Relativized Relation Algebras*

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Abstract

Relativization is one of the central topics in the study of algebras of relations (i.e. relation and cylindric algebras). Relativized representable relation algebras behave much nicer than the original class RRA: for instance, one obtains finite axiomatizability, decidability and amalgamation by relativization. The properties of the class obtained by relativizing RRA depend on the kind of element with which is relativized. We give a systematic account of all interesting choices of relativizing RRA, and show that relativizing with transitive elements forms the borderline where all above mentioned three properties switch from negative to positive.

In algebraic logic, relativized cylindric and relation algebras have been studied relatively deeply (cf. e.g., Henkin et al. (Henkin et al., 1981), Maddux (Maddux, 1982) and Resek–Thompson (Resek and Thompson, 1991)). The emphasis, however was different from the perspective we will take here. As the name "relativization" indicates, the non–standard algebras were viewed as being derived from the standard classical ones, while in our set–up the classical algebras appear as a very special case of the more basic (relativized) algebras. Recently, Andréka, van Benthem, Monk and Németi ((Andréka and Thompson, 1988), (Andréka et al., 1995), (van Benthem, 1996), (Monk, 1993), (Németi, 1991)) started promoting their study as structures which are interesting independently of their classical versions. Before, relativized algebras were not really studied in their own right[1], but as a tool to obtain results for the standard algebras[2]. Apart from that, one can view relativization as a way of *turning negative results into positive ones*, since several relativized versions of cylindric and relation algebras do have the nice properties discussed before which their classical counterparts lack.

In this paper we ask for all interesting choices of relativized RRA, whether (i) they are finitely axiomatizable, (ii) their equational theory is decidable, and (iii) they enjoy the amalgamation property. Our results are summarized in Theorem 1.3. We treat these three topics in sections 2 to 4. We finish with a discussion of reducts and the relativized versions of UR-algebras (unital residuated boolean groupoids (Jónsson and Tsinakis, 1993; Jipsen, 1992)).

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[1]As an illustration we mention the book (Henkin et al., 1981), from which it turns out that the authors know much about cylindric relativized set algebras, but they treat this knowledge as secondary, hence they do not include a large part of this knowledge in the book.

[2]Because of this, the technique of relativization was studied quite extensively (cf. (Henkin et al., 1971 1985) Chapter 2.2), but that is not what we are interested in now. We are interested in the relativized algebras in their own right.

1 Transitive vs non-transitive relativization

We start with defining relativized representable relation algebras, relativized RRA's for short. For binary relations R and S, their composition is denoted as $R \mid S$.

Definition 1.1 *Let W be a subset of the Cartesian square $U \times U$, for some set U. The operations $-^W$, $\cup^W$, $|^W$, $(\cdot)^{-1:W}$ and Id^W are defined on $\mathcal{P}(W)$ (the powerset of W) as the relativizations to W of the usual operations $-$, $\cup$, $|$, $(\cdot)^{-1}$ and Id respectively, i.e.,*

$$\begin{array}{lcl} -^W R & = & -R \cap W \\ R \mid^W S & = & (R \mid S) \cap W \\ etc. & & \end{array}$$

Any algebra of the form $\mathfrak{A} = (\mathcal{P}(W), -^W, \cup^W, |^W, (\cdot)^{-1:W}, Id^W)$ with $\mathcal{P}(W)$ and the operations as defined above, is called a relativized relational power set algebra; the class of such algebras is denoted by $\mathsf{FRA}_\emptyset$.

Let $H \subseteq \{R, S, T\}$ where R stands for 'reflexive', S for 'symmetric' and T for 'transitive'. We say that W is an H-relation *if W has the properties mentioned in H. For example, if $H = \{R, S\}$ then W is an H-relation if it is reflexive and symmetric.* FRA_H *is the subclass of* $\mathsf{FRA}_\emptyset$ *in which the algebras have an H-relation W as their top element. The class* RRA_H *of H-relativized representable relation algebras is defined as* $\mathbf{IS}\mathsf{FRA}_H$ *(isomorphic copies of subalgebras of H-relativized relational power set algebras).*

The notation we use is not standard in the literature. The usual way[3] of denoting RRA_H is $\mathbf{SRl}_H\mathsf{RRA}$, where RRA denotes the class of representable relation algebras. The last notation emphasizes the fact that the class of H-relativized RRA's can be obtained from the class RRA by H-relativization and taking subalgebras. Above we defined the classes RRA_H directly without referring to RRA at all. In fact, it is easy to see that $\mathsf{RRA} = \mathsf{RRA}_{\{R,S,T\}}$.

The classes just defined can be ordered as given in figure 1, where $X \to Y$ denotes $X \supseteq Y$. It is easy to see that all the inclusions are strict except the ones labelled with $=$.

To the set theoretic operations Id, "$|$", $(\cdot)^{-1}$ we associate so called abstract operation symbols $1'$, $;$, $\smile$, respectively. The reasons for doing this are explained in e.g. (Henkin et al., 1971 1985).

Proposition 1.2 $\mathsf{RRA}_{ST} = \mathsf{RRA}_{RST} = \mathsf{RRA}$. *These are the only equivalences occurring in Figure 1.*

Proof. Transitivity and symmetry together imply reflexivity, and every algebra in FRA_{RST} is a direct product of full powerset RRA's.

The differences between the classes can be indicated by the following equations; the reader is invited to complete the proof.

for reflexivity: $1';x = x$, ($1'$ denotes identity, ; is composition)
for symmetry: $(x^\smile)^\smile = x$ ($^\smile$ denotes converse), and
for transitivity: $(x;y);z = x;(y;z)$. QED

Our main results are conveniently summarized in the next theorem.

[3](Maddux, 1982) however uses $\mathbf{SRl}\mathsf{RRA}$ to denote $\mathbf{SRl}_{\{R,S\}}\mathsf{RRA}$, but that is the only relativized class of RRA considered there.

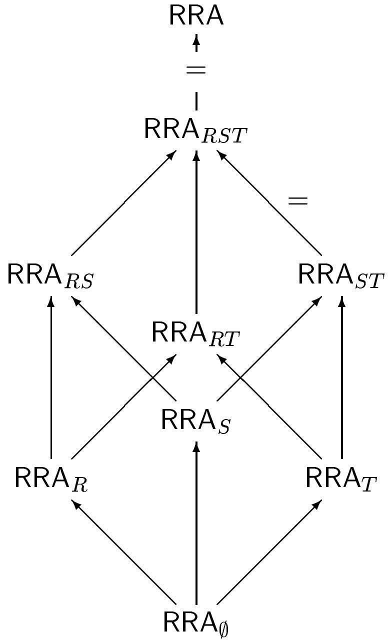


Figure 1: A hierarchy of relativized RRA's.

Theorem 1.3 *Let $H \subseteq \{R, S, T\}$. Then*
(i) RRA_H *is finitely axiomatizable iff $T \notin H$;*
(ii) The equational theory of RRA_H *is decidable iff $T \notin H$;*
(iii) RRA_H *enjoys the amalgamation property iff $T \notin H$.*

On the proof: we prove the theorem and give additional information in sections 2 to 4. In these sections we usually prove theorems which are stronger but not so neatly to formulate.

Arrow frames. In proving Theorem 1.3 we will frequently use the notion of an arrow frame (in (Jónsson and Tarski, 1951) arrow frames are called "relational atom structures"). An *arrow frame* is a structure $\mathfrak{F} = (W, C, F, I)$ where W is a set, $C \subseteq W \times W \times W$, $F \subseteq W \times W$, and $I \subseteq W$. The elements of W will be called *arrows*. Every arrow frame $\mathfrak{F}$ gives rise to a relational type power set algebra $\mathfrak{F}^+ = (\mathcal{P}(W), \cup, -, ;, ^\smile, 1')$ in which the extra-Boolean operations are defined as follows:

$$\begin{array}{lcl} R;S & = & \{x \in W \mid (\exists y, z \in W) : Cxyz \ \& \ y \in R \ \& \ z \in S\} \\ R^\smile & = & \{x \in W \mid (\exists y \in W) : Fxy \ \& \ y \in R\} \\ 1' & = & \{x \in W \mid Ix\}. \end{array}$$

Powerset algebras obtained from frames are called *complex algebras* (cf. (Henkin et al., 1971 1985; Goldblatt, 1989)). For K a class of arrow frames, K^+ denotes the class $\{\mathfrak{F}^+ \mid \mathfrak{F} \in \mathsf{K}\}$.

Let AF denote the class of all arrow frames and set $\mathsf{BAO} = \mathbf{IS}\mathsf{AF}^+$. By Jónsson and Tarski's well-known representation theorem, BAO (which stands for Boolean algebras with operators) is axiomatizable by the Boolean equations plus equations which say that ; and $^\smile$ are normal additive operations (Jónsson and Tarski, 1951). Whenever we mention axiomatizations in the sequel we will always silently assume these axioms as well. We will also use BAO for the set of equations axiomatizing that class.

Note that any H-relativized relational power set algebra with top element W is a complex algebra of the arrow frame (W, C_W, F_W, I_W) in which the relations are defined as

$$\begin{array}{lcl} C_W xyz & \iff & x_0 = y_0, y_1 = z_0 \ \& \ x_1 = z_1, \\ F_W xy & \iff & x_0 = y_1 \ \& \ x_1 = y_0, \\ I_W x & \iff & x_0 = x_1. \end{array}$$

Such arrow frames will be called *pair frames.* When we draw pictures of pair frames, we will denote a pair (x, y) as an arrow going from x to y.

2 Axiomatizability

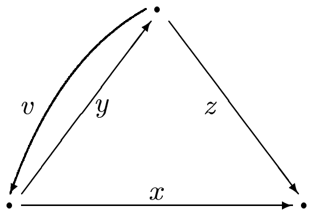
In this section we deal with the first part of our main theorem, that is for $H \subseteq \{R, S, T\}$,

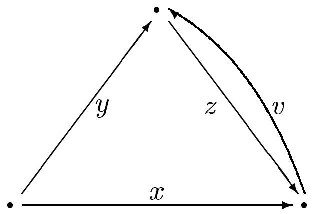
$$\mathsf{RRA}_H \text{ is finitely axiomatizable iff } T \notin H.$$

The negative results about finite axiomatizability can be found in (Andréka, 1991; Andréka et al., 1994b). The positive results are due to Kramer and Maddux (cf. (Kramer, 1991; Maddux, 1982)). A simplified proof and simpler axiomatization can be found in (Marx et al., 1996a). The axiomatizations given here are proven complete in (Marx et al., 1996a).

Axioms and and arrow frames. In the next proposition we give a list of axioms plus the properties of arrow frames which they characterize. All axioms are Sahlqvist equations (cf. (de Rijke and Venema, 1995)); this implies by that paper that they are canonical[4] and they correspond to first-order conditions on arrow frames. The first twelve are *valid* on all relativized RRA's. Equation *AL13* is valid on *symmetric*-relativized RRA's (i.e., relativized RRA's with a symmetric top element), and *AL14* and *AL15* on *reflexive* ones.

Their meaning is most easily grasped by checking these validities using the proposed way of drawing pair frames. We briefly go through the list. The meaning of *NL1* and *NL10* is easy: every arrow has at most one converse and the converse relation is symmetric (i.e., if we look at the relation F as a partial function f, it says that if fx is defined, then so is ffx and it equals x). *NL13* says that every arrow has a converse. Conditions *NL2* and *NL3* state that an identity arrow is its own converse, and can be decomposed into itself. The meaning of *NL4* and *NL5* becomes clear by the following pictures:





Condition *NL6* states that if an identity arrow is decomposed into y and z, then y and z are converses. Conditions *NL7* – *NL9* express that arrows which "meet" share the same identity arrow at this meeting–point. If an arrow is a pair (u, v), then we call (u, u) its domain, and (v, v) its range. Conditions *NL14* and *NL15* state that for every arrow its domain and range are defined. The meaning of conditions 11 and 12 is obvious.

[4] A BAO-equation is called canonical if it is preserved under taking canonical extensions (i.e., ultrafilter extensions).

Proposition 2.1 *For $1 \leq i \leq 15$, every equation ALi given below is canonical, and*

$$\mathfrak{F} \models NLi \iff \mathfrak{F}^+ \models ALi, \text{ for every arrow frame } \mathfrak{F}.$$

The correspondence of 2, and 7–9 hold only when 11 and 12 are assumed.

NL1	$\forall xy(Fxy \Rightarrow Fyx)$	*AL1*	$x \cdot y^{\smile} \leq (x^{\smile} \cdot y)^{\smile}$
NL2	$\forall x(Ix \Rightarrow Cxxx)$	*AL2*	$1' \leq 1';1'$
NL3	$\forall x(Ix \Rightarrow Fxx)$	*AL3*	$x \cdot 1' \leq x^{\smile}$
NL4	$\forall xyzv(Cxyz \& Fyv \Rightarrow Czvx)$	*AL4*	$x^{\smile}; -(x;y) \leq -y$
NL5	$\forall xyzv(Cxyz \& Fzv \Rightarrow Cyxv)$	*AL5*	$-(x;y);y^{\smile} \leq -x$
NL6	$\forall xyz(Cxyz \ \& \ Ix \Rightarrow Fzy)$	*AL6*	$x; -(x^{\smile}) \leq -1'$
NL7	$\forall xyzv(Cxyz \& Cxvx \& Iv \iff Cxyz \& Cyvy \& Iv)$	*AL7*	$[(x \cdot 1');y];z = (x \cdot 1');[y;z]$
NL8	$\forall xyzv(Cxyz \& Cyyv \& Iv \iff Cxyz \& Czvz \& Iv)$	*AL8*	$[x;(y \cdot 1')];z = x;[(y \cdot 1');z]$
NL9	$\forall xyzv(Cxyz \& Czzv \& Iv) \iff Cxyz \& Cxxv \& Iv$	*AL9*	$[x;y];(z \cdot 1') = x;[y;(z \cdot 1')]$
NL10	$\forall xyz(Fxy \& Fxz \Rightarrow y = z)$	*AL10*	$x^{\smile} \cdot y^{\smile} \leq (x \cdot y)^{\smile}$
NL11	$\forall xyz(Cxyz \& Iy \Rightarrow x = z)$	*AL11*	$1';x \leq x$
NL12	$\forall xyz(Cxyz \& Iz \Rightarrow x = y)$	*AL12*	$x;1' \leq x$
NL13	$\forall x \exists y(Fxy)$	*AL13*	$1^{\smile} = 1$
NL14	$\forall x \exists yz(Iy \& Cxyz)$	*AL14*	$1';1 = 1$
NL15	$\forall x \exists yz(Iy \& Cxzy)$	*AL15*	$1;1' = 1$

PROOF. Straightforward, since all equations are in Sahlqvist form. QED

The above proposition implies that the class defined by a set of the above equations ALi is a complex variety $\mathbf{IS}\mathsf{K}^+$, where K is the class of arrow frames defined by the corresponding conditions NLi.

Theorem 2.2 *The varieties* RRA_H, *for* $H \subseteq \{R, S\}$ *can be axiomatized by the following equations (on top of the* BAO*-equations):*

$$\begin{aligned}
\mathsf{RRA}_\emptyset &= \mathbf{Var}(AL1 - AL12);\\
\mathsf{RRA}_S &= \mathbf{Var}(AL1 - AL13);\\
\mathsf{RRA}_R &= \mathbf{Var}(AL1 - AL12, AL14, AL15);\\
\mathsf{RRA}_{R,S} &= \mathbf{Var}(AL1 - AL15).
\end{aligned}$$

PROOF. For a proof of this theorem we refer to (Marx et al., 1996a). There it is shown that every arrow frame satisfying conditions $NL1 - NL12$ is a bounded morphic image (see the proof of Theorem 4.2 for the definition of bounded morphisms) of a pair frame. The representation theorem can now be derived as follows: let $\mathfrak{A} \in \mathbf{Var}(AL1 - AL12)$. By the embedding theorem, $\mathfrak{A}$ is a subalgebra of $\mathfrak{Em}\mathfrak{A}$ –the canonical embedding algebra of $\mathfrak{A}$ (Jónsson and Tarski, 1951). Since all equations are canonical, $\mathfrak{Em}\mathfrak{A} \in \mathbf{Var}(AL1 - AL12)$. Now, $\mathfrak{Em}\mathfrak{A}$ is the complex algebra of an arrow frame satisfying $NL1 - NL12$ by proposition 2.1. Because this arrow frame is a bounded morphic image of a pair frame $\mathfrak{F}$, by standard duality theory ((Goldblatt, 1989)), $\mathfrak{Em}\mathfrak{A}$ is a subalgebra of the complex algebra of $\mathfrak{F}$, which is in $\mathsf{RRA}_\emptyset$ by definition. So $\mathfrak{A} \in \mathsf{RRA}_\emptyset$. The same argument can be applied for the other classes as well. QED

Remarks 2.3 The non-finite axiomatizability result for $\mathsf{RRA} = \mathsf{RRA}_{\{R,S,T\}}$ originates with Monk (Monk, 1969). This result has considerably been strengthened by various authors, cf. (Andréka, 1991) for the most recent version. Infinite axiomatizations of RRA are known, cf. for instance (Lyndon, 1950). Venema (Venema, 1992) provided a finite non-equational description of the equational theory of RRA, by adding a non-structural rule to the rules of equational logic (the price being that this rule is not sound in the sense of equational logic).

The axioms for $\mathsf{RRA}_{R,S}$ can be given in a much simpler way: all RA-axioms, but with the associativity axiom replaced by *AL7* (Maddux, 1982). Arrow frames originate under the name "atom structures" with (Jónsson and Tarski, 1951). In (Lyndon, 1950) one can find the first-order conditions on atom structures corresponding to the RA-equations.

In the related field of *cylindric algebras* we find very similar results with respect to relativization. The cylindric algebraic analogue of transitivity is the *patchwork property*; there are classes of cylindric relativized set algebras, like D_α and G_α, in which this property fails and which are finitely (schema) axiomatizable (Andréka and Thompson, 1988; Andréka and Németi, 1995; Henkin et al., 1971 1985).

3 Decidability

We prove a decidability theorem and sketch the proof of an undecidability theorem. Having done this we apply these two to our main theorem 1.3. We start with some preliminaries concerning our proof-technique.

Theorem 3.1 *Let* $\mathsf{BAO} \cup \{AL1 - AL12\} \subseteq \Delta \subseteq \mathsf{BAO} \cup \{AL1 - AL15\}$. *Then the universal theory of* Δ *is decidable, and every non-valid universal sentence can be falsified in a finite algebra from* $\mathbf{Var}(\Delta)$. *The variety* $\mathbf{Var}(\Delta)$) *is generated by its finite members.*

On the proof: we will prove decidability using a technique which is well–known from modal logic: *filtration.* In items 3.2–3.5 below we briefly recall this method, but we will recall the lemmas without proofs. For the missing proofs we refer to (Hughes and Creswell, 1984) or (Marx, 1995). In the sequel, we will call algebra valuation pairs $(\mathfrak{A}, \mathsf{v})$ *models.*

Definition 3.2 (Filtration) *Let* $\mathfrak{M} = (\mathfrak{F}^+, \mathsf{v})$ *be a model,* $\mathfrak{F} = \langle W, C, F, I\rangle$, *and* Σ *a set of relational type terms which is closed under taking sub–terms and under the Boolean operators. Define an equivalence relation* $\equiv_\Sigma \subseteq W \times W$ *as follows:*

$$(\forall w, v \in W) : w \equiv_\Sigma v \stackrel{\text{def}}{\iff} (\forall \tau \in \Sigma)\ [w \in [\![\tau]\!]_{\mathfrak{M}} \iff v \in [\![\tau]\!]_{\mathfrak{M}}].$$

Let $\overline{w}$ *denote the equivalence class* $w/{\equiv_\Sigma}$, *and equate equivalence classes* $\overline{w}$ *with the sets of terms* $\{\tau \in \Sigma : w \in [\![\tau]\!]_{\mathfrak{M}}\}$.
We call a model $\mathfrak{M}^* = (\mathfrak{F}^{*+}, \mathsf{v}^*)$ *a* filtration *of* $\mathfrak{M}$ through Σ *if conditions (i)–(iv) below hold.*

(i) $W^* \stackrel{\text{def}}{=} \{\overline{w} : w \in W\}$;
(ii) $\mathfrak{F}^* = (W^*, C^*, F^*, I^*)$ *is an arrow frame;*
(iii) $\mathsf{v}^*(x) \stackrel{\text{def}}{=} \{\overline{w} \in W^* : x \in \overline{w}\}$, *for all* variables $x \in \Sigma$;
(iv) **min** *and* **max** , *given below, hold.*

$$\begin{array}{ll}
\textbf{min} & Cxyz \Rightarrow C^*\overline{x}\,\overline{y}\,\overline{z} \\
& Fxy \Rightarrow F^*\overline{x}\,\overline{y} \\
& Ix \iff I^*\overline{x} \\
\textbf{max} & (\forall \tau;\sigma \in \Sigma) : ((C^*\overline{x}\,\overline{y}\,\overline{z} \ \ \& \ \ \tau \in \overline{y} \ \ \& \ \ \sigma \in \overline{z}) \Rightarrow \tau;\sigma \in \overline{x}) \\
& (\forall \tau^{\smile} \in \Sigma) : ((F^*\overline{x}\,\overline{y} \ \ \& \ \ \tau \in \overline{y}) \Rightarrow \tau^{\smile} \in \overline{x}).
\end{array}$$

We call a filtration *minimal* if all relations are defined minimally, as in

$$C^*\overline{x}\,\overline{y}\,\overline{z} \overset{\text{def}}{\iff} (\exists x' \equiv_\Sigma x, y' \equiv_\Sigma y, z' \equiv_\Sigma z) : Cx'y'z'.$$

As their names indicate, **min** provides a lower–bound, and **max** an upper–bound for the relations. They are designed to make the following lemma true. The proof is by a straightforward induction on the complexity of terms.

Lemma 3.3 (Truth lemma) *Let $\mathfrak{M}^* = (\mathfrak{F}^{*+}, \mathsf{v}^*)$ be a filtration of $\mathfrak{M} = \langle \mathfrak{F}^+, \mathsf{v} \rangle$ through Σ. For any universal formula φ constructed from terms in Σ,*

$$(\forall \varphi \in \Sigma) : \mathfrak{M} \models \varphi \iff \mathfrak{M}^* \models \varphi. \tag{1}$$

Definition 3.4 *Let K be a class of arrow frames. We say that K admits filtrations if for any finite set of terms X and any model $\mathfrak{M} = \langle \mathfrak{F}^+, \mathsf{v} \rangle$ where $\mathfrak{F} \in \mathsf{K}$, there exists a set $\Sigma \supseteq X$ and a filtration $(\mathfrak{F}^{*+}, \mathsf{v}^*)$ of $\mathfrak{M}$ through Σ such that $\mathfrak{F}^*$* is finite and it belongs to K.

We will call the set $\Sigma \supseteq X$ as it appears in the above definition the *closure set* of X. It will, by definition, be closed under taking sub–terms and under the Boolean operations.

Lemma 3.5 *Let K be a finitely axiomatizable class of frames which admits filtrations. Then (i)–(iii) below hold.*
(i) $\mathbf{Univ}(\mathsf{SK}^+)$ *is decidable;*
(ii) $\mathbf{Univ}(\mathsf{SK}^+) = \mathbf{Univ}(\mathbf{Fin}\mathsf{SK}^+)$*;*
(iii) The variety generated by K^+ is generated by its finite members.

Define the following abbreviations: $\mathsf{h}x = 1;(1' \cdot x)$ and $\mathsf{t}x = (1' \cdot x);1$. We are ready for the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Since all classes $\mathbf{Var}(\Delta)$ from the theorem are complex varieties SK^+, by the above lemma it is sufficient to show that the corresponding frame-classes admit filtrations. We will be able to show the last if we close the closure set under taking t, h and $^{\smile}$. The next claim states that we can do that without losing finiteness.

Claim 1 Let X be a finite set of terms closed under sub–terms and let $CL(X)$ be the smallest set containing $X \cup \{1'\}$ which is closed under the Boolean operators, t, h and $^{\smile}$. Then the set $CL(X)$ is finite modulo Δ-equivalence.

PROOF OF CLAIM. By Theorem 2.2, $\mathbf{Var}(\Delta) \subseteq \mathsf{RRA}_\emptyset$. The claim now follows from the easily verifiable fact that every finitely Boolean-plus-$\{1', ^{\smile}, \mathsf{t}, \mathsf{h}\}$-generated subalgebra in $\mathsf{RRA}_\emptyset$ is finite. ◀

Let $\mathfrak{F}^+ = (W, C, F, I)^+ \models \Delta$ and $\mathfrak{M} = \langle \mathfrak{F}^+, \mathsf{v} \rangle$ be a model and let X be a finite set of terms. Let the closure set $CL(X)$ be the smallest set containing $X \cup \{1'\}$ which is closed under sub–terms, $\mathsf{h}, \mathsf{t}, {}^{\smile}$ and the Booleans. Let $(\mathfrak{F}^{*+}, \mathsf{v}^*)$, where $\mathfrak{F}^* = \langle W^*, C^*, F^*, I^* \rangle$, be the *minimal* filtration of $\mathfrak{M}$ through $CL(X)$. The next claim implies that the class $\{\mathfrak{F} = (W, C, F, I) : \mathfrak{F} \models \Delta\}$ admits filtrations.

Claim 2 (i) W^* is finite, and (ii) $\mathfrak{F}^* \models \Delta$.

Proof of Claim. (i). Claim 1 implies that the closure set is finite modulo Δ–equivalence. So there are only finitely many equivalence classes, and W^* is finite.
(ii). For efficiency we introduce another claim. From now on we suppress the subscript $_{CL(X)}$ in $\equiv_{CL(X)}$.

Claim 3 The following statements hold for the above given $\mathfrak{F}$ and $\equiv$.

$$Ix \ \& \ x \equiv x' \ \Rightarrow \ Ix' \tag{2}$$
$$Fxy \ \& \ Fx'z \ \& \ x \equiv x' \ \Rightarrow \ y \equiv z \tag{3}$$
$$Fxy \ \& \ x \equiv x' \ \Rightarrow \ (\exists z) : Fx'z \tag{4}$$
$$Cxyx \ \& \ Iy \ \& \ x \equiv x' \ \Rightarrow \ (\exists z) : Cx'zx' \ \& \ Iz \ \& \ y \equiv z \tag{5}$$
$$Cxxy \ \& \ Iy \ \& \ x \equiv x' \ \Rightarrow \ (\exists z) : Cx'x'z \ \& \ Iz \ \& \ y \equiv z. \tag{6}$$

Proof of Claim. (2). Immediate since $1' \in CL(X)$.
(3). Suppose the antecedent. We compute:
$\tau \in \overline{y} \iff$ (since $CL(X)$ is closed under ${}^{\smile}$) $\tau^{\smile} \in \overline{x} \iff \tau^{\smile} \in \overline{x'} \iff \tau \in \overline{z}$.
(4). Suppose the antecedent. Then $1^{\smile} \in \overline{x}$, hence $1^{\smile} \in \overline{x'}$, so $\exists z : Fx'z$.
(5). Suppose the antecedent. Then $\mathsf{t}1 \stackrel{\text{def}}{=} 1';1 \in \overline{x}$, hence also in $\overline{x'}$, so there exists a z such that $Cx'zx'$ & Iz. To show that $y \equiv z$, suppose that $\tau \in \overline{y}$. We compute:
$\tau \in \overline{y} \iff$ (since $CL(X)$ is closed under taking t) $\mathsf{t}\tau \in \overline{x} \iff \mathsf{t}\tau \in \overline{x'} \iff \tau \in \overline{z}$.
(6) is similar to (5); now use closure under h. ◀

Now we are ready to prove that $\mathfrak{F}^*$ validates the conditions *NL1* – *NL15*, if $\mathfrak{F}$ validated them already. Conditions *NL1* – *NL3* are immediate because of the minimal filtration. For *NL4* suppose $C^*\overline{x}, \overline{y}, \overline{z}$ & $F^*\overline{y}, \overline{v}$, then (by the *minimal* definition of C^* and F^*) there are $(x' \equiv x, z' \equiv z, v' \equiv v$ & $y' \equiv y'' \equiv y)$ such that $Cx'y'z'$ & $Fy''v'$. By (3) and (4) we find $Fy'v''$ & $v'' \equiv v'$ for some v''. Then, by *NL4*, $Cz'v''x'$, so, by definition of C^*, $C^*\overline{z}, \overline{v}, \overline{x}$. Condition *NL5* can be shown similarly. For *NL6* use (2). Conditions *NL7* – *NL9* are all similar; we show the $\Rightarrow$ side of *NL8* as an example. Suppose the antecedent, then there are $(x' \equiv x, y''' \equiv y'' \equiv y' \equiv y, z' \equiv z, v'' \equiv v' \equiv v)$ such that $Cx'y'z'$ & $Cy''y'''v'$ & Iv''. By (2) also Iv', so by *NL12* we have $y''' = y''$ and by (6) and (2), $(\exists v''' \equiv v') : Cy'y'v'''$ & Iv'''. By *NL8* we obtain $Cz'v'''z'$ and, by definition of C^*, $C^*\overline{z}, \overline{v}, \overline{z}$. Condition *NL10* is immediate by (3) and conditions *NL11* and *NL12* by (2). Conditions *NL13* – *NL15* finally are immediate by the minimal filtration. QED

Theorem 3.6 ((Andréka et al., 1996)) *Let* K *be a class of relational type algebras such that (i)* $\mathsf{RRA} \subseteq \mathsf{K} \subseteq \mathsf{BAO}$ *and (ii)* $\mathsf{K} \models x;(y;z) = (x;y);z$. *Then the equational theory of* K *is undecidable.*

PROOF. Cf. (Andréka et al., 1996). We outline the main steps of the argument. Define the following term:

$$\Diamond x = x + 1;x + x;1 + 1;x;1.$$

Define $\Box x = -\Diamond - x$. The undecidability argument now uses the following three key steps:

(i) The quasi-equational theory of the class of semigroups (SG) is undecidable.

(ii) For every quasi-equation q in the language of semigroups, we have

$$\mathsf{SG} \models q \iff \mathsf{K} \models q.$$

(iii) For every quasi-equation $q \approx (\sigma_1 = \tau_1 \& \ldots \& \sigma_n = \tau_n \Rightarrow \sigma_0 = \tau_0)$ in the language of semigroups, we have

$$\mathsf{K} \models q \iff \mathsf{K} \models \Box(\sigma_1 \leftrightarrow \tau_1 \cdot \ldots \cdot \sigma_n \leftrightarrow \tau_n) \leq \sigma_0 \leftrightarrow \tau_0,$$

where $x \leftrightarrow y$ is defined as $(-x + y) \cdot (-y + x)$.

QED

Proof of Theorem 1.3.(ii) The negative part of the theorem follows from Theorem 3.6, since ; is associative on relativized RRA's with a transitive top element, and for any H, $\mathsf{RRA} \subseteq \mathsf{RRA}_H$.

The positive part follows from Theorem 3.1 together with the axiomatization theorem 2.2.

Remarks 3.7 Undecidability of RRA originates with Tarski; (Andréka et al., 1996) contains a vast number of strengthenings of Theorem 3.6 and related undecidability theorems. Decidability of the universal theory of $\mathsf{RRA}_{R,S}$ was first proved in (Németi, 1987), using Maddux' axioms.

4 Amalgamation

We start this section with the definition of amalgamation. Then we show which classes have the amalgamation property and which don't, and finally we apply these results to our main Theorem 1.3.

Definition 4.1 *Let* K *be a class of algebras.*

1. K *has the* Embedding Property (EP) *if, for any* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$*, and embeddings* f, h *such that* $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$*, there exists* $\mathfrak{D} \in \mathsf{K}$*, and embeddings* m, n *such that* $\mathfrak{B} \overset{m}{\rightarrowtail} \mathfrak{D} \overset{n}{\leftarrowtail} \mathfrak{C}$*, viz.*

$$\begin{tikzcd}
& \mathfrak{D} & \\
\mathfrak{B} \arrow[ur, "m"] & & \mathfrak{C} \arrow[ul, "n"'] \\
& \mathfrak{A} \arrow[ul, "f"] \arrow[ur, "h"'] &
\end{tikzcd}$$

2. K *has the* Amalgamation Property (AP) *if 1 can be strengthened*[5] *by requiring that* $m \circ f = n \circ h$.

3. K *has the* Strong Amalgamation Property (SAP) *if 2 can be strengthened by requiring that*

$$(\forall x \in B, \forall y \in C) : m(x) = n(y) \Rightarrow (\exists z \in A) : x = f(z) \ \ \& \ \ y = h(z).$$

Kiss et al. (Kiss et al., 1983) contains a vast amount of information about EP, AP, SAP and other related concepts, together with a very extended bibliography of the field.

Theorem 4.2 *Let* V_Δ *be any variety of relational type algebras which is defined by a set* $\mathsf{BAO} \subseteq \Delta \subseteq \{AL1 - AL15\}$. *Then* V_Δ *enjoys the strong amalgamation property.*

PROOF. In the proof we will use standard duality theory between frames and (complex) algebras (cf. (Goldblatt, 1989)). We recall the definition of a *bounded morphism.* Let $\mathfrak{F}_0 = (W_0, C_0, F_0, I_0)$ and $\mathfrak{F}_1 = (W_1, C_1, F_1, I_1)$ be two arrow frames. A function $f : W_0 \longrightarrow W_1$ is called a bounded morphism if f is a homomorphism (i.e., C_0xyz implies $C_1fxfyfz$, etc.) and f satisfies the following "zag"-conditions:

$$\begin{array}{lcl} \text{if } C_1fxyz & \text{then} & (\exists y', z' \in W_0) : C_0xy'z' \text{ and } fy' = y \text{ and } fz' = z, \\ \text{if } F_1fxy & \text{then} & (\exists y' \in W_0) : F_0xy' \text{ and } fy' = y, \\ I_1fx & \text{iff} & I_0x \end{array}$$

We only need to show that varieties defined by a subset of $\{AL1 - AL12\}$ have SAP, since any extension of such a class with equations using only constants will also have SAP. Because all are canonical equations, we can without loss of generality assume that the algebras involved are full powerset algebras over some arrow frame. So assume we have $\mathfrak{F}^+, \mathfrak{G}^+, \mathfrak{H}^+ \in \mathsf{V}_\Delta$ such that $\mathfrak{G}^+ \overset{f}{\hookleftarrow} \mathfrak{F}^+ \overset{h}{\rightarrowtail} \mathfrak{H}^+$.

Define a function $f^+ : G \longrightarrow F$ as $f^+x = y$ iff $x \in f(y)$, and similarly define $h^+ : H \longrightarrow F$. Then, by standard duality theory (Goldblatt, 1989), f^+ and h^+ are surjective bounded morphisms.

Define the frame $\mathfrak{J}$ as the model-theoretic product of $\mathfrak{G}$ and $\mathfrak{H}$ restricted to the set $B = \{(x, y) \in G \times H \mid f^+x = h^+y\}$, i.e., $\mathfrak{J} = (\mathfrak{G} \times \mathfrak{H})_{\restriction B}$. Let π_0 and π_1 denote the projections from J onto G and H, respectively.
We claim that:

(i) For any set $\Delta \subseteq \{NL1 - NL12\}$, if $\mathfrak{G} \models \Delta$ and $\mathfrak{H} \models \Delta$, then $\mathfrak{J} \models \Delta$.

(ii) $f^+ \circ \pi_0 = h^+ \circ \pi_1$.

(iii) π_0 and π_1 are surjective bounded morphisms.

(i) is immediate, since $\{NL1 - NL12\}$ are universal Horn sentences and they are preserved under taking subdirect products. The diagram commutes because we restricted the universe of $\mathfrak{J}$ to pairs with a common image in $\mathfrak{F}$; this proves (ii). By definition the projections are homomorphisms; they are surjective since f^+ and h^+ are surjective. For the zag-part, assume that

[5] We use this to abbreviate that, if we strengthen 1 by adding the requirement $m \circ f = n \circ h$ to the conclusion, then the stronger version of 1 will still hold in K.

$C^{\mathfrak{G}}\pi_0(x,x')yz$ holds. We have to show that we can find y',z' such that $C^{\mathfrak{J}}(x,x')(y,y')(z,z')$ holds. First, since f^+ is a homomorphism, we find $C^{\mathfrak{F}}f^+\pi_0(x,x')f^+yf^+z$. Since the diagram commutes this is the same as $C^{\mathfrak{F}}h^+\pi_1(x,x')f^+yf^+z$. Since h^+ is a bounded morphism, we find $y',z' \in H$ such that $C^{\mathfrak{H}}\pi_1(x,x')y'z'$, and $h^+y' = f^+y$ and $h^+z' = f^+z$. But then, by definition of $\mathfrak{J}$, we have $C^{\mathfrak{J}}(x,x')(y,y')(z,z')$. Clearly this argument works for any relation in the frames.

To finish the proof, define $m : \mathfrak{G}^+ \longrightarrow \mathfrak{J}^+$ and $n : \mathfrak{H}^+ \longrightarrow \mathfrak{J}^+$ as $m(x) = \{y \in J \mid \pi_0(y) \in x\}$ and $n(x) = \{y \in J \mid \pi_1(y) \in x\}$. By standard duality, m and n are embeddings, and, by (ii), the amalgamation diagram commutes. By (i) above $\mathfrak{J}^+ \models \Delta$, so rest us to show that the amalgamation is also strong. But that follows by a simple computation from the fact that for any $x \in G$ and $y \in H$ such that $f^+x = h^+y$ we have included (x,y) in J. QED

Theorem 4.3 *Let* K *be a class of relational type algebras such that (i)* $\mathsf{RRA} \subseteq \mathsf{K} \subseteq \mathsf{BAO}$ *and (ii)* $\mathsf{K} \models (x;1);1 \leq x;1$. *Then the embedding property fails in* K.

Proof. We refer the reader for a proof to (Marx, 199). QED

Proof of Theorem 1.3.(iii) The negative part follows from Theorem 4.3, since T-relativized RRA's clearly validate this equation. The positive part follows from Theorem 4.2, together with Theorem 2.2.

Remarks 4.4 Amalgamation is interesting for a number of reasons, one is its connection with *interpolation* (cf. for instance (Pigozzi, 1972; Andréka et al., 1994b)). The classes in Theorem 4.2 enjoy even a stronger form of amalgamation –super-amalgamation, a concept originating with (Maksimova, 1991). This can be shown by the same construction as used above to show SAP, cf (Marx, 1995). (Madarász, 1998b) proved for a broad class of logics that arrow interpolation (whenever $\models \varphi \to \psi$, there exists an interpolant θ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$) is equivalent with super-amalgamation. This holds for instance for any logic whose algebraic counterpart consists of (not necessarily normal) BAO's (cf. also (Madarász, 1998a)).

Németi showed that $\mathsf{RRA}_{R,S}$ has SAP (Sain, 1990), while Németi–Sain showed that the related concept of *epimorphisms are surjective* (ES) fails for RRA_H whenever $T \in H$. ES corresponds to the *Beth definability property* in logic (Németi, 1983), and is related to amalgamation via "ES+AP=SAP" (Kiss et al., 1983).

5 Reducts of relativized relation algebras

In this section, we use the standard notation for H-relativized RRA's, because it is more convenient here. So RRA_H will now be denoted by $\mathbf{SRl}_H\mathsf{RRA}$. If $H = \emptyset$ we merely write $\mathbf{SRl}\mathsf{RRA}$.

We look at subreducts of $\mathbf{SRl}_H\mathsf{RRA}$, for each $H \subseteq \{R,S,T\}$. For $Q \subseteq \{1', ^\smile, ;\}$ and $H \subseteq \{R,S,T\}$, we denote the $(\{\vee,\wedge,-,0,1\} \cup Q)$–subreduct of the class $\mathbf{SRl}_H\mathsf{RRA}$ by $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$. This notation is warranted, because (cf. (Henkin et al., 1971 1985)):

$$\mathbf{SRd}_Q\mathbf{SRl}_H\mathsf{RRA} = \mathbf{SRd}_Q\mathbf{Rl}_H\mathsf{RRA} = \mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}. \tag{7}$$

Remark 5.1 Andréka–Németi (Andréka et al., 1994a) showed that whenever $T \in H$ and $; \in Q$, the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is neither finitely axiomatizable nor decidable. For this reason, we concentrate on the case without T. The case where $T \in H$ and $; \notin Q$ is uninteresting, because transitivity does not influence the behaviour of the operators except composition. Whenever $Q \subseteq \{1', {}^\smile\}$, 8 below holds (cf. (Andréka et al., 1994a), Thm 2.1.64).

$$\mathbf{SRl}_{RS}\mathbf{Rd}_Q\mathsf{RRA} = \mathbf{SRl}_{RST}\mathbf{Rd}_Q\mathsf{RRA} = \mathbf{SRd}_Q\mathsf{RRA}. \tag{8}$$

We start our investigation with an easy theorem.

Theorem 5.2 *For any $Q \subseteq \{1', {}^\smile, ;\}$ and $H \subseteq \{R, S, T\}$, the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is a quasi–variety.*

Proof. This follows from the universal algebraic facts (cf. (Henkin et al., 1971 1985)) that (1) RRA is a variety, hence it is closed under $\mathbf{SPUp}$, (2) the operator $\mathbf{SPUp}$ commutes with $\mathbf{SRl}$ and $\mathbf{SRd}$, (3) 7 above, and (4) every class closed under $\mathbf{SPUp}$ is a quasi–variety. QED

In virtue of the above theorem we will investigate for all interesting choices of Q and H whether the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is a variety. It turns out that, for every Q, the class $\mathbf{SRlRd}_Q\mathsf{RRA}$ is a finitely axiomatizable canonical variety. If we consider subreducts for other choices of H, the situation is not so uniform. Some are not varieties, some are, and for some, we don't know the answer. Table 1 lists the results we do have. As a contrast, we add the results for subreducts of RRA (recall that $\mathsf{RRA} = \mathbf{SRl}_{RST}\mathsf{RRA}$) in the fifth column. These results can be found in (Andréka et al., 1994a). For completeness' sake, we add the results for the full language at the bottom. Table 1 should be read as follows. In the left column, we list the operators of the subreduct involved. The next four columns stand for the four different relativizations we have studied in the previous sections. Each item in the table stands for the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ in which Q is given by the row and H by the column. A V means that the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is a finitely axiomatizable canonical variety. We give the axiomatizations in table 2 below (the BAO-axioms are assumed as well). In one case we need an extra axiom, this is the following (canonical) equation:

$$AL16 \quad (x;(1' \cdot -(y;x)));y \leq -1'.$$

With V^* we denote that the class is a variety, but it is not axiomatizable by finitely many equations. A QV means that the class is not a variety, but it is a finitely axiomatizable quasi–variety, and a $\mathsf{QV}^?$ denotes that we only know that it is a quasi–variety (it might still be a variety).

Theorem 5.3 *Let* K *be any item occurring in Table 1. Then*

(i) if it is labelled by V^*, it is a non-finitely axiomatizable variety;

(ii) if it is labelled by V*, it is a canonical finitely axiomatizable variety whose axioms are given in Table 2;*

(iii) if it is labelled by QV*, it is a finitely axiomatizable quasi-variety, but not a variety;*

(iv) if it is labelled by $\mathsf{QV}^?$, it is a quasi-variety, and might still be a variety.

Proof. For (i) and (iii) we refer to (Andréka et al., 1994a). For an explicit proof of (ii) we refer to (Marx, 1995). These axiomatization results can also be obtained by a straightforward adaptation of the proofs given in (Marx et al., 1996a). (iv) follows from Theorem 5.2. QED

	any relation $\mathbf{SRl}\mathsf{RRA}$	reflexive relation $\mathbf{SRl}_R\mathsf{RRA}$	symmetric relation $\mathbf{SRl}_S\mathsf{RRA}$	reflexive and symmetric rel. $\mathbf{SRl}_{RS}\mathsf{RRA}$	refl., symm. and transitive rel. $\mathbf{SRl}_{RST}\mathsf{RRA} = \mathsf{RRA}$
$1'$	V	QV	V	QV	QV
$\smile$	V	QV	V	QV	QV
$\smile, 1'$	V	QV	V	QV	QV
$;$	V	$\mathsf{QV}^?$	$\mathsf{QV}^?$	$\mathsf{QV}^?$	V^*
$;, \smile$	V	$\mathsf{QV}^?$	V	$\mathsf{QV}^?$	V^*
$;, 1'$	V	V	$\mathsf{QV}^?$	$\mathsf{QV}^?$	V^*
$;, \smile, 1'$	V	V	V	V	V^*

Table 1: Subreducts of relativized RRA's.

Variety	Axiomatizable by axioms
$\mathbf{SRlRd}_{\{1'\}}\mathsf{RRA}$	$\emptyset$
$\mathbf{SRl}_S\mathbf{Rd}_{\{1'\}}\mathsf{RRA}$	$\emptyset$
$\mathbf{SRlRd}_{\{\smile\}}\mathsf{RRA}$	$AL1, AL10$
$\mathbf{SRl}_S\mathbf{Rd}_{\{\smile\}}\mathsf{RRA}$	$AL1, AL10, AL13$
$\mathbf{SRlRd}_{\{\smile,1'\}}\mathsf{RRA}$	$AL1, AL3, AL10$
$\mathbf{SRl}_S\mathbf{Rd}_{\{\smile,1'\}}\mathsf{RRA}$	$AL1, AL3, AL10, AL13$
$\mathbf{SRlRd}_{\{;\}}\mathsf{RRA}$	$\emptyset$
$\mathbf{SRlRd}_{\{;,\smile\}}\mathsf{RRA}$	$AL1, AL4, AL5, AL10$
$\mathbf{SRl}_S\mathbf{Rd}_{\{;,\smile\}}\mathsf{RRA}$	$AL1, AL4, AL5, AL10, AL13$
$\mathbf{SRlRd}_{\{;,1'\}}\mathsf{RRA}$	$AL2, AL7 - AL9, AL11, AL12, AL16$
$\mathbf{SRl}_R\mathbf{Rd}_{\{;,1'\}}\mathsf{RRA}$	$AL2, AL7 - AL9, AL11, AL12, AL14, AL15, AL16$

Table 2: Axiomatizations of subreducts of Relativized RRA's.

Decidability. Above, we have stated that whenever $T \in H$ and $; \in Q$, the equational theory of the resulting class is undecidable. For all other classes in table 1, the results are positive.

Theorem 5.4 *Let* K *be any class occurring in table 1. The universal theory of* K *is decidable if and only if* K *is not marked with* V^*.

Proof. The undecidability results follow from (the proof of) Theorem 3.6. Here, we prove decidability for all classes not marked with V^*. For the three quasi–varieties in the last column, the claim follows from (8) above and the argument given below. So we are left with the first four columns. By 3.1, we know that the universal theories of the first four classes in the bottom row are decidable. The following chain of reasoning shows that we can decide universal sentences in the reduct–language, using a decision procedure for the full language. Let φ be a universal sentence in the reduct $Q \subseteq \{1', \smile, ;\}$. Then:

$$\begin{array}{llll} \mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA} & \models \varphi & \iff & \text{(by equality (7) above)} \\ \mathbf{SRd}_Q\mathbf{SRl}_H\mathsf{RRA} & \models \varphi & \iff & \text{(universal sentences are preserved under } \mathbf{S}) \\ \mathbf{Rd}_Q\mathbf{SRl}_H\mathsf{RRA} & \models \varphi & \iff & (\varphi \text{ is in the language of } Q) \\ \mathbf{SRl}_H\mathsf{RRA} & \models \varphi & & \end{array}$$

QED

Amalgamation.

Theorem 5.5 *All classes in Table 1 which are labelled with* V *enjoy the strong amalgamation property.*

Proof. (The proof of) Theorem 4.2 gives us positive amalgamation results for all classes in Table 1 labelled with V, since all axioms involved, except *AL16*, are treated in that theorem. The canonical equation *AL16* corresponds to

$$NL16 \quad \forall xyzv((Cxyz \& Ix \& Cyyv \& Iv) \rightarrow Cvzy),$$

whence, by the proof of Theorem 4.2, also this equation is preserved under the construction given there. For the other reducts of relativized RRA's the amalgamation problem is still open. QED

Remarks 5.6 There is also an opposite direction possible. Instead of taking reducts, we can expand the language with operators like the cylindrifications c_0 and c_1, or the universal modality E which are term–definable in RRA (though not always when we relativize). We can also expand the language with operations which are not even RRA definable, such as the Kleene *. These expansions are all studied in (Marx, 1995). For all of them we see the same pattern as in Theorem 1.3: the properties finite axiomatizability, decidability and amalgamation hold whenever we allow relativized RRA's with non-transitive top-elements.

In (Marx et al., 1995) the language is expanded with the *difference operator*, which plays such an important part in Venema's approach to describing the equational theory of RRA. It has the following definition on full powerset algebras FRA_H with top W:

$$\mathsf{D}R = \{y \in W \mid (\exists x \neq y) : x \in R\}.$$

It is shown that the class $\mathsf{RRA}_{R,S}$ expanded with D is a finitely axiomatizable variety; moreover its equational theory is decidable (Mikulás, 1995). The embedding property however fails (Marx, 1995).

(Mikulás and Marx, 1997) show that there are also expansions with RRA-definable operators which lead to undecidability in the relativized cases. Undecidability can occur when we are able to express that a relation is a partial function. We can do this for instance with a "coordinate-wise difference operator D_1", defined on full powerset algebras FRA_H with top W as

$$\mathsf{D}_1 R = \{(x,y) \in W \mid (\exists y' \neq y) : (x,y') \in R\}.$$

The expansion of $\mathsf{RRA}_\emptyset$ with D_1 is undecidable (Mikulás and Marx, 1997). For the cylindric algebraic version of this, cf. (Németi, 1991; Németi, 1995; Mikulás and Marx, 1997).

6 Residuals and relativization

We briefly look at a similarity type in which we drop the converse operator $\smile$ and instead have the residuals $\backslash$ and $/$. The similarity type $\{;,/,\backslash\}$ (with and without $1'$) received quite a lot of attention (Pratt, 1990; Jipsen, 1992; Jónsson and Tsinakis, 1993). On relativized relational power set algebras the residuals are defined as follows:

$$\begin{aligned} x \in R\backslash S &\overset{\text{def}}{\iff} (\forall yz) : (x_0 = y_1, y_0 = z_0, z_1 = x_1 \ \&\ y \in R) \Rightarrow z \in S \\ x \in R/S &\overset{\text{def}}{\iff} (\forall yz) : (x_0 = y_0, x_1 = z_0, z_1 = y_1 \ \&\ z \in S) \Rightarrow y \in R. \end{aligned}$$

Instead of $\backslash$ and $/$ the conjugates of ;, denoted by $\lhd$ and $\rhd$ are taken as primitives. The slashes can then be defined by

$$R\backslash S = -(R \rhd -S) \quad \text{and} \quad R/S = -(-R \lhd S).$$

The presently discussed approach is often called "residuated Boolean monoid approach", and it was extensively investigated in Birkhoff's lattice theory book already in the 1950's, cf. also (Jipsen and Maddux, 1997) where the emphasis is on relativization (just as in the present paper).

Definition 6.1 *(i)* $\mathsf{RRA}_\lhd$ *denotes the class of* RRA*'s with* $\{;,\lhd,\rhd,1'\}$ *as basic extra-Boolean operations.*

(ii) Let $H \subseteq \{R,S,T\}$*. Then* $\mathsf{RRA}_{\lhd,H}$ *is defined in a way completely analogous with the definition of* RRA_H*, cf. Definition 1.1.*

The relation between the two similarity types becomes clear by the following

Proposition 6.2 *(i) If* $R \in H$*, then in* $\mathsf{RRA}_{\lhd,H}$*,* $\smile$ *is definable in terms of* $1'$ *and* $\lhd$*.*
(ii) If $S \in H$*, then in* RRA_H*,* $\lhd$ *and* $\rhd$ *are definable in terms of* ; *and* $\smile$*.*
(iii) Assume $\{R,S\} \subseteq H$*. Then* $\mathsf{RRA}_{\lhd,H}$ *and* RRA_H *are term-definitionally equivalent varieties.*

Proof. For (i), define converse from the triangles as in $R^\smile = (1' \lhd R)$. To show that this definition works, one only needs reflexivity. For (ii), define the triangles from converse as in $R \lhd S = R;S^\smile$. Now only symmetry is needed. (iii) follows from (i) and (ii). QED

We note that for some choices of H, $\lhd$ and $\rhd$ are not term-definable in RRA_H. Among others, this justifies the study of $\mathsf{RRA}_{\lhd,H}$ (independently of studying RRA_H).

Theorem 6.3 *Let $H \subseteq \{R, S\}$ be such that $(S \in H \Rightarrow R \in H)$. Then*

$$\mathsf{RRA}_{\lhd,H} \text{ is a finitely axiomatizable variety.} \tag{9}$$

$$\text{The equational theory of } \mathsf{RRA}_{\lhd,H} \text{ is decidable}^6. \tag{10}$$

$$\mathsf{RRA}_{\lhd,H} \text{ has SAP.} \tag{11}$$

We will return to the ideas of the proof later.

Open problem. What is the status of items (9)–(11) for the case $H = \{S\}$? (Of course, (10) remains true.)

Remark 6.4 Assume $H \subseteq \{R, S, T\}$.

1. $T \in H$ implies that $\mathsf{RRA}_{\lhd,H}$ is not finitely axiomatizable.
2. $T \in H$ implies that $\mathsf{RRA}_{\lhd,H}$ does not have the amalgamation property.
3. $T \in H$ implies that the equational theory of $\mathsf{RRA}_{\lhd,H}$ is not decidable.

The first two items are results of Andréka and Németi, while the last is a corollary of the result in (Andréka et al., 1996) quoted here as Theorem 3.6. Here we do not include a proof of these items. The present remark shows that the conditions of Theorem 6.3 are best possible if we disregard the subcondition $S \in H \Rightarrow R \in H$.

Let us turn to discussing the ideas behind the proofs of Theorem 6.3. For (9), there are three cases: $H = \{R, S\}$, $H = \{R\}$ and $H = \emptyset$. The first case follows immediately from Proposition 6.2.(iii) and Theorem 2.2. For the other two cases, consider the following axioms:

(L_1) $x;(x\backslash y) \leq y$
(L_2) $y \leq x\backslash(x;y)$
(L_3) $(x/y);y \leq x$
(L_4) $x \leq (x;y)/y$
(L_5) $(x \rhd 1') \cdot (y \rhd 1') \leq (x \cdot y) \rhd 1'$
(L_6) $(1' \lhd x) \cdot (1' \lhd y) \leq 1' \lhd (x \cdot y)$.

$(L_1) - (L_4)$ are the standard equations saying that ;, $\lhd$ and $\rhd$ are conjugates. (L_5) and (L_6) give $x \rhd 1'$ and $1' \lhd x$, respectively their "converse" behaviour (e.g., (L_5) corresponds on arrow frames –assuming $(L_1) - (L_4)$– to $\forall xyy'z((Ix\&Cxyz\&Cxy'z) \Rightarrow y = y'))$.

We claim that

Every algebra $\mathfrak{A} = (A, ;, \lhd, \rhd, 1')$ satisfying $(L_1) - (L_6)$ plus the equations which axiomatize the converse–free reduct of $\mathsf{RRA}_\emptyset$ (i.e., the axiomatization of the variety $\mathbf{SRlRd}_{\{;,1'\}}\mathsf{RRA}$ given in table 2) is a subalgebra of a relativized algebra $(\mathcal{P}(W), |^W, \lhd^W, \rhd^W, Id^W)$, for W a subset of $U \times U$ for some set U. If in addition $\mathfrak{A} \models 1;1' = 1';1 = 1$, then W is reflexive.

[6] The equational theory of $\mathsf{RRA}_{\lhd,S}$ is decidable too.

Let $\mathfrak{A}$ be as in the claim. In (Marx, 1996) it is described how to represent the $1'$-free reduct of $\mathfrak{A}$, which satisfies only $L_1 - L_4$ as a relativized algebra. When we combine this proof with the proof for the converse-free reduct of $\mathsf{RRA}_{\emptyset}$ (cf. Section 5), the result follows. *This shows* (9).

Whenever, $S \in H$, $\mathsf{RRA}_{\lhd,H}$ can be seen as a subreduct of RRA_H by Proposition 6.2.(ii), whence decidability follows by the arguments in Section 5. Decidability for the cases $H = \emptyset$ and $H = R$, can be proved in a very similar way as we did before; the obvious change is that where before we closed our set of terms under converse, we now require that for any τ we also have $1' \lhd \tau$ and $\tau \rhd 1'$. *This shows* (10).

By (9), the three classes from Theorem 6.3 are all axiomatizable by Sahlqvist equations which correspond to Universal Horn sentences or are variable free. But then strong amalgamation follows with the techniques from section 4. *This shows* (11).

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References

Andréka, H. (1991). Complexity of the equations valid in algebras of relations. Thesis for D.Sc. (a post–habilitation degree) with Math. Inst. Hungar. Ac. Sci. Budapest. Part I: Strong non–finitizability in *Annals of Pure and Applied Logic* **89**:149–209, 1997. Part II: Finite axiomatizations, forthcoming in APAL.

Andréka, H., Kurucz, Á., Németi, I., Sain, I., and Simon, A. (1996). Causes and remedies for undecidability in arrow logics and in multi-modal logics. In (Marx et al., 1996b), pages 63–100.

Andréka, H., Monk, J. D., and Németi, I., editors (1991). *Algebraic Logic (Proc. Conf. Budapest 1988)*, Amsterdam. Colloq. Math. Soc. J. Bolyai, North-Holland.

Andréka, H. and Németi, I. (1995). Finite base property in algebraic logic. Math. Inst. Hungar. Acad. Sci., Budapest.

Andréka, H., Németi, I., and Sain, I. (1994a). Algebras of relations and algebraic logic (an introduction). manuscript, Math. Inst. Hung. Ac. Sci. Budapest. Lecture notes for a university course.

Andréka, H., Németi, I., and Sain, I. (1994b). On interpolation, amalgamation, universal algebra and Boolean algebras with operators. manuscript, Math. Inst. Hungar. Acad. Sci., Budapest.

Andréka, H. and Thompson, R. J. (1988). A Stone-type representation theorem for algebras of relations of higher rank. *Transactions of the American Mathematical Society*, 309(2):671–682.

Andréka, H., van Benthem, J., and Németi, I. (1995). Back and forth between modal logic and classical logic. *Bulletin of the Interest Group in Pure and Applied Logics*, 3(3):685–720.

de Rijke, M. and Venema, Y. (1995). Sahlqvist's theorem for Boolean algebras with operators. *Studia Logica*, LIV:61–78.

Goldblatt, R. (1989). Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44:173–242.

Henkin, L., Monk, J. D., and Tarski, A. (1971 & 1985). *Cylindric Algebras, Parts I & II.* North-Holland.

Henkin, L., Monk, J. D., Tarski, A., Andréka, H., and Németi, I. (1981). *Cylindric Set Algebras.* Number 883 in Lecture Notes in Mathematics. Springer-Verlag, Berlin.

Hughes, G. and Creswell, M. (1984). *A Companion to Modal Logic.* Methuen.

Jipsen, P. (1992). *Computer Aided Investigations of Relation Algebras.* PhD thesis, Vanderbilt University, Nashville, Tennessee.

Jipsen, P. and Maddux, R. (1997). Nonrepresentable sequential algebras. *Logic Journal of the IGPL*, 5(4):565–574.

Jónsson, B. and Tarski, A. (1951). Boolean algebras with operators. Parts I and II. *American Journal of Mathematics*, 73:891–939. and 74:127–162, 1952.

Jónsson, B. and Tsinakis, C. (1993). Relation algebras as residuated Boolean algebras. *Algebra Universalis*, 30:469–478.

Kiss, E., Márki, L., Prőhle, P., and Tholen, W. (1983). Categorical algebraic properties. Compendium on amalgamation, congruence extension, epimorphisms, residual smallness and injectivity. *Studia Sci. Math. Hungar.*, 18.

Kramer, R. (1991). Relativized relation algebras. In (Andréka et al., 1991), pages 293–349.

Lyndon, R. C. (1950). The representation of relation algebras. *Annals of Mathematics*, 51:707–729. Part II appeared in volume 63 (1956) 294–307.

Madarász, J. (1998a). Interpolation and amalgamation. *Studia Logica*, 61:171–205,301–319.

Madarász, J. (1998b). Interpolation in algebraizable logic; semantics for non-normal modal logic. *Journal of Applied Nonclassical Logic*, 8(1–2):67–105.

Maddux, R. (1982). Some varieties containing relation algebras. *Transactions of the American Mathematical Society*, 272:501–526.

Maksimova, L. (1991). Amalgamation and interpolation in normal modal logics. *Studia Logica*, L(3/4):457–471.

Marx, M. (199?). Amalgamation in relation algebras. *Journal of Symbolic Logic.* Forthcoming. Available on WWW:
`http://theory.doc.ic.ac.uk/ mm28/pub/abstracts/jsl96.html`.

Marx, M. (1995). *Algebraic Relativization and Arrow Logic.* PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, University of Amsterdam. ILLC Dissertation Series 1995–3.

Marx, M. (1996). Multi-dimensional semantics for modal logics. *Notre Dame Journal of Formal Logic*, 37(1):25–34.

Marx, M., Mikulás, S., and Németi, I. (1995). Taming logic. *Journal of Logic, Language and Information*, 4:207–226.

Marx, M., Mikulás, S., Németi, I., and Sain, I. (1996a). Investigations in arrow logic. In (Marx et al., 1996b), pages 35–62.

Marx, M., Pólos, L., and Masuch, M., editors (1996b). *Arrow Logic and Multimodal Logics*, Studies in Logic, Language and Information, Stanford. CSLI Publications.

Mikulás, S. (1995). *Taming Logics.* PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, University of Amsterdam. ILLC Dissertation Series 95-12.

Mikulás, S. and Marx, M. (1997). Undecidable relativizations of algebras of relations. *Journal of Symbolic Logic.* forthcoming.

Monk, J. (1993). Lectures on cylindric set algebras. In Rauszer, C., editor, *Algebraic Methods in Logic and in Computer Science*, volume 28, pages 253–290. Banach Center Publications, Polish Academy of Sciences, Warsaw.

Monk, J. D. (1969). Nonfinitizability of classes of representable cylindric algebras. *Journal of Symbolic Logic*, 34:331–343.

Németi, I. (1983). Beth definability is equivalent with surjectiveness of epis in general algebraic logic. Technical report, Math. Inst. Hungar. Acad. Sci., Budapest.

Németi, I. (1987). Decidability of relation algebras with weakened associativity. *Proc. Amer. Math. Soc.*, 100, 2:340–344.

Németi, I. (1991). Algebraizations of quantifier logics: an overview (shortened version). *Studia Logica*, L(3/4):485–569. Full version, Math. Inst. Budapest, Preprint, 1996.

Németi, I. (1995). A fine-structure analysis of fisrt-order logic. In (Marx et al., 1996b), pages 221–247.

Pigozzi, D. (1972). Amalgamation, congruence extension and interpolation properties in algebras. *Algebra Universalis*, 1:269–349.

Pratt, V. (1990). Action logic and pure induction. In van Eijck, J., editor, *Logics in AI*, volume 478 of *Lecture Notes in Artificial Intelligence*, pages 97–120. Springer–Verlag.

Resek, D. and Thompson, R. (1991). An equational characterization of relativized cylindric algebras. In (Andréka et al., 1991), pages 519–538.

Sain, I. (1990). Beth's and Craig's properties via epimorphisms and amalgamation in algebraic logic. In Bergman, C. H., Maddux, R. D., and Pigozzi, D. L., editors, *Algebraic logic and universal algebra in computer science*, volume 425 of *Lecture Notes in Computer Science*, pages 209–226. Springer-Verlag, Berlin.

van Benthem, J. (1996). Modal foundations for predicate logic. In Orlowska, E., editor, *Memorial Volume for Elena Rasiowa*, Studia Logica Library. Kluwer Academic Publishers, Dordrecht.

Venema, Y. (1992). *Many–Dimensional Modal Logic*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, Universiteit van Amsterdam.