) \;=\; \mathtt{x}_\mathtt{i} \leq \mathtt{x}$$

It follows from (c) that $x_j \leq x$, for $j = 1, \ldots, i-1$. Moreover, in view of the above defined translation of contextualizations and of the compositionality of τ_1 with respect to negations, the translation of the *fine* membership of $\vec{\mathtt{x}}$ to $\vec{\mathtt{T}}^\mathtt{i}$, will result in $x_i \leq x$ and $x < x_{i+1}$.

Upward and downward projections. Atomic formulae of forms (3) and (4) relate (the values of) $\vec{\mathtt{x}}, \vec{\mathtt{y}}$ possibly belonging to different domains. Their translation is more complex, because neither the domain of $\vec{\mathtt{x}}$ nor the domain of $\vec{\mathtt{y}}$ are known in advance, and therefore the translation must encompass all possible cases. First, (the value of) $\vec{\mathtt{x}}$ can belong to any domain. Moreover, if (the value of) $\vec{\mathtt{x}}$ belongs to $\vec{T}^i$ and (the value of) $\vec{\mathtt{y}}$ belongs to its upward (resp. downward) projection, then (the value of) $\vec{\mathtt{y}}$ can belong to any domain $\vec{T}^j$ coarser (resp. finer) than $\vec{T}^i$. The translations of $\vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})$ and $\vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}})$ are therefore defined as follows:

$$\begin{aligned}
\tau_1(\vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})) &= \exists \mathtt{i}, \mathtt{j}(\mathtt{x}_\mathtt{i} \leq \mathtt{x} < \mathtt{x}_{\mathtt{i}+1} \wedge \mathtt{y}_\mathtt{j} \leq \mathtt{y} < \mathtt{y}_{\mathtt{j}+1} \wedge \mathtt{j} \leq \mathtt{i} \wedge \mathtt{x}_\mathtt{j} = \mathtt{y}_\mathtt{j}), \\
\tau_1(\vec{\mathtt{y}} \in \downarrow(\vec{\mathtt{x}})) &= \exists \mathtt{i}, \mathtt{j}(\mathtt{x}_\mathtt{i} \leq \mathtt{x} < \mathtt{x}_{\mathtt{i}+1} \wedge \mathtt{y}_\mathtt{j} \leq \mathtt{y} < \mathtt{y}_{\mathtt{j}+1} \wedge \mathtt{i} \leq \mathtt{j} \wedge \mathtt{x}_\mathtt{i} = \mathtt{y}_\mathtt{i}),
\end{aligned}$$

where both formulae are shorthands for finite disjunctions as usual.

Local orderings and congruences. Atomic formulae of the form (5) constrain the ordering of (the values of) $\vec{\mathtt{x}}, \vec{\mathtt{y}}$ finely belonging to the same temporal domain $\vec{T}^i$. The ordering relation between $\vec{\mathtt{x}}$ and $\vec{\mathtt{y}}$ is translated into an ordering relation between the starting points of the corresponding intervals:

$$\tau_1(\vec{\mathtt{x}} \vec{\leq}_{\mathtt{i}} \vec{\mathtt{y}}) \;=\; \mathtt{x}_{\mathtt{i}} \leq \mathtt{y}_{\mathtt{i}}.$$

Atomic formulae of the form (6) constrain (the values of) $\vec{\mathtt{x}}, \vec{\mathtt{y}}$ finely belonging to the same temporal domain $\vec{T}^i$ to belong to the same modulo-d congruence class with respect to $\vec{T}^i$. The translation constrains the starting points of the corresponding intervals to belong to the same modulo-$(d \cdot cf_{in})$ congruence class with respect to $\mathbf{N}$:

$$\tau_1(\vec{\mathtt{x}} \equiv_{\mathtt{i,d}} \vec{\mathtt{y}}) \;=\; \mathtt{x}_{\mathtt{i}} \equiv_{\mathtt{d}\cdot cf_{\mathtt{i,n}}} \mathtt{y}_{\mathtt{i}}$$

Remark. We assume that $\vec{+}_i 1, \vec{\leq}_i$, and $\vec{\equiv}_{i,d}$ are applied to variables of the proper type. Notice, however, that we do not need to check the domains of the arguments of $\vec{+}_i 1, \vec{\leq}_i$, and $\vec{\equiv}_{i,d}$, whenever they are generated by the expansion of $\vec{+}1, \vec{\leq}$, and $\vec{\equiv}_d$, respectively. This fact guarantees that type constraints are satisfied.

Predicates. Atomic formulae of the form (7) state the truth of a predicate $\mathtt{p}_{\mathtt{k}}$ at $\vec{\mathtt{x}}$. As we have already noticed, it may happen that, for example, there exist two domains $\vec{T}^i, \vec{T}^j$, with $i < j$, and a predicate $\mathtt{p}_{\mathtt{k}}$ such that $\mathtt{p}_{\mathtt{k}}$ holds at a given $\vec{x} \in' \vec{T}^i$ and $\mathtt{p}_{\mathtt{k}}$ does not hold at any $\vec{y} \in' \vec{T}^j$ such that $\vec{y} \in' \downarrow_j(\vec{x})$. As a consequence, for each predicate symbol $\mathtt{p}_{\mathtt{k}}$ we need to introduce n distinct predicate symbols $\mathtt{p}_{\mathtt{k},1}, \ldots \mathtt{p}_{\mathtt{k,n}}$ to model the truth of $\mathtt{p}_{\mathtt{k}}$ with respect to the sets $\vec{T}^1 \setminus \vec{T}^2, \ldots, \vec{T}^n$, respectively.[3]

Besides replacing the predicate symbol $\mathtt{p}_{\mathtt{k}}$ by the n predicate symbols $\mathtt{p}_{\mathtt{k},1}, \ldots \mathtt{p}_{\mathtt{k,n}}$, the translation states that there exists an index i such that $\mathtt{x}$ is greater than or equal to $\mathtt{x}_{\mathtt{i}}$ and $\mathtt{p}_{\mathtt{k,i}}$ holds at $\mathtt{x}$:

$$\tau_1(\mathtt{p}_{\mathtt{k}}(\vec{\mathtt{x}})) \;=\; \exists \mathtt{i}(\mathtt{x}_{\mathtt{i}} \leq \mathtt{x} \wedge \mathtt{p}_{\mathtt{k,i}}(\mathtt{x})),$$

where the translation is a shorthand for a finite disjunction.

Quantifiers and logical connectives. To generalize the translation function to any $\mathcal{L}^2_{nLM}$ sentence, we must define its behavior on quantifiers and logical connectives. Each quantification of individual variables $\forall \vec{\mathtt{x}}$ (resp. $\exists \vec{\mathtt{x}}$) is split into n quantifications $\forall \mathtt{x}_1, \ldots, \forall \mathtt{x}_{\mathtt{n}}$ (resp. $\exists \mathtt{x}_1, \ldots, \exists \mathtt{x}_{\mathtt{n}}$). Moreover, a nested existential quantification of the variable $\mathtt{x}$ is added. Finally, the formula $\xi(\mathtt{x}_1, \ldots, \mathtt{x}_{\mathtt{n}}, \mathtt{x})$ is inserted to restrict the set of admissible values for $\mathtt{x}_1, \ldots, \mathtt{x}_{\mathtt{n}}, \mathtt{x}$.

The translation of quantified formulae is thus defined as follows:

$$\begin{aligned}\tau_1(\forall \vec{\mathtt{x}} \phi) &= \forall \mathtt{x}_1, \ldots, \forall \mathtt{x}_{\mathtt{n}} \exists \mathtt{x}((\xi(\mathtt{x}_1, \ldots, \mathtt{x}_{\mathtt{n}}, \mathtt{x}) \rightarrow \tau_1(\phi)),\\ \tau_1(\exists \vec{\mathtt{x}} \phi) &= \exists \mathtt{x}_1, \ldots, \exists \mathtt{x}_{\mathtt{n}} \exists \mathtt{x}(\xi(\mathtt{x}_1, \ldots, \mathtt{x}_{\mathtt{n}}, \mathtt{x}) \wedge \tau_1(\phi)).\end{aligned}$$

[3] The addition of the consistency rule would make such a splitting of $\mathtt{p}_{\mathtt{k}}$ unnecessary.

Remark. The translation of quantifications over individual variables provides us with the set of all variables that can possibly occur in the translation of the formula in their scope. Which ones of these variables will actually come into play in the translation of the quantified formula depends on the contextualizations contained in the formula (if any).

Each quantification of predicate variables $\exists \mathtt{p_k}$ (resp. $\forall \mathtt{p_k}$) is split into n quantifications $\exists \mathtt{p_{k,1}}, \dots, \exists \mathtt{p_{k,n}}$ (resp. $\forall \mathtt{p_{k,1}}, \dots, \forall \mathtt{p_{k,n}}$). The corresponding translation of quantified formulae (existential case) is defined as follows:

$$\tau_1(\exists \mathtt{p_k} \phi) \;=\; \exists \mathtt{p_{k,1}}, \dots, \exists \mathtt{p_{k,n}} \tau_1(\phi).$$

Finally, the translation distributes over the logical connectives.

The definition we have adopted for the validity of a given predicate $\mathtt{p_k}$ on a given vector $\vec{\mathtt{x}}$ hides an existential quantifier ranging over the components of $\vec{\mathtt{x}}$ ($\mathtt{p_k}(\vec{\mathtt{x}})$ holds if and only if for some j, $\mathtt{p_k}$ holds on the j-th component of $\vec{\mathtt{x}}$). It is useful to compare the translations of $\mathtt{p_k}(\vec{\mathtt{x}})$ and $\neg\mathtt{p_k}(\vec{\mathtt{x}})$ to see how τ_1 deals with the different strength of positive and negative assertions. In the first case, the resulting formula says that there exists i such that $\mathtt{p_{k,i}}$ holds at $\mathtt{x}$ and $\mathtt{x_i} \leq \mathtt{x}$; in the second case, the resulting formula says that, for all i, either $\mathtt{x} < \mathtt{x_i}$ or $\mathtt{p_{k,i}}$ does not hold at $\mathtt{x}$ (or both). Therefore, the only way to say that there exists i such that $\neg\mathtt{p_{k,i}}$ holds at $\mathtt{x}$ and $\mathtt{x_i} \leq \mathtt{x}$ is replacing $\neg\mathtt{p_k}(\vec{\mathtt{x}})$ by $\mathtt{nonp_k}(\vec{\mathtt{x}})$ in the formula $\neg\mathtt{p_k}(\vec{\mathtt{x}})$, where $\mathtt{nonp_k}$ is a new predicate such that, for all $\vec{\mathtt{x}}$, $\mathtt{nonp_k}$ holds at $\vec{\mathtt{x}}$ if and only if $\mathtt{p_k}$ does not hold at $\vec{\mathtt{x}}$.

Let us now prove that τ_1 preserves the satisfiability (validity) of sentences of $\mathcal{L}^2_{nLM}$. By induction on formulae we prove a more general preservation result for generic formulae, instead of just sentences. We need a semantic counterpart of the translation function τ_1, mapping interpretations $\mathcal{I}$ for $\mathcal{L}^2_{nLM}$ into interpretations $\mathcal{J} = \tau_1(\mathcal{I})$ for $\mathcal{L}^2_M$.

4.3.1. Definition. Let $\mathcal{I}$ be an interpretation for $\mathcal{L}^2_{nLM}$. The interpretation $\tau_1(\mathcal{I})$ for $\mathcal{L}^2_M$ is defined as follows. For all free predicate symbols $\mathtt{p_k}$ in $\mathcal{L}^2_{nLM}$,

$$p_{k,i}^{\tau_1(\mathcal{I})} = \{x \mid \lfloor x/cf_{i,n} \rfloor \in p_{k,i}^{\mathcal{I}}\}$$

where $p_{k,i}^{\tau_1(\mathcal{I})}$ is the interpretation of the predicate $\mathtt{p_{k,i}} \in \mathcal{L}^2_M$, and $p_{k,i}^{\mathcal{I}}$ is the restriction of the interpretation of the predicate $\mathtt{p_k} \in \mathcal{L}^2_{nLM}$ to the domain $\vec{T}^i \setminus \vec{T}^{i+1}$.

In the following we prove that the only $\mathcal{L}^2_M$-interpretations we need to consider in order to check satisfiability/validity are those of the form $\tau_1(\mathcal{I})$, for some interpretation $\mathcal{I}$ for $\mathcal{L}^2_{nLM}$.

First of all, we show that for any sentence $\phi \in \mathcal{L}^2_{nLM}$ and any interpretation $\mathcal{I}$ for ϕ, $\mathcal{I}$ satisfies ϕ if and only if $\tau_1(\mathcal{I})$ satisfies $\tau_1(\phi) \in \mathcal{L}^2_M$.

4.3.2. Lemma. *Let $\mathcal{I}$ be an interpretation for the formula $\phi \in \mathcal{L}^2_{nLM}$, with free individual variables $\vec{x}^1, \dots, \vec{x}^l$. It holds that:*

$$\mathcal{I} \models \phi(\vec{\mathtt{x}}^1, \dots, \vec{\mathtt{x}}^1) \quad \textit{iff} \quad \tau_1(\mathcal{I}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1, \dots, \mathtt{x}^\mathtt{h}_\mathtt{n}, \mathtt{x}^\mathtt{h}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \dots, \vec{\mathtt{x}}^1)).$$

Proof. The proof is by induction on ϕ. The case of atomic formulae of the form (1) is straightforward. If ϕ is an atomic formula of form (2–7), it is sufficient to observe that for any satisfying assignment

$$\mathcal{I}(\vec{\mathtt{x}}) = \vec{t^1} = (t^1_1, \ldots, t^1_i)\ ,\ \mathcal{I}(\vec{\mathtt{y}}) = \vec{t^2} = (t^2_1, \ldots, t^2_j),$$

for ϕ with respect to $\mathcal{I}$, the assignment

$$\begin{array}{rcl} \tau_1(\mathcal{I})(\mathtt{x_1}) & = & t^1_1 \cdot cf_{1,n}, \\ & \ldots & \\ \tau_1(\mathcal{I})(\mathtt{x_i}) & = & \tau_1(\mathcal{I})(\mathtt{x}) = t^1_i \cdot cf_{i,n} \\ \tau_1(\mathcal{I})(\mathtt{x_{i+1}}) & = & \ldots = \tau_1(\mathcal{I})(\mathtt{x_n}) = \tau_1(\mathcal{I})(\mathtt{x_i}) + cf_{i,n}, \\ \tau_1(\mathcal{I})(\mathtt{y_1}) & = & t^2_1 \cdot cf_{1,n} \\ & \ldots & \\ \tau_1(\mathcal{I})(\mathtt{y_j}) & = & \tau_1(\mathcal{I})(\mathtt{y}) = t^2_j \cdot cf_{j,n} \\ \tau_1(\mathcal{I})(\mathtt{y_{j+1}}) & = & \ldots = \tau_1(\mathcal{I})(\mathtt{y_n}) = \tau_1(\mathcal{I})(\mathtt{y_j}) + cf_{j,n}, \end{array}$$

satisfies $\xi(\mathtt{x_1}, \ldots, \mathtt{x_n}, \mathtt{x}) \wedge \xi(\mathtt{y_1}, \ldots, \mathtt{y_n}, \mathtt{y}) \wedge \tau_1(\phi)$ with respect to $\tau_1(\mathcal{I})$.

As examples, consider the cases in which ϕ is an atom of the form either $\mathtt{p_k}(\vec{\mathtt{x}})$ or $\vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})$. In the first case, $\mathcal{I} \models \mathtt{p_k}(\vec{\mathtt{x}})$ is equivalent to say that there exists a vector $\vec{t} = (t_1, \ldots, t_i, \ldots, t_h)$ such that $t_i \in p^{\mathcal{I}}_{k,i}$; therefore, by definition of $\tau_1(\mathcal{I})$, $p^{\tau_1(\mathcal{I})}_{k,i}(t_i \cdot cf_{i,n})$ holds, and hence the formula $\xi(\mathtt{x_1}, \ldots, \mathtt{x_n}, \mathtt{x}) \wedge \exists \mathtt{i}(\mathtt{x_i} \leq \mathtt{x} \wedge \mathtt{p_{k,i}}(\mathtt{x}))$ is satisfied with respect to $\tau_1(\mathcal{I})$ by the above defined assignment. Let us now consider the case of $\phi = \vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})$. From $\mathcal{I} \models \vec{\mathtt{y}} \in \uparrow(\vec{\mathtt{x}})$, it follows that there exist $\vec{t^1} \in' T^i$ and $\vec{t^2} \in' T^j$ such that $j \leq i$ and the assignment $\mathcal{I}(\vec{\mathtt{x}}) = \vec{t^1}$, $\mathcal{I}(\vec{\mathtt{y}}) = \vec{t^2}$ satisfies ϕ. In these hypotheses, $t^2_j = \lfloor t^1_i / cf_{j,i} \rfloor$. To see that the translated formula

$$\begin{array}{l} \xi(\mathtt{x_1}, \ldots, \mathtt{x_n}, \mathtt{x}) \wedge \xi(\mathtt{y_1}, \ldots, \mathtt{y_n}, \mathtt{y}) \wedge \\ \quad \exists \mathtt{i}, \mathtt{j}(\mathtt{x_i} \leq \mathtt{x} \wedge \mathtt{x} < \mathtt{x_{i+1}} \wedge \mathtt{y_j} \leq \mathtt{y} \wedge \mathtt{y} < \mathtt{y_{i+1}} \wedge \mathtt{j} \leq \mathtt{i} \wedge \mathtt{x_j} = \mathtt{y_j}), \end{array}$$

is satisfied by the above defined assignment, recall that $\tau_1(\mathcal{I})(\mathtt{x}) = \tau_1(\mathcal{I})(\mathtt{x_i}) = t^1_i \cdot cf_{i,n}$ and $\tau_1(\mathcal{I})(\mathtt{y}) = \tau_1(\mathcal{I})(\mathtt{y_j}) = t^2_j \cdot cf_{j,n}$. It is straightforward to prove that the formula is satisfied. In particular, notice that

$$t^1_j \cdot cf_{j,n} = \lfloor t^1_i \cdot cf_{i,n} / cf_{j,n} \rfloor \cdot cf_{j,n} = \lfloor t^1_i / cf_{j,i} \rfloor \cdot cf_{j,n} = t^2_j \cdot cf_{j,n}.$$

Conversely, given a satisfying assignment

$$\begin{array}{rcl} \tau_1(\mathcal{I})(\mathtt{x_1}) & = & a_1 \\ & \vdots & \\ \tau_1(\mathcal{I})(\mathtt{x_n}) & = & a_n \\ \tau_1(\mathcal{I})(\mathtt{x}) & = & a \end{array}$$

$$\begin{aligned}\tau_1(\mathcal{I})(\mathtt{y_1}) &= b_1\\ &\vdots\\ \tau_1(\mathcal{I})(\mathtt{y_n}) &= b_n\\ \tau_1(\mathcal{I})(\mathtt{y}) &= b\end{aligned}$$

for $\xi(\mathtt{x_1},\ldots,\mathtt{x_n},\mathtt{x}) \wedge \xi(\mathtt{y_1},\ldots,\mathtt{y_n},\mathtt{y}) \wedge \tau_1(\phi)$ with respect to $\tau_1(\mathcal{I})$, a satisfying assignment for ϕ with respect to $\mathcal{I}$ can be obtained as follows. Let i and j be such that $a_i \leq a < a_{i+1}$ and $b_j \leq b < b_{j+1}$; the vectors $\vec{t^1}$ and $\vec{t^2}$ defined as

$$\vec{t^1} = (a_1/cf_{1,n},\ldots,a_i/cf_{i,n}) \ , \ \vec{t^2} = (b_1/cf_{1,n},\ldots,b_j/cf_{j,n}),$$

satisfy ϕ with respect to $\mathcal{I}$.

For existential quantification over individual variables notice that $\mathcal{I} \models \exists \vec{\mathtt{x}}^1 \phi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$ is equivalent to $\mathcal{I} \models \phi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$, and the thesis follows directly from the inductive hypothesis and the definition of τ_1.

For existential quantification on predicate variables we must show that

$$\mathcal{I} \models \exists \mathtt{p_k} \psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l}) \ \text{ iff } \ \tau_1(\mathcal{I}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \wedge \tau_1(\exists \mathtt{p_k}\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})). \qquad (*)$$

If the left-hand side of $(*)$ holds, then $\mathcal{I}^{p_k} \models \psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$ for some $\mathcal{I}^{p_k}$ extension of $\mathcal{I}$ to the predicate symbol $\mathtt{p_k}$. Hence, by inductive hypothesis, we have

$$\tau_1(\mathcal{I}^{p_k}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \wedge \tau_1(\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})),$$

from which it is easy to see that the right-hand side of $(*)$ holds.

Conversely, if the right-hand side of $(*)$ holds, then there exists an extension of the interpretation $\tau_1(\mathcal{I})$ to the predicates $\mathtt{p_{k,1}},\ldots,\mathtt{p_{k,n}}$, that we can denote by $\tau_1(\mathcal{I})^{p_k}$, such that $\tau_1(\mathcal{I})^{p_k} \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \wedge \tau_1(\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l}))$. In this case, we can conclude that $\tau_1(\mathcal{I}^{p_k}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \wedge \tau_1(\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l}))$, where $\mathcal{I}^{p_k}$ is the extension of $\mathcal{I}$ to the predicate symbol $\mathtt{p_k}$ (to see this, one can use Lemmas 4.3.4 and 4.3.6 below, and observe that $\epsilon(\pi(\tau_1(\mathcal{I})^{p_k})) = \tau_1(\mathcal{I}^{p_k})$). Now, by inductive hypothesis, $\mathcal{I}^{p_k} \models \psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$, and hence $\mathcal{I} \models \exists \mathtt{p_k}\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$.

If $\phi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$ is of the form $\neg\psi(\vec{\mathtt{x}}^1,\ldots,\vec{\mathtt{x}}^\mathtt{l})$, from the inductive hypothesis, it follows that $\mathcal{I} \models \phi$ if and only if $\tau_1(\mathcal{I}) \not\models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \ \wedge \ \tau_1(\psi(\mathtt{x}^1,\ldots,\mathtt{x}^\mathtt{l}))$, namely, $\tau_1(\mathcal{I}) \models \bigvee_{\mathtt{h}=1}^{\mathtt{l}} \neg\xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \ \vee \ \tau_1(\neg\psi(\mathtt{x}^1,\ldots,\mathtt{x}^\mathtt{l}))$. However, since $\tau_1(\mathcal{I}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1, \ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h})$, we have that the above is equivalent to

$$\tau_1(\mathcal{I}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^\mathtt{h}_1,\ldots,\mathtt{x}^\mathtt{h}_\mathtt{n},\mathtt{x}^\mathtt{h}) \ \wedge \ \tau_1(\neg\psi(\mathtt{x}^1,\ldots,\mathtt{x}^\mathtt{l})).$$

Finally, the case of conjunctions of formulae follows easily from the inductive hypothesis and the fact that τ_1 distributes over conjunctions. This concludes the proof[4]. $\dashv$

[4]Notice that, if the $\mathcal{L}^2_{nLM}$-formula ϕ is a sentence, the corresponding $\mathcal{L}^2_M$-formula is the sentence $\tau_1(\phi)$ (no free variables occur in ϕ).

On the ground of the previous result, we have that τ_1 preserves satisfiability. In order to prove that also validity is preserved, we show that for any sentence $\phi \in \mathcal{L}^2_{nLM}$ and any interpretation $\mathcal{J}$ for $\tau_1(\phi)$, there exist an interpretation $\mathcal{J}'$ for $\tau_1(\phi)$ and an interpretation $\mathcal{I}$ for ϕ such that $\mathcal{J} \models \tau_1(\phi)$ iff $\mathcal{J}' \models \tau_1(\phi)$ and $\mathcal{J}' = \tau_1(\mathcal{I})$. We will prove that, for any formula $\phi \in \mathcal{L}^2_{nLM}$, with free individual variables $\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}}$,

$$\mathcal{J} \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})) \text{ iff}$$
$$\mathcal{J}' \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})).$$

For any given interpretation $\mathcal{J}$, we build the corresponding interpretation $\mathcal{J}'$ in two steps. In the first step, we map the interpretation $\mathcal{J}$ into an interpretation $\pi(\mathcal{J})$ (projection on the starting point) defined as follows.

4.3.3. Definition. Let $\mathcal{J}$ be an interpretation for $\mathcal{L}^2_M$ satisfying the formula

$$\psi = \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})).$$

This implies that, for $1 \leq h \leq l$, there exists i such that either $1 \leq i < n$, $\mathcal{J}(\mathtt{x}^{\mathtt{h}}_{\mathtt{i}}) \leq \mathcal{J}(\mathtt{x}^{\mathtt{h}})$ and $\mathcal{J}(\mathtt{x}^{\mathtt{h}}_{\mathtt{i}+1}) > \mathcal{J}(\mathtt{x}^{\mathtt{h}})$, or $i = n$ and $\mathcal{J}(\mathtt{x}^{\mathtt{h}}) = \mathcal{J}(\mathtt{x}^{\mathtt{h}}_{\mathtt{i}})$. The interpretation $\pi(\mathcal{J})$ for ψ assigns to the free individual variables of ψ the same values as $\mathcal{J}$, and for $j = 1, \ldots i$ and $k = 1, \ldots, m$, $\pi(\mathcal{J})(\mathtt{x}^{\mathtt{h}}_{\mathtt{j}}) \in p^{\pi(\mathcal{J})}_{k,j}$ if and only if $\mathcal{J}(\mathtt{x}^{\mathtt{h}}) \in p^{\mathcal{J}}_{k,j}$.

For all the other elements x of the domain (including $\mathcal{J}(\mathtt{x}^1), \ldots \mathcal{J}(\mathtt{x}^{\mathtt{l}})$), $x \in p^{\pi(\mathcal{J})}_{k,i}$ if and only if $x \in p^{\mathcal{J}}_{k,i}$, for $k = 1, \ldots, m$ and $i = 1, \ldots, n$.

4.3.4. Lemma. *For each formula* $\bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}}))$ *and each interpretation* $\mathcal{J}$*,*

$$\mathcal{J} \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})) \quad \textit{iff}$$
$$\pi(\mathcal{J}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})).$$

The proof is straightforward and is left to the reader.

To obtain the desired interpretation, each π-interpretation $\mathcal{J}$ is then mapped into an interpretation $\epsilon(\mathcal{J})$ (expansion over the whole interval) defined as follows.

4.3.5. Definition. Let $\mathcal{J}$ be an interpretation for $\mathcal{L}^2_M$ satisfying the formula

$$\psi = \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}^{\mathtt{h}}_1, \ldots, \mathtt{x}^{\mathtt{h}}_{\mathtt{n}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})).$$

The interpretation $\epsilon(\mathcal{J})$ for ψ assigns to the free individual variables of ψ the same values as $\mathcal{J}$ and for $k = 1, \ldots, m$ and $i = 1, \ldots, n$,

$$p_{k,i}^{\epsilon(\mathcal{J})} = \{x \mid \lfloor x/cf_{i,n} \rfloor \cdot cf_{i,n} \in p_{k,i}^{\mathcal{J}}\}. \tag{4.1}$$

$\epsilon(\mathcal{J})$ is the interpretation that, for every $x \in [q \cdot cf_{i,n}, (q+1) \cdot cf_{i,n})$, sets the truth value of $p_{k,i}$ on x equal to the truth value of $p_{k,i}$ on $q \cdot cf_{i,n}$.

The following lemma holds:

4.3.6. Lemma. *For each formula $\bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}_1^{\mathtt{h}}, \ldots, \mathtt{x}_{\mathtt{n}}^{\mathtt{h}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}}))$ and each interpretation $\mathcal{J}$ for it,*

$$\pi(\mathcal{J}) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}_1^{\mathtt{h}}, \ldots, \mathtt{x}_{\mathtt{n}}^{\mathtt{h}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})) \quad \textit{iff}$$
$$\epsilon(\pi(\mathcal{J})) \models \bigwedge_{\mathtt{h}=1}^{\mathtt{l}} \xi(\mathtt{x}_1^{\mathtt{h}}, \ldots, \mathtt{x}_{\mathtt{n}}^{\mathtt{h}}, \mathtt{x}^{\mathtt{h}}) \wedge \tau_1(\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}})).$$

Moreover, there exists an interpretation $\mathcal{I}$ for $\phi(\vec{x}^1, \ldots, \vec{x}^l) \in \mathcal{L}_{nLM}^2$ such that $\tau_1(\mathcal{I}) = \epsilon(\pi(\mathcal{J}))$.

Proof. The formulae we are interested in only constrain the truth values of predicates at $x^1, \ldots, x^l$. On the other hand, from the definition of ϵ we have that, for every $p_{k,i}$, the truth-value of $\epsilon(\pi(\mathcal{J}))$ and $\pi(\mathcal{J})$ at $x^1, \ldots, x^l$ is the same, since $\pi(\mathcal{J})$ assigns the same truth values at $x^1, \ldots, x^l$ and $\lfloor x^1/cf_{i,n} \rfloor \cdot cf_{i,n}, \ldots, \lfloor x^l/cf_{i,n} \rfloor \cdot cf_{i,n}$, respectively. Therefore, the thesis follows from the definitions of π and ϵ.

Furthermore, let $\mathcal{I}$ an interpretation for $\phi(\vec{\mathtt{x}}^1, \ldots, \vec{\mathtt{x}}^{\mathtt{l}}) \in \mathcal{L}_{nLM}^2$ such that:

$$p_{k,i}^{\mathcal{I}} = \{x \mid x \cdot cf_{i,n} \in p_{k,i}^{\epsilon(\pi(\mathcal{J}))}\}.$$

It follows that $\tau_1(\mathcal{I}) = \epsilon(\pi(\mathcal{J}))$. $\dashv$

Notice that every ϵ-interpretation is a $\tau_1(\mathcal{I})$ interpretation, for some interpretation $\mathcal{I}$ for $\mathcal{L}_{nLM}^2$.

Now, our main preservation result follows from the previous lemmas.

4.3.7. Theorem. *For every sentence ϕ of $\mathcal{L}_{nLM}^2$, with free predicate symbols $p_1, \ldots, p_m$, there exists a sentence $\psi (= \tau_1(\phi))$ of $\mathcal{L}^M$, with free predicate symbols $p_{1,1}, \ldots, p_{m,n}$, such that ϕ is valid (satisfiable) in T_{nLM} if and only if ψ is valid (satisfiable) in T_M. Furthermore, if $\phi \in \mathcal{L}_{nLM}$, then $\psi \in \mathcal{L}_M$.*

Proof. On the one hand, from Lemma 4.3.2, it follows that, for any sentence $\phi \in \mathcal{L}_{nLM}^2$, if ϕ is satisfiable, then $\tau_1(\phi) \in \mathcal{L}_M^2$ is satisfiable, and, conversely, if $\tau_1(\phi)$ is valid, then ϕ is valid. On the other hand, Lemmas 4.3.4 and 4.3.6 prove that if ϕ is valid, then $\tau_1(\phi)$ is valid, and, conversely, if $\tau_1(\phi)$ is satisfiable, then ϕ is satisfiable. $\dashv$

4.3.3 Coding metric information

The second step of the translation is the mapping of $\mathcal{L}^2_M$ formulae into $\mathcal{L}^2$ ones. It is performed by a function τ_2 that reduces each formula $\psi \in \mathcal{L}^2_M$ to a formula $\tau_2(\psi) \in \mathcal{L}^2$ devoid of occurrences of the successor function and of congruence predicates. Moreover, τ_2 does not change the set of free individual variables of ϕ, so that if ϕ does not contain any free individual variable, no free individual variables occur in $\tau_2(\phi)$.

Before entering into the details of the definition of τ_2, we point out that at this stage we could simply use the same technique employed in [2] to map $\mathcal{L}^2_M$ formulae into $\mathcal{L}^2$ formulae. Even if the theory of metric temporal structures T_M does not support an explicit notion of state distinct from time[5], it can be easily reformulated in terms of a particular two-sorted second-order theory of timed state sequences whose time function is the identity function. Nevertheless, we will introduce and briefly discuss τ_2, mainly because it turns out to be a (rather elegant and) essentially *compositional* translation for our setting.

Terms of $\mathcal{L}^2_M$ are defined as follows: (i) the zero constant $\mathtt{0}$ is a term; (ii) each variable $\mathtt{x}$ is a term; (iii) if $\mathtt{t}$ is a term, then $\mathtt{t} + \mathtt{1}$ is a term; (iv) nothing else is a term. In the following, we will use $+\mathtt{n}$ as a shorthand for n superpositions of $+\mathtt{1}$. As in the case of $\mathcal{L}^2_{nLM}$-formulae, we assume without loss of generality that terms appearing in atomic formulae of $\mathcal{L}^2_M$ which are not equalities are variables. Atomic formulae are of the forms $\mathtt{t_1} = \mathtt{t_2}, \mathtt{x} \leq \mathtt{y}, \mathtt{x} \equiv_{\mathtt{d}} \mathtt{y}$, and $\mathtt{p_k}(\mathtt{t})$, where $\mathtt{t_1}, \mathtt{t_2}$ are terms, $\mathtt{x}, \mathtt{y}$ are variables, $\leq$ is the binary ordering relation, $\equiv_{\mathtt{d}}$ is a binary congruence relation, and $\mathtt{p_k}$ is an uninterpreted unary predicate symbol. Compound $\mathcal{L}^2_M$-formulae can be obtained by means of logical connectives and quantifications over individual and predicate variables. In particular, inequalities ($\neq$) and strict inequalities ($<$) can be defined in terms of $=$ and $\leq$ in the usual way.

With regard to compound $\mathcal{L}^2_M$-formulae, τ_2 distributes over quantifiers, negation, and conjunction. Therefore, we only need to define τ_2 on atomic formulae. We first consider atomic $\mathcal{L}^2_M$-formulae of the form $\mathtt{y} = \mathtt{x} + \mathtt{n}$. We will show that they can be reduced to $\mathcal{L}^2$-formulae involving first-order quantification over $n+1$ time variables and devoid of any occurrence of $+1$. Moreover, on the basis of the definition of the successor function, it is straightforward to prove that formulae of the form $\mathtt{x} + \mathtt{n} = \mathtt{y} + \mathtt{m}$, with $m, n > 0$, can be reduced either to formulae of the form $\mathtt{x} = \mathtt{y} + \mathtt{m'}$ or to formulae of the form $\mathtt{x} + \mathtt{n'} = \mathtt{y}$, with $m', n' \geq 0$ and $\mathtt{x} + \mathtt{0}$ to be read as $\mathtt{x}$. Let us start with the case $n = 1$. Let ψ be the $\mathcal{L}^2_M$-formula $\mathtt{y} = \mathtt{x} + \mathtt{1}$. The translation function τ_2 transforms it into an equivalent formula devoid of occurrences of $+1$:

$$\tau_2(\mathtt{y} = \mathtt{x} + \mathtt{1}) = \exists \mathtt{x_1}(\mathtt{x} < \mathtt{x_1} \wedge \mathtt{y} = \mathtt{x_1} \wedge \forall \bar{\mathtt{x}}(\mathtt{x} \leq \bar{\mathtt{x}} \leq \mathtt{x_1} \rightarrow (\bar{\mathtt{x}} = \mathtt{x} \vee \bar{\mathtt{x}} = \mathtt{x_1}))).$$

It is easy to generalize this transformation to any $\mathcal{L}^2_M$-formula $\mathtt{y} = \mathtt{x} + \mathtt{n}$, with $n > 1$:

$\tau_2(\mathtt{y} = \mathtt{x} + \mathtt{n}) =$

[5] It is worth noting that a differentiation between the notions of state and time can be recovered using granularity. Upward projection can indeed map two time instants which are distinct with respect to the domain they finely belong to into the same time instant of a coarser domain. With respect to the coarser domain, the original time instants can be viewed as an ordered pair of simultaneous states.

$$\exists \mathtt{x_1}, \ldots, \mathtt{x_n}(\mathtt{x} < \mathtt{x_1} < \ldots < \mathtt{x_n} \wedge \mathtt{y} = \mathtt{x_n} \wedge \forall \bar{\mathtt{x}}(\mathtt{x} \leq \bar{\mathtt{x}} \leq \mathtt{x_n} \rightarrow (\bar{\mathtt{x}} = \mathtt{x} \vee \ldots \vee \bar{\mathtt{x}} = \mathtt{x_n}))).$$

The case of atomic $\mathcal{L}^2_M$-formulae of the form $\mathtt{y} = \mathtt{n}$, where $\mathtt{n}$ stands for $\mathtt{0} + \mathtt{n}$, is analogous, and thus omitted. Equalities of the form $\mathtt{x} = \mathtt{y}$ as well as atomic formulae of the form $\mathtt{x} \leq \mathtt{y}$ and $\mathtt{p_k}(\mathtt{x})$ are left unchanged. Let us consider now atomic formulae of the form $\mathtt{x} \equiv_\mathtt{d} \mathtt{y}$. Each binary congruence relation $\equiv_\mathtt{d}$ partitions the set of time instants into d disjoint classes. For each class of time instants which are congruent modulo-d with i, with $0 \leq i \leq d-1$, τ_2 introduces a monadic predicate of the form $Tcong_{\mathtt{d,i}}$. It is defined as follows:

$$\tau_2(\mathtt{x} \equiv_\mathtt{d} \mathtt{y}) = \bigwedge_{\mathtt{i}=0}^{\mathtt{d}-1} (Tcong_{\mathtt{d,i}}(\mathtt{x}) \leftrightarrow Tcong_{\mathtt{d,i}}(\mathtt{y}))$$

where $\bigwedge_{\mathtt{i}=0}^{\mathtt{d}-1}$ denotes the usual shorthand. Since for every congruence relation $\equiv_\mathtt{d}$ the corresponding predicates $Tcong_{\mathtt{d},0}, \ldots, Tcong_{\mathtt{d,d}-1}$ are *uninterpreted* monadic predicate symbols, the following conditions must be added:

(a) for each congruence relation $\equiv_d$ (in ψ), $Tcong_{\mathtt{d},0}$ holds at time instant 0 (in $\tau_2(\psi)$);
(b) for each congruence relation $\equiv_d$, and each time instant x, there exists one and only one index i, with $0 \leq i \leq d-1$, such that $Tcong_{\mathtt{d,i}}$ holds at x;
(c) for each congruence relation $\equiv_d$, each index i, with $0 \leq i \leq d-1$, and each time instant x, if $Tcong_{\mathtt{d,i}}$ holds at time instant x, then $Tcong_{\mathtt{d,i}+1 mod \mathtt{d}}$ holds at time instant $x+1$.

Condition (a) links time-congruence predicates corresponding to different congruence relations (it provides a sort of initial synchronization); (b) and (c) link time-congruence predicates corresponding to the same congruence relation. Formally, for each congruence relation $\equiv_\mathtt{d}$ in ψ, let $\chi(Tcong_{\mathtt{d},0}, \ldots, Tcong_{\mathtt{d,d}-1})$ be the formula:

$$\begin{aligned} Tcong_{\mathtt{d},0}(\mathtt{0}) \quad \wedge \quad & \forall x(\bigvee_{i=0}^{d-1}(Tcong_{d,i}(x) \\ \wedge \quad & \bigwedge_{j \neq i} \neg Tcong_{d,j}(x)) \\ \wedge \quad & \bigwedge_{i=0}^{d-1}(Tcong_{d,i}(x) \rightarrow Tcong_{d,i+1 mod d}(x+1))) \end{aligned}$$

where the usual shorthands have been used.

The translation of $\mathcal{L}^2_M$-formulae ψ is thus defined by adding, for each distinct congruence relation $\equiv_\mathtt{d}$ occurring in ψ, the corresponding conjunct $\chi(Tcong_{\mathtt{d},0}, \ldots, Tcong_{\mathtt{d,d}-1})$. The resulting formula belongs to $\mathcal{L}^2$. Therefore, in order to prove that the validity (satisfiability) problem for $\mathcal{L}^2_M$ is decidable, we only need to show that a sentence ψ is valid (satisfiable) in T_M if and only if $\tau_2(\psi)$ is valid (satisfiable) in *S1S*.

4.3.8. THEOREM. *For every formula ψ of $\mathcal{L}^2_M$, there exists a formula θ of $\mathcal{L}^2$, which contains the additional time-congruence predicates* $\mathtt{Tcong}_{\mathtt{d_1},0}$, ..., $\mathtt{Tcong}_{\mathtt{d_1},\mathtt{d_1}-1}$, ..., $\mathtt{Tcong}_{\mathtt{d_1},0}$, ..., $\mathtt{Tcong}_{\mathtt{d_1},\mathtt{d_1}-1}$, *such that ψ is valid (satisfiable) in T_M if and only if θ is valid (satisfiable) in S1S. Furthermore, if $\psi \in \mathcal{L}_M$, then $\theta \in \mathcal{L}$.*

Take $\tau_2(\psi)$ as θ. The proof is similar to the one given in [2], and thus omitted.

On the basis of Theorems 4.3.7 and 4.3.8, we can conclude that the following holds:

4.3.9. THEOREM. *For every formula ϕ of $\mathcal{L}^2_{nLM}$, there exists a formula θ (i.e. $\tau_2(\tau_1(\phi))$) of $\mathcal{L}^2$ such that ϕ is valid (satisfiable) in T_{nLM} if and only if θ is valid (satisfiable) in S1S. Furthermore, if $\phi \in \mathcal{L}_M$, then $\theta \in \mathcal{L}$.*

Hence, from the the decidabiity of *S1S*, the decidability of T_{nLM} follows:

4.3.10. COROLLARY. *The theory of finitely-layered metric temporal structures is decidable.*

Theorem 4.3.9 actually states that metric and layered temporal structures, provided with a *finite* number of temporal domains, can be embedded (by means of a rather complex mapping) into flat metric temporal structures. This allows us to import decidability results, as well as other logical results about (sound and complete) axiomatizations and executability, from metric temporal logics to finitely-layered ones. Now, the natural question is: can we generalize this result to *ω-layered* metric temporal structures? Are there meaningful decidable theories of ω-layered metric temporal structures? If yes, can we still prove their decidability through a reduction to $S1S$?

The second part of this chapter will answer to these questions: such decidable theories exist but, to prove their decidability, we exploit engines more powerful than $S1S$.

The next two sections provide background knowledge and preliminary results about tree automata and extensions of $S1S$ that will be used to decide ω-layered metric temporal structures.

4.4 Systolic and Rabin tree automata

4.4.1 Systolic tree automata on ω-words

In this section, we give a brief description of *k-ary Systolic Tree Automata* (*k-STA* for short). A detailed systematic presentation can be found in [95, 96].

Throughout this chapter, let Σ denote a finite alphabet and let Σ^ω denote the set of *ω-words* over Σ. The symbols $\alpha, \beta, \ldots$ are used for ω-words and $L, L', \ldots$ for sets of ω-words. For an ω-word α, $\alpha(i)$, with $i \in \mathbb{N}$, denotes the i-th element of α, and $\alpha(m, n)$, with $m, n \in \mathbb{N}$, denotes the segment $\alpha(m) \ldots \alpha(n)$ of α. The symbol $\cdot$ denotes concatenation on strings. A systolic automaton consists of an infinite number of nodes which can be interpreted as memoryless processors. Nodes are linked among them and the resulting structure is an (infinite) leafless perfectly balanced k-ary tree. In order to process a word w, whose length $|w|$ is equal to k^i, the i-th level of the tree is chosen. Now, the automaton is fed in such a way that adjacent processors at level i-th are fed with adjacent symbols of w, and that the leftmost processor is fed with the first symbol of w. All the processors at level i-th synchronously output a symbol belonging to the *state alphabet* Q, according to the *input relation*. Each processor at level $(i-1)$-th receives k states output by its k children and it synchronously (with respect to processors at the same level) outputs a symbol belonging to Q according to the *transition relation*. Therefore, information flows bottom-up, in parallel and synchronously, level by level.

Let us introduce now a notion of step-wise systolic computation on ω-words. Consider an ω-word α. At each computation step, the automaton process a segment of α whose length increases by a factor of k step by step. In particular, an ω-word on the set of states Q stores at the i-th position the state q resulting from processing the prefix $\alpha(0, k^i - 1)$ of α. The state resulting from processing the next prefix $\alpha(0, k^{i+1} - 1)$ is obtained from q and from the states $q_2, \ldots, q_k$ output by the systolic automaton fed with $\alpha(k^i, 2k^i - 1), \ldots, \alpha((k-1)k^i, k^{i+1} - 1)$, respectively, according to the transition relation f. In Figure 4.1, we graphically describe the way in which a binary systolic tree automaton processes an ω-word α. The left-hand side edge of the tree structure consists of nodes associated with states obtained by processing prefixes of α whose length is a power of k. Such a sequence of states is called a *systolic run*.

Formally, a K-ary systolic tree automata is defined as follows.

4.4.1. Definition. *A* systolic automaton *is a tuple* $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$*, where*

- Σ *is the finite* input alphabet*;*
- Q *is the finite set of* states*;*
- $in \subseteq \Sigma \times Q$ *is the* input relation*;*
- $f \subseteq Q^{k+1}$ *is the* transition relation*;*
- $F \subseteq Q$ *is the set of* final states.

4.4.2. Definition. *The* finitary *computation of an automaton $\mathcal{A}$ over a (finite) word w of length k^m is a binary relation $O_{\mathcal{A}} \subseteq \Sigma^* \times Q$ recursively defined as follows:*

- *if $|w| = 1$, then $\langle w, q \rangle \in O_{\mathcal{A}}$ if and only if $\langle w, q \rangle \in in$;*
- *if $|w| = k^m$, with $m > 0$, then $\langle w, q \rangle \in O_{\mathcal{A}}$ if and only if $\langle q_1, \ldots, q_k, q \rangle \in f$, where q_i, for $1 \leq i \leq k$, is such that $\langle w_i, q_i \rangle \in O_{\mathcal{A}}$, with $|w_i| = k^{m-1}$ and $w_1 \cdot w_2 \ldots \cdot w_k = w$.*

4.4.3. Definition. *A* systolic run *of $\mathcal{A}$ on an ω-word $\alpha \in \Sigma^\omega$ is an ω-word $\sigma \in Q^\omega$ such that*

- $\langle \alpha(0), \sigma(0) \rangle \in in$*;*
- $\langle \sigma(i-1), q_2, \ldots, q_k, \sigma(i) \rangle \in f$*, with* $\langle \alpha((j-1)k^{i-1}, jk^{i-1} - 1), q_j \rangle \in O_{\mathcal{A}}$*, for* $2 \leq j \leq k$.

A systolic run σ is *successful* if and only if some state of F occurs infinitely often in σ. An automaton $\mathcal{A}$ *accepts* an ω-word α if and only if there exists a successful systolic run σ on α. The ω-language recognized by $\mathcal{A}$, denoted by $\mathcal{L}_\omega(\mathcal{A})$, is the set

$$\{\alpha \in \Sigma^\omega : \mathcal{A} \text{ accepts } \alpha\}.$$

4.4.4. Theorem. *Let $\mathcal{L}_\omega$(k-STA) be the class of ω-languages recognized by k-ary systolic tree automata. The following properties hold:*

(i) the class of regular ω-languages is strictly contained in $\mathcal{L}_\omega$(k-STA);

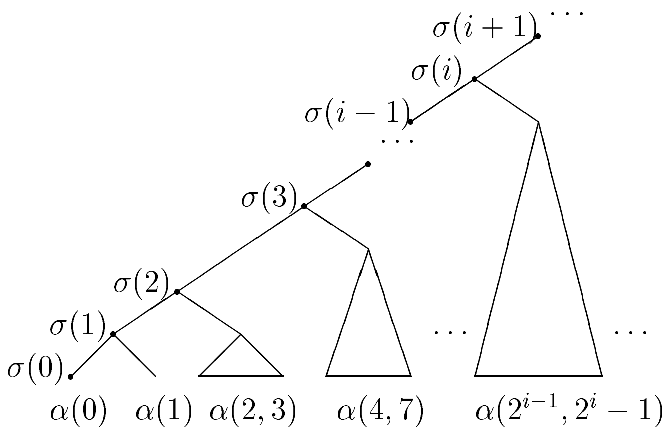

Figure 4.1: A systolic run σ on α.

(ii) the class $\mathcal{L}_\omega$(k-STA) is closed under union, intersection, projection, and complementation;

(iii) the emptiness problem for $\mathcal{L}_\omega$(k-STA) is decidable.

The proof of Theorem 4.4.4 is given in [95, 96].

4.4.5. Example. Let us consider the ω-language $L = \{a^{2^i} \cdot \{b\}^\omega : i \geq 0\}$. L is clearly non-regular and it is recognized by the (binary) systolic tree automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ defined as follows:

- $\Sigma = \{a, b\}$; $Q = \{q_1, q_2, q_3\}$; $F = \{q_3\}$;
- $in = \{\langle a, q_1 \rangle, \langle b, q_2 \rangle\}$;
- $f = \{\langle q_1, q_1, q_1 \rangle, \langle q_2, q_2, q_2 \rangle, \langle q_1, q_2, q_3 \rangle, \langle q_3, q_2, q_3 \rangle\}$.

4.4.6. Example. Another example of non-regular ω-language is $L' = \{s_0 \cdot s_1 \cdot \ldots \cdot s_i \cdot \ldots\}$, with $s_i = ba^{2^i - 1}$ for $i \geq 0$. L' is recognized by the (binary) systolic tree automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ defined as follows:

- $\Sigma = \{a, b\}$; $Q = \{q_1, q_2, q_3, q_4\}$; $F = \{q_3\}$;
- $in = \{\langle a, q_2 \rangle, \langle b, q_1 \rangle\}$;
- $f = \{\langle q_1, q_1, q_3 \rangle, \langle q_2, q_1, q_4 \rangle, \langle q_3, q_4, q_3 \rangle, \langle q_2, q_2, q_2 \rangle, \langle q_2, q_4, q_4 \rangle\}$.

In Section 4.5, we will define a proper extension of $S1S$ that allows us to transfer the above results from automata theory to logic.

4.4.2 Rabin tree automata and the theory SkS

In this section, we briefly describe Rabin tree automata, focusing on their relationships with SkS, the monadic second-order theory of k successors. An extensive presentation can be found in [118].

Given an alphabet Σ, a k-ary Σ-valued tree t is specified by its set of nodes, called the

domain of t and denoted by $dom(t)$, and by a valuation $v_t : dom(t) \mapsto \Sigma$. The domain $dom(t)$ is a nonempty subset of $\{0, \ldots, k-1\}^*$, closed under prefixes, and such that for all $w \in \{0, \ldots, k-1\}^*$ and $i, j \in \{0, \ldots, k-1\}$, if $w \cdot j \in dom(t)$ and $i < j$, then $w \cdot i \in dom(t)$. The proper prefix relation over $\{0, \ldots, k-1\}^*$ is denoted by $<_P$. A path through t is a maximal subset of $dom(t)$ linearly ordered by $<_P$.
For the sake of simplicity, in the sequel we will restrict our presentation to binary trees, and to the corresponding theory $S2S$ of two successors. Notation and results can be easily generalized to k-ary trees and to the corresponding theory SkS.

Let T^ω_Σ be the set of infinite Σ-valued binary trees (with domain $\{0,1\}^*$). Rabin tree automata operate over infinite tree, and are formally defined as follows:

4.4.7. Definition. *A* Rabin tree automaton *is a tuple* $\mathcal{A} = \langle \Sigma, Q, q_0, \Delta, \Omega \rangle$*, where*

- Σ *is the finite* input alphabet*;*
- Q *is the finite set of* states*;*
- $q_0 \in Q$ *is the* initial state*;*
- $\Delta \subseteq Q \times \Sigma \times Q \times Q$ *is the* transition relation*;*
- $\Omega = \{(L_1, U_1), \ldots, (L_n, U_n)\}$*, with* $L_i, U_i \subseteq Q$*, is a collection of* accepting pairs *of state sets.*

Unlike systolic tree automata, Rabin tree automata support a top-down computation. Given an infinite Σ-valued binary tree t, the automaton starts its computation at the root of t in the initial state q_0 and then simultaneously works down the paths of the tree level by level. The transition relation Δ determines which pairs of states (q_1, q_2) can be associated with the two children of a node, given the node's state and value. Formally, a run of a Rabin tree automaton is defined as follows.

4.4.8. Definition. *A* run *of a Rabin tree automaton* $\mathcal{A}$ *on* $t \in T^\omega_\Sigma$ *is a map* $\sigma : dom(t) \mapsto Q$ *such that*

- $\sigma(\epsilon) = q_0$*;*
- *for each* $w \in dom(t)$*,* $(\sigma(w), v_t(w), \sigma(w \cdot 0), \sigma(w \cdot 1)) \in \Delta$.

The automaton accepts the tree if there is a run built up in this way which is successful. The notion of *successful run* of a Rabin tree automaton is formally defined as follows.

4.4.9. Definition. *Let* $t \in T^\omega_\Sigma$ *and* r *be a run of a* Rabin tree automaton $\mathcal{A} = \langle \Sigma, Q, q_0, \Delta, \Omega \rangle$ *on* t*. The run* r *is* successful *if for all paths* π *through* t*, there exists* $i \in \{1, \ldots, n\}$ *such that* $In(r|\pi) \cap L_i = \emptyset$ *and* $In(r|\pi) \cap U_i \neq \emptyset$*, where* $r|\pi$ *denotes the restriction of the run* r *to the path* π *and* $In(r|\pi)$ *returns the set of states that occur infinitely many times in* $r|\pi$*.*

A tree $t \in T^\omega_\Sigma$ is *accepted* by Rabin tree automaton $\mathcal{A}$ if there exists a successful run of $\mathcal{A}$ on it. A set $T \subseteq T^\omega_\Sigma$ is *Rabin recognizable* if it consists of the trees accepted by a Rabin tree automaton.

4.4.10. Theorem. *Let* $T(\mathcal{A})$ *be the set of trees accepted by a Rabin tree automaton* $\mathcal{A}$*. The following properties hold:*

(i) the emptiness problem for Rabin tree automata is decidable;
(ii) for any Rabin tree automaton $\mathcal{A}$, there is a Rabin tree automaton $\mathcal{A}'$ recognizing $T^\omega_\Sigma \setminus T(\mathcal{A})$.

Let us consider now the logical counterpart of Rabin tree automata. We first introduce a representation of trees as model-theoretic structures. Let Σ be the alphabet $\{0,1\}^n$. A tree $t \in T^\omega_\Sigma$ can be codified by a model $\underline{t}$ of the form:

$$\underline{t} = \langle \{0,1\}^*, \epsilon, succ_0, succ_1, <_P, P_1, \ldots, P_n \rangle,$$

where $succ_0$ and $succ_1$ are the two successor functions over $\{0,1\}^*$, with $succ_0(w) = w \cdot 0$ and $succ_1(w) = w \cdot 1$, $<_P$ is the proper prefix relation over $\{0,1\}^*$, and $P_1, \ldots, P_n$ are subsets of $\{0,1\}^*$, where for each $1 \leq i \leq n$, $w \in P_i$ if and only if the i-th component of $v_t(w) = 1$.

4.4.11. Definition. *The (monadic) second-order theory of two successors $S2S$ includes individual variables $x, y, \ldots$ and first-order variables $p_1, p_2, \ldots$, ranging over elements and subsets of $\{0,1\}^*$, respectively. Terms are built up from the individual variables and the constant ϵ by applying the successor functions $succ_0$ and $succ_1$. Atomic formulae are of the forms $t = t'$, $t <_P t'$, and $p(t)$, where t, t' are terms and p is a first-order variable. Arbitrary formulae are obtained from atomic ones by using boolean connectives and the quantifier $\exists, \forall$ over both individual and first-order variables.*

Notice that ϵ and $<_P$ can actually be defined in terms of $succ_0$ and $succ_1$, and thus removed.

Let $\phi(p_1, \ldots p_n)$ be a $S2S$-formula, where at most the n first-order variables $p_1, \ldots, p_n$ occur free, and $\underline{t}$ be a tree model. We write $\underline{t} \models \phi(p_1, \ldots p_n)$ if ϕ is satisfied in $\underline{t}$, with P_i as interpretation of p_i (for $1 \leq i \leq n$). Moreover, let $T(\phi) = \{t \in T^\omega_\Sigma : \underline{t} \models \psi(p_1, \ldots p_n)\}$. If $T = T(\phi)$ for some $S2S$-formula ϕ, T is said definable in $S2S$.

The following theorem links the notions of $S2S$-definable and Rabin recognizable.

4.4.12. Theorem. *A set $T \subseteq T^\omega_\Sigma$ is definable in $S2S$ if and only if T is Rabin recognizable.*

Putting together Theorem 4.4.10 and Theorem 4.4.12, it immediately follows that:

4.4.13. Corollary. *The theory $S2S$ is decidable.*

4.5 The theory *S1S*^{k}

In Section 4.4.1, we introduced systolic ω-languages from an operational point of view. In this section, we will define a suitable (decidable) extension of the second-order theory of one successor $S1S$, called $S1S^k$, which can be viewed as the logical counterpart of the operational definition of systolic ω-languages. In the next section, we will show how $S1S^k$ can be used to decide the theory of ω-layered, k-refinable, metric temporal structures consisting of an infinite number of arbitrarily coarse temporal domains.

$S1S^k$ extends the sequential calculus $S1S$ by adding a unary function symbol $\overset{k}{\leftarrow}$, called *power function*, and a unary predicate symbol L. For any natural number $x > 0$, the power function computes the natural number $x-x'$, where x' is the least power of k (with non-null coefficient) in the k-ary representation of x. The predicate L holds for a natural number x if and only if the least power of k (with non-null coefficient) in the k-ary representation of x has coefficient $k-1$.

4.5.1. DEFINITION. *The* power function $\stackrel{k}{\leftarrow}: \mathbb{N}^+ \to \mathbb{N}$ *is such that* $y = \stackrel{k}{\leftarrow}(x)$ *if and only if*

$$\begin{aligned} x &= a_n k^n + a_{n-1}k^{n-1} + \ldots + a_m k^m, \\ &\quad \text{with } 0 \leq a_i \leq k-1,\ m \leq n \text{ and } a_m \neq 0, \text{ and} \\ y &= a_n k^n + a_{n-1}k^{n-1} + \ldots + (a_m - 1)k^m. \end{aligned}$$

The predicate L holds at x if and only if

$$\begin{aligned} x &= a_n k^n + a_{n-1}k^{n-1} + \ldots + a_m k^m, \\ &\quad \text{with } 0 \leq a_i \leq k-1,\ m \leq n \text{ and } a_m = k-1. \end{aligned}$$

It is worth noting that the predicate "*is a power of k*" can be easily expressed in terms of the power function $\stackrel{k}{\leftarrow}$ as follows: x "*is a power of k*" if and only if $0 = \stackrel{k}{\leftarrow}(x)$. Hence, $S1S^k$ is at least as expressive as the well-known extension of $S1S$ with the predicate "*is a power of k*".

Let $\mathcal{L}^2_{S1S^k}$ be the second-order language with (uninterpreted) predicate symbols, the unary function symbol $\stackrel{\mathtt{k}}{\leftarrow}$, the unary predicate symbol $\mathtt{L}$, the binary predicate symbol $\leq$, and quantification over individual variables and unary predicate symbols. We interpret $\mathcal{L}^2_{S1S^k}$ over the natural numbers $\mathbb{N}$, with $\stackrel{\mathtt{k}}{\leftarrow}$ being interpreted as the power function $\stackrel{k}{\leftarrow}$, $\mathtt{L}$ as the predicate L and $\leq$ as the usual linear ordering. $\mathcal{L}^2_{S1S^k}$ is the language underlying $S1S^k$.

In the following, we will restrict our attention to formulae devoid of free individual variables. Therefore, given a formula $\phi \in \mathcal{L}^2_{S1S^k}$, with free predicate symbols $\mathtt{p}_1, \ldots, \mathtt{p}_n$, an interpretation $\mathcal{I}$ for ϕ specifies n sets $p_1^{\mathcal{I}}, \ldots, p_n^{\mathcal{I}} \subseteq \mathbb{N}$. The interpretation $\mathcal{I}$ can be equivalently viewed as an infinite sequence of subsets $s_i \subseteq \{p_1, \ldots, p_n\}$, with $i \geq 0$, such that $p_j \in s_i$ if and only if $i \in p_j^{\mathcal{I}}$.

Let us now get a finite alphabet Σ, with $|\Sigma| = 2^n$, and define a bijection $cod : 2^{\{p_1,\ldots,p_n\}} \mapsto \Sigma$ mapping each $s \subseteq \{p_1, \ldots, p_n\}$ into a symbol $cod(s)$ of Σ. On the one hand, such a correspondence allows us to represent any interpretation $\mathcal{I}$ for ϕ by an ω-word α over Σ such that $\alpha(i) = a$, with $a \in \Sigma$, if and only if $a = cod(\{p_{i_1}, \ldots, p_{i_m}\})$ (with $m \leq n$ and $p_{i_j} \in \{p_1, \ldots, p_n\}$, for $1 \leq j \leq m$), $i \in p^{\mathcal{I}}$ for all $p \in \{p_{i_1}, \ldots, p_{i_m}\}$, and $i \notin p^{\mathcal{I}}$ for all $p \in \{p_1, \ldots, p_n\} \setminus \{p_{i_1}, \ldots, p_{i_m}\}$. On the other hand, each ω-word α naturally induces an interpretation as formally stated by the following definition.

4.5.2. DEFINITION. *An ω-word $\alpha \in \Sigma^\omega$* induces *an interpretation $\underline{\alpha}$ (the* canonical interpretation under α*) having the form*

$$\underline{\alpha} = \langle \mathbb{N}, \stackrel{k}{\leftarrow}, L, \leq, (Q_a)_{a\in\Sigma} \rangle,$$

where $\langle \mathbb{N}, \leq \rangle$ *is the structure of natural numbers with the usual linear ordering,* $\overset{k}{\leftarrow}$ *is the power function,* $L \subseteq \mathbb{N}$ *such that for all* $x \in \mathbb{N}$, $x \in L$, $\mathtt{L}$ *holds at* x, *and* Q_a *(for* $a \in \Sigma$*) is the set* $\{i \in \mathbb{N} : \alpha(i) = a\}$.

For any given formula ϕ, with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_n}$ (or, equivalently, for any finite alphabet Σ, with $|\Sigma| = 2^n$), we can thus define the "interpreted" theory $S1S^k_\Sigma$, whose language is defined as follows.

4.5.3. Definition. *For any given alphabet* Σ, *the language for the interpreted theory* $S1S^k_\Sigma$ *is built up as follows:*

- terms *are freely constructed from first-order variables by application of the power function* $\overset{k}{\leftarrow}$*;*
- atomic formulae *are of the form* $L(t)$, $X(t)$, $Q_a(t)$ *(for* $a \in \Sigma$*), and* $t \leq t'$, *where* t *and* t' *are terms and* X *is a predicate variable;*
- $S1S^k_\Sigma$-formulae *are freely constructed from atomic formulae by using the usual boolean connectives, and the quantifiers* $\exists$ *and* $\forall$ *acting on both individual and predicate variables.*

Let ϕ be a $S1S^k_\Sigma$-formula. We write $\phi(p_1, \ldots p_m)$ to indicate that at most the m predicate variables $p_1, \ldots, p_m$ occur free in ϕ. Formulae without free variables are called *sentences*. Given $\alpha \in \Sigma^\omega$ and a sentence ϕ, we write $\underline{\alpha} \models \phi$ if ϕ is satisfied in $\underline{\alpha}$ (the notion "ϕ is satisfied in $\underline{\alpha}$" is standard and is not formally defined here). The *ω-language defined by* a $S1S^k_\Sigma$- sentence ϕ is $\mathcal{L}(\phi) = \{\alpha \in \Sigma^\omega : \underline{\alpha} \models \phi\}$.

Let us prove now that the extended calculus $S1S^k$ can be viewed as the logical counterpart of systolic ω-languages. This result is a (non-trivial) generalization of the one given in [95, 96] for $k = 2$. Some preliminary definitions are needed.

First, notice that the power function, together with the predicate L, allow us to associate a set of natural numbers, structured as a perfectly balanced k-ary tree, with each natural number. Actually, we will associate such trees only with natural numbers having 1 as the least non-null coefficient in their k-ary representation (notice that for $k = 2$, this condition is true for every natural number (greater than 0), and thus a binary tree is associated with each natural number).

Formally, given a natural number

$$y = a_n k^n + a_{n-1} k^{n-1} + \ldots + a_m k^m, \text{ with } m > 0 \text{ and } a_m = 1$$

the j-th child x of y, with $0 \leq j \leq k-1$, is the natural number

$$x = a_n k^n + a_{n-1} k^{n-1} + \ldots + j k^m + 1 k^{m-1}.$$

As an example, the trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$) are shown in Figure 4.2, where we assumed that the 0-th child is the leftmost child in the representation of the tree (hereafter, we will always make this assumption). It is worth noting that the structure imposed in this way on the considered subset of natural numbers

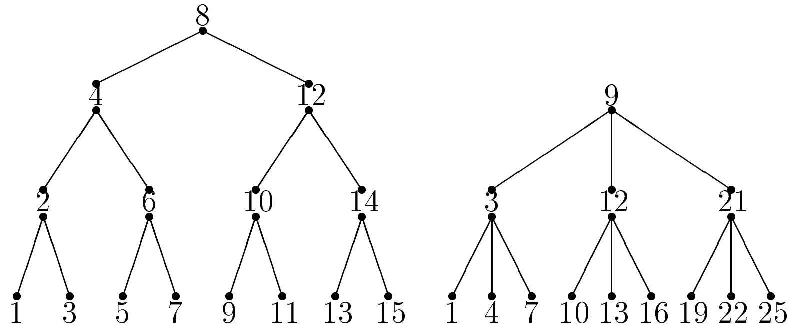

Figure 4.2: The binary trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$).

is analogous to the "structure" of a systolic computation, i.e. a systolic run (cf. Figure 4.1), on an ω-word.

By exploiting the power function $\overset{k}{\leftarrow}$ and the predicate L, we are able to define $k-1$ auxiliary predicates L_j, with $1 \leq j \leq k-1$, such that $L_j(x)$ holds if and only if j is the least non-null coefficient in the k-ary representation of x (notice that $L_{k-1} = L$). For $j = 1, \ldots, k-1$, $\mathtt{L_j(x)}$ can be defined as follows:

$$\mathtt{L_j(x)} \text{ iff } \exists \mathtt{x_{k-1}}, \ldots, \mathtt{x_j}(\mathtt{L(x_{k-1})} \wedge \mathtt{x} = \mathtt{x_j} \wedge \bigwedge_{\mathtt{i=j+1}}^{\mathtt{k-1}} \mathtt{x_{i-1}} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x_i})). \tag{4.2}$$

It is not difficult to see that for any pair of natuaral numbers x and y such that both $L_1(x)$ and $L_1(y)$ hold, x is the 0-th child of y ($y \overset{0}{\to} x$ for short) if and only if

$$x = max\{w : w < y \wedge \overset{k}{\leftarrow} (w) = \overset{k}{\leftarrow} (y)\}.$$

Therefore, $y \overset{0}{\to} x$ can be defined as follows:

$$\begin{gathered}\mathtt{L_1(y)} \wedge \mathtt{L_1(x)} \wedge \mathtt{x} < \mathtt{y} \wedge \overset{\mathtt{k}}{\leftarrow} (\mathtt{y}) = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x}) \wedge \\ \forall \mathtt{w}((\mathtt{L_1(w)} \wedge \mathtt{w} < \mathtt{y} \wedge \overset{\mathtt{k}}{\leftarrow} (\mathtt{y}) = \overset{\mathtt{k}}{\leftarrow} (\mathtt{w})) \to \mathtt{w} \leq \mathtt{x}).\end{gathered} \tag{4.3}$$

Analogously, for any pair of natuaral numbers x and y such that both $L_1(x)$ and $L_1(y)$ hold, x is the j-th child of y ($y \overset{j}{\to} x$ for short), with $1 \leq j \leq k-1$, if and only if

$$L_j(\overset{k}{\leftarrow} (x)) \wedge y = (\overset{k}{\leftarrow})^j(x).$$

Therefore, $y \overset{j}{\to} x$, with $0 < j \leq k-1$) can be defined as follows

$$\begin{gathered}\mathtt{L_1(y)} \wedge \mathtt{L_1(x)} \wedge (\exists \mathtt{x_j}, \ldots, \mathtt{x_1}(\mathtt{x_j} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x}) \wedge \\ \mathtt{L_j(x_j)} \wedge \mathtt{x_1} = \mathtt{y} \wedge \bigwedge_{\mathtt{i=2}}^{\mathtt{j}} \mathtt{x_{i-1}} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x_i}))).\end{gathered} \tag{4.4}$$

4.5.4. THEOREM. *An ω-language is definable in $S1S^k$ if and only if it belongs to $\mathcal{L}_\omega$(k-STA).*

Proof. ($\Leftarrow$) Let $\mathcal{A} = \langle \Sigma, Q, in, f, F\rangle$ be a systolic automaton. We prove that there exists a sentence ϕ such that $\mathcal{L}(\phi) = \mathcal{L}_\omega(\mathcal{A})$. In order to simulate a systolic run, we define a partial function from $\mathbb{N}$ to Q that associates a state with each natural number x for which $L_1(x)$ holds. Formally, for any state q, let $p_q \subseteq \mathbb{N}$ be the set of natural numbers the state q is associated with. Moreover, for $q \neq q'$, $p_q \cap p_{q'} = \emptyset$ (cf. subformula (4.5) of the definition of ϕ). Let us consider a systolic run on an ω-word α. First, the leaves of the hierarchical structure on $\mathbb{N}$ imposed by the set of relations $\stackrel{j}{\rightarrow}$, with $0 \leq j \leq k-1$, act as input nodes. More precisely, a natural number x is an input node if and only if $\mathtt{Input(x)}$ holds, where $\mathtt{Input(x)}$ stands for:

$$\mathtt{L_1(x)} \wedge \neg\exists \mathtt{y}(\mathtt{x} \mathrel{=\!\stackrel{\mathtt{k}}{\leftarrow}} (\mathtt{y})).$$

A state q is associated with an input node x if and only if

$$\langle \alpha(x-1), q_1\rangle, \langle \alpha(x), q_2\rangle, \ldots, \langle \alpha(x+k-2), q_k\rangle \in in \text{ and } \langle q_1, \ldots, q_k, q\rangle \in f.$$

(cf. subformula (4.6) of the definition of ϕ). Notice that in such a way we associate with each leaf the state resulting from a computation step over the k states output by the systolic automaton fed with a substring of lenght k, and not the state that the input relation in associates with each input symbol. Next, if the natural numbers $x_1, \ldots, x_k$ are the children of z and $q_1, \ldots, q_k$ are the states associated with them, then q can be associated with z if $\langle q_1, \ldots, q_k, q\rangle \in f$ (cf. subformula (4.7) of the definition of ϕ). Finally, if the node x is a power of k, then the state associated with x is a state which results from processing the prefix $\alpha(0, kx-1)$ of α. So, the acceptance condition can be expressed by requiring that an infinite number of powers of k is related with a final state (cf. subformula (4.8) of the definition of ϕ).

A sentence ϕ such that $\mathcal{L}(\phi) = \mathcal{L}_\omega(\mathcal{A})$ is the existential closure, with respect to first-order variables of the form p_q (with $q \in Q$), of the following formula:

$$\bigwedge_{\mathtt{q}\neq\mathtt{q}'} \neg\exists \mathtt{y}\ (\mathtt{p_q(y)} \wedge \mathtt{p_{q'}(y)}) \wedge \tag{4.5}$$

$$\begin{aligned}\forall \mathtt{x}\ \Bigg(\mathtt{Input(x)} \rightarrow \Bigg(\bigwedge_{\mathtt{a_1,\ldots,a_k}\in\Sigma} \Bigg(\bigwedge_{\mathtt{i}=1}^{\mathtt{k}} \mathtt{Q_{a_i}(x+i-2)}\Bigg) \rightarrow \\ \bigvee_{\{\mathtt{q}:\langle \mathtt{q_1,\ldots,q_k,q}\rangle\in\mathtt{f},\langle \mathtt{a_j,q_j}\rangle\in\mathtt{in},1\leq \mathtt{j}\leq \mathtt{k}\}} \mathtt{p_q(x)}\Bigg)\Bigg) \wedge\end{aligned} \tag{4.6}$$

$$\forall \mathtt{z}\ \Bigg(\bigwedge_{\mathtt{q_1,\ldots,q_k}\in\mathtt{Q}} \Bigg(\bigwedge_{\mathtt{i}=0}^{\mathtt{k}-1} (\mathtt{z} \stackrel{\mathtt{i}}{\rightarrow} \mathtt{q_i} \wedge \mathtt{p_{q_i}(x_i)}) \rightarrow \bigvee_{\{\mathtt{q}:\langle \mathtt{q_1,\ldots,q_k,q}\rangle\in\mathtt{f}\}} \mathtt{p_q(z)}\Bigg)\Bigg) \wedge \tag{4.7}$$

$$\bigvee_{\mathtt{q}\in\mathtt{F}} \forall \mathtt{x}\exists \mathtt{y}\ (\mathtt{x} < \mathtt{y} \wedge 0 \mathrel{=\!\stackrel{\mathtt{k}}{\leftarrow}} (\mathtt{y}) \wedge \mathtt{p_q(y)}). \tag{4.8}$$

($\Rightarrow$) The opposite implication is proved by exploiting the same technique used in [118]. For technical convenience, predicate symbols Q_a, for $a \in \Sigma$, are replaced by free set variables. So, formulae $\phi(p_1, \ldots, p_n)$ are considered, where no symbol Q_a occurs, that are interpreted over ω-words over the special alphabet $\{0,1\}^n$. If $\alpha \in (\{0,1\}^n)^\omega$, then $\underline{\alpha} \models p_k(x)$ if and only if the x-th symbol of α has 1 in its k-th component. For a suitable n, symbols in Σ can be binary encoded, and any atomic formula $Q_a(x)$ can be replaced by a finite conjunction including either $p_k(x)$ and $\neg p_k(x)$, for $1 \leq k \leq n$. Hence, given a sentence ϕ, the thesis can be proved for the corresponding formula $\phi(p_1, \ldots, p_n)$ interpreted over ω-words over $\{0,1\}^n$. We preliminary reduce each formula $\phi(p_1, \ldots, p_n)$ of $S1S^k$ to an equivalent formula of a formalism simpler than $S1S^k$, denoted by $S1S^k_0$, whose terms are the first-order variables of the original formalism and whose atomic formulae take one of the following forms:

1. $p_i \subseteq p_j$ ("p_i is a subset of p_j");
2. $Succ(p_i, p_j)$ ("p_i, p_j are the singletons $\{x\}$, $\{y\}$, resp., with $x + 1 = y$");
3. $Power(p_i, p_j)$ ("p_i, p_j are the singletons $\{y\}$, $\{x\}$, resp., with $y = \overset{2}{\leftarrow} (x)$");
4. $L(p)$ ("p is a singleton $\{x\}$ and $L(x)$").

The reduction from $S1S^k$ to $S1S^k_0$ is a trivial extension of the reduction from $S1S$ to $S1S_0$ (we refer to [118] for the details). We show now, by induction on the structure of the $S1S^k_0$-formula $\phi(p_1, \ldots, p_n)$, that there exists a k-STA $\mathcal{A}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi(p_1, \ldots, p_n))$.
(Base case). Let us consider the atomic formula $\phi(p) = L(p)$. An automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi)$ can be defined as follows:

- $\Sigma = \{0,1\}$; $Q = \{0,1,2\}$ and $F = \{2\}$;
- in is the identity function;
- $f = \{\langle q_1, \ldots, q_k, 0\rangle : q_i = 0 \text{ for } 1 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_{k-1}, 1, 2\rangle : q_i = 0 \text{ for } 1 \leq i \leq k-1\} \cup$
  $\{\langle 1, q_2, \ldots, q_k, 1\rangle : q_i = 0 \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_k, 2\rangle : q_i = 2 \text{ for some } 1 \leq i \leq k \text{ and } q_j = 0 \text{ for all } j \neq i\}$.

It is possible to check that $\alpha \in \mathcal{L}_\omega(\mathcal{A})$ if and only if there exists one (and only one) i such that $L(i)$ holds and $\alpha(i) = 1$, and $\alpha(j) = 0$, for all $j \neq i$.
Let us consider the atomic formula $\phi(p_1, p_2) = Power(p_1, p_2)$. An automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi)$ can be defined as follows:

- $\Sigma = \{0,1\}^2$; $Q = \{0,1\}^2 \cup \{1\}$ and $F = \{1\}$;
- in is the identity function;
- $f = \{\langle q_1, \ldots, q_k, \langle 0,0\rangle\rangle : q_i = \langle 0,0\rangle \text{ for } 1 \leq i \leq k\} \cup$
  $\{\langle\langle 1,0\rangle, q_2, \ldots, q_k, \langle 1,0\rangle\rangle : q_i = \langle 0,0\rangle \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle\langle 0,1\rangle, q_2, \ldots, q_k, \langle 0,1\rangle\rangle : q_i = \langle 0,0\rangle \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_k, 1\rangle : q_i = \langle 1,0\rangle, q_{i+1} = \langle 0,1\rangle, \text{ for some } 1 \leq i \leq k-1 \text{ and } q_j = \langle 0,0\rangle$
  $\text{for all } j \neq i, j \neq i+1\} \cup$
  $\{\langle q_1, \ldots, q_k, 1\rangle : q_i = 1 \text{ for some } 1 \leq i \leq k \text{ and } q_j = \langle 0,0\rangle, , \text{ for all } j \neq i\}$.

It is easy to check that $\alpha \in \mathcal{L}_\omega(\mathcal{A})$ if and only if $\alpha(i) = \langle 1, 0\rangle$ and $\alpha(j) = \langle 0, 1\rangle$, for i and j such that $i =\overset{k}{\leftarrow} (j)$, and $\alpha(k) = \langle 0, 0\rangle$, for all k such that $k \neq i$ and $k \neq j$.
Any other kind of atomic $S1S_0^k$-formula ϕ is a $S1S_0$-formula. Therefore, by Büchi Theorem, $\mathcal{L}(\phi)$ is a regular ω-language and, by Theorem 4.4.4, we have that $\mathcal{L}(\phi) \in \mathcal{L}_\omega(k\text{-}STA)$.
(Inductive step). For the inductive step, it suffices to consider the connectives $\neg$ and $\vee$, and the existential quantifier $\exists$. The thesis for these three cases is apparent by closure of $\mathcal{L}_\omega(k\text{-}STA)$ under complementation, union, and projection, respectively (cf. Theorem 4.4.4). $\dashv$

4.5.5. COROLLARY. *The theory $S1S^k$ is decidable.*

Proof. It follows immediately from the decidability of the emptiness problem for $\mathcal{L}_\omega(k\text{-}STA)$ (cf. Theorem 4.4.4) and Theorem 4.5.4. $\dashv$

4.6 Decidable theories of ω-layered temporal structures

In this section, we discuss two theories of ω-layered metric temporal structures, namely, the theory of ω-layered structures consisting of an infinite number of *arbitrarily coarse* infinite temporal domains (called upward unbounded layered structures), and the theory of ω-layered structures consisting of an infinite number of *arbitrarily fine* infinite temporal domains (called downward unbounded layered structures). We first introduce the second-order language $\mathcal{L}^2_{\omega LM^k}$ for ω-layered (k-refinable) metric temporal structures, and show how it can be interpreted over the class of upward unbounded layered structures as well as over the class of downward unbounded ones. Then, we prove that the theory of upward unbounded (k-refinable) layered structures is decidable, through its reduction to the theory $S1S^k$. Successively, we first show that $S1S^k$ cannot be exploited to prove the decidability of the theory of downward unbounded (k-refinable) layered structures, and then we prove that such a theory is decidable, through its reduction to SkS.

4.6.1 The basic language for ω-layered metric temporal structures

Let $\mathcal{L}^2_{\omega LM^k}$ be the second-order language for ω-layered (k-refinable) metric temporal structures, including individual variables, the binary function symbol $\downarrow$, (uninterpreted) unary predicate symbols, the binary relational symbol $\leq$, and quantification of individual variables and (uninterpreted) unary predicate symbols. We restrict ourselves to formulae that contain no free individual variables. The language $\mathcal{L}^2_{\omega LM^k}$ is formally defined as follows.

4.6.1. DEFINITION. (Basic language) *Let V and $\mathcal{V}$ be sets of individual and first-order variable symbols, respectively. Terms and formulae of $\mathcal{L}^2_{\omega LM^k}$ are built up as follows:*

- (Terms) *First-order variables $\mathtt{x} \in \mathtt{V}$ are terms. If $\mathtt{t}$ is a term, then $\downarrow(\mathtt{j}, \mathtt{t})$, with $0 \leq j \leq k-1$, is a term.*

- (Atomic formulae) *If* $\mathtt{t}$ *and* $\mathtt{t}'$ *are terms and* $\mathtt{p} \in \mathcal{V}$*, then* $\mathtt{p}(\mathtt{t})$ *and* $\mathtt{t} \leq \mathtt{t}'$ *are atomic formulae.*
- (Formulae) *Atomic formulae are formulae. If* ϕ *and* ψ *are formulae,* $\mathtt{x} \in \mathtt{V}$ *and* $\mathtt{p} \in \mathcal{V}$*, then* $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$, $\exists\mathtt{x}\phi$, $\exists\mathtt{p}\phi$, $\forall\mathtt{x}\phi$ *and* $\forall\mathtt{p}\phi$ *are formulae.*

In Sections 4.6.2 and 4.6.3, we will show how such a language can be interpreted over upward and downward unbounded layered structures, respectively.

4.6.2 Decidability of upward unbounded layered structures

The theory of upward unbounded layered structures.

In the following, we first formally define upward unbounded layered structures, and then show how to interpret $\mathcal{L}^2_{\omega LM^k}$ over them.

4.6.2. Definition. (Upward unbounded layered structure) *An upward unbounded ω-layered k-refinable metric temporal structure is a triplet* $\langle \bigcup_{i \geq 0} T^i, \downarrow, \leq \rangle$*, where*

- $\{T^i\}_{i \geq 0}$ *are pairwise disjoint enumerable sets;*
- $\downarrow$: $\{0, \ldots k-1\} \times \bigcup_{i \geq 1} T^i \mapsto \bigcup_{i \geq 0} T^i$ *is a diadic function such that*
  - $\downarrow|_{T^i}$: $\{0, \ldots k-1\} \times T^i \mapsto T^{i-1}$*, for* $i > 0$*;*
  - *for each* $t \in T^{i-1}$ *(with* $i > 0$*), there exist* $t' \in T^i$ *and* $0 \leq j \leq k-1$ *such that* $t = \downarrow (j, t')$*;*
- $\leq$ *is a total ordering of* $\bigcup_{i \geq 0} T^i$ *such that*
  - $\langle T^i, \leq|_{T^i \times T^i} \rangle$ *is isomorphic to* $\langle \mathbb{N}, \leq \rangle$*, for each* $i \geq 0$*;*
  - $\downarrow (0, t) < t$ *and* $t < \downarrow (j, t)$*, for* $1 \leq j \leq k-1$*;*
  - $\downarrow (j, t) < \downarrow (j+1, t)$*, for* $0 \leq j \leq k-2$*;*
  - $\downarrow (k-1, t) < \downarrow (0, +_i 1(t))$*, for all* $i \geq 1$ *and* $t \in T^i$*,*

  where $\downarrow|_{T^i}$ *and* $\leq|_{T^i \times T^i}$ *denote the restrictions of* $\downarrow$ *to* T^i *and of* $\leq$ *to* $T^i \times T^i$*, respectively,* $t < t'$ *is a shorthand for* $t \leq t'$ *and* $t' \not\leq t$*, and for all* $i \geq 0$ *and* $t \in T^i$*,* $+_i 1(t)$ *denotes the element* $t' \in T^i$ *such that* $t < t'$ *and* $\neg\exists t''(t'' \in T^i \wedge t < t'' < t')$*.*

Notice that, for each $t \in T^i$, with $i \geq 1$, and $0 \leq j \leq k-1$, $\downarrow (\mathtt{j}, \mathtt{t})$ associates with t the $(j+1)$-th element of its k-decomposition with respect to T^{i-1}.

Let $\mathcal{I}$ be an interpretation of the language $\mathcal{L}^2_{\omega LM^k}$ over upward unbounded layered structures. As usual, let us denote by $c^{\mathcal{I}}$ the element of the domain $\bigcup_{i \geq 0} T^i$ associated with the constant symbol c by $\mathcal{I}$. This notation is extended in a natural way to ground terms and atoms.

We restrict ourselves to interpretations $\mathcal{I}$ that satisfy the following conditions:

$$(\downarrow (\mathtt{j}, \mathtt{t}))^{\mathcal{I}} = \begin{cases} \downarrow (j, t^{\mathcal{I}}) & \text{if } t^{\mathcal{I}} \in \bigcup_{i > 0} T^i \\ \perp & \text{otherwise;} \end{cases}$$

$$(\leq)^{\mathcal{I}} = \leq,$$

where $\perp$ stands for *undefined*.

We now define the notion of satisfaction of a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ by an interpretation $\mathcal{I}$. As usual, the definition is recursive on the structure of the formula.

4.6.3. DEFINITION. (Satisfiability relation) *Let $\mathcal{I}$ be an interpretation for the language $\mathcal{L}^2_{\omega LM^k}$, and let $\mu : V \to \bigcup_{i\geq 0} T^i$ and $\nu : \mathcal{V} \to 2^{\bigcup_{i\geq 0} T^i}$ be the valuations of individual and first-order variables, respectively. We write $(\phi)^{\mathcal{I},\mu,\nu} = true$ to indicate that $\mathcal{I}$ satisfies ϕ under μ and ν. The satisfiability relation is formally defined as follows:*

- $(\mathtt{p}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = true \Leftrightarrow t^{\mathcal{I},\mu} \in \nu(p)$;
- $(\mathtt{t}_1 \leq \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = true \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \leq^{\mathcal{I}}$;
- *boolean connectives and quantifiers are dealt with in the usual way.*

Notice that, from the definition of satisfiability relation, it follows that (atomic) formulae evaluate to false whenever at least one of their arguments evaluates to $\perp$.

Since we are interested in formulae ϕ that contain no individual variables, two distinct interpretations may differ from each other only in the values they assign to free first-order variables. Formally, let ϕ be a formula of $\mathcal{L}^2_{\omega LM^k}$, with free predicate symbols $\mathtt{p}_1, \ldots, \mathtt{p}_\mathtt{m}$, that contains no free individual variables. An interpretation $\mathcal{I}$ for ϕ, under valuations μ and ν for individual and first-order variables, is given by m sets $p_1^{\mathcal{I}}, \ldots, p_m^{\mathcal{I}} \subseteq \bigcup_{i\geq 0} T^i$, where, for all $t \in \bigcup_{i\geq 0} T^i$ and $1 \leq j \leq m$, $t \in p_j^{\mathcal{I}}$ if and only if $\mathtt{p}_\mathtt{j}$ holds at t.

Mapping $\mathcal{L}^2_{\omega LM^k}$ into $\mathbf{S1S^k}$.

In this section, we prove that the theory of upward unbounded layered structures is decidable, by defining a translation function τ that maps $\mathcal{L}^2_{\omega LM^k}$-formulae into equisatisfiable *S1S*^{k}-formulae.

First of all, observe that upward unbounded layered structures may differ from each other only in the names of their elements, that is, they are isomorphic. Formally, given a pair of upward unbounded layered structures $\mathcal{T}' = \langle \bigcup_{i\geq 0} T'^i, \downarrow', \leq' \rangle$ and $\mathcal{T}'' = \langle \bigcup_{i\geq 0} T''^i, \downarrow'' , \leq'' \rangle$, the mapping $f : \bigcup_{i\geq 0} T'^i \to \bigcup_{i\geq 0} T''^i$ that associates the j-th element of T'^i with the j-th element of T''^i is a bijection which preserves projection and ordering. Therefore, it follows that a formula ϕ is satisfiable under an interpretation $\mathtt{p}_1^{\mathcal{I}}, \ldots \mathtt{p}_\mathtt{m}^{\mathcal{I}} \subseteq \bigcup_{\mathtt{i}\geq 0} \mathtt{T}'^\mathtt{i}$ (where $\mathtt{p}_1, \ldots, \mathtt{p}_\mathtt{m}$ are the free predicates symbols occurring in ϕ) if and only if ϕ is satisfiable under the interpretation $\mathtt{f}(\mathtt{p}_1^{\mathcal{I}}), \ldots, \mathtt{f}(\mathtt{p}_\mathtt{m}^{\mathcal{I}}) \subseteq \bigcup_{\mathtt{i}\geq 0} \mathtt{T}''^\mathtt{i}$. This property allows us to replace the class of upward unbounded layered structures by a suitable single *concrete* structure which can be easily encoded into $S1S^k$.

4.6.4. DEFINITION. *The* concrete *upward unbounded layered structure is the triplet:*

$$\mathcal{C} = \langle \bigcup_{i\geq 0} T^{i\mathcal{C}}, \downarrow^{\mathcal{C}}, \leq^{\mathcal{C}} \rangle,$$

where

- $T^{i\mathcal{C}} = \{k^i + nk^{i+1} : n \geq 0\}$;

- $\downarrow^{\mathcal{C}}$: $\{0,\ldots,k-1\}\times\bigcup_{i\geq 1}T^{i\mathcal{C}}\to\bigcup_{i\geq 0}T^{i\mathcal{C}}$ *maps the pair* (j,k^i+nk^{i+1}) *into* $k^{i-1}+(j+nk)k^i$, *for* $0\leq j\leq k-1$ *and* $i\geq 0$;
- $\leq^{\mathcal{C}}$ *is the restriction to* $\bigcup_{i\geq 0}T^{i\mathcal{C}}$ *of the usual ordering on natural numbers.*

In particular, from the above definition, it follows that the least element of the i-th layer $0_i^{\mathcal{C}}$ is equal to k^i and that the successor function $+_i^{\mathcal{C}}1$ maps k^i+nk^{i+1} into $k^i+(n+1)k^{i+1}$, for each $i\geq 0$.

The following proposition proves that the *concrete* upward unbounded layered structure is well-defined.

4.6.5. Proposition. *$\mathcal{C}$ is an upward unbounded layered structure.*

Proof. For $i,j\geq 0$, with $i\neq j$, $T^{i\mathcal{C}}$ and $T^{j\mathcal{C}}$ are disjoint. In fact, if $t\in T^{i\mathcal{C}}$ and the k-ary representation of t is $a_nk^n+\ldots+a_0k^0$, then $a_i=1$ and $a_l=0$ for $0\leq l<i$. For $t=k^i+nk^{i+1}$, we have that:

- $\downarrow^{\mathcal{C}}(0,t)=k^{i-1}+nk^{i+1}<k^i+nk^{i+1}=t$;
- $t=k^i+nk^{i+1}<k^{i-1}+(j+nk)k^i=\downarrow^{\mathcal{C}}(j,t)$, for $j>0$;
- $\downarrow^{\mathcal{C}}(j,t)=k^{i-1}+(j+nk)k^i<k^{i-1}+(j+1+nk)k^i=\downarrow^{\mathcal{C}}(j+1,t)$;
- $\downarrow^{\mathcal{C}}(k-1,t)=k^{i-1}+(k-1+nk)k^i<k^{i-1}+(n+1)k^{i+1}=\downarrow^{\mathcal{C}}(0,+_i^{\mathcal{C}}(t))$.

⊣

It is worth noting that, for $k=2$, there exists a unique concrete structure, while alternative definitions are possible for $k>2$.

The next step consists in showing that the function $\downarrow^{\mathcal{C}}$ can be expressed in $S1S^k$ by defining a suitable mapping $\tau:\mathcal{L}^2_{\omega LM^k}\to S1S^k$. Such a mapping is inductively defined as follows:

- if ϕ is an atomic formula devoid of any occurrence of terms of the form $\downarrow(\mathtt{j},\mathtt{t})$, then
  $\tau(\phi)=\phi$;
- if ϕ is an atomic formula and $\downarrow(\mathtt{j_1},\mathtt{x_1}),\ldots,\downarrow(\mathtt{j_n},\mathtt{x_n})$ are the n innermost occurrences of $\downarrow$ in ϕ, with $n\leq 2$, then
  $\tau(\phi)=\tau(\phi[\mathtt{z_1}\backslash\downarrow(\mathtt{j_1},\mathtt{x_1}),\ldots,\mathtt{z_n}\backslash\downarrow(\mathtt{j_n},\mathtt{x_n})])\wedge\bigwedge_{\mathtt{1}\leq\mathtt{i}\leq\mathtt{n}}\mathtt{x_i}\xrightarrow{\mathtt{j_i}}\mathtt{z_i}$,
  where $\mathtt{z_1},\ldots,\mathtt{z_n}$ are fresh variables and $\mathtt{x}\xrightarrow{\mathtt{j}}\mathtt{z}$ is a shorthand for the formula of Equation 4.3 (if $j=0$) or of Equation 4.4 (if $j>0$);
- if $\phi=\neg\psi$, then
  $\tau(\phi)=\neg\tau(\psi)$;
- if $\phi=\psi\wedge\theta$ (resp. $\phi=\psi\vee\theta$), then
  $\tau(\phi)=\tau(\psi)\wedge\tau(\theta)$ (resp. $\tau(\phi)=\tau(\psi)\vee\tau(\theta)$);
- if $\phi=\exists\mathtt{x}\psi$ (resp. $\forall\mathtt{x}\psi$), then
  $\tau(\phi)=\exists\mathtt{x}(\mathtt{L_1}(\mathtt{x})\wedge\tau(\psi))$ (resp. $\tau(\phi)=\forall\mathtt{x}(\mathtt{L_1}(\mathtt{x})\to\tau(\psi))$;
- if $\phi=\exists\mathtt{p}\psi$ (resp. $\forall\mathtt{p}\psi$), then
  $\tau(\phi)=\exists\mathtt{p}(\forall\mathtt{y}(\mathtt{p}(\mathtt{y})\to\mathtt{L_1}(\mathtt{y}))\wedge\tau(\psi))$
  (resp. $\tau(\phi)=\forall\mathtt{p}(\forall\mathtt{y}(\mathtt{p}(\mathtt{y})\to\mathtt{L_1}(\mathtt{y}))\to\tau(\psi)))$,
  where $\mathtt{L_1}$ is a shorthand for the formula of Equation 4.2.

4.6.6. LEMMA. *For any $\phi \in \mathcal{L}^2_{\omega LM^k}$, ϕ is satisfiable with respect to the concrete structure $\mathcal{C}$ if and only if $\tau(\phi)$ is satisfiable with respect to $\langle \mathbb{N}, \overset{k}{\leftarrow}, L, \leq \rangle$.*

Proof. Hereinafter, let $\mathcal{I}$ and $\mathcal{J}$ denote interpretations for $\mathcal{L}^2_{\omega LM^k}$ and $S1S^k$, respectively. For the sake of simplicity, we assume that the sets of individual and first-order variables of $\mathcal{L}^2_{\omega LM^k}$ and $S1S^k$ coincide.

($\Rightarrow$) Let ϕ be a formula, $\mathcal{I}$ be an intepretation for $\mathcal{L}^2_{\omega LM^k}$, μ be the valuation of individual variables, and ν be the valuation for first-order variables, and suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. We prove that there exist an interpretation $\mathcal{J}$ for $S1S^k$ and a valuation μ' such that $(\tau(\phi))^{\mathcal{J},\mu',\nu} = true$, where μ' differs from μ at most in the valuation of the fresh variables introduced by τ. The proof is by induction on the structure of ϕ.

(Base case). Let ϕ be an atomic formula. The proof is by induction on the maximum number n of nested occurrences of a term having the form $\downarrow(j,t)$ in ϕ (*nesting degree*).

Let $n = 0$. By the definition of τ, $\tau(\phi) = \phi$. Since $\bigcup_{i\geq 0} T^{i\mathcal{C}} \subseteq \mathbb{N}$ and $\tau(\phi)$ $(= \phi)$ does not include any occurrence of the function $\downarrow$, it immediately follows that $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$, for any interpretation $\mathcal{J}$.

Let $n > 0$. Let $\downarrow(j_1,x_1),\ldots,\downarrow(j_m,x_m)$ be the m innermost occurrences of $\downarrow$ in ϕ and let $z_1,\ldots,z_m$ be the corresponding m fresh variables. Let μ' be the map such that $\mu'(z_i) = (\downarrow(j_i,x_i))^{\mathcal{I},\mu}$, for $1 \leq i \leq m$, and $\mu' = \mu$ elsewhere. If $(\phi)^{\mathcal{I},\mu,\nu} = true$, then the formula $\phi' = \phi[z_1\backslash\downarrow(j_1,x_1),\ldots,z_m\backslash\downarrow(j_m,x_m)]$ is an atomic formula with nesting degree $n-1$ such that $(\phi')^{\mathcal{I},\mu',\nu} = true$. By the induction hypothesis, there exist $\mathcal{J}$ and μ'' such that $\tau(\phi')^{\mathcal{J},\mu'',\nu} = true$, where μ'' coincides with μ' at least on all individual variables occurring in ϕ'. Now, it is easy to check that $(\bigwedge_{1\leq i\leq m} x_i \overset{j_i}{\to} z_i)^{\mathcal{I},\mu'',\nu} = true$, and hence $(\tau(\phi))^{\mathcal{I},\mu'',\nu} = true$.

(Inductive step). The proofs for the boolean connectives $\phi = \neg\psi$, $\phi = \psi \wedge \theta$ and $\phi = \psi \vee \theta$ are straightforward, and thus omitted.

As for the quantifiers, let us consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x \psi$. The other quantifications are dealt with in a similar way.

Suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. Since $\phi = \exists x\psi$, this means that there exists $\overline{\mu}$ that differs from μ at most in the value it assigns to x such that $(\psi)^{\mathcal{I},\overline{\mu},\nu} = true$. By the induction hypothesis, there exist $\mathcal{J}$, μ', and ν' such that $(\tau(\psi))^{\mathcal{J},\mu',\nu'} = true$ and $\mu'(x) = \overline{\mu}(x)$. From $\mu'(x) = \overline{\mu}(x)$, it follows that $(L_1(x))^{\mathcal{J},\mu',\nu'} = true$, and therefore $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

($\Leftarrow$) Let ϕ be a formula, $\mathcal{J}$ be an intepretation for $S1S^k$, μ be the valuation of individual variables, and ν be the valuation for first-order variables, and suppose that $\tau(\phi)^{\mathcal{J},\mu,\nu} = true$. We prove that there exist an interpretation $\mathcal{I}$ for $\mathcal{L}^2_{\omega LM^k}$, and two valuations μ' and ν' such that $(\phi)^{\mathcal{I},\mu',\nu'} = true$, where μ' and ν' are defined as follows:

$$\mu'(x) = max\{v : v \in \bigcup_{i\geq 0} T^{i\mathcal{C}}, v \leq \mu(x)\} \tag{4.9}$$

$$\nu'(p) = \{v : \overline{v} \in \nu(p)\} \wedge v = max\{\overline{\overline{v}} : \overline{\overline{v}} \leq \overline{v}\}\}. \tag{4.10}$$

As for the opposite implication, the proof is by induction on the structure of ϕ.

(Base case). Let ϕ be an atomic formula. The proof is by induction on the nesting degree of ϕ.

If $n = 0$, then ϕ has either the form $p(x)$ or the form $x \leq x'$. From the definitions of μ' and ν', it immediately follows that $(\phi)^{\mathcal{I},\mu',\nu'} = true$, for any interpretation $\mathcal{I}$.

If $n > 0$, then from $\tau(\phi)^{\mathcal{J},\mu,\nu} = true$ it follows that both

$$(\tau(\phi[z_1 \backslash \downarrow (j_1, x_1), \ldots, z_m \backslash \downarrow (j_m, x_m)]))^{\mathcal{J},\mu,\nu} = true$$

and

$$(\bigwedge_{1 \leq i \leq m} x_i \stackrel{j_i}{\rightarrow} z_i)^{\mathcal{J},\mu,\nu} = true.$$

By the induction hypothesis, it follows that there exists $\mathcal{I}$ such that $(\phi[z_1 \backslash \downarrow (j_1, x_1), \ldots, z_m \backslash \downarrow (j_m, x_m)])^{\mathcal{I},\mu',\nu'} = true$. Since $(x_i \stackrel{j_i}{\rightarrow} z_i)^{\mathcal{J},\mu,\nu} = true$, for $1 \leq i \leq m$, it follows that $\mu(x_i), \mu(z_i) \in \bigcup_{i \geq 0} T^{i\mathcal{C}}$ and thus $\mu'(x_i) = \mu(x_i)$, $\mu'(z_i) = \mu(z_i)$ and $\mu'(z_i) = (\downarrow (j_i, x_i))^{\mathcal{I},\mu'}$. This allows us to conclude that $(\phi)^{\mathcal{I},\mu',\nu'} = true$.

(Inductive step). The proofs for the boolean connectives are straightforward, and thus omitted. As for the quantifiers, we only consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x \psi$. Let $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$. By the induction hypothesis, it follows that there exists $\mathcal{I}$ such that $(\psi)^{\mathcal{I},\mu',\nu'} = true$, with μ' and ν' defined as in Equation 4.9 and Equation 4.10, respectively. Since $(L_1(x))^{\mathcal{J},\mu,\nu} = true$, by the definition of μ' we have that $\mu'(x) \in \bigcup_{i \geq 0} T^{i\mathcal{C}}$ and thus $(\exists x \psi)^{\mathcal{I},\mu',\nu'} = true$. ⊣

4.6.7. THEOREM. *The theory of upward unbounded layered structures is decidable.*

Proof. It follows immediately from Lemma 4.6.6 and the decidability of the theory $S1S^k$ (cf. Theorem 4.5.5). ⊣

4.6.3 Decidability of downward unbounded layered structures

In the following sections, we define the theory of downward unbounded layered structures, and prove its decidability by reduction to the decidable theory SkS.

The theory of downward unbounded layered structures.

As for the upward case, we first formally define downward unbounded layered structures, and then show how to interpret $\mathcal{L}^2_{\omega LM^k}$ over them.

4.6.8. DEFINITION. (Downward unbounded layered structure) *A downward unbounded ω-layered k-refinable metric temporal structure is a triplet $\langle \bigcup_{i \geq 0} T^i, \downarrow, \leq \rangle$, where*

- $\{T^i\}_{i \geq 0}$ *are pairwise disjoint enumerable sets;*
- $\downarrow$: $\{0, \ldots k-1\} \times \bigcup_{i \geq 0} T^i \mapsto \bigcup_{i \geq 0} T^i$ *is a diadic function such that*
  - $\downarrow\!|_{T^i}$: $\{0, \ldots k-1\} \times T^i \mapsto T^{i+1}$, *for* $i \geq 0$*;*
  - *for each* $t \in T^i$ *(with* $i \geq 1$*), there exists* $t' \in T^{i-1}$ *such that* $t = \downarrow (j, t')$*;*
- $\leq$ *is a total ordering of* $\bigcup_{i \geq 0} T^i$ *such that*

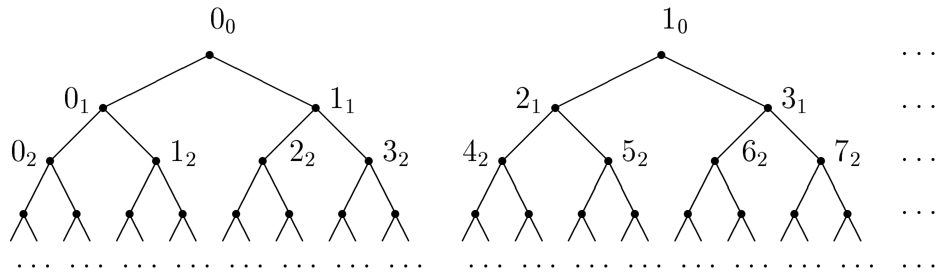

Figure 4.3: A downward unbounded (2-refinable) structure (i_j stands for $(+_j1)^i(0_j)$).

- $\langle T^i, \leq|_{T^i \times T^i}\rangle$ *is isomorphic to* $\langle \mathbb{N}, \leq\rangle$, *for each* $i \geq 0$;
- $t <\downarrow (j,t)$, *for* $0 \leq j \leq k-1$;
- $\downarrow (j,t) <\downarrow (j+1,t)$, *for* $0 \leq j \leq k-2$;
- *if* $t < t'$, *then* $\downarrow (j,t) < t'$, *for* $0 \leq j \leq k-1$,

where $\downarrow|_{T^i}$, $\leq|_{T^i \times T^i}$, *and* $+_i1$ *are defined as in Definition 4.6.2.*

An example of downward unbounded layered structure is shown in Figure 4.3.

From Definition 4.6.8, we can easily derive the truth of the following proposition:

4.6.9. Proposition. *For any* $t \in T^i$,

- $\downarrow (j+1,t) = +_{i+1}1(\downarrow (j,t))$
- $\downarrow (0, +_i1(t)) = +_{i+1}1(\downarrow (k-1,t))$.

Interpretations $\mathcal{I}$ of the language $\mathcal{L}^2_{\omega LM^k}$ over downward unbounded layered structures differ from the interpretations over upward unbounded layered structures only in the definition of the semantic clause for $\downarrow$:

$$(\downarrow (\mathtt{j},\mathtt{t}))^{\mathcal{I}} = \downarrow (\mathtt{j},\mathtt{t}^{\mathcal{I}}).$$

Unlike the case of upward unbounded layered structures, the interpretation of $(\downarrow (\mathtt{j},\mathtt{t}))$ over downward unbounded layered structures is indeed always defined.

The satisfability relation is defined exactly as in the case of upward unbounded layered structures.

Mapping $\mathcal{L}^2_{\omega LM^k}$ into SkS.

In this section, we prove that the theory of downward unbounded layered structures is decidable, by showing that any formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ can be transformed into an equisatisfiable formula of SkS. In particular, for each $0 \leq j \leq k-1$, the function $\downarrow (j,.)$ acts as $succ_j$ (i.e. the j-th successor of SkS), and the ordering $\leq$ of $\mathcal{L}^2_{\omega LM^k}$ can be expressed by using the prefix order $\leq_P$ of SkS. In Figure 4.4, we show how a downward unbounded layered structure can be encoded into the domain of interpretation of SkS (i.e., $\{0,\ldots,k-1\}^\star$).

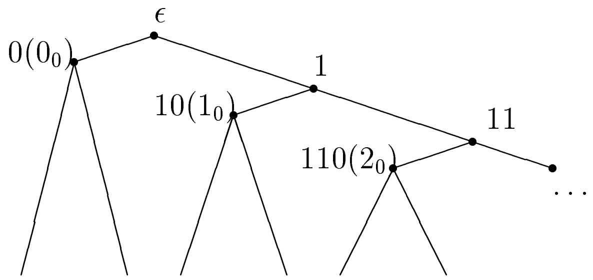

Figure 4.4: The encoding of a downward unbounded (2-refinable) structure into $\{0,1\}^\star$.

More formally, we consider a downward unbounded layered structure

$$\mathcal{R} = \langle \bigcup_{i\geq 0} T^{i\mathcal{R}}, \downarrow^{\mathcal{R}}, \leq^{\mathcal{R}} \rangle,$$

where

- $T^{i\mathcal{R}} = \{v \cdot 0 \cdot w : v \in \{k-1\}^\star, w \in \{0,\ldots,k-1\}^\star, |w| = i\}$;
- $\downarrow^{\mathcal{R}} (j,v) = v \cdot j (= succ_j(v))$, for all $v \in \bigcup_{i\geq 0} T^{i\mathcal{R}}$;
- $\leq^{\mathcal{R}}$ is the restriction to $\bigcup_{i\geq 0} T^{i\mathcal{R}}$ of the lexicographic ordering over $\{0,\ldots,k-1\}^\star$.

It is easy to verify that $\mathcal{R}$ is a downward unbounded layered structure.

4.6.10. Proposition. *Let $\mathcal{S} = \langle \bigcup_{i\geq 0} T^i, \downarrow, \leq \rangle$ be a downward unbounded layered structure. For each $i \geq 0$, $t \in T^i$, there exists $j \in \mathbb{N}$ such that $t = (+_i 1)^j(0_i)$. The map $cod : \bigcup_{i\geq 0} T^i \mapsto \bigcup_{i\geq 0} T^{i\mathcal{R}}$ such that*

$$cod(t) = \begin{cases} (k-1)^j \cdot 0 & \text{if } i = 0; \\ (k-1)^q \cdot 0 \cdot r & \text{otherwise,} \end{cases}$$

where $j = k^i q + r$, with $0 \leq r < k$, is an isomorphism of downward unbounded layered structures.

The proof is straightforward, and thus omitted.

In order to map a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ into an equisatisfiable formula of SkS, we define two auxiliary predicates `R` and `Dom` of SkS. These predicates are defined in such a way that `R(x)` holds if and only if $x^{\mathcal{I}} \in \{k-1\}^\star$, and `Dom(x)` holds if and only if $x^{\mathcal{I}} \in \bigcup_{i\geq 0} T^{i\mathcal{R}}$ (which is a proper subset of $\{0,\ldots,k-1\}^\star \setminus \{k-1\}^\star$).
Formally, `R(x)` is a shorthand for the formula:

$$\begin{aligned} \exists p(p(\epsilon) \wedge p(x) \wedge \forall y,z(\quad & p(y) \rightarrow p(succ_{k-1}(y)) \wedge \\ & (p(y) \wedge \textstyle\bigvee_{i=0}^{k-1} succ_i(z) = y) \rightarrow p(z) \wedge \\ & (p(z) \wedge p(y)) \rightarrow \neg \textstyle\bigvee_{i=0}^{k-2} succ_i(z) = y)), \end{aligned}$$

while $\mathtt{Dom(x)}$ is a shorthand for the formula:

$$\begin{aligned}\exists p(p(x) \wedge \forall y, z(\quad & R(y) \rightarrow (\neg p(y) \wedge p(succ_0(y)) \wedge \textstyle\bigwedge_{j=1}^{k-1} \neg p(succ_j(y))) \wedge \\ & p(y) \rightarrow \textstyle\bigwedge_{j=0}^{k-1} p(succ_j(y)) \wedge \\ & (p(y) \wedge \neg R(z) \wedge \textstyle\bigvee_{j=1}^{k-1} succ_j(z) = y) \rightarrow p(z)))\end{aligned}$$

The transformation of a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ into a formula of SkS is defined as follows:

if $\phi = \mathtt{t} \leq \mathtt{t}'$, then
$\tau(\phi) = \mathtt{Dom(t_1)} \wedge \mathtt{Dom(t_2)} \wedge (\mathtt{t_1} \leq_{\mathtt{P}} \mathtt{t_2} \vee$
$\exists x(\bigvee_{l=0}^{k-1}(succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2)))$,
where $\mathtt{t_1}$ and $\mathtt{t_2}$ are the terms obtained from $\mathtt{t}$ and $\mathtt{t}'$, respectively, by replacing any occurrence of a term of the form $\downarrow(j, t'')$ by $succ_j(t'')$;
if $\phi = \mathtt{p(t)}$, then
$\tau(\phi) = \mathtt{Dom(t')} \wedge \mathtt{p(t')}$,
where $\mathtt{t}'$ is the term obtained from $\mathtt{t}$ by replacing any occurrence of a term $\downarrow(j, t'')$ by $succ_j(t'')$;
if $\phi = \neg\psi$, then
$\tau(\phi) = \neg\tau(\psi)$;
if $\phi = \psi \wedge \theta$ (resp. $\phi = \psi \vee \theta$), then
$\tau(\phi) = \tau(\psi) \wedge \tau(\theta)$ (resp. $\tau(\phi) = \tau(\psi) \vee \tau(\theta)$);
if $\phi = \exists \mathtt{x}\psi$ (resp. $\forall \mathtt{x}\psi$), then
$\tau(\phi) = \exists \mathtt{x}(\mathtt{Dom(x)} \wedge \tau(\psi))$ (resp. $\tau(\phi) = \forall \mathtt{x}(\mathtt{Dom(x)} \rightarrow \tau(\psi))$;
if $\phi = \exists \mathtt{p}\psi$ (resp. $\forall \mathtt{p}\psi$), then
$\tau(\phi) = \exists \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \wedge \tau(\psi))$
(resp. $\tau(\phi) = \forall \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \rightarrow \tau(\psi)))$.

The relationship between the original formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ and the resulting formula $\phi' \in SkS$ is formally stated by the following lemma.

4.6.11. LEMMA. *For any $\phi \in \mathcal{L}^2_{\omega LM^k}$, ϕ is satisfiable with respect to the class of downward unbounded layered structures if and only if $\tau(\phi)$ is satisfiable with respect to $\langle\{0, \ldots, k-1\}^*, \epsilon, succ_0, \ldots, succ_{k-1}, <_P\rangle$.*

Proof. The proof is by induction on the structure of ϕ. Hereinafter, let $\mathcal{I}$ and $\mathcal{J}$ denote interpretations for the languages $\mathcal{L}^2_{\omega LM^k}$ and SkS, respectively. For the sake of simplicity, we assume that the sets of individual and first-order variables of $\mathcal{L}^2_{\omega LM^k}$ and SkS coincide.

($\Rightarrow$) Let $\mathcal{I}$ be an intepretation for $\mathcal{L}^2_{\omega LM^k}$, μ be a valuation of individual variables, and ν be a valuation of first-order variables, and suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. We prove that there exists an interpretation $\mathcal{J}$ for SkS such that if we take the valuations μ' and ν', with $\mu'(x) = cod(\mu(x))$, for any individual variable x, and $\nu'(p) = \{cod(t) : t \in \nu(p)\}$, for any first-order variable p, then $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.
(Base case). The only non-trivial case is $\phi = t \leq t'$. If $(t)^{\mathcal{I},\mu,\nu} \leq (t')^{\mathcal{I},\mu,\nu}$, then there are three possible cases:

(1) $(t)^{\mathcal{I},\mu,\nu}$ is an ancestor of $(t')^{\mathcal{I},\mu,\nu}$. This implies that there are $v_1, \ldots, v_n \in \bigcup_{i\geq 0} T^i$ such that $v_1 = (t)^{\mathcal{I},\mu,\nu}$, $v_n = (t')^{\mathcal{I},\mu,\nu}$, and $v_{i+1} =\downarrow (j_i, v_i)$, for each $1 \leq i \leq n-1$. Let t_1 and t_2 be the terms obtained from t and t' by replacing any occurrence of the operator $\downarrow (j,.)$ by the j-th successor of SkS. Since *cod* is an isomorphism, we have that $(t_1)^{\mathcal{J},\mu',\nu'} = cod((t)^{\mathcal{I},\mu,\nu})$ and $(t_2)^{\mathcal{J},\mu',\nu'} = cod((t)^{\mathcal{I},\mu,\nu}) \cdot j_1 \cdot \ldots \cdot j_{n-1}$. From the definitions of *cod* and *Dom*, it immediately follows that both $(Dom(t_1))^{\mathcal{J},\mu',\nu'}$ and $(Dom(t_2))^{\mathcal{J},\mu',\nu'}$ hold. Moreover, from the definition of $\leq_P$, it is easy to see that also $(t_1)^{\mathcal{J},\mu',\nu'} \leq_P (t_2)^{\mathcal{J},\mu',\nu'}$ holds, and thus $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

(2) There exists $v \in \bigcup_{i\geq 0} T^i$ such that v is an ancestor of both $(t)^{\mathcal{I},\mu,\nu}$ and $(t')^{\mathcal{I},\mu,\nu}$. If we take the valuation μ'', which differs from μ' at most in the value $cod(v)$ it assigns to x, we have that $(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2)^{\mathcal{J},\mu'',\nu'} = true$, where t_1 and t_2 are defined as in (1). It follows that $(\exists x(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2))^{\mathcal{J},\mu',\nu'} = true$, and thus the thesis.

(3) There exist $v, v' \in \bigcup_{i\geq 0} T^i$ such that v is an ancestor $(t)^{\mathcal{I},\mu,\nu}$ and v' is an ancestor of $(t')^{\mathcal{I},\mu,\nu}$, with $v \leq v'$. If we take the valuation μ'', which differs from μ' at most in the value $(k-1)^m$ it assigns to x, with $cod(v) = (k-1)^m \cdot 0$, then we have that $(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2)^{\mathcal{J},\mu'',\nu'} = true$, where t_1 and t_2 are defined as in (1), and thus the thesis.

(Inductive step). The proofs for the boolean connectives $\phi = \neg\psi$, $\phi = \psi \wedge \theta$ and $\phi = \psi \vee \theta$ are straightforward, and thus omitted.
As for the quantifiers, let us consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x\psi$. The other quantifications are dealt with in a similar way. Assume that $(\phi)^{\mathcal{I},\mu,\nu} = true$. This means that there exists $\overline{\mu}$ that differs from μ at most in the value it assigns to x such that $(\psi)^{\mathcal{I},\overline{\mu},\nu} = true$. By the induction hypothesis, there exists $\mathcal{J}$ such that $(\tau(\psi))^{\mathcal{J},\overline{\mu}',\nu'} = true$. Now, by construction, $\overline{\mu}'(x) = cod(\overline{\mu}(x))$, and thus $(Dom(x))^{\mathcal{J},\overline{\mu}',\nu'} = true$. This allows us to conclude that $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

($\Leftarrow$) The proof of the opposite implication is only sketched. Assume that $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$, for given $\mathcal{J}$, μ, and ν. It can be easily proved, by induction on the structure of ϕ, that there exist two valuations $\overline{\mu}$ and $\overline{\nu}$ such that, for each individual variable x and each first-order variable p, $\overline{\mu}(x) \in cod(\bigcup_{i\geq 0} T^i)$, $\overline{\nu}(p) \subseteq cod(\bigcup_{i\geq 0} T^i)$, and $(\tau(\phi))^{\mathcal{J},\overline{\mu},\overline{\nu}} = true$. Let us take two valuations μ' and ν' such that, for each individual variable x and each first-order variable p, $\mu'(x) = cod^{-1}(\overline{\mu}(x))$ and $\nu'(p) = \{cod^{-1}(w) : w \in \overline{\nu}(p)\}$. It is not difficult to show, by induction on the structure of ϕ, that there exists $\mathcal{I}$ such that $(\phi)^{\mathcal{I},\mu',\nu'} = true$. $\dashv$

4.6.12. Theorem. *The theory of downward unbounded layered structures is decidable.*

Proof. It follows immediately from Lemma 4.6.11 and the decidability of the theory SkS (cf. [118]). $\dashv$

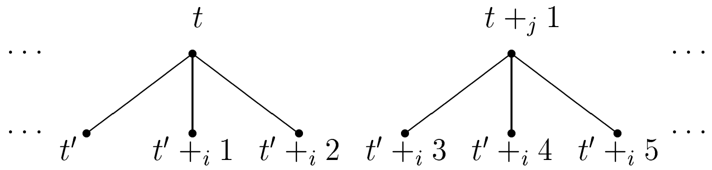

Figure 4.5: A 3-refinable temporal structure.

4.6.4 On the ordering relations

In this section, we briefly discuss the basic features of the total ordering relation $\leq$ defined over the domain $\bigcup_{i\geq 0} T^i$ of downward (resp. upward) unbounded layered structures.

In both the downward and the upward case, the relation $\leq$ is defined in such a way that the pair $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$ as well as each pair $\langle T^i, \leq|_{T^i \times T^i} \rangle$, with $i \geq 0$, are isomorphic to the pair $\langle \mathbb{N}, \leq \rangle$. Moreover, in both cases the zero element for $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$ coincides with the zero element for $\langle T^0, \leq|_{T^0 \times T^0} \rangle$.

It is worth noting that we cannot use the same definition of $\leq$ for both classes of structures: ordering upward (resp. downward) unbounded layered structures as downward (resp. upward) ones, we indeed would have no zero element for $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$. This is due to the fact that the topologies of upward and downward unbounded layered structures are intrinsically different.

A number of natural orderings over the domain of both downward and upward unbounded layered structures can be easily defined on the basis of the given one.

As an example, consider a possible interpretation of differently-grained temporal domains as different ways of partitioning the real (rational) time axis. Accordingly, we can interpret each time instant of a given domain as a suitable time interval over such an axis. More precisely, let each time instant be associated with a time interval open on the left and closed on the right. We can easily define an ordering relation $\leq_e$ over time instants such that for all $t, t' \in \bigcup_{i\geq 0} T^i$, $t \leq_s t'$ if and only if the ending point of the interval corresponding to t precedes the ending point of the interval for t' with respect to the standard ordering relation over real (rational) numbers. For k-refinable structures, this implies that any time instant t coincides with its rightmost child $\downarrow (k-1, t)$ and strictly precedes all the others, namely, $\downarrow (0, t)$, $\ldots$, $\downarrow (k-2, t)$. In Figure 4.5, we depict (a little part of) a 3-refinable structure, where the time instant t (resp. $+_j 1(t)$), belonging to the domain T^j, is decomposed into the three time instants t', $+_i 1(t')$, and $(+_i 1)^2(t')$ (resp. $(+_i 1)^3(t')$, $(+_i 1)^4(t')$, and $(+_i 1)^5(t')$), belonging to the finer domain T^i. For the sake of readability, for each $k \geq 0$, $n > 1$, and x, we denote $(+_k 1)^n(x)$ by $x +_k n$. Given the temporal structure of Figure 4.5, the interpretation that maps each time instant into a time interval open on the left and closed on the right is represented in Figure 4.6.

$$\ldots\ (t, \qquad\qquad\qquad\qquad\qquad\qquad t +_j 1]\,(t +_j 1, \qquad\qquad \ldots$$

$$\ldots\ (t', \qquad t' +_i 1]\,(t' +_i 1, t' +_i 2]\,(t' +_i 2, t' +_i 3]\,(t' +_i 3, t' +_i 4] \quad \ldots$$

Figure 4.6: A possible interval interpretation - case 1.

$$\ldots\ [t, \qquad\qquad\qquad\qquad\qquad\qquad t +_j 1)\,[t +_j 1, \qquad\qquad \ldots$$

$$\ldots\ [t', \qquad t' +_i 1)\,[t' +_i 1, t' +_i 2)\,[t' +_i 2, t' +_i 3)\,[t' +_i 3, t' +_i 4) \quad \ldots$$

Figure 4.7: A possible interval interpretation - case 2.

Similarly, we can associate each time instant with a time interval closed on the left and open on the right, and define an ordering relation $\leq_s$ over time instants such that for all $t, t' \in \bigcup_{i \geq 0} T^i$, $t \leq_s t'$ if and only if the starting point of the interval corresponding to t precedes the starting point of the interval for t'. Accordingly, any time instant t coincides with its leftmost child $\downarrow(0, t)$ and strictly precedes all the others, namely, $\downarrow(1, t)$, $\ldots$, $\downarrow(k-1, t)$. Such an interpretation for the temporal structure given in Figure 4.7 is graphically depicted in Figure 4.7.

Both downward and upward unbounded layered structures allow one to define the ordering relations $\leq_e$ and $\leq_s$ in terms of the basic ordering relation $\leq$.

Let us consider first downward unbounded layered structures. The relation $\leq_e$ can be defined as follows:

$$\mathtt{t} \leq_{\mathtt{e}} \mathtt{t'}$$

$$\text{iff}$$

$$\mathtt{ancestor(t', t)} \lor (\neg\mathtt{ancestor(t', t)} \land \neg\mathtt{ancestor(t, t')} \land \mathtt{t} \leq \mathtt{t'}) \lor \mathtt{ancestor_{k-1}(t, t')},$$

where $\mathtt{ancestor(t', t)}$ stands for the fact that t belongs to the tree rooted at t'. In formulae:

$$\exists \mathtt{p}(\mathtt{p(t')} \land \mathtt{p(t)} \land \forall \mathtt{w}(\mathtt{p(w)} \to \bigwedge_{\mathtt{i=0}}^{\mathtt{k-1}} \mathtt{p}(\downarrow(\mathtt{i}, \mathtt{w})))) \land \forall \mathtt{w}, \mathtt{z}(\mathtt{z} \neq \mathtt{t'} \land$$

$$\land \mathtt{p(z)} \land \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i}, \mathtt{w}) = \mathtt{z} \to \mathtt{p(w)}) \land \forall \mathtt{w}(\bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i}, \mathtt{w}) = \mathtt{t'} \to \neg\mathtt{p(w)}) \land$$

$$\land \forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i}, \mathtt{w}) = \mathtt{z} \land \mathtt{z} \neq \mathtt{t'} \to \neg\mathtt{p(z)}),$$

and $\mathtt{ancestor_{k-1}(t, t')}$ stands for the fact that t can be reached from t' following its rightmost branch ($(k-1)$-th children). In formulae:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}') \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{w}(\mathtt{p}(\mathtt{w}) \rightarrow \mathtt{p}(\downarrow(\mathtt{k}-1,\mathtt{w}))) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge \mathtt{p}(\mathtt{w}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow$$

$$\rightarrow \downarrow(\mathtt{k}-1,\mathtt{w}) = \mathtt{z}) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{z} \neq \mathtt{t}' \wedge \mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{w}(\downarrow(\mathtt{k}-1,\mathtt{w}) = \mathtt{t}' \rightarrow$$

$$\rightarrow \neg \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}_1 \rightarrow \neg \mathtt{p}(\mathtt{z}))).$$

while the relation $\leq_s$ is simply defined as:

$$\mathtt{t} \leq_{\mathtt{s}} \mathtt{t}' \text{ iff } \mathtt{t} \leq \mathtt{t}' \vee \mathtt{ancestor}_0(\mathtt{t}',\mathtt{t}),$$

where $\mathtt{ancestor}_0(\mathtt{t}',\mathtt{t})$ stands for the fact that t can be reached from t' following its leftmost branch (0-th children), and it can be obtained from $\mathtt{ancestor}_{\mathtt{k}-1}(\mathtt{t},\mathtt{t}')$ by substituting $\mathtt{0}$ for $\mathtt{k}-\mathtt{1}$ everywhere. In formulae:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}') \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{w}(\mathtt{p}(\mathtt{w}) \rightarrow \mathtt{p}(\downarrow(0,\mathtt{w}))) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge \mathtt{p}(\mathtt{w}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow$$

$$\rightarrow \downarrow(0,\mathtt{w}) = \mathtt{z}) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{z} \neq \mathtt{t}' \wedge \mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{w}(\downarrow(0,\mathtt{w}) = \mathtt{t}' \rightarrow$$

$$\rightarrow \neg \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}_1 \rightarrow \neg \mathtt{p}(\mathtt{z}))).$$

The definitions of $\leq_e$ and $\leq_s$ for upward unbounded layered structures are exactly the same, except for the fact that the definitions of $\mathtt{ancestor}$, $\mathtt{ancestor}_{\mathtt{k}-1}$, and $\mathtt{ancestor}_0$ do not include the last conjunct:

$$\forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}' \rightarrow \neg \mathtt{p}(\mathtt{z})).$$

Many other natural orderings over the domain of layered structures can be defined on the basis of the basic ordering relation $\leq$. A further example will be given in the next section.

4.7 High-level languages for ω-layered temporal structures

In this section, we demonstrate that the key features of metric and layered temporal logic can actually be expressed in $\mathcal{L}^2_{\omega LM^k}$. To this end, we first define a high-level second-order language for the theory of upward (resp. downward) unbounded layered structures, including suitable primitives for modeling contextualization, projection, and displacement, and then show that the resulting language is in fact as expressive as the basic one.

4.7.1 The language for upward unbounded layered structures

Let $\mathcal{UL}^2_{\omega LM^k}$ be the high-level second-order language for the theory of upward unbounded layered structures, including individual variables, the constant symbols $\mathtt{0_i}$ (local zero elements), the unary function symbols $\mathtt{+_i1}$ (local successors), the binary function symbol $\downarrow$ (projection), (uninterpreted) unary predicate symbols, the unary predicate symbols $\mathtt{T^i}$ (vertical contextualizations or y-contextualizations) and $\mathtt{\Delta_i}$ (horizontal contextualizations or x-contextualizations), the binary relational symbols $\leq$ (ordering), $\preceq$ (approximate ordering) and $\equiv_{\mathtt{i,j}}$ (congruences), and quantification of individual variables and (uninterpreted) unary predicate symbols (with $i \geq 0$). We restrict ourselves to formulae that contain no free individual variables. $\mathcal{UL}^2_{\omega LM^k}$ is formally defined as follows.

4.7.1. Definition. (Language) *Let V and $\mathcal{V}$ be sets of individual and first-order variable symbols, respectively. Terms and formulae of $\mathcal{UL}^2_{\omega LM^k}$ are built up as follows:*

- *(*Terms*) First-order variables $\mathtt{x} \in \mathtt{V}$ and constants $\mathtt{0_i}$ are terms. If $\mathtt{t}$ is a term, then $\mathtt{+_i1(t)}$ and $\downarrow(\mathtt{j},\mathtt{t})$, with $0 \leq j \leq k-1$, are terms.*
- *(*Atomic formulae*) If $\mathtt{t}$ and $\mathtt{t}'$ are terms and $\mathtt{p} \in \mathcal{V}$, then $\mathtt{T^i(t)}$, $\mathtt{\Delta_i(t)}$, $\mathtt{p(t)}$, $\mathtt{t} \leq \mathtt{t}'$, $\mathtt{t} \preceq \mathtt{t}'$ and $\mathtt{t} \equiv_{\mathtt{i,j}} \mathtt{t}'$ are atomic formulae.*
- *(*Formulae*) Atomic formulae are formulae. If ϕ and ψ are formulae, $\mathtt{x} \in \mathtt{V}$ and $\mathtt{p} \in \mathcal{V}$, then $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$, $\exists \mathtt{x}\phi$, $\exists \mathtt{p}\phi$, $\forall \mathtt{x}\phi$ and $\forall \mathtt{p}\phi$ are formulae.*

An intepretation $\mathcal{I}$ for the language $\mathcal{L}^2_{\omega LM^k}$ over upward unbounded layered temporal structures is also an interpretation for the high-level language $\mathcal{UL}^2_{\omega LM^k}$ if and only if it satisfies the following further conditions:

$\mathtt{0_i}^{\mathcal{I}} = t$, where t is the (unique) element of T^i such that $\forall t'(t' \in T^i \to t \leq t')$;

$$(\mathtt{+_i1(t)})^{\mathcal{I}} = \begin{cases} +_i1(t^{\mathcal{I}}) & \text{if } t^{\mathcal{I}} \in T^i \\ \perp & \text{otherwise,} \end{cases}$$

$(\mathtt{T^i})^{\mathcal{I}} = T^i$;
$(\mathtt{\Delta_i})^{\mathcal{I}} = \{(+_j1)^i(0_j^{\mathcal{I}}) : j \geq 0\}$;
$(\preceq)^{\mathcal{I}} = \{(t,t') : ancestor(t,t') \vee ancestor(t',t) \vee t \leq t'\}$,
where, for all $t,t' \in \bigcup_{i\geq 0} T^i$, $ancestor(t,t')$ if and only if $\exists j(t = \downarrow(j,t') \vee \exists j(t'' = \downarrow(j,t') \wedge ancestor(t,t'')))$;
$(\equiv_{\mathtt{i,j}})^{\mathcal{I}} = \{(t,t') : t \in T^i \wedge t' \in T^i \wedge t = (+_i1)^m(0_i^{\mathcal{I}}) \wedge t' = (+_i1)^n(0_i^{\mathcal{I}}) \wedge m \equiv_j n\}$.

For each $i \geq 0$, $\mathtt{0_i}$ denotes the origin of the domain T^i and $\mathtt{+_i1}$ is interpreted as the successor function over T^i. y-contextualizations $\mathtt{T^i(t)}$ and x-contextualizations $\mathtt{\Delta_i(t)}$ respectively restrict the range of possible values of the term t, constraining it to be interpreted over the domain T^i and over the set of elements at distance i from the origin of the domain they belong to. It is worth noting that, for each $i \geq 0$, the set of elements satisfying $\mathtt{\Delta_i}$, together with (the obvious restriction of) $\leq$, is isomorphic to the structure $\langle \mathbb{N}, \leq \rangle$. In this respect, x-contextualization and y-contextualization act in a perfectly symmetric way. For each $t,t' \in \bigcup_{i\geq 0} T^i$, $\preceq$ relates t to t' whenever $\mathtt{t} \leq \mathtt{t}'$ holds or either t' belongs to the projection of t over the domain t' belongs to or t belongs to the projection of t' over the domain t belongs to. Finally, for all $i \geq 0$ and $j \geq 2$, $\equiv_{\mathtt{i,j}}$ denotes the congruence modulo-j over T^i.

4.7.2. Example. We describe a property whose specification exploits the expressive power of the operators of x-contextualization and projection of upward unbounded layered structures, and cannot be expressed in the theory of finitely-layered ones. For the sake of simplicity, we take $k = 2$. Consider a process of decay, where the elapsing time between two successive occurrences of a given phenomenon exponentially increases. It can be modeled in $\mathcal{UL}^2_{\omega LM^k}$ as follows. Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates that respectively detect occurrences and not occurrences of the considered phenomenon. With respect to each temporal domain, we assume that the phenomenon occurs at the first two time instants (that is, with delay 0); the phenomenon is not detected at the next time instant; it occurs again at the fourth instant (that is, with delay 2^0); then it is not detected for 2 time instants, but it surely happens at least at the end of an interval of length 2 (that is, with delay greater than or equal to 2^1 and less than 2^2); then it is not detected for 4 time instants, but it surely happens at least at the end of an interval of length 4 (that is, with delay greater than or equal to 2^2 and less than 2^3); and so on. In such a way, if an occurrence of the phenomenon is detected at a time instant $i \in T^j$, then it occurs at least in its rightmost child, while if it is not detected at $i \in T^j$, then it is not detected at each of its children.

Formally, the process is specified by the following $\mathcal{UL}^2_{\omega LM^k}$-formula:

$$\forall \mathtt{x}((\Delta_0(\mathtt{x}) \vee \Delta_1(\mathtt{x})) \rightarrow \mathtt{p}(\mathtt{x}) \wedge \Delta_2(\mathtt{x}) \rightarrow \mathtt{q}(\mathtt{x}) \wedge \mathtt{p}(\mathtt{x}) \leftrightarrow \neg \mathtt{q}(\mathtt{x}) \wedge$$

$$\wedge \mathtt{q}(\mathtt{x}) \rightarrow (\mathtt{q}(\downarrow(0,\mathtt{x})) \wedge \mathtt{q}(\downarrow(1,\mathtt{x}))) \wedge \mathtt{p}(\mathtt{x}) \rightarrow \mathtt{p}(\downarrow(1,\mathtt{x}))).$$

It is easy to verify that each model of the above formula associates with each layer T^j an ω-sequence of the form $s_0 \cdot s_1 \cdot \ldots \cdot s_i \cdot \ldots$, where $s_0 = \{p\}$ and

$$s_i = \{p\} \cdot \{q\}^{2^{i-1}} \cdot w_i, \text{ with } w_i \in \{p,q\}^*, |w_i| = 2^{i-1} - 1.$$

It is possible to prove that the set of ω-sequences fulfilling the above condition is a non-regular ω-language over the alphabet $\Sigma = \{p, q\}$ belonging to the class $\mathcal{L}_\omega(k\text{-}STA)$.

The notion of satisfaction of a formula $\phi \in \mathcal{UL}^2_{\omega LM^k}$ by an interpretation $\mathcal{I}$ is a straightforward generalization of the corresponding notionf for $\mathcal{UL}^2_{\omega LM^k}$.

4.7.3. Definition. (Satisfiability relation) *Let $\mathcal{I}$ be an interpretation for the language $\mathcal{UL}^2_{\omega LM^k}$, and let μ and ν be the valuations of individual and first-order variables, respectively. The satisfiability relation is formally defined as follows:*

- $(\mathtt{p}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = true \Leftrightarrow t^{\mathcal{I},\mu} \in \nu(p)$;
- $(\mathtt{T}^{\mathtt{i}}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = true \Leftrightarrow t^{\mathcal{I},\mu} \in T^{i^{\mathcal{I}}}$;
- $(\Delta_{\mathtt{i}}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = true \Leftrightarrow t^{\mathcal{I},\mu} \in \Delta_i{}^{\mathcal{I}}$;
- $(\mathtt{t}_1 \leq \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = true \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \leq^{\mathcal{I}}$;
- $(\mathtt{t}_1 \preceq \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = true \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \preceq^{\mathcal{I}}$;
- $(\mathtt{t}_1 \equiv_{\mathtt{i,j}} \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = true \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \equiv^{\mathcal{I}}_{i,j}$;

- *boolean connectives and quantifiers are dealt with in the usual way.*

We conclude this section by proving that $\mathcal{UL}^2_{\omega LM^k}$ is as expressive as $\mathcal{L}^2_{\omega LM^k}$. More precisely, we show that for each $i \geq 0$, the constant $\mathtt{0_i}$, the successor function $+_\mathtt{i}\mathtt{1}$, the vertical and horizontal contextualizations $\mathtt{T^i}$ and $\mathtt{\Delta_i}$, the congruences $\equiv_{\mathtt{i,j}}$, with $j \geq 2$, and the approximate ordering $\preceq$ can be expressed in terms of $\downarrow$ and $\leq$.

Let $\phi \in \mathcal{UL}^2_{\omega LM^k}$. The following steps transforms ϕ into an equisatisfiable formula $\phi' \in \mathcal{L}^2_{\omega LM^k}$.

Step 1: removal of constants $\mathtt{0_i}$.

For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i}$ (if any) by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_{i-1}} (\forall \mathtt{y} \;\; (\mathtt{x_0} \leq \mathtt{y}) \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} \downarrow (\mathtt{0}, \mathtt{x_j}) = \mathtt{x_{j-1}}).$$

Step 2: removal of successor functions $+_\mathtt{i}\mathtt{1}$.

Since terms may contain nested occurrences of successor functions, we cannot remove all their occurrences in a single step. At each single step, we can only remove the innermost occurrences of successor functions, that is, each occurrence of $+_\mathtt{i}\mathtt{1}$, with $i \geq 0$, applied to a term $\mathtt{t}$ built up using first-order variables and the binary function $\downarrow$ only.

Let $+_\mathtt{i}\mathtt{1}(\mathtt{t})$ be a term of the above specified form. We replace it by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\mathtt{T^i}(\mathtt{x_i}) \wedge \mathtt{T^i}(\mathtt{t}) \wedge \mathtt{t} < \mathtt{x_i} \wedge \forall \mathtt{y} (\mathtt{T^i}(\mathtt{y}) \wedge \mathtt{t} < \mathtt{y} \rightarrow \mathtt{x_i} \leq \mathtt{y}).$$

We iterate such a replacement until the resulting formula is devoid of occurrences of successor functions.

Step 3: removal of y-contextualizations $\mathtt{T^i}$.

We replace each occurrence of atomic formulae of the form $\mathtt{T^i}(\mathtt{t})$, with $i \geq 0$, by the formula:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_i} (\mathtt{t} = \mathtt{x_i} \wedge \forall \mathtt{y} \;\; \neg \downarrow (\mathtt{0}, \mathtt{x_0}) = \mathtt{y} \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} (\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \downarrow (\mathtt{l}, \mathtt{x_j}) = \mathtt{x_{j-1}})).$$

For $i = 0$, the above formula reduces to:

$$\exists \mathtt{x_0} (\mathtt{t} = \mathtt{x_0} \wedge \forall \mathtt{y} \;\; \neg \downarrow (\mathtt{0}, \mathtt{x_0}) = \mathtt{y}).$$

Step 4: removal of x-contextualizations $\mathtt{\Delta_i}$.

The removal of x-contextualizations $\mathtt{\Delta_i}$ is performed in two steps. First, for each $i > 0$, we replace each occurrence of atomic formulae of the form $\mathtt{\Delta_i}(\mathtt{t})$ by a formula which contains an atomic formula of the form $\mathtt{\Delta_0}(\mathtt{x})$; then, we remove from the resulting formula each occurrence of atomic formulae of the form $\mathtt{\Delta_0}(\mathtt{t})$ (possibly added at the previous step).

For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We replace each occurrence of atomic formulae of the form $\mathtt{\Delta_i}(\mathtt{t})$ by the formula:

$$\exists \mathtt{x_{n+1}}, \ldots, \mathtt{x_0}(\Delta_0(\mathtt{x_{n+1}}) \wedge \mathtt{x_0} = \mathtt{t} \wedge \bigwedge_{\mathtt{j}=0}^{\mathtt{n}} \downarrow (\mathtt{a_j}, \mathtt{x_{j+1}}) = \mathtt{x_j}).$$

Then, we replace each occurrence of atomic formulae of the form $\Delta_0(\mathtt{t})$ by the formula:

$$\exists \mathtt{p}(\mathtt{p}(0_0) \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{y}, \mathtt{z}(\mathtt{p}(\mathtt{y}) \wedge \downarrow (0, \mathtt{z}) = \mathtt{y} \to \mathtt{p}(\mathtt{z})) \wedge$$

$$\forall \mathtt{y}, \mathtt{z}(\mathtt{p}(\mathtt{y}) \wedge \mathtt{p}(\mathtt{z}) \to \bigwedge_{\mathtt{i}=1}^{\mathtt{k}-1} \neg \downarrow (\mathtt{i}, \mathtt{z}) = \mathtt{y})),$$

and then we remove the constant 0_0 by executing the operations specified in step 1.

Step 5: removal of congruences $\equiv_{\mathtt{i},\mathtt{j}}$.

We replace each occurrence of atomic formulae of the form $\mathtt{t_1} \equiv_{\mathtt{i},\mathtt{j}} \mathtt{t_2}$, with $i \geq 0$ and $j \geq 2$, by the formula:

$$\bigvee_{\mathtt{l}=0}^{\mathtt{j}-1} (\mathtt{Cong}_{\mathtt{i}}^{\mathtt{l},\mathtt{j}}(\mathtt{t_1}) \wedge \mathtt{Cong}_{\mathtt{i}}^{\mathtt{l},\mathtt{j}}(\mathtt{t_2}))$$

where $\mathtt{Cong}_{\mathtt{i}}^{\mathtt{m},\mathtt{n}}(\mathtt{t})$, with $m < n$, stands for:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}) \wedge \mathtt{p}((+_{\mathtt{i}}1)^{\mathtt{m}}(0_{\mathtt{i}})) \wedge \bigwedge_{\mathtt{l}=0,\mathtt{l}\neq\mathtt{m}}^{\mathtt{n}-1} \neg \mathtt{p}((+_{\mathtt{i}}1)^{\mathtt{l}}(0_{\mathtt{i}})) \wedge$$

$$\wedge \forall \mathtt{y}(\mathtt{T}^{\mathtt{i}}(\mathtt{y}) \wedge (+_{\mathtt{i}}1)^{\mathtt{n}}(0_{\mathtt{i}}) \leq \mathtt{y} \to (\mathtt{p}(\mathtt{y}) \leftrightarrow \exists \mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge (+_{\mathtt{i}}1)^{\mathtt{n}}(\mathtt{z}) = \mathtt{y})))),$$

and then we remove the occurrences of 0_i, $+_i1$ and T^i by executing the operations specified in steps 1, 2 and 3, respectively.

Step 6: removal of the approximate ordering $\mathtt{t_1} \preceq \mathtt{t_2}$.

We replace each occurrence of atomic formulae of the form $\mathtt{t_1} \preceq \mathtt{t_2}$ by the formula:

$$\neg \mathtt{ancestor}(\mathtt{t_1}, \mathtt{t_2}) \wedge \neg \mathtt{ancestor}(\mathtt{t_2}, \mathtt{t_1}) \to \mathtt{t_1} \leq \mathtt{t_2},$$

where $\mathtt{ancestor}(\mathtt{t_1}, \mathtt{t_2})$ is defined as in Section 4.6.4.

Step 7: universal closure.

The formula ϕ' is the universal closure of the formula obtained by the sequential execution of steps 1-6.

The relationship between the original formula $\phi \in \mathcal{UL}^2_{\omega LM^k}$ and the resulting formula $\phi' \in \mathcal{SUL}^2_{\omega LM^k}$ is formally stated by the following lemma.

4.7.4. Lemma. *Let ϕ be an $\mathcal{UL}^2_{\omega LM^k}$-formula and ϕ' be the $\mathcal{SUL}^2_{\omega LM^k}$-formula obtained from ϕ after the execution of the transformation steps* 1-7. *ϕ is satisfiable if and only if ϕ' is satisfiable.*

The proof of the Lemma 4.7.4 is straightforward, and thus omitted.

4.7.2 The language for downward unbounded layered structures

The formulae of the high-level second-order language $\mathcal{DL}^2_{\omega LM^k}$ for the theory of downward unbounded layered structures differ from the ones of $\mathcal{UL}^2_{\omega LM^k}$ only for the fact that atomic formulae having the form $\mathtt{\Delta_i(t)}$ are replaced by atomic formulae having the form $\mathtt{\Delta_{i,j}(t)}$, for $i, j \geq 0$.

Interpretations $\mathcal{I}$ of $\mathcal{DL}^2_{\omega LM^k}$ differ from $\mathcal{UL}^2_{\omega LM^k}$-interpretations only in the definition of the semantic clause for $\downarrow$, and in the replacement of the semantic clause for Δ_i by the following one:

$(\mathtt{\Delta_{i,j}})^{\mathcal{I}} = \{(+_k 1)^i((\downarrow)^k(0, (+_0 1)^j(0_0^{\mathcal{I}}))) : k \geq 0\}.$

This semantic clause states that for any $k \geq 0$ and any $t \in T^k$, $\Delta_{i,j}$ holds at t if and only if t is at distance i from the (unique) element $t' \in T^k$ belonging to the leftmost branch of the tree rooted at the (unique) element $t'' \in T^0$ which is at distance j from 0_0. It is easy to see that the predicate Δ_i of $\mathcal{UL}^2_{\omega LM^k}$ is equivalent to $\Delta_{i,0}$.

The satisfability relation is defined exactly as in the case of $\mathcal{UL}^2_{\omega LM^k}$, except for the obvious replacement of Δ_i by $\Delta_{i,j}$.

4.7.5. EXAMPLE. Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates (denoting a pair of events or of states). We show how the properties of downward unbounded layered structures can be exploited to constrain $\mathtt{p}$ and $\mathtt{q}$ to be locally indistinguishable (resp. distinguishable). We say that two predicates $\mathtt{p}$ and $\mathtt{q}$ are *locally indistinguishable* with respect to a time instant t if both $\mathtt{p}$ and $\mathtt{q}$ hold at t, and there exists a child t' of t such that $\mathtt{p}$ and $\mathtt{q}$ are locally indistinguishable with respect to t'. Two predicates are *locally distinguishable* with respect to a time instant t if they are not locally indistinguishable with respect to it.

Formally, the condition that $\mathtt{p}$ and $\mathtt{q}$ are locally indistinguishable with respect to (a time instant t denoted by) x can be expressed as follows:

$$\exists \mathtt{r}(\mathtt{path}(\mathtt{x},\mathtt{r}) \wedge \forall \mathtt{y}(\mathtt{r}(\mathtt{y}) \rightarrow \mathtt{p}(\mathtt{y}) \wedge \mathtt{q}(\mathtt{y}))),$$

where $path(x, r)$ stands for:

$$\mathtt{r(x)} \wedge \forall \mathtt{y}((\mathtt{r(y)} \rightarrow \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \mathtt{r}(\downarrow(\mathtt{i},\mathtt{y}))) \wedge \forall \mathtt{i},\mathtt{j}(\mathtt{r}(\downarrow(\mathtt{i},\mathtt{y})) \wedge \mathtt{r}(\downarrow(\mathtt{j},\mathtt{y})) \rightarrow \mathtt{i}=\mathtt{j}) \wedge$$

$$\wedge \forall \mathtt{z}(\mathtt{y} \neq \mathtt{x} \wedge \mathtt{r(y)} \wedge \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i},\mathtt{z}) = \mathtt{y} \rightarrow \mathtt{r(z)}) \wedge (\bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i},\mathtt{y}) = \mathtt{x} \rightarrow \neg \mathtt{r(y)}) \wedge$$

$$\wedge(\neg \exists \mathtt{z} \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow(\mathtt{i},\mathtt{z}) = \mathtt{y} \wedge \mathtt{y} \neq \mathtt{x} \rightarrow \neg \mathtt{r(y)})),$$

where $\forall \mathtt{i},\mathtt{j}(\mathtt{r}(\downarrow(\mathtt{i},\mathtt{y})) \wedge \mathtt{r}(\downarrow(\mathtt{j},\mathtt{y})) \rightarrow \mathtt{i}=\mathtt{j})$ is the obvious shorthand.

The condition that $\mathtt{p}$ and $\mathtt{q}$ are locally distinguishable with respect to (a time instant t denoted by) x can be expressed as follows:

$$\forall \mathtt{r}(\mathtt{path}(\mathtt{x},\mathtt{r}) \rightarrow \exists \mathtt{y}(\mathtt{r(y)} \wedge ((\mathtt{p(y)} \wedge \neg \mathtt{q(y)}) \vee (\neg \mathtt{p(y)} \wedge \mathtt{q(y)}) \vee (\neg \mathtt{p(y)} \wedge \neg \mathtt{q(y)})))).$$

As in the upward unbounded case, it is possible to show that $\mathcal{DL}^2_{\omega LM^k}$ is as expressive as $\mathcal{L}^2_{\omega LM^k}$. Let $\phi \in \mathcal{DL}^2_{\omega LM^k}$. The following steps transforms ϕ into an equisatisfiable formula $\phi' \in \mathcal{SDL}^2_{\omega LM^k}$.

Step 1: removal of constants $\mathtt{0_i}$.

For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i}$ (if any) by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_{i-1}}(\forall \mathtt{y}\ \ (\mathtt{x_0} \leq \mathtt{y}) \wedge \bigwedge_{\mathtt{j=0}}^{\mathtt{i-1}} \downarrow (\mathtt{0}, \mathtt{x_j}) = \mathtt{x_{j+1}}).$$

Step 2: removal of successor functions $+_{\mathtt{i}}\mathtt{1}$.

It is exactly as in the case of upward unbounded layered structures.

Step 3: removal of y-contextualizations $\mathtt{T^i}$.

We replace each occurrence of atomic formulae of the form $\mathtt{T^i(t)}$, with $i \geq 0$, by the formula:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_i}(\mathtt{t} = \mathtt{x_i} \wedge \forall \mathtt{y}\ \ \neg(\bigvee_{\mathtt{j=0}}^{\mathtt{k-1}} \downarrow (\mathtt{j}, \mathtt{y}) = \mathtt{x_0}) \wedge \bigwedge_{\mathtt{j=0}}^{\mathtt{i-1}} (\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \downarrow (\mathtt{l}, \mathtt{x_j}) = \mathtt{x_{j+1}})).$$

Step 4: removal of x-contextualizations $\mathtt{\Delta_{i,j}}$.

The removal of x-contextualizations $\mathtt{\Delta_i}$ is performed recursively.
We replace each occurrence of atomic formulae of the form $\mathtt{\Delta_{0,j}(t)}$ by the formula:

$$\exists \mathtt{p}(\mathtt{p}((+_0\mathtt{1})^{\mathtt{j}}(\mathtt{0_0})) \wedge \mathtt{p(t)} \wedge \forall \mathtt{y}, \mathtt{z}(\mathtt{p(y)} \wedge \downarrow (\mathtt{0}, \mathtt{y}) = \mathtt{z} \rightarrow \mathtt{p(z)}) \wedge$$

$$\forall \mathtt{y}, \mathtt{z}(\mathtt{p(y)} \wedge \mathtt{p(z)} \rightarrow \bigwedge_{\mathtt{i=1}}^{\mathtt{k-1}} \neg \downarrow (\mathtt{i}, \mathtt{z}) = \mathtt{y}))$$

where $(+_0 1)^j(0_0)$ is replaced as shown at steps 1 and 2.
For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We replace each occurrence of atomic formulae of the form $\mathtt{\Delta_{i,j}(t)}$ by the formula:

$$\exists \mathtt{x_{n+1}}, \ldots, \mathtt{x_0}(\mathtt{\Delta_{0,j}}(\mathtt{x_{n+1}}) \wedge \mathtt{x_0} = \mathtt{t} \wedge (\bigwedge_{\mathtt{j=0}}^{\mathtt{n}} \downarrow (\mathtt{a_j}, \mathtt{x_{j+1}}) = \mathtt{x_j})) \vee$$

$$\bigvee_{\mathtt{l=0}}^{\lfloor \log_{\mathtt{k}}(\mathtt{i}) \rfloor} (\mathtt{T_l(t)} \wedge \mathtt{\Delta_{r_l, j+q_l}(t)}), \ \ \mathtt{with}\ \ \mathtt{i} = \mathtt{q_l k^l} + \mathtt{r_l}$$

where $T_l(t)$ is replaced as shown at step 4.

Step 5: removal of congruences $\equiv_{\mathtt{i,j}}$.

It is exactly as in the case of upward unbounded layered structures.

Step 6: removal of the approximate ordering $\mathtt{t_1} \preceq \mathtt{t_2}$.

We replace each occurrence of atomic formulae of the form $\mathtt{t_1} \preceq \mathtt{t_2}$ by the formula:

$$\neg \mathtt{ancestor}(\mathtt{t_1}, \mathtt{t_2}) \wedge \neg \mathtt{ancestor}(\mathtt{t_2}, \mathtt{t_1}) \rightarrow \mathtt{t_1} \leq \mathtt{t_2},$$

where $\mathtt{ancestor}(\mathtt{t}_1, \mathtt{t}_2)$ is defined as in Section 4.6.4.

Step 7: universal closure.

The formula ϕ' is the universal closure of the formula obtained by the sequential execution of steps 1-6. The relationship between the original formula $\phi \in \mathcal{DL}^2_{\omega LM^k}$ and the resulting formula $\phi' \in \mathcal{SDL}^2_{\omega LM^k}$ is formally stated by the following lemma.

4.7.6. LEMMA. *Let ϕ be an $\mathcal{DL}^2_{\omega LM^k}$-formula and ϕ' be the $\mathcal{L}^2_{\omega LM^k}$-formula obtained from ϕ after the execution of the transformation steps* 1-7. *ϕ is satisfiable if and only if ϕ' is satisfiable.*

As for Lemma 4.7.4, the proof is straightforward, and thus omitted.

Concluding remark. In Sections 4.7.1 and 4.7.2, we have shown how to express the basic functionalities of metric and layered temporal logic in $\mathcal{L}^2_{\omega LM^k}$. We want to point out that, however, there exist significant properties of ω-layered structures that cannot be expressed in $\mathcal{L}^2_{\omega LM^k}$. As an example, it is not possible to define a binary predicate *same_layer* such that, for all $t, t' \in \bigcup_{i \geq 0} T^i$,

$$same\_layer(t, t') \quad \text{iff} \quad \exists i(t \in T^i \wedge t' \in T^i).$$

The quantificational prefix $\exists i$ stands for an infinite disjunctions that cannot be expressed in $\mathcal{L}^2_{\omega LM^k}$. In fact, extending $\mathcal{L}^2_{\omega LM^k}$ with the predicate *same_layer* would make the theory of downward unbounded layered structures undecidable, a result which follows from the undecidability of the extension of $S2S$ with the *equal_level* predicate E given by $E(u, v)$ if and only if $|u| = |v|$, with $u, v \in \{0, 1\}^*$ (cf. [77]). We conjecture that a similar undecidability result holds for the theory of upward unbounded ones.

Concluding remarks

In this chapter, we have first proved the decidability of the theory of finitely-layered metric temporal structures through its reduction to the decidable theory *S1S*. Then, we considered the case of structures provided with an infinite number of either arbitrarily coarse or arbitrarily fine temporal layers. We first showed that the satisfiability and validity problems for the theory of upward unbounded layered structures are decidable by reducing them, through coding into the theory $S1S^k$, to the decidable problem of determining whether or not the ω-language recognized by a given systolic tree automaton is empty. Then, we obtained the same result for the theory of downward unbounded layered structures through coding into the theory SkS. We are currently exploring the natural generalization to layered structures which are both upward and downward unbounded.

It is worth noting that the questions whether or not we can decide upward unbounded layered structures using Büchi automata and whether or not we can decide downward unbounded layered structures using k-ary systolic tree automata are still open questions. Our conjecture is that the proposed reductions are actually the minimal ones. More generally, we are currently investigating the relations between $S1S^i$ and SjS, with $i, j > 1$, as well as the relations between $S1S^i$ and $S1S^j$, with $i \neq j$, in order to formally characterize the relationships between their temporal logic counterparts.

Finally, notice that the above results directly hold for metric and layered temporal logics non-axiomatically defined. Indeed, we identified relevant classes of temporal structures, we defined the corresponding theories, and we showed that such theories can be reduced to decidable ones. A temporal logic axiomatic counterpart of these theories can be obtained extending a simplified variant of $TPTL$ (real-time propositional temporal logic), where state variables are replaced by time variables and $\bigcirc$ is interpreted as the successor over time, with contextual and projection operators of $MLTL$. Moreover, since the validity problem is non-elementary already for the classical first-order theory of natural numbers with linear order and monadic predicates, it is obviously non-elementary also for the considered theories. Nevertheless, we expect that temporal logic counterparts of the proposed theories corresponding to elementary, yet expressively complete, fragments of $\mathcal{L}^2_{nLM}$ and $\mathcal{L}^2_{\omega LM^k}$ can be identified.

Chapter 5

Executing metric temporal logic

5.1 Introduction

In this chapter, we propose a novel set-theoretic translation method (hereafter, the $\Box$-as-*Pow* translation) to support derivability in propositional modal logic, whose basic idea is to map modal formulae into set-theoretic terms, and show how it can be used to execute metric temporal logics (and all metric and layered temporal logics that can be reduced to them, e.g., finitely-layered metric temporal logics). Most inference systems for modal logic are defined in the style of sequent or tableaux calculi, e.g. [48, 124]. As an alternative, a number of *translation* methods for modal logic into classical first-order logic have been proposed in the literature (for an up-to-date survey see [100]). Such methods allow the use of Predicate Calculus mechanical theorem provers to implement modal theorem provers. Compared with the direct approach of finding a proof algorithm for a specific class of modal logics, the translation methods have the advantage of being *independent* of the particular modal logic under consideration: a single theorem prover may be used for any translatable modal logic.

In the standard approach, the first-order language $\mathcal{L}$ into which the translation is carried out contains a constant τ denoting the initial world in the frame, a binary relation $R(x, y)$ denoting the accessibility relation, and a denumerable number of unary predicates $\mathtt{P_i}(x)$. The translation function π is defined by induction on the structural complexity of the modal formula as follows:

- $\pi(P_j, x) \equiv \mathtt{P_j}(x)$;
- $\pi(-, x)$ commutes with the boolean connectives;
- $\pi(\Box\, \psi, x) \equiv \forall y(xRy \to \pi(\psi, y))$.

Let H be a normal modal logic and ϕ be a modal formula. H is *first-order complete* if there exists a first-order sentence $Axiom_H$, involving only equality and the binary relational symbol $R(x, y)$, such that ϕ is derivable from H if and only if ϕ is true in the initial world

τ of all generated frames satisfying $Axiom_H$ [7, 69]. For these logics the following holds:

$$\vdash_H \phi \;\Leftrightarrow \vdash Axiom_H \rightarrow \pi(\phi, \tau),$$

where $\vdash$ stands for derivability in classical Predicate Calculus. Hence, as long as we have $Axiom_H$, a classical theorem prover can be used as a theorem prover for H.

Efficiency concerns have motivated further investigations on the above (relational) translation method. Such studies (e.g. [99]) suggested a "functional" semantics for modal logic and resulted in a family of more efficient and general translation methods. From the computational point of view, the functional translation may still cause some problem when using a first-order theorem prover, due to the presence of equalities in $Axiom_H$. A method for limiting the complexity induced by the introduction of equality using a mixed relational/functional translation is proposed in [98].

A common feature of all the methods mentioned above is that, in order to be applied directly, the underlying modal logic must have a first-order semantics: insofar as we are aware, all attempts to deal with logics not having a first-order semantics have required *ad-hoc* techniques. Moreover, if the logic has a first-order semantics, but it is only specified by Hilbert axioms, a preliminary step is necessary to find the corresponding first-order axioms. The question of automatically solving this last problem has been extensively studied and algorithms have been proposed, e.g. [7, 54].

One of the main motivations of the present work was to find a translation applicable to all complete modal logics, regardless of the first-order axiomatizability of their semantics. The $\Box$-as-Pow translation we propose works for all normal complete finitely axiomatizable modal logics. In particular, our method also works if the modal logic under consideration is only specified by Hilbert axioms.

The basic idea is to represent any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$[1]. Given a modal formula $\phi(P_1, \ldots, P_n)$ we define its translation as the *set-theoretic term* $\phi^*(x, x_1, \ldots, x_n)$, with variables $x, x_1, \ldots, x_n$, built using $\cup, \setminus$, and Pow. Intuitively, $\phi^*(x, x_1, \ldots, x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, \ldots, x_n)$ is rather straightforward except for the case of $\Box\,\phi$, whose translation is defined as: $(\Box\,\phi)^* \equiv Pow(\phi^*)$ (see Section 5.2 for details).

In order to achieve a *computationally valid result*, we want to refer to a *finitely* (first-order) axiomatizable set theory. We succeeded in carrying out our translation in a very weak set theory called Ω (compare this theory with more classical finite axiomatizations of Set Theory, such as NBG [80]).

We prove that for any normal modal logic

$$H = K + \psi(\alpha_{j_1}, \ldots, \alpha_{j_m}),$$

where $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ is an axiom schema, the following holds:

$$\vdash_H \phi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, x_1, \ldots x_n)))$$

[1]It is worth mentioning that a similar idea has been recently exploited by Barwise and Moss to "model" non-standard set theories (e.g., non-well-founded set theory) as infinitary modal logics [4].

and

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n))) \;\Rightarrow\; \psi \models \phi,$$

where $Trans(x)$ and $Axiom_H(x)$ stand for

$$\forall y\ (y \in x \to y \subseteq x) \text{ and } \forall x_{j_1}, \ldots, \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, \ldots, x_{j_m})),$$

respectively, and $\models$ represents frame logical consequence. In the case of frame-complete theories H, the proposed translation captures exactly the notion of H-derivability.

Instead of translating Hilbert axioms a set-theoretic semantics for H can be used, whenever such a semantics is available. We consider the case of G as an example of this approach.

Then, we generalize the $\Box$-as-Pow translation to polymodal logics. Such a generalization is not only the first step towards the definition of a set-theoretic translation method for *MTL*, but it is also an independent modal result. It involves a revision of the definition of the translation function to cope with a finite set of distinct modal operators instead of a single one. The technique we employ presents some similarities with the one introduced by Thomason in [120]; the use of a set-theoretic language, however, greatly simplifies Thomason's approach and turns out to be completely symmetric. It is worth noting that, since it is possible to reduce frame validity in tense logic to that in polymodal logic (cf. [121]), the composition of the $\Box$-as-Pow translation for polymodal logics with such a reduction allow us to deal with tense logic[2]. Next, we consider the problem of applying the $\Box$-as-Pow translation to *MTL*. On the basis of its rewriting as a PDL-like logic (cf. Chapter 2), *MTL* can be viewed as a polymodal logic provided with an arbitrary number of modal operators. To handle this (possibly infinite) set of operators, we consider a suitable modification of the set-theoretic translation for polymodal logics, and prove its soundness and completeness with respect to derivability in *MTL*.

The translation method we propose here may also be considered from a more abstract point of view as a means to analyze general deduction for modal formulae. Some results in this direction are reported in the last part of the chapter, together with a comparison with standard translation (a complete treatment can be found in [10, 11, 12]). We also briefly describe the application of set T-resolution techniques to support derivability in Ω. In order to apply such techniques, it is necessary to guarantee the decidability, with respect to Ω, of the class of ground formulae written in any language which extends the one in which the axioms of Ω are written with Skolem functions. We succeeded in providing such a decidability result in a suitable variant of Ω, and the main steps of the proof are outlined in Section 5.6 (the details of the proof can be found in [36, 37, 39]).

The chapter is organized as follows. In Section 5.2, we introduce the basic features of the $\Box$-as-Pow translation and show how to apply it to the modal logic G. In this case, the proofs are simple and a clear description of the main features of the translation method

[2]As an alternative, a specific translation method for tense logic can be devised that slightly extends Ω, but keeps the translation simpler. This method is defined in [11]).

is possible; moreover, G provides an example of how the method applies to a logic with a non-first-order semantics. In Section 5.3, we consider the general case and exploit the possibility of translating the Hilbert axioms of the logic. The proof of soundness of the translation is carried out using a particular universe of non-well-founded sets and applies to a large class of extensions of Ω. In Section 5.4, we generalize the proposed method to polymodal logics using a set-theoretic counterpart of Thomason's technique for translating polymodal logics into monomodal ones [120, 121]. In Section 5.5, we revise the $\Box$-as-Pow translation for polymodal logics to apply it to MTL. Finally, in Section 5.6 we briefly discuss some related work.

5.2 A set-theoretic translation of G

We first consider the case of the propositional modal logic G obtained by adding the Löb's axiom schema $\Box(\Box\alpha \to \alpha) \to \Box\alpha$ to K. Our goal is to find a translation of G formulae in the language of set theory and a finitely axiomatizable theory Ω such that, for any modal formula ϕ, $\vdash_G \phi$ if and only if Ω proves the translation of ϕ.

We consider the theory Ω specified by the following axioms in the language with relational symbols $\in, \subseteq$, and functional symbols $\cup, \setminus, Pow$:

$x \in y \cup z \leftrightarrow x \in y \vee x \in z$;
$x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z$;
$x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y)$;
$x \in Pow(y) \leftrightarrow x \subseteq y$.

Notice that neither the extensionality axiom nor the axiom of foundation are in Ω. In the next section, we will make an essential use of the latter fact: since we will model the accessibility relation by the membership relation, we will be forced to work in universes containing non-well-founded sets. As a matter of fact, it will be convenient to use universes satisfying AFA [1]. However, in the case of G a standard (well-founded) model of set theory is sufficient to carry out the proof of the soundness of the translation.

Given a modal formula $\phi(P_1, \ldots, P_n)$, its translation is the *set-theoretic term* $\phi^*(x, x_1, \ldots, x_n)$, with variables $x, x_1, \ldots, x_n$, inductively defined as follows:

- $P_i^* \equiv x_i$;
- $(\phi \vee \psi)^* \equiv \phi^* \cup \psi^*$;
- $(\phi \wedge \psi)^* \equiv \phi^* \cap \psi^*$;
- $(\neg\phi)^* \equiv x \setminus \phi^*$;
- $(\phi \to \psi)^* \equiv (x \setminus \phi^*) \cup \psi^*$;
- $(\Box\phi)^* \equiv Pow(\phi^*)$,

where x is different from x_i for $i = 1, \ldots, n$, $\phi^* \cap \psi^*$ stands for $\phi^* \setminus (\phi^* \setminus \psi^*)$, and $\Diamond$ is translated as $\neg\Box\neg$.

We will show that:

$$\vdash_G \phi \Leftrightarrow \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))),$$

where $Trans(x)$ stands for $\forall y(y \in x \to y \subseteq x)$ (x is transitive), and $Axiom_G(x)$ represents the conjunction of $\forall y(y \subseteq x \wedge \exists z(z \in y) \to \exists s(s \in y \wedge \forall v(v \notin s \cap y)))$ and $\forall z \forall w \forall y(z \in x \wedge w \in x \wedge y \in x \wedge z \in w \wedge w \in y \to z \in y)$ (x is well-founded and $\in$ restricted to x is transitive, respectively).

We prove that the proposed translation is complete and sound. The proof of completeness is straightforward; the proof of soundness relies on the characterization of G using the class of all finite trees.

5.2.1. THEOREM. **(Completeness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_G \phi \ \Rightarrow\ \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Proof. The proof is by induction on the derivation of $\vdash_G \phi(P_1, \ldots, P_n)$. The cases of tautologies and closure under modus ponens do not present any difficulty, and thus they are left to the reader (a proof can be found in [36]). We explicitly prove the result for K and Löb's axiom schemata, and for closure under necessitation. We first consider the axiom schema K:

$$\Box(\alpha \to \beta) \to (\Box \alpha \to \Box \beta).$$

Without loss of generality, we suppose that α and β involve n propositional variables $P_1, \ldots, P_n$, and show that Ω derives its translation, namely:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq (\Box(\alpha \to \beta) \to (\Box \alpha \to \Box \beta))^*)).$$

By definition, $(\Box(\alpha \to \beta) \to (\Box \alpha \to \Box \beta))^* \equiv (x \setminus t) \cup s$, where $t = Pow((x \setminus \alpha^*) \cup \beta^*)$ and $s = (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$ are terms on the variables $x, x_1, \ldots, x_n$. We have to prove that $\forall z(z \in x \to z \in (x \setminus t) \cup s)$, or, equivalently, that $\forall z(z \in x \wedge z \in t \to z \in s)$. By replacing t and s by their definitions, we may rewrite the last condition as: $z \in x$ and $z \subseteq (x \setminus \alpha^*) \cup \beta^*$ implies that $z \in (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$. To prove it, it suffices to show that $z \in x$, $z \subseteq (x \setminus \alpha^*) \cup \beta^*$, and $z \subseteq \alpha^*$ implies that $z \subseteq \beta^*$. Since $z \subseteq \alpha^*$, for each s, if $s \in z$, then $s \in \alpha^*$; from $z \subseteq (x \setminus \alpha^*) \cup \beta^*$, it follows that $s \in \beta^*$. Notice that we never used the hypothesis that x satisfies $Trans(x)$ and $Axiom_G(x)$.

Consider now the closure under necessitation: if $\vdash_G \phi$, then $\vdash_G \Box \phi$. In this case, we suppose that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*)),$$

and prove that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq Pow(\phi^*))).$$

For each x satisfying $Trans(x)$ and $Axiom_G(x)$, we prove that $\forall x_1, \ldots, \forall x_n(x \subseteq Pow(\phi^*))$, that is, for each z, if $z \in x$, then $z \in Pow(\phi^*)$ or, equivalently, $z \subseteq \phi^*$. Suppose that $z \in x$ and $t \in z$. From the validity of $Trans(x)$, it follows that $z \subseteq x$ and thus $t \in x$. The conclusion $t \in \phi^*$ directly follows from the hypothesis that $x \subseteq \phi^*$.

Finally, let us show that Ω proves the translation of Löb's axiom, that is, if $P_1, \ldots, P_n$ are the n variables occurring in ϕ, then

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq (\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi)^*)).$$

The proof is nothing but the formalization in Ω of the proof of the validity of Löb's axiom schema in any well-founded transitive frame (cf., e.g., [114]).

By definition, $(\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi)^* \equiv (x \setminus t) \cup Pow(\phi^*)$, where t stands for the term $Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$. We want to prove that, if x satisfies $Trans(x) \wedge Axiom_G(x)$, then $\forall s(s \in x \wedge s \in t \rightarrow s \in Pow(\phi^*))$. This is equivalent to showing that there exists no set belonging to the subset y of x with $y = x \cap t \setminus Pow(\phi^*)$. We consider the formula:

$$\forall y(\forall s(s \in y \rightarrow \exists v(v \in s \cap y)) \rightarrow (y \subseteq x \rightarrow \forall z(z \notin y))),$$

which can be derived from the axiom stating the well-foundedness of x, and show that for $y = x \cap t \setminus Pow(\phi^*)$ the formula $\forall s(s \in y \rightarrow \exists v(v \in s \cap y))$ holds. Since $y \subseteq x$, this proves the result.

If $s \in y$, then $s \in x$, $s \in t$, with $t = Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, and $s \notin Pow(\phi^*)$. From the last conjunct, we derive that $\exists v(v \in s \wedge v \notin \phi^*)$. Since x satisfies $Trans(x)$ and $Axiom_G(x)$ (in particular, the transitivity of $\in$ with respect to x holds) and $s \in x$, from $v \in s$, it follows that $v \subseteq s$. Now, from $s \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$ and $v \subseteq s$, it follows that $v \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, that is, $v \in t$. Finally, from $v \in s$, $s \subseteq (x \setminus Pow(\phi^*)) \cup \phi^*$, and $v \notin \phi^*$, it follows that $v \in x \setminus Pow(\phi^*)$, and then $v \notin Pow(\phi^*)$. From $v \in x$, $v \in t$, and $v \notin Pow(\phi^*)$, we can conclude that $v \in x \cap t \setminus Pow(\phi^*) = y$, and that proves the result. $\dashv$

It is worth noting that all the set-theoretic principles involved in the proof of completeness are those expressed by the (extremely simple) axioms of Ω.

The proof of soundness exploits the (frame) characterization theorem for G stating that $\vdash_G \phi$ if and only if ϕ is valid in every finite tree, where by a finite tree is meant a frame (W, R, r) in which W is a finite set containing the element r (the root), R is transitive and asymmetric, and the set of $R-$predecessors of any element contains r and is linearly ordered by R (see [114] for details).

5.2.2. Theorem. **(Soundness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))) \;\Rightarrow \vdash_G \phi.$$

Proof. Let HF^A be the structure for the language of Ω consisting of all the hereditarily finite sets built from atoms in $A = \{a_0, a_1, \ldots\}$, with the natural set-theoretic interpretation of the relational and functional symbols $\in, \subseteq, \cap, \cup, \setminus$, and Pow. HF^A is a model for Ω [70]. Therefore, for every term $t(x_0, \ldots, x_n)$ and for every $h_0, \ldots, h_n$ in HF^A, we may consider the element $t^{HF^A}(h_0, \ldots, h_n)$ in HF^A. Moreover, if $\phi(P_1, \ldots, P_n)$ is a modal formula, the evaluation of the term $\phi^*(x, x_1, \ldots, x_n)$ over the elements $h_0, \ldots, h_n$ results in an element of HF^A.

Given a finite tree (W, R, r), we determine an element W^* of HF^A such that:

1. $Trans(W^*) \wedge Axiom_G(W^*)$ holds in the model HF^A, and
2. given a modal formula $\phi(P_1, \ldots, P_n)$, if $\forall x_1 \ldots \forall x_n(W^* \subseteq \phi^*(W^*, x_1, \ldots, x_n))$ holds in HF^A, then $\phi(P_1, \ldots, P_n)$ is valid in (W, R, r).

Fix an injection π from the leaves of W (i.e. nodes without any successor) to A. We define W^* in HF^A as follows: for every node $w \in W$, let

$$w^* = \begin{cases} \pi(w) & \text{if } w \text{ is a leaf of } W, \\ \{v^* : wRv\} & \text{otherwise.} \end{cases}$$

Let W^* be r^*. For every $w \in W$, $w^* \in HF^A$; moreover, it is not difficult to see that $Trans(W^*)$ and $Axiom_G(W^*)$ hold in HF^A.

Let $\models$ be a valuation of the propositional variables $P_1, \ldots, P_n$ on W and, for $i = 1, \ldots n$, let $P_i^* = \{w^* \in W^* : w \models P_i\}$. Since W is finite, we have that $P_1^*, \ldots, P_n^*$ belong to HF^A.

If the elements w^* and v^* are equal in HF^A, then $w = v$ (by induction on the height $h(w)$ of the node w in the tree (W, R, r)). This fact will be useful in proving the following lemma.

5.2.3. Lemma. *For all $w \in W$ and for any formula $\phi(P_1, \ldots, P_n)$,*

$$w \models \phi(P_1, \ldots, P_n) \ \Leftrightarrow \ w^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*) \text{ holds in } HF^A.$$

Proof. By induction on the structural complexity of the formula $\phi(P_1, \ldots, P_n)$.

If $\phi(P_1, \ldots, P_n) \equiv P_i$ and $w \models P_i$, then, by definition of P_i^*, $w^* \in P_i^*$. Vice versa, if $w^* \in P_i^*$, then $z \models P_i$ for some $z \in W$ with $w^* = z^*$; hence, as we observed, $w = z$ and therefore $w \models P_i$.

The case of boolean connectives is straightforward.

Now consider the formula $\Box\, \phi(P_1, \ldots, P_n)$:

$$\begin{array}{rl} w \models \Box\, \phi(P_1, \ldots, P_n) \Leftrightarrow & \\ \forall z \in W\ (wRz \to z \models \phi(P_1, \ldots, P_n)) \Leftrightarrow & \\ \forall z \in W\ (wRz \to z^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*)) \Leftrightarrow & \\ \{z^* : wRz\} \subseteq \phi^*(W^*, P_1^*, \ldots, P_n^*) \Leftrightarrow & w^* \subseteq \phi^*(W^*, P_1^*, \ldots, P_n^*) \Leftrightarrow \\ w^* \in Pow^*(\phi^*(W^*, P_1^*, \ldots, P_n^*)) \Leftrightarrow & w^* \in (\Box\, \phi)^*(W^*, P_1^*, \ldots, P_n^*). \end{array}$$

$\dashv$

From Lemma 5.2.3, we have that $\phi(P_1, \ldots, P_n)$ is valid in the model $(W, R, \models)$ if and only if the corresponding set W^* in HF^A is a subset of $\phi^*(W^*, P_1^*, \ldots, P_n^*)$. From this, item 2 above easily follows.

To conclude the proof of Theorem 5.2.2, suppose that

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

If (W, R, r) is a finite tree, then the corresponding set W^* in HF^A satisfies $Trans(W^*) \wedge Axiom_G(W^*)$. Hence, from $\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*))$, it follows that, for all elements $h_1, \ldots, h_n$ in HF^A, we have that $W^* \subseteq \phi^*(W^*, h_1, \ldots, h_n)$. In particular, for all valuations $\models$ of the propositional variables $P_1, \ldots, P_n$ on W, the above is true for the sets $P_1^*, \ldots, P_n^*$ defined as in Lemma 5.2.3. From the same lemma, one deduces that $\phi(P_1, \ldots, P_n)$ is valid in the model $(W, R, \models)$ and, from the Finite Tree Completeness Theorem [114], it follows that $\vdash_G \phi$. $\dashv$

5.3 The $\Box$-as-Pow translation method

In this section we generalize the translation method to any normal finitely axiomatizable modal logic, possibly specified by Hilbert axioms only.

Let $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ be an axiom schema and H be the modal logic obtained by adding $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ to K. The completeness of the translation will be shown with respect to derivability in H, while soundness holds with respect to logical consequence. More formally, we will prove that, for any formula ϕ involving n propositional variables $P_1, \ldots, P_n$,

$$\vdash_H \phi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n)))$$

and

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n))) \;\Rightarrow\; \psi \models \phi,$$

where $Trans(x)$ is the formula $\forall y\ (y \in x \to y \subseteq x)$ and $Axiom_H(x)$ is the formula $\forall x_{j_1}, \ldots, \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, \ldots, x_{j_m}))$.

In case H is complete, the notions of $\vdash_H$ and $\psi \models$ coincide and modal derivability of a given formula in H is equivalent to first-order derivability of the translated formula in Ω.

5.3.1. THEOREM. **(Completeness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_H \phi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Proof. The proof follows the same path of the proof of Theorem 5.2.1, except for the verification of the case in which the formula ϕ is an instance of the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For this case it is easy to check that the term $(\psi(\alpha_{j_1}, \ldots, \alpha_{j_m}))^*$ is syntactically equal to the term $\psi^*(x_1/\alpha_{j_1}^*, \ldots, x_m/\alpha_{j_m}^*)$, and the result follows from $Axiom_H(x)$ and simultaneous substitution in Ω. $\dashv$

5.3.2. THEOREM. **(Soundness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))) \Rightarrow \psi \models \phi$$

Proof. Hereafter, let $\mathcal{U}$ denote a universe of hypersets satisfying all the axioms of $ZF - FA$ (ZF except the foundation axiom) and AFA. In $\mathcal{U}$, for any graph (W, R), there is a (unique) function d such that, for every $w \in W$, the following holds (see [1] for details):

$$d(w) = \{d(v) | v \in W \wedge wRv\}.$$

Actually, it can be seen that the use of AFA is not essential for this proof and a model falsifying foundation "whenever needed" could be used in its place. However, as we will see, the use of AFA will simplify our argument making the construction more uniform.

We begin proving the following lemma:

5.3.3. LEMMA. *Let α be an ordinal, V_α be the set of all well-founded sets of rank less than α, and $\mathcal{U} \setminus V_\alpha$ be the universe of all hypersets not belonging to V_α.*

The structure for the language of Ω with support (domain) $\mathcal{U} \setminus V_\alpha$ and interpretation function $(.)'$ defined as follows[3]*:*

$$\begin{array}{rcl} x \in' y & \textit{iff} & x \in y; \\ x \cup' y & = & x \cup y; \\ x \subseteq' y & \textit{iff} & x \setminus V_\alpha \subseteq y; \\ x \setminus' y & = & \left\{ \begin{array}{ll} x \setminus y & \textit{if } x \setminus y \notin V_\alpha \\ V_\alpha & \textit{otherwise.} \end{array} \right. ; \\ Pow'(y) & = & \{x : x \setminus V_\alpha \subseteq y\}. \end{array}$$

is a model of Ω.

Proof. We first show that $\setminus'$, Pow', and $\cup'$ are well-defined over $\mathcal{U} \setminus V_\alpha$.

The proof for $\setminus'$ follows directly from its definition since either $x \setminus y$ does not belong to V_α and then $\setminus'$ is equal to $x \setminus y$, or it is actually equal to V_α which does not belong to V_α.

The case of Pow' is also straightforward: proceeding by contradiction, suppose that $y \notin V_\alpha$ and $Pow'(y) \in V_\alpha$. By definition, it follows that $y \in Pow'(y)$ and, from the hypothesis, $Pow'(y) \subseteq V_\alpha$ since V_α is transitive. Hence $y \in V_\alpha$, while we assumed $y \notin V_\alpha$.

Finally, for $x \cup' y$ notice that $x \cup' y$ is equal to $x \cup y$ by definition, and $x \cup y \in V_\alpha$ if and only if $x \in V_\alpha$ and $y \in V_\alpha$.

To complete the proof we must show that the proposed interpretation verifies the axioms of Ω.

Since $x \cup' y$ and $\in'$ are defined as $x \cup y$ and $\in$, respectively, the verification of the first axiom is trivial.

Now consider the second axiom. Let x, y, z belong to $\mathcal{U} \setminus V_\alpha$. If $y \setminus z \in V_\alpha$ then $y \setminus' z = V_\alpha$, and thus there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y \setminus' z$. Since from $y \setminus z \in V_\alpha$ it follows that $y \setminus z \subseteq V_\alpha$, there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y$ and $x \notin' z$. In case $y \setminus z \notin V_\alpha$ we have that $y \setminus' z$ is equal to $y \setminus z$ and therefore the axiom is verified.

For the third axiom, suppose that x, y, z belong to $\mathcal{U} \setminus V_\alpha$. By definition, $x \subseteq' y$ if and only if $x \setminus V_\alpha \subseteq y$, which is equivalent to saying that for all z in $\mathcal{U} \setminus V_\alpha$, if $z \in x$ then $z \in y$, namely that $\forall z(z \in' x \rightarrow z \in' y)$ holds in $\mathcal{U} \setminus V_\alpha$.

Finally consider the fourth axiom. Let x belong to $\mathcal{U} \setminus V_\alpha$. From the definition of $Pow'(y)$, it follows that $x \in' Pow'(y)$ if and only if $x \setminus V_\alpha \subseteq y$; but $x \setminus V_\alpha \subseteq y$ if and only if $x \subseteq' y$ and therefore the axiom is verified. $\dashv$

[3] We denote the defined interpretation of symbols $\in, \cup, \setminus, \subseteq, Pow$ in $\mathcal{U} \setminus V_\alpha$ by $\in', \cup', \setminus', \subseteq', Pow'$, and the *standard* interpretation in $\mathcal{U}$ simply by $\in, \cup, \setminus, \subseteq, Pow$.

Given a frame (W, R), we want to embed it into the universe $\mathcal{U} \setminus V_\alpha$, for some suitable α. Let us associate a set $a\downarrow$ in $\mathcal{U}$ with each world $a \in W$. From *AFA* it follows that, for each $a \in W$, there exists a unique labeled decoration * such that $a^* = \{b^* : aRb\} \cup a\downarrow$ (cf. [1]). Moreover, it is possible to define $a\downarrow$ in such a manner that, for each a, b in W, $a^* \notin b\downarrow$ and $a \neq b$ in W implies $a^* \neq b^*$. For this purpose, let us consider a set $\widetilde{W}$ in $\mathcal{U}$, whose elements are well-founded sets of the same rank α, and such that there exists a bijection between $\widetilde{W}$ and W. For each $a \in W$, we denote the image of a in $\widetilde{W}$ by $\tilde{a}$, and define $a\downarrow = \{\tilde{a}\}$. The following lemma can be easily proved.

5.3.4. Lemma. *For each a, b in W,*

(i) $a^* \notin b\downarrow$;
(ii) $a \neq b$ implies $a^* \neq b^*$;
(iii) $a^* \notin V_{\alpha+1}$ and $a^* \setminus V_{\alpha+1} = \{b^* : aRb\}$.

Proof. (i) If $a^* \in b\downarrow$, then $b\downarrow = \{\tilde{b}\}$ implies that $a^* = \tilde{b}$. Since $\tilde{a} \in a\downarrow$ and $a\downarrow \subseteq a^*$, it follows that $\tilde{a} \in \tilde{b}$, which is impossible because $\tilde{a}$ and $\tilde{b}$ have the same rank α. (ii) If $a^* = b^*$, then $\tilde{a} \in b^* = \{c^* : bRc\} \cup b\downarrow$. If $\tilde{a} \in b\downarrow$, then $\tilde{a} = \tilde{b}$, contradicting $a \neq b$. If $\tilde{a} = c^*$, for a given c such that bRc, then $\tilde{c} \in \tilde{a}$ (contradiction). (iii) Immediate. ⊣

Now consider a valuation $\models$ of the propositional variables $P_1, \ldots, P_n$ over the frame (W, R), and let W^* be equal to $\{a^* : a \in W\}$, where * is the labeled decoration previously introduced. W^* does not belong to $V_{\alpha+1}$ because $V_{\alpha+1}$ is transitive and, for each $a \in W$, $a^* \notin V_{\alpha+1}$. Furthermore, let P_i^* be equal to $\{a^* \in W^* : a \models P_i\}$ if this set is not empty, and to $V_{\alpha+1}$ otherwise. The following lemma holds:

5.3.5. Lemma. *For each $a \in W$ and each formula $\phi(P_1, \ldots, P_n)$*

$$a \models \phi(P_1, \ldots, P_n) \;\Leftrightarrow\; a^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*) \text{ in the universe } \mathcal{U} \setminus V_{\alpha+1}.$$

Proof. By induction on the structural complexity of the formula $\phi(P_1, \ldots, P_n)$.

The cases of propositional variables and boolean combinations of formulae are left to the reader.

In the case of a formula of the form $\Box\,\phi(P_1, \ldots, P_n)$, we have that:

$$\begin{array}{rl}
a \models \Box\,\phi(P_1, \ldots, P_n) & \Leftrightarrow \\
\forall b \in W\ (aRb \rightarrow b \models \phi(P_1, \ldots, P_n)) & \Leftrightarrow \\
\forall b \in W\ (aRb \rightarrow b^* \in' \phi^*(W^*, P_1^*, \ldots, P_n^*)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \phi^*(W^*, P_1^*, \ldots, P_n^*) & \Leftrightarrow \\
a^* \in' Pow'(\phi^*(W^*, P_1^*, \ldots, P_n^*)) & \Leftrightarrow \\
a^* \in' (\Box\,\phi)^*(W^*, P_1^*, \ldots, P_n^*). &
\end{array}$$

⊣

From Lemma 5.3.5, it follows that a formula $\phi(P_1, \ldots, P_n)$ is valid in a model $(W, R, \models)$ if and only if W^* is a subset of $\phi^*(W^*, P_1^*, \ldots, P_n^*)$ in the model $\mathcal{U} \setminus V_{\alpha+1}$. This result can be generalized to frames.

5.3.6. LEMMA. *A formula $\phi(P_1, \ldots, P_n)$ is valid in the frame (W, R) if and only if, for the corresponding hyperset W^*,*

$$\forall x_1, \ldots, x_n (W^* \subseteq' \phi^*(W^*, x_1, \ldots, x_n))$$

holds in $\mathcal{U} \setminus V_{\alpha+1}$.

Proof. First of all, we show that, for each $a \in W$ and $x_1, \ldots, x_n \in \mathcal{U} \setminus V_{\alpha+1}$,

$$a^* \in' \phi^*(W^*, x_1, \ldots, x_n) \;\Leftrightarrow\; a^* \in' \phi^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^*).$$

The proof is by induction on the structural complexity of the formula ϕ. We only report the proof of the inductive step for $\phi \equiv \Box\,\beta$, leaving the remaining cases to the reader (complete details can be found in [36]):

$$\begin{array}{rl}
a^* \in' (\Box\,\beta)^*(W^*, x_1, \ldots, x_n) & \Leftrightarrow \\
a^* \in' Pow'(\beta^*(W^*, x_1, \ldots, x_n)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*, x_1, \ldots, x_n) & \Leftrightarrow \\
\forall b \in W \;(aRb \to b^* \in' \beta^*(W^*, x_1, \ldots, x_n)) & \Leftrightarrow \\
\forall b \in W \;(aRb \to b^* \in' \beta^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^*)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^* & \Leftrightarrow \\
a^* \in' (\Box\,\beta)^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^*). &
\end{array}$$

Given n hypersets $x_1, \ldots, x_n$ in $\mathcal{U} \setminus V_{\alpha+1}$, let $\models$ be a valuation of $P_1, \ldots, P_n$ such that, for each $a \in W$, $a \models P_i$ if and only if $a^* \in' x_i \cap' W^*$.

It is straightforward to see that, if $P_1^*, \ldots, P_n^*$ are the hypersets defined in Lemma 5.3.5 on the basis of the valuation $\models$, then P_i^* and $x_i \cap' W^*$ have the same elements in the model $\mathcal{U} \setminus V_{\alpha+1}$. From this, it is easy to verify by induction on ϕ that $\phi^*(W^*, P_1^*, \ldots, P_n^*)$ and $\phi^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^*)$ have the same elements.

If a formula ϕ is valid in the frame (W, R), then it is also valid in the model $(W, R, \models)$, and from Lemma 5.3.5 it follows that, for all $a \in W$, $a^* \in' \phi^*(W^*, x_1 \cap' W^*, \ldots, x_n \cap' W^*)$. This allows us to conclude that $a^* \in' \phi^*(W^*, x_1, \ldots, x_n)$, and, therefore, for all hypersets $x_1, \ldots, x_n$ in $\mathcal{U} \setminus V_{\alpha+1}$, $W^* \subseteq' \phi^*(W^*, x_1, \ldots, x_n)$.

The converse can easily be proved by associating the hypersets $P_1^*, \ldots, P_n^*$ (where P_i^* is equal to $\{a^* \in' W^* : a \models P_i\}$ if this set is not empty, and to $V_{\alpha+1}$ otherwise) with each valuation $\models$ of $P_1, \ldots, P_n$. $\dashv$

To conclude the proof of Theorem 5.3.2, let us suppose that

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1, \ldots, \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Let (W, R) be a frame in which the formula $\psi(P_{j_1}, \ldots, P_{j_m})$ is valid; from Lemma 5.3.6, it follows that the formula $\forall x_{j_1}, \ldots, \forall x_{j_m}(W^* \subseteq' \psi^*(W^*, x_{j_1}, \ldots, x_{j_m}))$ is true in the universe $\mathcal{U} \setminus V_{\alpha+1}$. Furthermore, it is easy to prove that $Trans(W^*)$ holds as well. Since $\mathcal{U} \setminus V_{\alpha+1}$ is an Ω-model, from the hypotheses it follows that the formula $\forall x_1, \ldots, \forall x_n \;(W^* \subseteq' \phi^*(W^*, x_1, \ldots, x_n))$ is true in $\mathcal{U} \setminus V_{\alpha+1}$, and thus, again by Lemma 5.3.6, that the formula ϕ is valid in (W, R). $\dashv$

Remark. From the preceding proof, it should be clear that the proposed translation method works for any theory Ω^* extending Ω, provided that the model $\mathcal{U} \setminus V_\alpha$ of Lemma 5.3.3 is a model of Ω^*. This fact will play an essential role in Section 5.6, where we will discuss the decidability results needed to apply the machinery of T-theorem proving to a theory Ω', somehow stronger than Ω, having $\mathcal{U} \setminus V_\alpha$ as a model. One could observe that this remark does not apply to theories containing the extensionality and/or the foundation axioms. As far as theories with extensionality are concerned, it is possible to show that we can deal with such theories by a minor technical change in the definition of the translation function $(\cdot)^*$. The status of the axiom of foundation is more delicate, in the sense that, at least as long as one wants to represent the accessibility relation using the membership relation, some form of *anti-foundation* does seem to be the best possible choice.

5.4 The generalization to polymodal logics

In this section we generalize the $\Box$-as-Pow translation to polymodal logics. Our approach presents some analogies with Thomason's technique for reducing tense logic to modal logic, e.g. the replacement of a set of accessibility relations with a single one and the addition of a corresponding number of copies of the domain. Nevertheless, the (completely symmetric) set-theoretic reduction of polymodal logics we propose turns out to be much easier of Thomason's one.

The key problem is to map a polymodal frame, consisting of a set U endowed with k accessibility relations $\lhd_1, \dots, \lhd_k$, with $k > 1$, into a set provided with the membership relation only. Our main concern is thus with getting the membership relation $\in$ to mimick the work of some finite bunch of them. To this end, we preliminary provide polymodal logics with an alternative semantics that transforms the plurality of accessibility relations $\lhd_1, \dots, \lhd_k$ into a single accessibility relation R together with k subsets $U_1, \dots, U_k$ of U.

5.4.1 An alternative semantics for polymodal logics

Let us introduce an alternative semantics for polymodal logics, called *p-semantics*, and the relevant notions of frame, valuation, and validity. To distinguish such notions from the standard ones, we add the prefix p to the usual terms (e.g. p-valuation, p-model, p-frame).

5.4.1. Definition. *A p-frame $\mathcal{F}$ is a $(k+2)$-tuple $(U, U_1, \dots, U_k, R)$, where $U, U_1, \dots, U_k$ are sets and R is a binary relation on $U \cup U_1 \cup \ldots \cup U_k$, such that, for all u, v, t in $U \cup U_1 \cup \ldots \cup U_k$, if $u \in U$, uRv and vRt, then $t \in U$ (we will denote this property by $Trans^2(U)$).*

A p-valuation assigns a truth value to propositional variables only at worlds belonging to U. Formally

5.4.2. Definition. *A p-valuation $\models_p$ is a subset of $U \times \Phi$, where Φ is the set of propositional variables.*

In the case of boolean combinations, the p-valuation $\models_p$ may be lifted to the set of all polymodal formulae in the canonical fashion. In the case of $\Box_i$, with $i = 1, \ldots, k$, for all $u \in U$ we put

$$u \models_p \Box_i \phi \Leftrightarrow \forall v (uRv \wedge v \in U_i \to \forall t (vRt \to t \models_p \phi)).$$

5.4.3. DEFINITION. *A polymodal formula ϕ is p-valid in a p-frame $(U, U_1, \ldots, U_k, R)$ if and only if for all p-valuations $\models_p$ and all worlds $u \in U$, $u \models_p \phi$ holds.*

On the basis of the above definitions, the following lemma holds:

5.4.4. LEMMA. *Given a p-frame $(U, U_1, \ldots, U_k, R)$, there exists a classical polymodal frame $(U, \lhd_1, \ldots, \lhd_k)$, based on the set U, that validates all and only the formulae ϕ which are p-valid in $(U, U_1, \ldots, U_k, R)$.*

Proof. Let $\lhd_1, \ldots, \lhd_k$ be defined as follows:

$$u \lhd_i v \;\; \Leftrightarrow \;\; \exists t (t \in U_i \wedge uRt \wedge tRv).$$

Any p-valuation $\models_p$ on the p-frame $(U, U_1, \ldots, U_k, R)$ may be interpreted as a valuation on $(U, \lhd_1, \ldots, \lhd_k)$, and vice versa.

For any $u \in U$ and any polymodal formula ϕ, we show that:

$$u \models_p \phi \Leftrightarrow u \models \phi.$$

The proof is by induction on ϕ. We confine ourselves to the case of $\Box_i$ operators (the proof in the other cases is straightforward).

Suppose that $u \models_p \Box_i \psi$. We want to prove that $u \models \Box_i \psi$, that is, $\forall w (u \lhd_i w \to w \models \psi)$. Consider a world w such that $u \lhd_i w$. By definition of $\lhd_i$, we have that $\exists t (t \in U_i \wedge uRt \wedge tRw)$. Since $u \models_p \Box_i \psi$ is defined as $\forall v (uRv \wedge v \in U_i \to \forall t (vRt \to t \models_p \psi))$, it follows that $w \models_p \psi$ and hence $w \models \psi$ by induction.

Suppose now that $u \models \Box_i \psi$. If $v \in U_i$ is such that uRv, then, for all t such that vRt, it follows that $u \lhd_i t$. From the hypothesis, we have that $t \models \psi$ and, by induction, $t \models_p \psi$.

If the formula ϕ is p-valid in $(U, U_1, \ldots, U_k, R)$, then, given any classical valuation $\models$ on $(U, \lhd_1, \ldots, \lhd_k)$, it follows that, for all $u \in U$, $u \models_p \phi$ holds in the corresponding p-model $(U, U_1, \ldots, U_k, R, \models_p)$, and thus $u \models \phi$ in $(U, \lhd_1, \ldots, \lhd_k)$. Since this is true for all $u \in U$ and all valuations $\models$, it follows that ϕ is classically valid in the frame $(U, \lhd_1, \ldots, \lhd_k)$. Symmetrically, it is possible to prove that if ϕ is valid in $(U, \lhd_1, \ldots, \lhd_k)$, then it is p-valid in the p-frame $(U, U_1, \ldots, U_k, R)$. $\dashv$

5.4.5. LEMMA. *For every classical polymodal frame $(U, \lhd_1, \ldots, \lhd_k)$ there exists a p-frame $(U, U_1, \ldots, U_k, R)$ that p-validates exactly the formulae which are valid in $(U, \lhd_1, \ldots, \lhd_k)$.*

Proof. Let $U_1, \ldots, U_k, R$ be defined as follows:

- let $U_1, \ldots, U_k$ be pairwise disjoint sets isomorphic to U, each one disjoint from U (let us denote by $u \mapsto u_i$ a fixed correspondence between U and U_i);
- for $i = 1, \ldots, k$ and $u, v \in U$, let $u_i R v$ if and only if $u \lhd_i v$;

- for all $u \in U$ and $i = 1, \ldots, k$, let uRu_i.

It is easy to show that $Trans^2(U)$ holds in $(U, U_1, \ldots, U_k, R)$. Moreover, any valuation $\models$ on $(U, \lhd_1, \ldots, \lhd_k)$ can be seen as a p-valuation $\models_p$ on $(U, U_1, \ldots, U_k, R)$.

For any $u \in U$ and any polymodal formula ϕ, the following holds:

$$u \models \phi \Leftrightarrow u \models_p \phi$$

The verification for boolean combinations is left to the reader.

Let us consider the case in which ϕ is of the form $\Box_i\psi$. If $u \models \Box_i\psi$, then, to prove that $u \models_p \Box_i\psi$, take v, t in $U \cup U_1 \cup \ldots \cup U_k$ such that $v \in U_i$, uRv, and vRt. By definition of R, there exists $u_i \in U_i$ such that $v = u_i$ (uRv and u is different from any u_i), and thus vRt can be rewritten as u_iRt. By definition of R, u_iRt if and only if $u \lhd_i t$. Therefore, by the hypothesis, it follows that $t \models \psi$, and, by induction, $t \models_p \psi$.

Now suppose that $u \models_p \Box\,\psi$, and let v be $\lhd_i$-related to u, that is, $u \lhd_i v$. By definition of R, it follows that uRu_i and u_iRv; thus, by definition of $\models_p$, it follows that $v \models_p \psi$, and, by induction, $v \models \psi$.

The result easily follows as in Lemma 5.4.4. $\dashv$

Lemmas 5.4.4 and 5.4.5 together show that any p-frame $\mathcal{F} = (U, U_1, \ldots, U_k, R)$ can be reduced to a p-frame $\mathcal{F}' = (U, U'_1, \ldots, U'_k, R')$ such that $U, U'_1, \ldots, U'_k$ are pairwise disjoint.

From the previous two lemmas we have:

5.4.6. THEOREM. *If ψ, ϕ are polymodal formulae, then*

$$\psi \models \phi \Leftrightarrow \phi \textit{ is p-valid in all p-frames in which } \psi \textit{ is p-valid.}$$

Proof. Apply Lemmas 5.4.5 and 5.4.4. $\dashv$

5.4.2 A set-theoretic translation method for polymodal logics

As in the soundness proof for the monomodal case, we interpret any p-frame $(U, U_1, \ldots, U_k, R)$ as a $(k+1)$-tuple $U^*, U_1^*, \ldots, U_k^*$ of "sets" in a particular Ω-model such that, for all elements t^* of the model which are $\in$-related to $U^* \cup U_1^* \cup \ldots \cup U_k^*$, we have:

$$t^* = \{s^* : tRs\}.$$

As in the monomodal case, every p-valuation of $P_1, \ldots, P_n$ on the p-frame is interpreted in terms of n subsets $P_1^*, \ldots, P_n^*$ of U^*. Moreover, for each polymodal formula ϕ, we define its translation as a term $\phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)$ such that, for all $u \in U$,

$$u \models_p \phi \Leftrightarrow u^* \in \phi^*(U^*, U_1^*, \ldots, U_k^*, P_1^*, \ldots P_n^*).$$

Under this constraint, the definition of the translation of $\Box_i\phi$ directly follows from the definition of $\models_p$ (and induction):

$u \models_p \Box_i \phi$ iff $\forall v(uRv \wedge v \in U_i \to \forall t(vRt \to t \models_p \phi))$ iff $\forall v(v^* \in u^* \wedge v^* \in U_i^* \to \forall t(t^* \in v^* \to t^* \in \phi^*))$ iff $u^* \cap U_i^* \subseteq Pow(\phi^*)$ iff $u^* \subseteq ((U^* \cup U_1^* \cup \ldots \cup U_k^*) \setminus U_i^*) \cup Pow(\phi^*)$ iff $u^* \in Pow(((U^* \cup U_1^* \cup \ldots \cup U_k^*) \setminus U_i^*) \cup P(\phi^*))$.

Thus, the translation of the term $\Box_i \phi(P_1, \ldots, P_n)$ is defined as follows:

$$(\Box_i \phi)^* \equiv Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\phi^*)).$$

Now let us prove the soundness and completeness of the translation method for polymodal logics. Hereafter, we will refer to polymodal logics provided with k distinct accessibility relations as *k-dimensional polymodal logics*[4].

5.4.7. THEOREM. *(***Soundness of the translation method***) Let H be a k-dimensional polymodal logic extending $K \otimes \ldots \otimes K$ with the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For any polymodal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\begin{aligned} \Omega \ \vdash \ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \to \\ & \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n))) \end{aligned}$$

$$\Rightarrow \psi \models \phi$$

where $Axiom_H(x, y_1, \ldots, y_k)$ stands for $\forall x_1 \ldots \forall x_m (x \subseteq \psi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_m)))$, and $Trans^2(x)$ stands for $\forall y \forall z\ (y \in z \wedge z \in x \to y \subseteq x)$, that is, $x \subseteq Pow(Pow(x))$.

Proof. To show that $\psi \models \phi$ it is sufficient to prove that ϕ is p-valid in all p-frames in which ψ is p-valid (Theorem 5.4.6).

Let $(U, U_1, \ldots, U_k, R)$ be a p-frame in which ψ is p-valid. Then, we proceed as in the monomodal case to prove that in a model of Ω there are $k+1$ sets $U^*, U_1^*, \ldots, U_k^*$ such that $Trans^2(U^*)$ holds and, for any polymodal formula $\alpha(P_1, \ldots, P_n)$,

$$\forall x_1, \ldots, x_n (U^* \subseteq \alpha^*(U^*, U_1^*, \ldots, U_k^*, x_1, \ldots, x_n))$$

holds in the model if and only if $\alpha(P_1, \ldots, P_n)$ is p-valid in $(U, U_1, \ldots, U_k, R)$.

Hence, $Axiom_H(U^*, U_1^*, \ldots, U_k^*)$ holds in the model, and, by the hypothesis, it follows that $\forall x_1, \ldots, x_n (U^* \subseteq \phi^*(U^*, U_1^*, \ldots, U_k^*, x_1, \ldots, x_n))$. This allows us to conclude that ϕ is p-valid in $(U, U_1, \ldots, U_k, R)$. $\dashv$

5.4.8. THEOREM. *(***Completeness of the translation method***) Let H be defined as in Theorem 5.4.7. For each polymodal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_H \phi \Rightarrow$$

$$\begin{aligned} \Omega \ \vdash \ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \to \\ & \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n))) \end{aligned}$$

[4] To prevent misunderstandings, we remark that *k-dimensional polymodal logics* are evaluated at a single world, and not at k-tuple of worlds.

Proof. The proof is by induction on the length of a derivation of ϕ in H.

The cases of tautologies, closure under the substitution rule and modus ponens, and closure under the axiom H are as in the monomodal case.

We prove the closure under the axiom K and necessitation rule of the modalities $\Box_i$. For the axiom K ($\Box_i(\alpha \to \beta) \to (\Box_i\alpha \to \Box_i\beta)$), we show that Ω proves the formula

$$\forall x \forall y_1 \dots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \dots, y_k) \to$$

$$\forall x_1 \dots \forall x_n (x \subseteq (x \setminus u) \cup (x \setminus v) \cup Pow(((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\beta^*)))$$

where u and v stand for $Pow(((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow((x \setminus \alpha^*) \cup \beta^*))$ and $Pow(((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\alpha^*))$, respectively.

Let $x_1, \dots, x_n$ be fixed and consider $x, y_1 \dots, y_k$ such that both $Axiom_H(x, y_1, \dots, y_k)$ and $Trans^2(x)$ hold. Suppose that $t \in x$, $t \in u$, and $t \in v$. We prove that $t \subseteq ((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\beta^*)$, that is, if $s \in t$, then either $s \in (x \cup y_1 \cup \dots \cup y_k) \setminus y_i$ or $s \in Pow(\beta^*)$. If $s \notin (x \cup y_1 \cup \dots \cup y_k) \setminus y_i$, then $t \in u$ implies $s \subseteq (x \setminus \alpha^*) \cup \beta^*$, and $t \in v$ implies $s \subseteq \alpha^*$.

In order to prove the closure under necessitation rule, we show that from

$$\begin{array}{lll} \Omega & \vdash & \forall x \forall y_1 \dots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \dots, y_k) \to \\ & & \forall x_1 \dots \forall x_n (x \subseteq \phi^*(x, y_1, \dots, y_k, x_1, \dots, x_n))) \end{array}$$

it follows that

$$\begin{array}{lll} \Omega & \vdash & \forall x \forall y_1 \dots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \dots, y_k) \to \\ & & \forall x_1 \dots \forall x_n (x \subseteq (\Box_i\phi)^*(x, y_1, \dots, y_k, x_1, \dots, x_n))). \end{array}$$

Let $x_1, \dots, x_n$ be fixed and consider $x, y_1 \dots, y_k$ such that both $Axiom_H(x, y_1, \dots, y_k)$ and $Trans^2(x)$ hold. By the hypothesis, $x \subseteq \phi^*(x, y_1, \dots, y_k, x_1, \dots, x_n)$. Hence, $Pow(x) \subseteq Pow(\phi^*) \subseteq ((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\phi^*)$ and therefore $Pow(Pow(x)) \subseteq Pow(((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\phi^*))$. From $Trans^2(x)$, it follows that $x \subseteq Pow(Pow(x))$, and, therefore, $x \subseteq Pow(((x \cup y_1 \cup \dots \cup y_k) \setminus y_i) \cup Pow(\phi^*)) = (\Box_i\phi)^*$. $\dashv$

Remark. As in the case of monomodal logics, if H is complete then by Theorems 5.4.7 and 5.4.8 modal derivability of a given formula in H is equivalent to first-order derivability of the translated formula in Ω.

5.5 A set-theoretic translation of metric temporal logic

In this section, we consider Metric Temporal Logic (*MTL* for short) [72, 92] as a polymodal logic provided with an arbitrary number of accessibility relations. This interpretation directly follows from the reformulation of *MTL* as a PDL-like logic. In the following, we report the descritpion of the basic features of (the *PDL*-like version of) *MTL*, that we called *MTL*-as-*PDL*, given in Chapter 2.

The language for *MTL* distinguishes between the algebraic and the temporal component. The algebraic part is built up from a non-empty set A of *constants* denoting group elements (temporal displacement). The set of terms over A, $T(A)$, is the smallest set such that (1) $A \subseteq T(A)$, and (2) if α, $\beta \in T(A)$ then $(\alpha+\beta)$, $(-\alpha), 0 \in T(A)$. Next, the temporal part of the language is built from a non-empty set Φ of *proposition letters*. The set of *MTL*-formulae over Φ and A, $F(\Phi, A)$, is the smallest set such that (1) $\Phi \subseteq F(\Phi, A)$, and (2) if ϕ, $\psi \in F(\Phi, A)$ and $\alpha \in T(A)$, then $\neg\phi$, $\phi \wedge \psi$, $\nabla_\alpha \phi$ (and its dual $\Delta_\alpha \phi := \neg\nabla_\alpha \neg\phi$), $\perp \in F(\Phi, A)$.

Each modal operator ∇_α is interpreted as a distinct accessibility relation R_α. Frames $\mathfrak{F}$ and models $\mathfrak{M}$ are defined as $\mathfrak{F} = (T, \{R_\alpha : \alpha \in T(A)\})$ and $\mathfrak{M} = (\mathfrak{F}, V)$, respectively. Moreover, the forcing relation will satisfy $\mathfrak{M}, i \Vdash \nabla_\alpha \phi \Leftrightarrow \forall j (R_\alpha(i, j) \Rightarrow \mathfrak{M}, j \Vdash \phi)$.

MTL is interpreted on *standard* frames. Let $\circ$ and $^{-1}$ be respectively a binary and a unary operation over the set $\{R_\alpha : \alpha \in Term(A)\}$ defined as (i) $R_\alpha \circ R_\beta = \{(i, k) : \exists j (R_\alpha(i, j) \wedge R_\beta(j, k))\}$, and (ii) $(R_\alpha)^{-1} = \{(j, i) : R_\alpha(i, j)\}$. Moreover, let I be the identity relation over T, that is, $I = \{(i, i) : i \in T\}$.

We define a frame $\mathfrak{F}$ to be standard if it satifies the following conditions: the structure $\langle\{R_\alpha : \alpha \in T(A)\}, \circ\rangle$ is a commutative group and $R_{\alpha+\beta} = R_\alpha \circ R_\beta, R_{-\alpha} = (R_\alpha)^{-1}$, and $R_0 = I,$. A standard model $\mathfrak{M}$ is simply a model based on a standard frame It is possible to show that, in any standard frame $\mathfrak{F}$, transitivity directly follows from the definition of $\circ$. Moreover, it is worth noting that, in order to define a standard frame $\mathfrak{M}$, it suffices to provide each modal operator ∇_α, indexed by an *atomic* displacement $\alpha \in A$, with an interpretation R_α, that is, to specify the structure: $(T, \{R_\alpha : \alpha \in A\}, V)$. The relations R_α, for all non-atomic displacements α, can then be inductively defined according to the above conditions giving the intended meaning of $+, -$, and 0.

MTL can be axiomatized as follows:

$$
\begin{aligned}
&\nabla_{\alpha+\beta} p \leftrightarrow \nabla_{\beta+\alpha} p \\
&\nabla_{\alpha+(\beta+\gamma)} p \leftrightarrow \nabla_{(\alpha+\beta)+\gamma} p \\
&\nabla_{\alpha+0} p \leftrightarrow \nabla_\alpha p \\
&\nabla_{\alpha+(-\alpha)} p \leftrightarrow \nabla_0 p \\
&\nabla_\alpha (p \to q) \to (\nabla_\alpha p \to \nabla_\alpha q) \\
&p \to \nabla_\alpha \Delta_{-\alpha} p \\
&\nabla_{\alpha+\beta} p \to \nabla_\alpha \nabla_\beta p \\
&\nabla_0 p \leftrightarrow p \\
&\nabla_\alpha p \to \Delta_\alpha p
\end{aligned}
$$

together with the rules:

(Rep)	$\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi \implies \vdash \nabla_{[\alpha/x]\delta} \phi \leftrightarrow \nabla_{[\beta/x]\delta} \phi$	(replacement)
	where $[\alpha/x]$ denotes substitution of α for the variable x	
(Sub)	$\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi \implies \vdash \nabla_{[\delta/x]\alpha} \phi \leftrightarrow \nabla_{[\delta/x]\beta} \phi$	(substitution)
(NEC)	$\vdash \phi \implies \vdash \nabla_\alpha \phi$	(necessitation rule for ∇_α),

plus modus ponens and replacement (REP) and uniform substitution (SUB) of propositional variables

(REP) $\vdash \phi \leftrightarrow \psi \implies \vdash \chi(\phi/p) \leftrightarrow \chi(\psi/p)$ (replacement)
where (ϕ/p) denotes substitution of ϕ for the variable p

(SUB) $\vdash \phi \leftrightarrow \psi \implies \vdash \phi(\chi/p) \leftrightarrow \psi(\chi/p)$ (uniform substitution).

As the reader may have noticed, the above introduced definition of *MTL* closely resembles the well-known definition of Propositional Dynamic Logic (PDL). Howeve, unlike PDL, *MTL* does not encompass any operation corresponding to the PDL program (term) $?\phi$, which is mapped into an accessibility relation $R_{?\phi}$ whose definition depends on the considered model. This explains why we expressed the semantics of *MTL* in terms of standard frames instead of standard models, a standard model $\mathfrak{M}$ simply being a model based on a standard frame. Moreover, *MTL* has no infinitary operations (like the operation $(.)^*$ of PDL), but its finitary structure is richer (e.g., the operation $+$ of *MTL* satisfies the properties of inverse and commutativity). In Chapter 2, we proved that the above given axiomatization of *MTL* is sound and complete with respect to the class of standard frames.

As in the case of polymodal logics, in order to define the set-theoretic counterpart of *MTL*-derivability, we replace the set of distinct accessibility relations of *MTL* by a single one. However, unlike frames for polymodal logics, *MTL* frames $\mathcal{F} = (T, \{R_\alpha : \alpha \in T(A)\}$ generally encompass an infinite number of distinct accessibility relations, each one corresponding to a different program $\alpha \in T(A)$. For this reason, we modify the notion of p-frame introduced for polymodal logics as follows.

5.5.1. DEFINITION. *A MTL p-frame is a triplet $(\bar{T}, T(A), R)$, where $\bar{T}$ is equal to $T \cup \bigcup_{\alpha \in T(A)} T_\alpha$, $T(A)$ is the set of programs and R is a binary relation on $\bar{T} \cup T(A)$, such that*

(i) for all $s \in T, t \in \bar{T} \setminus T$ and $x \in \bar{T}$, if sRt and tRx, then $x \in T$;
(ii) for all $s \in T$ and $\alpha \in T(A)$, not $sR\alpha$ and there exists $t \in \bar{T} \setminus T$ such that sRt and $tR\alpha$;
(iii) for all $s \in T, t, u \in \bar{T} \setminus T$ and $\alpha \in T(A)$, if sRt, sRu, $tR\alpha$ and $uR\alpha$, then $t = u$;
(iv) for all $s \in \bar{T} \setminus T$ and $\alpha, \beta \in T(A)$, if $sR\alpha$ and $sR\beta$, then $\alpha = \beta$.

The conjunction of properties (i)-(iv) plays in the case of MTL the same role that $Trans^2(U)$ plays in the polymodal case. We will denote it by $Trans^2(T, \bar{T}, T(A))$.

It is worth noting that condition (iii) is actually unnecessary (it will not play any role in the proof of the soundness and completeness of the translation) and therefore could be removed.

A p-valuation $\models_p$ assigns a truth value to proposition letters only at worlds belonging to T. As far as boolean operators are concerned, $\models_p$ may be lifted to the set of all *MTL* formulae in the canonical way. In the case of $[\alpha]\phi$, with $\alpha \in T(A)$, for all $s \in T$ we put

$$s \models_p [\alpha]\phi \Leftrightarrow \forall t(sRt \wedge tR\alpha \rightarrow \forall u(tRu \wedge u \neq \alpha \rightarrow u \models_p \phi)).$$

The notion of validity in *MTL* p-frames (p-validity) is defined in the usual manner. It is possible to prove that for any pair of *MTL* formulae ψ, ϕ, ϕ is a logical consequence of ψ (notationally, $\psi \models \phi$) if and only if ϕ is p-valid in all p-frames in which ψ is p-valid.

The above definition of *MTL* semantics allows us to embed each *MTL* frame into a model of a suitable set theory. To this aim, it is convenient to work with a set theory based on a language extended with all the terms in $T(A)$. There will be no axioms in the underlying set theory constraining the behaviour of these additional terms; it will be governed by (the translation of) *MTL* axioms.

First, the considered model interprets each $\alpha \in T(A)$ as a set (element of the domain of support of the model). Then, as usual, each R-literal sRt is replaced by the corresponding $\in$-literal $t \in s$. Hence, each $s \in T$ is mapped into a set whose elements are of the form s_α, with $\alpha \in T(A)$; moreover, each s_α is unique w.r.t. α and contains all the R_α successors of s in the *MTL* frame plus α itself. Schematically, we have

$$s = \{s_\alpha : \alpha \in T(A)\} \text{ and } s_\alpha = \{t : sR_\alpha t\} \cup \{\alpha\}$$

Let $(.)^{*,[\cdot]}$ be the extension of the $\Box$-as-Pow translation to *MTL*.We must define the set-theoretic counterpart of each modal operator $[\alpha]$, with $\alpha \in T(A)$, in such a way that the following condition holds:

$$w \in ([\alpha]\phi)^{*,[\cdot]} \;\;\Leftrightarrow\;\; \forall t(t \in s \wedge \alpha \in t \to \forall u(u \in t \setminus \{\alpha\} \to u \in \phi^{*,[\cdot]}).$$

Let Ω^{MTL} be the theory having $=, \in$, and $\subseteq$ as predicate symbols, $\{\}$ and Rng as unary functional symbols, and $\cup, \setminus$, and $\times_\in$ as binary functional symbols. The axioms for Ω^{MTL} are the identity axioms and the axioms, already in Ω, describing $\subseteq, \cup$, and $\setminus$ in terms of $\in$, plus the following axioms defining $\{\}, Rng$, and $\times_\in$:

$$\begin{array}{rcl} t \in \{x\} & \leftrightarrow & t = x; \\ t \in x \times_\in y & \leftrightarrow & \exists a \in x \exists b \in y \; (t = \langle a, b\rangle \wedge a \in b); \\ t \in Rng(x) & \leftrightarrow & \exists s \; (\langle s, t\rangle \in x). \end{array}$$

where $\langle x, y\rangle$ is the usual a shorthand for $\{\{x\}\} \cup \{\{x\} \cup \{y\}\}$. Inductively, we denote by $\langle x_1, \ldots, x_n\rangle$ the pair $\langle x_1, \langle x_2, \ldots, x_n\rangle\rangle$.

It is not difficult to verify that the following definition meets the above condition:

$$([\alpha]\phi)^{*,[\cdot]} = Pow(\bar{T} \setminus T_\alpha \;\cup\; Pow(\phi^{*,[\cdot]} \cup \{\alpha\})),$$

where $T_\alpha = \{s \in \bar{T} : s \cap T(A) = \{\alpha\}\} = Rng(\{\alpha\} \times_\in \bar{T})$.

Therefore, the weakest set theory for which we can prove the soundness and completeness of the above defined translation is:

$$\Omega^{MTL} \;\;=\;\; \Omega + \{\} + \times_\in + Rng.$$

Let $Trans^2(x, y, z)$ be the formula

$$\forall x_1 \forall x_2 \forall x_3 (x_1 \in x \wedge x_2 \in y \setminus x \wedge x_3 \in y \wedge x_2 \in x_1 \wedge x_3 \in x_2 \to x_3 \in x) \wedge$$

$$\wedge\forall x_1\forall\alpha(x_1\in x\wedge\alpha\in z\rightarrow\alpha\notin x_1\wedge\exists x_2(x_2\in y\setminus x\wedge x_2\in x_1\wedge\alpha\in x_2))\wedge$$

$$\wedge\forall x_1\forall x_2\forall x_3\forall\alpha(x_1\in x\wedge x_2\in y\setminus x\wedge x_3\in y\setminus x\wedge\alpha\in z\wedge x_2\in x_1\wedge x_3\in x_1\wedge\alpha\in x_2\wedge$$

$$\wedge\alpha\in x_3\rightarrow x_2=x_3)\wedge\forall x_1\forall\alpha\forall\beta(x_1\in y\setminus x\wedge\alpha\in z\wedge\beta\in z\wedge\alpha\in x_1\wedge\beta\in x_1\rightarrow\alpha=\beta)$$

and let $Axiom_{MTL}(x,y,z)$ be the formula

$$\forall x_1\forall x_2\forall\alpha\forall\beta(x_1\in x\wedge x_2\in x\wedge\alpha\in z\wedge\beta\in z\wedge x\subseteq\psi^*(x,y,x_1,x_2,\alpha,\beta)),$$

where ψ is the logical conjunction of MTL axioms.

It is possible to prove the following theorem.

5.5.2. THEOREM. *(***Soundness and completeness of the translation method***) For any MTL formula ϕ involving n propositional variables $P_1,\ldots,P_n$,*

$$\Omega^{MTL}\vdash\forall x\forall y\forall z(Trans^2(x,y,z)\wedge Axiom_{MTL}(x,y,z)\rightarrow$$

$$\forall x_1\ldots\forall x_n(x\subseteq\phi^*(x,y,x_1,\ldots,x_n)))\ \Leftrightarrow\vdash_{MTL}\phi$$

where x,y and z play the role of T, $\bar{T}$ and $T(A)$, respectively.

Proof. As in the case of polymodal logics, the proof of soundness is semantic: given an MTL p-model that falsifies an MTL-formula ϕ, that is, that satisfies $\neg\phi$, we show how to obtain a model of Ω^{MTL} that satisfies the formula $\exists x\exists y\exists z(Trans^2(x,y,z)\wedge Axiom_{MTL}(x,y,z)\wedge\exists x_1\ldots\exists x_n(x\not\subseteq\phi^*(x,y,x_1,\ldots,x_n)))$.

The only technical difference between the proof for polymodal logics and the one for MTL lies in the choice of the ordinal α (cf. [38]): in the case of MTL, we must choose α in such a way that it is possible to associate a distinct set of rank α to each world $s\in S$ and to each displacement $\alpha\in T(A)$.

The proof of completeness is by induction on the length of a derivation of ϕ in MTL. The cases of tautologies, closure under the substitution rule and modus ponens, and closure under MTL axioms are as in the standard polymodal case.

We prove the closure under the axiom K and necessitation rule of each modality $[\alpha]$. As for the axiom $[\alpha](p\rightarrow q)\rightarrow([\alpha]p\rightarrow[\alpha]q)$, we show that Ω^{MTL} proves the formula:

$$\forall x\forall y\forall z(Trans^2(x,y,z)\wedge Axiom_{MTL}(x,y,z)\rightarrow\forall x_1\forall x_2(x\subseteq(x\setminus(u\cup Pow((x\setminus x_1)\cup$$

$$x_2\cup\{\alpha\})))\cup(x\setminus(u\cup Pow(x_1\cup\{\alpha\})))\cup(u\cup Pow(x_2\cup\{\alpha\})))$$

where $u=y\setminus Rng(\{\alpha\}\times_{\in}y)$.

Consider x,y,z such that $Trans^2(x,y,z)$ and $Axiom_{MTL}(x,y,z)$ hold, and suppose that $t\in x$, $t\in u\cup Pow((x\setminus x_1)\cup x_2\cup\{\alpha\})$, and $t\in u\cup Pow(x_1\cup\{\alpha\})$. We must prove that $t\in u\cup Pow(x_2\cup\{\alpha\})$. If $t\in u$, then the thesis is trivially true. If $t\notin u$, then we must prove that if $t\subseteq(x\setminus x_1)\cup x_2\cup\{\alpha\}$ and $t\subseteq x_1\cup\{\alpha\}$, then $t\subseteq x_2\cup\{\alpha\}$. Let $s\in t$. The case $s=\alpha$ is immediate. If $s\neq\alpha$, then s must belong to x_2 and the thesis follows.

In order to prove the closure under necessitation rule, we must show that from

$$\Omega^{MTL}\vdash\forall x\forall y\forall z(Trans^2(x,y,z)\wedge Axiom_{MTL}(x,y,z)\rightarrow$$

$$\forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y, x_1, \ldots, x_n))),$$

it follows that

$$\Omega^{MTL} \vdash \forall x \forall y \forall z (Trans^2(x, y, z) \wedge Axiom_{MTL}(x, y, z) \rightarrow$$
$$\forall x_1 \ldots \forall x_n (x \subseteq ([\alpha]\phi)^*(x, y, x_1, \ldots, x_n))).$$

Let $x_1, \ldots, x_n$ be fixed and consider x, y, z such that both the antecedents of the implication ($Axiom_{MTL}(x, y, z)$ and $Trans^2(x, y, z)$) hold. By the hypothesis, $x \subseteq \phi^*(x, y, x_1, \ldots, x_n)$ and thus

$$Pow(y \setminus Rng(\{\alpha\} \times_{\in} y) \cup Pow(x \cup \{\alpha\}))$$
$$\subseteq Pow(y \setminus Rng(\{\alpha\} \times_{\in} y) \cup Pow(\phi^*(x, y, x_1, \ldots, x_n) \cup \{\alpha\})),$$

where

$$Pow(y \setminus Rng(\{\alpha\} \times_{\in} y) \cup Pow(\phi^*(x, y, x_1, \ldots, x_n) \cup \{\alpha\})) = ([\alpha]\phi)^*(x, y, x_1, \ldots, x_n).$$

To complete the proof, it suffices to show that

$$x \subseteq Pow(y \setminus Rng(\{\alpha\} \times_{\in} y) \cup Pow(x \cup \{\alpha\})).$$

Let $s \in x$. For all $t \in s$, $t \notin T(A)$ (by condition (ii) of $Trans^2(x, y, z)$). We distinguish two cases, depending on whether or not $\alpha \in t$. If $\alpha \in t$, then $t \notin x$ (again, by condition (ii) of $Trans^2(x, y, z)$). Hence $t \in y \setminus x$. By conditions (i) and (iv) of $Trans^2(x, y, z)$, it follows that if $u \in t$, then either $u = \alpha$ or $u \in x$ and thus $t \in Pow(x \cup \{\alpha\})$. If $\alpha \notin t$, then $t \in y \setminus Rng(\{\alpha\} \times_{\in} y)$. This allows us to conclude that $s \in Pow(y \setminus Rng(\{\alpha\} \times_{\in} y) \cup Pow(x \cup \{\alpha\}))$. $\dashv$

Given the translation for $[\alpha]$, we capture MTL via the translation of MTL-axioms that is introduced in the antecedent of the translating formula. The general method is therefore parametric with respect to the terminological component of the language, and can be seen as a form of "deduction theorem" for (MTL-like) extensions of modal logic. From this point of view, the translation behaves as the $\Box$-as-Pow translation for modal logic.

Notice that the above argument rests on a frame-completeness result, that holds for the considered logics. If the modal logic to translate is not frame-complete, the above translation cannot be applied. For example, it cannot be used to translate PDL, which is complete with respect to a particular class of models (*standard* models) not characterizable in terms of frames. For logics as PDL, a different semantic argument, calling models into play, is needed.

5.6 Related work

5.6.1 Set-theoretic translations for extended modal logics

In Sections 5.4 and 5.5, we have shown how the $\Box$-as-Pow translation can be generalized to modal logics that replace the single accessibility relation of monomodal logics by a set of accessibility relations. In this section, we briefly show how the field of applicability of the $\Box$-as-Pow translation can actually be further extended. The proofs of the stated results are given in [12, 91]. First, we consider another important family of extended modal logics, namely, modal logics that impose stronger constraints on the (single) accessibility relation. More specifically, we will discuss two paradigmatic cases: the *modal logic of inequality* [109] and the so-called *irreflexivity rule* [52]. Then, we will show that the technical work necessary to prove the correctness and completeness of the modified versions of the translation for extended modal logics can be seen as a particular instance of the more general problem of set-theoretically model the notion of general frame closed under L_0-definitions. The notion of general frame closed under L_0-definitions was introduced in [5] and it was shown to be equivalent to derivability in a weak system of second-order logic called L_2. We show that validity in general frames closed under L_0-definitions can be captured via (a suitable adaptation of) the $\Box$-as-Pow translation, when the underlying set theory is the extension of Ω with closure axioms for Gödel constructible operations [3, 60]. The resulting theory Ω_c turns out to be somewhat minimal for the pourpose and hence can be considered, modulo the $\Box$-as-Pow translation, as a set-theoretic counterpart of L_2.

Let us start with the modal logic of inequality. Consider the difference operator D whose Kripke frame semantics can be expressed as follows:

$$w \models D\varphi \;\;\Leftrightarrow\;\; \exists w'(w' \neq w \;\wedge\; w' \models \varphi).$$

The introduction of the D operator allows one to develop a modal theory of inequality and has been extensively studied. Our reference for technical aspects relative to the modal theory of D is [109], where the interested reader can find more details and proofs of the modal results mentioned in this section.

An axiomatization of the basic modal logic of inequality DL is given by the axioms and inference rules of propositional logic plus the axioms:

A_1 $\bar{D}(p \to q) \to (\bar{D}p \to \bar{D}q)$ (normality),
A_2 $p \to \bar{D}Dp$ (symmetry),
A_3 $DDp \to (p \vee Dp)$ (pseudotransitivity),

and the rules of inference:

R_1 $\vdash \varphi \;\Rightarrow\; \vdash \bar{D}\varphi$,
R_2 $\vdash (p \wedge \bar{D}\neg p) \to \phi$ and p is not in $\varphi \;\Rightarrow\; \vdash \varphi$,

where $\bar{D}$ is the dual of D, that is, $\bar{D} = \neg D \neg$.

Let $\mathcal{L}(D)$ be the language containing D as the only modal operator and let $\varphi \in \mathcal{L}(D)$. The logic DL is complete with respect to $\mathcal{L}(D)$ as well as to all its extensions (even though this is not the case if R_2 is eliminated). Also the logic $DL_m = K + DL + (p \wedge \bar{D}p \to \Box p)$ in the language $\mathcal{L}(\Box, D)$ is complete and will be considered as basic hereafter.

Our first aim is to introduce a term which will allow us to extend the $\Box$-as-Pow translation to the D operator. The term corresponding to D must guarantee the following fact:

$$w \in (D\varphi)^{*,D} \;\;\Leftrightarrow\;\; \exists w'(w' \neq w \;\wedge\; w' \in (\varphi)^{*,D}),$$

where $(.)^{*,D}$ is the extension of the function $(.)^*$ that we are seeking. Equivalently, given a frame x, the following must hold:

$$(D\varphi)^{*,D} \quad = \quad \begin{cases} \varphi^{*,D} & \text{if } \varphi^{*,D} = \emptyset, \\ x \setminus \varphi^{*,D} & \text{if } x \cap \varphi^{*,D} = \{w\}, \\ x & \text{otherwise.} \end{cases}$$

Let Ω^D be the theory having $=, \in$, and $\subseteq$ as predicate symbols, $\{\}$ and Dom as unary functional symbols, and $\cup, \setminus, \times$ and $\times_=$ as binary functional symbols. The axioms for Ω^D are the identity axioms and the axioms describing $\subseteq, \cup, \setminus$ and $\{\}$, plus the following axioms defining Dom, $\times$ and $\times_=$:

$$\begin{aligned} t \in x \times y \quad &\leftrightarrow \quad \exists a \in x \exists b \in y \ (t = \langle a, b\rangle); \\ t \in x \times_= y \quad &\leftrightarrow \quad \exists a \in x \exists b \in y \ (t = \langle a, b\rangle \wedge a = b)); \\ t \in Dom(x) \quad &\leftrightarrow \quad \exists s \ (\langle t, s\rangle \in x). \end{aligned}$$

The following definition is immediately seen to satisfy our requirements:

$$(D\varphi)^{*,D} \quad = \quad Dom(x \times_{\neq} \varphi^{*,D}),$$

where $A \times_{\neq} B = \{\langle a, b\rangle \mid a \in A \wedge b \in B \wedge a \neq b\}$ can be defined in terms of our original operators as $A \times B \setminus A \times_= B$.

The following theorem shows that the translation $(.)^{*,D}$ is sound and complete with respect to Ω^D (the proof can be found in [91]).

5.6.1. THEOREM. *Let φ and ψ be formulae in the language $\mathcal{L}(\Box, D)$,*

(completeness) $\varphi \vdash_{DL_m} \psi \ \Rightarrow \ \Omega^D \vdash \forall x(\forall \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^{*,D}(x, \vec{z}))),$

(soundness) $\Omega^D \vdash \forall x(\forall \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^{*,D}(x, \vec{z}))) \ \Rightarrow \ \varphi \models_f \psi.$

Being able to treat DL_m has many interesting by-products coming from the fact that important notions, which cannot be expressed in $\mathcal{L}(\Box)$, become expressible in $\mathcal{L}(\Box, D)$. As an example, consider the case of irreflexivity, that corresponds to $\bar{D}p \rightarrow \Box p$.

An alternative approach to the treatment of irreflexivity in frames consists in adding the following *irreflexivity rule* (IRR)

$$\vdash (\neg p \wedge \Box p \rightarrow \varphi) \text{ with } p \notin \varphi \quad \Rightarrow \quad \vdash \varphi,$$

that allows to capture logical consequence in irreflexive frames via modal deduction (cf. [52]).

The $\Box$-as-Pow translation characterizes exactly logical consequence in irreflexive frames as long as the underlying set theory is well-founded. As a matter of facts a very weak form of foundation is necessary, namely the requirement that no set belongs to itself. In the following theorem let $(.)^*$ denote the $\Box$-as-Pow translation relative to the language $\mathcal{L} = \{\cup, \setminus, Pow\}$ (cf. [38]).

5.6.2. THEOREM. *Let φ and ψ be formulae in the language $\mathcal{L}(\Box)$ and let $\models_f^{irr}$ denote logical consequence in irreflexive frames.*

(completeness) $\varphi \vdash_{K+IRR} \psi \;\Rightarrow\; \Omega + \forall y(y \notin y) \vdash \forall x(Trans(x) \wedge \forall \vec{y}(x \subseteq \varphi^*(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z})))$,

(correctness) $\Omega + \forall y(y \notin y) \vdash \forall x(Trans(x) \wedge \forall \vec{y}(x \subseteq \varphi^*(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))) \;\Rightarrow\; \varphi \models_f^{irr} \psi$.

Notice that the theory $\Omega + \forall y(y \notin y)$ used in the above theorem does not contain any more comprehension than Ω (as a matter of fact, moving the axiom $\forall y(y \notin y)$ to the antecedent of the translation as $\forall y \in x(y \notin y)$, one could work with Ω). On the other hand, we have seen that the set theory in which one can carry out the translation for the minimal modal logic of inequality is $\Omega^D = \Omega + \{\} + \times_{\neq} + Dom$. This is not surprising, as we know that irreflexivity can be expressed using the D operator, and it also tells us that the comprehension we need to capture the notion of irreflexivity is as much as we need for the basic machinery of the translation and strictly less of what we need for D.

Let us show now how the technique employed for applying the $\Box$-as-Pow translation to extended modal logics can in fact be generalized to capture a larger fragment of the (non r.e.) notion of modal logical consequence, namely, the fragment corresponding to weak second-order logic (cf. [5]). Let L_2 be a second-order language containing a binary predicate R and equality, plus a countable number of unary predicates $\mathtt{P}_1, \mathtt{P}_2, \ldots$. In [5], axiomatic theories for deduction in the L_2-language are introduced. In particular, we will consider a system defined by means of a suitable form of substitution. Let an L_0-formula be an L_2-formula without occurrences of second-order quantifiers. Given an L_2-formula α and an L_0-formula $\gamma(x)$, denote by $\alpha(\gamma|\mathtt{P})$ the L_2-formula obtained from α by replacing subformulae of the form $\mathtt{P}(u)$ by $\gamma(u|x)$ (modulo some technicalities about free variables [5]). L_0-substitution is expressed by the following schemata, where α is an L_2-formula and γ is an L_0-formula:

$$\forall \mathtt{P}\alpha \rightarrow \alpha(\gamma|P).$$

Weak second-order logic contains a set of axioms complete for first-order predicate logic, plus L_0-substitution. A semantic counterpart of deducibility in weak second-order logic by means of closure under L_0-definitions in general frames. A general frames $(F, \mathcal{W})$ is *closed under L_0-definitions* if, for all L_0-formulae γ with free world-variables $x, x_1, \ldots, x_n$, free set-variables $X_1, \ldots, X_m$, and for all $w_1, \ldots, w_n \in W$, $A_1, \ldots, A_m \in \mathcal{W}$, the set $\{w \in W : F \models \gamma(w, w_1, \ldots, w_n, A_1, \ldots, A_m)\}$ belongs to $\mathcal{W}$. $\alpha \vdash_{L_2} \beta$ is equivalent to say that for all general frames $(F, \mathcal{W})$ closed under L_0-definitions, if $(F, \mathcal{W}) \models \alpha[f]$, then $(F, \mathcal{W}) \models \beta[f]$ (where f is an assignment of worlds in W to individual variables and set of worlds in $\mathcal{W}$ to unary predicate variables).

In [10], we show that validity in general frames closed under L_0-definitions can be captured via (a suitable adaptation of) the $\Box$-as-Pow translation, provided that Ω is replaced by a set theory whose axioms are closely related to the so-called Gödel operations for defining the constructible universe. We introduce functions defining the singleton operator, suitable cartesian products $(\times, \times_=, \times_\in)$ together with their projections (Dom, Rng), plus some operations allowing us to manipulate argument positions in ordered sequences (C_1, C_2).

Let us call Ω_c the resulting theory. Its language has $=, \in$, and $\subseteq$ as predicate symbols, $\{\}$, Dom, and Rng as unary functional symbols, and $\cup, \setminus, \times, \times_\in, \times_=, C_1$, and C_2 as binary functional symbols. The axioms for Ω_c are the identity axioms and the axioms describing $\subseteq, \cup, \setminus, \{\}, \times, \times_\in, \times_=, Dom$, and Rng (cf. the theories Ω^D and Ω^{MTL} introduced before), plus the following two axioms defining C_1, and C_2:

$$\begin{array}{lcl} t \in C_1(x, y) & \leftrightarrow & \exists a \exists b \exists c\ (\langle a, b\rangle \in x \wedge c \in y \wedge t = \langle a, \langle b, c\rangle\rangle); \\ t \in C_2(x, y) & \leftrightarrow & \exists a \exists b \exists c\ (\langle a, c\rangle \in x \wedge b \in y \wedge t = \langle a, \langle b, c\rangle\rangle). \end{array}$$

The translation function $(\cdot)^*$ is the standard one except for the $\Box$ operator, and this is obviously the case, since we do not have Pow among the symbols in the language of Ω_c. Moreover, $Trans(x)$ disappears from the antecedent of the translated sentence. To justify our approach, it can be easily checked that, in the case of Ω, it would have made no difference to work with $(\Box\phi)^*$ defined either as $Pow(\phi^*)$ or as $Pow(\phi^*) \cap x$. Actually, we chose the first alternative only to maintain the translated terms simpler. It is also easy to see that $Pow(\phi^*) \cap x = \{y \in x : y \cap x \subseteq \phi^*\}$, whenever x is transitive; as a matter of fact, we will see that the set $\{y \in x : y \cap x \subseteq \phi^*\}$ can always be used to translate $\Box\phi$ (even in the case in which x is not transitive). Moreover, the following holds

$$\{y \in x : y \cap x \subseteq \phi^*\} = x \setminus Rng((x \setminus \phi^*) \times_\in x),$$

and hence we put:

$$(\Box\phi)^* = x \setminus Rng((x \setminus \phi^*) \times_\in x).$$

5.6.3. THEOREM. *(Soundness and completeness of the translation method) For any pair of modal formulae ϕ, ψ,*

$$\vdash_{L_2} \overline{ST}(\phi) \to \overline{ST}(\psi) \Leftrightarrow \Omega_c \vdash \forall x (\forall \vec{z}\ (x \subseteq \phi^*(x, \vec{z})) \to \forall \vec{z}\ (x \subseteq \psi^*(x, \vec{z}))).$$

where $\overline{ST}(\phi)$ denotes the closed standard translation of a formula ϕ [7].

Concluding remark. The fact that L_2-derivability can be translated into derivability from a set theory like Ω_c hints at the possibility of sistematically mapping (reasonable) modal operators into set-theoretic terms. The specific cases analized in this section give interesting and rather natural examples of how such a translation goes, and also point out the possibility of tailoring the background set theory precisely—below Ω_c. A strong argument in favour of the existence of a sistematic mapping technique, is the fact that that the constructible operators certainly offer a great—and 'tunable'—variety of possibilities for searching the set-theoretic counterparts of a given modal operator together with its semantics. The literature on extensions of modal logics offers a wealth of cases to study: counting modalities, terminological logics, etc..

5.6.2 A Comparison with the standard translation method

In Section 5.3, using a set-theoretic translation method, we proved that derivability in the minimal modal logic K_s corresponds precisely to derivability in a weak, computationally attractive set theory Ω. In [11], this approach was shown equivalent to working with standard first-order translations of modal formulae in a theory of general frames. More precisely, we completely cleared up the relationship between the original set-theoretic translation and the standard modal translation, and we managed to extend the adequacy result to all modal logics (frame-complete or not) under a slightly modified translation, using a standard model-theoretic proof not involving any non-well-founded set theory.

The *standard translation* of modal formulae into formulae of a second-order language containing equality, a binary constant R and unary predicate variables (L_2-formulae [7, 6]) is defined as follows:

- $ST(P_i) = \mathtt{P_i}(x)$,
- $ST(\phi \vee \psi) = ST(\phi) \vee ST(\psi)$,
- $ST(\neg\phi) = \neg ST(\phi)$,
- $ST(\Box\phi) = \forall y\ (xRy \to ST(\phi)(y|x))$,

where $y|x$ denotes uniform substitution of the variable y for the variable x.

The *closed standard translation* $\overline{ST}(\phi)$ of a modal formula ϕ is defined as the L_2-sentence $\forall \mathtt{P_1} \ldots \mathtt{P_n} \forall x ST(\phi)$, and it is is easy to see that

$$\phi \models_f \psi \Longleftrightarrow \overline{ST}(\phi) \models \overline{ST}(\psi),$$

where the $\models$ on the right-hand side denotes second-order logical consequence.

Standard modal deduction via the standard translation is supported by a simple two-sorted first-order theory μ of general frames. Consider a two-sorted first-order language, with *worlds* (denoted by small letters) and *sets* (denoted by capital letters), with binary predicates R (on worlds) and $\in$ (between worlds and sets), as well as operations $-, \cup$ and $\Box$ on the set sort. This language is adequate for the description of the general frame semantics. The minimal logic describing it is obtained by considering the following theory μ:

- First-Order Principles for both sorts;
- $\forall \mathtt{P} \forall w (w \in -\mathtt{P} \leftrightarrow \neg w \in \mathtt{P})$;
- $\forall \mathtt{PQ} \forall w (w \in \mathtt{P} \cup \mathtt{Q} \leftrightarrow w \in \mathtt{P} \vee w \in \mathtt{Q})$;
- $\forall \mathtt{P} \forall w (w \in \Box\mathtt{P} \leftrightarrow \forall v (wRv \to v \in \mathtt{P}))$
- $\forall \mathtt{PQ} (\forall w (w \in \mathtt{P} \leftrightarrow w \in \mathtt{Q}) \to \mathtt{P} = \mathtt{Q})$ (extensionality).

Comparing μ with Ω, we established an effective equivalence between the latter and a suitable extension of μ, that we call μ^+. Such an extension is obtained adding to μ an axiom reflecting the "uniformity" of our set-theoretic approach which uses just $\in$ for both set-membership and the accessibility relation. Formally, μ^+ is the theory obtained by adding to μ the axiom

- $\forall w \exists \mathtt{P} \forall v\ (v \in \mathtt{P} \leftrightarrow wRv)$

that links R and $\in$, guaranteeing that, for each world w, there exists the set of its R-successors..

In [11], we showed that the standard translation method with respect to μ^+ and the set-theoretic one with respect to Ω are equivalent. More precisely, we proved that, for all modal formulae ϕ, ψ,

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \iff \Omega \vdash \forall x\ (Trans(x) \wedge$$

$$\wedge \forall x_1 \dots x_m (x \subseteq \phi^*(x, x_1, \dots, x_m)) \rightarrow \forall x_1 \dots x_n (x \subseteq \psi^*(x, x_1, \dots, x_n))).$$

We then proved that μ^+ is non-conservative over μ. On the ground of the equivalence between Ω and μ^+, this allows us to conclude that the $\Box$-as-Pow translation method given in Section 5.3 is not adequate for logics which are not frame complete. By varying our translation, however, we managed to obtain a full equivalence between general modal deduction in K_s and Ω-deduction. This is the desired general result, which is based on standard set-theoretic methods, including a suitable adaptation of Fraenkel-Mostowski permutations of the universe [74].

Remark. In this dissertation, we concentrate our attention on the application of the proposed set-theoretic translation method to metric temporal logics. The study of the general relations between (standard) set theory and modal logic, which is of great interest on its own, is the main research interest of Giovanna D'Agostino, whose dissertation investigates the alternative approaches briefly described in this chapter, and their connections, in detail.

5.6.3 On the application of set T-resolution

As we said in Section 5.1, on the basis of the results presented in the preceding sections it is possible to automatically test modal derivability–from modal theories in the specified class–using a classical first-order theorem prover.

Recently a more specialized technique (called T-theorem proving) for automated theorem proving in first-order theories has been proposed (cf. [104]). Based on the translation method introduced above, a suitable application of T-theorem proving in which the underlying theory T is Ω (or one similar to it) can now be considered as an alternative for automatically testing modal derivability. In this section we briefly discuss the problem of applying *set* T-resolution together with our translation method.

A prerequisite to employing T-resolution in the context of a given theory T, is the decidability, with respect to T, of the class of ground formulae written in any language which extends the one in which the axioms of T are written with Skolem (uninterpreted) function symbols. In [104], it was shown that the satisfiability problem with respect to any theory T of ground formulae on a given language $\mathcal{L}^*$ obtained from $\mathcal{L}(T)$ by adding an arbitrary number of functional and constant symbols, is equivalent to the T-satisfiability of the class of purely existential formulae written in $\mathcal{L}(T)$. Therefore we are interested in this last problem in the case of Ω, whose language ($\mathcal{L}(\Omega)$ from now on) consists of the symbols $\emptyset, \cup, \setminus, \subseteq, \in$, and Pow.

Before commenting on the above mentioned problem, notice that the decidability of classes very similar to the one we want to deal with has already been proved by Cantone, Schwartz, and Ferro [23, 24, 25]. Unfortunately, the results mentioned–among the most complex in the field of computable set theory–cannot be applied to our context, the problem being the underlying set theory on which they rest. Our theory Ω is very weak; in fact it can easily be verified that the proofs in [23, 25] make an essential use of assumptions such as regularity, existence of the transitive closure of sets, extensionality, etc., which are certainly not derivable in Ω.

We succeeded in providing a proof of the decidability result we need for a theory Ω' slightly stronger than Ω (but having essentially the same language). The main difference between Ω and Ω' is that Ω' contains as axioms some simple consequences–not derivable in Ω–of Cantor's theorem on the number of subsets of a given set [36, 39].

The proof is based on a technique first introduced in [26, 102]. The main idea is the following: in order to establish whether there exists a model of Ω' satisfying a formula $\varphi(x_1, \ldots, x_n)$ (an unquantified formula written in $\mathcal{L}(\Omega')$), we assume that there exists a model M of Ω' such that $M \models \varphi(x_1, \ldots, x_n)$, and we concentrate our attention on n elements $a_1, \ldots, a_n$ in the support of M satisfying φ. The goal is to show that under this hypothesis we can build another (*simpler*) n-tuple $a_1^*, \ldots, a_n^*$ of elements in the support of a model M' of Ω' still satisfying φ. The elements $a_1^*, \ldots, a_n^*$ are completely described by a graph G whose size is bounded by a function of n, in the sense that, in order to test the existence of (M' and) $a_1^*, \ldots, a_n^*$, it is sufficient to test the existence of G, and this result guarantees the decidability.

The problem of determining $a_1^*, \ldots, a_n^*$ is combinatorially non-trivial. First of all, notice that if in the formula $\varphi(x_1, \ldots, x_n)$ we had no conjuncts of the form $Pow(x_i) = x_j$, then we could define $a_1^*, \ldots, a_n^*$ simply as a n-tuple satisfying $a_i^* = \{a_j^* \mid a_j \in^M a_i\}$, and it would be easy to check that all our requirements are satisfied (recall that we do not have to deal with the extensionality axiom). As a matter of fact we can think of the map * as a way of *marking* some of the elements in each of the a_i (the marked elements being those of the form a_j) and then take a_i^* as the set of (images with respect to $*$ of) marked elements in a_i. In order to deal with a literal of the form $Pow(x_i) = x_j$, we need to mark more elements: at least all those elements which are subsets of the set of marked elements in a_i. Notice that if one simply does so and marks all such elements (subsets of a_i) without "care", new elements can turn out marked in a_i, and the marking process may not terminate. We solved the above problem processing the a_i's in an order compatible with their size and applying the simple consequences of Cantor's theorem that were forced to hold in Ω' precisely for this purpose (the details of the proof are given in [39]).

It may be interesting to note that it is still an open problem whether the class of purely existential formulae of $\mathcal{L}(\Omega)$ is decidable with respect to Ω. In other words it is not known whether T-theorem proving can be applied directly to Ω; hence, up to this point, despite its simplicity, Ω seems to be a less suitable theory for computational purposes, than a more complex one (i.e. Ω').

Concluding remarks

In this chapter, we proposed a new translation method mapping polymodal formulae into set-theoretic terms of the very week set theory Ω, and we showed how it can be exploited to execute propositional metric temporal logics (and all metric and layered temporal logics that can be reduced to them). The application of the translation method to a larger class of metric and layered temporal logics is part of our ongoing research.

The proposed method can be used for any normal finitely axiomatizable polymodal logic, possibly specified with Hilbert axioms only, and applies to a large class of theories extending Ω. An important and interesting line of investigation in this respect is the generalization of the proposed method to first-order polymodal logics (including quantified systems of MTL).

We are also investigating the possibility of exploiting our translation method to reduce undecidable decision problems for particular propositional polymodal logics, e.g. [73], to the derivability problem with respect to Ω of formulae of type $\forall^*\exists$, thereby showing the undecidability of the latter problem. In the meanwhile, we are investigating the possibility of obtaining decidability results for relevant classes of (poly)modal logics through their reduction to decidable classes of set-theoretic formulae.

Finally, we are considering two possible developments of the application of the set-theoretic translation to extended modal logics [12, 91]. On the modal side, one can try to explicitly characterize the class of extended modal logics which are embeddable in L_2. On the set-theoretic side, one can try to establish whether the set theories obtained by tailoring Ω_c, e.g. Ω^D for the modal logic of inequality and Ω^{MTL} for metric temporal logic, are somehow *minimal*. An interesting consequence of such a minimality result would be the ability of comparing the relative expressive strength of (translatable) extended modal logics.

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Samenvatting

Dit proefschrift behandelt het ontwerp van temporele logica's die kunnen werken met veranderingen in temporele grootteorde. Deze grootteorde (of 'granulariteit') kan worden omschreven als het onderscheidend vermogen dat past bij de temporele aard van een bewering. Als we een temporeel formalisme voorzien van zo'n granulariteit, dan kunnen we informatie specificeren met betrekking tot verschillende tijdsdomeinen (maanden, dagen, enzovoorts) binnen eenzelfde model. We krijgen dan wel te maken met kwesties van de juiste betekenis voor beweringen die diverse tijdschalen combineren, en de juiste overgangen tussen fijnere en ruwere temporele grootteordes.

De oorspronkelijke motivering voor dit werk was de opzet van een temporele logica voor het specificeren van concrete real-time systemen waarvan de componenten zich op verschillende tijdschalen ontwikkelen. Niettemin zijn er opmerkelijke overeenkomsten tussen deze problemen en recent meer algemeen logisch onderzoek naar veranderingen in semantische contexten en perspectieven. Het hier beschreven soort logica's past dus in een breder gebied tussen logica, informatica, computationele taalkunde, en kunstmatige intelligentie. In het byzonder introduceren wij technieken voor combinatie van logica's op afzonderlijke temporele domeinen, en voor het bewijzen van meta-logische eigenschappen van de resulterende systemen, zoals volledigheid en beslisbaarheid.

We stellen een metrische gelaagde temporele logica voor die granulariteit kan verantwoorden, en laten zien hoe hiermee real-time systemen zijn te specificeren. Daartoe beschouwen we eerst een zuiver metrische logica. Dit is een tweesoortig systeem van temporele posities en verplaatsingen, waarin vele bestaande metrische temporele logica's zijn in te bedden. Met dit systeem analyzeren we volledigheidsvragen voor zulke logica's, slechts gedeeltelijk bestudeerd in de literatuur. Vervolgens definieren we een gelaagde metrische temporele logica, met verschillende temporele grootteordes. We bepalen de voornaamste functies die zo'n systeem moet dienen, en de randvoorwaarden waaraan het moet voldoen. In het byzonder betreffen deze de relaties tussen temporele entiteiten op verschillende niveaus. Vervolgens definieren we axiomatisch een systeem dat hieraan voldoet, met speciale aandacht voor beslisbaarheid. Voor relevante speciale klassen van metrische gelaagde tem-

porele structuren bewijzen we beslisbaarheid: met name voor eindig-gelaagde en aftelbaar-oneindig-gelaagde structuren. Dit geeft ons ook precies inzicht in de mogelijke reducties tussen gelaagde en vlakke temporele structuren. In het laatste deel van het proefschrift bestuderen we de computationele executie van gelaagde metrische temporele logica's. In plaats van een specifiek rekensysteem voor dit doel, geven we een reductiestrategie door middel van vertaling in zwakke verzamelingstheorieen waarvoor reeds een algorithme bekend is. Deze strategie is van belang op zich, ook buiten het gebied van de temporele logica.

Abstract

This dissertation is about the design of temporal logics that deal with changing time granularities. *Time granularity* can be defined as the resolution power of the temporal qualification of a statement. Providing a formalism with the concept of time granularity makes it possible to specify time information with respect to differently-grained temporal domains. This does not merely mean that one can use different time units–say, months and days–to represent time quantities in a *unique flat temporal model*, but it involves more difficult semantic issues related to the problem of assigning a proper meaning to the association of statements with the different temporal domains of a *layered temporal model* and of switching from one domain to a coarser/finer one.

The original motivation of the work was the design of a temporal logic suitable for the specification of real-time systems whose components evolve according to different time scales (*granular real-time systems*). Nevertheless, there are significant similarities between the problems it addressed and those dealt with by the current research on logics that deal with changing contexts and perspectives. The design of these types of logics is emerging as a relevant research topic in the broader area of combination of logics, theories, and structures, at the intersection of logic with artificial intelligence, computer science, and computational linguistics. In this dissertation, we devised suitable combination techniques both to define temporal logics and to prove logical properties of these logics, such as completeness and decidability.

We proposed a metric and layered temporal logic for time granularity, and we showed how to use it to specify granular real-time systems. We started by considering the purely metric fragment in isolation. We defined a general two-sorted framework where a number of metric temporal logics, having a different expressive power, can be defined as suitable combinations of a temporal component and an algebraic one. Then, we exploited the proposed framework to study completeness issues for the various systems of metric temporal logic. Despite their relevance, these issues have been ignored or only partially addressed in the literature. The next step was the definition of a many-layer metric temporal logic, embedding the notion of time granularity. We identified the main functionalities a logic for time

granularity must support and the constraints it must satisfy. In particular, we identified the set of properties constraining the relations between time instants belonging to different layers. Then, we axiomatically defined a metric and layered temporal logic, viewed as the combination of a number of differently-grained (single-layer) metric temporal logics, and we studied its logical properties. We devoted a special attention to the decidability problem. We identified relevant classes of metric and layered temporal structures, and showed that the corresponding theories are decidable. More precisely, we proved the decidability of the validity and satisfiability problems for the theory of finitely-layered metric temporal structures, and for two relevant theories of ω-layered metric temporal structures. These decidability results provide useful insights about the relations between many-layer and flat metric temporal systems, e.g., they answer the natural question whether, and under which conditions, many-layer temporal systems can be reduced to flat ones. In the last part of the dissertation, we concentrated on the problem of executing metric and layered temporal logics. However, instead of proposing any specific-system oriented solution, we devised a general set-theoretic translation method which has a value of its own, and whose range of applicability is not restricted to temporal logics.

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