Logics and Provability

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Logics and Provability

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Amsterdam
September, 2001.

Katsumi Sasaki

Chapter 1

Introduction

All of the logics in this thesis are related to or connected with the provability logic **GL**.

Provability logic **GL** is one of the normal modal logics, which is obtained from the smallest normal modal logic **K** by adding Löb's axiom $\Box(\Box p \supset p) \supset \Box p$. The name "*provability logic*" derives from Solovay's theorem in Solovay [Sol76]. He proved that **GL** is complete for the formal provability interpretation in Peano arithmetic **PA**. So, **GL** has been considered as one of the most important modal logics. Let us briefly explain Solovay's theorem, following Chagrov and Zakharyaschev [CZ97].

All syntactical constructions of the arithmetic language can be effectively coded by natural numbers; the code $\ulcorner \phi \urcorner$ of an arithmetic formula ϕ is called the *Gödel number* of ϕ. Gödel constructed a formula $Pr(x)$ with a single free variable x such that, for any natural number n,

$$\vdash_{PA} Pr(\overline{n}) \text{ iff}^{1}\ \overline{n} = \ulcorner \phi \urcorner \text{ and } \vdash_{PA} \phi \text{ for some arithmetic formula } \phi,$$

where $\overline{n}$ is the term representing the number n. In other words, $Pr(\ulcorner \phi \urcorner)$ asserts that the formula ϕ is provable in **PA**. By an arithmetic interpretation of the language of modal logic we mean any mapping $*$ from the set of modal formulas to the set of arithmetic sentences such that

$\bot^*$ is $\overline{0} = \overline{1}$;
$(A \wedge B)^* = A^* \wedge B^*$
$(A \vee B)^* = A^* \vee B^*$
$(A \supset B)^* = A^* \supset B^*$
$(\Box A)^* = Pr(\ulcorner A^* \urcorner)$.

Solovay proved the following arithmetic completeness theorem

$$A \in \mathbf{GL} \text{ iff } A^* \text{ is provable in } \mathbf{PA} \text{ for any arithmetic interpretation } *.$$

[1] Iff is the standard abbreviation for "if and only if".

Thus, **GL** is the logic of formal provability of **PA**. For example, the formula $\Box\neg\Box\bot \supset \Box\bot$ is provable in **GL**. This formula expresses Gödel's second incompleteness theorem, i.e., the statement: "if it is provable that **PA** is consistent, then **PA** is inconsistent", is provable in **PA**. This makes **GL** into an interesting research topic. For example, the de Jongh-Sambin fixed point theorem (see Sambin [Sam76] and Smoryński [Smo78]) was proved for **GL**. An extensive overview on the subject can be found in Boolos [Boo93], Smoryński [Smo84] and Smoryński [Smo85]; for a short survey, see Boolos and Sambin [BS91].

The normal modal logic **K4** is a sublogic of **GL**, which is obtained from **K** by adding the transitivity axiom $\Box p \supset \Box\Box p$. **K4** is much easier to deal with than **GL**. Although the difficulty of **GL** is to be expected in view of the additional axioms of **K4** and **GL**, i.e., the transitivity axiom and Löb's axiom, we can also give two concrete examples. One concerns Kripke semantics. Completeness and finite model property for **K4** are obtained by the standard method, i.e., the canonical model and filtration introduced in Lemmon and Scott [LS77], while the corresponding properties for **GL** cannot be obtained in this way (see Gabbay [Gab70]). The other example concerns cut-free sequent systems. A cut-elimination theorem for **K4** can be proved by the standard method due to Gentzen [Gen35] using *degree* and *rank* as induction parameters, while the proof for **GL** first given in Valentini [Val83] uses another parameter *width* (see also Avron [Avr84]).

GL is also obtained by adding Löb's axiom to **K4**. So, as was asserted in [Smo84], the knowledge of **K4** is useful for the discussion of **GL**. Smoryński treated **K4** as a preliminary for the study of **GL**, where he used the name "*Basic modal logic*" instead of **K4**. Here, in chapter 2 and chapter 6, we first discuss a logic corresponding to **K4**, and then a logic corresponding to **GL**.

We now introduce the logics that will be treated here in the following three sections 1.1, 1.2 and 1.3.

1.1 A propositional logic having the formal provability interpretation

Gödel's translation τ is the translation from a propositional non-modal formula A to the modal formula obtained by attaching the modal operator $\Box$ to each subformula of A (cf. Orlov [Orl28], Gödel [Göd33]). Using this translation every intermediate propositional logic, a logic between intuitionistic propositional logic (**IPL**) and classical propositional logic, is embedded into a modal logic between **S4** and **S5** (cf. McKinsey and Tarski [MT48], Dummett and Lemmon [DL59], Zakharyaschev [Zak91]). For example, it was shown that for any non-modal formula A,

$$A \in \mathbf{IPL} \text{ iff } \tau(A) \in \mathbf{S4}.$$

So, it is natural to conjecture that the propositional logic $\mathbf{L}$ satisfying for any non-modal formula A,

$$A \in \mathbf{L} \text{ iff } \tau(A) \in \mathbf{GL}$$

has the formal provability interpretation.

Visser characterized the propositional logics that are embedded into **GL** and **K4**. We call those logics "*formal propositional logic*" and "*Visser's propositional logic*[2]", **FPL** and **VPL** for short. (Not only are **GL** and **K4** among important modal logics that do not include **S4**, but also many other extensions of **K**. Some corresponding propositional logics were considered in Corsi [Cor87], Došen [Dos93] and Wansing [Wan97].) Visser gave natural deduction systems and proved a Kripke completeness for **FPL** and **VPL**. Also he proved a fixed point theorem and an arithmetic completeness for **FPL**, e.g., using Solovay's theorem and the equivalence

$$A \in \mathbf{FPL} \text{ iff } \tau(A) \in \mathbf{GL},$$

for any non-modal formula A.

As was argued in [Vis81], however, the arithmetic interpretation obtained in the above paragraph is not the only one which yields **FPL** and possibly not even the most interesting one. **FPL** turned out to be also the logic of the Σ^0_1-sentences of **PA** by the translation f below:

$f(\bot)$ is $\overline{0} = \overline{1}$;
$f(A \wedge B) = f(A) \wedge f(B)$
$f(A \vee B) = f(A) \vee f(B)$
$f(A \supset B) = Pr(\ulcorner f(A) \supset f(B) \urcorner)$,

and the arithmetic completeness

$$\Gamma \vdash_F A \text{ iff } \mathbf{PA} + \{f(B) | B \in \Gamma\} \vdash f(A),$$

which was proved in [Vis81].

Considering the consequence relation $\vdash_V$ of **VPL**, there is a strange fact, $\{p, p \supset q\} \nvdash_V q$, in particular, $\{\top \supset q\} \nvdash_V q$. In short, the consequence relation $\vdash_V$ of **VPL** does not obey modus ponens in general. This is the essential difference between the consequence relation of **IPL** and $\vdash_V$. This difference can be found in Visser's natural deduction system. His system is obtained from Gentzen's natural deduction system $\vdash_{NJ}$ for **IPL** by replacing the implication elimination rule

$$\frac{A \quad A \supset B}{B} (\supset E)$$

[2]Visser gave the name "*Basic propositional logic*" in view of the fact that **K4** is sometimes called basic modal logic, e.g., [Smo84]

by the following three inference rules

$$\frac{A \supset B \quad A \supset C}{A \supset B \wedge C}(I\wedge_f) \qquad \frac{A \supset C \quad B \supset C}{A \vee B \supset C}(E\vee_f) \qquad \frac{A \supset B \quad B \supset C}{A \supset C}(Tr),$$

which hold in $\vdash_{NJ}$. From the construction of the system, we can confirm the fact $\{p, p \supset q\} \not\vdash_V q$. Also we can see that the formula $(\top \supset A) \supset A$ is not provable in **VPL**, while it is provable in **IPL**.

Visser treated **VPL** as a preliminary for a study of **FPL**. **VPL**, however, was also motivated by a revision of the Brouwer-Heyting-Kolmogorov (BHK) proof interpretation introduced in Ruitenburg [Rui91] and Ruitenburg [Rui92] (see also Kolmogorov [Kol32] and Heyting [Hey56]). Ruitenburg's interpretation for $A \supset B$ is

> A proof of $A \supset B$ is a construction that uses the assumption A to produce a proof of B

while the standard BHK interpretation looks like

> A proof of $A \supset B$ is a construction that converts proofs of A into proofs of B.

Ruitenburg argued that using assumption A, rather than a proof of A, to produce a proof of B avoids the need for converting proofs as in the BHK interpretation. It also makes it harder to prove B, since less information is provided. Under his interpretation, the formula $(\top \supset A) \supset A$ is not provable.

Also Ruitenburg [Rui99] described the relation between propositional formulas in **VPL** and first order formulas with one variable. Predicate extensions of **VPL** are discussed in Ardeshir [Ard99] and Ruitenburg [Rui98].

Sequent style systems for **VPL** were given in Ardeshir [Ard95], and Ardeshir and Ruitenburg [AR99]. Although [Ard95] proved the cut-elimination theorem for his system, a subformula property has not been given. His system corresponds to Visser's natural deduction system, and therefore, contains the inference rule

$$\frac{\Gamma \to A \supset B \quad \Gamma \to B \supset C}{\Gamma \to A \supset C}$$

corresponding to the rule (Tr). This rule makes it difficult to prove the subformula property. Another cut-free sequent system $\mathbf{GVPL}^+$ for **VPL** will be given in this thesis in chapter 2. A subformula property for the system is obtained in the usual way, and what is more, this system can be extended to a cut-free system $\mathbf{GFPL}^+$ for **FPL**. The proof of the cut-elimination theorem for $\mathbf{GFPL}^+$ will be obtained using a new induction parameter *width*, which was used in Valentini [Val83] for the proof of the cut-elimination theorem of **GL**.

A Hilbert style formalization for **VPL** was given by Y. Suzuki and H. Ono in [SO98] using modus ponens and 12 axioms. Since the consequence relation $\vdash_V$ does not obey modus ponens, one might doubt whether they may use modus ponens; but we have

$$A, A \supset B \in \mathbf{VPL} \text{ implies } B \in \mathbf{VPL}.$$

So, they can use modus ponens without hypothesis, as an admissible rule, which is the one inference rule in the usual Hilbert style formalization of a theory. In order to give a Hilbert style formalization of $\vdash_V$, however, we cannot use the rule

$$\Gamma \vdash_V A \text{ and } \Gamma \vdash_V A \supset B \text{ imply } \Gamma \vdash_V B,$$

which is the only inference rule in the usual Hilbert style formalization of a consequence relation. So, it seems difficult to give a Hilbert style formalization of $\vdash_V$, and this difficulty was pointed out by Y. Suzuki, F. Wolter and M. Zakharyaschev in [SWZ98]. In chapter 3, we consider this problem using restricted modus ponens.

Extensions of $\vdash_V$ were treated in [AR99] and [SWZ98]. As extensions of **IPL**, we can consider axiomatic extensions of $\{A \mid \emptyset \vdash_V A\}$. However, in the papers just mentioned, not only extensions of $\{A \mid \emptyset \vdash_V A\}$ but also extensions of the consequence relation $\vdash_V$ were treated. [SWZ98] pointed out that it is not enough to consider extensions as a set of formulas. There are some natural classes of Kripke frames that cannot be formalized by means of extensions as a set of formulas. They also described that a possible solution of this problem is to consider extensions of the consequence relation $\vdash_V$. So, in this thesis, we also treat extensions of $\vdash_V$ rather than of $\{A \mid \emptyset \vdash_V A\}$, i.e., additional rules instead of only additional axioms. In chapter 4, we consider a property of Löb's axiom in those extensions. Using the property and a cut-free system for **VPL**, another proof of the cut-elimination theorem of $\mathbf{FPL}^+$ will also be given. This method can be used to give a cut-free system for interpretability logic that will be introduced in section 1.3 and chapter 6.

1.2 An intuitionistic modal logic

The problem of presenting an intuitionistic concept of modality was faced in Fitch [Fit49] and Prior [Pri57]. Prior proposed a modal extension of **IPL** which turns out to be **S5** once the axiom of excluded middle is added. They were also considered as counterparts of classical modal logics in Božić and Došen [BD84] and Fischer Servi [Fis77]. In [BD84], the intuitionistic modal logic **IntK** was introduced as the smallest set of formulas including the standard axioms of **IPL** and the axiom:

$$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$$

and closed under modus ponens, substitution and necessitation. An intuitionistic modal logic is a set of formulas including **IntK** and closed under modus ponens, substitution and necessitation.

The relationship between intuitionistic modal logics and other logics has been discussed in the literature. In [OS87] and [Suz89], H. Ono and N.-Y. Suzuki investigated the relationship to intermediate predicate logics. Wolter and Zakharyaschev [WZ97] argued that the intuitionistic modal logics are much more closely related to classical bimodal logics than to the usual monomodal ones, and discussed their relation.

Another relation, namely between intuitionistic modal logics and extensions of the consequence relation $\vdash_V$ of **VPL**, was given by Y. Suzuki, F. Wolter and M. Zakharyaschev in [SWZ98]. They used Kripke semantics to establish that relationship, but here we describe it in an axiomatic way, since our treatment in this thesis is mainly axiomatic. First the authors of [SWZ98] introduced a new binary operator $\supset_I$, which is intended to denote the implication in **IPL**. They defined an extension $\vdash_U$ of $\vdash_V$ in the extended language with $\supset_I$, and proved that

$$\Gamma \vdash_U A \supset_I B \text{ iff } \Gamma \cup \{A\} \vdash_V B.$$

($\Gamma \vdash_U A \supset_I B$ is also equivalent to $(A \supset B)^+ \in \mathbf{GVPL}^+$, where the expression $(A \supset B)^+$ will be introduced in chapter 2 to define $\mathbf{GVPL}^+$.) In other words, it was shown that the consequence relation $\vdash_V$ could be interpreted as a binary logical connective. Furthermore, a translation τ^* from the extended propositional language with $\supset_I$ to modal language was defined as follows.

$\tau^*(p) = p,$
$\tau^*(A \wedge B) = \tau^*(A) \wedge \tau^*(B),$
$\tau^*(A \vee B) = \tau^*(A) \vee \tau^*(B),$
$\tau^*(A \supset B) = \Box(\tau^*(A) \supset \tau^*(B)),$
$\tau^*(A \supset_I B) = \tau^*(A) \supset \tau^*(B).$

Finally, they proved that for any propositional formula A in extended language with $\supset_I$,

$$\emptyset \vdash_U A \text{ iff } \tau^*(A) \in \mathbf{U},$$

where $\mathbf{U}$ is the intuitionistic modal logic obtained from **IntK** by adding the axioms $p \supset \Box p$ and $\Box p \supset (q \vee (q \supset p))$. Also a translation σ from the modal language to the extended propositional language was defined as:

$\sigma(p) = p,$
$\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B),$
$\sigma(A \vee B) = \sigma(A) \vee \sigma(B),$
$\sigma(A \supset B) = \sigma(A) \supset_I \sigma(B),$
$\sigma(\Box A) = \top \supset \sigma(A);$

and it was proved that for any modal formula A,

$$\emptyset \vdash_U \sigma(A) \text{ iff } A \in \mathbf{U}.$$

This intuitionistic modal logic $\mathbf{U}$ was considered in Goldblatt [Gol81], and Wolter and Zakharyaschev [WZ97].

The intuitionistic modal logic treated here is related to the logic **U**, but has the axiom $\Box\Box p \supset \Box p$ instead of $\Box p \supset (q \vee (q \supset p))$; it was called propositional lax logic (**PLL**) in Fairtlough and Mendler [FM95]. In other words, **PLL** is obtained from **IntK** by adding the axioms[3]

$$T_c : p \supset \Box p \text{ and } 4_c : \Box\Box p \supset \Box p.$$

PLL is not a logic for provability. However, **PLL** has other interesting interpretations. Benton, Bierman and de Paiva [BBP98], [Gol81] and [FM95] considered this logic with different motivations.

[BBP98] showed that the logic corresponds to the computational typed lambda calculus introduced in Moggi [Mog89] by the Curry-Howard isomorphism (cf. Curry and Feys [CF58], Girard [Gir89] and Howard [How80]). Moggi's computational typed lambda calculus is a metalanguage for denotational semantics which more faithfully models real programming language features such as non-termination, various evaluation strategies, non-determinism and side-effects than does the ordinary simply typed lambda calculus. The starting point for Moggi's work is an explicit semantic distinction between *computations* and *values*. If A is an object which interprets the values of a particular type, then $T(A)$ is the object which models computations of that type A. This constructor T corresponds to the modality[4] $\Box$ just as the constructors $\rightarrow$ and $\times$ in the ordinary typed lambda calculus correspond to $\supset$ and $\wedge$ in propositional formulas. They gave a natural deduction system for **PLL** and prove a strong normalization theorem by using the method in Prawitz [Pra97] (see also Tait [Tai67] and Troelstra [Tro73]).

[Gol81] argued for an application of the logic in *Grothendieck's topology*. He extracted the principle

$(*)$ A is *locally true* at α iff A is true at all points close to α

For instance, two functions f and g are said to be equivalent, or to have the same germ, at a point α in the intersection of their domains if there is a neighborhood of α on which f and g assign the same values. Thus f and g have the same germ at α when the statement "$f = g$" is *locally true* at α, i.e., true throughout some neighborhood of α. Intuitively this conveys the idea that f and g assign the same values to points "close" to α. On the other hand, he introduced a frame structure $F = \langle W, \leq, R\rangle$, where $\langle W, \leq\rangle$ is a partially ordered set and R is a binary relation on W such that

$$\text{if } \alpha \leq \beta \text{ and } \beta R \gamma, \text{ then } \alpha R \gamma,$$

for any $\alpha, \beta, \gamma \in W$. Also a model $M = \langle F, \models\rangle$ based on a frame F is defined as in the definition of Kripke model $\langle W, \leq, \models\rangle$ for **IPL**, while the truth of $\Box A$ at a point α is defined as follows[5]:

[3]The name T_c and 4_c are used in Chellas [Che80] (See also Bull and Segerberg [BS84]).

[4]They used the symbol $\Diamond$ instead of $\Box$.

[5]He used the symbol ∇ instead of $\Box$.

$$(M, \alpha) \models \Box A \text{ iff } (M, \beta) \models A \text{ for any } \beta \in \{\gamma \mid \alpha R \gamma\}.$$

It is easily seen that the axiom K is valid in every model. He argued that the above clause formalizes the principle $(*)$, and that models on which the axioms T_c and 4_c are valid are basic models for the logic of *Grothendieck topologies*. In [Gol81], he proved a Kripke completeness for **PLL**.

[FM95] treated **PLL** as the logic with applications to the formal verification of hardware, in particular, they argued that it is convenient to reason about the static behavior of combinational circuits in terms of high or low voltage and to abstract away from propagation delays. The intuitive interpretation of $\Box A$ is[6]

for some constraint c, formula A holds under c.

For example, the modality $\Box$ was used to account for the stability and timing constraints. The generic interpretation leads to the axioms T_c, 4_c, and

$$K' : (p \supset q) \supset (\Box p \supset \Box q),$$

where **PLL** can also be formalized by K' instead of K. The axiom T_c says "if p holds outright, then it holds under a (trivial) constraint"; 4_c says "if under some constraint, p holds under another constraint, then p holds under a (combined) constraint"; finally K' says "if p implies q, then if p holds under a constraint, q holds under a (the same) constraint." They gave a cut-free sequent system for **PLL** and proved completeness with respect to Kripke constraint models defined by them.

Extensions of **PLL** were also considered in [Gol81], [FM95] and [WZ97].

In this thesis, in chapter 5, we discuss the set of formulas constructed from the propositional variables $p_1, \cdots, p_n$ and the constant $\perp$ using $\supset$ and $\Box$ in **PLL**. The non-modal formulas of this kind were first considered in Diego [Die66]. He showed that the set of such non-modal formulas contains only finitely many pairwise non-equivalent in **IPL**. More precisely, he showed that the quotient set

$$\mathbf{I}(p_1, \cdots, p_n)/ \equiv$$

is finite, where $\mathbf{I}(p_1, \cdots, p_n)$ is the set of formulas constructed from $p_1, \cdots, p_n$ by using only implication, and $A \equiv B$ iff $A \supset B, B \supset A \in$ **IPL**.

Let $\mathbf{I}_{p_i}(p_1, \cdots, p_n)$ be the set of formulas of the form

$$A_1 \supset (\cdots (A_n \supset p_i) \cdots)$$

in the set $\mathbf{I}(p_1, \cdots, p_n)$. Urquhart [Urq74] clarified the construction of the ordered sets

$$(\mathbf{I}(p_1, \cdots, p_n)/ \equiv, \leq)$$

[6] They used the symbol $\bigcirc$ instead of $\Box$.

and

$$(\mathbf{I}_{p_i}(p_1, \cdots, p_n)/\equiv, \leq),$$

where $[A] \leq [B]$ iff $A' \supset B' \in \mathbf{IPL}$ for some $A' \in [A]$ and $B' \in [B]$, in particular, he proved that $(\mathbf{I}_{p_i}(p_1, \cdots, p_n)/\equiv, \leq)$ is Boolean, and the number of generators of the Boolean algebra is 23 if $n = 3$.

Hendriks [Hen96] calculated the numbers of such generators and equivalence classes for $n \leq 4$. He also gave a method how to construct the canonical representatives of the equivalent classes[7]. He investigated not only the implicational fragment, but also fragments containing $\wedge, \neg$, and so on.

We treat the disjunction free fragment with only the propositional variables $p_1, \cdots, p_n$ of **PLL**, and extend their results.

1.3 Interpretability logics

The idea of interpretability logics arose in Visser [Vis90]. He introduced the logics as extensions of the provability logic **GL** with a binary modality $\rhd$. The arithmetic realization of $A \rhd B$ in a theory T will be that T plus the realization of B is interpretable in T plus the realization of A ($T + A$ *interprets* $T + B$). More precisely, there exists a function f (the relative interpretation) on the formulas of the language of T such that $T + B \vdash C$ implies $T + A \vdash f(C)$.

The basic interpretability logic **IL** is the smallest set of formulas containing **GL** and axioms

$J1 : \Box(p \supset q) \supset (p \rhd q)$,
$J2 : (p \rhd q) \wedge (q \rhd r) \supset (p \rhd r)$,
$J3 : (p \rhd r) \wedge (q \rhd r) \supset ((p \vee q) \rhd r)$,
$J4 : (p \rhd q) \supset (\Diamond p \supset \Diamond q)$,
$J5 : \Diamond p \rhd p$,

and closed under modus ponens, substitution and necessitation. The principles of **IL** are arithmetically sound for a wide class of theories and for various interpretations of its main connective $\rhd$. The theory is not arithmetically complete for any known interpretation. The motivation for studying this specific set of formulas lies in its modal simplicity and elegance.

The modality $\rhd$ has more than one interpretation. Another most salient interpretation is Π_1-*conservativity*. More precisely, the arithmetic realization of $A \rhd B$ in a theory T, containing $\mathbf{I\Sigma_1}$, will be that T plus the realization of B is Π_1-conservative over T plus the interpretation of A. In Berarducci [Ber90] and Shavrukov [Sha88], it was proved that the interpretability logic (**ILM**) obtained by adding Montagna's axiom

$$M : (p \rhd q) \supset (p \wedge \Box r) \rhd (q \wedge \Box r)$$

[7] [Sas97a] also gave the same method, independently.

to **IL** is complete for this arithmetic interpretation in **PA**, and hence for interpretability as well, since over **PA** interpreatbility and Π_1-conservativity are equivalent. This was extended with regard to Π_1-conservativity by Hájek and Montagna (cf. [HM90] and [HM92]) to all theories containing $I\Sigma_1$.

The interpretability logic **ILP**, an extension of **IL** by adding the axiom

$$P : (p \rhd q) \supset \Box(p \rhd q),$$

is also complete for another arithmetic interpretation. P is valid for interpretations in finitely axiomatized arithmetical theories extending, say, $I\Delta_0 + \Omega_1$, and an arithmetic completeness for this interpretation was proved in [Vis90].

The completeness with respect to Kripke semantics due to Veltman was, for **IL**, **ILM** and **ILP**, proved in de Jongh and Veltman [JV90]. The fixed point theorem of **GL** can be extended to **IL** and hence **ILM** and **ILP** (de Jongh and Visser [JV91]). The unary pendant "T interprets $T + A$" is much less expressive and was studied in de Rijke [Rij92]. For an overview of interpretability logic, see Visser [Vis97], and Japaridze and de Jongh [JJ98].

This thesis gives, in chapter 6, a cut-free system for the basic interpretability logic **IL**. First, we give a cut-free system for a sublogic **IK4**, whose $\rhd$-free fragment is the modal logic **K4**. Using the system and a property of Löb's axiom, which will be presented in chapter 4, we obtain a cut-free system for **IL**[8].

1.4 Overview of the thesis

In chapter 2, we give cut-free sequent systems for **VPL** and **FPL**. The result for **VPL** was published in Nanzan Management review (cf. [Sas98a]). Also these results appeared in [Sas01b] in datail.

In chapter 3, we consider Hilbert style formalization for the consequence relation $\vdash_V$ of **VPL**. Using restricted modus ponens and adjunction, we give a formalization for $\vdash_V$. This result has been published in Reports on Mathematical Logic (cf. [Sas99b], see also [Sas98b]).

In chapter 4, we consider a property of Löb's axiom in extensions of $\vdash_V$. The results in this chapter appeared in [Sas97b], [Sas98c] and [Sas01a].

In chapter 5, we discuss the formulas without disjunction and conjunction in propositional lax logic. The results in this chapter appeared in [Sas99a] and [Sas01c].

In chapter 6, we give a cut-free sequent system for the smallest interpretability logic **IL**. The result was accepted in Studia Logica(cf. [Sas01d]).

[8] In a similar way, we can give a cut-free system for **ILP**(see [Sas01f]).

Chapter 2

Cut-elimination theorems for Visser's propositional logic and formal propositional logic

In this chapter, we consider cut-free sequent systems for Visser's propositional logic (**VPL**) and formal propositional logic (**FPL**). Although a cut-free sequent system for **VPL** was given in [Ard95], a subformula property has not been proved. Here we give another cut-free sequent system for **VPL**, which does satisfy the subformula property. A decision procedure for **VPL** is easily derived from our cut-free system. We also give a cut-free sequent system for **FPL** by modifying the system for **VPL**.

2.1 Preliminaries

We use lower case Latin letters p, q, r, possibly with suffixes, for propositional variables. Formulas are defined, as usual, from the propositional variables and the logical constant $\perp$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (implication). We assume $\wedge$ and $\vee$ to connect stronger than $\supset$ and omit those brackets that can be recovered according to this priority of connectives. We use upper case Latin letters $A, B, C, \cdots$, possibly with suffixes, for formulas. By **WFF**, we mean the set of formulas. The expression $\top$ is an abbreviation for $\perp \supset \perp$. We use Greek letters, possibly with suffixes, for finite sets of formulas.

As we mentioned in section 1.1, the first axiomatization for **VPL** was given in natural deduction style in [Vis81]. His natural deduction system $\vdash_V$ for **VPL** consists of the following inference rules.

$$(\bot E)\frac{\bot}{A}$$

$$(\wedge I)\frac{A \quad B}{A \wedge B} \qquad (\wedge E_1)\frac{A \wedge B}{A}$$

$$(\wedge E_2)\frac{A \wedge B}{B}$$

$$(\vee I_1)\frac{A}{A \vee B} \qquad (\vee E)\frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$(\vee I_2)\frac{B}{A \vee B}$$

$$(\supset I)\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}$$

$$(\wedge I_f)\frac{A \supset B \quad A \supset C}{A \supset B \wedge C} \qquad (\vee E_f)\frac{A \supset C \quad B \supset C}{A \vee B \supset C} \qquad (Tr)\frac{A \supset B \quad B \supset C}{A \supset C}$$

The consequence relation $\vdash_V$ is defined by the axiom

(1) $\Gamma \vdash_V A$ if $A \in \Gamma$

and the inference rules above, inductively.

Here we can see that the system is obtained from Gentzen's natural deduction system $\vdash_{NJ}$ for the intuitionistic propositional logic by replacing

$$(\supset E)\frac{A \quad A \supset B}{B}$$

by three inference rules $(\wedge I_f)$, $(\vee E_f)$ and (Tr). Using inference rules corresponding to the system above, [Ard95] gave a cut-free sequent style system for **VPL**. A cut-free system usually gives a subformula property and thereby a decision procedure in the usual way. However, his cut-free system includes the inference rule

$$(Tr)\frac{\Sigma \to A \supset B \quad \Sigma \to B \supset C}{\Sigma \to A \supset C}$$

corresponding to the inference rule (Tr) in $\vdash_V$. Here we immediately find that this inference rule makes it difficult to prove subformula property.

In the next section, we introduce another sequent system $\mathbf{GVPL}^+$ and prove a cut-elimination theorem and subformula property. In section 2.3, we show the equivalence between $\vdash_V$ and $\mathbf{GVPL}^+$. Section 2.4 is devoted to giving a cut-free sequent system for **FPL** by modifying $\mathbf{GVPL}^+$.

2.2 The system GVPL$^+$

First, we introduce a new expression $A \supset^+ B$, which is intended to denote the implication of A and B in intuitionistic propositional logic. In [SWZ98], an implication in intuitionistic logic was treated as an additional logical connective. However, we use it as an auxiliary expression in order to give a sequent style system.

2.2.1. Notation. We put

$$\mathbf{WFF}^+ = \mathbf{WFF} \cup \{A \supset^+ B \mid A, B \in \mathbf{WFF}\}$$

If there is no confusion, we also call an element of $\mathbf{WFF}^+$ *a formula*, and use upper case Latin letters $X, Y, Z, \cdots$, possibly with suffixes, for elements of $\mathbf{WFF}^+$.

2.2.2. Definition. The degree $d(X)$ of a formula $X \in \mathbf{WFF}^+$ is defined inductively as follows:
(1) $d(p) = 0$,
(2) $d(\bot) = 0$,
(3) $d(A \wedge B) = d(A \vee B) = d(A \supset^+ B) = d(A) + d(B) + 1$,
(4) $d(A \supset B) = d(A) + d(B) + 2$.

We also use Greek letters for finite sets of formulas in $\mathbf{WFF}^+$, especially, we use Δ, possibly with suffixes, for a set that contains at most one $\mathbf{WFF}^+$-formula.

2.2.3. Notation. We put

$$\Gamma_X = \Gamma - \{X\},$$

$$\Gamma^+ = (\Gamma - \{A \supset B \mid A \supset B \in \Gamma\}) \cup \{A \supset^+ B \mid A \supset B \in \Gamma\}.$$

By a sequent, we mean an expression $\Gamma \to \Delta$. For brevity's sake, we write

$$X_1, \cdots, X_n, \Gamma_1, \cdots, \Gamma_m \to$$

and

$$X_1, \cdots, X_n, \Gamma_1, \cdots, \Gamma_m \to Y$$

instead of

$$\{X_1, \cdots, X_n\} \cup \Gamma_1 \cup \cdots \cup \Gamma_m \to \emptyset$$

and

$$\{X_1, \cdots, X_n\} \cup \Gamma_1 \cup \cdots \cup \Gamma_m \to \{Y\},$$

respectively.

The system $\mathbf{GVPL}^+$ is defined from the following axioms and inference rules in the usual way.

Axioms of $\mathbf{GVPL}^+$

$$A \to A \qquad \text{and} \qquad \bot \to$$

Inference rules of $\mathbf{GVPL}^+$

$$(T \to)\frac{\Gamma_X \to \Delta}{X, \Gamma_X \to \Delta} \qquad\qquad (\to T)\frac{\Gamma \to}{\Gamma \to X}$$

$$(\text{cut})\ \frac{\Gamma \to X \quad X, \Pi \to \Delta}{\Gamma, \Pi_X \to \Delta}$$

$$(\wedge \to_1)\frac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \qquad\qquad (\to \wedge)\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}$$

$$(\wedge \to_2)\frac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$$

$$(\vee \to)\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta} \qquad\qquad (\to \vee_1)\frac{\Gamma \to A}{\Gamma \to A \vee B}$$

$$(\to \vee_2)\frac{\Gamma \to B}{\Gamma \to A \vee B}$$

$$(\supset^+\to)\frac{\Gamma \to A \quad B, \Gamma \to \Delta}{A \supset^+ B, \Gamma \to \Delta} \qquad\qquad (\to\supset^+)\frac{A, \Gamma \to B}{\Gamma \to A \supset^+ B}$$

$$(\to\supset)\frac{A, \Gamma^+ \to B}{\Gamma \to A \supset B}$$

2.2.4. Definition. A proof figure in $\mathbf{GVPL}^+$ for a sequent $\Gamma \to \Delta$ is defined as follows:

(1) if $\Gamma \to \Delta$ is an axiom in $\mathbf{GVPL}^+$, then $\Gamma \to \Delta$ is a proof figure for $\Gamma \to \Delta$,

(2) if P_1 is a proof figure for $\Gamma_1 \to \Delta_1$ and $\dfrac{\Gamma_1 \to \Delta_1}{\Gamma \to \Delta}$ is an inference rule in $\mathbf{GVPL}^+$, then $\dfrac{P_1}{\Gamma \to \Delta}$ is a proof figure for $\Gamma \to \Delta$,

(3) if P_1 and P_2 are proof figures for $\Gamma_1 \to \Delta_1$ and $\Gamma_2 \to \Delta_2$, and $\dfrac{\Gamma_1 \to \Delta_1 \quad \Gamma_2 \to \Delta_2}{\Gamma \to \Delta}$ is an inference rule in $\mathbf{GVPL}^+$, then $\dfrac{P_1 \quad P_2}{\Gamma \to \Delta}$ is a proof figure for $\Gamma \to \Delta$.

We say that $\Gamma \to \Delta$ is provable in $\mathbf{GVPL}^+$, and write $\Gamma \to \Delta \in \mathbf{GVPL}^+$, if there exists a proof figure for $\Gamma \to \Delta$. We use P, Q, possibly with suffixes, for proof figures.

Let P be a proof figure for $\Gamma \to \Delta$. In order to emphasize the end sequent of P, we also use the expressions

$$P \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta \end{array} \right. \qquad \text{and} \qquad \left. \begin{array}{c} \vdots \\ \Gamma \to \Delta \end{array} \right\} P$$

instead of P.

2.2.5. Definition. A set $\mathsf{SubFig}(P)$ of a proof figure P is defined as follows:
(1) $\mathsf{SubFig}(P) = \{P\}$ if P is an axiom,
(2) $\mathsf{SubFig}(\dfrac{P_1}{\Gamma \to \Delta}) = \mathsf{SubFig}(P_1) \cup \{P\}$,
(3) $\mathsf{SubFig}(\dfrac{P_1 \quad P_2}{\Gamma \to \Delta}) = \mathsf{SubFig}(P_1) \cup \mathsf{SubFig}(P_2) \cup \{P\}$.

We call an element of $\mathsf{SubFig}(P)$ a subfigure of P and an element of $\mathsf{SubFig}(P) - \{P\}$ a proper subfigure of P. As to the other terminology concerning the system, we mainly follow [Gen35].

Our main purpose in this section is to prove

2.2.6. Theorem. *If $\Gamma \to \Delta \in \mathbf{GVPL}^+$, then there exists a cut-free proof figure for $\Gamma \to \Delta$.*

In order to prove the theorem above, we mainly use the method in [Gen35]. The only one essential difference between $\mathbf{GVPL}^+$ and the system **LJ** for intuitionistic propositional logic provided in [Gen35] is the inference rule $(\to\supset)$ in $\mathbf{GVPL}^+$. So, we only show the cases concerning $(\to\supset)$. The other cases can be shown in the usual way. To show our new cases, we provide some preparations.

Let P be a proof figure for $\Gamma \to \Delta$. By $len(P)$, we mean the largest number of consecutive sequents in a path so that the lowest of these sequents is the end sequent of P and the succedent of each sequent is Δ.

2.2.7. Lemma. *Let P be a cut-free proof figure for $\Gamma, \Pi \to X$. Then there exists a cut-free proof figure P^+ for $\Gamma, \Pi^+ \to X$ such that $len(P) = len(P^+)$.*

Proof. Without loss of generality, we can assume that Π is a non-empty set of formulas of the form $A \supset B$. We use an induction on P.

Basis(P is an axiom $X \to X$): We note

$$\Gamma = \emptyset \text{ and } \Gamma \cup \Pi = \{X\}.$$

Hence, if $X = A \supset B$ for some A and B, then the following proof figure P^+ is a cut-free proof figure for $\Gamma, \Pi^+ \to X$ such that $len(P^+) = len(P) = 1$:

$$\dfrac{\dfrac{A \to A \qquad \dfrac{B \to B}{A, B \to B}}{A, A \supset^+ B \to B}}{A \supset^+ B \to A \supset B},$$

if not, we have $\Pi^+ = \{X\}^+ = \{X\} = \Pi$, and so, P is also a proof figure for $\Gamma, \Pi^+ \to X$.

Induction step(P is not axiom): Suppose that the lemma holds for any proper subfigure of P. Since P is not axiom, there exists an inference rule I that introduces the end sequent $\Gamma, \Pi \to \Delta$ in P. We divide into the following cases.

The case that I is $(\vee \to)$: P is of the form

$$\dfrac{P_1\left\{\begin{array}{c}\vdots\\ A, \Gamma_1, \Pi \to X\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ B, \Gamma_1, \Pi \to X\end{array}\right\}P_2}{A \vee B, \Gamma_1, \Pi \to X}$$

where $\{A \vee B\} \cup \Gamma_1 = \Gamma$. We note that

$$len(P) = \max\{len(P_1), len(P_2)\} + 1.$$

By the induction hypothesis, there exist proof figures P_1^+ for the sequent $A, \Gamma_1, \Pi^+ \to X$ and P_2^+ for $B, \Gamma_1, \Pi^+ \to X$ such that $len(P_1) = len(P_1^+)$ and $len(P_2) = len(P_2^+)$. Using P_1^+, P_2^+ and $(\vee \to)$, we have the following proof figure P^+:

$$\dfrac{P_1^+\left\{\begin{array}{c}\vdots\\ A, \Gamma_1, \Pi^+ \to X\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ B, \Gamma_1, \Pi^+ \to X\end{array}\right\}P_2^+}{A \vee B, \Gamma_1, \Pi^+ \to X}$$

From the proof figure above, we note

$$len(P^+) = \max\{len(P_1^+), len(P_2^+)\} + 1 = \max\{len(P_1), len(P_2)\} + 1 = len(P).$$

The case that I is $(\wedge \to_i)(i = 1, 2)$ can be shown similarly.

The case that I is $(\supset^+ \to)$: P is of the form

$$\dfrac{P_1\left\{\begin{array}{c}\vdots\\ \Gamma_1, \Pi \to A\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ B, \Gamma_1, \Pi \to X\end{array}\right\}P_2}{A \supset^+ B, \Gamma_1, \Pi \to X}$$

where $\{A \supset^+ B\} \cup \Gamma_1 = \Gamma$. By the induction hypothesis, there exist cut-free proof figures P_1^+ for $\Gamma_1, \Pi^+ \to A$ and P_2 for $B, \Gamma_1, \Pi^+ \to X$ such that $len(P_1) = len(P_1^+)$ and $len(P_2) = len(P_2^+)$. Using P_1^+, P_2^+ and $(\supset^+ \to)$, we have the following proof figure P^+:

$$\cfrac{P_1\left\{\begin{array}{c}\vdots\\ \Gamma_1,\Pi^+\to A\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ B,\Gamma_1,\Pi^+\to X\end{array}\right\}P_2}{A\supset^+ B,\Gamma_1,\Pi^+\to X}$$

If $X = A$, we have

$$len(P^+) = \max\{len(P_1^+), len(P_2^+)\} + 1 = \max\{len(P_1), len(P_2)\} + 1 = len(P),$$

if not,

$$len(P^+) = len(P_2^+) + 1 = len(P_2) + 1 = len(P).$$

Hence we obtain the lemma.

The case that I is $(T\to)$: I is either one of the forms

$$\frac{\Gamma_Y,\Pi\to X}{Y,\Gamma_Y,\Pi\to X} \text{ and } \frac{\Gamma,\Pi_Y\to X}{Y,\Gamma,\Pi_Y\to X}.$$

If I is of the first one, then we obtain the lemma similarly to the two cases above. So, we assume that I is of the second one. By the induction hypothesis, there exists a cut-free proof figure P_1^+ for $\Gamma,(\Pi_Y)^+\to X$ such that $len(P_1^+) = len(P) - 1$. Using P_1 and $(T\to)$, we have a cut-free proof figure P^+:

$$\cfrac{P_1\left\{\begin{array}{c}\vdots\\ \Gamma_1,(\Pi_Y)^+\to A\end{array}\right.}{\Gamma,(\{Y\}\cup\Pi_Y)^+\to X}$$

From the inference rules above, we have $len(P^+) = len(P) = len(P_1) + 1$. So, we obtain the lemma.

The case that I is $(\to\supset)$: I is of the form

$$\frac{A,\Gamma^+,\Pi^+\to B}{\Gamma,\Pi\to A\supset B}.$$

Let P^+ be the figure obtained from P by replacing the end sequent by $\Gamma,\Pi^+\to A\supset B$. We note that P^+ is also a proof figure since

$$\frac{A,\Gamma^+,\Pi^+\to B}{\Gamma^+,\Pi\to A\supset B}$$

is an inference rule $(\to\supset)$ in **GVPL**$^+$. Also we can easily see that $len(P^+) = len(P) = 1$. Hence we obtain the lemma.

The case that I is $(\to\supset^+)$: I is of the form

$$\frac{A,\Gamma,\Pi\to B}{\Gamma,\Pi\to A\supset^+ B}.$$

By the induction hypothesis, there exists a cut-free proof figure for $A, \Gamma, \Pi^+ \to B$. Using $(\to\supset^+)$, we have a cut-free proof figure P^+ for $\Gamma, \Pi^+ \to A \supset^+ B$ such that $len(P^+) = len(P) = 1$.

The case that I is either one of the inference rules $(\to \wedge)$, $(\to T)$ and $(\to \vee_i)(i = 1, 2)$ can be shown similarly. $\dashv$

As is known, Theorem 2.2.6 follows from the following lemma.

2.2.8. LEMMA. *Let P_l be a cut-free proof figure for $\Gamma \to X$ and P_r be a cut-free proof figure for $X, \Pi \to \Delta$. Let P be the proof figure*

$$\frac{P_l\left\{\begin{array}{c}\vdots\\ \Gamma \to X\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ X, \Pi \to \Delta\end{array}\right\}P_r}{\Gamma, \Pi_X \to \Delta}.$$

Then there exists a cut-free proof figure for the end sequent of P.

Proof. The degree $d(P)$ of P is defined as $d(X)$. The left rank $R_l(P)$ and the right rank $R_r(P)$ of P are defined as usual. We use an induction on $R_l(P) + R_r(P) + \omega d(P)$. We only treat the following two cases.

The case that P is of the form

$$\frac{\dfrac{P'_l\left\{\begin{array}{c}\vdots\\ C, \Gamma^+ \to D\end{array}\right.}{\Gamma \to C \supset D} \qquad \dfrac{\left.\begin{array}{c}\vdots\\ A, C \supset^+ D, \Pi^+ \to B\end{array}\right\}P'_r}{C \supset D, \Pi \to A \supset B}}{\Gamma, \Pi_{C\supset D} \to A \supset B}:$$

Using two cut-free proof figures P'_l and P'_r and cut, we obtain the following proof figure P_1:

$$\frac{\dfrac{P'_l\left\{\begin{array}{c}\vdots\\ C, \Gamma^+ \to D\end{array}\right.}{\Gamma^+ \to C \supset^+ D} \qquad \left.\begin{array}{c}\vdots\\ A, C \supset^+ D, \Pi^+ \to B\end{array}\right\}P'_r}{\Gamma^+, A, (\Pi^+)_{(C\supset^+ D)} \to B}$$

From the definition of degree, we have $d(C \supset^+ D) < d(C \supset D)$, and so, $d(P_1) < d(P)$. By the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_1. We note that $(\Pi^+)_{(C\supset^+ D)} \subseteq (\Pi_{C\supset D})^+$. So, using $(T \to)$, possibly several times, we have a cut-free proof figure for

$$A, \Gamma^+, (\Pi_{C\supset D})^+ \to B.$$

Using $(\to\supset)$, we obtain a cut-free proof figure for

$$\Gamma, \Pi_{C \supset D} \to A \supset B.$$

The case that P is of the form

$$\dfrac{P_l \left\{ \begin{array}{c} \vdots \\ \Gamma \to X \end{array} \right. \qquad \dfrac{\left.\begin{array}{c} \vdots \\ A, X, \Pi^+ \to B \end{array}\right\} P_r'}{X, \Pi \to A \supset B}}{\Gamma, \Pi_X \to A \supset B}$$

where X is not of the form $C \supset D$. It is easily seen that $len(P_l) = R_l(P)$.

By Lemma 2.2.7, there exists a cut-free proof figure P_l^+ for $\Gamma^+ \to X$ such that $len(P_l^+) = len(P_l) = R_l(P)$. By P_l^+ and P_r' and cut, we obtain the following proof figure P_1:

$$\dfrac{P_l^+ \left\{ \begin{array}{c} \vdots \\ \Gamma^+ \to X \end{array} \right. \qquad \left.\begin{array}{c} \vdots \\ A, X, \Pi^+ \to B \end{array}\right\} P_r'}{\Gamma^+, (\{A\} \cup \Pi^+)_X \to B}$$

By $len(P_l^+) = len(P_l)$, we have $R_l(P_1) = R_l(P)$. Also we have $R_r(P_1) < R_r(P)$ and $d(P_1) = d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_1. On the other hand, since X is not of the form $C \supset D$, we have $(\{A\} \cup \Pi^+)_X \subseteq \{A\} \cup (\Pi_X)^+$. So, using $(T \to)$, possibly several times, we have a cut-free proof figure for

$$A, \Gamma^+, (\Pi_X)^+ \to B.$$

Using $(\to\supset)$, we obtain a cut-free proof figure for

$$\Gamma, \Pi_X \to A \supset B.$$

$\dashv$

2.2.9. Lemma. *Let Λ be a set that contains at most one* **WFF***-formula and let P be a cut-free proof figure for $\Gamma \to \Lambda$. If a sequent S occurs in P, then the succedent of S contains at most one* **WFF***-formula.*

Proof. Let S be a sequent occurring in P. By $\#(S)$, we mean the smallest number of consecutive sequents in a path so that the lowest of these sequents is the end sequent of P and the highest is S. We use an induction on $\#(S)$.

If $\#(S) = 1$, then S is the end sequent $\Gamma \to \Lambda$, hence we obtain the lemma.

Suppose that $\#(S) > 1$ and that the lemma holds for any sequent S^* in P such that $\#(S^*) < \#(S)$. By $\#(S) > 1$, there exists an inference rule I and S is an upper sequent of I. By the induction hypothesis, the succedent of the lower sequent of I contains at most one **WFF**-formula. Since P is cut-free, I is not cut, hence we can easily check the succedent of each upper sequent of I contains at most one **WFF**-formula. Since S is an upper sequent of I, we obtain the lemma. $\dashv$

By Theorem 2.2.6 and Lemma 2.2.9, we have the following corollary.

2.2.10. COROLLARY. *Let Λ be a set that contains at most one* **WFF***-formula. If $\Gamma \to \Lambda \in$* **GVPL**$^+$*, then there exists a cut-free and $(\to\supset^+)$-free proof figure for $\Gamma \to \Lambda$.*

For a formula $A \in$ **WFF**, $\mathsf{Sub}(A)$ denotes the set of subformulas of A. We put

$$\mathsf{Sub}^+(A) = \mathsf{Sub}(A) \cup \{B \supset^+ C \mid B \supset C \in \mathsf{Sub}(A)\}$$

and

$$\mathsf{Sub}^+(A \supset^+ B) = \mathsf{Sub}^+(A \supset B) - \{A \supset B\}.$$

The following corollary follows from Theorem 2.2.6.

2.2.11. COROLLARY. *Let it be that $\Gamma \to \Delta \in$* **GVPL**$^+$*. Then there exists a proof figure P satisfying the following condition for any $X \in$* **WFF**$^+$*:*

if X occurs in P, then $X \in \mathsf{Sub}^+(Y)$ for some formula Y occurring in $\Gamma \to \Delta$.

It is easily seen that $\mathsf{Sub}^+(X)$ is finite. So, in the usual way, we obtain a decision procedure for **GVPL**$^+$ in an axiomatic way.

By Theorem 2.2.6, we can also show a result concerning the disjunction property. We say that a logic **L** has the disjunction property(cf. [CZ97]) if

$$A \vee B \in \mathbf{L} \text{ implies either } A \in \mathbf{L} \text{ or } B \in \mathbf{L}.$$

2.2.12. COROLLARY. *Every logic obtained by adding an axiom schema $\to \mathsf{A} \supset \mathsf{B}$ to* **GVPL**$^+$ *has the disjunction property.*

Proof. Let **L** be a logic obtained by adding an axiom schema $\to \mathsf{A} \supset \mathsf{B}$ to **GVPL**$^+$ and let it be that $C \vee D \in \mathbf{L}$. Then there exist a finite number of axioms

$$A_1 \supset B_1, \cdots, A_n \supset B_n$$

of the form $\mathsf{A} \supset \mathsf{B}$ such that

$$A_1 \supset B_1, \cdots, A_n \supset B_n \to X \vee Y \in \mathbf{GVPL}^+.$$

Using Theorem 2.2.6, there exists a cut-free proof figure P for the sequent above. We note that there exists a lowest logical inference rule I in P, which must be $(\to \vee)$. Considering the upper sequent of I, we have either

$$A_1 \supset B_1, \cdots, A_n \supset B_n \to C \in \mathbf{GVPL}^+$$

or

$$A_1 \supset B_1, \cdots, A_n \supset B_n \to D \in \mathbf{GVPL}^+.$$

Hence we have either

$$C \in \mathbf{L} \text{ or } D \in \mathbf{L}.$$

$\dashv$

2.3 Visser's system and $\mathbf{GVPL}^+$

Here we show the equivalence between $\vdash_V$ and $\mathbf{GVPL}^+$. We use Σ for a finite set of formulas in **WFF** and we use Λ for a set that contains at most one **WFF**-formula.

2.3.1. Notation. We put

(1) $\Gamma^- = (\Gamma - \{A \supset^+ B \mid A \supset^+ B \in \Gamma\}) \cup \{A \supset B \mid A \supset^+ B \in \Gamma\}$,

(2) $f(\Delta) = \begin{cases} X & \text{if } \Delta = \{X\}, \\ \bot & \text{if } \Delta = \emptyset. \end{cases}$

The main theorem in this section is

2.3.2. Theorem. $\Sigma \to \Lambda \in \mathbf{GVPL}^+$ *iff* $\Sigma \vdash_V f(\Lambda)$.

In order to prove the theorem above, we provide some lemmas.

2.3.3. Lemma. $\Sigma \vdash_V f(\Lambda)$ *implies* $\Sigma \to \Lambda \in \mathbf{GVPL}^+$.

Proof. It is sufficient to show that every inference rule in $\vdash_V$ holds in $\mathbf{GVPL}^+$. First, we show that $(\wedge I_f)$ holds in $\mathbf{GVPL}^+$. Suppose that $\Sigma \to B \supset C$ and $\Sigma \to B \supset D$ are provable in $\mathbf{GVPL}^+$. Using the proof figure

$$\cfrac{\cfrac{\cfrac{B \to B \quad C \to C}{B, B \supset^+ C \to C} \quad \cfrac{B \to B \quad D \to D}{B, B \supset^+ D \to D}}{B, B \supset^+ C, B \supset^+ D \to C \wedge D}}{B \supset C, B \supset D \to B \supset C \wedge D},$$

and cut, we obtain $\Sigma \to B \supset C \wedge D \in \mathbf{GVPL}^+$. Similarly, we can show that $(\vee E_f)$ and (Tr) hold in $\mathbf{GVPL}^+$ using the following two proof figures, respectively:

$$\cfrac{\cfrac{\cfrac{B \to B \quad D \to D}{B, B \supset^+ D \to D} \quad \cfrac{C \to C \quad D \to D}{C, C \supset^+ D \to D}}{B \vee C, B \supset^+ D, C \supset^+ D \to D}}{B \supset D, C \supset D \to B \vee C \supset D}$$

$$\dfrac{B \to B \qquad \dfrac{C \to C \quad D \to D}{C, C \supset^{+} D \to D}}{\dfrac{B, B \supset^{+} C, C \supset^{+} D \to D}{B \supset C, C \supset D \to B \supset D}}$$

Other cases can be shown in the usual way. ⊣

The following lemma is almost obvious.

2.3.4. Lemma. (1) *if* $\Sigma_1 \subseteq \Sigma_2$ *and* $\Sigma_1 \vdash_V A$*, then* $\Sigma_2 \vdash_V A$*,*
(2) *if* $\Sigma_1 \vdash_V A$ *and* $\Sigma_2 \vdash_V B$*, then* $\Sigma_1 \cup (\Sigma_2 - \{A\}) \vdash_V B$*.*

2.3.5. Lemma. *If there exists a cut-free and* $(\to\supset^{+})$*-free proof figure* P *for* $\Gamma \to \Lambda$*, then* $(\Gamma - \{A_1, \cdots, A_n\})^{-} \vdash_V A_1 \wedge \cdots \wedge A_n \supset f(\Lambda)$ *for* $n \geq 1$*.*

Proof. For brevity's sake, we put $A = A_1 \wedge \cdots \wedge A_n$ and $\mathbf{A} = \{A_1, \cdots, A_n\}$. We use an induction on P.

Basis(P is an axiom): Since $\Gamma \to \Lambda$ is an axiom, we have $\Gamma = \{f(\Lambda)\} \subseteq \mathbf{WFF}$.

If $\Gamma - \mathbf{A} = \Gamma$, then $f(\Lambda) \in \Gamma = \Gamma - \mathbf{A} = (\Gamma - \mathbf{A})^{-}$, and so, $(\Gamma - \mathbf{A})^{-} \vdash_V f(\Lambda)$. Using Lemma 2.3.4(1), $\{A\} \cup (\Gamma - \mathbf{A})^{-} \vdash_V f(\Lambda)$. Using $(\supset I)$, we obtain the lemma.

If $\Gamma - \mathbf{A} = \emptyset$, then $f(\Lambda) = \{A_k\}$ and $(\Gamma - \mathbf{A})^{-} = \emptyset$. Using $(\wedge E)$ and $(\supset I)$, $\emptyset \vdash_V A \supset f(\Lambda)$, and so, $(\Gamma - \mathbf{A})^{-} \vdash_V A \supset f(\Lambda)$.

Induction step(P is not axiom): Suppose that the lemma holds for any proper subfigure of P. Since P is not axiom, there exists an inference rule I that introduces the end sequent of P. We divide into the following cases.

The case that I is $(\to T)$: The upper sequent of I is $\Gamma \to$. So, by the induction hypothesis, we have

$$(\Gamma - \mathbf{A})^{-} \vdash_V A \supset \bot.$$

On the other hand, by $(\bot E)$ and $(\supset I)$, we have

$$(\Gamma - \mathbf{A})^{-} \vdash_V \bot \supset f(\Lambda).$$

Using (Tr), we obtain the lemma.

The case that I is $(\to \vee_i)$ can be shown similarly using $(\Gamma - \mathbf{A})^{-} \vdash_V B_i \supset B_1 \vee B_2$.

The case that I is $(\to \wedge)$: I is of the form

$$\frac{\Gamma \to B_1 \quad \Gamma \to B_2}{\Gamma \to B_1 \wedge B_2},$$

where $B_1 \wedge B_2 = f(\Lambda)$. By the induction hypothesis, we have

$$(\Gamma - \mathbf{A})^- \vdash_V A \supset B_1 \text{ and } (\Gamma - \mathbf{A})^- \vdash_V A \supset B_2.$$

Using $(\wedge I_f)$, we obtain the lemma.

The case that I is $(\rightarrow\supset)$: I is of the form

$$\frac{B_1, \Gamma^+ \rightarrow B_2}{\Gamma \rightarrow B_1 \supset B_2},$$

where $B_1 \supset B_2 = f(\Lambda)$. By the induction hypothesis, we have

$$(\Gamma^+)^- \vdash_V f(\Lambda).$$

Since $(\Gamma^+)^- = \Gamma^-$, we have

$$\Gamma^- \vdash_V f(\Lambda).$$

On the other hand, by $(\wedge E_i)$, $\{A\} \vdash_V A_k$ for any k. Using Lemma 2.3.4(2),

$$\{A\} \cup (\Gamma - \mathbf{A})^- \vdash_V f(\Lambda).$$

Using $(\supset I)$, we obtain the lemma.

The case that I is $(T \rightarrow)$: Let Π be the antecedent of the upper sequent of I. We note that $\Pi \subseteq \Gamma$. So, using the induction hypothesis and Lemma 2.3.4(1), we obtain the lemma.

The case that I is $(\wedge \rightarrow_i)$: I is of the form

$$\frac{C_i, \Gamma_1 \rightarrow \Lambda}{C_1 \wedge C_2, \Gamma_1 \rightarrow \Lambda},$$

where $\{C_1 \wedge C_2\} \cup \Gamma_1 = \Gamma$.

If $C_1 \wedge C_2 \notin \mathbf{A}$, then $\Gamma - \mathbf{A} = \{C_1 \wedge C_2\} \cup (\Gamma_1 - \mathbf{A})$. By the induction hypothesis, we have

$$((\{C_i\} \cup \Gamma_1) - \mathbf{A})^- \vdash_V A \supset f(\Lambda).$$

Using $\{C_1 \wedge C_2\} \vdash_V C_i$ and Lemma 2.3.4(2), we have

$$(\{C_1 \wedge C_2\} \cup (\Gamma_1 - \mathbf{A}))^- \vdash_V A \supset f(\Lambda).$$

Hence we obtain the lemma.

If $C_1 \wedge C_2 \in \mathbf{A}$, then $\Gamma - \mathbf{A} = \Gamma_1 - \mathbf{A}$. By the induction hypothesis, we have

$$(\Gamma_1 - \mathbf{A})^- \vdash_V A \wedge C_i \supset f(\Lambda).$$

On the other hand, it is easily seen that

$$(\Gamma_1 - \mathbf{A})^- \vdash_V A \supset A \wedge C_i.$$

Using (Tr), we obtain the lemma.

The case that I is $(\vee \rightarrow)$: I is of the form

$$\frac{C_1, \Gamma_1 \rightarrow \Lambda \quad C_2, \Gamma_1 \rightarrow \Lambda}{C_1 \vee C_2, \Gamma_1 \rightarrow \Lambda},$$

where $\{C_1 \vee C_2\} \cup \Gamma_1 = \Gamma$.

If $C_1 \vee C_2 \notin \mathbf{A}$, then $\Gamma - \mathbf{A} = \{C_1 \vee C_2\} \cup (\Gamma_1 - \mathbf{A})$. By the induction hypothesis, we have

$$((\{C_1\} \cup \Gamma_1) - \mathbf{A})^- \vdash_V A \supset f(\Lambda) \text{ and } ((\{C_2\} \cup \Gamma_1) - \mathbf{A})^- \vdash_V A \supset f(\Lambda).$$

Using $(\vee E)$, we obtain the lemma.

If $C_1 \vee C_2 \in \mathbf{A}$, then $\Gamma - \mathbf{A} = \Gamma_1 - \mathbf{A}$. By the induction hypothesis, we have

$$(\Gamma_1 - \mathbf{A})^- \vdash_V A \wedge C_1 \supset f(\Lambda) \text{ and } (\Gamma_1 - \mathbf{A})^- \vdash_V A \wedge C_2 \supset f(\Lambda).$$

Using $(\vee E_f)$, we have

$$(\Gamma_1 - \mathbf{A})^- \vdash_V (A \wedge C_1) \vee (A \wedge C_2) \supset f(\Lambda). \tag{2.1}$$

On the other hand, from the proof figure

$$\dfrac{[C_1 \vee C_2] \quad \dfrac{\dfrac{[A] \quad [C_1]^1}{A \wedge C_1}}{(A \wedge C_1) \vee (A \wedge C_2)} \quad \dfrac{\dfrac{[A] \quad [C_2]^1}{A \wedge C_2}}{(A \wedge C_1) \vee (A \wedge C_2)}}{(A \wedge C_1) \vee (A \wedge C_2)}\,1,$$

we have

$$\{C_1 \vee C_2, A\} \vdash_V (A \wedge C_1) \vee (A \wedge C_2).$$

Since $\{A\} \vdash_V C_1 \vee C_2$,

$$\{A\} \vdash_V (A \wedge C_1) \vee (A \wedge C_2).$$

Using $(\supset I)$,

$$\emptyset \vdash_V A \supset (A \wedge C_1) \vee (A \wedge C_2).$$

Using (2.1) and (Tr), we obtain the lemma.

The case that I is $(\supset^+ \rightarrow)$: I is of the form

$$\frac{\Gamma_1 \rightarrow C_1 \quad C_2, \Gamma_1 \rightarrow \Lambda}{C_1 \supset^+ C_2, \Gamma_1 \rightarrow \Lambda},$$

where $\{C_1 \supset^+ C_2\} \cup \Gamma_1 = \Gamma$. By the induction hypothesis, we have

$$(\Gamma_1 - \mathbf{A})^- \vdash_V A \supset C_1 \text{ and } (\Gamma_1 - \mathbf{A})^- \vdash_V A \wedge C_2 \supset f(\Lambda).$$

Using $\{C_1 \supset C_2\} \vdash_V C_1 \supset C_2$ and (Tr),

$$\{C_1 \supset C_2\} \cup (\Gamma_1 - \mathbf{A})^- \vdash_V A \supset C_2$$

Using $\emptyset \vdash_V A \supset A$ and $(\wedge I_f)$,

$$\{C_1 \supset C_2\} \cup (\Gamma_1 - \mathbf{A})^- \vdash_V A \supset A \wedge C_2$$

Using $(\Gamma_1 - \mathbf{A})^- \vdash_V A \wedge C_2 \supset f(\Lambda)$ and (Tr), we obtain the lemma. ⊣

2.3.6. Lemma. $\Sigma \to \Lambda \in$ **GVPL**$^+$ *implies* $\Sigma \vdash_V f(\Lambda)$.

Proof. By Corollary 2.2.10, there exists a cut-free and $(\to\supset^+)$-free proof figure P for $\Sigma \to \Lambda$. We use an induction on P.

Basis(P is an axiom): Since $\Sigma \to \Lambda$ is an axiom, $f(\Lambda) \in \Sigma$. Hence we obtain the lemma.

Induction step(P is not axiom): Suppose that the lemma holds for any proper subfigure of P. Since P is not axiom, there exists an inference rule I that introduces the end sequent of P. We note that I is neither $(\supset^+\to)$ nor $(\to\supset^+)$. If I is $(\to\supset)$, then $\Delta = \{A \supset B\}$ and the upper sequent of I is $A, \Sigma^+ \to B$, and so, we obtain the lemma by Lemma 2.3.5. Otherwise, the lemma can be shown using the induction hypothesis. ⊣

Now, Theorem 2.3.2 follows from Lemma 2.3.3 and Lemma 2.3.6.

2.4 The system GFPL$^+$

In this section, we define a sequent system for **FPL** and prove a cut-elimination theorem. [AR99] showed that formal propositional logic is obtained from Visser's propositional logic by adding Löb's axiom

$$L(p) = ((\top \supset p) \supset p) \supset (\top \supset p).$$

2.4.1. Definition. By **GVPL**$^+$ + $L(p)$, we mean the system obtained from **GVPL**$^+$ by adding an axiom schema $\to L(A)$. It is easily seen that for any formula $B \in$ **WFF**

$$B \in \mathbf{FPL} \text{ iff } \to B \in \mathbf{GVPL}^+ + L(p).$$

By **GFPL**$^+$, we mean the system obtained from **GVPL**$^+$ by replacing $(\to\supset)$ by the following inference rule:

$$(\to\supset_f)\frac{A, A \supset B, \Gamma^+ \to B}{\Gamma \to A \supset B}.$$

The formula $A \supset B$ is called the *diagonal formula* of the inference rule above.

2.4.2. Lemma. $\Gamma \to \Delta \in \mathbf{GVPL}^+ + L(p)$ *iff* $\Gamma \to \Delta \in \mathbf{GFPL}^+$.

Proof. From the following proof figure, we have $\to L(A) \in \mathbf{GFPL}^+$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \supset A \to \top \supset A \qquad \dfrac{A \to A}{\top \supset A, A \to A}}{(\top \supset A) \supset^+ A, \top \supset A \to A}}{\top \supset A, (\top \supset A) \supset^+ A \to A}}{\dfrac{\top, \top \supset A, (\top \supset A) \supset^+ A \to A}{(\top \supset A) \supset A \to \top \supset A}}}{\to ((\top \supset A) \supset A) \supset (\top \supset A)}$$

It is easily seen that $L(A) \to \{L(A)\}^+ \in \mathbf{GVPL}^+$. So, by the following figure, $(\to\supset_f)$ holds in $\mathbf{GVPL}^+ + L(p)$:

$$\dfrac{\dfrac{\to L(D) \quad L(D) \to \{L(D)\}^+}{\to \{L(D)\}^+} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A, \Gamma^+ \to \top \quad A, D, \Gamma^+ \to B}{A, \top \supset^+ D, \Gamma^+ \to B}}{\top \supset D, \Gamma^+ \to D}}{\Gamma \to (\top \supset D) \supset D} \qquad \dfrac{\dfrac{A, \Gamma^+ \to \top \quad A, D, \Gamma^+ \to B}{A, \top \supset^+ D, \Gamma^+ \to B}}{\top \supset D, \Gamma \to D}}{\{L(D)\}^+, \Gamma \to D}}{\Gamma \to D},$$

where $D = A \supset B$. ⊣

Our main purpose in this section is to prove

2.4.3. Theorem. *If* $\Gamma \to \Delta \in \mathbf{GFPL}^+$, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$.

[Val83] defined the width w of proof figures in his sequent system for the modal logic **GL** in order to prove the cut-elimination theorem. Here we use his technique. So, first we define the width $w(P)$ of a proof figure P in our system.

2.4.4. Definition. Let P be a proof figure for $\Gamma \to A$ and let J be an inference rule $(\to\supset_f)$ occurring in P. We say that J is 1-*ary* if its diagonal formula is A and the segment in P from the end sequent $\Gamma \to A$ to the lower sequent of J does not contain $(\to\supset_f)$. We say that J is 2-*ary* if there exists an 1-ary inference rule J' below J such that the segment from the upper sequent of J' to the lower sequent of J does not contain $(\to\supset_f)$ and every antecedent of sequents in the segment contains A. By $\mathbf{SI}(P)$, we mean the set of 2-ary inference rules in P. The *width* $w(P)$ of P is defined as the cardinality of $\mathbf{SI}(P)$.

It is easily seen that $w(P) = 0$ if A is not of the form $B \supset C$.

2.4.5. COROLLARY. *Let P be a cut-free proof figure for $\Gamma \to A$ with an inference rule I introducing the end sequent of P. Let P_1 be a subfigure of P whose end sequent $\Gamma_1 \to A_1$ is an upper sequent of I. If $A = A_1$, then $w(P) \geq w(P_1)$.*

The following lemma can be shown similarly to Lemma 2.2.7.

2.4.6. LEMMA. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in* **GFPL**$^+$*. Then there exists a cut-free proof figure P^+ for $\Gamma^+ \to \Delta$ such that $len(P) = len(P^+)$.*

2.4.7. LEMMA. *Let P_1 be a cut-free proof figure for $A, A \supset B, \Gamma^+ \to B$ and let P be the proof figure* $$\dfrac{P_1\left\{\begin{array}{c}\vdots\\ A, A \supset B, \Gamma^+ \to B\end{array}\right.}{\Gamma \to A \supset B}.$$ *If $w(P) = 0$, then for any sequent $\Pi \to \Delta$ in P_1, there exists a cut-free proof figure for $\Pi_{A \supset B}, \Gamma^+ \to \Delta$.*

Proof. Since $\Pi \to \Delta$ occurs in P_1, there exists a subfigure Q of P whose end sequent is $\Pi \to \Delta$. We use an induction on Q.

If $A \supset B \notin \Pi$, then the lemma is obvious.

If $A \supset B \in \Delta$, then there exists a cut-free proof figure for $\Gamma^+ \to \Delta$ using P and Lemma 2.4.6. Using $(T \to)$, possibly several times, we obtain the lemma.

So, we assume $A \supset B \in \Pi - \Delta$. Then Q is not axiom, and so, there exists an inference rule I that introduces the end sequent of Q. Also we suppose that the lemma hold for any proper subfigure of Q.

If $A \supset B$ is a principal formula of I, then I is $(T \to)$ or $(\to \supset_f)$. By $w(P) = 0$, I is not $(\to \supset_f)$, and so, I is $(T \to)$ of the form

$$\frac{\Pi_{A \supset B} \to \Delta}{A \supset B, \Pi_{A \supset B} \to \Delta}.$$

By the induction hypothesis, there exists a cut-free proof figure for $\Pi_{A \supset B}, \Gamma^+ \to \Delta$.

If $A \supset B$ is an auxiliary formula of I and belongs to the antecedent of an upper sequent of I, then I is either one of the following inference rules:

$$(\supset^+ \to), (\to \supset^+), (\wedge \to), (\vee \to).$$

We only show the case that I is $(\supset^+\rightarrow)$. The other cases can be shown similarly. Since I is $(\supset^+\rightarrow)$, it is of the form

$$\frac{\Pi_1 \rightarrow C \quad A \supset B, \Pi_1 \rightarrow \Delta}{C \supset^+ (A \supset B), \Pi_1 \rightarrow \Delta},$$

where $\{C \supset^+ (A \supset B)\} \cup \Pi_1 = \Pi$. By the induction hypothesis, there exists a cut-free proof figure for

$$(\Pi_1)_{A \supset B}, \Gamma^+ \rightarrow \Delta.$$

Hence we obtain a cut-free proof figure for

$$C \supset^+ (A \supset B), (\Pi_1)_{A \supset B}, \Gamma^+ \rightarrow \Delta.$$

That is,

$$\Pi_{A \supset B}, \Gamma^+ \rightarrow \Delta.$$

Otherwise, we can obtain a cut-free proof figure for $\Pi_{A \supset B}, \Gamma^+ \rightarrow \Delta$ by using the induction hypothesis and the same kind of inference rule as I. ⊣

2.4.8. Corollary. *Let P_1 be a cut-free proof figure for $A, A \supset B, \Gamma^+ \rightarrow B$. If $w(\dfrac{P_1}{\Gamma \rightarrow A \supset B}) = 0$, then there exists a cut-free proof figure for $A, \Gamma^+ \rightarrow B$.*

2.4.9. Definition. We define a mapping g_{Γ^+} on the set of cut-free proof figures in $\mathbf{GFPL}^+$ as follows.

(1) $g_\emptyset(B \rightarrow B) = B \rightarrow B$, $g_\emptyset(\perp \rightarrow) = \perp \rightarrow$,

(2) $g_{(\{A\} \cup \Gamma_A)^+}(B \rightarrow B) = \dfrac{g_{\Gamma_A^+}(B \rightarrow B)}{B, (\{A\} \cup \Gamma_A)^+ \rightarrow B}$,

(3) $g_{(\{A\} \cup \Gamma_A)^+}(\perp \rightarrow) = \dfrac{g_{\Gamma_A^+}(\perp \rightarrow)}{\perp, (\{A\} \cup \Gamma_A)^+ \rightarrow}$,

(4) $g_{\Gamma^+}(\dfrac{P_1}{\Pi \rightarrow \Delta}) = \dfrac{g_{\Gamma^+}(P_1)}{\Pi, \Gamma^+ \rightarrow \Delta}$,

(5) $g_{\Gamma^+}(\dfrac{P_1 \quad P_2}{\Pi \rightarrow \Delta}) = \dfrac{g_{\Gamma^+}(P_1) \quad g_{\Gamma^+}(P_2)}{\Pi, \Gamma^+ \rightarrow \Delta}$.

It is easily seen that for any cut-free proof figure P for $\Pi \rightarrow \Delta$, $g_{\Gamma^+}(P)$ is a cut-free proof figure for $\Pi, \Gamma^+ \rightarrow \Delta$. Also we have

2.4.10. Corollary. *Let P_1 be a cut-free proof figure for $A, A \supset B, \Gamma^+ \rightarrow B$ and let P and P' be the following proof figures, respectively:*

$$
\dfrac{P_1\left\{\begin{array}{c}\vdots\\ A, A \supset B, \Pi^+ \to B\end{array}\right.}{\Pi \to A \supset B}
\qquad
\dfrac{g_{\Gamma^+}(P_1)\left\{\begin{array}{c}\vdots\\ A, A \supset B, \Pi^+, \Gamma^+ \to B\end{array}\right.}{\Pi, \Gamma \to A \supset B}.
$$

Then

(1) $w(P) = w(P')$,

(2) $\dfrac{C, C \supset D, \Phi^+ \to D}{\Phi \to C \supset D} \in \mathbf{SI}(P)$ *implies* $\dfrac{C, C \supset D, \Phi^+, \Gamma^+ \to D}{\Phi, \Gamma^+ \to C \supset D} \in \mathbf{SI}(P')$.

2.4.11. Lemma. *Let P_l be a cut-free proof figure for $\Gamma \to X$ and P_r be a cut-free proof figure for $X, \Pi \to \Delta$. Let P be the proof figure*

$$
\dfrac{P_l\left\{\begin{array}{c}\vdots\\ \Gamma \to X\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ X, \Pi \to \Delta\end{array}\right\}P_r}{\Gamma, \Pi_X \to \Delta}.
$$

Then there exists a cut-free proof figure for the end sequent of P.

Proof. We define $d(P)$, $R_l(P)$ and $R_r(P)$ in the same way as in Lemma 2.2.8. Here it is also seen that $len(P_l) = R_l(P)$. We use an induction on $R_l(P)+R_r(P)+\omega w(P_l)+\omega^2 d(P)$. By Corollary 2.4.5, our new parameter $w(P_l)$ does not influence the usual proof for the other cases, and so, the lemma can be shown in the usual way except the following two cases.

The case that P is of the form

$$
\dfrac{\dfrac{P_l'\left\{\begin{array}{c}\vdots\\ C, C \supset D, \Gamma^+ \to D\end{array}\right.}{\Gamma \to C \supset D} \qquad \dfrac{\left.\begin{array}{c}\vdots\\ A, A \supset B, C \supset^+ D, \Pi^+ \to B\end{array}\right\}P_r'}{C \supset D, \Pi \to A \supset B}}{\Gamma, \Pi_{C \supset D} \to A \supset B}:
$$

If $w(P_l) = 0$, then we have a cut-free proof figure P_1 for $C, \Gamma^+ \to D$ by Corollary 2.4.8. Using P_r', we obtain the following proof figure P_2.

$$
\dfrac{\dfrac{P_1\left\{\begin{array}{c}\vdots\\ C, \Gamma^+ \to D\end{array}\right.}{\Gamma^+ \to C \supset^+ D} \qquad \left.\begin{array}{c}\vdots\\ A, A \supset B, C \supset^+ D, \Pi^+ \to B\end{array}\right\}P_r'}{\Gamma^+, (A, A \supset B, \Pi^+)_{C \supset^+ D} \to B}
$$

Here we note that $d(P_2) < d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_2. Using $(T \to)$, possibly several times, we have a cut-free proof figure for

$$A, A \supset B, \Gamma^+, (\Pi_{C \supset D})^+ \to B.$$

Using $(\rightarrow\supset_f)$, we obtain the lemma.

If $w(P_l) > 0$, then there exists $J \in \mathbf{SI}(P_l)$ of the form

$$\frac{E, E \supset F, \Phi^+ \rightarrow F}{\Phi \rightarrow E \supset F}(J).$$

Let P_1 be the proof figure

$$\frac{g_{E\supset^+F}(P_l')}{E \supset F, \Gamma^+ \rightarrow C \supset D}.$$

Then, by Corollary 2.4.10, we have $w(P_l) = w(P_1)$ and there exists $J_1 \in \mathbf{SI}(P_1)$ of the form

$$\frac{E, E \supset F, E \supset^+ F, \Phi^+ \rightarrow F}{E \supset^+ F, \Phi \rightarrow E \supset F}(J_1).$$

On the other hand, we have the following proof figure P_2 for for $\Phi, E \supset^+ F \rightarrow E \supset F$.

$$\cfrac{\cfrac{\cfrac{\cfrac{E \rightarrow E \qquad \cfrac{F \rightarrow F}{F, E \rightarrow F}}{E, E \supset^+ F \rightarrow F}}{E, E \supset F, E \supset^+ F \rightarrow F}}{E \supset^+ F \rightarrow E \supset F}}{\cfrac{\text{using } (T \rightarrow),\ \text{possibly several times}}{E \supset^+ F, \Phi \rightarrow E \supset F}}$$

Let P_3 be the figure obtained from P_1 by replacing the subfigure for the lower sequent of J_1 by P_2. It is easily seen that P_3 is a cut-free proof figure for

$$E \supset F, \Gamma^+ \rightarrow C \supset D$$

and no 2-ary inference rule in P_3 occurs in P_2. Using $J_1 \in \mathbf{SI}(P_l)$, we have $w(P_3) < w(P_1) = w(P_l)$. By P_3 and P_l', we have the following proof figure P_4:

$$\frac{P_3\left\{\begin{array}{c}\vdots\\ E \supset F, \Gamma^+ \rightarrow C \supset D\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ C, C \supset D, \Gamma^+ \rightarrow D\end{array}\right\}P_l'}{E \supset F, C, \Gamma^+ \rightarrow D}.$$

We note that $w(P_3) < w(P_l)$ and $d(P_4) = d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure P_5 for the end sequent of P_4. Let P_6 be the subfigure of P_l' for the upper sequent of J. Then we have the following proof figure P_7:

$$\frac{\cfrac{P_5\left\{\begin{array}{c}\vdots\\ C, E \supset F, \Gamma^+ \rightarrow D\end{array}\right.}{E \supset F, \Gamma^+ \rightarrow C \supset^+ D} \qquad \left.\begin{array}{c}\vdots\\ E, E \supset F, \Phi^+ \rightarrow F\end{array}\right\}P_6}{E \supset F, \Gamma^+, (\{E, E \supset F\} \cup \Phi^+)_{C\supset^+D} \rightarrow F}.$$

Here we have $d(P_7) < d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_7. Using $(T \to)$, possibly several times, we have a cut-free proof figure for

$$E, E \supset F, (\Phi_{C \supset D})^+, \Gamma^+ \to F.$$

Using $(\to\supset_f)$, we have a cut-free proof figure P'_8 for

$$\Phi_{C \supset D}, \Gamma^+ \to E \supset F.$$

Here we note that $C \supset D$ does not belong to the antecedent above. Using $(T \to)$, we have a cut-free proof figure P_8 for

$$\Phi, \Gamma^+ \to E \supset F.$$

Let P_9 be the proof figure

$$\frac{g_{\Gamma^+}(P'_l)}{\Gamma^+ \to C \supset D}.$$

By Corollary 2.4.10, we have $w(P_l) = w(P_9)$ and there exists $J_2 \in \mathbf{SI}(P_9)$ of the form

$$\frac{E, E \supset F, \Phi^+, \Gamma^+ \to F}{\Phi, \Gamma^+ \to E \supset F}(J_2).$$

Let P_{10} be the figure obtained from P_9 by replacing the subfigure for the lower sequent of J_2 by P_8. It is easily seen that P_{10} is a cut-free proof figure for

$$\Gamma^+ \to C \supset D.$$

Also, considering the end sequent of P'_8, no 2-ary inference rule in P_{10} occurs in P_8. Using $J_2 \in \mathbf{SI}(P_9)$, we have $w(P_{10}) < w(P_9) = w(P_l)$. By P_{10} and P'_l, we have the following proof figure P_{11}:

$$\frac{P_{10}\left\{\begin{array}{c}\vdots\\ \Gamma^+ \to C \supset D\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ C, C \supset D, \Gamma^+ \to D\end{array}\right\}P'_l}{C, \Gamma^+ \to D}.$$

We note that $w(P_{10}) = w(P_l)$ and $d(P_{11}) = d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_{11}. Using $(\to\supset^+)$, we have a cut-free proof figure P_{12} for

$$\Gamma^+ \to C \supset^+ D.$$

Using P'_r, we obtain the following proof figure P_{13}:

$$\frac{P_{12}\left\{\begin{array}{c}\vdots\\ \Gamma^+ \to C \supset^+ D\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ A, A \supset B, C \supset^+ D, \Pi^+ \to B\end{array}\right\}P'_r}{\Gamma^+, (\{A, A \supset B\} \cup \Pi^+)_{C \supset^+ D} \to B}.$$

Here we have $d(P_{13}) < d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_{13}. Using $(T \rightarrow)$, possibly several times, we have a cut-free proof figure for

$$A, A \supset B, \Gamma^+, (\Pi_{C \supset D})^+ \rightarrow B.$$

So, using $(\rightarrow\supset_f)$, we obtain the lemma.

The case that P is of the form

$$\dfrac{P_l \left\{ \begin{array}{c} \vdots \\ \Gamma \rightarrow X \end{array} \right. \qquad \dfrac{\left.\begin{array}{c} \vdots \\ A, A \supset B, X, \Pi^+ \rightarrow B \end{array}\right\} P_r'}{X, \Pi \rightarrow A \supset B}}{\Gamma, \Pi_X \rightarrow A \supset B},$$

where X is not of the form $C \supset D$. By Lemma 2.4.6, there exists a cut-free proof figure P_l^+ for $\Gamma^+ \rightarrow X$ such that $len(P_l^+) = len(P_l) = R_l(I)$. Using P_r' we obtain the following proof figure P_1:

$$\dfrac{P_l^+ \left\{ \begin{array}{c} \vdots \\ \Gamma^+ \rightarrow X \end{array} \right. \qquad \left.\begin{array}{c} \vdots \\ A, A \supset B, \Pi^+ \rightarrow B \end{array}\right\} P_r'}{\Gamma^+, (\{A, A \supset B\} \cup \Pi^+)_X \rightarrow B}.$$

By $len(P_l^+) = len(P_l)$, we have $R_l(P_1) = R_l(P)$. Since X is not of the form $C \supset D$, $w(P_l^+) = w(P_l) = 0$. And we have $R_r(P_1) < R_r(P)$ and $d(P_1) = d(P)$. So, by the induction hypothesis, there exists a cut-free proof figure for the end sequent of P_1. Here we note that $(\Pi^+)_X \subseteq (\Pi_X)^+$ since X is not of the form $C \supset D$. Using $(T \rightarrow)$, possibly several times, we have a cut-free proof figure for

$$A, A \supset B, \Gamma^+, (\Pi_X)^+ \rightarrow B.$$

So, using $(\rightarrow\supset_f)$, we obtain a cut-free proof figure for

$$\Gamma, \Pi_X \rightarrow A \supset B.$$

$\dashv$

Now, Theorem 2.4.3 follows from Lemma 2.4.11 in the usual way. The following two corollaries can be shown similarly to Corollary 2.2.10 and Corollary 2.2.11.

2.4.12. COROLLARY. *If* $\Gamma \rightarrow \Lambda \in \mathbf{GFPL}^+$, *then there exists a cut-free and* $(\rightarrow\supset^+)$*-free proof figure for* $\Gamma \rightarrow \Lambda$.

2.4.13. COROLLARY. *Let it be that* $\Gamma \rightarrow \Delta \in \mathbf{GFPL}^+$. *Then there exists a proof figure* P *satisfying the following condition for any* $X \in \mathbf{WFF}^+$*:*

if X *occurs in* P, *then* $X \in \mathsf{Sub}^{+}(Y)$ *for some* Y *occurring in* $\Gamma \to \Delta$.

2.4.14. Corollary. *Let* Σ *be a finite set of formulas in* $\mathbf{WFF}$ *and let* A *be an implication free formula in* $\mathbf{WFF}$. *Then*

$$\Sigma \to A \in \mathbf{GFPL}^{+} \text{ iff } \Sigma \to A \in \mathbf{GVPL}^{+}.$$

Proof. Suppose that $\Sigma \to A \in \mathbf{GFPL}^{+}$. By Corollary 2.4.12, there exists a cut-free and $(\to\supset^{+})$-free proof figure P for $\Sigma \to A$ in $\mathbf{GFPL}^{+}$. By an induction on the number of inference rules in P, we can easily see that there is neither $(\supset^{+}\to)$ nor $(\to\supset_{f})$ in P. So, every inference rule in P is also an inference rule in $\mathbf{GVPL}^{+}$. Then the sequent is provable in $\mathbf{GVPL}^{+}$.

The "if" part follows from Lemma 2.4.2. $\dashv$

Chapter 3

Formalizations for the consequence relation of Visser's propositional logic

In this chapter, we consider Hilbert style formalizations of the consequence relation $\vdash_V$ of Visser's propositional logic (**VPL**). As we mentioned in section 1.1, it seems difficult to give a finite Hilbert style formalization for $\vdash_V$ because $\vdash_V$ does not obey modus ponens. Here we introduce a restricted modus ponens, which mostly has the same role as modus ponens in a Hilbert style formalization for the consequence relation of intuitionistic propositional logic. Using the restricted modus ponens and adjunction, we give a formalization for $\vdash_V$, and at the same time we show that $\vdash_V$ cannot be formalized by any system with a restricted modus ponens as its only inference rule.

3.1 Kripke semantics for $\vdash_V$

[Vis81] showed the completeness theorem for $\vdash_V$ using Kripke models. Since his results are useful for our investigations, we show them below.

A Kripke model is a triple $M = \langle W, R, P \rangle$, where R is a transitive binary relation on a set $W \neq \emptyset$ and P is a mapping from the set of all propositional variables to the set

$$\{S \in 2^W \mid \text{if } \alpha R \beta \text{ and } \alpha \in S, \text{ then } \beta \in S\}.$$

The truth valuation $\models$ is defined in the following way:

(K1) $(M, \alpha) \models p$ iff $\alpha \in P(p)$,

(K2) $(M, \alpha) \not\models \bot$,

(K3) $(M, \alpha) \models A \wedge B$ iff $(M, \alpha) \models A$ and $(M, \alpha) \models B$,

(K4) $(M, \alpha) \models A \vee B$ iff $(M, \alpha) \models A$ or $(M, \alpha) \models B$, (K5) $(M, \alpha) \models A \supset B$ iff for any $\beta \in \{\gamma \in W \mid \alpha R \gamma\}$, $(M, \beta) \models A$ implies $(M, \beta) \models B$.

The expression $M \models A$ denotes $(M, \alpha) \models A$ for every $\alpha \in W$. We write $(M, \alpha) \models \Gamma$ if $(M, \alpha) \models A$ for every $A \in \Gamma$. We put $\alpha\!\upharpoonleft = \{\beta \in W \mid \alpha R \beta\}$.

3.1.1. Lemma. ([Vis81])
(1)**VPL** $= \{A \mid$ *for every Kripke model* $M, M \models A\}$.
(2)$\Gamma \vdash_V A$ *iff for every* M *and every* $\alpha \in W$,

$$(M, \alpha) \models \Gamma \textit{ implies } (M, \alpha) \models A.$$

From Lemma 3.1.1, we can see that $\{p, p \supset q\} \nvdash_V q$ by a Kripke model $\langle \{\alpha\}, \emptyset, P \rangle$, where $P(p) = \{\alpha\}, P(q) = \emptyset$. So, in $\vdash_V$, modus ponens does not hold in general.

The following lemma is useful for chapter 4.

3.1.2. Lemma. ([SWZ98]) $(M, \alpha) \models A$ *implies* $(M, \beta) \models A$ *for any* $\beta \in \alpha\!\upharpoonleft$.

3.2 A formalization of the consequence relation of VPL

In this section, we give a formalization for $\vdash_V$. A Hilbert style formalization for **VPL** has been given in [SO98] as follows.

3.2.1. Lemma. *The closure under modus ponens and substitution of the set of the following 12 axioms coincides with* **VPL**:

$$
\begin{array}{ll}
(\supset_1) & p \supset p, \\
(\supset_2) & p \supset (q \supset p), \\
(\supset_3) & (q \supset r) \wedge (p \supset q) \supset (p \supset r), \\
(\wedge_1) & p \wedge q \supset p, \\
(\wedge_2) & p \wedge q \supset q, \\
(\wedge_3) & (r \supset p) \wedge (r \supset q) \supset (r \supset p \wedge q), \\
(\wedge_4) & p \supset (q \supset p \wedge q), \\
(\vee_1) & p \supset p \vee q, \\
(\vee_2) & q \supset p \vee q, \\
(\vee_3) & (p \supset r) \wedge (q \supset r) \supset (p \vee q \supset r), \\
(\vee_4) & p \wedge (q \vee r) \supset (p \wedge q) \vee (p \wedge r), \\
(\perp) & \perp \supset p.
\end{array}
$$

By **A**, we mean the set of all substitution instances of axioms described in Lemma 3.2.1 except ($\wedge_4$).

3.2.2. Definition. We define the consequence relation $\vdash_{V^*}$ inductively as follows:

(axi) if $A \in \mathbf{A}$, then $\Gamma \vdash_{V^*} A$,
(asp) if $A \in \Gamma$, then $\Gamma \vdash_{V^*} A$,
(rmp) if $\Gamma \vdash_{V^*} A$ and $\emptyset \vdash_{V^*} A \supset B$, then $\Gamma \vdash_{V^*} B$,
(adj) if $\Gamma \vdash_{V^*} A$ and $\Gamma \vdash_{V^*} B$, then $\Gamma \vdash_{V^*} A \wedge B$.

Our main theorem in this section is

3.2.3. Theorem. $\Gamma \vdash_{V^*} A$ *iff* $\Gamma \vdash_V A$.

In order to prove the theorem above, we show some lemmas.

3.2.4. Lemma. (1) *if* $\Sigma \subseteq \Gamma$ *and* $\Sigma \vdash_{V^*} A$, *then* $\Gamma \vdash_{V^*} A$,
(2) *if* $\Gamma \vdash_{V^*} A$ *and* $\Sigma \cup \{A\} \vdash_{V^*} B$, *then* $\Gamma \cup \Sigma \vdash_{V^*} B$.

Proof. (1) is trivial. We show only (2). We use an induction on the number of inference rules used in the proof of $\Sigma \cup \{A\} \vdash_{V^*} B$.

If $B \in \mathbf{A} \cup \Sigma$, then (2) is obvious.

If $B = A$, then we have $\Gamma \vdash_{V^*} B$. So, $\Gamma \cup \Sigma \vdash_{V^*} B$.

If $\Sigma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Sigma \cup \{A\} \vdash_{V^*} C \text{ and } \emptyset \vdash_{V^*} C \supset B$$

for some C by (rmp) then, by the induction hypothesis, we have $\Gamma \cup \Sigma \vdash_{V^*} C$. Using (rmp), we obtain the lemma.

If $\Sigma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Sigma \cup \{A\} \vdash_{V^*} C \text{ and } \Sigma \cup \{A\} \vdash_{V^*} D$$

for some C and D such that $B = C \wedge D$ by (adj) then, by the induction hypothesis, we have $\Gamma \cup \Sigma \vdash_{V^*} C$ and $\Gamma \cup \Sigma \vdash_{V^*} D$. Using (adj), we obtain the lemma. $\dashv$

3.2.5. Lemma. $\Gamma \vdash_{V^*} A$ *implies* $\Gamma \vdash_V A$.

Proof. From Lemma 3.2.1, $\mathbf{A} \subseteq \mathbf{VPL}$ and we can easily check that (rmp) and (adj) hold in every Kripke model. Using Lemma 3.1.1, we obtain the lemma. $\dashv$

We put $(\wedge\emptyset) = \top$ and by $(\wedge\Gamma)$, we mean the conjunction of all the formulas in Γ if $\Gamma \neq \emptyset$.

3.2.6. LEMMA. *$\emptyset \vdash_{V^*} (\wedge\Gamma) \supset B$ implies $\Gamma \vdash_{V^*} B$.*

Proof. If $\Gamma = \emptyset$, then we have $\Gamma \vdash_{V^*} (\wedge\Gamma)$ from the axiom ($\supset_1$). If not, we also have $\Gamma \vdash_{V^*} (\wedge\Gamma)$ using (adj), possibly several times. Using $\emptyset \vdash_{V^*} (\wedge\Gamma) \supset B$ and (rmp), we obtain the Lemma. $\dashv$

By this lemma, Lemma 3.2.4(2) and the axioms ($\supset_2$), ($\supset_3$), ($\wedge_3$) and ($\vee_3$), we have

3.2.7. COROLLARY. *The following rules hold in $\vdash_{V^*}$:*
($R\supset_2$) *if $\Gamma \vdash_{V^*} A$, then $\Gamma \vdash_{V^*} B \supset A$,*
($R\supset_3$) *if $\Gamma \vdash_{V^*} B \supset C$ and $\Gamma \vdash_{V^*} A \supset B$, then $\Gamma \vdash_{V^*} A \supset C$,*
($R\wedge_3$) *if $\Gamma \vdash_{V^*} C \supset A$ and $\Gamma \vdash_{V^*} C \supset B$, then $\Gamma \vdash_{V^*} C \supset A \wedge B$,*
($R\vee_3$) *if $\Gamma \vdash_{V^*} A \supset C$ and $\Gamma \vdash_{V^*} B \supset C$, then $\Gamma \vdash_{V^*} A \vee B \supset C$.*

3.2.8. LEMMA. *If $\Gamma \cup \{A\} \vdash_{V^*} B$, then $\Gamma \vdash_{V^*} A \supset B$.*

Proof. We use an induction on the number of inference rules used in the proof of $\Gamma \cup \{A\} \vdash_{V^*} B$.

If $B = A$, then $A \supset B \in \mathbf{A}$. So, we have $\Gamma \vdash_{V^*} A \supset B$.

If $B \in \mathbf{A} \cup \Gamma$, then we have $\Gamma \vdash_{V^*} B$. So, using ($R\supset_2$), we obtain the lemma.

If $\Gamma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Gamma \cup \{A\} \vdash_{V^*} C \text{ and } \emptyset \vdash_{V^*} C \supset B$$

for some C by (rmp) then, by the induction hypothesis and Lemma 3.2.4(1), we have

$$\Gamma \vdash_{V^*} A \supset C \text{ and } \Gamma \vdash_{V^*} C \supset B.$$

Using ($R\supset_3$), we obtain the lemma.

If $\Gamma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Gamma \cup \{A\} \vdash_{V^*} C \text{ and } \Gamma \cup \{A\} \vdash_{V^*} D$$

for some C and D such that $B = C \wedge D$ by (adj) then, by the induction hypothesis, we have

$$\Gamma \vdash_{V^*} A \supset C \text{ and } \Gamma \vdash_{V^*} A \supset D.$$

Using ($R\wedge_3$), we obtain the lemma. $\dashv$

Here we can see $\emptyset \vdash_{V^*} p \supset (q \supset p \wedge q)$ by (adj) and the lemma above. Hence we confirm that ($\wedge_4$) does not necessarily belong to $\mathbf{A}$.

3.2.9. LEMMA. *$A \in \Gamma$ implies $(\wedge\Gamma) \vdash_{V^*} A$.*

Proof. Using the axioms $(\wedge_1)$ and $(\wedge_2)$ and (rmp), possibly several times, we obtain the Lemma. ⊣

3.2.10. LEMMA. *$\Gamma \vdash_V A$ implies $\Gamma \vdash_{V^*} A$.*

Proof. We use an induction on the number of inference rules used in the proof of $\Gamma \vdash_V A$.

If $A \in \Gamma$, then the lemma is trivial.

Suppose that $\Gamma \vdash_V A$ is proved using at least one inference rule. Let I be the inference rule that introduces $\Gamma \vdash_V A$. If I is either one of the inference rules

$$(\bot E), (\wedge E_1), (\wedge E_2), (\vee I_1) \text{ and } (\vee I_2),$$

then we obtain the lemma by the induction hypothesis, (rmp) and the corresponding axioms

$$(\bot), (\wedge_1), (\wedge_2), (\vee_1) \text{ and } (\vee_2),$$

respectively. If I is $(\wedge I)$, then the lemma follows from (adj) and the induction hypothesis. If I is $(\supset I)$, then the lemma follows from Lemma 2.7 and the induction hypothesis. If I is either one of the inference rules $(\wedge I_f), (\vee E_f)$ and (Tr), then we obtain the lemma by the induction hypothesis and inference rules in Corollary 3.2.7.

The remaining case is that I is $(\vee E)$. I is of the form $\dfrac{B \vee C \quad \begin{matrix}[B] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[C] \\ \vdots \\ A\end{matrix}}{A}$. By the induction hypothesis, we have

$$\Gamma \cup \{B\} \vdash_{V^*} A.$$

On the other hand, from Lemma 3.2.9, we have

$$\{(\wedge\Gamma) \wedge B\} \vdash_{V^*} D$$

for any $D \in \Gamma \cup \{B\}$. Using Lemma 3.2.4(2),

$$\{(\wedge\Gamma) \wedge B\} \vdash_{V^*} A.$$

Using Lemma 3.2.8,

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge B \supset A.$$

Similarly, we have

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge C \supset A.$$

Using $(R\vee_3)$,

$$\emptyset \vdash_{V^*} ((\wedge\Gamma) \wedge B) \vee ((\wedge\Gamma) \wedge C) \supset A.$$

By $(\vee_4)$, we also have

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge (B \vee C) \supset ((\wedge\Gamma) \wedge B) \vee ((\wedge\Gamma) \wedge C).$$

Using $(R\supset_3)$,

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge (B \vee C) \supset A.$$

Using Lemma 3.2.6,

$$\Gamma \cup \{B \vee C\} \vdash_{V^*} A.$$

By the induction hypothesis, we also have

$$\Gamma \vdash_{V^*} B \vee C.$$

Hence, using Lemma 3.2.4(2),

$$\Gamma \vdash_{V^*} A.$$

$\dashv$

Now, Theorem 3.2.3 follows from Lemma 3.2.5 and Lemma 3.2.10.

3.2.11. Corollary. $\{A \mid \emptyset \vdash_V A\} = \{A \mid \emptyset \vdash_{V^*} A\}$.

By modifying the system $\vdash_{V^*}$, we can easily define a system $\vdash_{V^{**}}$ for the consequence relation of **VPL** with only one inference rule.

3.2.12. Definition. We define the consequence relation $\vdash_{V^{**}}$ inductively as follows:

(axi) if $A \in \mathbf{A}$, then $\Gamma \vdash_{V^{**}} A$,
(asp) if $A \in \Gamma$, then $\Gamma \vdash_{V^{**}} A$,
(rmp*) if $\Gamma \vdash_{V^{**}} A_1, \Gamma \vdash_{V^{**}} A_2$ and $\emptyset \vdash_{V^{**}} A_1 \wedge A_2 \supset B$, then $\Gamma \vdash_{V^{**}} B$.

3.2.13. Lemma. *$\Gamma \vdash_{V^{**}} A$ iff $\Gamma \vdash_{V^*} A$.*

Proof. We show "if" part. It is sufficient to show that (rmp) and (adj) hold in $\vdash_{V^{**}}$. By $\emptyset \vdash_{V^{**}} A \wedge B \supset A \wedge B$ and (rmp*), we can see that (adj) holds in $\vdash_{V^{**}}$. From the following proof, we can also see that (rmp) holds in $\vdash_{V^{**}}$.

(1) $\Gamma \vdash_{V^{**}} A$ assumption,
(2) $\emptyset \vdash_{V^{**}} A \supset B$ assumption,
(3) $\emptyset \vdash_{V^{**}} A \wedge A \supset A$ $(\wedge_1)$,
(4) $\emptyset \vdash_{V^{**}} (A \supset B) \wedge (A \wedge A \supset A) \supset (A \wedge A \supset B)$ $(\supset_3)$,
(5) $\emptyset \vdash_{V^{**}} A \wedge A \supset B$ (2),(3),(4), (rmp*),
(6) $\Gamma \vdash_{V^{**}} B$ (1),(5) (rmp*). $\dashv$

3.3 Restricted modus ponens and $\vdash_V$

In the previous section, we show that $\vdash_V$ can be formalized by the inference rules (rmp), a restricted modus ponens, and (adj). Also by (rmp*) alone. Here we prove that $\vdash_V$ cannot be formalized by any restricted modus ponens as only one inference rule. As a corollary, we find that (adj) is not redundant in $\vdash_{V^*}$.

First of all, we have to make the meaning of "restricted modus ponens" clear. By a restricted modus ponens, we mean an inference rule obtained from modus ponens, i.e.,

(mp) for any pair (A, B) of formulas,

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash A \supset B, \text{ then } \Gamma \vdash B$$

by restricting the domain of the pair (A, B) of variables. For instance, the inference rule

(rmp$'$) for any pair $(A, B) \in \{(C, D) \mid C \supset D \in \mathbf{VPL}\}$,

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash A \supset B, \text{ then } \Gamma \vdash B$$

is a restricted modus ponens. Since the inference rule (rmp) in Definition 2.1 is equivalent to the inference rule above, we might as well say that (rmp) is a restricted modus ponens.

3.3.1. Definition. Let $\mathbf{S}$ be a set of formulas and let $\mathbf{MP}$ be a set of pairs of formulas. We define the consequence relation $\vdash_{\mathbf{S},\mathbf{MP}}$ inductively as follows:

(AXI) if $A \in \mathbf{S}$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$,

(ASP) if $A \in \Gamma$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$,

(RMP) for any pair $(A, B) \in \mathbf{MP}$,

$$\text{if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A \text{ and } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A \supset B, \text{ then } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B.$$

Our main theorem in this section is

3.3.2. Theorem. *There exists no pair* $(\mathbf{S}, \mathbf{MP})$ *satisfying that for any* Γ *and any* A*,*

$$\Gamma \vdash_V A \textit{ iff } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A.$$

To prove the theorem above, we provide some preparations. It is easily seen that if $\Sigma \subseteq \Gamma$ and $\Sigma \vdash_{\mathbf{S},\mathbf{MP}} A$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$.

3.3.3. Lemma. *Let* $\mathbf{MP}_1 \subseteq \mathbf{MP}_2$ *and* $\mathbf{S}_1 \subseteq \mathbf{S}_2$*. Then*

$$\Gamma \vdash_{\mathbf{S}_1,\mathbf{MP}_1} A \textit{ implies } \Gamma \vdash_{\mathbf{S}_2,\mathbf{MP}_2} A.$$

Proof. Every axiom in $\vdash_{\mathbf{S}_1,\mathbf{MP}_1}$ is also an axiom in $\vdash_{\mathbf{S}_2,\mathbf{MP}_2}$. And the inference rule in $\vdash_{\mathbf{S}_1,\mathbf{MP}_1}$ holds in $\vdash_{\mathbf{S}_2,\mathbf{MP}_2}$. $\dashv$

Let us consider the consequence relation $\vdash_{\mathbf{VPL},\mathbf{MP}_V}$, where

$$\mathbf{MP}_V = \{(A, B) \mid \text{for any } \Gamma \text{ if } \Gamma \vdash_V A \text{ and } \Gamma \vdash_V A \supset B, \text{ then } \Gamma \vdash_V B\}.$$

The following lemma is almost immediate.

3.3.4. LEMMA. *$\Gamma \vdash_{\mathbf{VPL},\mathbf{MP}_V} A$ implies $\Gamma \vdash_V A$.*

3.3.5. LEMMA. *$\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} A \supset B$.*

Proof. It is easily seen that $\emptyset \vdash_V A \supset \top$ and $\emptyset \vdash_V \bot \supset B$. Using (Tr) twice, we have

(1) $\{\top \supset \bot\} \vdash_V A \supset B$.

Using Lemma 2.3(2), $\Gamma \vdash_V \top \supset \bot$ implies $\Gamma \vdash_V A \supset B$, and so, we have $(\top \supset \bot, A \supset B) \in \mathbf{MP}_V$. On the other hand, by (1) and $(\supset I)$, we have $(\top \supset \bot) \supset (A \supset B) \in \mathbf{VPL}$. So, we have

$$\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} (\top \supset \bot) \supset (A \supset B).$$

We also have $\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} \top \supset \bot$. Using (RMP), we have $\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} A \supset B$. $\dashv$

3.3.6. LEMMA. *If $\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C$, then either*

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C \text{ or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C.$$

Proof. We use an induction on the number of inference rules used in the proof for $\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C$.

If $C \in \mathbf{VPL} \cup \{A, B, \top \supset \bot\}$, then the lemma is trivial.

Suppose that there exists a formula D such that $(D, C) \in \mathbf{MP}_V$,

$$\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \text{ and } \{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \supset C.$$

By the induction hypothesis, we have either

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \text{ or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D.$$

On the other hand, by Lemma 3.3.5, we have

$$\{\top \supset \bot, E\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \supset C, \text{ for any } E \in \{A, B\}.$$

Since $(D, C) \in \mathbf{MP}_V$, we can use (RMP). Hence, we have either

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C \text{ or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C.$$

$\dashv$

3.3.7. LEMMA. $\{p, q\} \nvdash_{\mathbf{VPL},\mathbf{MP}_V} p \wedge q$.

Proof. Suppose that $\{p, q\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} p \wedge q$. Then, $\{\top \supset \bot, p, q\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} p \wedge q$. By Lemma 3.3.6, we have either

$$\{\top \supset \bot, p\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} p \wedge q \text{ or } \{\top \supset \bot, q\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} p \wedge q.$$

Using Lemma 3.3.4, we have either

$$\{\top \supset \bot, p\} \vdash_V p \wedge q \text{ or } \{\top \supset \bot, q\} \vdash_V p \wedge q.$$

However, using a Kripke model, we can easily show

$$\{\top \supset \bot, p\} \nvdash_V p \wedge q \text{ and } \{\top \supset \bot, q\} \nvdash_V p \wedge q.$$

This is a contradiction. $\dashv$

Proof of Theorem 3.3.3. Suppose that there exists a pair $(\mathbf{S}, \mathbf{MP})$ satisfying that for any Γ and any A,

$$\Gamma \vdash_V A \textit{ iff } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A.$$

If $\mathbf{S} \not\subseteq \mathbf{VPL}$, then there exists a formula $B \in \mathbf{S} - \mathbf{VPL}$. So, we have $\emptyset \vdash_{\mathbf{S},\mathbf{MP}} B$ and $\emptyset \nvdash_V B$. This is a contradiction.

If $\mathbf{MP} \not\subseteq \mathbf{MP}_V$, then there exists a pair $(B, C) \in \mathbf{MP} - \mathbf{MP}_V$. By $(B, C) \notin \mathbf{MP}_V$, there exists a set Σ of formulas such that $\Sigma \vdash_V B$, $\Sigma \vdash_V B \supset C$ and $\Sigma \nvdash_V C$. Using Lemma 3.2.4(2), $\Sigma \cup \{B, B \supset C\} \nvdash_V C$. On the other hand, we have $(B, C) \in \mathbf{MP}$. So, for any Γ,

$$\text{if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B \text{ and } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B \supset C, \text{ then } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} C.$$

By replacing Γ by $\Sigma \cup \{B, B \supset C\}$, we have $\Sigma \cup \{B, B \supset C\} \vdash_{\mathbf{S},\mathbf{MP}} C$. This is a contradiction.

So, we assume that $\mathbf{S} \subseteq \mathbf{VPL}$ and $\mathbf{MP} \subseteq \mathbf{MP}_V$. By Lemma 3.3.3, $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} B$ implies $\Gamma \vdash_{\mathbf{VPL},\mathbf{MP}_V} B$. Using Lemma 3.3.7, we have $\{p, q\} \nvdash_{\mathbf{S},\mathbf{MP}} p \wedge q$. However, by $(\wedge I)$, we have $\{p, q\} \vdash_V p \wedge q$. This is a contradiction.

Hence, we obtain the theorem. $\dashv$

From this proof, we have

3.3.8. COROLLARY. *If* $\vdash_{\mathbf{S},\mathbf{MP}} \subseteq \vdash_V$, *then* $\{p, q\} \nvdash_{\mathbf{S},\mathbf{MP}} p \wedge q$.

Chapter 4

Löb's axiom in propositional logics

In this chapter, we consider Löb's axiom in extensions of $\vdash_V$. [Vis81] axiomatized the consequence relation $\vdash_F$ of formal propositional logic by adding Löb's inference rule $\dfrac{(\top \supset A) \supset A}{\top \supset A}$ to $\vdash_V$. $\vdash_F$ is also obtained by adding Löb's axiom $((\top \supset p) \supset p) \supset (\top \supset p)$ to $\vdash_V$ (cf. [AR99] and [SWZ98]). However most of the extensions of $\vdash_V$ obtained by adding an inference rule to $\vdash_V$ cannot obtained by adding the corresponding axiom to $\vdash_V$. For instance, the consequence relation $\vdash_I$ of intuitionistic propositional logic is obtained by adding the inference rule $\dfrac{\top \supset A}{A}$ to $\vdash_V$, while it cannot be obtained by adding the axiom $(\top \supset p) \supset p$ to $\vdash_V$. So, it is natural to ask what axiomatization have such a property as the axiomatization by Löb's axiom (or inference rule), and what extension has an axiomatization with this property. Here we consider this problem. We prove that if an extension has an axiomatization with the property, then so does every axiomatization of the extension, and that the maximum one among such extensions is $\vdash_F$. We end up with some other results about the extensions with the property.

4.1 Extensions of $\vdash_V$

There are two possible axiomatic ways to extend a consequence relation $\vdash_L$; one is by adding an axiom, and the other by adding an inference rule. First, we define extensions of $\vdash_V$ in these two different ways.

4.1.1. Definition. By $\vdash_{L+A}$, we mean the consequence relation obtained by adding an axiom A to $\vdash_L$. By $\vdash_{L+A/B}$, we mean the consequence relation obtained from $\vdash_L$ by adding an inference rule $\dfrac{\mathsf{A}}{\mathsf{B}}$, where A and B are schemas obtained from formulas A and B by substituting all the propositional variables a_i occurring in

A or B by formulas A_i, respectively.

There are two important extensions of $\vdash_V$. One is the consequence relation $\vdash_F$ of formal propositional logic, and the other the consequence relation $\vdash_I$ of intuitionistic propositional logic. These extensions are obtained from $\vdash_V$ by adding Löb's inference rule

$$LR(A) = \frac{(\top \supset A) \supset A}{\top \supset A}$$

and the rule of modus ponens

$$\frac{\top \supset A}{A},$$

respectively, and so, they are expressed as follows:

$$\vdash_F = \vdash_{V+LR(p)},$$

$$\vdash_I = \vdash_{V+\top\supset p/p} .$$

[AR99] showed that $\vdash_F$ is also obtained by adding Löb's axiom

$$L(p) = ((\top \supset p) \supset p) \supset (\top \supset p)$$

to $\vdash_V$. In other words,

$$\vdash_F = \vdash_{V+L(p)} .$$

Hence, we have

4.1.2. Lemma.

$$\vdash_F = \vdash_{V+L(p)} = \vdash_{V+LR(p)}$$

On the other hand, considering the intermediate propositional logics, we immediately have

$$\vdash_{I+A/B} = \vdash_{I+A\supset B} .$$

So, Lemma 4.1.2 seems to be obvious. However, considering the extensions of $\vdash_V$, it is not obvious. There is a pair of extensions of $\vdash_V$ such that

$$\vdash_{V+A/B} \neq \vdash_{V+A\supset B} .$$

For instance, we can show

4.1.3. LEMMA. ([SWZ98])

$$\vdash_I \neq \vdash_{V+(\top\supset p)\supset p}$$

Proof. We note that every implication is true at α in $\langle\{\alpha\},\emptyset,P\rangle$ for any P. Let it be that $P(p)=\emptyset$. Then we have $(\langle\{\alpha\},\emptyset,P\rangle,\alpha)\models\{(\top\supset A)\supset A,\top\supset p\}$ for any A, and $(\langle\{\alpha\},\emptyset,P\rangle,\alpha)\not\models p$. ⊣

So, we may well say that Löb's axiom or rule has a nice property. Also it is natural to ask what consequence relations can be axiomatized by adding an axiom or a rule with such property as Löb's one has. In this chapter, we consider this problem. In other words, we investigate the set of consequence relations

$$\mathcal{R}=\{\vdash \mid \vdash = \vdash_{V+A/B} = \vdash_{V+A\supset B},\ \text{for some } A,B\in\mathbf{WFF}\}.$$

First, we show some examples of consequence relations in $\mathcal{R}$ and not in $\mathcal{R}$. We immediately confirm

$$\vdash_V\in\mathcal{R} \text{ and } \vdash_F\in\mathcal{R}.$$

Using the same proof as for $\vdash_F\in\mathcal{R}$,

$$\vdash_{V+LR(A)}\in\mathcal{R}$$

is also true for any formula A. However,

$$\vdash_I\notin\mathcal{R}$$

is not clear. It is true that

$$\vdash_I=\vdash_{V+\top\supset p/p}\neq\vdash_{V+(\top\supset p)\supset p},$$

but there might exist another axiomatization $\vdash_{V+A/B}$ for $\vdash_I$ such that

$$\vdash_I=\vdash_{V+A/B}=\vdash_{V+A\supset B}\ .$$

From this, we note that it is not easy to give an example of a consequence relation not in $\mathcal{R}$. We prove the following theorem in order to give such examples.

4.1.4. THEOREM. $\vdash_{V+A/B}\in\mathcal{R}$ *iff* $\vdash_{V+A/B}=\vdash_{V+A\supset B}$.

Since some previous papers gave useful results, there are several possibilities to prove the theorem. We can use the proof of Theorem 1.9 in [Vis81], Proposition 4.1.4 in [AR99] or sequent system $\mathbf{GVPL}^+$ for $\vdash_V$ introduced in chapter 2. Here we use the system $\mathbf{GVPL}^+$, because it is useful not only for the proof of Theorem 4.1.4 but also for other results, which will be described below.

We also introduce extensions of $\mathbf{GVPL}^+$.

4.1.5. Definition. By

$$\mathbf{GVPL}^+ + A_1, \cdots, A_n \to A_0,$$

we mean the system obtained by adding the new axiom $\mathsf{A}_1, \cdots, \mathsf{A}_n \to \mathsf{A}_0$ to $\mathbf{GVPL}^+$, where each A_i is a schema obtained from A_i by substituting all the propositional variables $a_{i,j}$ occurring in A_i by formulas $B_{i,j}$, respectively.

For brevity's sake, we write $\mathbf{GVPL}^+ + A$ instead of $\mathbf{GVPL}^+ + \to A$.

4.1.6. Corollary. (1) $\Sigma \vdash_{L+A} f(\Lambda)$ *iff* $\Sigma \to \Lambda \in \mathbf{GVPL}^+ + A$,
(2) $\Sigma \vdash_{V+A/B} f(\Lambda)$ *iff* $\Sigma \to \Lambda \in \mathbf{GVPL}^+ + A \to B$.

4.1.7. Lemma. $\Sigma \vdash_{V+A/B} f(\Lambda)$ *iff* $\Sigma \to \Lambda \in \mathbf{GVPL}^+ + (A \supset B)^+$.

Proof. By Corollary 4.1.6, it is sufficient to show

$$\Sigma \to \Lambda \in \mathbf{GVPL}^+ + A \to B \text{ iff } \Sigma \to \Lambda \in \mathbf{GVPL}^+ + (A \supset B)^+.$$

The following two proof figures in $\mathbf{GVPL}^+ + A \to B$ and in $\mathbf{GVPL}^+ + (A \supset B)^+$ convince us of the equivalence:

$$\dfrac{A \to B}{\to (A \supset B)^+} \qquad\qquad \dfrac{\to (A \supset B)^+ \qquad \dfrac{A \to A \quad B \to B}{(A \supset B)^+, A \to B}}{A \to B}$$

⊣

Let X be a formula in $\mathbf{WFF}^+$. By $\mathsf{Subst}(X)$, we mean the set of formulas obtained from X by substituting each propositional variable in X by a formula in $\mathbf{WFF}$.

Now, we prove Theorem 4.1.4.

Proof of Theorem 4.1.4. The "if" part is obvious. We show the "only if" part. Suppose that $\vdash_{V+A/B} \in \mathcal{R}$. Then there exist formulas C and D such that

$$\vdash_{V+A/B} = \vdash_{V+C/D} = \vdash_{V+C \supset D} .$$

So, we have $\{C\} \vdash_{V+A/B} D$. Using Lemma 4.1.7,

$$C \to D \in \mathbf{GVPL}^+ + (A \supset B)^+.$$

Hence, there exist $(A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+ \in \mathsf{Subst}((A \supset B)^+)$ such that

$$(A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+, C \to D \in \mathbf{GVPL}^+.$$

Using $(\to\supset)$,

$$A_1 \supset B_1, \cdots, A_n \supset B_n \to C \supset D \in \mathbf{GVPL}^+.$$

Since $A_1 \supset B_1, \cdots, A_n \supset B_n \in \mathsf{Subst}(A \supset B)$,

$$\to C \supset D \in \mathbf{GVPL}^+ + A \supset B.$$

Using Lemma 4.1.7,

$$\emptyset \vdash_{V+A\supset B} C \supset D.$$

Hence,

$$\vdash_{V+A/B} = \vdash_{V+C\supset D} \subseteq \vdash_{V+A\supset B} .$$

On the other hand, it is easily seen that

$$\vdash_{V+A/B} \supseteq \vdash_{V+A\supset B} .$$

Hence, we obtain the theorem. $\dashv$

From the theorem above, we have

$$\vdash_I \notin \mathcal{R}.$$

We also have the following lemma in a way similar to the proof of Theorem 4.1.4.

4.1.8. LEMMA. $\vdash_{V+A\supset B} \in \mathcal{R}$ *iff* $\vdash_{V+A/B} = \vdash_{V+A\supset B}$.

Proof. The outline of the proof is similar to the proof of Theorem 4.1.4. All we have to do is to show that

$$A_1 \supset B_1, \cdots, A_n \supset B_n \to C \supset D \in \mathbf{GVPL}^+$$

implies

$$(A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+, C \to D \in \mathbf{GVPL}^+.$$

If $C \supset D = A_i \supset B_i$, then this is obtained by $(\supset^+\to)$ and $(T \to)$, if not, it is derived from Lemma 4.1.6. $\dashv$

4.1.9. COROLLARY. $\mathcal{R} = \{\vdash | \vdash = \vdash_{V+A} = \vdash_{V+\top\supset A},$ *for some* $A \in \mathbf{WFF}\}$.

Proof. It is sufficient to note that $\vdash_{V+A} = \vdash_{V+\top/A}$. $\dashv$

4.2 The maximum in $\mathcal{R}$

Our main theorem in this section is

4.2.1. Theorem. *$\vdash_F$ is the maximal consequence relation in $\mathcal{R}$.*

In order to prove the theorem above, we provide some preparations.

4.2.2. Lemma. *Let α be a world in a Kripke model $M = \langle W, R, P\rangle$ and let it be that $\{A\} \vdash_{V+A\supset B} B$. If $(M, \alpha) \not\models C$ for some $C \in \mathsf{Subst}(A \supset B)$, then there exists a world $\beta \in \alpha\!\uparrow$ such that $(M, \beta) \not\models D$ for some $D \in \mathsf{Subst}(A \supset B)$.*

Proof. Suppose that $(M, \alpha) \not\models A^* \supset B^*$ for some $A^* \supset B^* \in \mathsf{Subst}(A \supset B)$. So, there exists a world $\beta \in \alpha\!\uparrow$ satisfying the condition

(1) $(M, \beta) \models A^*$ and $(M, \beta) \not\models B^*$.

By $\{A\} \vdash_{V+A\supset B} B$, we have $\{A^*\} \vdash_{V+A\supset B} B^*$. So, for any finite set $\Sigma \subseteq \mathsf{Subst}(A \supset B)$,

$$(M, \beta) \models \{A^*\} \cup \Sigma \text{ implies } (M, \beta) \models B^*.$$

Using (1), we have

$$(M, \beta) \not\models D \text{ for some } D \in \Sigma \subseteq \mathsf{Subst}(A \supset B).$$

So, we obtain the lemma. ⊣

4.2.3. Lemma. *$\{A\} \vdash_{V+A\supset B} B$ implies $\emptyset \vdash_F A \supset B$.*

Proof. Suppose that

$$\{A\} \vdash_{V+A\supset B} B \text{ and } \emptyset \nvdash_F A \supset B.$$

[Vis81] showed that for any finite set Σ and for any A, the following two conditions are equivalent:

(i) $\Sigma \vdash_F C$,

(ii) for any finite irreflexive Kripke model $M = \langle W, R, P\rangle$ and for any $\alpha \in W$, $(M, \alpha) \models \Sigma$ implies $(M, \alpha) \models C$.

So, by $\emptyset \nvdash_F A \supset B$, there exists a finite irreflexive Kripke model $M = \langle W, R, P\rangle$ and $\alpha \in W$ satisfying the condition

$$(M, \alpha) \not\models A \supset B.$$

Since M is finite, we can take n as the number of worlds in W.

Let it be that $\gamma \in W$. By $C(\gamma)$, we mean the condition

$$(M, \gamma) \not\models D, \text{ for some } D \in \mathsf{Subst}(A \supset B).$$

We note that

(1) the condition $C(\alpha)$ holds.

Let β be a world in W. Using Lemma 4.2.2,

(2) if $C(\beta)$ holds, then there exists a world $f(\beta) \in \beta\Uparrow$ satisfying $C(f(\beta))$.

Using (1) and (2) n times, we obtain the sequence

$$\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n,$$

where $\alpha_0 = \alpha, \alpha_{k+1} = f(\alpha_k)$ and $C(\alpha_k)$. Since W has only n worlds, there exists a pair (i, j) such that $0 \leq i < j \leq n$ and $\alpha_i = \alpha_j$. On the other hand, from $f(\beta) \in \beta\Uparrow$, we have $\alpha_k R \alpha_{k+1}$. So, using transitivity of M, we have $\alpha_i R \alpha_j$. Hence, $\alpha_i R \alpha_i$. This is in contradiction with the irreflexivity of M. $\dashv$

4.2.4. LEMMA. *$\vdash \in \mathcal{R}$ implies $\vdash \subseteq \vdash_F$.*

Proof. Suppose that $\vdash \in \mathcal{R}$. So, there exist formulas A and B such that

$$\vdash \ = \ \vdash_{V+A/B} \ = \ \vdash_{V+A\supset B} .$$

Since $\{A\} \vdash_{V+A/B} B$, we have $\{A\} \vdash_{V+A\supset B} B$. Using Lemma 4.2.3, $\emptyset \vdash_F A \supset B$. So, $\vdash_{V+A\supset B} \subseteq \vdash_F$. Hence, $\vdash \subseteq \vdash_F$.
$\dashv$

Now, Theorem 4.2.1 follows from Lemma 4.1.2 and Lemma 4.2.4.

4.2.5. COROLLARY.

(1) $\mathcal{R} \subseteq \{\vdash | \vdash_V \subseteq \vdash \subseteq \vdash_F\}$,

(2) $\min \mathcal{R} = \vdash_V$,

(3) $\max \mathcal{R} = \vdash_F$.

Although one might conjecture that the converse of Lemma 4.2.4 also holds, the following lemmas provide counterexamples.

4.2.6. LEMMA. *Let it be that $B = ((\top \supset p) \supset p) \vee L(p)$. Then*

$$\vdash_{V+B} \subseteq \vdash_F \text{ and } \vdash_{V+B} \notin \mathcal{R}.$$

Proof. Since $\emptyset \vdash_F L(p)$, we have $\vdash_{V+B} \subseteq \vdash_F$. Let it be that

$$W = \{\alpha, \beta, \gamma\}, R = \{(\alpha, \beta), (\alpha, \gamma), (\beta, \beta)\}, P(p) = \emptyset, M = \langle W, R, P \rangle.$$

We can easily check that

$$(M,\alpha) \models \mathsf{Subst}(\top \supset B),\ (M,\alpha) \models \top \text{ and } (M,\alpha) \not\models B.$$

So, we have

$$\{\top\} \cup \mathsf{Subst}(\top \supset B) \not\vdash_V B.$$

Hence, $\vdash_{V+\top\supset B} \neq \vdash_{V+\top/B}$. Using Theorem 4.1.4, we obtain $\vdash_{V+\top/B} \notin \mathcal{R}$. Hence, $\vdash_{V+B} \notin \mathcal{R}$. ⊣

Similarly, we have the following example with A and B having only the connective $\supset$.

4.2.7. LEMMA. *$\vdash_{V+A/B} \subset \vdash_F$ and $\vdash_{V+A/B} \notin \mathcal{R}$, where $A = ((\top \supset p) \supset p) \supset q, B = (L(p) \supset q) \supset (\top \supset q)$.*

4.3 Kripke semantics for extensions of $\vdash_V$

In section 4.2, we obtained that

$$\{\vdash_{V+L(A)} \mid A \in \mathbf{WFF}\} \subseteq \mathcal{R}.$$

Also we note that every examples of consequence relations in $\mathcal{R}$ in the previous sections can be axiomatized as $\vdash_{V+L(A)}$ for some A. In addition, the maximal consequence relation of $\{\vdash_{V+L(A)} \mid A \in \mathbf{WFF}\}$ is $\vdash_F$ and the minimum one is $\vdash_V$. So, it is natural to conjecture that

$$\{\vdash_{V+L(A)} \mid A \in \mathbf{WFF}\} = \mathcal{R}.$$

Using Proposition 4.1.21 in [AR99], we obtain

$$\vdash_{V+L(A)} = \vdash_{V+A}$$

if $T \supset A \vdash_V A$. So, if we can prove

$$\vdash_{V+A} \in \mathcal{R} \text{ implies } \top \supset A \vdash_V A \cdots (1),$$

then the conjecture is trivial. However, it is difficult to show (1). It is true that if $\vdash_{V+A} \in \mathcal{R}$, then $\vdash_{V+\top\supset A} A$, and so,

$$\{\top \supset A_1, \cdots, \top \supset A_n\} \vdash_V A$$

for some substitution instances $A_1, \cdots, A_n$ of A, but it does not mean

$$\{\top \supset A\} \vdash_V A.$$

In this section, we do not give the answer to the conjecture above. We consider relations between $\mathcal{R}$ and finite Kripke models and show the difficulty to give a counterexample of the conjecture.

The main theorem in this section is

4.3.1. THEOREM. *Let it be that* $\vdash_{V+A} \in \mathcal{R}$. *Then for any finite Kripke model* M,

$$M \models A \textit{ iff } M \models L(A).$$

The theorem says that for any $\vdash_{V+A} \in \mathcal{R}$, there exists no finite Kripke model that distinguishes A from $L(A)$ even if $\vdash_{V+A}$ does not equal $\vdash_{V+L(A)}$. So, it is difficult to give an example $\vdash \in \mathcal{R}$ such that $\vdash \neq \vdash_{V+L(A)}$ for any A.

In order to prove we provide some preparations.

4.3.2. NOTATION. Let $M = \langle W, R, P \rangle$ be a Kripke model. For any $\alpha \in W$, we put

$R_\alpha = R \cap (\alpha\!\uparrow \times \alpha\!\uparrow)$,
$P_\alpha(a) = P(a) \cap \alpha\!\uparrow$,
$M_\alpha = \langle \alpha\!\uparrow, R_\alpha, P_\alpha \rangle$.

4.3.3. LEMMA. *Let* α *be a world in* W *and let* β *be a world in* $\alpha\!\uparrow$. *Then for any formula* A,

$$(\langle W, R, P \rangle, \beta) \models A \textit{ iff } (\langle \alpha\!\uparrow, R_\alpha, P_\alpha \rangle, \beta) \models A.$$

4.3.4. LEMMA. *Let it be that* $\emptyset \vdash_{V+\top\supset A} A$ *and let* $M = \langle W, R, P \rangle$ *be a finite Kripke model. If* $M \not\models A$, *then there exists a substitution instance* $A_1 \in \mathsf{Subst}(A)$ *and worlds* $\alpha \in W$ *and* $\beta \in \alpha\!\uparrow$ *such that*

(1) $\beta R \beta$,
(2) $(M, \beta) \not\models A_1$,
(3) *for every* $\gamma \in \alpha\!\uparrow$, $\beta \notin \gamma\!\uparrow$ *implies* $(M, \gamma) \models A_1$.

Proof. We use an induction on the number $\#(W)$ of elements in W.

Basis($\#(W) = 1$): We can put $W = \{\alpha\}$. By $\emptyset \vdash_{V+\top\supset A} A$, there exist $A_1, \cdots, A_n \in \mathsf{Subst}(A)$ such that $\{\top \supset A_1, \cdots, \top \supset A_n\} \vdash_V A$. Using $(M, \alpha) \not\models A$, we have $(M, \alpha) \not\models \top \supset A_i$ for some $i = 1, \cdots, n$. Without loss of generality, we assume that $i = 1$. Then there exists $\beta \in \alpha\!\uparrow = \{\alpha\}$ such that $(M, \beta) \not\models A_1$. Since $\beta = \alpha$, we obtain (1), (2) and (3).

Induction step($\#(W) > 0$): Suppose that the lemma holds for any W^* such that $\#(W^*) < \#(W)$. Similarly as in the Basis, there exists $A_1 \in \mathsf{Subst}(A)$ and $\beta \in \alpha\!\uparrow$ such that $(M, \beta) \not\models A_1$.

If $\alpha \neq \beta$, then $\#(\beta\!\uparrow) < \#(W)$. By Lemma 4.3.3, $(M_\beta, \beta) \not\models A_1$. Also by $A_1 \in \mathsf{Subst}(A)$ and $\emptyset \vdash_{V+\top\supset A} A$, we have $\emptyset \vdash_{V+\top\supset A_1} A_1$. So, using the induction hypothesis, there exists a substitution instance $A_2 \in \mathsf{Subst}(A_1) \subseteq \mathsf{Subst}(A)$ and worlds $\beta_1 \in \beta\!\uparrow \subseteq \alpha\!\uparrow$ and $\beta_2 \in \beta_1\!\uparrow$ such that

(4) $\beta_2 R \beta_2$,

(5) $(M_\beta, \beta_2) \not\models A_2$,

(6) for every $\gamma \in \beta_1\!\uparrow$, $\beta_2 \notin \gamma\!\uparrow$ implies $(M_\beta, \gamma) \models A_2$.

By Lemma 4.3.3 and (5), we have $(M, \beta_2) \not\models A_2$. Hence, we obtain the lemma.

If $\alpha = \beta$ and (3) holds, then we also obtain the lemma.

So, we assume that $\alpha = \beta$ and that (3) does not hold. Then there exists $\gamma \in \alpha\!\uparrow$ such that $\beta \notin \gamma\!\uparrow$ and $(M, \gamma) \not\models A_1$. Since $\beta \in \alpha\!\uparrow$ and $\beta \notin \gamma\!\uparrow$, we have $\alpha \neq \gamma$. So, we have $\#(\gamma\!\uparrow) < \#(W)$. Hence, we obtain the lemma as in the proof of the case that $\alpha \neq \beta$. $\dashv$

4.3.5. Lemma. *Let it be that $\emptyset \vdash_{V+\top\supset A} A$ and let $M = \langle W, R, P\rangle$ be a finite Kripke model. If $M \not\models A$, then $M \not\models L(A_1)$ for some $A_1 \in \mathsf{Subst}(A)$.*

Proof. By Lemma 4.3.4, there exists a substitution instance $A_1 \in \mathsf{Subst}(A)$ and worlds $\alpha \in W$ and $\beta \in \alpha\!\uparrow$ such that

(1) $\beta R \beta$,

(2) $(M, \beta) \not\models A_1$,

(3) for every $\gamma \in \alpha\!\uparrow$, $\beta \notin \gamma\!\uparrow$ implies $(M, \gamma) \models A_1$.

Let γ be a world in $\alpha\!\uparrow$. By (3), if $\beta \notin \gamma\!\uparrow$, then $(M, \gamma) \models A_1$. By (2), if $\beta \in \gamma\!\uparrow$, then $(M, \gamma) \not\models \top \supset A_1$.

Hence, we have either $(M, \gamma) \not\models \top \supset A_1$ or $(M, \gamma) \models A_1$ for any $\gamma \in \alpha\!\uparrow$, which means $(M, \alpha) \models (\top \supset A_1) \supset A_1$. On the other hand, by (2) and $\alpha R \beta$, we have $(M, \alpha) \not\models \top \supset A_1$. Hence, $(M, \alpha) \not\models L(A_1)$. $\dashv$

4.3.6. Corollary. *Let it be that $\emptyset \vdash_{V+\top\supset A} A$ and let $M = \langle W, R, P\rangle$ be a finite Kripke model. Then*

$$M \models \mathsf{Subst}(L(A)) \text{ implies } M \models A.$$

Now, Theorem 4.3.1 follows Corollary 4.3.6 and $\{A\} \vdash_V L(A)$.

4.4 Cut-elimination theorem

In section 2.4, sequent system $\mathbf{GFPL}^+$ for formal propositional logic was introduced. The cut-elimination theorem for the system was proved using the method in [Val83]. Here we give another proof of the theorem using a property of Löb's axiom[1].

4.4.1. DEFINITION. The expression $\top^n A$ is defined inductively as follows:
(1) $\top^0 A = A$,
(2) $\top^{k+1} A = \top \supset \top^k A$.

Also the expression $(\top^{k+1} A)^+$ denotes $\top \supset^+ \top^k A$.

By Corollary 4.1.9, it is true that $\vdash_{V+\top\supset L(p)} L(p)$, but Löb's axiom has the following stronger property.

4.4.2. LEMMA. *$\top^n L(A) \to L(A) \in$ **GVPL**$^+$, for any $n \geq 0$.*

Proof. If $n = 0$, then the lemma is obvious. Suppose that $n > 0$ and $\top^{n-1} L(A) \to L(A) \in$ **GVPL**$^+$. It is easily seen that $\top^1 L(A) \to L(A) \in$ **GVPL**$^+$. On the other hand, by the following figure, we have that for any $k \geq 0$, $\top^{k+1} L(A) \to \top^k L(A) \in$ **GVPL**$^+$ implies $\top^{k+2} L(A) \to \top^{k+1} L(A) \in$ **GVPL**$^+$:

$$\dfrac{\dfrac{\top \to \top \qquad \dfrac{\top^{k+1} L(A) \to \top^k L(A)}{\top, \top^{k+1} L(A) \to \top^k L(A)}}{\top, (\top^{k+2} L(A))^+ \to \top^k L(A)}}{\top^{k+2} L(A) \to \top^{k+1} L(A)}$$

Hence, we have $\top^n L(A) \to \top^{n-1} L(A) \in$ **GVPL**$^+$. Using the induction hypothesis and cut, we obtain the lemma. $\dashv$

By Lemma 2.4.2 and Lemma 4.4.2, we have

4.4.3. LEMMA. *For any $n \geq 0$,*

$$\Gamma \to \Delta \in \mathbf{GFPL}^+ \text{ iff } \Gamma \to \Delta \in \mathbf{GVPL}^+ + \top^n L(p).$$

Our main purpose in this section is to give another proof to the following theorem using the lemma above (cf. Theorem 2.4.3). The method in this section is also useful in chapter 6.

4.4.4. THEOREM. *If $\Gamma \to \Delta \in$ **GFPL**$^+$, then there exists a cut-free proof figure for $\Gamma \to \Delta$.*

In order to prove the theorem above, we provide some preparations.

[1]Using the same method, [Sas01e] gives a proof of the cut-elimination theorem of **GL**.

4.4.5. Definition. By $\mathbf{GFPL}^*$, we mean the system obtained from $\mathbf{GFPL}^+$ by adding the inference rule $(\rightarrow\supset)$ in $\mathbf{GVPL}^+$.

4.4.6. Definition. Let P be a cut-free proof figure in $\mathbf{GFPL}^*$. We define $dep_I(P)$ as follows:

(1) $dep_I(D \rightarrow D) = dep_I(\perp \rightarrow) = 0$,

(2) $dep_I(\dfrac{P_1 \quad P_2}{\Gamma \rightarrow \Delta}) = \max\{dep_I(P_1), dep_I(P_2)\}$,

(3) $dep_I\left(\dfrac{P_1\left\{\begin{array}{c}\vdots\\ \Gamma_1 \rightarrow \Delta_1\end{array}\right.}{\Gamma \rightarrow \Delta}\right)$

$$= \begin{cases} dep_I(P_1)+1 & \text{if } \dfrac{\Gamma_1 \rightarrow \Delta_1}{\Gamma \rightarrow \Delta} \text{ is either } (\rightarrow\supset_f) \text{ or } (\rightarrow\supset) \\ dep_I(P_1) & \text{otherwise.} \end{cases}$$

4.4.7. Notation. We put

$$\mathsf{Sub}^+(A_1, \cdots, A_n \rightarrow \Delta) = \bigcup_{1\le i\le n} \mathsf{Sub}^+(A_i) \cup \mathsf{Sub}^+(f(\Delta)).$$

4.4.8. Lemma. *Let Σ_1 and Σ_2 be finite sets of formulas in* **WFF** *and let P be a cut-free proof figure for*

$$\{\top^n A \mid A \in \Sigma_1\}, \{(\top^{n+1}B)^+ \mid B \in \Sigma_2\}, \Gamma \rightarrow \Delta$$

in $\mathbf{GFPL}^*$, *where $n \geq 1$. If $dep_I(P) < n$ and $(\Sigma_1 \cup \Sigma_2) \cap \mathsf{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in* $\mathbf{GFPL}^*$.

Proof. We use an induction on P.

Basis(P is an axiom): By $(\Sigma_1 \cup \Sigma_2) \cap \mathsf{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset$, $\Gamma \rightarrow \Delta$ is an axiom, and so, we obtain the lemma.

Induction step(P is not axiom): Suppose that the lemma holds for any proper subfigure of P. Since P is not axiom, there exists an inference rule I that introduces the end sequent of P. We show only the following two typical cases.

The case that I is $(\rightarrow\supset_f)$: We have $0 < dep_I(P) < n$ and P is of the form

$$\dfrac{P_1\left\{\begin{array}{c}\vdots\\ C, C \supset D, \{(\top^n A)^+ \mid A \in \Sigma_1\}, \{(\top^{n+1}B)^+ \mid B \in \Sigma_2\}, \Gamma^+ \rightarrow D\end{array}\right.}{\{\top^n A \mid A \in \Sigma_1\}, \{(\top^{n+1}B)^+ \mid B \in \Sigma_2\}, \Gamma \rightarrow C \supset D}.$$

Another expression of the upper sequent of I is

$$C, C \supset D, \{(\top^{(n-1)+1}B)^+ \mid B \in \Sigma_1 \cup \{\top B' \mid B' \in \Sigma_2\}\}, \Gamma^+ \rightarrow D$$

Since $0 < dep_I(P) < n$, we have $n \geq 2$, and so, $n-1 \geq 1$. By $(\Sigma_1 \cup \Sigma_2) \cap \mathsf{Sub}^+(\Gamma \to \Delta) = \emptyset$, we have

$$(\Sigma_1 \cup \{\top B \mid B \in \Sigma_2\}) \cap \mathsf{Sub}^+(C, C \supset D, \Gamma^+ \to D) = \emptyset.$$

Also we have

$$dep_I(P_1) = dep_I(P) - 1 < n - 1.$$

So, by the induction hypothesis, there exists a cut-free proof figure for

$$C, C \supset D, \Gamma^+ \to D.$$

Using $(\to\supset_f)$, we obtain the lemma.

The case that the principal formula of I is $(\top^{n+1} B_1)^+$ for some $B_1 \in \Sigma_2$: P is of the form

$$\frac{P_1 \left\{ \begin{array}{c} \vdots \\ \Sigma_1^*, \Sigma_2^*, \Gamma \to \top \end{array} \right. \qquad \left. \begin{array}{c} \vdots \\ \top^n B_1, \Sigma_1^*, \Sigma_2^*, \Gamma \to \Delta \end{array} \right\} P_2}{(\top^{n+1} B_1)^+, \Sigma_1^*, \Sigma_2^*, \Gamma \to \Delta}.$$

where $\Sigma_1^* = \{\top^n A \mid A \in \Sigma_1\}$ and $\Sigma_2^* = \{(\top^{n+1} B)^+ \mid B \in \Sigma_2 - \{B_1\}\}$. Another expression of the right upper sequent of I is

$$\{\top^n A \mid A \in \Sigma_1 \cup \{B_1\}\}, \{(\top^{n+1} B)^+ \mid B \in \Sigma_2 - \{B_1\}\}, \Gamma \to \Delta.$$

By $(\Sigma_1 \cup \Sigma_2) \cap \mathsf{Sub}^+(\Gamma \to \Delta) = \emptyset$, we have

$$(\Sigma_1 \cup \{B_1\} \cup (\Sigma_2 - \{B_1\})) \cap \mathsf{Sub}^+(\Gamma \to \Delta) = \emptyset.$$

Also we have

$$dep_I(P_2) \leq dep_I(P) < n.$$

So, by the induction hypothesis, there exists a cut-free proof figure for

$$\Gamma \to \Delta.$$

$\dashv$

4.4.9. Notation. By $\mathcal{P}(A \supset B)$, we mean the set of each cut-free proof figure P such that the inference rule introducing the end sequent of P is either $(\to\supset)$ or $(\to\supset_f)$ and its principal formula in the succedent is $A \supset B$.

4.4.10. Definition. We define a mapping $h_{C\supset^+ D}$ on the set of cut-free proof figures in $\mathbf{GFPL}^*$ as follows:

(1) $h_{C\supset^+ D}(A \to A) = \dfrac{A \to A}{C \supset^+ D, A \to A}$,

(2) $h_{C\supset^+ D}(\bot \to) = \dfrac{\bot \to}{C \supset^+ D, \bot \to}$,

(3) $h_{C\supset^+ D}(\dfrac{P_1}{\Gamma \to \Delta})$

$$= \begin{cases} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to C \qquad D \to D}{C \supset^+ D, C \to D}}{C, C \supset D, C \supset^+ D \to D}}{C \supset^+ D \to C \supset D}}{\text{using } (T \to)\text{, possibly several times}}}{C \supset^+ D, \Gamma \to C \supset D} & \text{if } \dfrac{P_1}{\Gamma \to \Delta} \in \mathcal{P}(C \supset D) \\ \\ \dfrac{h_{C\supset^+ D}(P_1)}{C \supset^+ D, \Gamma \to \Delta} & \text{otherwise} \end{cases}$$

(4) $h_{C\supset^+ D}(\dfrac{P_1 \quad P_2}{\Gamma \to \Delta}) = \dfrac{h_{C\supset^+ D}(P_1) \quad h_{C\supset^+ D}(P_2)}{C \supset^+ D, \Gamma \to \Delta}$.

4.4.11. Corollary. *Let P be a cut-free proof figure for $\Gamma \to \Delta$. Then $h_{C\supset^+ D}(P)$ is a cut-free proof figure for $C \supset^+ D, \Gamma, \to \Delta$ such that $dep_I(P) \geq dep_I(h_{C\supset^+ D}(P))$.*

Proof. Using an induction on P. ⊣

4.4.12. Notation. By $\#_I(P)$, we mean the sum of the number of inference rule $(\to\supset)$ in P and the number of inference rule $(\to\supset_f)$ in P.

4.4.13. Lemma. *Let P be a cut-free proof figure. If there exists a subfigure $Q \in \mathcal{P}(A \supset B)$ of P satisfying $dep_I(Q) \geq 2$, then $\#_I(P) > \#_I(h_{A\supset^+ B}(P))$.*

Proof. We use an induction on P.

If $P \in \mathcal{P}(A \supset B)$, then $\#_I(h_{A\supset^+ B}(P)) = 1$. Since there exists a subfigure Q of P such that $dep_I(Q) \geq 2$, $\#_I(P) \geq 2$. Hence $\#_I(P) \geq 2 > 1 = \#_I(h_{A\supset^+ B}(P))$.

Suppose that $P \notin \mathcal{P}(A \supset B)$ and the lemma holds for any proper subfigure of P. We only show the case that P is of the form

$$\dfrac{P_1 \left\{ \begin{array}{c} \vdots \\ C, \Gamma \to \Delta \end{array} \right.}{C \wedge D, \Gamma \to \Delta}.$$

By the induction hypothesis, $\#_I(P_1) > \#_I(h_{A\supset^+B}(P_1))$. Since $h_{A\supset^+B}(P)$ is $\dfrac{h_{A\supset^+B}(P_1)}{A\supset^+ B, C\wedge D, \Gamma\to\Delta}$, we obtain

$$\#_I(P_1) > \#_I(h_{A\supset^+B}(P_1)) = \#_I(h_{A\supset^+B}(P)).$$

The other cases can be shown similarly. ⊣

4.4.14. Definition. We define a mapping $h_{C\supset D}$ on the set of cut-free proof figures in **GFPL*** as follows:

(1) $h_{C\supset D}(A\to A) = \begin{cases} \dfrac{A\to A}{C\supset D, A\to A} & \text{if } A\neq C\supset D \\ \\ A\to A & \text{otherwise} \end{cases}$,

(2) $h_{C\supset D}(\perp\to) = \dfrac{\perp\to}{C\supset D,\perp\to}$,

(3) $h_{C\supset D}(\dfrac{P_1}{\Gamma\to\Delta})$

$$= \begin{cases} \dfrac{\dfrac{C\supset D\to C\supset D}{\text{using } (T\to)\text{, possibly several times}}}{C\supset D,\Gamma\to C\supset D} & \text{if } \dfrac{P_1}{\Gamma\to\Delta}\in\mathcal{P}(C\supset D) \\ \\ \dfrac{h_{C\supset^+D}(P_1)}{C\supset D,\Gamma\to\Delta} & \text{if } \dfrac{P_1}{\Gamma\to\Delta}\in\mathcal{P}(E\supset F) \\ & \text{for some } E\supset F\neq C\supset D \\ \dfrac{h_{C\supset D}(P_1)}{C\supset D,\Gamma\to\Delta} & \text{otherwise} \end{cases},$$

(4) $h_{C\supset D}(\dfrac{P_1\quad P_2}{\Gamma\to\Delta}) = \dfrac{h_{C\supset D}(P_1)\quad h_{C\supset D}(P_2)}{C\supset D,\Gamma\to\Delta}$.

4.4.15. Corollary. *Let P be a cut-free proof figure for $\Gamma\to\Delta$. Then $h_{C\supset D}(P)$ is a cut-free proof figure for $C\supset D,\Gamma,\to\Delta$ such that $dep_I(P)\geq dep_I(h_{C\supset D}(P))$.*

Proof. Using an induction on P and Corollary 4.4.11. ⊣

4.4.16. Lemma. *Let P be a cut-free proof figure. If there exists a subfigure $Q\in\mathcal{P}(A\supset B)$ of P satisfying $dep_I(Q)\geq 2$, then $\#_I(P) > \#_I(h_{A\supset B}(P))$.*

Proof. We use an induction on P.

Most of the cases can be shown as in Lemma 4.4.13. Only one case we should show is that $P \in \mathcal{P}(C \supset D)$ for $C \supset D \neq A \supset B$, but using Lemma 4.4.13 instead of the induction hypothesis, we also obtain the lemma. $\dashv$

4.4.17. Lemma. *Let Σ_1 and Σ_2 be finite sets of formulas in* **WFF** *and let P be a cut-free proof figure for*

$$\{\top^{2n+3}A \mid A \in \Sigma_1\}, \{(\top^{2n+4}B)^+ \mid B \in \Sigma_2\}, \Gamma \to \Delta$$

in $\mathbf{GFPL}^*$, *where n is the number of elements in $\{A \supset B \mid A \supset B \in \mathsf{Sub}^+(\Gamma \to \Delta)\}$. Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in* $\mathbf{GFPL}^*$.

Proof. We use an induction on $\#_I(P)+\omega(dep_I(P))$. We note that $dep_I(P) \leq \#_I(P)$. Also the end sequent of P is

$$\{\top^{n+2}A \mid A \in \{\top^{n+1}A' \mid A' \in \Sigma_1\}\}, \{(\top^{n+3}B)^+ \mid B \in \{\top^{n+1}B' \mid B' \in \Sigma_2\}, \Gamma \to \Delta$$

and

$$(\{\top^{n+1}A' \mid A' \in \Sigma_1\} \cup \{\top^{n+1}B' \mid B' \in \Sigma_2\}) \cap \mathsf{Sub}^+(\Gamma \to \Delta) = \emptyset.$$

If $dep_I(P) < n+2$, we obtain the lemma by Lemma 4.4.8. Suppose that $dep_I(P) \geq n+2$ and the lemma holds for any P^* such that $\#_I(P^*)+\omega(dep_I(P^*)) < \#_I(P)+\omega(dep_I(P))$. Since $dep_I(P) \geq n+2$, there exists a sequence of subfigures of P

$$P_1, P_2, \cdots, P_{n+1}, P_{n+2}, \cdots, P_{dep_I(P)}$$

such that

(1) P_{i+1} is a proper subfigure of P_i,

(2) $P_i \in \mathcal{P}(C_i \supset D_i)$ for some C_i and D_i.

We note that if $i \leq n+1$, then the sum of the number of inference rules $(\to\supset)$ and $(\to\supset_f)$ on the path from the end sequent to the lower sequent of P_i is $i-1$. On the other hand, logical inference rules whose principal formula is of the form $A \supset B$ are only $(\to\supset)$ and $(\to\supset_f)$. So, using an induction, we can easily show that the succedent of each sequent on the path contains only elements of $\{\top\} \cup \mathsf{Sub}^+(\Gamma \to \Delta)$. Hence we have $P_i \in \mathcal{P}(C_i \supset D_i)$ for some $C_i \supset D_i \in \mathsf{Sub}^+(\Gamma \to \Delta)$. Since n is the number of elements in $\{A \supset B \mid A \supset B \in \mathsf{Sub}^+(\Gamma \to \Delta)\}$, there exist i, j and $C \supset D \in \mathsf{Sub}^+(\Gamma \to \Delta)$ such that $P_i, P_j \in \mathcal{P}(C \supset D)$ and $1 \leq i < j \leq n+1$. Using $dep_I(P) \geq n+2$, we have $dep_I(P_j) \geq 2$. Let P_i' be the subfigure of P_i whose end sequent is the upper sequent of the inference rule introducing the end sequent of P_i. Then by Lemma 4.4.15, $dep_I(P_i') \geq dep_I(h_{C\supset D}(P_i'))$ and by Lemma 4.4.16, $\#_I(P_i') > \#_I(h_{C\supset D}(P_i'))$. Let Q_i be the figure

$$\frac{h_{C\supset D}(P_i')}{\Pi \to C \supset D}.$$

We note that Q_i is a cut-free proof figure satisfying $\#_I(P_i) > \#_I(Q_i)$ and $dep_I(P_i) \geq dep_I(Q_i)$. Let Q be a figure obtained from P by replacing P_i by Q_i. Then Q is a cut-free proof figure for the end sequent of P satisfying $\#_I(P) > \#_I(Q)$ and $dep_I(P) \geq dep_I(Q)$. By the induction hypothesis, we obtain the lemma. $\dashv$

4.4.18. Lemma. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in* **GFPL***. *Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in* **GFPL**$^+$.

Proof. By replacing each inference rule

$$\frac{A, \Gamma^+ \to B}{\Gamma \to A \supset B}$$

by

$$\cfrac{\cfrac{A, \Gamma^+ \to B}{A, A \supset B, \Gamma^+ \to B}}{\Gamma \to A \supset B}$$

we obtain a cut-free proof figure in **GFPL**$^+$. $\dashv$

Proof of Theorem 4.4.4. Suppose that $\Gamma \to \Delta \in$ **GFPL**$^+$. Using Lemma 4.4.3, we have

$$\Gamma \to \Delta \in \mathbf{GVPL}^+ + \top^{2n+3} L(p),$$

where n is the number of elements in $\{A \supset B \mid A \supset B \in \mathsf{Sub}^+(\Gamma \to \Delta)\}$. So, there exist formulas $A_1, \cdots, A_m$ such that

$$\top^{2n+3} L(A_1), \cdots, \top^{2n+3} L(A_m), \Gamma \to \Delta \in \mathbf{GVPL}^+.$$

Using Theorem 2.2.6, there exists a cut-free proof figure P for the sequent above in **GVPL**$^+$. It is easily seen that P is also a proof figure in **GFPL***. Using Lemma 4.4.17, there exists a cut-free proof figure Q for

$$\Gamma \to \Delta$$

in **GFPL***. Using Lemma 4.4.18, we obtain the theorem. $\dashv$

4.5 Other results

In this section, we show some other results concerning $\mathcal{R}$.

We say that a consequence relation $\vdash$ has the disjunction property if

$$\emptyset \vdash A \vee B \text{ implies either } \emptyset \vdash A \text{ or } \emptyset \vdash B$$

(cf. [CZ97]).

4.5.1. PROPOSITION. *Every consequence relation in $\mathcal{R}$ has the disjunction property.*

Proof. By Corollary 2.2.12, it was proved that for any formula $C \supset D$, $\vdash_{V+C\supset D}$ has disjunction property. So, using Corollary 4.1.9, we obtain the proposition. ⊣

By a superintuitionistic logic, we mean a set of formulas containing intuitionistic propositional logic closed under modus ponens and substitution. We also consider the cardinality of $\mathcal{R}$ by comparing it with the set $\mathcal{SI}$ of all the finite axiomatizable superintuitionistic logics.

4.5.2. LEMMA. *$\mathcal{R}$ is homomorphic to $\mathcal{SI}$.*

Proof. It suffices to provide an example of a homomorphism from $\mathcal{R}$ to $\mathcal{SI}$. We define a mapping f from $\mathcal{R}$ to $\mathcal{SI}$ as follows:

$$f(\vdash_{V+A}) = \vdash_{I+A} .$$

In other words,

$$f(\vdash_{V+A}) = \vdash_{V+A+\top\supset p/p} .$$

It is easily seen that

$$\vdash_1 \subseteq \vdash_2 \text{ implies } f(\vdash_1) \subseteq f(\vdash_2)$$

and so, we confirm that f is a mapping from $\mathcal{R}$ to $\mathcal{SI}$. Hence, all we have to do is to show that f is a surjection. Since

$$((\top \supset A) \supset A)^+, L(A) \to A \in \mathbf{GVPL}^+,$$

we have

$$\vdash_{I+A} = \vdash_{I+L(A)} .$$

So,

$$f(\vdash_{V+L(A)}) = \vdash_{I+L(A)} = \vdash_{I+A} .$$

Since $\vdash_{V+L(A)} \in \mathcal{R}$, f is a surjection. ⊣

4.5.3. PROPOSITION. *There are infinitely many consequence relations in $\mathcal{R}$.*

Proof. It is known that there are infinitely many finitely axiomatizable super-intuitionistic logics(cf. [CZ97]). So, by Lemma 4.5.2, we obtain the proposition. $\dashv$

Also we have the following result.

4.5.4. PROPOSITION. *Let it be that*

$$\mathcal{IF} = \{\vdash \mid \vdash = \vdash_{V+A/B} \textit{ for some implication free formula } B\}.$$

Then

$$\mathcal{R} \cap \mathcal{IF} = \{\vdash_V\}.$$

Proof. It is easily seen that $\{\vdash_V\} \subseteq \mathcal{R} \cap \mathcal{IF}$. So, we have only to show $\{\vdash_V\} \supseteq \mathcal{R} \cap \mathcal{IF}$. Suppose that $\vdash_{V+A/B} \in \mathcal{R} \cap \mathcal{IF}$. By Theorem 4.1.4 and Theorem 4.2.1, we have

$$\vdash_{V+A/B} = \vdash_{V+A\supset B} \subseteq \vdash_F .$$

So, $\{A\} \vdash_F B$. Using Corollary 2.4.4, $\{A\} \vdash_V B$, and so, $\vdash_{V+A/B} = \vdash_{V+A\supset B} \subseteq \vdash_V$. $\dashv$

4.6 Corresponding results in modal logics

In this section, we extend the results in section 4.3 to normal modal logics. Results in the other previous sections in this chapter can also be extended in a similar way. The modal operator is denoted by $\Box$ (necessity). Modal formulas are defined, as usual. If there is no confusion, we simply call them formulas. A normal modal logic is a set of formulas containing all the tautologies of classical logic and

$$\Box(p \supset q) \supset (\Box p \supset \Box q),$$

which is closed under modus ponens, substitution and necessitation,

$$\frac{A}{\Box A}.$$

By **K**, we mean the smallest normal modal logic. Let **L** be a normal modal logic. The expression $\mathbf{L} + A$ denotes the closure under modus ponens, substitution and necessitation of $\mathbf{L} \cup \{A\}$. The normal modal logics **K4** and **GL** are defined as follows:

$$\mathbf{K4} = \mathbf{K} + \Box p \supset \Box\Box p \text{ and } \mathbf{GL} = \mathbf{K4} + L^{\Box}(p).$$

where $L^{\Box}(p) = \Box(\Box p \supset p) \supset \Box p$.

A Kripke frame for the modal language is a pair $\langle W, R\rangle$, in which R is a binary relation on a set $W \neq \emptyset$. A Kripke model for the modal language is a triple $M = \langle W, R, P\rangle$, where $\langle W, R\rangle$ is a Kripke frame and P is a mapping from the set of all propositional variables to the set 2^W. The truth valuation $\models$ differs from that for the non-modal propositional language in the following respects: (K5) in section 3.1 is replaced by

(K5)′ $(M, \alpha) \models A \supset B$ iff $(M, \alpha) \models A$ implies $(M, \alpha) \models B$,

and we add the condition

(K6) $(M, \alpha) \models \Box A$ iff for any $\beta \in \alpha\!\uparrow, (M, \beta) \models A$.

Similarly to the non-modal case, we use the expression $M \models A$.

4.6.1. Lemma. (cf. [CZ97])

$A \in \mathbf{K}$ *iff for any Kripke model M, $M \models A$,*

$A \in \mathbf{K4}$ *iff for any transitive Kripke model M, $M \models A$,*

$A \in \mathbf{GL}$ *iff for any finite irreflexive transitive Kripke model M, $M \models A$.*

Now, we consider the set:

$$\mathcal{ML}/\mathbf{L}_0 = \{\mathbf{L} \mid \mathbf{L} = \mathbf{L}_0 + A = \mathbf{L}_0 + \Box A, \text{ for some formula } A\},$$

which corresponds to $\mathcal{R}$ if $\mathbf{L}_0 = \mathbf{K4}$. Also the following lemma can be proved similarly to the proof of Lemma 4.2.4.

4.6.2. Lemma. *Let $\mathbf{L}_0$ be a normal modal logic contained in* $\mathbf{GL}$. *Then*

$$\mathbf{L}_0 + A \in \mathcal{ML}/\mathbf{L}_0 \textit{ implies } \mathbf{L}_0 + A \subseteq \mathbf{GL}.$$

4.6.3. Theorem. $\mathbf{GL}$ *is the maximal modal logic in* $\mathcal{ML}/\mathbf{K4}$.

Proof. By Lemma 4.6.2, it is sufficient to prove $\mathbf{GL} \in \mathcal{ML}/\mathbf{K4}$. It is easily seen that for any transitive Kripke frame M, $M \models \Box L^{\Box}(p) \supset L^{\Box}(p)$. So, by Lemma 4.6.1, we have $\Box L^{\Box}(p) \supset L^{\Box}(p) \in \mathbf{K4}$. Using modus ponens, $L^{\Box}(p) \in \mathbf{K4} + \Box L^{\Box}(p)$. By necessitation, we also have $\Box L^{\Box}(p) \in \mathbf{K4} + L^{\Box}(p)$. Hence, we obtain the theorem.
$\dashv$

4.6.4. Corollary. (1) $\mathcal{ML}/\mathbf{K4} \subseteq \{\mathbf{L} \mid \mathbf{K4} \subseteq \mathbf{L} \subseteq \mathbf{GL}\}$,

(2) $\min \mathcal{ML}/\mathbf{K4} = \mathbf{K4}$,

(3) $\max \mathcal{ML}/\mathbf{K4} = \mathbf{GL}$.

However $\mathbf{GL}$ is not the maximal modal logic in $\mathcal{ML}/\mathbf{K}$, since $\mathbf{GL} \notin \mathcal{ML}/\mathbf{K}$.

Chapter 5

Disjunction free formulas in propositional lax logic

The logic treated here is the intuitionistic modal logic obtained from the smallest intuitionistic modal logic **IntK** by adding the axioms $T_c : p \supset \Box p$ and $4_c : \Box\Box p \supset \Box p$. This logic is called *propositional lax logic* (**PLL**) in [FM95]. We discuss the set $\mathcal{A}$ of formulas constructed from the propositional variables $p_1, \cdots, p_n$ and $\perp$ using $\wedge$, $\supset$ and a unary modal operator in **PLL**.

The set of these non-modal formulas in $\mathcal{A}$ was first considered in Diego [Die66] in intuitionistic propositional logic (**IPL**). He showed that the set of these non-modal formulas contains only finitely many equivalence classes (modulo intuitionistic provability). Urquhart [Urq74], de Bruijn [Bru75], Hendriks [Hen96] and Sasaki [Sas97a] gave a more precise description of this set.

Since the non-modal fragment of **PLL** is **IPL**, the results in the papers just mentioned are useful for our investigations, especially the exact models introduced in [Bru75]. With the help of exact models we can elucidate the structure of the set.

In section 5.1, we introduce exact models for fragments of **IPL**. In section 5.2, we define propositional lax logic and show some useful lemmas. Section 5.3 is devoted to giving the structure *Exm*, and the following three sections to proving that *Exm* is the exact model for our fragment of **PLL**. A method how to construct *Exm* will be clarified in section 5.7. Normal forms in the fragment are given in section 5.8 and we show what kind of modal formulas do we need to express such forms.

5.1 Exact models in IPL

In this section, we explain the notion of an exact model, and how it may be used to investigate disjunction free fragments with only finitely many propositional variables.

By $[\wedge, \supset, \bot]^n$, we mean the set of formulas constructed from the propositional variables $p_1, \cdots, p_n$ and $\bot$ using $\wedge$ and $\supset$. We write $A \equiv_L B$ if both of $A \supset B$ and $B \supset A$ are provable in a logic **L**. Then the purpose to clarify the fragment $[\wedge, \supset, \bot]^n$ in **IPL** is accomplished by investigating the following ordered set:

$$([\wedge, \supset, \bot]^n / \equiv_{IPL}, \leq_{IPL}),$$

where $[A] \leq_{IPL} [B]$ means that $A' \supset B' \in$ **IPL** for some $A' \in [A]$ and $B' \in [B]$.

An exact model introduced here corresponds to the ordered set $([\wedge, \supset, \bot]^n / \equiv_{IPL}, \leq_{IPL})$.

5.1.1. Definition. Let $\langle W, \leq \rangle$ be a finite partially ordered set.

(1) For a subset S of W, $\mathbf{Maxl}(S)$ denotes the set of maximal elements of S and $\mathbf{Minl}(S)$ denotes the set of minimal elements of S.

(2) For an element $\alpha \in W$, we put $\alpha\!\uparrow = \{\beta \in W \mid \alpha \leq \beta\}$.

(3) For any elements $\alpha, \beta \in W$, we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$, and we write $\alpha <_1 \beta$ if $\beta \in \mathbf{Minl}(\alpha\!\uparrow - \{\alpha\})$.

(4) A subset $W' \subseteq W$ is called *closed* if for any $\alpha, \beta \in W$,

$$\alpha \in W \text{ and } \alpha \leq \beta \text{ implies } \beta \in W.$$

By $\mathcal{P}^*(W)$, we mean the set of all closed subsets of W.

(5) The depth of a world $\alpha \in W$, write $\delta(\alpha)$, is defiend as follows:

$$\delta(\alpha) = \max(\{0\} \cup \{\delta(\beta) \mid \alpha < \beta\}) + 1.$$

Note that

$$\alpha < \beta \text{ implies } \delta(\alpha) > \delta(\beta),$$
$$\alpha \leq \beta \text{ iff } \beta \in \alpha\!\uparrow,$$
$$\alpha < \beta \text{ iff } \beta \in \alpha\!\uparrow - \{\alpha\},$$
$$\alpha <_1 \beta \text{ iff } \beta \in \mathbf{Minl}(\alpha\!\uparrow - \{\alpha\}).$$

5.1.2. Definition. A Kripke model for **IPL** is a structure $\langle W, \leq, P \rangle$, where $\langle W, \leq \rangle$ is a partially ordered set and P is a mapping from the set of propositional variables to $\mathcal{P}^*(W)$.

The truth valuation $\models$ for the non-modal propositional language is defined by the conditions (K1),(K2),(K3),(K4) and (K5) in section 3.1, but here we use $\leq$ instead of R. Using this valuation we extend the mapping P to the set of formulas as follows.

$$P(A) = \{\alpha \mid (M, \alpha) \models A\}.$$

It is known that $P(A) \in \mathcal{P}^*(W)$ and the following completeness for finite Kripke models (cf. [CZ97]).

5.1.3. Lemma. *$A \in \mathbf{IPL}$ iff $M \models A$ for every finite Kripke model M.*

5.1.4. Definition. A Kripke model $\langle W, \leq, P\rangle$ is said to be *exact* for a fragment F in **IPL** if the following two conditions hold:

(1) P maps F onto $\mathcal{P}^*(W)$,

(2) $A \supset B \in \mathbf{IPL}$ if $P(A) \subseteq P(B)$.

For the brevity's sake, an exact Kripke model is said to be an exact model. Note that the converse of (2) of the condition above follows from Lemma 5.1.3. Hence

5.1.5. Corollary. *For an exact model $\langle W, \leq, P\rangle$ for a fragment F in* **IPL**,

$$A \supset B \in \mathbf{IPL} \text{ iff } P(A) \subseteq P(B).$$

5.1.6. Lemma. *Let $M = \langle W, \leq, P\rangle$ be an exact model for a fragment F in* **IPL**. *Then $\langle \mathcal{P}^*(W), \subseteq\rangle$ is isomorphic to $\langle F/\equiv_{IPL}, \leq_{IPL}\rangle$.*

Proof. By Corollary 5.1.5, we have $P(A) = P(B)$ for each $B \in [A]$. So, we can define an one-to-one mapping f from $F/\equiv_{IPL}$ onto $\mathcal{P}^*(W)$ as follows:

$$f([A]) = P(A).$$

Again using Corollary 5.1.5 and the equivalence between $[A] \leq_{IPL} [B]$ and $A \supset B \in \mathbf{IPL}$, f is an isomorphism. $\dashv$

Hence by investigating the structure of the exact model for $[\wedge, \supset, \perp]^n$ in **IPL**, we can obtain information on

$$([\wedge, \supset, \perp]^n/\equiv_{IPL}, \leq_{IPL}).$$

5.2 Propositional lax logic

In this section, we introduce an intuitionistic modal logic, which was called *propositional lax logic* (**PLL**) in [FM95], and show some results shown in some previous papers about the logic. The logic contains the axiom $p \supset \Box p$, which is typical axiom for the modality of the possibility, while $\Box$ is often used as the modality of the necessity. Using $\Box$ as symbol for the modal operator might cause confusion, hence we follow [FM95] and write $\bigcirc$.

By an *atomic formula*, we mean a propositional variable or $\bot$. We use lower case Latin letters a, b, c, possibly with suffixes, for atomic formulas. Formulas are constructed, as usual, from atomic formulas using logical connectives $\wedge$, $\vee$, $\supset$ and $\bigcirc$. In particular, a formula of the form $\bigcirc A$ is called a *circled* formula.

5.2.1. DEFINITION. *The propositional lax logic* (**PLL**) is the smallest set of formulas containing all the theorems in **IPL** and the axioms

$K' : (p \supset q) \supset (\bigcirc p \supset \bigcirc q)$,

$T_c : p \supset \bigcirc p$,

$4_c : \bigcirc \bigcirc p \supset \bigcirc p$

and closed under modus ponens and substitution.

By T_c and modus ponens, we note that **PLL** is closed under the rule

$$\frac{A}{\bigcirc A},$$

which is indispensable in normal modal logics.

Similarly to section 5.1, an exact model for a fragment F in **PLL** are defined as follows.

5.2.2. DEFINITION. *A Kripke* **IM**-*model* is defined as a structure $\langle W, \leq, R, P\rangle$, where

(1) $\langle W, \leq\rangle$ is a partially ordered set,

(2) R is a binary relation on W such that $\leq \circ R = R$,

(3) P is a mapping from the set of propositional variables to $\mathcal{P}^*(W)$.

The truth valuation $\models$ for the non-modal propositional language is defined by the conditions in section 5.1, and that for the modal language we add the condition

(K6)$'$ $(M, \alpha) \models \bigcirc A$ iff for each $\beta \in \{\gamma \mid \alpha R \gamma\}, (M, \beta) \models A$.

We use the expression $M \models A$ similarly to section 3.1 and section 4.6.

The following lemma was shown in [WZ97].

5.2.3. LEMMA. *Let* $M = \langle W, \leq, R, P\rangle$ *be a Kripke* **IM**-*model. Then* $\{\alpha \mid (M, \alpha) \models A\} \in \mathcal{P}^*(W)$.

Hence, we can extend the mapping P in a Kripke **IM**-model $M = \langle W, \leq, R, P\rangle$ to the set of formulas as follows:

$$P(A) = \{\alpha \mid (M, \alpha) \models A\}.$$

In [Gol81], a Kripke semantics for **PLL** is introduced as follows.

5.2.4. DEFINITION. *A Kripke* **IM**-*model* $\langle W, \leq, R, P\rangle$ is called a Kripke **PLL**-model if the following two conditions hold:
(1) $R \subseteq \leq$,
(2) R is *dense*, i.e., if $\alpha R \beta$, then $\alpha R \gamma$ and $\gamma R \beta$ for some $\gamma \in W$.

5.2.5. LEMMA. $A \in$ **PLL** *iff* $M \models A$ *for every finite Kripke* **PLL**-*model* M ([Gol81]).

5.2.6. DEFINITION. A Kripke **IM**-model $\langle W, \leq, R, P\rangle$ is said to be *exact* for a fragment F in **PLL** if the following two conditions hold:
(1) P maps F onto $\mathcal{P}^*(W)$,
(2) $A \supset B \in$ **PLL** if $Q(A) \subseteq Q(B)$.

Similarly to section 5.1, an exact Kripke **IM**-model is simply said to be an exact model. Also there holds the following corollary and lemma similarly to Corollary 5.1.5 and Lemma 5.1.6.

5.2.7. COROLLARY. *For an exact model* $\langle W, \leq, R, P\rangle$ *for a fragment* F *in* **PLL**,

$$A \supset B \in \mathbf{PLL} \textit{ iff } P(A) \subseteq P(B).$$

5.2.8. LEMMA. *Let* $\langle W, \leq, R, P\rangle$ *be an exact model for a fragment* F *in* **PLL**. *Then* $\langle \mathcal{P}^*(W), \subseteq\rangle$ *is isomorphic to* $\langle F/\equiv_{PLL}, \leq_{PLL}\rangle$, *where* $[A] \leq_{PLL} [B]$ *means that* $A' \supset B' \in$ **PLL** *for some* $A' \in [A]$ *and* $B' \in [B]$.

5.3 An exact model in PLL

By $\mathbf{atom}^n$, we mean the set $\{\bot, p_1, \cdots, p_n\}$. By $[\wedge, \supset, \bigcirc, \bot]^n$, we mean the set of formulas constructed from atomic formulas in $\mathbf{atom}^n$ using $\wedge$, $\supset$ and $\bigcirc$. In the following sections of this chapter, we only treat formulas in the fragment $[\wedge, \supset, \bigcirc, \bot]^n$. We use $\#(\mathbf{S})$ for the number of elements in a finite set $\mathbf{S}$. In this section, section 5.4, section 5.5 and section 5.6, we give the exact model for $[\wedge, \supset, \bigcirc, \bot]^n$ in **PLL**. We define a structure Exm in this section. The following three sections are devoted to proving that Exm is the exact model. As in [Hen96], we first define an A-independent world and its semantic type. Using them, we define a structure Exm.

5.3.1. NOTATION. Let $M = \langle W, \leq, R, P\rangle$ be a Kripke **IM**-model. For a world $\alpha \in W$, we put
$atom(\alpha) = \{a \in \mathbf{atom}^n \mid \alpha \in P(a)\}$,
$th(\alpha) = \{A \mid (M, \alpha) \models A\} \cap [\wedge, \supset, \bigcirc, \bot]^n$.
The expression

$$\bigcap_{\alpha<\beta} th(\beta)$$

denotes the set $[\wedge, \supset, \bigcirc, \bot]^n$ if $\delta(\alpha) = 1$.

5.3.2. DEFINITION. Let $M = \langle W, \leq, R, P\rangle$ be a finite Kripke **IM**-model.
(1) We say that a world $\alpha \in W$ is *A-independent* if

$$A \notin th(\alpha) \text{ and } A \in \bigcap_{\alpha<\beta} th(\beta).$$

(2) We say that a world $\alpha \in W$ is *$\cap$-independent* if there exists an atomic or circled formula A such that α is A-independent.
(3) We put $W^\cap = \{\beta \in W \mid \beta \text{ is } \cap\text{-independent }\}$.

5.3.3. DEFINITION. Let $M = \langle W, \leq, R, P\rangle$ be a finite Kripke **IM**-model and let α be a world in W. We define $\tau(\alpha)$, the semantic type of α, as follows:
(1) if α is reflexive, i.e., $\alpha R \alpha$, then

$$\tau(\alpha) = \langle atom(\alpha), \{\tau(\beta) \mid \alpha < \beta, \beta \in W^\cap\}, \circ\rangle,$$

(2) if α is irreflexive, i.e., α is not reflexive, then

$$\tau(\alpha) = \langle atom(\alpha), \{\tau(\beta) \mid \alpha < \beta, \beta \in W^\cap\}, \bullet\rangle.$$

By $\mathbf{T}$, we mean the set of all semantic types for $\cap$-independent worlds in finite Kripke **PLL**-models.

5.3.4. NOTATION. For a triple $t = \langle e_1, e_2, e_3 \rangle$, $1st(t)$ denotes e_1, and similarly, $2nd(t)$ denotes e_2 and $3rd(t)$, e_3.

The symbols $\circ$ and $\bullet$ are intended to express the reflexivity and the irreflexivity of a world α, respectively. Clearly,

$$3rd(\tau(\alpha)) = \circ \text{ iff } \alpha \text{ is reflexive}$$

and

$$3rd(\tau(\alpha)) = \bullet \text{ iff } \alpha \text{ is irreflexive.}$$

5.3.5. DEFINITION. Let t_1 and t_2 be semantic types of some $\cap$-independent worlds. We write $t_1 \leq t_2$ if either $t_1 = t_2$ or $t_2 \in 2nd(t_1)$. Also we write $t_1 R t_2$ if there exists a semantic type t_3 such that $t_1 \leq t_3 \leq t_2$ and $3rd(t_3) = \circ$. We put $P^t(p) = \{t \in \mathbf{T} \mid p \in 1st(t)\}$.

5.3.6. REMARK. Let t_1 and t_2 be semantic types in $\mathbf{T}$. If $(3rd(t_1), 3rd(t_2)) \neq (\bullet, \bullet)$, then

$$t_1 R t_2 \text{ iff } t_1 \leq t_2.$$

To find the exact model for $[\wedge, \supset, \bigcirc, \perp]^n$, we first define a Kripke **IM**-model $\langle W, \leq, R, P \rangle$, and then, prove that $\langle W, \leq, R, P \rangle$ is the exact model for the fragment.

5.3.7. THEOREM. *The structure $\langle \mathbf{T}, \leq, R, P^t \rangle$ is a Kripke* **IM**-*model.*

The proof of the theorem needs some lemmas.

5.3.8. LEMMA. *Let α be a world in a finite Kripke* **PLL**-*model $\langle W, \leq, R, P \rangle$. If $A \notin th(\alpha)$, then there exists an A-independent world $\alpha_1 \in \alpha\!\uparrow$.*

Proof. We use an induction on $\delta(\alpha)$ in the ordered set $\langle W, \leq \rangle$. If $A \in \bigcap_{\alpha<\beta} th(\beta)$, then α is A-independent. Suppose that $A \notin \bigcap_{\alpha<\beta} th(\beta)$ and the lemma holds for any α^* such that $\delta(\alpha^*) < \delta(\alpha)$. Then by $A \notin \bigcap_{\alpha<\beta} th(\beta)$, there exists a world $\alpha_1 \in \alpha\!\uparrow - \{\alpha\}$ such that $A \notin th(\alpha_1)$. Using the induction hypothesis, we obtain the lemma. $\dashv$

5.3.9. Lemma. *Let α be a world in a finite Kripke* **PLL**-*model* $\langle W, \leq, R, P\rangle$.

(1) *if α is $B\wedge C$-independent, then it is either B-independent or C-independent,*

(2) *if α is $B \supset C$-independent, then it is C-independent and $B \in th(\alpha)$,*

(3) *if α is $\bigcirc B$-independent, then there exists B-independent world $\alpha_1 \in \alpha\!\uparrow$.*

Proof. For (1): Since α is $B \wedge C$-independent, we have

$$B \wedge C \notin th(\alpha) \text{ and } B \wedge C \in \bigcap_{\alpha<\beta} th(\beta).$$

By $B\wedge C \notin th(\alpha)$, we have either $B \notin th(\alpha)$ or $C \notin th(\alpha)$. By $B\wedge C \in \bigcap_{\alpha<\beta} th(\beta)$, we have $B \in \bigcap_{\alpha<\beta} th(\beta)$ and $C \in \bigcap_{\alpha<\beta} th(\beta)$. Hence α is either B-independent or C-independent.

For (2): Since α is $B \supset C$-independent, we have

$$B \supset C \notin th(\alpha) \text{ and } B \supset C \in \bigcap_{\alpha<\beta} th(\beta).$$

By $B \supset C \notin th(\alpha)$, there exists $\alpha_1 \in \alpha\!\uparrow$ such that $B \in th(\alpha_1)$ and $C \notin th(\alpha_1)$. By $C \notin th(\alpha_1)$ and Lemma 5.3.8, there exists C-independent world $\alpha_2 \in \alpha_1\!\uparrow$. Using $B \in th(\alpha_1)$, we have $B \in th(\alpha_2)$. If $\alpha = \alpha_2$, then we obtain (2). If $\alpha < \alpha_2$, then by $B \supset C \in \bigcap_{\alpha<\beta} th(\beta)$, we have $B \supset C \in th(\alpha_2)$, and thereby, $C \in th(\alpha_2)$. This is a contradiction.

For (3): Since α is $\bigcirc B$-independent, we have $\bigcirc B \notin th(\alpha)$. So, there exists $\alpha_1 \in \{\beta \mid \alpha R \beta\} \subseteq \alpha\!\uparrow$ such that $B \notin th(\alpha_1)$. Using Lemma 5.3.8, we obtain (3). ⊣

5.3.10. Corollary. *Let α be a world in a finite Kripke* **PLL**-*model* $\langle W,\leq,R,P\rangle$.

(1) *If α is A-independent for some formula A, then α is B-independent for some atomic or circled formula $B \in \mathsf{Sub}(A)$, and hence, it is $\cap$-independent.*

(2) *If $A \notin th(\alpha)$, then there exists an A-independent world $\alpha_1 \in \alpha\!\uparrow \cap W^{\cap}$.*

Proof. For (1): We use an induction on A. If A is either an atomic formula or a circled formula, then (1) holds. Suppose that A is neither an atomic formula nor a circled formula and (1) holds for any proper subformula of A. Then either $A = C\wedge D$ or $A = C \supset D$. By Lemma 5.3.9(1), Lemma 5.3.9(2) and the induction hypothesis, we obtain (1).

(2) follows from (1) and Lemma 5.3.8. ⊣

5.3.11. Lemma. *Let α be a $\bigcirc A$-independent world in a finite Kripke* **PLL**-*model* $M = \langle W,\leq,R,P\rangle$. *Then α is reflexive.*

Proof. Suppose that α is $\bigcirc A$-independent. Then

$$\bigcirc A \notin th(\alpha) \text{ and } \bigcirc A \in \bigcap_{\alpha<\beta} th(\beta).$$

By $\bigcirc A \notin th(\alpha)$, there exists a world β such that $\alpha R\beta$ and $A \notin th(\beta)$. By $\alpha R\beta$ and density of R, $\alpha R\gamma R\beta$ for some world γ. If $\gamma = \alpha$, then the lemma is trivial. Assume that $\gamma \neq \alpha$. Using $R \subseteq \leq$, we have $\alpha < \gamma$. Using $\bigcirc A \in \bigcap_{\alpha<\beta} th(\beta)$, we have $\bigcirc A \in th(\gamma)$. On the other hand, by $\gamma R\beta$ and $A \notin th(\beta)$, we have $\bigcirc A \notin th(\gamma)$. This is a contradiction. $\dashv$

5.3.12. LEMMA. *Let α and β be $\cap$-independent worlds in finite Kripke* **PLL**-*models. If $\tau(\alpha) = \tau(\beta)$, then $th(\alpha) = th(\beta)$.*

Proof. Suppose that $\tau(\alpha) = \tau(\beta)$. It is sufficient to show that for each A,

$$A \in th(\alpha) \text{ iff } A \in th(\beta). \tag{5.1}$$

To show (5.1), we use an induction on A.

If A is an atomic formula, then (5.1) follows from

$$atom(\alpha) = 1st(\tau(\alpha)) = 1st(\tau(\beta)) = atom(\beta).$$

Suppose that A is not atomic formula and (5.1) holds for any proper subformula of A. We only show the "if" part since the "only if" part can be shown similarly. We divide into the following cases.

(i) The case that $A = B \wedge C$: Suppose that $A \notin th(\alpha)$. Then we have either $B \notin th(\alpha)$ or $C \notin th(\alpha)$. Using the induction hypothesis, we have either $B \notin th(\beta)$ or $C \notin th(\beta)$, and thereby, $A \notin th(\beta)$.

(ii) The case that $A = B \supset C$: Suppose that $A \notin th(\alpha)$. Then by Corollary 5.3.10(2) and Lemma 5.3.9(2), there exists a world $\alpha_1 \in \alpha\!\uparrow \cap W^{\cap}$ such that $B \in th(\alpha_1)$ and $C \notin th(\alpha_1)$.

If $\alpha = \alpha_1$, then $B \in th(\alpha)$ and $C \notin th(\alpha)$. Using the induction hypothesis, $B \in th(\beta)$ and $C \notin th(\beta)$, and hence $A = B \supset C \notin th(\beta)$.

If $\alpha < \alpha_1$, then $\tau(\alpha_1) \in 2nd(\tau(\alpha)) = 2nd(\tau(\beta))$. So, there exists $\cap$-independent world $\beta_1 \in \beta\!\uparrow - \{\beta\}$ such that $\tau(\alpha_1) = \tau(\beta_1)$. Using the induction hypothesis, we have $B \in th(\beta_1)$ and $C \notin th(\beta_1)$. Since $\beta < \beta_1$, we obtain $A = B \supset C \notin th(\beta)$.

(iii) The case that $A = \bigcirc B$: Suppose that $A \notin th(\alpha)$. Then by Corollary 5.3.10(2), there exists a $\bigcirc B$-independent world $\alpha_1 \in \alpha\!\uparrow \cap W^{\cap}$. Using Lemma 5.3.11, α_1 is reflexive, i.e., $3rd(\tau(\alpha_1)) = \circ$. Since α_1 is a world in a Kripke **PLL**-model, $B \supset \bigcirc B \in th(\alpha_1)$, and thereby, $B \notin th(\alpha_1)$.

If $\alpha = \alpha_1$, then by the induction hypothesis, $B \notin th(\beta)$. Also

$$\circ = 3rd(\tau(\alpha_1)) = 3rd(\tau(\alpha)) = 3rd(\tau(\beta)),$$

it means β is reflexive. Hence $\bigcirc B = A \notin th(\beta)$.

If $\alpha < \alpha_1$, then $\tau(\alpha_1) \in 2nd(\tau(\alpha)) = 2nd(\tau(\beta))$. So, there exists a $\cap$-independent world $\beta_1 \in \beta\!\uparrow - \{\beta\}$ such that $\tau(\alpha_1) = \tau(\beta_1)$. By $B \notin th(\alpha_1)$ and the induction hypothesis, we have $B \notin th(\beta_1)$. By $\tau(\alpha_1) = \tau(\beta_1)$,

$$\circ = 3rd(\tau(\alpha_1)) = 3rd(\tau(\beta_1)),$$

it means β_1 is reflexive. Using $B \notin th(\beta_1)$, we have $\bigcirc B = A \notin th(\beta_1)$. Using $\beta < \beta_1$, we obtain $\bigcirc B = A \notin th(\beta)$. $\dashv$

5.3.13. LEMMA. *Let α and β be $\cap$-independent worlds in finite Kripke* **PLL**-*models. If $\tau(\alpha) \leq \tau(\beta)$, then there exists a $\cap$-independent world $\alpha_1 \in \alpha\!\uparrow$ such that $\tau(\alpha_1) = \tau(\beta)$.*

Proof. If $\tau(\alpha) = \tau(\beta)$, then the lemma is trivial. So, we assume that $\tau(\beta) \in 2nd(\tau(\alpha))$. Then there exists a $\cap$-independent world $\alpha_1 \in \alpha\!\uparrow - \{\alpha\}$ such that $\tau(\beta) = \tau(\alpha_1)$. $\dashv$

5.3.14. LEMMA. *Let α and β be $\cap$-independent worlds in finite Kripke* **PLL**-*models. If $\tau(\alpha) \leq \tau(\beta)$, then $th(\alpha) \subseteq th(\beta)$.*

Proof. By Lemma 5.3.13, there exists a $\cap$-independent world $\alpha_1 \in \alpha\!\uparrow$ such that $\tau(\alpha_1) = \tau(\beta)$. By Lemma 5.2.3 and Lemma 5.3.12, we obtain $th(\alpha) \subseteq th(\alpha_1) = th(\beta)$. $\dashv$

5.3.15. LEMMA. *Let $M = \langle W, \leq, R, P\rangle$ be a finite Kripke* **PLL**-*model and let α be an A-independent world. Then either one of the following two holds:*

(1) *α is a-independent for an atomic formula a,*

(2) *α is $\bigcirc B$-independent for a circled formula $\bigcirc B$ such that*

$$B \notin \bigcap_{\beta \in W^{\cap}, \tau(\beta) \in 2nd(\tau(\alpha))} th(\beta).$$

Proof. We use an induction on A. If A is an atomic formula, then (1) holds. Suppose that A is not an atomic formula and the lemma holds for any proper subformula of A. If either $A = C \wedge D$ or $A = C \supset D$, then by Corollary 5.3.10(1) and the induction hypothesis, we obtain the lemma. So, we assume that $A = \bigcirc C$. Then by Lemma 5.3.9(3), there exists C-independent world $\alpha_1 \in \alpha\!\uparrow \cap W^{\cap}$. Note that $C \notin th(\alpha)$. If $\alpha < \alpha_1$, then we have $\tau(\alpha_1) \in 2nd(\tau(\alpha))$. Hence we obtain (2). If $\alpha = \alpha_1$, then by the induction hypothesis, we obtain the lemma. $\dashv$

5.3.16. LEMMA. *For any semantic type $t \in \mathbf{T}$, $t \notin 2nd(t)$.*

Proof. Suppose that $t \in \mathbf{T}$ and $t \in 2nd(t)$. By $t \in \mathbf{T}$, there exists a $\cap$-independent world α in a finite Kripke **PLL**-model $M = \langle W, \leq, R, P\rangle$ such that $t = \tau(\alpha)$. By $t \in 2nd(t) = 2nd(\tau(\alpha))$, there exists a $\cap$-independent world $\beta \in \alpha\!\uparrow - \{\alpha\}$ such that $t = \tau(\beta)$.

If α is a-independent for some atomic formula a, then $a \notin atom(\alpha)$ and $a \in atom(\beta)$. So, we have $a \notin 1st(\tau(\alpha)) = 1st(t)$ and $a \in 1st(\tau(\beta)) = 1st(t)$. This is a contradiction.

If α is not a-independent for any atomic formula a, then by Lemma 5.3.15, α is $\bigcirc B$-independent and

$$B \notin \bigcap_{\alpha_1 \in W^{\cap}, \tau(\alpha_1) \in 2nd(\tau(\alpha))} th(\alpha_1).$$

for some B. By the $\bigcirc B$-independency of α and $\alpha < \beta$, we have $\bigcirc B \in th(\beta)$. By the $\bigcirc B$-independency of α and Lemma 5.3.11, α is reflexive, i.e., $3rd(\tau(\alpha)) = \circ$. So, $3rd(\tau(\beta)) = 3rd(t) = 3rd(\tau(\alpha)) = \circ$. Hence β is also reflexive. Using $\bigcirc B \in th(\beta)$, we have $B \in th(\beta)$.

On the other hand, by $B \notin \bigcap_{\alpha_1 \in W^{\cap}, \tau(\alpha_1) \in 2nd(\tau(\alpha))} th(\alpha_1)$, there exists a world $\alpha_1 \in W^{\cap}$ such that $B \notin th(\alpha_1)$ and $\tau(\alpha_1) \in 2nd(\tau(\alpha))$. By $\tau(\alpha_1) \in 2nd(\tau(\alpha)) = 2nd(t) = 2nd(\tau(\beta))$, we have $\tau(\beta) \leq \tau(\alpha_1)$. Using Lemma 5.3.14, $th(\beta) \subseteq th(\alpha_1)$. Since $B \in th(\beta)$, we have $B \in th(\alpha_1)$, but $B \notin th(\alpha_1)$. This is a contradiction. $\dashv$

5.3.17. LEMMA. *The structure $\langle \mathbf{T}, \leq \rangle$ is a partially ordered set.*

Proof. Let it be that $t_1, t_2, t_3 \in \mathbf{T}$. Then it is sufficient to show the following three,

(1) $t_1 \leq t_1$,
(2) $t_1 \leq t_2$ and $t_2 \leq t_3$ implies $t_1 \leq t_3$,
(3) $t_1 \leq t_2$ and $t_2 \leq t_1$ implies $t_1 = t_2$.

For (1): Trivial from the definition.

For (2): If either $t_1 = t_2$ or $t_2 = t_3$, then (2) is trivial. So, we assume that $t_2 \in 2nd(t_1)$ and $t_3 \in 2nd(t_2)$. By $t_1 \in \mathbf{T}$, there exists a $\cap$-independent world α for some finite Kripke **PLL**-model $\langle W, \leq, R, P\rangle$ such that $t_1 = \tau(\alpha)$. By $t_2 \in 2nd(t_1) = 2nd(\tau(\alpha))$, there exists $\beta \in W^{\cap}$ such that $t_2 = \tau(\beta)$ and $\alpha < \beta$. Using $t_3 \in 2nd(t_2) = 2nd(\tau(\beta))$, there exists $\gamma \in W^{\cap}$ such that $t_3 = \tau(\gamma)$ and $\beta < \gamma$. By $\alpha < \beta$ and $\beta < \gamma$, we have $\alpha < \gamma$, Using $t_3 = \tau(\gamma)$, we have $t_3 \in 2nd(\tau(\alpha)) = 2nd(t_1)$. Hence we have $t_1 \leq t_3$.

For (3): Suppose that $t_1 \leq t_2$, $t_2 \leq t_1$ and $t_1 \neq t_2$. Then we have $t_2 \in 2nd(t_1)$ and $t_1 \in 2nd(t_2)$. Similarly to the proof of (2), we have $t_1 \in 2nd(t_1)$. This is in contradiction with Lemma 5.3.16. ⊣

5.3.18. COROLLARY. *For any t and s in $\mathbf{T}$,*

$$t \leq s \text{ iff } s \in 2nd(t) \cup \{t\} = t\Uparrow,$$
$$t < s \text{ iff } s \in 2nd(t) = t\Uparrow - \{t\},$$
$$t <_1 s \text{ iff } s \in \mathbf{Minl}(2nd(t)).$$

Proof of Theorem 5.3.7. By Lemma 5.3.17, it is sufficient to show the following two:

(1) $t_1 \leq t_2$ and $t_2 R t_3$ implies $t_1 R t_3$, for each $t_1, t_2, t_3 \in \mathbf{T}$,
(2) $P^t(a) \in \mathcal{P}^*(\mathbf{T})$.

For (1): By $t_2 R t_3$ and definition of R, there exists $t_4 \in \mathbf{T}$ such that $t_2 \leq t_4 \leq t_3$ and $3rd(t_4) = \circ$. Using $t_1 \leq t_2$ and Lemma 5.3.17(2), we have $t_1 \leq t_4 \leq t_3$. Hence we obtain $t_1 R t_3$.

For (2): Suppose that $t_1 \in P^t(a)$ and $t_1 \leq t_2$. By $t_1 \in P^t(a)$, we have $t_1 \in \{t \mid a \in 1st(t)\}$, and so, $a \in 1st(t_1)$. By $t_1, t_2 \in \mathbf{T}$, there exist $\cap$-independent worlds α and β in some finite Kripke **PLL**-models such that $t_1 = \tau(\alpha)$ and $t_2 = \tau(\beta)$. By $\tau(\alpha) = t_1 \leq t_2 = \tau(\beta)$ and Lemma 5.3.14, $atom(\alpha) \subseteq atom(\beta)$, i.e., $1st(t_1) \subseteq 1st(t_2)$. Hence $a \in 1st(t_2)$, and thereby, $t_2 \in P^t(a)$. ⊣

By Lemma 5.2.3, we can extend the mapping P^t in a Kripke **IM**-model $\langle \mathbf{T}, \leq, R, P^t \rangle$ to $P^t : [\wedge, \supset, \bigcirc, \perp]^n \to \mathcal{P}^*(\mathbf{T})$ as follows.

$$P^t(A) = \{\alpha \in \mathbf{T} \mid (M, \alpha) \models A\}.$$

Now, we can define a structure that will be proved to be an exact model.

5.3.19. DEFINITION. $Exm = \langle \mathbf{T}, \leq, R, P^t \rangle$.

The main theorem in this chapter is

5.3.20. THEOREM. *Exm is a finite exact model for $[\wedge, \supset, \bigcirc, \perp]^n$ in* **PLL**.

To prove the theorem above, it is sufficient to show the following three:
(finiteness) Exm is finite, i.e., $\mathbf{T}$ is finite,
(soundness and completeness) $A \supset B \in$ **PLL** iff $P^t(A) \subseteq P^t(B)$,
(exactness) P^t maps $[\wedge, \supset, \bigcirc, \perp]^n$ onto $\mathcal{P}^*(\mathbf{T})$.

The following three sections are devoted to showing the three conditions above. In section 5.7 and section 5.8, we investigate the exact model Exm in detail.

5.4 Soundness and completeness of Exm

In this section, we prove the following theorem.

5.4.1. THEOREM. *$A \supset B \in$ **PLL** iff $P^t(A) \subseteq P^t(B)$.*

The proof of the theorem needs some lemmas.

5.4.2. LEMMA. *Exm is a Kripke **PLL**-model.*

Proof. It is sufficient to show that $R \subseteq \leq$ and the density of R. Suppose that t_1Rt_2. Then there exists t_3 such that $t_1 \leq t_3 \leq t_2$ and $3rd(t_3) = \circ$. It is easily seen that $t_1 \leq t_3 \leq t_3$ and $t_3 \leq t_3 \leq t_2$, Hence we have t_1Rt_3 and t_3Rt_2. Also by $t_1 \leq t_3 \leq t_2$ and Lemma 5.3.17, we obtain $t_1 \leq t_2$. ⊣

5.4.3. COROLLARY. *$A \supset B \in$ **PLL** implies $Exm \models A \supset B$.*

Proof. By Lemma 5.2.5 and Lemma 5.4.2. ⊣

5.4.4. COROLLARY. *$A \supset B \in$ **PLL** implies $P^t(A) \subseteq P^t(B)$.*

5.4.5. LEMMA. *Let t be a world in Exm and let α be a world in a finite Kripke **PLL**-model M. If $t = \tau(\alpha)$, then $th(t) = th(\alpha)$.*

Proof. It is sufficient to show that for any formula A,

$$A \in th(t) \text{ iff } A \in th(\alpha). \tag{5.2}$$

To show (5.2), we use an induction on A. If A is an atomic formula, then

$$atom(t) = \{a \mid t \in P^t(a)\} = \{a \mid t \in \{s \mid a \in 1st(s)\}\}$$

$$= \{a \mid a \in 1st(t)\} = 1st(t) = atom(\alpha).$$

Suppose that A is not an atomic formula and (5.2) holds for any proper subformula of A. We divide into the cases.

(i) The case that $A = B \wedge C$: Suppose that $B \wedge C \notin th(t)$. Then either $B \notin th(t)$ or $C \notin th(t)$. Using the induction hypothesis, $B \notin th(\alpha)$ or $C \notin th(\alpha)$. Hence $B \wedge C \notin th(\alpha)$.

Suppose that $B \wedge C \notin th(\alpha)$. Then by Lemma 5.3.10(2), there exists a $B \wedge C$-independent world $\alpha_1 \in \alpha\!\uparrow$. We note $t \leq \tau(\alpha_1)$ and $B \wedge C \notin th(\alpha_1)$. By $B \wedge C \notin th(\alpha_1)$, we have either $B \notin th(\alpha_1)$ or $C \notin th(\alpha_1)$. Using the induction hypothesis, either $B \notin th(\tau(\alpha_1))$ or $C \notin th(\tau(\alpha_1))$. Hence, $B \wedge C \notin th(\tau(\alpha_1))$. Using $t \leq \tau(\alpha_1)$, we obtain $B \wedge C \notin th(t)$.

(ii) The case that $A = B \supset C$: Suppose that $B \supset C \notin th(t)$. Then there exists $t_1 \in t\!\Uparrow$ such that $B \in th(t_1)$ and $C \notin th(t_1)$. By $t_1 \in t\!\Uparrow$ and Lemma 5.3.13, there exists a $\cap$-independent world $\alpha_1 \in \alpha\!\uparrow$ such that $t_1 = \tau(\alpha_1)$. Using the induction hypothesis, $B \in th(\alpha_1)$ and $C \notin th(\alpha_1)$. Hence $B \supset C \notin th(\alpha)$.

Suppose that $B \supset C \notin th(\alpha)$. Then by Lemma 5.3.10(2), there exists a $B \supset C$-independent world $\alpha_1 \in \alpha\!\uparrow$. Using Lemma 5.3.9(2), $B \in th(\alpha_1)$ and $C \notin th(\alpha_1)$. Using the induction hypothesis, $B \in th(\tau(\alpha_1))$ and $C \notin th(\tau(\alpha_1))$. Using $\tau(\alpha_1) \geq \tau(\alpha) = t$, we have $B \supset C \notin th(t)$.

(iii) The case that $A = \bigcirc B$: Suppose that $\bigcirc B \notin th(t)$. Then there exists t_1 such that $B \notin th(t_1)$ and tRt_1. By tRt_1, there exists a world t_2 such that $t \leq t_2 \leq t_1$ and $3rd(t_2) = \circ$. Using Lemma 5.3.13, there exists $\cap$-independent worlds $\alpha_2 \in \alpha\!\uparrow$ and $\alpha_1 \in \alpha_2\!\uparrow$ such that $t_1 = \tau(\alpha_1)$ and $t_2 = \tau(\alpha_2)$. Using the induction hypothesis, $B \notin th(\alpha_1)$. Using $\alpha_2 \leq \alpha_1$, $B \notin th(\alpha_2)$. Since $3rd(t_2) = 3rd(\tau(\alpha_2)) = \circ$, α_2 is reflexive, and hence, $\bigcirc B \notin th(\alpha_2)$. Using $\alpha \leq \alpha_2$, Hence $\bigcirc B \notin th(\alpha)$.

Suppose that $\bigcirc B \notin th(\alpha)$. by Lemma 5.3.10(2), there exists a $\bigcirc B$-independent world $\alpha_1 \in \alpha\!\uparrow$. Using Lemma 5.3.9(3), there exists a B-independent world $\alpha_2 \in \alpha\!\uparrow_1$. We note that $B \notin th(\alpha_2)$ and $t \leq \tau(\alpha_1) \leq \tau(\alpha_2)$. By $B \notin th(\alpha_2)$ and the induction hypothesis, we have $B \notin th(\tau(\alpha_2))$. On the other hand, by the $\bigcirc B$-independency of α_1 and Lemma 5.3.11, α_1 is reflexive, and hence, $3rd(\tau(\alpha_1)) = \circ$. Using $t \leq \tau(\alpha_1) \leq \tau(\alpha_2)$, we have $tR\tau(\alpha_2)$, Hence we obtain $\bigcirc B \notin th(t)$. $\dashv$

5.4.6. Lemma. *$A \notin$ **PLL** implies $Exm \not\models A$.*

Proof. Let it be that $A \notin$ **PLL**. Using Lemma 5.2.8, there exist a finite Kripke **PLL** model $M = \langle W, \leq, R, P\rangle$ and a world $\alpha \in W$ such that $(M, \alpha) \not\models A$, i.e., $A \notin th(\alpha)$. Using Lemma 5.3.10(2), there exists an A-independent world $\alpha_1 \in \alpha\!\uparrow \cap W^{\cap}$. By the A-independency of α_1, we have $A \notin th(\alpha_1)$.

The semantic type $\tau(\alpha_1)$ is a world in Exm, and by Lemma 5.4.5, $th(\tau(\alpha_1)) = th(\alpha_1)$. Hence $A \notin th(\tau(\alpha_1))$, i.e., $(Exm, \tau(\alpha_1)) \not\models A$. $\dashv$

5.4.7. Corollary. *$A \supset B \notin$ **PLL** implies $P^t(A) \not\subseteq P^t(B)$.*

Proof. Suppose that $A \supset B \notin$ **PLL**. Then by lemma 5.4.6, $Exm \not\models A \supset B$. So, there exists a world $t \in$ **T** such that $A \in th(t)$ and $B \notin th(t)$. Hence, $t \in P^t(A)$ and $t \notin P^t(B)$. Hence we obtain the lemma. $\dashv$

By Corollary 5.4.4 and Corollary 5.4.7, we obtain Theorem 5.4.1.

We also obtain the following lemmas, which is useful for the following sections.

5.4.8. LEMMA. *Let t_1, t_2 be semantic types in* $\mathbf{T}$.
(1) *If $t_2 \in 2nd(t_1)$, then $1st(t_1) \subseteq 1st(t_2)$.*
(2) *If $t_2 \in 2nd(t_1)$, then $2nd(t_2)$ is a proper subset of $2nd(t_1)$.*

Proof. By Lemma 5.3.14, we have (1). We show (2). Suppose that $s \in 2nd(t_2)$. Then we have $t_1 < t_2 < s$. So, $t_1 < s$, and hence $s \in 2nd(t_1)$.

On the other hand, by Lemma 5.3.16, we have $t_2 \notin 2nd(t_2)$, but $t_2 \in 2nd(t_1)$. Hence we obtain (2). $\dashv$

5.4.9. LEMMA. *Let t be a world in Exm.*
(1) *t is $\cap$-independent,*
(2) *$\tau(t) = t$.*

Proof. For (1): Since $t \in \mathbf{T}$, there exists a $\cap$-independent world α in a finite Kripke **PLL**-model such that $t = \tau(\alpha)$. Assume that α is A-independent for an atomic or circled formula A. Then $A \notin th(\alpha)$ and $A \in \bigcap_{\alpha<\beta} th(\beta)$. We show that t is A-independent. By $A \notin th(\alpha)$ and Lemma 5.4.5, we have $A \notin th(t)$. Let $t < t_1$, i.e., $t_1 \in 2nd(t)$. Then there exists a $\cap$-independent world $\beta \in \alpha\!\uparrow - \{\alpha\}$ such that $t_1 = \tau(\beta)$. Using $A \in \bigcap_{\alpha<\beta} th(\beta)$, we have $A \in th(\beta)$. Using Lemma 5.4.5, we have $A \in th(t_1)$. Hence $A \in \bigcap_{t<t_1} th(t_1)$. Hence t is A-independent.

To prove (2), it is sufficient to show the following three:
(2.1) $1st(\tau(t)) = 1st(t)$,
(2.2) $2nd(\tau(t)) = 2nd(t)$,
(2.3) $3rd(\tau(t)) = 3rd(t)$.
For (2.1):

$$1st(\tau(t)) = atom(t) = \{a \mid t \in P^t(a)\}$$
$$= \{a \mid t \in \{t \mid a \in 1st(t)\}\} = \{a \mid a \in 1st(t)\} = 1st(t).$$

For (2.3): Suppose that $3rd(\tau(t)) = \circ$. Then we have tRt. So, there exists $t_1 \in \mathbf{T}$ such that $t \le t_1 \le t$ and $3rd(t_1) = \circ$. By $t \le t_1$ and $t_1 \le t$, we have $t_1 = t$, and hence, $3rd(t) = \circ$.

Suppose that $3rd(t) = \circ$. Using $t \le t \le t$, we have tRt, and thereby, $3rd(\tau(t)) = \circ$.

For (2.2): We use an induction on $\#(2nd(t))$. If $2nd(t) = t\!\uparrow - \{t\} = \emptyset$, then

$$2nd(\tau(t)) = \{\tau(t_1) \mid t < t_1, t_1 \in \mathbf{T}^\cap\} = \{\tau(t_1) \mid t_1 \in 2nd(t), t_1 \in \mathbf{T}^\cap\} = \emptyset = 2nd(t).$$

Suppose that $2nd(t) = t\Uparrow - \{t\} \neq \emptyset$ and (2.2) holds for any t^* such that $\#(2nd(t^*)) < \#(2nd(t))$. By (1), we have $\mathbf{T} = \mathbf{T}^{\cap}$. By Lemma 5.4.8, we have $\#(2nd(t_1)) < \#(2nd(t))$ for any $t_1 \in 2nd(t)$. Using (2.1), (2.3) and the induction hypothesis, $\tau(t_1) = t_1$ for any $t_1 \in 2nd(t)$. Hence

$$2nd(\tau(t)) = \{\tau(t_1) \mid t < t_1, t_1 \in \mathbf{T}^{\cap}\} = \{t_1 \mid t < t_1\} = \{t_1 \mid t_1 \in 2nd(t)\} = 2nd(t).$$

$\dashv$

5.5 Finiteness of Exm

In this section, we prove the following theorem.

5.5.1. THEOREM. *There are only finitely many semantic types in* $\mathbf{T}$.

The proof of the theorem needs some preparations.

5.5.2. DEFINITION.

$$\begin{array}{ll}
\mathbf{T}^A = \{t \in \mathbf{T} \mid t \text{ is } A\text{-independent in } Exm\}, & \\
\mathbf{T}_k = \{t \in \mathbf{T} \mid \#(1st(t)) \geq k\}, & \\
\mathbf{T}^{atom} = \bigcup_{A \in \{p_1, \cdots, p_n, \bot\}} \mathbf{T}^A, & \mathbf{T}_k^{atom} = \mathbf{T}_k \cap \mathbf{T}^{atom}, \\
\mathbf{T}^{circ} = \mathbf{T} - \mathbf{T}^{atom}, & \mathbf{T}_k^{circ} = \mathbf{T}_k \cap \mathbf{T}^{circ}, \\
\mathbf{T}^{\bullet} = \{t \in \mathbf{T} \mid 3rd(t) = \bullet\}, & \mathbf{T}_k^{\bullet} = \mathbf{T}_k \cap \mathbf{T}^{\bullet}, \\
\mathbf{T}_{k,l}^{circ} = \{t \in \mathbf{T}_k^{circ} \mid \#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \leq l\}. &
\end{array}$$

By Lemma 5.4.9(1), we note the following:

$$\mathbf{T}^A \subseteq \mathbf{T},$$

$$\mathbf{T}^{circ} \subseteq \bigcup_{B \in [\wedge, \supset, \bigcirc, \bot]^n} \mathbf{T}^{\bigcirc B}.$$

5.5.3. LEMMA. $\mathbf{T}^{\bullet} \subseteq \mathbf{T}^{atom}$.

Proof. Suppose that $t \notin \mathbf{T}^{atom}$. Then t is $\bigcirc B$-independent. Using Lemma 5.3.11, t is reflexive. Hence, $3rd(t) = \circ$, and thereby, $t \notin \mathbf{T}^{\bullet}$ $\dashv$

5.5.4. LEMMA. *If $t \in \mathbf{T}_k^{circ}$, there exists a semantic type $t_1 \in 2nd(t)$ such that $\{s \mid t < s \leq t_1\} \subseteq \mathbf{T}_k^{\bullet}$.*

Proof. Suppose that $t \in \mathbf{T}_k^{circ}$. Then by Lemma 5.3.15, t is $\bigcirc B$-independent for some B such that

$$B \notin \bigcap_{t_1 \in \mathbf{T}^{\cap}, \tau(t_1) \in 2nd(\tau(t))} th(t_1).$$

Using Lemma 5.4.9,

$$B \notin \bigcap_{t_1 \in 2nd(t)} th(t_1).$$

So, there exists a world $t_1 \in 2nd(t)$ such that $B \notin th(t_1)$.

Suppose that $t < s \leq t_1$ and $s \notin \mathbf{T}_k^{\bullet}$. By $t < s$ and Lemma 5.4.8(1), we have $s \in \mathbf{T}_k - \mathbf{T}_k^{\bullet}$. So, $3rd(s) = \circ$, i.e., s is reflexive. By $t < s$ and the $\bigcirc B$-independency of t, we have $\bigcirc B \in th(s)$. Using the reflexivity of s, $B \in th(s)$. Using $B \notin th(t_1)$, we have $th(s) \not\subseteq th(t_1)$. On the other hand, by Lemma 5.4.9 and $s \leq t_1$, we have $\tau(s) \leq \tau(t_1)$. Using Lemma 5.3.14 we have $th(s) \subseteq th(t_1)$. This is a contradiction. Hence $\{s \mid t < s \leq t_1\} \subseteq \mathbf{T}_k^{\bullet}$. ⊣

5.5.5. LEMMA.

(1) $\mathbf{T}_{k,0}^{circ} = \emptyset$.

(2) *If $t \in \mathbf{T}_{k,l+1}^{circ}$, then $2nd(t) \subseteq \mathbf{T}_k^{atom} \cup \mathbf{T}_{k,l}^{circ}$.*

Proof. For (1): Suppose that $t \in \mathbf{T}_{k,0}^{circ}$. Then $\#(2nd(t) \cap \mathbf{T}_k^{\bullet}) = 0$. However, by Lemma 5.5.4, $\#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \geq 1$. This is a contradiction.

For (2): Suppose that $t \in \mathbf{T}_{k,l+1}^{circ}$ and $t_2 \in 2nd(t)$. If $t_2 \in \mathbf{T}_k^{atom}$, then we have $t_2 \in \mathbf{T}_k^{atom} \cup \mathbf{T}_{k,l}^{circ}$. Assume that $t_2 \notin \mathbf{T}_k^{atom}$. By Lemma 5.4.8 and $t_2 \in 2nd(t)$, we have $t_2 \in \mathbf{T}_k$. Hence $t_2 \in \mathbf{T}_k^{circ}$, and thereby, $t_2 \notin \mathbf{T}_k^{\bullet}$.

On the other hand, by $t \in \mathbf{T}_{k,l+1}^{circ}$ and Lemma 5.5.4, there exists semantic type $t_1 \in 2nd(t)$ such that $\{s \mid t < s \leq t_1\} \subseteq \mathbf{T}_k^{\bullet}$. Using $t_2 \in 2nd(t)$, i.e., $t < t_2$, we have $t_1 \notin 2nd(t_2)$, i.e., $t_2 \not< t_1$.

By $t_2 \in 2nd(t)$ and Lemma 5.4.8, we have $2nd(t_2) \subseteq 2nd(t)$, and so,

$$2nd(t_2) \cap \mathbf{T}_k^{\bullet} \subseteq 2nd(t) \cap \mathbf{T}_k^{\bullet}.$$

Using $t_1 \in 2nd(t) \cap \mathbf{T}_k^{\bullet}$ and $t_1 \notin 2nd(t_2)$.

$$\#(2nd(t_2) \cap \mathbf{T}_k^{\bullet}) < \#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \leq l + 1.$$

Using $t_2 \in \mathbf{T}_k^{circ}$, we have $t_2 \in \mathbf{T}_{k,l}^{circ}$, and thereby, $t_2 \in \mathbf{T}_{k,l}^{circ} \cup \mathbf{T}_k^{atom}$. ⊣

5.5.6. LEMMA. *If $\mathbf{T}_k^{atom}$ has only finitely many semantic types, then so does $\mathbf{T}_k^{circ}$.*

Proof. By the finiteness of $\mathbf{T}_k^{atom}$ and Lemma 5.5.3, we can put $\#(\mathbf{T}_k^{\bullet}) = m$. By Lemma 5.4.8(1), for any $t \in \mathbf{T}_k$.

$$2nd(t) \subseteq \mathbf{T}_k,$$

and so,

$$2nd(t) \cap \mathbf{T}_k^{\bullet} \subseteq \mathbf{T}_k^{\bullet},$$

therefore,

$$\#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \leq \#(\mathbf{T}_k^{\bullet}) = m.$$

Hence

$$\mathbf{T}_k^{circ} = \{t \in \mathbf{T}_k^{circ} \mid \#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \leq m\} = \mathbf{T}_{k,m}^{circ}$$

Hence it is sufficient to show the finiteness of the set

$$\mathbf{T}_{k,l}^{circ}$$

for any $l = 0, \cdots, m$. We use an induction on l.

If $l = 0$, then by Lemma 5.5.5(1), $\mathbf{T}_{k,0}^{circ} = \emptyset$, which has only finitely many semantic types.

Suppose that $l > 0$ and the finiteness of $\mathbf{T}_{k,l^*}^{circ}$ for any $l^* < l$. By Lemma 5.5.5(2),

$$\bigcup_{t \in \mathbf{T}_{k,l}^{circ}} 2nd(t) \subseteq \mathbf{T}_{k,l-1}^{circ} \cup \mathbf{T}_k^{atom}.$$

Hence for any $t \in \mathbf{T}_{k,l}^{circ}$,

$$1st(t) \in \{atom \mid atom \subseteq \{p_1, \cdots, p_n\}, \#(atom) \geq k\},$$

$$2nd(t) \in \mathcal{P}^*(\mathbf{T}_{k,l-1}^{circ} \cup \mathbf{T}_k^{atom}),$$

$$3rd(t) \in \{\bullet, \circ\}.$$

By the induction hypothesis and the finiteness of $\mathbf{T}_k^{atom}$, every components of $t \in \mathbf{T}_{k,l}^{circ}$ is a member of finite sets. Hence $\mathbf{T}_{k,l}^{circ}$ is finite. $\dashv$

5.5.7. LEMMA. *If $t \in \mathbf{T}_k^{atom}$, then $2nd(t) \subseteq \mathbf{T}_{k+1}$.*

Proof. By $t \in \mathbf{T}_k^{atom}$, t is a-independent for an atomic formula a. Hence for any $t_1 \in 2nd(t)$, we have $atom(t) \cup \{a\} \subseteq atom(t_1) = 1st(t_1)$. Also we note that $a \notin atom(t)$. Hence $k + 1 \leq \#(atom(t) \cup \{a\}) \leq \#(1st(t_1))$, and thereby, $t_1 \in \mathbf{T}_{k+1}$. $\dashv$

5.5.8. LEMMA. $\mathbf{T}_k^{atom}$ *has only finitely many semantic types.*

Proof. We use an induction on $n-k$. If $k > n$, we note that $\mathbf{T}_k^{atom} = \emptyset$, which is finite. Suppose that $k \leq n$ and the lemma holds for any $k' > k$. By Lemma 5.5.7, for any $t \in \mathbf{T}_k^{atom}$,

$$1st(t) \in \{atom \mid atom \subseteq \{p_1, \cdots, p_n\}, \#(atom) \geq k\},$$

$$2nd(t) \in \mathcal{P}^*(\mathbf{T}_{k+1}^{circ} \cup \mathbf{T}_{k+1}^{atom}),$$
$$3rd(t) \in \{\bullet, \circ\}.$$

By the induction hypothesis, we obtain the finiteness of $\mathbf{T}_{k+1}^{atom}$, and using Lemma 5.5.6, that of $\mathbf{T}_{k+1}^{circ}$. Hence we obtain the lemma. ⊣

5.5.9. COROLLARY. *There are only finitely many semantic types in* $\mathbf{T}_k$.

Proof. By Lemma 5.5.8 and Lemma 5.5.6. ⊣

We note that $\mathbf{T}_0 = \mathbf{T}$. Hence we obtain Theorem 5.5.1.

5.5.10. COROLLARY. *If* $t \in \mathbf{T}_{k,l}^{circ}$, *there exists a semantic type* $t_1 \in \mathbf{T}_k^{\bullet}$ *such that* $t <_1 t_1$.

Proof. By Lemma 5.5.4 and Theorem 5.5.1. ⊣

5.6 Exactness of *Exm*

In this section, we prove the following theorem.

5.6.1. THEOREM. P^t *maps* $[\wedge, \supset, \bigcirc, \bot]^n$ *onto* $\mathcal{P}^*(\mathbf{T})$.

The proof of the theorem needs some preparations.

5.6.2. LEMMA. *Let* α *be an a-independent world in a finite Kripke* **PLL**-*model* $M = \langle W, \leq, R, P\rangle$ *for an atomic formula* a. *If* $3rd(\tau(\alpha)) = \bullet$, *then* $\bigcirc a \in th(\alpha)$.

Proof. It is sufficient to show that $a \in th(\alpha_1)$ for any $\alpha_1 \in \{\beta \mid \alpha R \beta\}$. Let α_1 be a world in $\{\beta \mid \alpha R \beta\}$, i.e., $\alpha R \alpha_1$. By $3rd(\tau(\alpha)) = \bullet$, we have α is irreflexive, and thereby, $\alpha \neq \alpha_1$. Since M is a finite Kripke **PLL**-model, we have $R \subseteq \leq$. Hence $\alpha < \alpha_1$. Using the a-independency of α, we obtain $a \in th(\alpha_1)$. ⊣

5.6.3. Lemma. *Let α and β be $\cap$-independent worlds in some finite Kripke* **PLL**-*models. If $th(\alpha) = th(\beta)$, then*
(1) $1st(\tau(\alpha)) = 1st(\tau(\beta))$,
(2) $3rd(\tau(\alpha)) = 3rd(\tau(\beta))$.

Proof. For (1): Since $th(\alpha) = th(\beta)$, we have $atom(\alpha) = atom(\beta)$, and thereby, $1st(\tau(\alpha)) = 1st(\tau(\beta))$.

For (2): Suppose that $3rd(\tau(\alpha)) = \bullet$: Then α is irreflexive. Using Lemma 5.3.11, α is a-independent for some atomic formula a. Hence $a \notin th(\alpha)$. Also using Lemma 5.6.2, we have $\bigcirc a \in th(\alpha)$. Using $th(\alpha) = th(\beta)$, we have

$$a \notin th(\beta) \text{ and } \bigcirc a \in th(\beta).$$

Hence β is irreflexive, and thereby $3rd(\tau(\beta)) = \bullet = 3rd(\tau(\alpha))$.

Similarly we have that $3rd(\tau(\beta)) = \bullet$ implies $3rd(\tau(\alpha)) = \bullet$. ⊣

5.6.4. Lemma. $[\wedge, \supset, \bigcirc, \perp]^n / \equiv_{PLL}$ *is finite.*

Proof. By Theorem 5.5.1 and Theorem 5.4.1. ⊣

By the lemma above, we can define a formula below.

5.6.5. Definition. For a set $\mathbf{S}$ of formulas in $[\wedge, \supset, \bigcirc, \perp]^n$, $\bigwedge(\mathbf{S}/\equiv)$ is defined as a conjunction of all the canonical representatives of the quotient set $\mathbf{S}/\equiv_{PLL}$.

5.6.6. Lemma. *Let α and β be A-independent worlds in some finite Kripke* **PLL**-*models. Then $\tau(\alpha) \leq \tau(\beta)$ implies $\tau(\alpha) = \tau(\beta)$.*

Proof. Suppose that $\tau(\alpha) \leq \tau(\beta)$ and $\tau(\alpha) \neq \tau(\beta)$. Then we have $\tau(\beta) \in 2nd(\tau(\alpha))$. So, there exists a $\cap$-independent world $\alpha_1 \in \alpha\!\uparrow - \{\alpha\}$ such that $\tau(\beta) = \tau(\alpha_1)$. Using Lemma 5.3.12, we have $th(\beta) = th(\alpha_1)$. Since α is A-independent, $A \in th(\alpha_1) = th(\beta)$. This is in contradiction with the A-independency of β. ⊣

5.6.7. LEMMA. *Let α and β be $\cap$-independent worlds in some finite Kripke* **PLL**-*models.*

(1) $th(\alpha) = th(\beta)$ *implies* $\tau(\alpha) = \tau(\beta)$,

(2) $th(\alpha) \subseteq th(\beta)$ *implies* $\tau(\alpha) \leq \tau(\beta)$.

Proof. We use an induction on $\#(2nd(\tau(\alpha))) + \#(2nd(\tau(\beta)))$.

Basis:

For (1): Suppose that $\#(2nd(\tau(\alpha))) + \#(2nd(\tau(\beta))) = 0$. Then we have $2nd(\tau(\alpha)) = 2nd(\tau(\beta)) = \emptyset$. By Lemma 5.6.3, we have $1st(\tau(\alpha)) = 1st(\tau(\beta))$ and $3rd(\tau(\alpha)) = 3rd(\tau(\beta))$. Hence we obtain (1).

Induction step:

For (2): Suppose that (2) holds for any α^* and β^* such that $\#(2nd(\tau(\alpha^*))) + \#(2nd(\tau(\beta^*))) < \#(2nd(\tau(\alpha))) + \#(2nd(\tau(\beta)))$, and that (1) holds. Since β is $\cap$-independent, β is B-independent for an atomic or circled formula B. Then we have $\bigwedge(th(\beta)/\equiv) \supset B \notin th(\beta)$. Using $th(\alpha) \subseteq th(\beta)$, we have $\bigwedge(th(\beta)/\equiv) \supset B \notin th(\alpha)$. Using Corollary 5.3.10(2), there exists a $\bigwedge(th(\beta)/\equiv) \supset B$-independent world $\alpha_1 \in \alpha\!\uparrow$. Using Lemma 5.3.9, we have

$$\bigwedge(th(\beta)/\equiv) \in th(\alpha_1) \text{ and } \alpha_1 \text{ is } B\text{-independent.}$$

Hence $th(\beta) \subseteq th(\alpha_1)$.

If $\alpha = \alpha_1$, then $th(\beta) = th(\alpha)$, and by (1), we obtain (2). So, we assume that $\alpha < \alpha_1$. Then $\tau(\alpha_1) \in 2nd(\tau(\alpha))$. Using Lemma 5.4.8, $\#(2nd(\tau(\alpha_1))) < \#(2nd(\tau(\alpha)))$. Then by the induction hypothesis, $\tau(\beta) \leq \tau(\alpha_1)$. Using Lemma 5.6.6, we have $\tau(\beta) = \tau(\alpha_1) \in 2nd(\tau(\alpha))$.

For (1): Suppose that (1) and (2) holds for any α^* and β^* such that $\#(2nd(\tau(\alpha^*))) + \#(2nd(\tau(\beta^*))) < \#(2nd(\tau(\alpha))) + \#(2nd(\tau(\beta)))$. By Lemma 5.6.3, it is sufficient to show

$$2nd(\tau(\alpha)) = 2nd(\tau(\beta)).$$

Let t be a semantic type in $2nd(\tau(\alpha))$. Then there exist an atomic or circled formula A and an A-independent world $\alpha_1 \in \alpha\!\uparrow - \{\alpha\}$ such that $t = \tau(\alpha_1)$. So, we have $\bigwedge(th(\alpha_1)/\equiv) \supset A \notin th(\alpha_1)$, and thereby, it does not belong to $th(\alpha)$, neither does $th(\beta)$. Using Corollary 5.3.10(2) and Lemma 5.3.9(2), there exists an A-independent world $\beta_1 \in \beta\!\uparrow$ such that $\bigwedge(th(\alpha_1)/\equiv) \in th(\beta_1)$. Hence we have

$$th(\alpha_1) \subseteq th(\beta_1).$$

By $\tau(\alpha_1) \in 2nd(\tau(\alpha))$ and Lemma 5.4.8(2), we have $\#(2nd(\tau(\alpha_1))) < \#(2nd(\tau(\alpha)))$. Also by $\tau(\beta_1) \in 2nd(\tau(\beta)) \cup \{\beta\}$ and Lemma 5.4.8(2), we have $\#(2nd(\tau(\beta_1))) \leq \#(2nd(\tau(\beta)))$. So, by the induction hypothesis, we have

$$\tau(\alpha_1) \leq \tau(\beta_1).$$

Using Lemma 5.6.6, we have $\tau(\beta_1) = \tau(\alpha_2)$, and thereby, $t = \tau(\alpha_1) = \tau(\beta_1) \in 2nd(\tau(\beta)) \cup \{\beta\}$. Since α is $\cap$-independent, it is C-independent for some C. So, $C \notin th(\alpha)$. Using $th(\alpha) = th(\beta)$, we have $C \notin th(\beta)$. Also by $\alpha < \alpha_1$, we have $C \in th(\alpha_1)$. Using $th(\alpha_1) \subseteq th(\beta_1)$ we have $C \in th(\beta_1)$. Hence $\beta_1 \neq \beta$, and thereby, $\beta < \beta_1$. Hence $t = \tau(\beta_1) \in 2nd(\tau(\beta))$. ⊣

5.6.8. Definition. For any set $X \in \mathcal{P}^*(\mathbf{T})$, we put

$$\phi(X) = \bigwedge((\bigcap_{t \in X} th(t))/\equiv).$$

Note that $\phi(X) \in th(t)$ for any $t \in X$.

5.6.9. Lemma. *For any set $X \in \mathcal{P}^*(\mathbf{T})$,*

$$P^t(\phi(X)) = X.$$

Proof. If $t \in X$, then $\phi(X) \in th(t)$, and so,

$$t \in \{s \mid \phi(X) \in th(s)\} = P^t(\phi(X)).$$

Suppose that $t \in P^t(\phi(X))$, i.e., $\phi(X) \in th(t)$. Then

$$\bigcap_{s \in X} th(s) \subseteq th(t).$$

By Lemma 5.4.9(1), t is $\cap$-independent, and thereby, it is A-independent for an atomic or circled formula A. Hence

$$\bigwedge(th(t)/\equiv) \supset A \notin th(t).$$

Using $\bigcap_{s \in X} th(s) \subseteq th(t)$, there exists a world $s \in X$ such that

$$\bigwedge(th(t)/\equiv) \supset A \notin th(s).$$

Using Lemma 5.3.10(2), there exists a $\bigwedge(th(t)/\equiv) \supset A$-independent world $s_1 \in s\!\uparrow$. Using Lemma 5.3.9(2), s_1 is A-independent and $th(t) \subseteq th(s_1)$. Using Lemma 5.6.7, $\tau(t) \leq \tau(s_1)$. Using Lemma 5.6.6, $\tau(t) = \tau(s_1)$. Using Lemma 5.4.9(2), $t = s_1$, and hence $s \leq t$. Since X is a closed subset, we have $t \in X$. ⊣

By the lemma above, we obtain Theorem 5.6.1. Hence we obtain Theorem 5.3.20 by Theorem 5.5.1, Theorem 5.4.1 and Theorem 5.6.1.

5.7 An inductive definition of Exm

In the previous sections, we proved that Exm is the exact model for the fragment $[\wedge, \supset, \bigcirc, \bot]^n$ in **PLL**. The structure Exm is defined by semantic types of $\cap$-independent worlds, i.e., A_i-independent worlds for atomic or circled formulas A_i. However, we have infinitely many possible selections of circled formulas. So, we have not clarified the structure of Exm, yet. In this section, we show a method to construct Exm in an inductive way. We define the set E_0 of triples inductively, and prove that $E_0 = \mathbf{T}$. The structure of E_0 is perspicuous, and thus the structure of $\mathbf{T}$ is elucidated.

5.7.1. Definition.

(1) For $k > n$, $E_k = E_k^{atom} = E_k^{circ} = \emptyset$.

(2) For $0 \leq k \leq n$,

$$
\begin{aligned}
E_k^{\bullet} = \{\langle atom, S, \bullet\rangle \mid {} & \\
& \#(atom) \geq k, atom \subseteq \mathbf{atom}^n, \\
& atom \subseteq \bigcap_{e_1 \in S} 1st(e_1) \neq atom, \\
& \bigcup_{e_1 \in S} 2nd(e_1) \subseteq S \subseteq E_{k+1}\}, \\
E_k^{atom} = E_k^{\bullet} \cup \{\langle & atom, S, \circ\rangle \mid \langle atom, S, \bullet\rangle \in E_k^{\bullet}\}, \\
E_{k,0}^{circ} = \emptyset, & \\
E_{k,l+1}^{circ} = \{\langle atom, & \{e\} \cup S, \circ\rangle \mid \\
& \#(atom) \geq k, atom \subseteq \mathbf{atom}^n, \\
& atom \subseteq \bigcap_{e_1 \in \{e\} \cup S} 1st(e_1), \\
& \bigcup_{e_1 \in \{e\} \cup S} 2nd(e_1) \subseteq \{e\} \cup S \subseteq E_k^{atom} \cup E_{k,l}^{circ}, \\
& \#((\{e\} \cup S) \cap E_k^{\bullet}) \leq l + 1, \\
& e \in E_k^{\bullet}, \{e_1 \in \{e\} \cup S \mid e \in 2nd(e_1)\} = \emptyset\}, \\
E_k^{circ} = E_{k,\#(E_k^{\bullet})}^{circ}, & \\
E_k = E_k^{atom} \cup E_k^{circ}. &
\end{aligned}
$$

We will see in below that e occurring the definition of $E_{k,l+1}^{circ}$ means a direct irreflexive successor of a type (see Corollary 5.5.10). Also the sets defined above correspond to the sets defined in Definition 5.5.2 (see Corollary 5.7.19).

5.7.2. Fact.

$$E_n^{\bullet} = \{\langle \mathbf{atom}^n, \emptyset, \bullet\rangle\},$$

$$E_n^{atom} = \{\langle \mathbf{atom}^n, \emptyset, \bullet\rangle, \langle \mathbf{atom}^n, \emptyset, \circ\rangle\},$$

$$E_{n,1}^{circ} = E_n^{circ} = \{\langle \mathbf{atom}^n, E_n^{\bullet}, \circ\rangle, \langle \mathbf{atom}^n, E_n^{atom}, \circ\rangle\},$$
$$E_n = \{\langle \mathbf{atom}^n, \emptyset, \bullet\rangle, \langle \mathbf{atom}^n, \emptyset, \circ\rangle, \langle \mathbf{atom}^n, E_n^{\bullet}, \circ\rangle, \langle \mathbf{atom}^n, E_n^{atom}, \circ\rangle\}.$$

The main theorem in this section is

5.7.3. THEOREM. $E_0 = \mathbf{T}$.

The proof of the theorem needs some preparations.

5.7.4. LEMMA. *For any $e \in E_0$, $e \notin 2nd(e)$.*

Proof. If $e \in E_k^{atom}$, then we have $1st(e) \subseteq \bigcap_{e_1 \in 2nd(e)} 1st(e_1) \neq 1st(e)$. Hence for any $e_1 \in 2nd(e)$, $1st(e)$ is a proper subset of $1st(e_1)$. Hence $e \notin 2nd(e)$.

If $e \in E_k^{circ}$, then there exists $e_1 \in 2nd(e) \cap E_k^{\bullet}$ such that

$$\{e_2 \in 2nd(e) \mid e_1 \in 2nd(e_2)\} = \emptyset.$$

Using $e_1 \in 2nd(e)$, we have $e \notin 2nd(e)$. ⊣

5.7.5. LEMMA.

(1) *If $e \in E_k^{atom}$, then $2nd(e) \subseteq E_{k+1}$.*

(2) *If $e \in E_{k,l+1}^{circ}$, then $2nd(e) \subseteq E_k^{atom} \cup E_{k,l}^{circ}$.*

(3) *There are only finitely many triples in E_k.*

Proof. (1) and (2) follow from the definition. (3) can be shown by an induction on $n - k$ using (1) and (2). ⊣

5.7.6. COROLLARY. *There are only finitely many triples in E_0.*

5.7.7. LEMMA. *Let e_1 be a triple in E_0. Then*

$$1st(e) \subseteq \bigcap_{e_3 \in 2nd(e)} 1st(e_3) \text{ and } \bigcup_{e_3 \in 2nd(e_1)} 2nd(e_3) \subseteq 2nd(e_1),$$

especially, $1st(e)$ is a proper subset if $e \in E_k^{atom}$.

Proof. By Definition 5.7.1. ⊣

5.7.8. Corollary. *Let e_1 and e_2 be triples in E_0. If $e_1 \in 2nd(e_2)$, then*

$$1st(e_2) \subseteq 1st(e_1) \text{ and } 2nd(e_1) \subseteq 2nd(e_2),$$

especially, $1st(e_1)$ is a proper subset if $e_1 \in E_k^{atom}$.

5.7.9. Definition. Let e_1 and e_2 be triples in E_0. We write $e_1 \leq^e e_2$ if either $e_1 = e_2$ or $e_2 \in 2nd(e_1)$. Also write $e_1 R^e e_2$ if there exists a triple $e_3 \in E_0$ such that $e_1 \leq^e e_3 \leq^e e_2$ and $3rd(e_3) = \circ$. We put $P^e(a) = \{e \mid a \in 1st(e)\}$.

5.7.10. Lemma. *The structure $\langle E_0, \leq^e, R^e, P^e \rangle$ is a finite Kripke* **PLL**-*model.*

Proof. By Corollary 5.7.6, the structure is finite. So, it is sufficient to show the following seven properties, for any $e_1, e_2, e_3 \in E_0$,

(1) $e_1 \leq^e e_1$,
(2) $e_1 \leq^e e_2$ and $e_2 \leq^e e_3$ implies $e_1 \leq^e e_3$,
(3) $e_1 \leq^e e_2$ and $e_2 \leq^e e_1$ implies $e_1 = e_2$,
(4) $e_1 \leq^e e_2$ and $e_2 R^e e_3$ implies $e_1 R^e e_3$,
(5) $P^e(a) \in \mathcal{P}^*(E_0)$,
(6) $e_1 R^e e_2$ implies $e_1 \leq^e e_2$,
(7) $e_1 R^e e_2$ implies $e_1 R^e e_4 R^e e_2$ for some $e_4 \in E_0$.

For (1): Trivial from the definition.

For (2): If either $e_1 = e_2$ or $e_2 = e_3$, then (2) is trivial. So, we assume that $e_2 \in 2nd(e_1)$ and $e_3 \in 2nd(e_2)$. By Corollary 5.7.8, $2nd(e_2) \subseteq 2nd(e_1)$. Using $e_3 \in 2nd(e_2)$, we obtain $e_3 \in 2nd(e_1)$, and thereby, $e_1 \leq^e e_3$.

For (3): Suppose that $e_1 \leq^e e_2$, $e_2 \leq^e e_1$ and $e_1 \neq e_2$. Then we have $e_2 \in 2nd(e_1)$ and $e_1 \in 2nd(e_2)$. Similarly to the proof of (2), we have $e_1 \in 2nd(e_1)$. This is in contradiction with Lemma 5.7.4.

For (4): By $e_2 R^e e_3$, there exists $e_4 \in E_0$ such that $e_2 \leq^e e_4 \leq^e e_3$ and $3rd(e_4) = \circ$. Using $e_1 \leq^e e_2$ and (2), we have $e_1 \leq^e e_4 \leq^e e_3$. Hence we obtain $e_1 R^e e_3$.

For (5): Suppose that $e_1 \in P^e(a)$ and $e_1 \leq^e e_2$. If $e_1 = e_2$, we have $e_2 \in P^e(a)$. Assume that $e_2 \in 2nd(e_1)$. Using Corollary 5.7.8, we have $1st(e_1) \subseteq 1st(e_2)$. By $e_1 \in P^e(a)$, we have $e_1 \in \{e \mid a \in 1st(e)\}$, and so, $a \in 1st(e_1)$. Using $1st(e_1) \subseteq 1st(e_2)$, we have $a \in 1st(e_2)$, and thereby, $e_2 \in P^e(a)$.

For (6) and (7): Suppose that $e_1 R^e e_2$. Then there exists $e_3 \in E_0$ such that $e_1 \leq e_3 \leq e_2$ and $3rd(e_3) = \circ$. Using (2), we obtain (6). Since $e_1 \leq e_3 \leq e_3$ and $e_3 \leq e_3 \leq e_2$, we have $e_1 R^e e_3$ and $e_3 R^e e_2$. Hence we obtain (7). $\dashv$

5.7.11. LEMMA. *Let $e \in E_0$ be a world in the Kripke* **PLL**-*model* $\langle E_0, \leq^e, R^e, P^e \rangle$. *Then*

$$eR^e e \text{ iff } 3rd(e) = \circ.$$

Proof. If $3rd(e) = \circ$, then using $e \leq^e e \leq^e e$, we obtain $eR^e e$. Suppose that $eR^e e$. Then there exists a triple e_1 such that $e \leq^e e_1 \leq^e e$ and $3rd(e_1) = \circ$. By $e \leq^e e_1 \leq^e e$ and Lemma 5.7.10, we have $e = e_1$. Using $3rd(e_1) = \circ$, we have $3rd(e) = \circ$. ⊣

5.7.12. LEMMA. *Let $e \in E_k$ be a world in the Kripke* **PLL**-*model* $\langle E_0, \leq^e, R^e, P^e \rangle$.

(1) *If $e \in E_k^{atom}$, then e is a-independent for some atomic formula a and $\tau(e) = e$.*

(2) *If $e \in E_{k,l}^{circ}$, then there exists an a-independent world $e_1 \in \mathbf{Minl}(2nd(e)) \cap E_k^{\bullet}$ for some atomic formula a such that e is $\bigcirc(\wedge(th(e_1)/ \equiv_{PLL}) \supset a)$-independent and $\tau(e) = e$.*

Proof. We use an induction on $n - k$. If $k > n$, then $E_k = \emptyset$, hence we obtain the lemma. Suppose that $1 \leq k \leq n$ and the lemma holds for any $k^* \geq k$.

For (1): By Lemma 5.7.7, we have $1st(e)$ is a proper subset of $\bigcap_{e_3 \in 2nd(e)} 1st(e_3)$. Hence there exists an atomic formula $a \notin 1st(e)$ such that $a \in \bigcap_{e_3 \in 2nd(e)} 1st(e_3)$. Hence $a \notin atom(e)$ and $a \in \bigcap_{e_3 \in 2nd(e)} atom(e_3)$, i.e., e is a-independent.

On the other hand, we have

$$1st(\tau(e)) = atom(e) = 1st(e).$$

By Lemma 5.7.11,

$$3rd(\tau(e)) = \circ \text{ iff } eR^e e \text{ iff } 3rd(e) = \circ.$$

Hence, we have only to show

$$2nd(\tau(e)) = 2nd(e).$$

By the definition,

$$2nd(\tau(e)) = \{\tau(e_1) \mid e < e_1, e_1 \in E_0^{\cap}\}.$$

In other words,

$$2nd(\tau(e)) = \{\tau(e_1) \mid e_1 \in 2nd(e), e_1 \in E_0^{\cap}\}.$$

Let it be that $e_1 \in 2nd(e)$. Then by Corollary 5.7.8, $k \leq \#(atom(e)) < \#(atom(e_1))$, and hence $e_1 \in E_{k+1}$. By the induction hypothesis, e_1 is $\cap$-independent and $\tau(e_1) = e_1$. Hence

$$2nd(\tau(e)) = \{\tau(e_1) \mid e_1 \in 2nd(e), e_1 \in E_0^{\cap}\} = \{e_1 \mid e_1 \in 2nd(e)\} = 2nd(e).$$

Hence we obtain (1).

For (2): From the definition, there exists $e_1 \in \mathbf{Minl}(2nd(e)) \cap E_k^{\bullet}$. Using (1), e_1 is a-independent for some a. Hence $\bigwedge(th(e_1)/\equiv) \supset a \notin th(e_1)$. Using $e_1 \in 2nd(e)$, $\bigwedge(th(e_1)/\equiv) \supset a \notin th(e)$. By $e \in E_k^{circ}$, we have $3rd(e) = \circ$. Using Lemma 5.7.11, we have $eR^e e$. Hence

$$\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a) \notin th(e).$$

Suppose that

$$\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a) \notin th(e_2)$$

for some $e_2 \in e{\uparrow} - \{e\}$. Using Lemma 5.3.10(2), there exists a $\bigcirc(\bigwedge(th(e_1)/\equiv)\supset a)$-independent world $e_3 \in e_2{\uparrow}$. Using Lemma 5.3.9(3) and Lemma 5.3.9(2), there exists a a-independent world $e_4 \in e_3{\uparrow}$ such that $\bigwedge(th(e_1)/\equiv) \in th(e_4)$. By $\bigwedge(th(e_1)/\equiv) \in th(e_4)$, we have $th(e_1) \subseteq th(e_4)$. Using Lemma 5.6.7, we have

$$\tau(e_1) \leq \tau(e_4).$$

By $e_1 \in E_k^{\bullet}$ and Lemma 5.7.11, e_1 is irreflexive. Using Lemma 5.6.2 and the a-independency of e_1, we have $\bigcirc a \in th(e_1) \subseteq th(e_4)$. Hence e_4 is also irreflexive. Using Lemma 5.7.11 and $e_4 \in e{\uparrow}$, we have $e_4 \in E_k^{\bullet}$. Hence $e_1, e_4 \in E_k^{atom}$. Using (1), $\tau(e_1) = e_1$ and $\tau(e_4) = e_4$. Using $\tau(e_1) \leq \tau(e_4)$, we obtain $e_4 \in 2nd(e_1) \cup \{e_1\}$. Using the a-independency of e_1, we have $e_1 = e_4$, and thereby, $e < e_2 \leq e_3 \leq e_1$. Hence $e_2 = e_3 = e_1 \in \mathbf{Minl}(2nd(e)) \cap E_k^{\bullet}$. Using Lemma 5.7.11, e_3 is irreflexive.

On the other hand, by the $\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a)$-independency of e_3 and Lemma 5.3.11, we have e_3 is reflexive. This is a contradiction. Hence for any $e_2 \in e{\uparrow} - \{e\}$,

$$\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a) \in th(e_2).$$

Using $\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a) \notin th(e)$, e is $\bigcirc(\bigwedge(th(e_1)/\equiv) \supset a)$-independent.

Similarly to the proof of (1), we have

$$1st(\tau(e)) = 1st(e) \text{ and } 3rd(\tau(e)) = 3rd(e).$$

We show

$$2nd(\tau(e)) = 2nd(e). \tag{5.3}$$

By $e \in E_k^{circ}$, we have $e \in E_{k,l}^{circ}$ for some l. We use an induction on l. If $l = 0$, then $E_{k,l}^{circ} = \emptyset$. Suppose that $l > 0$ and (5.3) holds for any $e^* \in E_{k,l^*}^{circ}$ such that $l^* < l$. By the definition, we have

$$2nd(\tau(e)) = \{\tau(e_5) \mid e_5 \in 2nd(e), e_5 \in E_0^{\cap}\}.$$

By Lemma 5.7.5(2),

$$2nd(e) \subseteq E_k^{atom} \cup E_{k,l-1}^{circ}.$$

By the induction hypothesis and (1), for any $e_5 \in 2nd(e)$,

$$e_5 \in E_0^{\cap} \text{ and } \tau(e_5) = e_5.$$

Hence

$$2nd(\tau(e)) = \{e_5 \mid e_5 \in 2nd(e)\} = 2nd(e).$$

Hence we obtain (5.3), and hence we obtain (2). $\dashv$

5.7.13. Corollary.
(1) $E_k^{atom} \subseteq \mathbf{T}_k^{atom}$.
(2) $E_k^{\bullet} \subseteq \mathbf{T}_k^{\bullet}$.
(3) $E_{k,l}^{circ} \subseteq \mathbf{T}_k$.
(4) $E_0 \subseteq \mathbf{T}$.

5.7.14. Lemma. $\mathbf{T}_k^{atom} \subseteq E_k^{atom}$ *implies* $\mathbf{T}_k^{\bullet} \subseteq E_k^{\bullet}$.

Proof. $\mathbf{T}_k^{\bullet} = \mathbf{T}_k^{atom} \cap \{t \mid 3rd(t) = \bullet\} \subseteq E_k^{atom} \cap \{t \mid 3rd(t) = \bullet\} = E_k^{\bullet}$. $\dashv$

5.7.15. Lemma. $\mathbf{T}_k^{atom} \subseteq E_k^{atom}$ *implies* $\mathbf{T}_{k,l}^{circ} \subseteq E_{k,l}^{circ}$.

Proof. We use an induction on l. If $l = 0$, then by Lemma 5.5.5, $\mathbf{T}_{k,l}^{circ} = \emptyset$. Suppose that $l > 0$ and the lemma holds for any $l^* < l$. Let it be that $t \in \mathbf{T}_{k,l}^{circ}$. To show $t \in E_{k,l}^{circ}$, it is sufficient to show the following five:
(1) $\#(1st(t)) \geq k$,
(2) $2nd(t) \subseteq E_k^{atom} \cup E_{k,l-1}^{circ}$,
(3) $1st(t) \subseteq \bigcap_{t_1 \in 2nd(t)} 1st(t_1)$,
(4) $\bigcup_{t_1 \in 2nd(t)} 2nd(t_1) \subseteq 2nd(t)$,
(5) there exists $t_1 \in 2nd(t) \cap E_k^{\bullet}$ such that $t <_1 t_1$,
(6) $\#(2nd(t) \cap E_k^{\bullet}) \leq l$.

For (1): By $t \in \mathbf{T}_k$.
For (2): By Lemma 5.5.5, $2nd(t) \subseteq \mathbf{T}_k^{atom} \cup \mathbf{T}_{k,l-1}^{circ}$. By $\mathbf{T}_k^{atom} \subseteq E_k^{atom}$ and the induction hypothesis, $2nd(t) \subseteq E_k^{atom} \cup E_{k,l-1}^{circ}$.
For (3): Let it be that $t_1 \in 2nd(t)$. Then by Lemma 5.7.8, we have $1st(t) \subseteq 1st(t_1)$. Hence we obtain (3).
For (4): By Lemma 5.4.8.
For (5): By Corollary 5.5.10, there exists $t_1 \in 2nd(t) \cap \mathbf{T}_k^{\bullet}$ such that $t <_1 t_1$. Using Lemma 5.7.14, we obtain (5).
For (6): From the definition $\#(2nd(t) \cap \mathbf{T}_k^{\bullet}) \leq l$. Using Corollary 5.7.13(2), we obtain (6). $\dashv$

5.7.16. LEMMA. $\mathbf{T}_k^{atom} \subseteq E_k^{atom}$ *implies* $\mathbf{T}_k^{circ} \subseteq E_k^{circ}$.

Proof. By Lemma 5.7.15 and Lemma 5.7.14,

$$\mathbf{T}_k^{circ} = \mathbf{T}_{k,\#(\mathbf{T}_k^\bullet)}^{circ} \subseteq \mathbf{T}_{k,\#(E_k^\bullet)}^{circ} \subseteq E_{k,\#(E_k^\bullet)}^{circ} \subseteq E_k^{circ}.$$

$\dashv$

5.7.17. LEMMA. $\mathbf{T}_k^{atom} \subseteq E_k^{atom}$.

Proof. We use an induction on $n-k$. If $k > n$, then we obtain the lemma by $\mathbf{T}_k^{atom} = \emptyset$. Suppose that $k \leq n$ and the lemma holds for any $k^* > k$. Let it be that $t \in \mathbf{T}_k^{atom}$. To show $t \in E_k^{atom}$, it is sufficient to show the following four:
(1) $\#(1st(t)) \geq k$,
(2) $2nd(t) \subseteq E_{k+1}$,
(3) $1st(t)$ is a proper subset of $\bigcap_{t_1 \in 2nd(t)} 1st(t_1)$,
(4) $\bigcup_{t_1 \in 2nd(t)} 2nd(t_1) \subseteq 2nd(t)$.

For (1): By $t \in \mathbf{T}_k$.
For (2): By Lemma 5.5.7, $2nd(t) \subseteq \mathbf{T}_{k+1}$. By the induction hypothesis and Lemma 5.7.16, $2nd(t) \subseteq E_{k+1}$.
For (3): By Lemma 5.4.8, we have $1st(t) \subseteq \bigcap_{t_1 \in 2nd(t)} 1st(t_1)$. Since $t \in \mathbf{T}_k^{atom}$, t is a-independent for some atomic formula a. Hence $a \notin th(t)$, but $a \in \bigcap_{t_1 \in 2nd(t)} th(t_1)$. Hence $a \notin 1st(t)$ and $a \in \bigcap_{t_1 \in 2nd(t)} 1st(t_1)$. Hence we obtain (3).
For (4): By Lemma 5.4.8. $\dashv$

5.7.18. COROLLARY.
(1) $\mathbf{T}_k^\bullet \subseteq E_k^\bullet$,
(2) $\mathbf{T}_{k,l}^{circ} \subseteq E_{k,l}^{circ}$,
(3) $\mathbf{T}_k^{circ} \subseteq E_k^{circ}$,
(4) $\mathbf{T} \subseteq E_0$.

From Corollary 5.7.13 and Corollary 5.7.18, we obtain Theorem 5.7.3. Also we have

5.7.19. COROLLARY.
(1) $\mathbf{T}_k^\bullet = E_k^\bullet$,
(2) $\mathbf{T}_k^{atom} = E_k^{atom}$,
(3) $\mathbf{T}_{k,l}^{circ} = E_{k,l}^{circ} - E_k^{atom}$,
(4) $\mathbf{T}_k^{circ} = E_k^{circ} - E_k^{atom}$.

Proof. We only show (3). Suppose that $t \in \mathbf{T}^{circ}_{k,l}$. Then $t \notin \mathbf{T}^{atom}_k$. Using (2), $t \notin E^{atom}_k$. On the other hand, by Corollary 5.7.18, $t \in E^{circ}_{k,l}$. Hence $t \in E^{circ}_{k,l} - E^{atom}_k$. Suppose that $e \in E^{circ}_{k,l} - E^{atom}_k$. Then $e \in E^{circ}_{k,l}$ and $e \notin E^{atom}_k$. Using (2), $e \notin \mathbf{T}^{atom}_k$, and thereby, $e \in \mathbf{T}^{circ}_k$. On the other hand, by $e \in E^{circ}_{k,l}$, we have $\#(2nd(e) \cap E^{\bullet}_k) \leq l$. Using (2), $\#(2nd(e) \cap \mathbf{T}^{\bullet}_k) \leq l$. Hence we obtain $e \in \mathbf{T}^{circ}_{k,l}$. ⊣

Hence we can write the members of $\mathbf{T}$ if $n = 0$.

5.7.20. FACT. Let $n = 0$. Then

$$\mathbf{T}^{\bullet} = \{\langle \emptyset, \emptyset, \bullet \rangle\},$$

$$\begin{aligned}\mathbf{T}^{atom} &= \{\langle \emptyset, \emptyset, \bullet \rangle, \langle \emptyset, \emptyset, \circ \rangle\},\\ \mathbf{T}^{circ} &= \{\langle \emptyset, \mathbf{T}^{\bullet}, \circ \rangle, \langle \emptyset, \mathbf{T}^{atom}, \circ \rangle\},\\ \mathbf{T} &= \{\langle \emptyset, \emptyset, \bullet \rangle, \langle \emptyset, \emptyset, \circ \rangle, \langle \emptyset, \mathbf{T}^{\bullet}, \circ \rangle, \langle \emptyset, \mathbf{T}^{atom}, \circ \rangle\}.\end{aligned}$$

We can draw Hasse's diagrams of Exm, where we use the points $\bullet$ and $\circ$ to express the worlds whose third components are $\bullet$ and $\circ$, respectively; and we write each propositional variable p near the points α if $\alpha \in P^t(p)$ (see Figure 5.1 and Figure 5.2). In these diagrams, the relation R can be read by

$$\alpha R \beta \text{ iff there exists } \gamma \text{ such that } \alpha \leq \gamma \leq \beta \text{ and } 3rd(\gamma) = \circ.$$

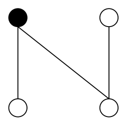

Figure 5.1: Hasse's diagram of $\langle \mathbf{T}, \leq, R, P^t \rangle$ for the case that $n = 0$

For the case that $n = 1$, there are many more semantic types in $\mathbf{T}$. We only list the members of $\mathbf{T}^{atom}$ and $\mathbf{T}^{circ}_{0,1}$.

5.7.21. FACT. Let it be that $n = 1$. Then

$$\mathbf{T}^{\bullet}_1 = \{\langle \{p_1\}, \emptyset, \bullet \rangle\},$$

$$\begin{aligned}\mathbf{T}^{atom}_1 &= \{\langle \{p_1\}, \emptyset, \bullet \rangle, \langle \{p_1\}, \emptyset, \circ \rangle\},\\ \mathbf{T}^{circ}_1 &= \{\langle \{p_1\}, \mathbf{T}^{\bullet}_1, \circ \rangle, \langle \{p_1\}, \mathbf{T}^{atom}_1, \circ \rangle\},\end{aligned}$$

$$\mathbf{T}_1 = \{\langle\{p_1\}, \emptyset, \bullet\rangle, \langle\{p_1\}, \emptyset, \circ\rangle, \langle\{p_1\}, \mathbf{T}_1^{\bullet}, \circ\rangle, \langle\{p_1\}, \mathbf{T}_1^{atom}, \circ\rangle\}.$$

We put

$t_1 = \langle\{p_1\}, \emptyset, \bullet\rangle$,

$t_2 = \langle\{p_1\}, \emptyset, \circ\rangle$,

$t_3 = \langle\{p_1\}, \mathbf{T}_1^{\bullet}, \circ\rangle = \langle\{p_1\}, \{t_1\}, \circ\rangle$,

$t_4 = \langle\{p_1\}, \mathbf{T}_1^{atom}, \circ\rangle = \langle\{p_1\}, \{t_1, t_2\}, \circ\rangle$.

Then

$$\mathcal{P}^*(\mathbf{T}_1) = \{\emptyset, \{t_1\}, \{t_1, t_2\}, \{t_2\}, \{t_1, t_2, t_3\}, \{t_1, t_3\}, \{t_1, t_2, t_3, t_4\}, \{t_1, t_2, t_4\}\},$$

$$\mathbf{T}_0^{\bullet} = \{t_1\} \cup \{\langle\emptyset, S, \bullet\rangle \mid S \in \mathcal{P}^*(\mathbf{T}_1)\},$$

$$\mathbf{T}^{atom} = \mathbf{T}_0^{atom} = \{t_1, t_2\} \cup \{\langle\emptyset, S, *\rangle \mid S \in \mathcal{P}^*(\mathbf{T}_1), * \in \{\bullet, \circ\}\},$$

$$\mathbf{T}_{0,1}^{circ} = \{t_3, t_4\} \cup \{\langle\emptyset, t{\uparrow} \cup S, \circ\rangle \mid t \in \{t_1, \langle\emptyset, \emptyset, \bullet\rangle, \langle\emptyset, \{t_2\}, \bullet\rangle\},$$
$$S \in \mathcal{P}^*(\{t_2, \langle\emptyset, \{t_2\}, \circ\rangle, \langle\emptyset, \emptyset, \circ\rangle\})\}\}.$$

See also Figure 5.2.

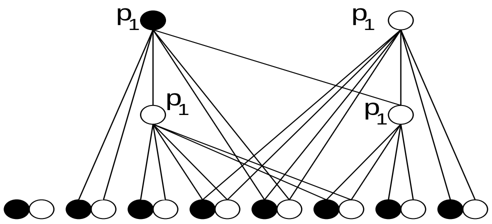

Figure 5.2: Hasse's diagram of $\langle\mathbf{T}_1 \cup \mathbf{T}^{atom}, \leq, R, P^t\rangle$ for the case that $n = 1$

5.8 Normal forms in $[\wedge, \supset, \bigcirc, \perp]^n$

In Definition 5.6.8, we define the formula $\phi(X)$ for $X \in \mathcal{P}^*(\mathbf{T})$. By Lemma 5.6.9, the formula has the following property.

5.8.1. Fact. Let it be that $X \in \mathcal{P}^*(\mathbf{T})$. Then

(1) $\phi(X) \in th(t)$ iff $t \in X$,

(2) for any formula $A \in [\wedge, \supset, \bigcirc, \perp]^n$, $A \equiv_{PLL} \phi(P^t(A))$.

However, $\phi(X)$ was defined by using unspecified canonical representatives of $(\bigcap_{t \in X} th(t))/ \equiv_{PLL}$. So, we do not know the form of $\phi(X)$. In this section we inductively define a formula equivalent to $\phi(X)$.

5.8.2. Definition. We fix the enumeration **ENU** of all formulas in $[\wedge, \supset, \bigcirc, \bot]^n$ whose first $n+1$ formulas are

$$\bot, p_1, \cdots, p_n.$$

5.8.3. Definition.

$$\mathbf{base} = \{p_1, \cdots, p_n, \bot, \bigcirc p_1, \cdots, \bigcirc p_n, \bigcirc\bot\}.$$

5.8.4. Definition. Let it be that $0 \leq k \leq n$ and $t \in \mathbf{T}_k - \mathbf{T}_{k+1}$. We put

$$\mathbf{base}(t) = \begin{cases} \mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{k+1}\} & \text{if } t \in \mathbf{T}^{atom} \\ \mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{k}\} & \text{if } t \in \mathbf{T}^{circ} \end{cases},$$

$$\mathbf{Tbase}(t) = th(t) \cap \mathbf{base}(t),$$

$$\mathbf{NTbase}(t) = \{A \in \mathbf{base}(t) \mid t \in \mathbf{T}^A\},$$

$$\Phi(t) = \mathbf{Tbase}(t) \cup \{A \supset B \mid A, B \in \mathbf{NTbase}(t)\} \cup \{\Psi(t_1) \supset A_t \mid t_1 \in \mathbf{Minl}(2nd(t))\}$$
$$\cup \{\Psi(t_1) \mid t_1 \in \mathbf{Maxl}(\{t_2 \mid \mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(t_2), t_2 \notin 2nd(t)\})\},$$

$$\Psi(t) = \bigwedge \Phi(t) \supset A_t,$$

where A_t is a member of $\mathbf{NTbase}(t)$ and is the first to occur in **ENU**.

5.8.5. Theorem. *For any world t in Exm,*

$$\{A \mid (\bigwedge \Phi(t)) \supset A \in \mathbf{PLL}\} = th(t).$$

The proof of theorem needs some preparations.

5.8.6. Lemma. *Let t be a world in Exm. Then*
(1) $\mathbf{Tbase}(t) \subseteq th(t)$,
(2) $\{A \supset B \mid A, B \in \mathbf{NTbase}(t)\} \subseteq th(t)$.

Proof. (1) is Trivial. We show (2). Suppose that there exists a formula $A, B \in \mathbf{NTbase}(t)$ such that $A \supset B \notin th(t)$. Then there exists a world $t_1 \in t{\Uparrow}$ such that $A \in th(t_1)$ and $B \notin th(t_1)$. By $B \in \mathbf{NTbase}(t)$, t is B-independent, and hence $B \in th(t_2)$ for each $t_2 \in t{\Uparrow} - \{t\}$. Hence $t_1 = t$, and thereby, $A \in th(t)$. However, by $A \in \mathbf{NTbase}(t)$, t is A-independent, and hence, $A \notin th(t)$. This is a contradiction. ⊣

5.8.7. LEMMA. *Let* t *be an* A*-independent world in* Exm*. If* $\bigcirc A \notin th(t)$*, then* t *is* $\bigcirc A$*-independent.*

Proof. Let t_1 be A world in $t\Uparrow - \{t\}$. Then by the A-independency of t, $A \in th(t_1)$. Using the axiom $p \supset \bigcirc p$, we have $\bigcirc A \in th(t_1)$. Hence $\bigcirc A \in \bigcap_{t<t_1} th(t_1)$. Using $\bigcirc A \notin th(t)$, t is $\bigcirc A$-independent. $\dashv$

5.8.8. DEFINITION. Let t be a world in Exm. We put

$$\#(t) = \omega \cdot \#(atom(t)) + \#(th(t) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\})).$$

5.8.9. LEMMA. *Let* t *and* s *be worlds in* Exm*.*
(1) $t < s$ *implies* $\#(t) < \#(s)$*,*
(2) $\mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(s)$ *implies* $\#(t) < \#(s)$*.*

Proof. For (1): By $t < s$, we have $th(t) \subseteq th(s)$, and thereby, $\#(atom(t)) \leq \#(atom(s))$. If $\#(atom(t)) < \#(atom(s))$, then (1) is obvious. Assume that $\#(atom(t)) = \#(atom(s))$. Then using $th(t) \subseteq th(s)$, we have

$$\begin{aligned} & th(t) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\}) \\ &= th(t) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}) \\ &\subseteq th(s) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}). \end{aligned}$$

Hence we have $\#(t) \leq \#(s)$.

On the other hand, by $t < s$ and $\#(atom(t)) = \#(atom(s))$, we have $t \in \mathbf{T}^{circ}$, and thereby,

$$\mathbf{base}(t) = \mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\}.$$

From the definition of A_t, $A_t \notin th(t)$ and $A_t \in th(s) \cap \mathbf{base}(t)$. Hence,

$$A_t \notin th(t) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\})$$

and

$$A_t \in th(s) \cap (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}).$$

Hence we have $\#(t) < \#(s)$.

For (2): By $\mathbf{Tbase}(t) \subseteq th(s)$, we have $\#(atom(t)) \leq \#(atom(s))$. If $\#(atom(t)) < \#(atom(s))$, then (2) is obvious. Assume that $\#(atom(t)) = \#(atom(s))$. Using $\mathbf{Tbase}(t) \cup \mathbf{NTbase} \subseteq th(s)$, we have $atom(t) \cup (\mathbf{NTbase} \cap \mathbf{atom}^n) = atom(s)$. Hence $\mathbf{NTbase} \cap \mathbf{atom}^n \subseteq atom(t)$. Since $atom(t) \cap \mathbf{NTbase} = \emptyset$, we have $\mathbf{NTbase} \cap \mathbf{atom}^n = \emptyset$. Hence $t \in \mathbf{T}^{circ}$, and thereby,

$$\mathbf{base}(t) = (\mathbf{base} \cup \{\bigcirc \Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\}).$$

Hence

$$\begin{aligned}
& th(t) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\}) \\
= \; & (th(t) \cap \mathbf{base}(t)) \cap \mathbf{base}(t) \\
= \; & \mathbf{Tbase}(t) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}) \\
\subseteq \; & th(s) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}).
\end{aligned}$$

Hence we have $\#(t) \leq \#(s)$.

By $t \in \mathbf{T}^{circ}$,

$$A_t \notin \mathbf{Tbase}(t) = th(t) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\})$$

and

$$\begin{aligned}
& A_t \in \mathbf{NTbase}(t) \cap \mathbf{base}(t) \\
& \subseteq th(s) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(t))}\}) \\
& = th(s) \cap (\mathbf{base} \cup \{\bigcirc\Psi(t_1) \mid t_1 \in \mathbf{T}^{\bullet}_{\#(atom(s))}\}).
\end{aligned}$$

Hence we have $\#(t) < \#(s)$. $\dashv$

5.8.10. Lemma. *Let t be a world in Exm.*

(1) $\Phi(t) \subseteq th(t)$.

(2) $A \in th(t)$ *implies* $\bigwedge \Phi(t) \supset A \in \mathbf{PLL}$.

Proof. We use an induction on $\#(t)$. Note that $\#(t) < \omega \cdot n + \#(\mathbf{T}^{\bullet}) + 2n + 2$. Suppose that the lemma holds for any world t^* in Exm such that $\#(t^*) > \#(t)$.

For (1): By Lemma 5.8.6, it is sufficient to show the following two:

(1.1) $\{\Psi(t_1) \supset A_t \mid t_1 \in \mathbf{Minl}(2nd(t))\} \subseteq th(t)$,

(1.2) $\{\Psi(t_1) \mid t_1 \in \mathbf{Maxl}(\{t_2 \mid \mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(t_2), t_2 \notin 2nd(t)\})\} \subseteq th(t)$.

For (1.1): Suppose that there exists a world $t_1 \in \mathbf{Minl}(2nd(t))$ such that $\Psi(t_1) \supset A_t \notin th(t)$. By Corollary 5.3.10 and Lemma 5.3.9, there exists an A_t-independent world $s \in t\Uparrow$ such that $\Psi(t_1) \in th(s)$. Since t is also A_t-independent, we have $t = s$, and thereby, $\Psi(t_1) \in th(t)$. Using $t_1 \in \mathbf{Minl}(2nd(t))$, we have $\Psi(t_1) \in th(t_1)$.

On the other hand, by Lemma 5.8.9(1), $\#(t) < \#(t_1)$. Using the induction hypothesis, we have $\Phi(t_1) \subseteq th(t_1)$. Hence $A_{t_1} \in th(t_1)$. This is in contradiction with the A_{t_1}-independency of t_1.

For (1.2): Suppose that there exists a world $t_1 \notin 2nd(t)$ such that $\mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(t_1)$ and $\Psi(t_1) \notin th(t)$. By Corollary 5.3.10 and Lemma 5.3.9, there exists an A_{t_1}-independent world $s \in t\Uparrow$ such that $\Phi(t_1) \subseteq th(s)$.

On the other hand, by Lemma 5.8.9(2), $\#(t) < \#(t_1)$. Using the induction hypothesis, $A \in th(t_1)$ implies $\bigwedge \Phi(t_1) \supset A \in \mathbf{PLL}$. Hence $th(t_1) \subseteq th(s)$. Using Lemma 5.4.9 and Lemma 5.6.7, we have $t_1 \leq s$. Since t_1 is A_{t_1}-independent, we have $t_1 = s \in t\Uparrow$. Using $t_1 \notin 2nd(t)$, we have $t_1 = t$. This is in contradiction with

$\mathbf{NTbase}(t) \subseteq th(t_1)$.

For (2); Suppose that $\bigwedge \Phi(t) \supset A \notin \mathbf{PLL}$. Then there exists a world s in Exm such that $\Phi(t) \subseteq th(s)$ and $A \notin th(s)$. We show the following three:

(2.1) for each $u \in s\!\uparrow$, $A_t \in th(u)$ implies $t < u$,

(2.2) for each A_t-independent world $u \in s\!\uparrow$, $t = u$,

(2.3) $A_t \notin th(s)$ implies $t = s$.

By $s \in s\!\uparrow$, (2.1) and (2.3), we have $t \leq s$. Using $A \notin th(s)$, we obtain $A \notin th(t)$. We needs (2.2) for the proof of (2.3).

For (2.1): Suppose that $u \in s\!\uparrow$, $A_t \in th(u)$ and $u \notin 2nd(t)$. Then $\{A_t\} \cup \Phi(t) \subseteq th(u)$. By $A_t \in \mathbf{NTbase}(t)$, $\{A_t \supset B \mid B \in \mathbf{NTbase}(t)\} \subseteq \Phi(t) \subseteq th(u)$, and hence, $\mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(u)$. Using $u \notin 2nd(t)$, u belongs to the set

$$\mathbf{U} = \{u_1 \mid \mathbf{Tbase}(t) \cup \mathbf{NTbase}(t) \subseteq th(u_1), u_1 \notin 2nd(t)\}.$$

Hence there exists a world $u_1 \in \mathbf{Maxl}(\mathbf{U}) \cap u\!\uparrow$, and thereby, $\Psi(u_1) \in \Phi(t) \subseteq th(u) \subseteq th(u_1)$. By $u_1 \in \mathbf{U}$ and Lemma 5.8.9(2), we have $\#(t) < \#(u_1)$. Using the induction hypothesis, we have $\Phi(u_1) \subseteq th(u_1)$. Using $\Psi(u_1) \in th(u_1)$, we have $A_{u_1} \in th(u_1)$. This is in contradiction with the A_{u_1}-independency of u_1.

For (2.2): Suppose that $u \in s\!\uparrow$ is A_t-independent. We show $2nd(u) = 2nd(t)$. Let u_1 be a world in $2nd(u)$. Then we have $\Phi(t) \subseteq th(u_1)$. Also by the A_t-independency of u, $A_t \in th(u_1)$. Using (2.1), we obtain $u_1 \in 2nd(t)$.

Suppose that $t_1 \in 2nd(t)$. Then $\Psi(t_1) \supset A_t \in \Phi(t) \subseteq th(u)$. Using the A_t-independency of u, we have $\Psi(t_1) \notin th(u)$. So, there exists a world $u_1 \in u\!\uparrow$ such that $\Phi(t_1) \subseteq th(u_1)$ and $A_{t_1} \notin th(u_1)$. On the other hand, by $t_1 \in 2nd(t)$ and Lemma 5.8.9(1), $\#(t_1) > \#(t)$. Using the induction hypothesis, $B \in th(t_1)$ implies $\bigwedge \Phi(t_1) \supset B \in \mathbf{PLL}$. Hence $th(t_1) \subseteq th(u_1)$. Using Lemma 5.4.9 and Lemma 5.6.7, we obtain $t_1 \leq u_1$. Using the A_{t_1}-independency of t_1 and $A_{t_1} \notin th(u_1)$, we have $t_1 = u_1 \in u\!\uparrow$. By $t_1 \in 2nd(t)$ and the A_t-independency of t, $A_t \in th(t_1)$, but by the A_t-independency of u, $A_t \notin th(u)$. Hence $t_1 \neq u$, and thereby, $t_1 \in 2nd(u)$.

We show $1st(u) = 1st(t)$. By $1st(t) \subseteq \Phi(t) \subseteq th(u)$, we obtain $1st(t) \subseteq 1st(u)$. Suppose that $a \in 1st(u)$ and $a \notin 1st(t)$. Then by $a \in 1st(u)$ we have $a \in \bigcap_{u_1 \in 2nd(u)} th(u_1)$. Using $2nd(u) = 2nd(t)$, we have $a \in \bigcap_{u_1 \in 2nd(t)} th(u_1)$. Using $a \notin 1st(t)$, t is a-independent, and thereby, $a \in \mathbf{NTbase}(t)$. Hence $a \supset A_t \in \Phi(t) \subseteq th(u)$. Using $a \in 1st(u)$, we have $A_t \in th(u)$. This is a contradiction.

We show $3rd(u) = 3rd(t)$. Suppose that $3rd(t) = \bullet$. Then by Lemma 5.3.11, $t \in \mathbf{T}^a - \mathbf{T}^{\bigcirc a}$ for an atomic formula a. Hence A_t is an atomic formula and we assume that $A_t = a$. Using Lemma 5.8.7, $\bigcirc a \in \mathbf{Tbase}(t) \subseteq \Phi(t) \subseteq th(u)$. Using the A_t-independency of u, $A_t = a \notin th(u)$. Hence u is irreflexive, i.e., $3rd(u) = \bullet$.

Suppose that $3rd(t) = \circ$. Then $\bigcirc A_t \notin th(t)$. If A_t is an atomic formula, then by Lemma 5.8.7, $A_t, \bigcirc A_t \in \mathbf{NTbase}(t)$, and thereby, $\bigcirc A_t \supset A_t \in \Phi(t) \subseteq th(u)$. Using the A_t-independency of u, $\bigcirc A_t \notin th(u)$. Using Lemma 5.8.7 and Lemma

5.3.11, u is $\bigcirc A_t$-independent and reflexive, and thereby, $3rd(u) = \circ$. If A_t is circled formula, then by Lemma 5.3.11 and A_t-independency of u, we have u is reflexive, and thereby, $3rd(u) = \circ$.

Hence we obtain (2.2).

For (2.3): Suppose that $A_t \notin th(s)$ and $t \neq s$. By Corollary 5.3.10, there exists an A_t-independent world $u \in s{\uparrow}$. By (2.2), we have $s \leq u = t$. Hence $atom(s) \subseteq atom(t)$. Also by $atom(t) \subseteq \Phi(t) \subseteq th(s)$, we have $atom(t) \subseteq atom(s)$. So, $atom(t) = atom(s)$. By $s \neq t$, we have $s < t$, and thereby, $s \in \mathbf{T}^{circ}$. Using Corollary 5.5.10, there exists $s_1 \in \mathbf{T}^{\bullet}$ such that $s <_1 s_1$.

If $A_t \in th(s_1)$, then by (2.1), $s_1 \in 2nd(t)$. Hence $s < t < s_1$. This is in contradiction with $s <_1 s_1$.

If $A_t \notin th(s_1)$, then by Corollary 5.3.10(2), there exists an A_t-independent world $s_2 \in s_1{\uparrow}$. Hence $s <_1 s_1 \leq s_2$. Using (2.2), we have $s_2 = t$. By $s_1 \in \mathbf{T}^{\bullet}_k$ and Lemma 5.5.3, s_1 is b-independent for some atomic formula b. So, we have $b \in th(t)$. Hence $b \in \Phi(t) \subseteq th(s) \subseteq th(s_1)$. This is in contradiction with the b-independency of s_1. ⊣

Hence we obtain Theorem 5.8.5. Also we have the following corollaries.

5.8.11. Corollary. *Let t be a world in Exm Then*

$$\bigwedge \Phi(t) \equiv_{PLL} \phi(t{\uparrow}),$$

$$\Psi(t) \equiv_{PLL} \phi(t{\uparrow}) \supset A_t.$$

5.8.12. Corollary. *Let X be a closed subset of $\mathbf{T}$ in Exm and let it be that $A_X = \bigwedge_{s \in \mathbf{Minl}(X)} A_s$. Then*

$$\bigwedge_{t \in \mathbf{Minl}(X)} (\bigwedge \Phi(t) \supset A_X) \supset A_X \equiv_{PLL} \phi(X),$$

$$\bigwedge_{t \in \mathbf{Minl}(X)} (\bigwedge \Phi(t) \supset A_X) \equiv_{PLL} \phi(X) \supset A_X,$$

especially, if $\mathbf{Minl}(X) \subseteq \mathbf{T}^{A_s}$ for some $s \in \mathbf{Minl}(X)$,

$$\bigwedge_{t \in \mathbf{Minl}(X)} \Psi(t) \supset A_s \equiv_{PLL} \phi(X),$$

$$\bigwedge_{t \in \mathbf{Minl}(X)} \Psi(t) \equiv_{PLL} \phi(X) \supset A_s.$$

5.8.13. COROLLARY. *Let A be a formula in $[\wedge, \supset, \bigcirc, \perp]^n$ and let it be that $B = \bigwedge_{s \in \mathbf{Minl}(P^t(A))} A_s$. Then*

$$A \equiv_{PLL} \bigwedge_{t \in \mathbf{Minl}(P^t(A))} (\bigwedge \Phi(t) \supset B) \supset B,$$

especially, if $\mathbf{Minl}(P^t(A)) \subseteq \mathbf{T}^{A_s}$ *for some* $s \in \mathbf{Minl}(P^t(A))$,

$$A \equiv_{PLL} \bigwedge_{t \in \mathbf{Minl}(P^t(A))} \Psi(t) \supset A_s,$$

5.8.14. COROLLARY. *For any formula $A \in [\wedge, \supset, \bigcirc, \perp]^n$, there exists a formula B constructed from the formulas in*

$$\mathbf{base} \cup \{\bigcirc \Psi(t) \mid t \in \mathbf{T}^\bullet\}$$

by using $\wedge$ and $\supset$ such that $A \equiv_{PLL} B$.

5.8.15. COROLLARY. *For any formula $A \in [\wedge, \supset, \bigcirc, \perp]^0$, there exists a formula B constructed from the formulas in*

$$\{\perp, \bigcirc\perp, \bigcirc(\bigcirc\perp \supset \perp)\}$$

by using $\wedge$ and $\supset$ such that $A \equiv_{PLL} B$.

Chapter 6

Interpretability logics

In this chapter, we give a cut-free sequent system for the interpretability logic **IL**. To begin with, we give a cut-free system for the sublogic **IK4** of **IL**, whose $\rhd$-free fragment is the modal logic **K4** in the sense of section 1.3. Using the system for **IK4** and a property of Löb's axiom, a cut-free system for **IL** can be given in the way given in section 4.4.

6.1 Introduction

As we mentioned in section 1.3, the language of interpretability logic contains two modal operators $\Box$ and $\rhd$. However, without using Löb's axiom, we can show the equivalence between $\Box A$ and $\neg A \rhd \bot$ in the logic **IL** introduced in section 1.3 (cf. [JJ98]). Hence, we do not have to treat $\Box$ as a primary operator. Systems for interpretability logics with two primary modal operators are much more complicated than the ones with one primary modal operator. So, in this chapter, we treat $\Box A$ as an abbreviation of $\neg A \rhd \bot$. To do so, however, we have to redefine formulas and reintroduce interpretability logics. This section is devoted to reintroducing a basic interpretability logic **IL** and its sublogic **IK4**.

First we redefine formulas. We also use **WFF** for the set of new formulas.

6.1.1. Definition. The set **WFF** of formulas are defined inductively as follows.

(1) a propositional variable belongs to **WFF**,
(2) $\bot \in$ **WFF**,
(3) $A, B \in$ **WFF** implies $A \wedge B, A \vee B, A \supset B, A \rhd B \in$ **WFF**.

An element of **WFF** is said to be a formula, especially a formula of the form $A \rhd B$ is said to be a $\rhd$-formula.

6.1.2. NOTATION. The expressions $\neg A$, $\Box A$ and $\Diamond A$ are abbreviations for $A \supset \bot, \neg A \rhd \bot$ and $\neg(A \rhd \bot)$, respectively.

As to the other terminology, we follow chapter 2.

6.1.3. DEFINITION. The degree $d(A)$ of a formula A is defined inductively as follows:

(1) $d(p) = 1$,
(2) $d(\bot) = 0$,
(3) $d(A \wedge B) = d(A \vee B) = d(A \supset B) = d(A \rhd B) = d(A) + d(B) + 1$.

Note that $d(A \rhd \bot) < d(A \rhd B)$ for each $B \neq \bot$.

6.1.4. DEFINITION. We define two modal logics.

(1) By **IK4**, we mean the smallest set of formulas containing all the tautologies and axioms

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,
$4 : \Box p \supset \Box\Box p$,
$J1 : \Box(p \supset q) \supset (p \rhd q)$,
$J2 : (p \rhd q) \wedge (q \rhd r) \supset (p \rhd r)$,
$J3 : (p \rhd r) \wedge (q \rhd r) \supset ((p \vee q) \rhd r)$,
$J5 : (\Diamond p) \rhd p$,

and closed under modus ponens, substitution and necessitation.

(2) By **IL**, we mean the smallest set of formulas containing all the theorems in **IK4** and Löb's axiom

$$L^{\Box}(p) = \Box(\Box p \supset p) \supset \Box p$$

and closed under modus ponens, substitution and necessitation.

We note that the axiom

$J4 : (p \rhd q) \supset (\Diamond p \supset \Diamond q)$

described in section 1.3 is not in the list of axioms above. Because it is provable in **IK4** defined in Definition 6.1.4. We show this provability, briefly. The expression of $J4$ above is an abbreviation of

$$(p \rhd q) \supset (\neg(p \rhd \bot) \supset \neg(q \rhd \bot)).$$

It is easily seen that the formula above is equivalent to

$$(p \rhd q) \wedge (q \rhd \bot) \supset (p \rhd \bot)$$

by tautologies, modus ponens and substitution. Also we find this formula is a substitution instance of the axiom $J2$, and so it is provable in **IK4**.

The aim of this chapter is to give sequent systems for **IL** and **IK4**. In the next section we give a sequent system **GIK4**. Section 6.3 is devoted to showing the equivalence between **IK4** and **GIK4**. Cut-elimination theorem is shown in section 6.4. In section 6.5 and section 6.6, we give a cut-free sequent system for **IL**.

6.2 The system GIK4

In this section we introduce a sequent system **GIK4**.

As we mentioned in chapter 2, we use Greek letters, possibly with suffixes, for finite sets of formulas. Here Δ and Λ are also for finite sets of formulas while they were used for sets containing at most one formula. In this chapter, we often use finite sets of $\rhd$-formulas. So, it is useful to prepare symbols for them and we use Σ, possibly with suffixes, for finite sets of $\rhd$-formulas. For each prefix $\odot \in \{\Box, \Diamond, \neg\}$, the expression $\odot\Gamma$ denotes the set $\{\odot A \mid A \in \Gamma\}$. Similarly, $\Gamma \rhd \bot$ denotes $\{A \rhd \bot \mid A \in \Gamma\}$. By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma^\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

Our system **GIK4** is defined from the following axioms and inference rules in the usual way.

Axioms of GIK4

$$A \to A$$

$$\bot \to$$

Inference rules of GIK4

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(T \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to T)$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi_A \to \Delta_A, \Lambda}(\text{cut})$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n \quad \Sigma \to Y_1 \rhd B \ \cdots \ \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to A \rhd B}(\rhd_{IK4})$$

where $n = 0, 1, 2, \cdots$.

A proof figure in **GIK4** for a sequent S is defined as in Definition 2.2.5. As to the other terminology concerning the system, we also follow section 2.2.

If $n = 0$, the inference rule $(\rhd_{IK4})$ has only one upper sequent and is of the following form:

$$\frac{A, B \rhd \bot \to B}{\Sigma \to A \rhd B}$$

Hence

6.2.1. LEMMA. *There exist cut-free proof figures for* $\to \bot \rhd A$ *and* $\to A \rhd A$ *in* **GIK4**.

6.3 Equivalence between IK4 and GIK4

The main theorem in this section is

6.3.1. THEOREM. $A \in$ **IK4** *iff* $\to A \in$ **GIK4**.

To prove the theorem above, we need some preparations.

6.3.2. DEFINITION. By **GK4**, we mean the system obtained from **GIK4** by replacing $(\rhd_{IK4})$ by

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box_{K4}).$$

By **GK4** $+ J$, we mean the system obtained from **GK4** by adding the following four axioms:

$(GJ1) : \Box(A \supset B) \to A \rhd B,$
$(GJ2) : A \rhd B, B \rhd C \to A \rhd C,$
$(GJ3) : A \rhd C, B \rhd C \to (A \vee B) \rhd C,$
$(GJ5) :\to \Diamond A \rhd A.$

It is known that **GK4** enjoys the cut-elimination theorem and that $\rightarrow A \in$ **GK4** iff $A \in$ **K4**. So, we have

6.3.3. Lemma. *$A \in$ **IK4** iff $\rightarrow A \in$ **GK4** $+ J$.*

6.3.4. Lemma. *$\rightarrow A \in$ **GK4** $+ J$ implies $\rightarrow A \in$ **GIK4**.*

Proof. It is sufficient to show that the four axioms $J1, J2, J3$ and $J5$ are provable in **GIK4** and $(\Box_{K4})$ holds in **GIK4**. This is shown by the following inference rules and Lemma 6.2.1.

$$\frac{p, q \rhd \bot, \neg(p \supset q) \rhd \bot \rightarrow q, \neg(p \supset q) \qquad \rightarrow \bot \rhd q}{\neg(p \supset q) \rhd \bot \rightarrow p \rhd q}$$

$$\frac{p, r \rhd \bot, p \rhd \bot \rightarrow r, p \qquad q \rhd r \rightarrow q \rhd r}{p \rhd q, q \rhd r \rightarrow p \rhd r}$$

$$\frac{p \vee q, r \rhd \bot, p \rhd \bot, q \rhd \bot \rightarrow r, p, q \qquad \rightarrow r \rhd r \qquad \rightarrow r \rhd r}{p \rhd r, q \rhd r \rightarrow p \vee q \rhd r}$$

$$\frac{\Diamond p, p \rhd \bot \rightarrow p}{\rightarrow \Diamond p \rhd p}$$

$$\frac{\neg A, \Box\Gamma, \bot \rhd \bot \rightarrow \neg\Gamma, \bot \qquad \rightarrow \bot \rhd \bot \quad \cdots \quad \rightarrow \bot \rhd \bot}{\Box\Gamma \rightarrow \Box A}$$

$\dashv$

6.3.5. Lemma. *The following rules hold in* **GK4**$+J$.

(1) $\Gamma \rightarrow \Box(A \supset B) \in$ **GK4** $+ J$ *implies* $\Gamma \rightarrow A \rhd B \in$ **GK4** $+ J$,

(2) *if* $\Gamma \rightarrow A \rhd B$ *and* $\Gamma \rightarrow B \rhd C$ *are provable in* **GK4** $+ J$, *then so is* $\Gamma \rightarrow A \rhd C$,

(3) *if* $\Gamma \rightarrow A \rhd C$ *and* $\Gamma \rightarrow B \rhd C$ *are provable in* **GK4** $+ J$, *then so is* $\Gamma \rightarrow A \vee B \rhd C$,

(4) $\rightarrow (B \vee \Diamond B) \rhd B \in$ **GK4** $+ J$,

(5) $\Gamma \rightarrow A \rhd (B \vee \Diamond B) \in$ **GK4** $+ J$ *implies* $\Gamma \rightarrow A \rhd B \in$ **GK4** $+ J$,

(6) $\Gamma \rightarrow A \rhd B \in$ **GK4** $+ J$ *implies* $\Gamma \rightarrow (A \vee \Diamond A) \rhd B \in$ **GK4** $+ J$.

Proof. We obtain (1), (2) and (3), from $(GJ1)$, $(GJ2)$ and $(GJ3)$, respectively. (5) and (6) are from (2) and (4).

So, we only show (4). By $(GJ1)$, it is easily seen that $\rightarrow B \rhd B \in$ **GK4** $+ J$. Using $(GJ5)$ and (3), we obtain (4). $\dashv$

6.3.6. LEMMA. $\rightarrow A \in \mathbf{GIK4}$ *implies* $\rightarrow A \in \mathbf{GK4} + J$.

Proof. It is sufficient to show that the rule ($\rhd_{IK4}$) holds in $\mathbf{GK4}+J$. Suppose
(1) $A, B \rhd \bot, X_1 \rhd \bot, \cdots, X_n \rhd \bot \rightarrow B, X_1, \cdots, X_n \in \mathbf{GIK4} + J$
and
(2) $\Sigma \rightarrow Y_i \rhd B \in \mathbf{GIK4} + J$ for $i = 1, \cdots, n$.
Clearly,

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow X_i \rhd Y_i \in \mathbf{GIK4} + J \text{ for } i = 1, \cdots, n.$$

Using (2) and Lemma 6.3.5(2),

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow X_i \rhd B \in \mathbf{GIK4} + J \text{ for } i = 1, \cdots, n.$$

Using Lemma 6.3.5(6),

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow (X_i \vee \Diamond X_i) \rhd B \in \mathbf{GIK4} + J \text{ for } i = 1, \cdots, n.$$

Using Lemma 6.3.5(3) and Lemma 6.3.5(4),

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow ((B \vee \Diamond B) \vee \bigvee_{i=1}^{n} (X_i \vee \Diamond X_i)) \rhd B \in \mathbf{GIK4} + J. \quad (6.1)$$

On the other hand, by (1), we have

$$A \rightarrow (B \vee \Diamond B) \vee \bigvee_{i=1}^{n} (X_i \vee \Diamond X_i) \in \mathbf{GIK4} + J.$$

Using ($\rightarrow\supset$) and ($\Box_{K4}$),

$$\rightarrow \Box(A \supset (B \vee \Diamond B) \vee \bigvee_{i=1}^{n} (X_i \vee \Diamond X_i)) \in \mathbf{GIK4} + J.$$

Using Lemma 6.3.5(1),

$$\rightarrow A \rhd ((B \vee \Diamond B) \vee \bigvee_{i=1}^{n} (X_i \vee \Diamond X_i)) \in \mathbf{GIK4} + J.$$

Using (6.1) and Lemma 6.3.5(2),

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow A \rhd B \in \mathbf{GIK4} + J.$$

$\dashv$

From Lemma 6.3.3, Lemma 6.3.4 and Lemma 6.3.6, we obtain Theorem 6.3.1.

6.4 Cut-elimination theorem for GIK4

In this section, we prove cut-elimination theorem for **GIK4**.

6.4.1. THEOREM. *If $\Gamma \to \Delta \in$ **GIK4**, then there exists a cut-free proof figure for $\Gamma \to \Delta$ in **GIK4**.*

To prove the theorem, we need some lemmas.

6.4.2. LEMMA. *Let P_1 and P_2 be cut-free proof figures for $\Sigma_1 \to A \rhd B$ and $\Sigma_2 \to B \rhd C$, respectively. Then there exists a cut-free proof figure for $\Sigma_1, \Sigma_2 \to A \rhd C$.*

Proof. We use an induction on P_1. If P_1 is an axiom, then $\Sigma_1 = \{A \rhd B\}$, and hence we have the following cut-free proof figure for $\Sigma_1, \Sigma_2 \to A \rhd C$.

$$\dfrac{\dfrac{\dfrac{A \to A}{\text{using } (T \to) \text{ twice, and } (\to T)}}{A, C \rhd \bot, A \rhd \bot \to C, A} \qquad \left.\begin{matrix} \vdots \\ \Sigma_2 \to B \rhd C \end{matrix}\right\} P_2}{A \rhd B, \Sigma_2 \to A \rhd C}$$

If P_1 is not axiom, then there exists an inference rule I that introduces the end sequent of P_1. We only show the case that I is $(\rhd_{IK4})$ since the other cases can be shown easily. The inference rule I is of the form

$$\dfrac{A, \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n \quad \Sigma_1' \to Y_1 \rhd B \quad \cdots \quad \Sigma_1' \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma_1' \to A \rhd B}$$

where $\Sigma_1 = \Sigma_1' \cup \{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n\}$. Clearly, there exist cut-free proof figures for the upper sequents of I. Using the induction hypothesis and P_2, there exists a cut-free proof figure for $\Sigma_1', \Sigma_2 \to Y_i \rhd C$ for each $i = 1, \cdots, n$. Using $(\rhd_{IK4})$ below, we obtain the lemma.

$$\dfrac{A, \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n \quad \Sigma_1', \Sigma_2 \to Y_1 \rhd C \quad \cdots \quad \Sigma_1', \Sigma_2 \to Y_n \rhd C}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma_1', \Sigma_2 \to A \rhd C}$$

$\dashv$

6.4.3. LEMMA. *If there exists a cut-free proof figure for $\Sigma \to A \rhd B$, then either one of the following two holds:*

(1) *there exists a cut-free proof figure for $\Sigma \to$,*

(2) *for some subsets Σ_1 and Σ_2 of Σ, there exist cut-free proof figures for*

$$A, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\} \to \{X \mid X \rhd Y \in \Sigma_1\}, B$$

and

$$\Sigma_2 \to Y \rhd B, \textit{ for each } Y \in \{Y' \mid X \rhd Y' \in \Sigma_1\}.$$

Proof. We use an induction on the cut-free proof figure P for $\Sigma \to A \rhd B$. If P is an axiom, then $\{A \rhd B\} = \Sigma$ and by Lemma 6.2.1, there exist cut-free proof figures for

$$A, B \rhd \bot, A \rhd \bot \to A, B \text{ and } \to B \rhd B.$$

Hence (2) holds.

If P is not axiom, then there exists an inference rule I that introduces the end sequent of P. If I is $(\to T)$, then (1) holds. If I is $(T \to)$, then by the induction hypothesis, we obtain the lemma. If I is $(\rhd_{IK4})$, then (2) holds. $\dashv$

It is easily seen that Theorem 6.4.1 follows from the following lemma.

6.4.4. Lemma. *Let P^ℓ be a cut-free proof figure for $\Gamma \to \Delta, X$ and P^r be a cut-free proof figure for $X, \Pi \to \Lambda$. Let P be the proof figure*

$$\dfrac{P^\ell\left\{\begin{array}{c}\vdots\\ \Gamma \to \Delta, X\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ X, \Pi \to \Lambda\end{array}\right\}P^r}{\Gamma, \Pi_X \to \Delta_X, \Lambda}.$$

Then there exists a cut-free proof figure for the end sequent of P.

Proof. The degree $d(P)$ of P is defined as $d(X)$. The left rank $R^\ell(P)$ and the right rank $R^r(P)$ of P are defined as usual. We use an induction on $R^\ell(P) + R^r(P) + \omega d(P)$. We only treat the case that P, P^ℓ and P^r are of the following forms.

P^ℓ:

$$\dfrac{P^\ell_0\left\{\begin{array}{c}\vdots\\ C, \mathbf{X}^\ell \rhd \bot \to \mathbf{X}^\ell\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ \Sigma^L \to Y^\ell_1 \rhd D\end{array}\right\}P^\ell_1 \cdots \quad \left.\begin{array}{c}\vdots\\ \Sigma^L \to Y^\ell_m \rhd D\end{array}\right\}P^\ell_m}{\Sigma^\ell, \Sigma^L \to C \rhd D}$$

P^r:

$$\dfrac{P^r_0\left\{\begin{array}{c}\vdots\\ A, \mathbf{X}^r \rhd \bot \to \mathbf{X}^r\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ \Sigma^R \to Y^r_1 \rhd B\end{array}\right\}P^r_1 \cdots \quad \left.\begin{array}{c}\vdots\\ \Sigma^R \to Y^r_n \rhd B\end{array}\right\}P^r_n}{C \rhd D, \Sigma^r, \Sigma^R \to A \rhd B}$$

P:

$$\dfrac{P^\ell \left\{ \begin{array}{c} \vdots \\ \Sigma^\ell, \Sigma^L \to C \rhd D \end{array} \right. \qquad \left. \begin{array}{c} \vdots \\ C \rhd D, \Sigma^r, \Sigma^R \to A \rhd B \end{array} \right\} P^r}{\Sigma^\ell, \Sigma^L, \Sigma^r_{C\rhd D}, \Sigma^R_{C\rhd D} \to A \rhd B}$$

where
$\Sigma^\ell = \{X^\ell_1 \rhd Y^\ell_1, \cdots, X^\ell_m \rhd Y^\ell_m\}$,
$\Sigma^r = \{X^r_1 \rhd Y^r_1, \cdots, X^r_n \rhd Y^r_n\}$,
$\mathbf{X}^\ell = \{X^\ell_1, \cdots, X^\ell_m, D\}$,
$\mathbf{X}^r = \{X^r_1, \cdots, X^r_n, B\}$
and $C \rhd D \in \Sigma^r \cup \Sigma^R$.

By P^ℓ and P^r_j, we have the following proof figure for each $j = 1, \cdots, n$:

$$\dfrac{P^\ell \left\{ \begin{array}{c} \vdots \\ \Sigma^\ell, \Sigma^L \to C \rhd D \end{array} \right. \qquad \left. \begin{array}{c} \vdots \\ \Sigma^R \to Y^r_j \rhd B \end{array} \right\} P^r_j}{\Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to Y^r_j \rhd B}$$

We note that the degree and the left rank of the figure above are the same as those of P and that the right rank is smaller. Using the induction hypothesis, we obtain a cut-free proof figure Q^r_j for the end sequent of the figure above.

If $C \rhd D \notin \Sigma^r$, then by Q^r_j, P^r_0 and $(\rhd_{IK4})$, we obtain a cut-free proof figure for the end sequent of P.

Assume that $C \rhd D \in \Sigma^r = \{X^r_1 \rhd Y^r_1, \cdots, X^r_n \rhd Y^r_n\}$. Without loss of generality, we also assume that $C \rhd D = X^r_1 \rhd Y^r_1 \notin \Sigma^r - \{X^r_1 \rhd Y^r_1\}$. It is sufficient to show the case that $C = \bot$ and the case that $C \neq \bot$.

The case that $C = \bot$: By P^r_0, we have the following proof figure Q_1:

$$\dfrac{\dfrac{\dfrac{\bot \to \bot}{\bot, \bot \rhd \bot \to \bot}}{\to \bot \rhd \bot} \qquad \left. \begin{array}{c} \vdots \\ A, \{B, \bot, X^r_2, \cdots, X^r_n\} \rhd \bot \to B, \bot, X^r_2, \cdots, X^r_n \end{array} \right\} P^r_0}{A, \{B, X^r_2, \cdots, X^r_n\} \rhd \bot \to B, \bot, X^r_2, \cdots, X^r_n}$$

If $D = \bot$, then $d(Q_1) = d(\bot \rhd \bot) = d(\bot \rhd D) = d(P)$, $1 = R^\ell(Q_1) = R^\ell(P)$ and $R^r(Q_1) < R^r(P)$; if not, $d(Q_1) = d(\bot \rhd \bot) < d(\bot \rhd D) = d(P)$. Using the induction hypothesis, we obtain a cut-free proof figure for the end sequent of the figure above. Using the axiom $\bot \to$, (cut) and the induction hypothesis, we obtain the proof figure for $A, \{B, X^r_2, \cdots, X^r_n\} \rhd \bot \to B, X^r_2, \cdots, X^r_n$. Using $Q^r_2, \cdots, Q^r_n$ and $(\rhd_{IK4})$, we have a cut-free proof figure for the end sequent of P.

The case that $C \neq \bot$: By P^ℓ_0, Lemma 6.2.1 and $(\rhd_{IK4})$, we have the following cut-free proof figure:

$$\dfrac{P^\ell_0 \left\{ \begin{array}{c} \vdots \\ C, \{D, X^\ell_1, \cdots, X^\ell_m\} \rhd \bot \to D, X^\ell_1, \cdots, X^\ell_m \end{array} \right. \qquad \begin{array}{c} \vdots \\ \to \bot \rhd \bot \end{array} \quad \cdots \quad \begin{array}{c} \vdots \\ \to \bot \rhd \bot \end{array}}{\{D, X^\ell_1, \cdots, X^\ell_m\} \rhd \bot \to C \rhd \bot}$$

Using P_0^r, we have the following proof figure P_1:

$$
\dfrac{\dfrac{P_0^\ell \quad \begin{array}{c}\vdots\\ \to \bot \rhd \bot\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \to \bot \rhd \bot\end{array}}{\mathbf{X}^\ell \rhd \bot \to C \rhd \bot} \qquad \left.\begin{array}{c}\vdots\\ A, \{B, C, X_2^r, \cdots, X_n^r\} \rhd \bot \to B, C, X_2^r, \cdots, X_n^r\end{array}\right\} P_0^r}{\{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot, A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot \to B, C, X_2^r, \cdots, X_n^r}
$$

If $D = \bot$, then $d(P_1) = d(C \rhd \bot) = d(C \rhd D) = d(P)$, $1 = R^\ell(P_1) = R^\ell(P)$ and $R^r(P_1) < R^r(P)$; if not, $d(P_1) = d(C \rhd \bot) < d(C \rhd D) = d(P)$. Using the induction hypothesis, we obtain a cut-free proof figure P_2 for the end sequent of the figure above. Using P_0^ℓ, again,

$$
\dfrac{P_2 \qquad \left.\begin{array}{c}\vdots\\ C, \{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot \to D, X_1^\ell, \cdots, X_m^\ell\end{array}\right\} P_0^\ell}{A, \{B, D, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r\} \rhd \bot \to B, D, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r}
$$

We note the degree of the figure above is smaller than that of P. Using the induction hypothesis, we obtain a cut-free proof figure P_3 for the end sequent of the figure above.

By Q_1^r and Lemma 6.4.3, either one of the following two holds:

(1) there exists a cut-free proof figure for $\Sigma^\ell, \Sigma^L, \Sigma^R_{C \rhd D} \to$,

(2) for some subsets Σ_1 and Σ_2 of $\Sigma^\ell \cup \Sigma^L \cup \Sigma^R_{C \rhd D}$, there exist cut-free proof figures for

$$
D, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\} \to \{X \mid X \rhd Y \in \Sigma_1\}, B
$$

and

$$
\Sigma_2 \to Y \rhd B, \text{ for each } Y \in \{Y' \mid X \rhd Y' \in \Sigma_1\}.
$$

If (1) holds, we obtain the lemma, immediately. Assume that (2) holds. Then by P_3 and (cut) whose cut formula is D, we have the following proof figure:

$$
\dfrac{P_3 \qquad \begin{array}{c}\vdots\\ D, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\} \to \{X \mid X \rhd Y \in \Sigma_1\}, B\end{array}}{A, D \rhd \bot, \Delta \rhd \bot \to B, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r, \{X \mid X \rhd Y \in \Sigma_1\}}
$$

where Δ is the succedent of the end sequent. We note that the degree of the proof figure above is $d(D) < d(C \rhd D) = d(P)$. Using the induction hypothesis, we have a cut-free proof figure P_4 for the end sequent of the figure above.

By (2), Lemma 6.2.1 and ($\rhd_{IK4}$), we have a cut-free proof figure for

$$
B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\} \to D \rhd \bot.
$$

Using P_4, we have the following proof figure:

$$\dfrac{\begin{array}{c}\vdots\\ B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\} \to D \rhd \bot \qquad P_4\end{array}}{A, \Delta \rhd \bot \to B, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r, \{X \mid X \rhd Y \in \Sigma_1\}}$$

Since $C \neq \bot$, the degree of the proof figure above is $d(D \rhd \bot) < d(C \rhd D) = d(P)$. Using the induction hypothesis, we have a cut-free proof figure P_5 for the end sequent of the figure above.

On the other hand, by P_i^ℓ, Q_1^r and Lemma 6.4.2, we obtain a cut-free proof figure Q_i^ℓ for $\Sigma^\ell, \Sigma^L, \Sigma^R_{C \rhd D} \to Y_i^\ell \rhd B$ for each $i = 1, \cdots, m$. Using P_5, $Q_2^r, \cdots, Q_n^r$, (2) and ($\rhd_{IK4}$), we obtain a cut-free proof figure for the end sequent of P. $\dashv$

6.5 The system GIL

In this section, we introduce a sequent system **GIL** for **IL**.

6.5.1. Definition. The system **GIL** is obtained from **GIK4** by replacing ($\rhd_{IK4}$) by the following inference rule:

$$\dfrac{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n \quad \Sigma \to Y_1 \rhd B \ \cdots \ \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to A \rhd B}(\rhd_{IL})$$

where $n = 0, 1, 2, \cdots$.

6.5.2. Theorem. *$A \in$ **IL** iff $\to A \in$ **GIL**.*

To prove the theorem above, we need some preparations.

6.5.3. Definition. By **GIK4** $+ L$, we mean the system obtained from **GIK4** by adding Löb's axiom

$$\to \Box(\Box A \supset A) \supset \Box A.$$

6.5.4. Corollary. *$A \in$ **IL** iff $\to A \in$ **GIK4** $+ L$.*

Proof. From Theorem 6.3.1. $\dashv$

6.5.5. Lemma. *$\to A \in$ **GIK4** $+ L$ implies $\to A \in$ **GIL**.*

Proof. By the following figures, we can see that Löb's axiom $\rightarrow \Box(\Box A \supset A) \supset \Box A$ is provable in **GIL** and $(\rhd_{IK4})$ holds in **GIL**.

$$\dfrac{\neg A, \Box A, \bot \rhd \bot, \Box(\Box A \supset A) \rightarrow \bot, \neg(\Box A \supset A) \qquad \rightarrow \bot \rhd \bot}{\neg(\Box A \supset A) \rhd \bot \rightarrow \neg A \rhd \bot}(\rhd_{IL})$$

$$\dfrac{\dfrac{A, \{B, X_1, \cdots, X_n\} \rhd \bot \rightarrow B, X_1, \cdots, X_n}{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot \rightarrow B, X_1, \cdots, X_n} \quad \Sigma \rightarrow Y_1 \rhd B \cdots \Sigma \rightarrow Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow A \rhd B}(\rhd_{IL})$$

⊣

6.5.6. Lemma. *$(A \wedge (A \rhd \bot)) \rhd B \rightarrow A \rhd B \in$ **GIK4** $+ L$.*

Proof. Immediately,

$$\Box(\Box\neg A \supset \neg A) \rightarrow \Box\neg A \in \mathbf{GIK4} + L.$$

The sequent is an abbreviation of

$$\neg((\neg\neg A \rhd \bot) \supset \neg A) \rhd \bot \rightarrow \neg\neg A \rhd \bot.$$

So,

$$(A \wedge (A \rhd \bot)) \rhd \bot \rightarrow A \rhd \bot \in \mathbf{GIK4} + L.$$

Using the axiom $A \rightarrow A$ and $(\rightarrow \wedge)$,

$$A, (A \wedge (A \rhd \bot)) \rhd \bot \rightarrow A \wedge (A \rhd \bot) \in \mathbf{GIK4} + L.$$

Using $(T \rightarrow)$ and $(\rightarrow T)$, we have

$$A, B \rhd \bot, (A \wedge (A \rhd \bot)) \rhd \bot \rightarrow B, A \wedge (A \rhd \bot) \in \mathbf{GIK4} + L.$$

Using Lemma 6.2.1 and $(\rhd_{IK4})$ below, we obtain the lemma.

$$\dfrac{A, B \rhd \bot, (A \wedge (A \rhd \bot)) \rhd \bot \rightarrow B, A \wedge (A \rhd \bot) \qquad \rightarrow B \rhd B}{(A \wedge (A \rhd \bot)) \rhd B \rightarrow A \rhd B}$$

⊣

6.5.7. Lemma. *$\rightarrow A \in$ **GIL** implies $\rightarrow A \in$ **GIK4** $+ L$.*

Proof. By the following figure, Lemma 6.5.6 and cut, the inference rule $(\rhd_{IL})$ holds in $\mathbf{GIK4} + L$.

$$\dfrac{\dfrac{\dfrac{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n}{A, A \wedge (A \rhd \bot), \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n}}{A \wedge (A \rhd \bot), \{B, X_1, \cdots, X_n\} \rhd \bot \to B, X_1, \cdots, X_n} \quad \Sigma \to Y_1 \rhd B \cdots \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to (A \wedge (A \rhd \bot)) \rhd B}$$

⊣

From Corollary 6.5.4, Lemma 6.5.5 and Lemma 6.5.7, we obtain Theorem 6.5.2.

6.6 Cut-elimination theorem for GIL

In this section, we prove

6.6.1. THEOREM. *If* $\Gamma \to \Delta \in \mathbf{GIL}$*, then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* $\mathbf{GIL}$.

To prove the theorem, we use the method in section 4.4.

6.6.2. DEFINITION. The expression $\Box^n A$ is defined inductively as follows:
(1) $\Box^0 A = A$,
(2) $\Box^{k+1} A = \Box(\Box^k A)$.

As in section 4.4, the following property of Löb's axiom is important.

6.6.3. LEMMA. $\Box^n L^\Box(A) \to L^\Box(A) \in \mathbf{GIK4}$*, for any* $n \geq 0$.

Proof. It can be shown that

$$\Box^{k+1} L^\Box(A) \to \Box^k L^\Box(A) \in \mathbf{GIK4}$$

for any $k \geq 0$. Using cut, possibly several times, we obtain the lemma. ⊣

6.6.4. COROLLARY. *For any* $n \geq 0$,

$$\Gamma \to \Delta \in \mathbf{GIL} \textit{ iff } \Gamma \to \Delta \in \mathbf{GIK4} + \Box^n L^\Box(p),$$

where $\mathbf{GIK4} + \Box^n L^\Box(p)$ *is the system obtained by adding* $\to \Box^n L^\Box(A)$ *to* $\mathbf{GIK4}$ *as an axiom.*

6.6.5. LEMMA. *Let P be a proof figure for $\Gamma \to \Delta$ in $\mathbf{GIK4} + \Box^{n+1} L^\Box(p)$. Then there exist formulas $A_1, \cdots, A_m$ such that*

$$\Box^{n+1} L^\Box(A_1), \cdots, \Box^{n+1} L^\Box(A_m), \Gamma \to \Delta \in \mathbf{GIK4}.$$

Proof. We use an induction on the number $\#(P)$ of axioms of the form $\to \Box^{n+1} L^\Box(A)$ in P. If $\#(P) = 0$, then P is a proof figure for $\Gamma \to \Delta$ in $\mathbf{GIK4}$. Suppose that $\#(P) > 0$ and the lemma holds for any P^* such that $\#(P^*) < \#(P)$. Then there exists an axiom $\to \Box^{n+1} L^\Box(A_1)$ in P for some A_1. For a subfigure Q of P, we define $h(Q)$ as follows:

(1) $h(A \to A) = \dfrac{A \to A}{\Box^{n+1} L^\Box(A_1), A \to A}$,

(2) $h(\bot \to) = \dfrac{\bot \to}{\Box^{n+1} L^\Box(A_1), \bot \to}$,

(3) $h(\to \Box^{n+1} L^\Box(A)) = \dfrac{\to \Box^{n+1} L^\Box(A)}{\Box^{n+1} L^\Box(A_1) \to \Box^{n+1} L^\Box(A)}$, where $A \neq A_1$,

(4) $h(\to \Box^{n+1} L^\Box(A_1)) = \Box^{n+1} L^\Box(A_1) \to \Box^{n+1} L^\Box(A_1)$,

(5) $h(\dfrac{P_1 \ \cdots \ P_k}{\Gamma \to \Delta})$

$$= \begin{cases} Q^* & \text{if the inference rule that introduces } \Gamma \to \Delta \text{ is } (\rhd_{IK4}) \\[2ex] \dfrac{h(P_1) \ \cdots \ h(P_k)}{\Box^{n+1} L^\Box(A_1), \Gamma \to \Delta} & \text{otherwise} \end{cases}$$

where Q^* is

$$\dfrac{\dfrac{h(P_1)}{A, \Lambda \rhd \bot \to \Lambda} \quad h(P_2) \ \cdots \ h(P_k) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \to}}{\bot \to B}}{\bot, B \rhd \bot \to B}}{\to \bot \rhd B}}{\dfrac{\text{using } (T \to), \text{ possibly several times}}{\Box^{n+1} L^\Box(A_1), \Sigma \to \bot \rhd B}}}{\Box^{n+1} L^\Box(A_1), X_1 \rhd Y_1, \cdots, X_k \rhd Y_k, \Sigma \to A \rhd B}$$

and $\Lambda = \{B, X_1, \cdots, X_k, \neg\Box^n L^\Box(A_1)\}$.

Note that $\Box^{n+1} L^\Box(A_1) = \neg\Box^n L^\Box(A_1) \rhd \bot$ and $h(P)$ is a proof figure for

$$\Box^{n+1} L^\Box(A_1), \Gamma \to \Delta$$

satisfying $\#(h(P)) < \#(P)$. Using the induction hypothesis, we obtain the lemma. $\dashv$

6.6.6. DEFINITION. By $\mathbf{GIL}^*$, we mean the system obtained from **GIL** by adding the inference rule $(\rhd_{IK4})$ in **GIK4**.

6.6.7. DEFINITION.

$$\mathsf{Sub}^*(\Gamma \to \Delta) = \mathsf{Sub}(\Gamma \to \Delta) \cup \{C \rhd D \mid C, D \in \mathsf{Sub}(\Gamma \to \Delta) \cup \{\bot\}\} \cup \{\bot\}$$

6.6.8. LEMMA. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GIL}^*$. Then every formula occurring in P belongs to $\mathsf{Sub}^*(\Gamma \to \Delta)$.*

Proof. By an induction on P. $\dashv$

6.6.9. DEFINITION. Let P be a cut-free proof figure in $\mathbf{GIL}^*$. We define $dep_{\rhd}(P)$ as follows:

(1) $dep_{\rhd}(D \to D) = dep_{\rhd}(\bot \to) = 0$,

(2) $dep_{\rhd}(\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta})$

$= \begin{cases} \max\{dep_{\rhd}(P_1)+1, dep_{\rhd}(P_2)\cdots, dep_{\rhd}(P_n)\} & \text{if } I \text{ is } (\rhd_{IK4}) \text{ or } (\rhd_{IL}) \\ \max\{dep_{\rhd}(P_1), \cdots, dep_{\rhd}(P_n)\} & \text{otherwise} \end{cases}$

where I is the inference rule that introduces $\Gamma \to \Delta$ in $\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta}$.

6.6.10. LEMMA. *Let P be a cut-free proof figure for*

$$\Box^n \Pi, \Gamma \to \Delta, \neg\Box^n \Lambda$$

in $\mathbf{GIL}^*$, where $n \geq 1$. If $dep_{\rhd}(P) < n$ and $(\Pi \cup \Lambda) \cap \mathsf{Sub}^*(\Gamma \to \Delta) = \emptyset$, then there exists a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GIL}^*$.

Proof. The lemma can be shown in the way similar to Lemma 4.4.8 by an induction on P. Here we only show the case that there exists an inference rule I that introduces the end sequent of P and I is $(\rhd_{IK4})$. Then sequents occurring I are of the following forms

upper sequents:

$A, \{B, X_1, \cdots, X_m\} \rhd \bot, \Box^{n-1}(\Box\Pi_1) \to \neg\Box^{n-1}\Pi_1, B, X_1, \cdots, X_m$

$\Box^n\Pi_2, \Sigma \to Y_1 \rhd B$

$\cdots$

$\Box^n\Pi_2, \Sigma \to Y_m \rhd B$

$\Box^n\Pi_2, \Sigma \to \bot \rhd B$

$\cdots$
$\Box^n\Pi_2, \Sigma \to \bot \rhd B$
lower sequent:
$\Box^n\Pi_1, \Box^n\Pi_2, X_1 \rhd Y_1, \cdots, X_m \rhd Y_m, \Sigma \to A \rhd B$

Let P_0 be the proof figure for the first upper sequent above in P and let P_i be the proof figure for $\Box^n\Pi_2, \Sigma \to Y_i \rhd B$. We note that

$$n > dep_{\rhd}(P) = \max\{dep_{\rhd}(P_0) + 1, dep_{\rhd}(P_1), \cdots, dep_{\rhd}(P_m), 1\},$$

and hence $n - 1 > dep_{\rhd}(P_0)$ and $n > dep_{\rhd}(P_i)$. Using the induction hypothesis, there exist cut-free proof figures for the following sequents:

$$A, \{B, X_1, \cdots, X_m\} \rhd \bot \to B, X_1, \cdots, X_m \quad \Sigma \to Y_1 \rhd B \ \cdots \ \Sigma \to Y_m \rhd B.$$

Using $(\rhd_{IK4})$, we obtain the lemma. ⊣

6.6.11. Notation. By $\mathcal{P}(A \rhd B)$, we mean the set of each cut-free proof figure P in $\mathbf{GIL}^*$ such that the inference rule introducing the end sequent of P is either $(\rhd_{IL})$ or $(\rhd_{IK4})$ and its principal formula in the succedent is $A \rhd B$.

6.6.12. Definition. We define a mapping $h_{C\rhd\bot}$ on the set of cut-free proof figures in $\mathbf{GIL}^*$ as follows:

(1) $h_{C\rhd\bot}(A \to A) = \dfrac{A \to A}{C \rhd \bot, A \to A}$,

(2) $h_{C\rhd\bot}(\bot \to) = \dfrac{\bot \to}{C \rhd \bot, \bot \to}$,

(3) $h_{C\rhd\bot}(\dfrac{P_1 \ \cdots \ P_n}{\Gamma \to \Delta})$

$$= \begin{cases} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to C}{C \to D, C}}{C, C \rhd \bot \to D, C}}{C, D \rhd \bot, C \rhd \bot \to D, C} \quad \dfrac{\dfrac{\dfrac{\bot \to}{\bot \to D}}{\bot, D \rhd \bot \to D}}{\to \bot \rhd D}}{C \rhd \bot \to C \rhd D}}{\text{using } (T \to)\text{, possibly several times}}}{C \rhd \bot, \Gamma \to C \rhd D} & \text{if } \dfrac{P_1 \ \cdots \ P_n}{\Gamma \to \Delta} \in \mathcal{P}(C \rhd D) \\ Q^* & \text{if } \dfrac{P_1 \ \cdots \ P_n}{\Gamma \to \Delta} \in \mathcal{P}(A \rhd B) \text{ is of the form } Q \text{ and } A \neq C \\ \dfrac{h_{C\rhd\bot}(P_1) \ \cdots \ h_{C\rhd\bot}(P_n)}{C \rhd \bot, \Gamma \to \Delta} & \text{otherwise} \end{cases}$$

where Q is

$$\dfrac{P_1\left\{\begin{array}{cc}\vdots & \vdots\\ \Pi\to B, X_1,\cdots,X_n & \Sigma\to Y_1\rhd B\end{array}\right\}P_2 \quad \cdots \quad \left.\begin{array}{c}\vdots\\ \Sigma\to Y_{n-1}\rhd B\end{array}\right\}P_n}{X_1\rhd Y_1,\cdots,X_{n-1}\rhd Y_{n-1},\Sigma\to A\rhd B}$$

and Q^* is

$$\dfrac{\dfrac{h_{C\rhd\bot}(P_1)}{C\rhd\bot,\Pi\to B,X_1,\cdots,X_n,C} \quad h_{C\rhd\bot}(P_2) \quad \cdots \quad h_{C\rhd\bot}(P_n) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot\to}{\bot\to B}}{\bot,B\rhd\bot\to B}}{\to\bot\rhd B}}{\text{using }(T\to)\text{, possibly several times}}}{C\rhd\bot,\Sigma\to\bot\rhd B}}{C\rhd\bot,X_1\rhd Y_1,\cdots,X_{n-1}\rhd Y_{n-1},\Sigma\to A\rhd B}$$

Note that Q^* above is a proof figure satisfying

$$\begin{aligned} dep_{\rhd}(Q^*) &= \max\{dep_{\rhd}(h_{C\rhd\bot}(P_1))+1, dep_{\rhd}(h_{C\rhd\bot}(P_2)),\cdots,dep_{\rhd}(h_{C\rhd\bot}(P_n)),1\}\\ &= \max\{dep_{\rhd}(h_{C\rhd\bot}(P_1))+1, dep_{\rhd}(h_{C\rhd\bot}(P_2)),\cdots,dep_{\rhd}(h_{C\rhd\bot}(P_n))\}.\end{aligned}$$

6.6.13. COROLLARY. *Let P be a cut-free proof figure for $\Gamma\to\Delta$. Then $h_{C\rhd\bot}(P)$ is a cut-free proof figure for $C\rhd\bot,\Gamma,\to\Delta$ such that $dep_{\rhd}(P)\geq dep_{\rhd}(h_{C\rhd\bot}(P))$.*

6.6.14. NOTATION. By $\#_{\rhd}(P)$, we mean the sum of the number of inference rule $(\rhd_{IK4})$ in P and the number of inference rule $(\rhd_{IL})$ in P.

Similarly to Lemma 4.4.13, we have

6.6.15. LEMMA. *Let P be a cut-free proof figure. If there exists a subfigure $Q\in\mathcal{P}(A\rhd B)$ of P satisfying $dep_{\rhd}(Q)\geq 2$, then $\#_{\rhd}(P)>\#_{\rhd}(h_{A\rhd\bot}(P))$.*

6.6.16. LEMMA. *Let P be a cut-free proof figure in* **GIL***. *Then there exists a sequence*

$$P_1,\cdots,P_{dep_{\rhd}(P)}$$

of subfigures of P satisfying

(1) *$P_i\in\mathcal{P}(C_i\rhd D_i)$ for some C_i and D_i,*

(2) *P_{i+1} is a subfigure of $P_{i,0}$, where* $P_i=\dfrac{P_{i,0}\quad\cdots\quad P_{i,n}}{S}$.

Proof. We use an induction on P. If P is an axiom, the lemma is clear. Suppose that P is not axiom and the lemma holds for any proper subfigure P^* of P. Since P is not axiom there exists an inference rule I that introduces the end sequent of P. We only show the case that I is $(\rhd_{IK4})$. P is of the form

$$\frac{P' \quad Q_1 \quad \cdots \quad Q_n}{S}.$$

If $dep_{\rhd}(P) = dep_{\rhd}(Q_i)$, then by the induction hypothesis, we obtain a sequence of subfigures of Q_i. The length of the sequence is $dep_{\rhd}(Q) = dep_{\rhd}(P)$ and each subfigure of Q_i is a subfigure of P. Hence we obtain the lemma.

If $dep_{\rhd}(P) = dep_{\rhd}(P') + 1$, then by the induction hypothesis, there exists a sequence

$$P_1, \cdots, P_{dep_{\rhd}(P)-1}$$

of subfigures of P' satisfying

(3) $P_i \in \mathcal{P}(C_i \rhd D_i)$ for some C_i and D_i,

(4) P_{i+1} is a subfigure of $P_{i,0}$ where $P_i = \dfrac{P_{i,0} \quad \cdots \quad P_{i,n}}{S}$.

Note that each subfigure of P' is a subfigure of P and P is a subfigure of P. Hence the sequence

$$P, P_1, \cdots, P_{dep_{\rhd}(P)-1}$$

satisfies the conditions. $\dashv$

6.6.17. Lemma. *Let P be a cut-free proof figure for*

$$\Box^{2n+3}\Pi, \Gamma \to \Delta$$

in $\mathbf{GIL}^*$, *where n is the number of elements in $\{C \rhd D \mid C \rhd D \in \mathsf{Sub}^*(\Gamma \to \Delta)\}$. Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in* $\mathbf{GIL}^*$.

Proof. We use an induction on $\#_{\rhd}(P) + \omega(dep_{\rhd}(P))$. We note that

$$\Box^{n+1}\Pi \cap \mathsf{Sub}^*(\Gamma \to \Delta) = \emptyset$$

and the end sequent of P can be expressed as

$$\Box^{n+2}(\Box^{n+1}\Pi), \Gamma \to \Delta.$$

If $dep_{\rhd}(P) < n+2$, then by Lemma 6.6.10, we obtain the lemma. Suppose that $dep_{\rhd}(P) \geq n+2$ and the lemma holds for any proper subfigure of P. Then by Lemma 6.6.16, there exists a sequence

$$P_1, \cdots, P_{n+2}, \cdots, P_{dep_{\rhd}(P)}$$

of subfigures of P satisfying

(1) $P_i \in \mathcal{P}(C_i \rhd D_i)$ for some C_i and D_i,

(2) P_{i+1} is a subfigure of $P_{i,0}$ where $P_i = \dfrac{P_{i,0} \quad \cdots \quad P_{i,n}}{\Sigma \to C_i \rhd D_i}$.

By Lemma 6.6.8, $C_i \rhd D_i \in \mathsf{Sub}^*(\Gamma \to \Delta)$. So, there exist i and j such that $C_i = C_j$ and $1 \leq i < j \leq n+1$. Since $j \leq n+1 < n+2 \leq dep_{\rhd}(P)$, we have $dep_{\rhd}(P_j) \geq 2$. By (2), P_j is a subfigure of $P_{i,0}$. Using Corollary 6.6.13 and Lemma 6.6.15, $h_{C_i \rhd \perp}(P_{i,0})$ is a cut-free proof figure such that $dep_{\rhd}(P_{i,0}) \geq dep_{\rhd}(h_{C_i \rhd \perp}(P_{i,0}))$ and $\#_{\rhd}(P_{i,0}) > \#_{\rhd}(h_{C_i \rhd \perp}(P_{i,0}))$. Using $(\rhd_{IL})$, we have the following cut-free proof figure P'_i

$$\frac{h_{C_i \rhd \perp}(P_{i,0}) \quad P_{i,1} \quad \cdots \quad P_{i,n}}{\Sigma \to C_i \rhd D_i}.$$

By P', we mean the figure obtained from P by replacing P_i by P'_i. P_i and P'_i have the same end sequent. So, P' is a cut-free proof figure for the end sequent of P such that $\#_{\rhd}(P) > \#_{\rhd}(P')$ and $dep_{\rhd}(P) \geq dep_{\rhd}(P')$. Using the induction hypothesis, we obtain the lemma. $\dashv$

6.6.18. Lemma. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in* **GIL****. Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in* **GIL**.

Proof. By replacing each inference rule $(\rhd_{IK4})$ in P by

$$\frac{\dfrac{A, \{B, X_1, \cdots, X_n\} \rhd \perp \to B, X_1, \cdots, X_n}{A, A \rhd \perp, \{B, X_1, \cdots, X_n\} \rhd \perp \to B, X_1, \cdots, X_n} \qquad \Sigma \to Y_1 \rhd B \quad \cdots \quad \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to A \rhd B}$$

we obtain a cut-free proof figure in **GIL**. $\dashv$

By Lemma 6.6.4, Lemma 6.6.5, Theorem 6.4.1, Lemma 6.6.17 and Lemma 6.6.18, we obtain Theorem 6.6.1 in the way similar to Theorem 4.4.4.

6.6.19. Corollary. *Let it be that $\Gamma \to \Delta \in$* **GIL***. Then there exists a cut-free proof figure P such that every formula occurring in P belongs to $\mathsf{Sub}^*(\Gamma \to \Delta)$.*

Index

List of symbols

Abstract

In this thesis, we treat three kinds of propositional logics. The first kind connects with a non-modal propositional logic, called *formal propositional logic* (**FPL**), another is an intuitionistic modal logic, and the third kind consists of interpretability logics. These logics are related to or connected with the provability logic **GL**, the normal modal logic obtained from the smallest normal modal logic **K** by adding Löb's axiom $\Box(\Box p \supset p) \supset \Box p$. The name "*provability logic*" derives from Solovay's completeness theorem. He proved that **GL** is complete for the formal provability interpretation in Peano arithmetic **PA**. So, **GL** has been considered as one of the most important modal logics.

FPL as well as interpretability logics also have a formal provability interpretation. **FPL** is the propositional logic embedded into **GL** by Gödel's translation τ. Interpretability logics are modal logics with a binary modal operator $\rhd$ including **GL**. We treat these two kinds of logics with this motivation in mind.

The normal modal logic **K4** is a sublogic of **GL**, which is obtained from **K** by adding the transitivity axiom $\Box p \supset \Box\Box p$. As is expected by the additional axioms of **K4** and **GL**, the transitivity axiom and Löb's axiom, **K4** is much easier to deal with than **GL**. So, as was stated by C. Smoryński, knowledge of **K4** is useful for the discussion of **GL**. Here we also treat Visser's propositional logic (**VPL**), the propositional logic embedded into **K4** by τ, before treating **FPL**, and the sublogic of the smallest interpretability logic **IL** whose $\rhd$-free fragment is **K4**, before **IL**. We consider the consequence relation of **VPL** and a property of Löb's axiom on **VPL**. To give cut-free sequent systems is one of the issues here. We first give such systems for **VPL** and the sublogic of **IL**, and then, using a property of Löb's axiom, for **FPL** and **IL**.

The remaining one among the logics treated here is the intuitionistic modal logic called propositional lax logic (**PLL**) by M. Fairtlough and M. Mendler. **PLL** is not a logic for provability. However, **PLL** has other interesting interpretations. For example, it corresponds to the computational typed lambda calculus introduced by E. Moggi by the Curry-Howard isomorphism. Here we discuss Diego's theorem in **PLL**, and elucidate the structure of sets of disjunction free formulas with only finitely many propositional variables.

Samenvatting

In dit proefschrift behandelen we drie soorten propositielogica's. De eerste is een niet-modale propositielogica, *formele propositielogica* (**FPL**) genaamd, een tweede is een intuitionistische modale logica, and de derde soort bestaat uit interpreteerbaarheidslogica's. Deze logica's zijn gerelateerd aan of verbonden met de bewijsbaarheidslogica **GL**, de normale modale logica verkregen uit de kleinste normale modale logica **K** door toevoeging van Löb's axioma $\Box(\Box p \supset p) \supset \Box p$. De naam "*bewijsbaarheidslogica*" komt van Solovay's volledigheidsstelling. Hij bewees dat **GL** volledig is met betrekking tot de formele bewijsbaarheidsinterpretatie in de Peano-rekenkunde **PA**. Om die reden wordt **GL** wel beschouwd als een van de belangrijkste modale logica's.

FPL and de interpreteerbaarheidslogica's hebben ook een formele bewijsbaarheidsinterpretatie. De formele bewijsbaarheidslogica is de propositielogica die door Gödel's vertaling τ wordt ingebed in **GL**. Interpreeerbaarheidslogica's zijn modale logica's met een binaire modale operator $\rhd$ die **GL** omvatten. We behandelen deze twee soorten logica's met deze motivering in gedachten.

De normale modal logica **K4** is de sublogica van **GL** die uit **K** verkregen wordt door toevoeging van het transitiviteitsaxioma $\Box p \supset \Box\Box p$. Zols te verwachten valt uit de additionele axioma's van **K4** and **GL**, het transitiviteitsaxioma en Löb's axioma, is **K4** veel eenvoudiger te behandelen dan **GL**. Om die reden is, zoals door C. Smoryński al werd gezegd, kennis van **K4** nuttig voor de discussie van **GL**. We behandelen hier ook Visser's propositielogica (**VPL**), de propositielogica die wordt ingebed in **K4** door τ alvorens **FPL** te behandelen, en de sublogica van de kleinste interpreteerbaarheidslogica **IL** waarvan het $\rhd$-vrije fragment **K4** is vóór **IL**. We beschouwen de gevolgtrekkingsrelatie van **VPL** en een eigenschap van Löb's axioma op **VPL**. Het verkrijgen van snedevrije sequentensystemen is hier de opgave. We geven een dergelijk systeem eerst voor **VPL** en de sublogica of **IL**, and daarna, onder gebruikmaking van een eigenscahp van Löb's axioma, voor **FPL** and **IL**.

De laatste logica die hier wordt behandeld is een intuitionistische modale logica, propositionele lax-logica (**PLL**) genoemd door M. Fairtlough en M. Mendler. **PLL** is geen logica voor bewijsbaarheid, maar heeft andere interessante interpretaties. Bijvoorbeeld, zij correspondeert met de computationele getypte lambdacalculus geïntroduceerd door E. Moggi via het Curry-Howard isomorfisme. Hier bediscussieren we Diego's stelling in **PLL**, and verhelderen de structuur van verzamelingen van disjunctievrije formules met slechts eindig veel propositievariabelen.

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