

Sums, Numbers and Infinity

Collections in Bolzano's Mathematics and Philosophy

Anna Bellomo

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Collections in Bolzano's Mathematics and Philosophy

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per Guillaume, Nonna Maria, e Nonno Peppo

Contents

Acknowledgments	xi
Bibliographical abbreviations	xiii
1 Introduction	1
1.1 Bernard Bolzano (1781-1848)	1
1.1.1 Bolzano's mature writings	2
1.2 Readings of Bolzano's work	4
1.3 Structure and themes	5
2 Collections	9
2.1 Introduction	9
2.2 Bolzano's collections	10
2.2.1 Definitions	10
2.2.2 Conceptual dependencies	13
2.2.3 Concept-first collections	16
2.2.4 Final picture	21
2.3 Sets, wholes and extensionality	22
2.3.1 Different ways to be extensional	23
2.3.2 Neither sets nor wholes?	25
2.3.3 Some sets, some wholes	28
2.3.4 Taking stock	31
2.4 Bolzano's collections and the search for a foundation	32
2.4.1 Maddy, or what foundations are for	32
2.4.2 Early Bolzano's foundational aspirations	34
2.4.3 The definitional turn and Elucidation	37
2.4.4 Generous Arena	42
2.5 Conclusion	46

3	Natural Number Concepts	49
3.1	Introduction	49
3.2	A misleading dilemma	50
3.2.1	Bolzano and Galileo's Paradox	50
3.2.2	Bolzano's texts	51
3.3	Ideas and the notion of size	56
3.3.1	A relative interpretation of Bolzano's collections	56
3.3.2	Part-whole for extensions	60
3.4	Bolzano's concept(s) of natural number	63
3.4.1	Bolzano's de nition(s) of the natural numbers	63
3.4.2	Subordination between number concepts in the letter	66
3.5	Conclusion	70
4	Measurable Numbers	71
4.1	Introduction	71
4.2	Real numbers and the arithmetization of analysis	73
4.3	<i>RZ VII</i> : Text and interpretations	77
4.3.1	Numbers, quantities or expressions?	77
4.3.2	Summary and segmentation	78
4.3.3	Interpretations and interpreters	81
4.3.4	Three problems	84
4.4	Measurable numbers and arithmetization	87
4.4.1	Closure under addition	88
4.4.2	Equality and equivalence	91
4.4.3	Cauchy convergence criterion and <i>RZ VII §107</i>	93
4.4.4	Bolzano and the arithmetization of analysis	96
4.5	Conclusion	97
5	The Mathematical Infinite	99
5.1	Introduction	99
5.2	The received view on the <i>PU</i>	101
5.2.1	Bolzano's collections, multitudes, and sums	106
5.2.2	Bolzano's commitment to part-whole in the <i>PU</i>	109
5.3	Bolzano's calculation of the in nite	112
5.3.1	Computing with in nite sums	113
5.3.2	Grandi's series	116
5.3.3	The sum of all squares	119
5.4	Modelling Bolzano's arithmetic of the in nite	124
5.4.1	The basic framework	124
5.4.2	Modelling Bolzano's results about in nite sums	127
5.4.3	Grandi's series	132
5.4.4	Comparisons with related work	136
5.5	Higher-order in nities	138

5.5.1	The product of two infinite quantities	138
5.5.2	Second-order infinities via an iterated ultrapower	142
5.6	Reassessing the <i>PU</i>	145
5.6.1	A theory of infinite sums	146
5.6.2	Part-whole reasoning in Bolzano's computations	147
5.6.3	Bolzano and early set theory	151
5.7	Conclusion	154
5.8	Appendix	154
6	Domain Extension and Ideal Elements in Mathematics	161
6.1	Introduction	161
6.2	Ideal elements	162
6.3	Manders's framework	164
6.4	Domain extensions and ideal elements	166
6.4.1	Infinitesimals as ideal elements	166
6.5	Domain extension according to Dedekind	169
6.5.1	The law of permanence of forms	169
6.5.2	Early Dedekind on domain extension	171
6.5.3	Formalising Dedekind's proposal	175
6.6	Extending the concept of number	176
6.6.1	Quaternions, octonions and other hypercomplex numbers	176
6.6.2	Dedekind's ideals and ideal elements	178
6.6.3	Good Dedekind-extensions?	180
6.7	Comparison	181
6.8	Conclusion	183
7	Concluding Remarks	185
	Bibliography	189
	Abstract	201
	Samenvatting	203

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Bibliographical abbreviations

- BBGA* Berg, Jan, Friedrich Kambartel, Jaromír Louzil, Bob van Rootseelaar and Eduard Winter, eds. 1969. *Bernard Bolzano Gesamtausgabe*. Stuttgart - Bad Canstatt: Frommann-Holzboog.
- BBGA 2A/12.2* Bolzano, Bernard. 1978. *Vermischte philosophische und physikalische Schriften 1832-1848*. Zweiter Teil. Vol. 2A/12.2 of *BBGA*, edited by Jan Berg. Stuttgart - Bad Cannstatt: Frommann-Holzboog.
- Beyträge* Bolzano, Bernard. 1810. *Beyträge zu einer begründeteren Darstellung der Mathematik: Erste Lieferung*. Prague: Caspar Widtmann. English translation: pp. 83-138 of Steve Russ, ed. and trans. 2004. *The Mathematical Works of Bernard Bolzano*. Oxford: Oxford University Press.
- EB* Bolzano, Bernard. 1975a. *Erste Begriffe der allgemeinen Größerenlehre*. In *Größerenlehre I*, edited by Jan Berg, 217{286. *BBGA*. Stuttgart - Bad Canstatt: Frommann-Holzboog.
- EGL* Bolzano, Bernard. 1975b. *Einleitung zur Größerenlehre*. In *Größerenlehre I*, edited by Jan Berg, 23{216. *BBGA*. Stuttgart - Bad Cannstatt: Frommann-Holzboog.
- EGL III* Bolzano, Bernard. 1975c. 'Vorkenntnisse'. In *Einleitung zur Größerenlehre*, edited by Jan Berg, 98{216. *BBGA*. Stuttgart - Bad Canstatt: Frommann-Holzboog.
- GL I* Bolzano, Bernard. 1975d. *Größerenlehre I*. Vol. 2A/7 of *BBGA*, edited by Jan Berg. Stuttgart - Bad Cannstatt: Frommann-Holzboog.
- PT* Bolzano, Bernard. 1979b. *Philosophische Tagebücher 1827-1844*. Zweiter Teil. Vol. 2B/18.2 of *BBGA*, edited by Jan Berg. Stuttgart - Bad Canstatt: Frommann-Holzboog.

- PU* Bolzano, Bernard. 1851. *Paradoxien des Unendlichen*. Edited by Frantisek Pr honsky. English translation: pp. 591-679 of Russ 2004; Bernard Bolzano. 1950. *Paradoxes of the Infinite*. Edited by Donald A. Steele and Frantisek Pr honsky. London: Kegan Paul.
- RAB* Bolzano, Bernard. 1817. *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewahren, wenigstens eine reelle Wurzel der Gleichung liege*. Prague: Gottlieb Haase. English translation: pp. 251-279 of 2004.
- RZ* Bolzano, Bernard. 1976. *Grundlehren II: Reine Zahlenlehre*. Vol. 2A/8 of *BBGA*, edited by Jan Berg. Stuttgart - Bad Cannstatt: Frommann-Holzboog.
- WL* Bolzano, Bernard. 1837. *Wissenschaftslehre*. Sulzbach: Seidel. Reprinted as Bernard Bolzano. 1969. *Wissenschaftslehre*. Edited by Jan Berg. 12 vols. *BBGA*. Stuttgart - Bad Cannstatt: Frommann-Holzboog.

Chapter 1

Introduction

1.1 Bernard Bolzano (1781-1848)

Bernard Bolzano was born in Prague in 1781 to a German-speaking mother and an Italian immigrant father.¹ He grew up and was educated his whole life in Prague, and when it came to attending university he studied theology and mathematics. In 1804 Bolzano competed both for the Mathematics and for the Theology Chair of Charles University (Prague), and eventually he took up the theology position. Later the same year he was appointed Chair of Religious Studies (*Lehrstuhl der Religionswissenschaft*) at the Charles University in Prague, where one of his duties was to deliver religious 'edifying speeches' (*Erbauungsreden*, sermons of sorts) meant for the students of the university but open to the general public as well. This however did not last. Bolzano's sermons quickly built a fame for being too progressive, both politically and theologically. Bolzano was eventually removed from his theology chair in 1819 and forbidden from teaching and disseminating his ideas.² From that point onwards, he dedicated himself almost exclusively to writing. In the decade that followed, he wrote his most ambitious work, the *Wissenschaftslehre* (Theory of Science), to which many more followed, including the *Große enlehre* (Theory of Quantity) and *Paradoxien des Unendlichen* (Paradoxes of the Infinite). Instrumental to his continued productivity from hereon was his friendship with Anna Hoffmann and her husband Josef, who provided for him and hosted him for long periods of time at their residence in Techobuz, near Prague, up to Anna's death in 1841 (Winter 1969, pp. 73-74, Rusnock and Sebestik 2019,

1. (Winter 1969) contains the official biography of Bolzano that is part of the *BBGA*. Winter (1969, p. 9) uses Bolzano's own correspondence as well as Bolzano's autobiography as sources for his reconstruction. Here I rely on Winter's work and the biographical notes that can be found in (Sebestik 1992; Rusnock 2000; Lapointe 2011; Rusnock and Sebestik 2019).

2. Lapointe (2011, p. 2) paints a vivid picture of the political context in which Bolzano's accusation and eventual removal came to pass, explaining how it was not exceptional given the routine repression of intellectuals in Hapsburg territories at the time. Cf. Winter (1969, pp. 38-40).

pp. 64-68). After that Bolzano moved back to Prague, where he stayed until his death in 1848.

Despite the imperial interdiction, Bolzano managed to cultivate intellectual friendships with various people, including Robert Zimmermann, Franz Exner and Frantisek Pr honsky (as Bolzano's letters attest). It is thanks to this small circle of friends that Bolzano managed to publish some of his works during his lifetime (cf. e.g. Rusnock and Sebest k 2019, pp. 68-70), and it is also to the same group of people, and especially Robert Zimmermann, that Bolzano entrusted all of his writings by the time of his death. Zimmermann took up an academic position in Vienna, and he took some of Bolzano's manuscripts with him, but these lay untouched in the Vienna university library until well after Zimmermann's death. Partly because of the prohibition to publish, partly because of Bolzano's collaborators' mismanagement of his writings after his death, most of Bolzano's work has only come to light during the 20th century, and mostly starting in the late '50s and '60s. Winter (1969, pp. 5-6) notes that there were quite a few earlier attempts at publishing Bolzano's work more widely, which culminated in partial editions of Bolzano's work, but what is now the most authoritative and exhaustive edition of Bolzano's writings is the *Bernard Bolzano Gesamtausgabe (BBGA)*, Frommann-Holzboog's multi-volume effort to publish all of Bolzano's writings in critical edition that began in the mid-1960s (Winter 1969, p. 6) and is still ongoing. As of September 2021, the *BBGA* consists of 105 published volumes and 27 in preparation.

1.1.1 Bolzano's mature writings

The main primary sources for the work in this dissertation are Bolzano's *Wissenschaftslehre (WL)*, *Gro enlehre (GL)* and *Paradoxien des Unendlichen (PU)*.³

The *WL* was first published in 1837, but based on Bolzano's notes it seems that he had completed a first draft already in 1830 (Winter 1969, p. 80). This is Bolzano's main work of theoretical philosophy and, while it contains developments of views that one can already glimpse in earlier works of his (Blok 2016; Russ 2004; Rusnock 2000), it is usually considered as the watershed work in Bolzano's intellectual development and the one which contains the methodological, epistemic and ontological insights which provide the blueprint and foundation for everything else Bolzano writes afterwards, especially the *GL*. The *WL* has subsequently been published in the early 1920s, with minimal differences from the 1837 edition, and then in the 1960s the *BBGA* edition started to be published (Winter 1969, p. 6). This was followed in the 1970s by two abridged translations of the *WL* (Bolzano 1972, 1973), which helped with putting Bolzano on the map for Anglophone philosophers. Nowadays there is also an integral translation due to Rolf George and Paul Rusnock (Bolzano 2014). In citing from the *WL*, I will follow the

3. A full list of the abbreviations used in this dissertation can be found on page xiv.

convention of writing *WL* §x, where 'x' is the relevant paragraph number.

The *GL* was planned to be a comprehensive treatment of all of mathematics according to the principles of proper science outlined in the *WL* (cf. e.g. Rusnock and Sebestik 2019, pp. 71-72). The *BBGA* has a total of five volumes carrying the title *Gr̃ enlehre*, but for this dissertation I have mostly used the first two, *Gr̃ enlehre I (GL I)* and *Gr̃ enlehre II. GL I* contains Bolzano's general introduction to the 'Theory of Quantity' (mathematics), i.e. an overview of key ideas from the *WL* that he will make use of throughout the *GL (Einleitung zur Gr̃ enlehre, EGL for short)*, and the 'first concepts' of general mathematics (*Erste Begr̃ e der allgemeinen Gr̃ enlehre, EB for short*). The *EGL* is further divided into three sections, 'Von dem Begr̃ e der Mathematik und ihren Theilen'⁴ (*EGL I*), 'Von der mathematischen Lehrart'⁵ (*EGL II*), and 'Vorkenntnisse'⁶ (*EGL III*). Since the numbering of the paragraphs starts from 1 at each section, I refer to passages from the *GL I* by specifying the title, *EGL* or *EB*, followed by the section number in the case of the *EGL*, and then paragraph number. In the rare cases where a paragraph has been left unnumbered in the edition or it runs over several pages, I have specified the page number of the relevant *BBGA* volume instead (*BBGA 2A/7*).

The *Gr̃ enlehre II*, titled *Reine Zahlenlehre (RZ for short)*, is divided into seven sections. The first one is about the definition of the natural numbers or, as Bolzano often calls them, the actual or real numbers (*wirkliche Zahlen*). The second and third concern addition and subtraction, and multiplication and division, while the fourth introduces the rational numbers, and the fifth deals with order relations between numbers. In the sixth section, Bolzano introduces 'rational numbers which can infinitely increase or decrease'. Finally, in the seventh section (which is the object of study for Chapter 4), Bolzano extends the domain of numbers and number expressions to that of infinite numbers and number expressions, and he introduces a special subclass of those, the measurable numbers. Here I also refer to the text by title, section, and paragraph number.

Finally, the *PU* is probably the most well-known of Bolzano's writings. Often touted as evidence of Bolzano's anticipation of Cantorian set theory (which is an important theme in Chapter 5), the *PU* is a booklet that was edited for publication by Frantisek Prhonsky, one of Bolzano's students/collaborators, immediately after Bolzano's death (1851). It was admirably cited by Cantor (1883) himself, among others, and it has received several German editions and foreign language translations (for French and Italian, see for instance Bolzano 1993, 1979a). Among the German editions, it is worth mentioning the 1920 Meiner edition, which contains Prhonsky's editorial note and table of contents from 1851 as well as a commentary by Hans Hahn. Among the English translations, Steele's was the first

4. English: On the concept of mathematics and its parts.

5. On the mathematical method.

6. Preliminaries.

(Bolzano 1950), and it still forms the basis for Russ's 2004. The *BBGA* edition of the *PU* however is yet to appear, which is why in this dissertation I made use of the *Meiner* edition (Bolzano 1920) and the brand new (Bolzano 2012). In the *PU* Bolzano aims to (dis)solve some classical paradoxes in mathematics, physics and metaphysics by first developing a rigorous definition of the infinite, and then applying it to various paradoxes to argue that they are not as threatening as one might think without the right definition at hand.

1.2 Readings of Bolzano's work

All three works mentioned above contain a presentation of Bolzano's mature theory of collections (as opposed to an early theory of collections, i.e. pre-*WL*, that has been studied by Blok (2016) and Centrone and Siebel (2018), most recently).

For a long time since the first publications of Bolzano's works in the twentieth century, it had been the orthodoxy to just describe Bolzano's theory of collections as an anticipation of Cantorian set theory. Even Sebestik (1992), who carefully describes each collection notion (Sebestik 1992, pp. 305-334), still does so under the banner of 'Bolzano's theory of sets' (*ensembles*). This unity of interpretation was broken with Krickel (1995), whose work sparked a debate concerning Bolzano's theory of collections and whether it should be seen as an anticipation of set theory. Krickel (1995), in opposition to the traditional reading of Bolzano, argues for a mereological interpretation of Bolzano's collections, and he also proposes a formalisation of his interpretation within an extension of general extensional mereology (Krickel 1995, pp. 305-306). Simons (1997) reacted to Krickel's interpretation by tempering the swing towards a mereological interpretation and favouring the view that Bolzano's collections are *sui generis* and cannot be reduced to either sets or mereological wholes. This is the position that several Bolzano scholars⁷ seem to have endorsed, whether explicitly or implicitly (by adopting Simons's suggestion regarding how to translate 'Menge', for instance).⁸

Thus, at least in some circles, it seems that the debate over the metaphysics of Bolzano's collections has been settled roughly in favour of Simons's view. Yet, beyond the confines of discussions on Bolzano's metaphysics, Bolzano's collections { or at least his Mengen { are still routinely identified with sets, or modelled as sets (see for example Mancosu 2016; Parker 2008, 2013; Zermelo 2010). It is as if the metaphysical literature has gone down one path, but the mathematical literature has not followed.

This indicates that the question of how to interpret Bolzano's collections in the

7. In addition to the works already mentioned, see also (Lapointe 2011) and (Rusnock 2000).

8. In the unabridged English translation of the *WL*, George and Rusnock decide to avoid translating Bolzano's 'Menge' as 'set' and opt for Simons's (1997) suggestion of 'multitude' instead (Bolzano 2014, p. xlix). This is also the choice professed by Russ (2004) in his translation of a selection of Bolzano's mathematical writings which includes the *PU*.

context of his philosophy of mathematics is yet to be given a satisfactory answer. With this dissertation I advance one proposal for such an answer, by expanding (and where applicable, course-correcting) the views expressed by Simons (1997) and Behboud (1997). Instead of only considering metaphysical arguments, however, I argue that a reexamination of Bolzano's mathematical goals (Chapter 2) and practice (Chapters 3 and 5) suggests that one cannot swap Bolzano's collections for sets without consequences. These consequences are a distorted and less charitable understanding of Bolzano's views on infinity, and a missed opportunity in truly understanding the role of Bolzano's theory of collections in his philosophy of mathematics. The new interpretation presented in this dissertation aims to clarify Bolzano's views on collections in mathematics, with an eye to appreciating his contributions to the discipline without trapping them into exceedingly presentist readings (cf. Chapter 7).

The next section offers a more detailed discussion of the structure and themes of this dissertation.

1.3 Structure and themes

This dissertation starts off with a treatment of the main topic, Bolzano's theory of collections (Chapter 2). Chapter 2 argues that while Bolzano's theory of collections has variously been presented as a forerunner of set theory, of mereology, or as something in between, its importance for Bolzano's mathematical goals has not sufficiently been argued for. In the first part of the chapter I compare Bolzano's collections to sets and mereological wholes and conclude that Bolzano's collections are not extensional like sets, but some specific kinds of collections are extensional like (some) mereological wholes. In the second part of the chapter I argue that Bolzano's collections also play the role of foundation for his mathematics, although this turns out to be significantly different from the way set theory can be said to be a foundation for modern mathematics.

Chapter 3 focuses on the problem of the interaction between Bolzano's theory of ideas on the one hand, and the question of how to measure the size of infinite collections of natural numbers on the other. The realisation that, for Bolzano at least, these collections are actually concept extensions is crucial to understand that Bolzano's answer to the measurement question cannot neatly be framed the way it often is, namely within the dilemma of Galileo's Paradox as a dilemma about how to extend size relations from finite to infinite collections of objects.⁹

9. Galileo's Paradox plagues the characters of his *Two New Sciences* (Galileo 1958) once they try to establish whether the collection of all natural numbers and that of all natural numbers squared share the same size. On the one hand, it is possible to establish a one-to-one correspondence between the two. So there seem to be as many natural numbers as there are squares. On the other, the collection of all natural number comprises infinitely many numbers which are not in the collection of all squared natural numbers, so the latter seem to be smaller. Each option seems equally plausible and intuitive, hence the paradox.

Our alternative framing of infinite collections of natural numbers as concept extensions also has the advantage of highlighting the overarching continuity between the *WL* and the *PU* when it comes to views on infinite collections.

Chapter 4 focuses on the seventh and last section of Bolzano's *Reine Zahlenlehre* (*RZ* for short), which was first published as 'Bolzano's Theory of Real Numbers' (*Bolzano's TRZ*) by Karel Rychlík in the early 1960s. *RZ VII* is the culmination of all preceding sections from the *RZ*, where each transition from one section to the next corresponds to an extension of either the number domain or of the operations that are allowed on the number domain at hand. I say that *RZ VII* is the culmination of the sections preceding it in the *RZ* because it is the only section that allows all operations introduced so far to be iterated infinitely many times, thus obtaining the infinite number expressions which are the topic of study of *RZ VII*. Each number expression corresponds to an infinite number, and among these infinite number expressions there are some that also exhibit the property of measurability.¹⁰ These are the ones that one might want to identify with real numbers. Bolzano goes on to prove that addition is closed over the measurable numbers, and he tries to extend the notion of equality in a way that is both coherent with the way operations on the measurable numbers work, and with the definition of equality proper.

Bolzano's *RZ VII* is a very interesting (if challenging) text, and much work remains to be done in trying to connect it to the rest of Bolzano's texts on the mathematical infinite. Our main focus for Chapter 4 is to situate Bolzano's attempt at defining the real numbers within the broader context of other 19th century attempts. We furnish Epple's (2003) framework with a fine-grained analysis of the notion of arithmetization to illustrate the specific ways in which Bolzano's measurable numbers are exemplary of his position as a mathematician caught between the traditional view of mathematics as the science of quantity and the arithmetization of analysis.

Chapter 5 presents several arguments why Bolzano's §§29-33 of the *PU* should not be interpreted as an attempt at measuring the size of infinite sets (or set-like collections). Drawing from a close reading of those paragraphs, we argue that the problem Bolzano was trying to solve was how to handle non-converging infinite series. Second, if we interpret Bolzano's 'calculations of the infinite' as involving infinite series, that is, infinite sums of sequences, a number of apparent inconsistencies disappear. Third, we argue that not just Bolzano's results, but Bolzano's way of proving those results, can be rather faithfully recreated in a surprisingly well-behaved structure of iterated ultrapowers of \mathbb{Z} , thus confirming that Bolzano's own arithmetic of the infinite is consistent. Because of these results, and because of the arguments provided in Chapter 2, we can conclude that the usual reconstruction of Bolzano's work in the *PU* as proto-set theoretic is by and large misleading. In its place, Bolzano's own proofs suggest that the closest

10. See Chapter 4 for the definition.

reconstruction of his computations and the reasoning behind them is by way of an iterated ultrapower construction in which we model Bolzano's infinite sums as equivalence classes of series $\{ \}$ not as sets. The upshot is that, on top of following the text more closely than a Cantorian interpretation of Bolzano, we can also prove that they are consistent in our modern sense, in that they have a model.

Chapter 5 touches on themes that run through each of the preceding chapters. Here I want to draw attention to a particularly important one that unifies Chapters 3 to 5. In the above discussion of Chapter 3 I referred to the problem underlying Galileo's Paradox as the problem of how to extend the domain of application of certain notions, or alternatively, how to expand the notion of size so that it may apply to infinite as well as finite collections. Similarly, the analysis of *RZ VII* in Chapter 4 reveals that one of the key problems Bolzano contends with therein is how to extend certain concepts without redefining or distorting them (equality and addition, respectively). Although I do not even attempt to articulate what Bolzano's views might have been on extending concepts from a narrower to a broader domain of application, it is clear that this is exactly what is at stake for him when defining his most general class of numbers, the measurable numbers.

The problem of how to extend operations and notions beyond their intended domain of application preoccupied more scholars than just Bolzano in the 19th century. Starting in the second half of the century, several mathematicians tried to formulate a principled way of extending notions, especially arithmetical notions, from one domain of objects to another. As a counterpoint to the in-depth analysis of Bolzano's practice on extending mathematical concepts and relations in the preceding chapters, Chapter 6 examines Dedekind's, in the context of assessing to what extent the modern proposal of Manders (1989) does actually capture the same phenomenon earlier mathematicians were interested in.

Sources of the material

Several of this dissertation's chapters are based on single- or co-authored work or papers, either published, submitted or just drafted. Here I explain in more detail provenance of the material and authors' contributions:

1. The paper that forms the basis of Chapter 3 is co-authored with Annapaola Ginammi. Both authors contributed equally, both at ideation and writing stage. As for the writing, Ginammi was the main author for Section 3.3 and Bellomo was the main author for Sections 3.2 and 3.4. Everything else was written collaboratively.
2. Chapter 4 is a paper currently in preparation, co-authored with Arianna Betti. Betti is the main author for Sections 4.1 and 4.2, Bellomo is the main author for Sections 4.3 to 4.5. Bellomo is the main writer of the version included in this dissertation.

3. Chapter 5 has been published in the *Review of Symbolic Logic* as 'Bolzano's Mathematical Infinite' (Bellomo and Massas 2021). The authors collaborated closely at all stages, but Bellomo was mainly responsible for writing Sections 5.1 to 5.3 and Massas for Sections 5.4 and 5.5. Section 5.6 was written collaboratively.
4. Chapter 6 has been published in *Philosophia Mathematica* as 'Domain Extension and Ideal Elements in Mathematics' (Bellomo 2021).

2.1 Introduction

Bolzano's theory of collections has often been presented as a set theory in disguise. But are Bolzano's collections really sets? This question can be given two different types of answer { a metaphysical answer, about what collections *really are* according to Bolzano, and a functional answer, about what collections are *supposed to do* for Bolzano.

The metaphysical answer is the one that has received the most attention over the years. Krickel (1995) has claimed that Bolzano's theory can be captured in an extension of Lesniewski's mereology, while Simons (1997) has argued that Bolzano's theory cannot be interpreted as either a set theory or a mereology. Despite these contributions, it is still considered an acceptable approximation to introduce Bolzano as a forefather of set theory (see for example Ferreiros 2007b; Sebestik 2017 and Ulrich Felgner's commentary in (Zermelo 2010, pp. 160-162)). I believe this is because the metaphysical answers provided so far do not address the functional concern, that is, whether Bolzano's collections play the same role sets do within mathematical foundations. I therefore propose two arguments for why Bolzano's collections are not functionally the same as sets.

I begin with an overview of Bolzano's definitions of the different kinds of collections he considers from the *WL* onwards (Section 2.2). A schematic representation of how the different collections relate to one another is given and contrasted with Simons's more usual picture of how Bolzano's collection kinds relate to one another (Section 2.2.2). Section 2.3 explores the question of whether Bolzano's collections can be said to be metaphysically the same as sets or as mereological wholes. This is the first argument against identifying Bolzano's collections with sets, in which I emphasise the difference between Bolzano's collections and sets on a point that is at the heart of the usefulness of sets in mathematical foundations, namely, extensionality. I argue that Bolzano's collections as they appear in his mathematical writings are not extensional in the way sets are extensional, yet they

can be interpreted as extensional in a precise mereological sense (Section 2.3.2, Section 2.3.3). Answering the question of whether Bolzano's collections are sets or mereological wholes does not amount to answering the question of whether Bolzano's collections play the same role as sets (or mereological wholes) in mathematics. One can agree that Bolzano's collections cannot be identified with sets, and yet hold that, for all intents and purposes, Bolzano's mathematical uses of his collections are well understood by simply swapping collections for sets. This is the attitude that we often see implicitly or explicitly woven into commentaries of Bolzano's work (besides the already mentioned Ferreiros 2007b; Sebestik 2017; and Felgner, see also Berg 1962, 1992; Sebestik 1992; Rusnock 2000; Parker 2008; Mancosu 2016), and in these works the metaphysical debate over Bolzano's collections simply seems to have left no mark. I therefore try to give my second argument for why Bolzano's collections do not play the same role as sets that is based on Maddy's (2017, 2019) characterisation of set theory as a foundation (Section 2.4). The outcome of this analysis is that Bolzano's collections are not just metaphysically different from sets. They are also functionally different from them.

2.2 Bolzano's collections

Bolzano develops a detailed taxonomy of collections (*Inbegriffe*), which was first published¹ in his lifetime in the *Wissenschaftslehre* (*WL* §§82-88). A second and a third version are found in the posthumous publications of the *Paradoxes of the Infinite* (*Paradoxien des Unendlichen*, *PU*) and of the first volume of the *Grundgesetze der Arithmetik* (*GL I*).

2.2.1 Definitions

Modulo some variation in terminology between these three presentations, Bolzano's definitions do not change across the three works. Below I summarise the definitions of those collection notions which are relevant to Bolzano's mathematics.

Collections (*Inbegriffe*) Anything that can be said to have parts or 'have compositeness' (*Zusammengesetztheit*) is a collection, as opposed to a simple (*einfach*) object (*WL* §82; *PU* §3; *EGL III* §6, p. 100). Bolzano's use of 'parts' here suggests that he only counts proper parts as such, and as a consequence, a collection needs to have at least two parts. Among collections, one can distinguish some for which the order of the parts (*Ordnung*, *Anordnung*) matters, and some for which it does not. While the former { the collections for which the order of

1. Strictly speaking, there is an earlier version of a theory of collections in Bolzano's published works. Given its substantial difference with what is considered Bolzano's mature theory, I do not consider it here. See Blok (2016) and Centrone and Siebel (2018) for recent treatments.

the parts matters { do not receive a specific name, the latter do: these are the Mengen, to which we now turn.

Mengen (*WL* §84; *EGL* III §88, pp. 151-152; *PU* §4) According to Bolzano, there are two aspects to a collection: the parts it is composed of, and the way these parts are connected among themselves { this is what Bolzano at times refers to as the 'order' (*Ordnung, Anordnung*) or 'manner of composition' (*Art der Verbindung, Verbindungsart*) of the collection. If this order or manner of composition is considered as 'of no consequence' (*gleichgültig*) for a certain collection, then the collection is said to be a Menge.

The definition of Menge from the *GL* I begins by noting that there are many kinds of collections (in Bolzano's words, there are many concepts that 'include that of collection as part').² But there is an important distinction to be drawn between collections such that how their parts link to one another³ is of no consequence for the purposes of the exposition in the *GL* I, and collections for which the how does make a difference. It is difficult to give a translation of Menge that truly feels neutral when it comes to interpretive issues: translators who want to stress the terminological and/or conceptual continuity with Cantor and Dedekind will prefer 'set', and for those who want to avoid 'set' the main alternative has become 'multitude' (thanks to Simons 1997, discussed in what follows). This however is not universally accepted as a translation, and using it can make comparisons across different Bolzano scholars unnecessarily hard to follow. I therefore choose to leave Menge untranslated here.

Vielheiten (*WL* §86; *EGL* III §§119, 121; *PU* §4) A special kind of Menge is one where each of its parts can be considered as a unit (*Einheit*), that is, it should be considered as not dividable into further simples. A Menge of such kind is then dubbed a Vielheit.⁴ Bolzano's way of defining the natural numbers, for example, suggests that these are Vielheiten (except for the number 1, which is a unit, hence a simple). This fact seems to prompt a minor readjustment over time in the classification of Vielheit with respect to other kinds of collection, and I discuss it in Section 2.2.3.

Summen For the kinds of collections seen so far, Bolzano leaves it undetermined whether the parts of the parts are also parts of the collection as a whole. There is a kind of collection for which the transitivity of the part relation is a given though,

2. For a recent exposition of a mereological reading of conceptual containment in Bolzano's work, see (Claas 2021).

3. German: 'Die Art, wie ihre Theille mit einander verbunden sind'. (*EGL* III §88)

4. I leave it untranslated because some authors use for Vielheit translations that other authors use for Menge. Vielheiten are explicitly defined as a kind of Menge only in the *PU*. In the *WL* and *GL* I Bolzano writes that sometimes Vielheiten are referred to as Mengen in common usage, but he does not explicitly define Vielheiten as Mengen.

namely, Summen. In the *GL I*, Bolzano summarises the 'sum-property', i.e. the property that all and only Summen have, as follows: $pA \ Bq \ C \ A \ B \ C$,⁵ where the whole in question is made out of proper parts $pA \ Bq; C$, and $pA \ Bq$ also has parts which count as parts of the whole, namely, A and B . Note that Summen are presented by Bolzano as a kind of Mengen, just like Vielheiten. So for all these three kinds of collections, the mode of combination or the order between the parts is 'a matter of indifference'. Concerning leaving 'Summe' untranslated: I prefer to keep the original German in order to distinguish Bolzano's notion from the mereological notion of sum.

Reihen Reihen are the only kind of collection Bolzano treats for which the way the parts relate to one another is part of what determines the collection (*WL* §85; *EGL III* §144; *PU* §7). First of all, characteristic of a Reihe in Bolzano's sense is that there is an order relation on the parts $\{$ which are called terms, *Glieder*, of the Reihe $\}$ and that each term of the Reihe is determined by its predecessor or by its successor by applying a determinate rule to it (the successor or predecessor), which applies to all terms of the Reihe. Bolzano is aware that the notion of Reihe is sometimes presented more narrowly than he does by his contemporaries, in that the 'rule' between terms is only allowed to be an arithmetic ratio, for example, or at most an arithmetic or a geometric ratio, with the result that one cannot talk of Reihe beyond the mathematical domain. Bolzano explicitly wants to give a definition of Reihe that encompasses things like the Reihe of all European capitals, ordered from West to East. Clearly, Bolzano's Reihen correspond to both sequences and series and as such play an important role in mathematics. Moreover, Bolzano defines the natural numbers essentially as a Reihe (*WL* §87.4).

As for translating Reihe, again there is no established uniform translation: Rusnock and George translate it as 'sequence' in their English translation of the *WL* (Bolzano 2014), whereas Russ tends to prefer series (Russ 2004), and other authors alternate between the two, depending on the context. To avoid confusion then, I prefer to keep the original once again.

Quantities (*Großen*) The last of Bolzano's collection kinds that I want to mention here is that of 'quantity' (*WL* §87, *EB* §1; *PU* §6). Something x is a quantity if it can be said to belong to a kind A such that it can be compared with any other object y of kind A in order to determine whether there is z (still of the same kind) such that $x \ y \ z$, or $y \ x \ z$, or $x \ y$. I mention this definition in this chapter because Bolzano defines mathematics as the theory of quantity, and by concluding this list of kinds of collections with quantities I wanted to stress the importance of Bolzano's collections to his conception of mathematics.

5. This comes from *PU* §5.

2.2.2 Conceptual dependencies

On the whole, Bolzano's definitions of these notions is rather stable between the *WL*, the *GL* and the *PU*. What clearly changes is the focus, in the sense that besides the notions considered here, each work also contains the definition of other kinds of collections which are not to be found in the other two works. For example, in the *WL* another kind of collection-idea is that of 'exceptive ideas'⁶ { ideas that result from 'think[ing] merely of the part of a collection that remains once one has *removed* the other parts belonging to it' (*WL* §88)} and in the *GL* Bolzano uses kinds of collections to define also equality (*Gleichheit, Gleichartigkeit*) and qualities (*Bescha enheiten*). The work with the most compressed treatment is without a doubt the *PU*, whereas the work with the most expansive treatment is the *GL* I. These subtleties aside, I have drawn below a diagram to illustrate how the various collection concepts we are interested in relate in Bolzano's work, based on the definitions one reads in the *WL*, *GL* and *PU*.

- 1 = Inbegriff (collection)
- 2 = Menge
- 3 = Vielheit (plurality)
- 4 = Summe (sum)
- 5 = Reihe (sequence/series)

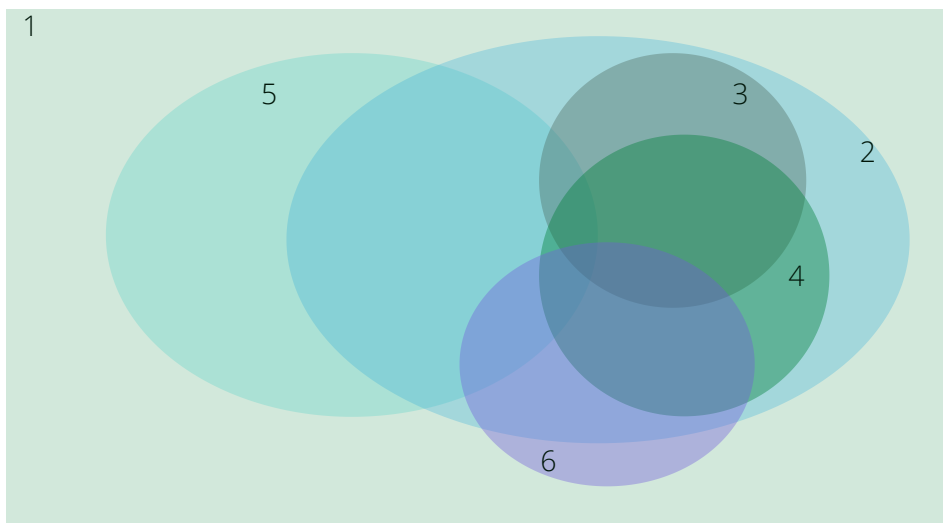


Figure 2.1: Bolzano's definitions of collection concepts

This diagram only includes concepts which play a significant role in Bolzano's mathematical and foundational writings, thus excluding the 'exceptive ideas' of the *WL*. I also decided not to include 'totalities' (*Allheit*) because although they

6. This is Rusnock and George's translation of *Ausnahmsvorstellungen* (Bolzano 2014, p. 297).

are mentioned by Bolzano in the *GL* they end up not playing as significant a role as the notions of Menge, Summe, quantity and so on.

Something which might surprise the reader on the basis of the definitions of Menge and Reihe we have seen in the previous section is to see that Mengen and Reihen are not presented as mutually exclusive concepts (unlike, say, what is suggested by Simons 1997, p. 99). This is because Bolzano himself rejects the idea in the *EGL* III §146 (p. 191):

Some will consider the Reihen to be collections of such a kind that they should be opposite to that of Mengen, i.e. in which the way in which their parts are connected, the order thereof, is regarded as something essential. One will retreat from such an opinion on one's own accord, once one realises that e.g. the propositions: *a* is *b*, *b* is *c*, *c* is *d* form a Reihe, although, as propositions in themselves, they do not need to stand in a certain connection, and their signs can be placed next to each other in whatever order. To me, then, the essence of a Reihe seems to consist merely in the relationship that must prevail between the terms of it.⁷

To understand this passage one first needs to clarify what Bolzano means by 'opposite' and 'essential'. For two concepts to be opposites is a technical notion: it means that each concept exhibits a property, while the other exhibits the negation of that property. In the case of Menge and Reihe, such property would be the order in which the parts are related to one another, and this property would be essential for a Reihe, while its negation is essential for a Menge.

The notion of an 'essential property' is also technical, and it is defined starting from the example of a true proposition '*A* has quality [*Beschaffenheit*] *b*'. If *A* is a pure concept, *b* is true of the objects falling under [*A*] in virtue of being objects of [*A*], which is to say, *b* is an essential property of the *As* (*WL* §111). What this means according to, e.g., Siebel (1997, p. 128), is that to be an essential property means to be in a ternary relation such that the three arguments of the relation are: the property in question, the object exhibiting the property, and the concept under which one takes the object to fall. In addition, the relation needs to be functional in its first argument, that is to say, once we fix object and concept, the class of essential properties the object has is also fixed.

So in the excerpt above Bolzano is denying the impossibility that something be simultaneously a Reihe and a Menge { that satisfying the essential properties of

7. Original German: Manche werden die Reihen für Inbegriffe von einer solchen Art halten, welche den Mengen entgegengesetzt werden müßten, d.h. bey denen die Art der Verbindung ihrer Theile, die Ordnung derselben als etwas Wesentliches angesehen werde. Allein von dieser Meinung wird man zurückkommen, wenn man bemerkt, daß z.B. die Sätze: *a* ist *b*, *b* ist *c*, *c* ist *d* eine Reihe bilden, obgleich sie als Sätze an sich, gar nicht in einer gewissen Verbindung zu stehen brauchen, und ihre Zeichen vollends in was immer für eine Ordnung neben einander neben einander gestellt werden können. Mir also dünkt das Wesentliche einer Reihe lediglich in dem Verhältnisse zu bestehen, daß zwischen den Gliedern derselben obwalten muß.

both concepts is incompatible. One can only infer from the example of the chain of propositions 'a is b', 'b is c' and 'c is d' that what he aims to do is drawing a distinction between the rule that is constitutive of every Reihe, and the order in which we are given the representations of terms of a Reihe. In the propositions example, then, what makes them a Reihe is not that 'a is b' is first, 'b is c' is second and 'c is d' is third, but the fact that the predicate of one becomes the subject of the other proposition.

Specifically on the point of the relationship between Mengen and Summen we can already see a disagreement between my analysis and Simons's, as his diagram illustrates: Simons puts series, i.e. Reihen, and Mengen, multitudes, on separate

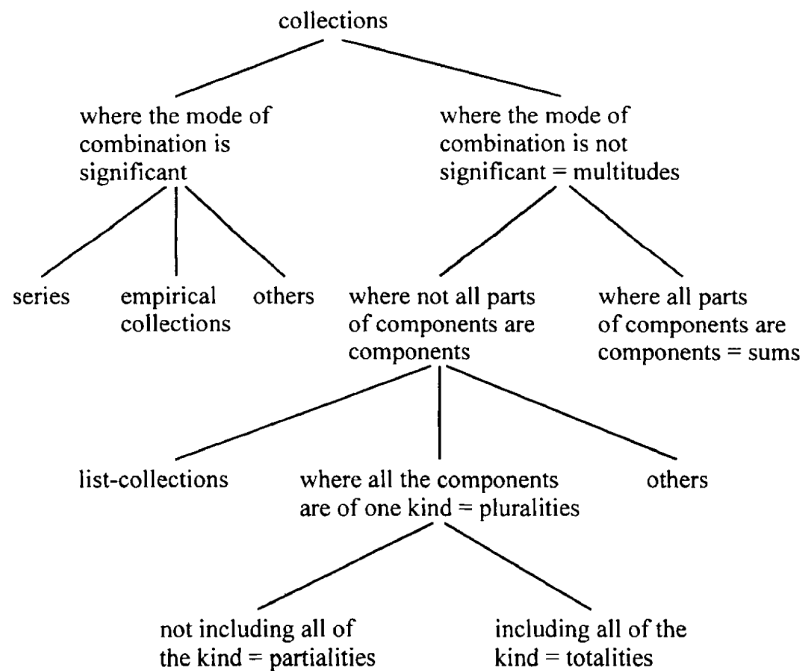


Figure 2.2: Simons's representation of Bolzano's collections

branches of a non-binary Porphyrian tree, meaning not just that Reihen and Mengen have no objects in common (their extensions do not overlap), but that the two concepts are incompatible, contradicting the passage quoted above. This should be an illustrative example of why a new account of Bolzano's collections is needed. Let me proceed then with describing what I believe is a helpful approach in developing one.

2.2.3 Concept-first collections

In this chapter I follow Krickel (1995)⁸ in defending an interpretation of Bolzano's theory of collections that one might call *concept-first*. This interpretation consists in the following two theses:

Kind Bolzano's theory of collections is primarily a theory about concept kinds, not object kinds;

Part The parthood relations of a collection are determined by the concept that collection is taken to fall under.

Kind is supported by how Bolzano introduces talk of his collection kinds in *WL* §§82-87, and how he refers to concepts under which collections may fall as 'viewpoints' (*Hinsicht*, *Rucksicht*) from which to consider (*betrachten*) a certain collection { for example, in *PU* §4 a glass can be considered as a drinking vessel or as just a mass of glass, or an object can be considered of a kind (*Gattung*) such that it counts as a quantity (*PU* §6). Both Krickel (1995) and Bellomo and Ginammi (2021) also note in support of **Kind** that Bolzano always answers questions of existence of collections by answering the question of whether a certain collection-concept has a non-empty extension.

As for **Part**, it is a consequence of how Bolzano characterises essential properties (or qualities) in *WL* §111, once we recognise that the part-relations of an object should also be counted among its essential properties. [*Menge*],⁹ [*Summe*] and [*Reihe*] are pure concepts according to Bolzano (*WL* §84 and §85) { since they do not contain intuitions { hence **Part** holds for the objects falling under them in virtue of the definition of essential property we find in (*WL* §111).

To illustrate **Kind** and **Part**, take the example of a panelled skirt. If considered as an object falling under the concept [*panelled skirt*], then it will be considered as having certain parts { including the panels, tassels of fabric from which this kind of skirt takes its name. **Part** just boils down to this argument, that the parts of a panelled skirt *qua* panelled skirt are such just in virtue of the concept under which we are subsuming the object in question. Now suppose however that the same skirt fits someone as a dress, not as a skirt, and they use it as such. Then the skirt, considered as a dress, will have certain properties (having a neckline, being sleeveless, etc.) that skirts simply do not have, and they will have parts (a neckline, a bust) that skirts do not have. **Kind** allows us in cases such as this to refrain from concluding that there are two objects where common sense dictates there is simply one.

Krickel (1995) holds both views (**Kind** and **Part**), but instead of calling his a concept-first interpretation of Bolzano's theory of collection, he calls it a relative

8. See also Chapter 3 for a more thorough presentation.

9. It is customary in Bolzano scholarship to use the [concept] notation for concepts, cf. Morscher as referred to by Claas (2021).

interpretation, because for any given collection, the kind of collection it is is relative to the concept it is taken to fall under, and also the parts it has depend on the concept it falls under. I eschew talk of relativity here because it might engender two confusions: first, that there is a kind of subjectivism lurking in Bolzano's theory, and second, that there is some sort of arbitrariness in the parts that can be ascribed to a collection. In addition, there are other ways { besides the one spelled out by **Part** { in which Bolzano's parthood relation has been taken to be relative in the literature. It is two of these relative readings that I examine now, with the goal of showing some undesirable consequences of these alternatives to **Part**.

Simons on relative parthood

(Simons 1997) is an article-length response to Krickel in which Simons argues contra Krickel that not all of Bolzano's collections can be treated as mereological wholes. Contra everyone else but Krickel, they also cannot be treated as sets. In spite of the apparent dialectic, Simons's position is close to Krickel's on important aspects of Bolzano's theory. Like Krickel in fact, Simons also recognises that, at least in some cases, the concept under which a collection falls plays a role in determining the parts. Accordingly, he calls *components* of a collection those parts that are recognised as parts of the collection in virtue of the concept under which the collection falls. 'Part' becomes the umbrella term for anything that makes up a composite object. To illustrate the distinction, consider the example of Titus, Caius and Sempronius (adapted from *WL* §83). They form a triumvirate, and each of them is a member of the triumvirate, that is, in Simons's terms, a *component* of the triumvirate. Caius's head, on the other hand, though it is a part of Caius and Caius is a component of the triumvirate it is not a component of the triumvirate. Here is Simons's own illustration of the distinction:

In the case of a list-collection the listed objects are the only parts according to Bolzano. But do not such collections have other parts? Of the collection of Cajus, Sempronius and Titus, are not the arms of Cajus or the subcollection Cajus and Titus also parts, or any collection composed of arbitrary parts of the three gentlemen? No, says Bolzano, parts of the parts or collections of the parts less than the whole have no claim on the title "part" in this case. "What is commonly said, that the parts of a part are also parts of the whole, holds only for collections of certain kind, of which we shall soon come speak." (GA 1/11, 201) Clearly Bolzano is using the word "part" in more than one sense in this passage. There is a very general sense in which the arms of Cajus are parts of him and the subcollection Cajus and Titus is a part of the whole collection. I shall continue to use the word "part" for this very broad sense. However for the sense in which only the three men are parts of the list-collection I shall use the word "component"

(*Bestandteil*). We may thus say that for Bolzano, it is not always true that a part of a component of a collection is a component of the collection, nor that every collection of components of a collection is itself a component of that collection.

Simons's distinction between parts in general and components in particular also affects the definition of *Summe*:

If [...] we have a multitude [*Menge*, author's addition] where every part of a component is a component, then Bolzano calls this a sum [*Summe*, author's addition].

Simons's interpretation of parthood singles out a certain kind of parthood relation, being a component of, that is relative in the sense of **Part**, namely that it depends on the concept under which the collection falls. The difference between Simons's interpretation of parthood and the one that I call concept-first is that, according to the latter, all of parthood is concept-dependent. There is more to be said about Simons's components and his analysis of Bolzano's collections, but I leave that for later (Section 2.3.2). I now turn to a second twist on the relative interpretation of parthood, as articulated by Behboud (1997).

Behboud on relative parthood

Behboud, like Simons, believes that Bolzano's notion of part is 'systematically ambiguous' (Behboud 1997, p. 111) because parthood is actually concept-dependent. Unlike Simons, however, Behboud makes parthood dependent on the concept under which the part falls (and unlike the concept-first view I advocate for, namely that parthood is dependent on the concept under which the object falls). Here I use Bolzano's example of the heap of coins (*Geldhaufen*) to illustrate the difference between Behboud's interpretation of how parthood works according to Bolzano, and the concept-first interpretation.

Bolzano uses a heap of coins as an example of a collection both in the *WL* (§84) and in the *GL* (*EGL* III §§88, 120). While in the *WL* and in the first occurrence in the *GL* the heap of coins is invoked to illustrate the difference between Mengen which are also Summen and Mengen which are not Summen, in the second occurrence in the *GL* the contrast is between Vielheiten, a particular kind of Summen, and Summen which are not Vielheiten.

Here is the excerpt from the *WL* regarding the heap of coins example (the translators prefer 'pile' to 'heap' for the German *Haufen*):

§84.2 The example of a pile of coins shows that with collections where the way the parts are combined is a matter of indifference there may be reasons that prevent us from looking upon the parts of the parts as parts of the whole, or from interchanging them. For if we replaced one of the coins in such a pile with the parts into which it can be reduced

by mechanical or chemical means, the value of the whole might well be altered. But there is no lack of collections where both are permitted, namely, where the way the parts are combined may be looked upon as a matter of indifference, and the parts of the parts looked upon as parts of the whole. An example is the length of any line. [...]

Behboud proposes to use his interpretation of the parthood relation as a ternary relation between parts, a whole and the concept (or idea) the parts fall under to give his reading of the passage above:

Let us see how Bolzano's examples fit into this scheme [Behboud's proposed scheme, my note]. Suppose we are given a certain collection *A* of coins. If we relativize the part-relation to the idea [*coin*], then *A* is a multitude [*Menge*, my note] w.r.t. [*coin*]-parts and the idea [*heap of coins*]. [...]

[O]ur heap of coins will not be a multitude w.r.t. the idea [*heap of coins*] { if we relativize the part-relation to the idea [*piece of metal*]: pressing the coins into a massive cubic block is a rearrangement of [*piece of metal*]-parts that is not a heap-of-coins, since, as Bolzano says, its monetary value might thereby have changed. (Behboud 1997, pp. 113-114)

Because in Behboud's account the concept (or idea) the collection falls under and the one the parts fall under are completely independent of one another, he is forced to say that the heap of coins as a heap of coins both is and is not a Menge, depending on what concept the parts are taken to fall under.

This then clearly shows that Behboud's interpretation does not fulfil **Kind** of the concept-first interpretation, because the concept [*heap of coins*] cannot be said to be a Menge concept according to him, given that the objects falling under it are sometimes Mengen, sometimes other kinds of collections depending on the parts they are deemed to have. Behboud's interpretation however does not follow Bolzano's words closely. Bolzano is explicit that the distinction he is drawing is about collection kinds, not different notions of parts applied to the same collection. In that sense, the concept-first approach is more faithful to Bolzano's thought: it is an essential property of the objects falling under [*heap of coins*] that their only parts are coins, not metal molecules obtained by electrolysis of a single coin. According to the concept-first interpretation, the heap of coins example is used to underscore the difference between two kinds of concepts and ideas, the Menge-concepts which are not Summen on the one hand, and the Menge-concepts which are Summen on the other.

The heap-of-coins concept though is not merely a Menge-concept, it is a Vielheit-concept, because a heap of coins is a collection made of (concrete) units of a certain kind, namely coins. Interestingly, the definition of Vielheit does not specify anything more than that, but in the *GL* (*EGL* III §120, p. 169) Bolzano

uses the heap of coins as an example again, but this time, he suggests that it be a Summe as well as a Vielheit:

Incidentally, I do not think I am making a mistake when I look for the concept of a Summe in that of a Vielheit, that is when I picture to myself that by a Vielheit we think of a collection for which the manner of composition of its parts is to be considered as of no consequence, and for which the parts of the parts are to be considered as parts of the whole. Because no one supposes that the how-many of a heap of gold pieces [that is, coins] changes if we group them together this or that way. It is essential for a Vielheit though that those parts that are units are treated by us as simple; even when from another point of view they appear to us to be composite. Just like the concepts of Two, Three can be fixed, so can one go on in the obvious way and determine the concept of every arbitrary number.¹⁰

This passage shows that according to the Bolzano of *EGL* III §120 Vielheiten 'contain in themselves also the concept of Summe'. Given that Bolzano proceeds to illustrate the idea by appealing to a heap of coins again, it seems that this passage is at odds with the other two passages regarding heaps of coins (*WL* §84 and *EGL* III §88), because here Bolzano suggests that the very concept that he had used to contrast with Summen is now a Summe-concept. Note that Behboud-style interpretations of parthood as relative to the concept of the parts, not the one of the collection, do not help here, because the whole (a heap of coins considered as a heap of coins), the parts (the pieces of metal) and the concept under which the parts fall (coins, or pieces of gold with monetary value) are the same across the three examples. For Behboud to be able to explain the difference we would need a different concept other than [*piece of gold*] to subsume the parts.

On the concept-first view, on the other hand, I see two plausible accounts of what has changed between *WL* §84 and *EGL* III §88 on the one hand, and *EGL* §120 on the other:

1. In the first two passages, where the heap of coins is said not to be a Summe, Bolzano already thinks that Vielheiten are Summen { it just so happens that he has not yet realised that the [*heap of coins*] concept is a Vielheit-concept (and the realisation comes in *EGL* III §120);

10. Ubrigens glaube ich nicht zu irren, wenn ich in dem Begriffe der Vielheit den einer Summe suche, d.h. mir vorstelle, daß wir bey einer Vielheit an einen Inbegriff denken, bey welchem die Art der Verbindung seiner Theile als etwas Gleichgültiges angesehen wird, und die Theile der Theile als Theile des Ganzen betrachtet werden. Denn Niemand vermeint, daß sich das Wieviel eines Haufens von Goldstücken ändere, wenn wir sie so oder anders zusammenstellen. Wesentlich aber ist es bey einer Vielheit, daß wir diejenigen Theile, die Einheiten sind, schon als einfach betrachten; selbst wenn sie in einer anderen Rücksicht uns als zusammengesetzt erscheinen. { Wie die Begriffe von Zwey, Drey festgesetzt wurden, so kann man begreiflicher Weise noch immer weiter gehen, und den Begriff jeder beliebigen Zahl bestimmen.

2. In *WL* §84 and *EGL* III §88 Bolzano already thinks that [*heap of coins*] is a Vielheit-concept, he just has not yet realised that Vielheiten are Summen as well.

Given that Bolzano himself treats a single coin as a unit { that is, as something with no parts { in *WL* §84.2 already, option 2 strikes me as more plausible than option 1. To regard [*heap of coins*] as a Vielheit-concept explains Bolzano's realisation in *WL* §84.2 that no matter how we group the individual coins, a heap of coins is still the same (because it is identified with its monetary value) as long as we consider the individual coins as units, that is, as simples, and none gets destroyed. More generally, this change that makes Vielheiten not merely a specific kind of Menge-concept, but a specific kind of Summe-concept, is just the natural endpoint both of the actual definition of Vielheiten and that of natural numbers (that is, positive integers). The fact that the natural numbers except for [*one*] are Summen, for example, immediately explains why it is possible to obtain one new number from a Summe of two (or more), and the fact that they are Vielheiten explains why we can use natural numbers for counting. There would be much more to say about the fruitfulness of these conceptual overlaps for mathematical applications, but for now what we need to pay attention to is that such fruitfulness is straightforward to explain on the concept-first interpretation, whereas an interpretation like Behboud's does not seem to have the tools to do so. For, a heap of coins is not both a Vielheit and a Summe in virtue of the different concepts under which we can subsume the parts of the heap { it is a Vielheit in virtue of the conceptual analysis of [*heap of coins*], and then a Summe in virtue of the conceptual analysis of [*Vielheit*].

2.2.4 Final picture

In the previous section (Section 2.2.3) I have discussed Simons's and Behboud's interpretations of parthood in Bolzano. To summarise:

1. Simons thinks parthood is a binary relation, but componenthood is a relation between an object, a part of that object and a concept under which the object is thought.
2. Behboud on the other hand thinks parthood is a ternary relation between two objects (the whole, that is, the collection, and the part) and a concept under which the part is thought.

Under my interpretation, parthood is a ternary relation between two objects (the collection and the part) and the concept under which the collection falls. This interpretation is the one that is an immediate consequence of the theses that 1) Parthood is an essential property of collections (Section 2.2.3) 2) Essential properties of an object depend on the concept under which the object falls (Section 2.2.2).

Having thus concluded the discussion of Bolzano's collection notions and how they are interpreted according to the concept-first approach, it is time to update the diagram from page 13.

- | | |
|-----------------------------|------------------------------|
| 1 = Inbegriff (collection) | 7 = Infinite |
| 2 = Menge | 8 = Number series |
| 3 = Vielheit (plurality) | 9 = Natural numbers except 1 |
| 4 = Summe (sum) | |
| 5 = Reihe (sequence/series) | |
| 6 = Quantity | |

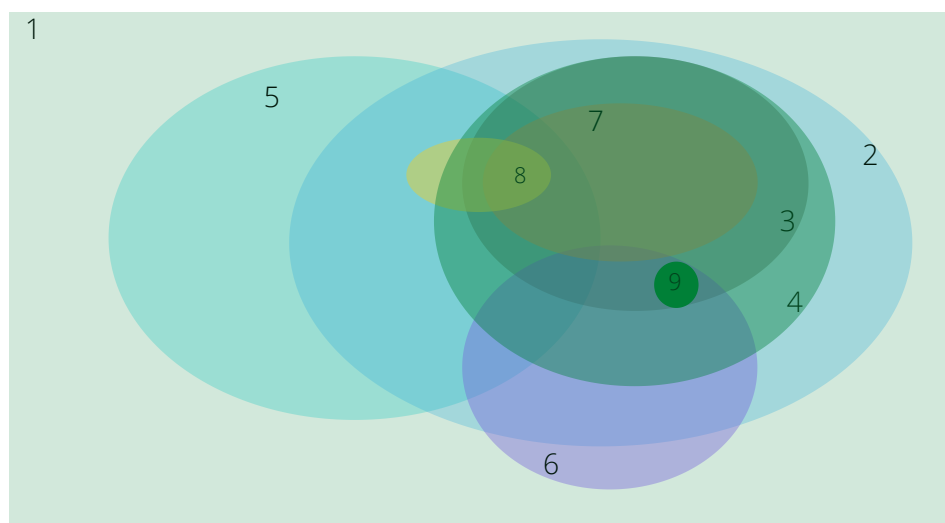


Figure 2.3: Bolzano's definitions, refined

In this version of the diagram I have adjusted the relative positions of Summe and Vielheit in light of the results from Section 2.2.3 and I have also included the concept of 'infinite' among Bolzano's collection notions, despite him introducing its definition as a special kind of Vielheit only in one of our texts, namely the *PU*. This definition of infinity has the upshot that whatever Bolzano says is infinite needs to be also a Vielheit, in some sense or other. Then we must have that some Reihen are Vielheiten, and some quantities, too, and also some Summen more generally. This allows me to point out how misleading a tree-like picture can be for Bolzano's collection notions, given that, on the basis of Bolzano's definitions alone, something may be a Menge, a Summe and a Reihe all at the same time.

2.3 Sets, wholes and extensionality

In the previous section we saw what Bolzano has to say about each kind of collections, including collections in general (*Inbegriffe*), I summarised the findings in a Venn-style diagram to illustrate the dependencies that emerge from the

definitions alone, and I specified that I take this taxonomy to be not about objects but about concepts. Now that I have laid the groundwork to compare Bolzano's collections to sets and mereological wholes, this comparison is what I develop in this section; the notion of extensionality plays a central role in said comparison. Before delving into the comparison proper, however, I need to clarify what is meant by 'extensionality' when it comes to Bolzano's collections, and that is where this section begins.

2.3.1 Different ways to be extensional

In this section I begin my analysis of extensionality in relation to Bolzano's theory of collections by rehearsing the different versions of extensionality Varzi presents in (Varzi 2008), which are versions of the extensionality principle for mereology. The discussion of different extensionality principles in mereology is followed by a discussion of extensionality for sets.

Varzi's discussion of extensionality

Varzi's paper contains three principles which he says capture the 'nominalistic dictum, no difference without a difference maker' (Varzi 2008, p. 108). The 'nominalistic dictum' Varzi mentions can be conceptualised as the informal counterpart to the various extensionality principles we are going to see in the next few paragraphs.

Varzi presents three extensionality principles in his chapter, the *extensionality of parthood (EP)*, *extensionality of sum (ES)*,¹¹ and *extensionality of composition (EC)*. These all express sufficient conditions for numerical identity providing a certain relation obtains between two composite objects x and y , and they read as follows:

(EP) If x and y are composite objects and they have exactly the same proper parts, then $x = y$.

(ES) If x and y are *sums of* exactly the same objects, then $x = y$.

(EC) If x and y are *composed of* exactly the same objects, then $x = y$.

While the notion of part can be regarded as primitive, hence undefined, being a sum and being (a) composite of certain z s is defined as follows:

x is a sum of the z s if and only if all the z s are parts of x and all parts of x have a part in common with a z .

11. Varzi calls this Uniqueness of Composition and labels the principle accordingly, but in the economy of this chapter I find it more helpful to change it to 'extensionality of sums'.

x is **composed of the** z s if and only if x is a sum of the z s and the z s are pairwise disjoint.

So to be a sum of the z s requires more than saying that all parts of x are z s, and being composed of the z s requires more than just being a sum of the z s. One might then expect a similar linear ordering according to strength among the three principles **(EP)**, **(ES)**, **(EC)**, but not so. Varzi quickly shows that in fact **(ES)** implies both **(EP)** and **(EC)**, but neither does **(EP)** imply **(EC)** nor conversely **(EC)** **(EP)**. When it comes to discussing extensionality for Bolzano's collections, I believe the most relevant of Varzi's principles is **(EC)**, because of a restriction on parthood that Simons calls the 'no-redundancy' condition, according to which in a list of the parts of a collection, no two expressions can pick out two overlapping parts { otherwise the whole collection is actually a spurious one (*PU* §3). Then it seems that Bolzano's parthood is actually closer to Varzi's composition, and this is why **(EC)** is the relevant principle of extensionality for Bolzano's collections.

To complete our three-way comparison between Bolzano's collections, sets, and mereological wholes, we also need to be able to compare mereological extensionality with set-theoretic extensionality. However, none of Varzi's three principles is directly comparable to the principle of extensionality for sets, which is usually expressed by the axiom of extensionality:

Ext If $x; y$ are sets, $x = y$ if and only if all elements of x are also elements of y , and vice versa.

If we translate the relation of elementhood of **Ext** into that of composition appropriately, we can compare Varzi's principles with it. **Ext** turns out to be equivalent to the conjunction of the following two principles:

unique composition Given a list of elements, there is exactly one set that is composed of precisely (the singletons of) those elements.

unique decomposition For any set x there is precisely one list of (singletons of) elements which gives the full decomposition of the set.

The equivalence can be derived if one follows Lewis's standard conversion (Burgess 2015a, p. 460) according to which given two sets x and y , x is a part of y if and only if $x \subseteq y$. Then a set x is composed (in Varzi's sense) exactly of the singletons of its elements, and by **Ext** if y is composed of the same singletons, then $x = y$, thus satisfying **(EC)**. This means that **(EC)** always holds for sets or, which amounts to the same, **unique composition** always holds for sets.

Note however that **(EC)** and full **Ext** fail to be equivalent, because **(EC)** does not express a necessary condition for the identity $x = y$. This is because a mereological whole admits for several decompositions { a glass can shatter in several different ways, and it can be decomposed into molecules, or into atoms. A

set by contrast does uniquely decompose into the singletons of its elements (on this point, cf. also Incurvati 2020, pp. 5-6).

2.3.2 Neither sets nor wholes?

The distinctions among different extensionality principles from Section 2.3.1 allow us to now target the metaphysical comparison between Bolzano's collections on one hand, and sets and mereological wholes on the other. I begin with listing the traits that sets have but Bolzano's collections lack, and I mostly follow Simons's suggestions on that score.

abstractness First of all, and this is something stressed by Behboud as well as Simons, sets are usually taken to be¹² abstract entities. By contrast, Bolzano's collections can be both abstract (like the collection whose parts are the concept [*equilateral triangle*] and the angle $\frac{2}{3}$) and concrete (to use one of Bolzano's examples, a heap of coins), or a mix of the two if some of the parts are concrete, some are abstract.

emptyset Bolzano does not recognise the existence of 'empty' collections. When it comes to dealing with ideas that have no objects falling under them, for example, there are no empty collections representing the extension of said ideas { the ideas themselves are said to be objectless.

singleton Bolzano does not distinguish between an object and the singleton of that object. This might not preempt modelling Bolzano's collections as sets, but if strict identification is what we are after, then it is enough to sink the project. For sets, on the other hand, given a set it is always possible to define the singleton of that set, thus obtaining a distinct¹³ one.

set-extensionality Sets are identical if and only if they share exactly the same elements. By contrast, some of Bolzano's collections can have exactly the same parts without being identical, and one and the same collection can be said to have different partitions (i.e. exhaustive lists of parts).

I have already given the reason why **emptyset** and **singleton** do not apply to Bolzano's theory { Bolzano flatly denies the existence of one (the empty collection)

12. Maddy (1980) might be taken to be an exception, but, first, Maddy does not quite deny that sets are abstract { she defends the view that humans can be said to perceive sets, partly because they comprise concrete elements, but that does not mean that the sets themselves are concrete. Moreover, in recent years Maddy herself has come to rebuke her early views and arguments for them.

13. Provided what one means by set theory is actually the so-called naive set theory, or something like ZFC. There are also less mainstream { though still well-known { set theories, such as Quine's NF that identify the singleton of a set with the set itself. (Thanks to Incurvati for pointing it out to me).

and lacks awareness about the other (a single object, and a collection whose only part is that object). The discussion (*WL* §84.1) on whether 'Menge' is an apt word-choice for a specific kind of collection contains a concession by Bolzano that a collection may be of as few as two parts, and yet still be a collection. A collection with one or no parts is never mentioned as a possibility.

As for **abstractness**, Simons's argument is simply that Bolzano's collections are not universally abstract. Similarly, the argument against **set-extensionality** is that Bolzano's collections do not universally satisfy its two conjuncts **unique composition** and **unique decomposition**. For a failure of **unique composition**, consider the collection that is the Vienna Philharmonic Orchestra (this is Simons's example). Suppose now that the same musicians that make up the Vienna Philharmonic Orchestra also play pro bono in elementary schools as the Vienna Philanthropic Orchestra. Because in principle the two orchestras have different persistence conditions (one of the two can cease to exist, without thus destroying the other), they are two distinct wholes and therefore they constitute a violation of **unique composition**. Conversely, the Vienna Philharmonic Orchestra can be decomposed into its individual musicians, but also into each instrument, for example. This suffices to conclude that **unique decomposition** also fails, and from these prima facie failures of **unique composition** and **unique decomposition** Simons infers that Bolzano's theory of collections is not extensional.

The strategy Simons deploys to argue that Bolzano's collections are not the collections of mereology is the same { he identifies certain defining characteristics for mereological collections, and then argues that some of Bolzano's collections do not share those characteristics. Simons singles out two kinds of mereological wholes, sums on the one hand and 'individuals' on the other. Thus, to be mereological wholes, Bolzano's collections should exhibit at least one of these two characteristics:

extensionality Mereological sums are 'extensional in their identity conditions' (Simons 1997, p. 101), and as a consequence they cannot easily accommodate collections such as orchestras, which, Simons maintains, are not extensional in that sense.

one-ness A collection exhibits this defining property of mereological individuals if it exists as long as its parts exist and it ceases to exist when the parts do so, too.

However, there is a third trait that one may consider indispensable for a theory to count as a mereology, and that is the transitivity of parthood (cf. Varzi 2016). Behboud cites Frege to make the point that transitivity is one of the defining characteristics of the part-whole relation, so as to contrast it to set-membership which in general fails to be transitive. So we add to our list:

transitivity For something to count as a mereological whole, any part of a part needs to also count as part of the whole { that is, parthood needs to

be transitive to be the parthood of mereology. But Bolzano's part-whole relations are not always transitive. It is not the case that if Caius's head is part of Caius and Caius is part of a triumvirate, then also Caius's head is part of the triumvirate according to Bolzano.

Thus, there seem to be enough reasons to disregard a full identification of Bolzano's collections with either sets or mereological wholes, but this conclusion does not provide us with a positive account of what Bolzano's collections are. Simons's proposal for a positive account of Bolzano's collections treats them as things which are determined by three ingredients: a *kind* they belong to, a *list of parts* that constitute them (Simons actually talks about the components of the collection, and the components are kind-dependent parts, not all the parts), and the *mode of combination* of the collection, if the collection is one of those for which the mode of combination makes a difference (i.e. if the collection is not a Menge). To illustrate that all three are needed to fully determine a collection, Simons (1997, pp. 103-105) proposes a thought experiment to his reader in which a list of four two-dimensional, impenetrable squares of equal side is shown to give rise to several collections, because many collections can be built starting from those four squares { or so Simons argues. Simons lists twenty-six collections 'based on' the four squares, and the difference between any two items on his list is one of the following:

1. The components are not the same.
2. The persistence conditions of the collections are not the same.
3. The spatial relations between the four squares are not the same.

Let us consider in turn what each difference actually achieves, and whether it yields a failure of extensionality. 1 is not a failure of extensionality in the sense of **(EC)**, because if two collections do not have the same components in Simons's sense, then they also do not have the same parts in Bolzano's sense. So they cannot constitute a counterexample to **(EC)**. Cases of the kind of 2 and 3 also do not necessarily spell doom for extensionality, but the explanation of how Bolzano can meet their challenge to extensionality requires a bit more work. Recall that by Bolzano's definition of an essential property, certain properties of a collection { the essential properties of that collection { are determined by the collection-concept under which the collection is thought. So it is possible for one and the same collection to exhibit incompatible properties, such as different persistence conditions or different spatial relations between the parts, depending on the concept under which it is considered at the time. Thus, the concept-first interpretation allows us to block putative counterexamples to extensionality of the 2 and 3 kinds. But there is more. The very set-up of Simons's thought-experiment is not permissible from a concept-first perspective, for it is not possible to present a list of things, such as Simons's four squares, without already considering it through

a collection-concept. The moment Simons gives his reader the four squares, he has given them the four squares under the collection-concept [*collection of four squares*]. So the only parts of this collection can be the four squares, and they have to be four. If we want to ascribe more properties or different properties to the collection, we need to consider it from the point of view of a different collection-concept, for example, the [*Menge of four squares*] concept, and so on. But this does not mean that we are multiplying entities by the number of concepts we can come up with. There is always just one collection.

Simons's analysis aimed at showing the inadequacy of both sets and mereological wholes to capture the full extent of Bolzano's theory of collections, because some of Bolzano's collections cannot be sets, and some cannot be mereological wholes. Simons's own proposal exhibits other problems, most notably, an in-built denial of extensionality (of composition, **(EC)**) that I argued does not follow from Bolzano's texts alone.

In the remainder of Section 2.3 I will set aside the broader question of what Bolzano's most general collection notion (*Inbegri*) is, and focus on the more specific collection notions that play an important role in his theory of collections. This allows me to revive the question of whether some of Bolzano's collections are sets or sums. I briefly thematise this question next.

2.3.3 Some sets, some wholes

Let me start once again with sets, and now instead of addressing the question of whether Bolzano's collections in general are sets, I am going to focus on the more substantiated hypothesis that Bolzano's *Mengen* may be sets. If we consider again the four defining characteristics of sets, *Mengen* seem to fare on them similarly to collections in general: *Mengen* can be concrete (cf. the heap of coins example, *WL*, §84) as well as abstract, so **abstractness** is not universally true of *Mengen*. We also do not have empty *Mengen* nor *Mengen* with just one proper part, since the minimum of two parts holds for all collections of all kinds, so **emptyset** and **singleton** fail for *Menge* as much as for collections in general. While these are genuine metaphysical differences, we might agree that Bolzano's mathematical *Mengen*, that is, *Mengen* whose parts are abstract mathematical objects, can be modelled by (abstract) sets and we might agree that in such a model we should also leave out all singleton sets and there should be no empty set either. This leaves us only with **unique composition** and **unique decomposition**. Many interpreters (notably, Sebestik 1992, p. 308) believe that Bolzano's *Mengen* are extensional in the sense that they satisfy **Ext**, and this is because of a passage to be found in the *GL* that reads:

§89 Theorem. The parts which constitute a *Menge* determine it, and they do so wholly and uniquely.

Proof. Because the way the parts are connected to one another is not

to be taken into account for a Menge as such, there is nothing to it then other than the parts themselves. Thus if these are given, then so is the Menge itself also given. The parts determine it and they do so wholly. But also they do so uniquely. For nothing changes in the Menge if we swap two or more of its parts, so that the part designated as *A* assumes the qualities of the one designated as *B* and the *B* part those of *A*; and so on. (*EGL* III §89, my trans.)¹⁴

Admittedly, the statement of the proposition is quite similar to some phrasings of extensionality. To begin with, however, there is only one direction of **Ext** mentioned in the proposition, namely the sufficiency for identity (this corresponds to **unique composition**). The proof should establish two things, that the parts of a Menge determine the Menge, and that they do so exhaustively and uniquely, that is to say a Menge is completely individuated by its parts, and the same parts can determine at most one Menge. The proof of the 'exhaustiveness' of the determination ordered by the parts boils down to an appeal to the definition of Menge: since by definition a Menge is just its proper parts (where the 'just' signals implicit contrast with collections where the way the parts relate is also constitutive of them) if we have the parts then we have the Menge. So the first part of the proof, despite the slogan, seems to be proving more something like 'a Menge is just its parts, not its parts and the way they are related'. Next comes the part of the proof that aims to establish the uniqueness of the Menge. The argument is quite peculiar. Bolzano asks the reader to picture two parts of the Menge, call them *A* and *B*. Then imagine to transfer all the qualities of part *A* on to part *B*, and vice versa, thus 'swapping', in some non-spatial sense, the two parts. We should conclude that the Menge is still the same it was before the swap, and therefore the parts do determine the Menge uniquely . . . up to swapping parts.

I do not think that this can be considered a proof of **Ext**. All Bolzano seems to be proving here is that, first, we have full information about a Menge once we know which parts it has, and second, a Menge is invariant under permutation of parts. Such facts can perhaps be taken to prove (or better, they can be taken as cues of Bolzano's intention to prove) extensionality of parthood (**EP**), by showing that if *x* and *y* are Mengen and they have the same proper parts, then they are the same Menge, for the only way they could differ is in the relative position of their parts, and the proof shows that by swapping parts a Menge remains the same.

14. §89 Lehrsatz. Die Theile, aus denen eine Menge bestehet, bestimmen sie, und zwar vollständig und alle auf einerley Art.

Beweis. Weil die Art, wie ihre Theile mit einander verbunden sind, bey einer Menge als einer solchen, gar nicht beachtet werden soll, so gibt es nichts an ihr, als jene Theile selbst zu bemerken. Sind also diese gegeben, so ist die Menge selbst gegeben. Die Theile bestimmen sie also und zwar vollständig. Aber auch alle auf einerley Art. Denn es ändert sich nichts in der Menge, wenn wir zwey oder mehrere ihrer Theile gegen einander vertauschen, so daß der unter *A* bezeichnete Theil die Beschaffenheiten des *B* und der unter *B* bezeichnete die Beschaffenheiten des *A* annimmt; u. dgl.

We should note that swapping parts only makes sense because of Bolzano's 'no overlap' requirement mentioned above (Section 2.3.1; see also *EGL* III §6, p. 101). This proof does secure **unique composition**, or **(EC)** in Varzi's terminology. This is because a consequence of the proof is that, if we are given a list of the (disjoint, proper) parts of a Menge, we can swap any two parts in that list and get the same Menge, which will be a composite of the given parts in Varzi's sense, thus satisfying **(EC)**.

Even so, this proof falls short of establishing **(Ext)**, and that is what we need if we want to argue that Bolzano's Mengen are sets. What is unequivocally missing from the proof is **unique decomposition**, and the problem is that other features of Bolzano's theory of collections make **unique decomposition** impossible, on pain of inconsistency. Recall (Section 2.2) that, for Bolzano, Summen are a kind of Mengen, and that a typical example of a Summe for Bolzano is that of a line, considered as a length (*WL* §84). It should be obvious that, given a segment, there is no single way in which we can decompose it into partial segments, even if we abide by the restriction that the partial segments should not overlap. So Summen do not exhibit uniqueness of decomposition. If Summen do not exhibit uniqueness of decomposition, and Summen are Mengen, then it is a good thing that Bolzano does not prove uniqueness of decomposition for Mengen. This makes the reading of Menge as sets extremely difficult to defend. I do think however that there is one kind of collection in Bolzano's theory which has both **unique composition** and **unique decomposition**, namely, Vielheiten. A Vielheit is a Menge, and so **(EC)** holds for it simply because it holds for any Menge. Moreover, it follows from the definition of a Vielheit that it uniquely decomposes into units. In that sense, then, a Vielheit¹⁵ is what comes closest to a set in Bolzano's framework.

Summen as wholes

Summen by contrast are the collection kind that offers the most hope to be identified with mereological wholes (more precisely, sums). This is because the two defining characteristics of sums we singled out in Section 2.3.2 are **extensionality** and **transitivity**. By definition, Summen are a kind of Menge such that parthood is transitive on them. As Mengen, they are extensional in the sense of **(EC)**, as we just saw. So they seem to completely fit the criteria for sums.

Rusnock (2013, p. 159), however, claims that we should not interpret Bolzano's Summen as Mengen for which parthood is transitive, because this interpretation of Bolzano's definition, together with what Simons's 'no-redundancy condition' (Simons 1997), leads to a counterintuitive result. Roughly, Rusnock's derivation of the result runs as follows. By various passages in Bolzano's works, we know that a collection cannot have a part appearing twice. This can be captured by the more

15. Given the result about extensionality and Vielheiten, the latter now start to sound just like modern pluralities, especially as characterised by Incurvati (2020, pp. 3-11). If so, then even Vielheiten cannot be said to just be sets.

precise notion of overlap: if for x and y there is a z such that z is both a part of x and a part of y , then we say that x and y overlap. Then the no-redundancy condition becomes: if x and y are (proper) parts of some whole t , then x and y do not overlap.

If we take the definition of Summe to mean that for any t a Summe, if x is a part of y and y is a part of t then x is a part of t , then clearly any Summe t is in violation of the no-redundancy condition as understood by Rusnock, because x and y are bound to overlap. This means that no collection can satisfy both the no-redundancy condition and the Summe definition, and given that for Bolzano universal statements cannot be vacuously true (Rusnock 2013, p. 159), this makes any universal proposition about Summen false, thus making it odd that Bolzano would write about them in the first place.

Since it seems that the no-redundancy condition, together with the interpretation of Summen as Mengen for which parthood is transitive, leads to the conclusion that there are no Summen, Rusnock resolves to show that the definition of Summen can be understood as stipulating something other than the transitivity of parthood. According to Rusnock, Bolzano's definition of Summe establishes a substitution rule for parts of a certain kind of collection (namely Summen).

Rusnock's solution however is not the only one available. In fact, if one looks at the passage that Simons cites to support his no-redundancy clause, one simply reads that 'for the existence of a collection or the unification of certain things it is only necessary that none of these things is already a part of the others' (*EGL* III §6, p. 101). This leaves open the possibility that one object may be decomposed into several different partitions, say, one with parts $A; B; C$, another with parts $D; E; F$. Bolzano's restriction is upheld even if A overlaps with D , provided there is no overlap within $A; B; C$ or $D; E; F$. I thus see no reason to read Bolzano's definition of Summe as anything but a Menge for which parthood is transitive. As a consequence, Bolzano's notion of Summe is equivalent¹⁶ to the *mereological* notion of sum.

2.3.4 Taking stock

In this section I have tackled the question of what Bolzano's collections are, specifically with respect to sets and mereological wholes. I have split the notion of extensionality into different principles, and this has allowed me a clearer analysis of the alleged arguments against the extensionality of Bolzano's collections. Overall I agree with Simons that not all of Bolzano's collections are extensional, but not because of the arguments Simons advances { the concept-first interpretation of Bolzano's collections, unlike Simons's interpretation, is compatible with the claim that Bolzano's collections are extensional. Rather, it is Bolzano's apparent need

16. I want to refrain from claiming outright that Bolzano's Summen just are mereological sums, because I do not want to commit to the view that Bolzano's theory of Summen is a mereology through and through. Such a topic deserves its own paper.

to establish extensionality (in the form of **unique composition**) exclusively for Mengen (and thus also for Summen and Vielheiten) that seems to imply that collections in general do not need to satisfy **unique composition**, or **(EC)** in mereological terms. The upshot of this analysis is that I agree with Simons that general collections are neither sets nor wholes, but I refine Simons's position by arguing that Vielheiten and Summen are close to sets and wholes, respectively.

2.4 Bolzano's collections and the search for a foundation

In the previous two sections we have explored the question of how to interpret Bolzano's theory of collections in general. The upshot of those sections is that Bolzano's theory is best interpreted as a theory of concept kinds, not object kinds, and that these concepts are not reducible to one another { their differences are genuine. In this section I build on the work done so far on the metaphysical and conceptual front to illustrate how Bolzano's collections can and do play a foundational role for his mathematics. Given the still influential claim¹⁷ that Bolzano's theory of collections is an anticipation of set theory, in Section 2.4.1 I will use Maddy's (2017, 2019) characterisation of set theory's foundational roles as a guide to what foundational roles Bolzano's theory may fulfil. Next, in Section 2.4.2, Section 2.4.3 and Section 2.4.4 I will use excerpts from Bolzano's programmatic and mathematical writings to establish which, if any, of the foundational roles played by set theory are also roles played by Bolzano's theory of collections.

2.4.1 Maddy, or what foundations are for

Maddy (2017) is not interested in giving a prescriptive list of what a foundation ought to do for mathematics, but of what set theory specifically does that makes it function as a foundation for mathematics. Based on what professional mathematicians have said on the topic, she isolates five foundational roles for set theory:

1. Set theory offers a **Meta-Mathematical Corral**. This means that set theory allows for mathematics to be treated as one object, and to prove results about mathematics as a whole (for example, where it is incomplete, nitely or at least recursively axiomatisable, etc.). Clearly, Maddy thinks set theory manages to fulfil such a role for classical mathematics.

17. See for example (Sebestik 2017), that claims his theory to be a proto-set theory, or the following claim made in the introduction to (Zermelo 1908, p. 160): 'It was Bernard Bolzano (1781{1848), the Bohemian-Austrian mathematician, philosopher, and theologian, who first systematically studied the concept of set.'

2. It provides a **Generous Arena** such that every part of mathematics can be developed within the confines of this shared foundation and potentially compared to one another. It is what allows to apply theorems discovered in area A to problems in area B , and vice versa.
3. When we use set theory to define a key concept more sharply than it was possible before, then the theory is shown to provide **Elucidation**. Maddy's example is Dedekind's definition of the real numbers as cuts: the definition of cuts cannot be made sense of without a set-theoretic framework, and it differs from traditional definitional attempts, but it opens up new possibilities { results that could not be established with previous characterisations of the reals are now within reach.
4. Set theory can be seen as providing **Risk Assessment** when mathematicians (or perhaps one should say, mathematical logicians) try to justify the trustworthiness of some axiomatic theory by establishing that if ZFC is consistent, then so is their new theory. Maddy uses Voevodsky's formulation of set theory as a 'benchmark of consistency' (Voevodsky as cited by Maddy 2017, p. 295) to explain how set theory can play this role.
5. Finally, Maddy claims that set theory nowadays plays the role of a **Shared Standard** (across mathematical disciplines, and among mathematicians) for proof, meaning that if a result can be proved from the axioms of set theory, with inferences recognised as valid within set theory, then that counts as a mathematical proof more generally. The question of whether something can be proved is safely reduced to the question of whether something can be proved within set theory.

Before discussing Bolzano's position with respect to each of these foundational roles at length, let me introduce and discuss briefly two roles that Maddy regards as spurious or overly ambitious roles mathematicians and philosophers have at times attempted to ascribe to set theory.

The most obvious perhaps is wanting set theory to provide us with a guide as per what mathematical objects exist, and how they exist. This would see set theory in the role of a guiding us **Metaphysical Insight**, which Maddy argues was never in the cards for such a theory. Similarly, to treat set theory as an **Epistemic Source** for mathematics, that is, to regard set theory as the source of one's mathematical knowledge, is misguided.

I think that **Epistemic Source** may be contested as a foundational role also for Bolzano, depending on one's interpretation of the *WL*. If we accept the *WL* to be Bolzano's attempt at giving an account of how one should acquire scientific knowledge, and if we think that he is mostly successful, then **Epistemic Source** may hold. Engaging with such an epistemic reading goes beyond the scope of this chapter however, so I find myself unable to settle the question here.

Regarding **Metaphysical Insight**, we need to clarify further what Maddy means by that before we can assess whether it applies to Bolzano. It emerges from the context of Maddy's discussion of **Metaphysical Insight** that by that she does not mean just any knowledge of mathematical objects. She is not claiming that set theory does not provide us with any information whatsoever about the metaphysics of mathematics. Rather she is claiming that set theory systematically cannot provide the kind of insight that would allow us to decide whether the von Neumann ordinals are the only true ordinals, or whether the Kuratowski ordered pair is how ordered pairs are 'really' structured (Maddy 2017, p. 292) { and this is just as it should be. Nevertheless some philosophers do expect this much of set theory { albeit mistakenly so, according to Maddy. If this is the bar to clear for **Metaphysical Insight**, it is a consequence of the concept-*rst* view advanced in Section 2.2 that Bolzano's theory of collection cannot clear it. This is because according to the concept-*rst* interpretation, any object, including a mathematical one, cannot be said to be one thing or the other, one kind of collection or the other, without *rst* choosing under what concept we are considering said object. As a consequence, it is impossible within Bolzano's framework to have the kind of **Metaphysical Insight** that can settle the question of which specific individual objects are the ordinal numbers, say.¹⁸

Finally, I will also skip an extended investigation of **Meta-mathematical Corral** in relation to Bolzano's theory of collections, for a reason that is perhaps obvious: **Meta-mathematical Corral** is the kind of role that set theory can play in virtue of its (*rst*-order) axiomatisation, because the very notions that make such an axiomatisation useful, that is, consistency and Gödel-like encoding, are simply not applicable to Bolzano's theory.

Instead, in the remainder of this section I will closely read passages from Bolzano's programmatic and mathematical writings (*Beytrage*; *RAB*; *PU*) to argue for Bolzano's commitment to versions of **Shared Standard**, **Risk Assessment**, and **Elucidation**. I conclude this section with a discussion of what prevents Bolzano from even considering **Generous Arena** as a foundational ideal worth striving for.

2.4.2 Early Bolzano's foundational aspirations

My thesis in this section is the following: something like **Risk Assessment** and **Shared Standard** is what motivated Bolzano's work in the foundations of mathematics in the *rst* place. To defend this thesis I will present some quotes from Bolzano's early writings that witness his interest in shoring up something like **Shared Standard** and **Risk Assessment** for his foundations.

Bolzano's early programmatic piece about mathematics is *Contributions to*

18. Which is not the same as saying that Bolzano does not have a metaphysics of natural numbers { cf. Chapter 3.

a better founded presentation of mathematics (*Beyträge*), first published in 1810 (cf. Russ 2004, p. x). This essay goes over what the proper definition of mathematics is, and then over what the right presentation is of mathematics in print.

Regarding the definition of mathematics, Bolzano breaks with tradition by defining mathematics not as the science of quantity, but as the science of forms for all objects. He writes:

I therefore think that mathematics could best be defined as a science which deals with the general laws (forms) to which things must conform [*sich richten nach*] in their existence [*Dasein*]. By the word 'things' I understand here not merely those which possess an objective existence independent of our consciousness, but also those which simply exist in our imagination, either as individuals (i.e. intuitions), or simply as general concepts, in other words, everything which can in general be an object of our capacity for representation [*Vorstellungsvermögens*]. Furthermore, if I say that mathematics deals with the laws to which these things conform in their existence, this indicates that our science is concerned not with the proof of the existence of these things but only with the conditions of their possibility. In calling these laws general, I mean it to be understood that mathematics never deals with a single thing as an individual but always with whole genera [*Gattungen*]. These genera can of course sometimes be higher and sometimes lower, and the classification of mathematics into individual disciplines will be based on this. (*Beyträge* I §8, Russ 2004, p. 94)

If this makes mathematics sound a lot like metaphysics, that's on purpose. Bolzano (in the 1810s at least) sees mathematics as the right arm of metaphysics, concerned with the existence conditions of objects in general:

Mathematics and metaphysics, the two main parts of our a priori knowledge would, by this definition, be contrasted with each other so that the former would deal with the general conditions under which the existence of things is possible; the latter, on the other hand, would seek to prove a priori the reality of certain objects (such as the freedom of God and the immortality of the soul). Or, in other words, the former concerns itself with the question, how must things be made in order that they should be possible? (*Beyträge* I §9, Russ 2004, p. 94)

It is clear from the rest of the *Beyträge* that Bolzano's 'mathematics' here actually encompasses a broader range of subdisciplines than what his later works suggest (see the definition of mathematics and its subdisciplines in the *Grundlehren*, by way of comparison). In the *Beyträge*, mathematics { and more specifically, that area of mathematics that concerns itself with the conditions of existence that apply to anything possible, *mathesis universalis* { includes the treatment of the

notions of *ground* and *consequence* (which he later will instead present in his logic treatise, the *WL*):

This part of mathematics contains the theorems of ground and consequence [*Grund und Folge*], some of which also used to be presented in ontology, e.g. that similar grounds have similar consequences. (*Beyträge* I §13, Russ 2004, p. 97)

This means that, for the Bolzano of the *Beyträge*, it is part of the duties of the foundations of mathematics, this *Mathesis Universalis*, to establish results concerning the objective dependencies of consequence from ground between concepts. Moreover, all of mathematics should be presented in a way that does not hide these objective relations, lest we commit a mistake of kind-crossing, that is, lest we use results concerning a less general mathematical kind to prove a theorem concerning a more general mathematical kind. Famously, this is the shortcoming Bolzano sees in the recourse to geometric notions when proving the intermediate value theorem, and it is this very issue that spurs him to present his own proof of the theorem, which instead will be 'purely analytic', that is, free from the synthetic reasoning of geometry (cf. *RAB*, Preface, trans. in Russ 2004, pp. 253-261)

This detour is to say that, while Bolzano's theory of collections in and of itself may not serve the purpose of **Shared Standard**, the need for providing a **Shared Standard** of proof, and not just for mathematics but for all sciences worthy of the name, was very much one of Bolzano's concerns, and it seems to be the one that guides his investigation of the relation of ground and consequence. Moreover, in the *Beyträge* Bolzano sees mathematics as ordered almost like in a Porphyrian tree from the most general mathematical disciplines to the particular ones.¹⁹ *Mathesis Universalis* is, as the name suggests, the most general, and the theory of ground and consequence is part of Aetiology, which in turn also belongs to *Mathesis Universalis*. If we consider this *Mathesis Universalis* as Bolzano's first attempt at a semi-formal foundation for mathematics, we can say that **Shared Standard** is one of the roles fulfilled by his foundations, insofar as it is fulfilled by the theory of ground and consequence. As for **Risk Assessment**, Maddy's formulation of it, just like her formulation of **Meta-mathematical Corral**,²⁰

19. These points have already been made in the literature on Bolzano's ground/consequence relation and its relation to Bolzano's understanding of proper science, see e.g. (de Jong 2001; de Jong and Betti 2010; Betti 2010; Roski 2017).

20. For the reader who is wondering exactly where **Meta-mathematical Corral** and **Risk Assessment** differ, here is Maddy:

Notice that **Risk Assessment** in either form isn't the same as **Meta-mathematical Corral**: the point isn't to round up all classical mathematical items into one simple package, so as to prove something about all of it all at once, but to assess a particular new, somehow dangerous or suspicious item to determine just how risky it is. (Maddy 2017, p. 295)

seems to make it a non-starter for a theory of collections such as Bolzano's, because it also makes use of concepts that are only applicable to fully formalised theories. Strictly speaking, there is no concern on Bolzano's part that by adding new results to mathematics we might make it inconsistent, that is, that we might introduce a contradiction. I am inclined to interpret what Bolzano says about the role of mathematics as the science of all possible entities as the expression of a concern that one might want to add to one's metaphysics something that in fact cannot exist. In that sense, mathematics as a whole would then be performing the role of a **Risk Assessment** tool not for itself, but for the science of what there is as a whole. It can be surprising to realise that even someone as careful as Bolzano could not fathom the possibility of a future piece of mathematics undoing the internal consistency of the whole enterprise. But I think this does not take into account exactly how foundational endeavours such as Bolzano's are supposed to bootstrap their own consistency. If the starting point of mathematics is the most general concepts, and then we add theorems in a stratified fashion as we need to add differentiae to the the most general concepts to generate more specialised concepts, and if our proofs are genuine scientific proofs, that is, they respect the order of ground and consequence at each step in their inference, there is no way of producing a piece of mathematics whose consistency with the extant body of mathematical knowledge needs to be checked. In conclusion, even though we may be unable to ascribe the foundational roles of **Risk Assessment** and **Shared Standard** to Bolzano's theory of collections in particular, I believe there is a case to be made that a form of both function as constraints on what Bolzano attempts to achieve with his investigations in mathematical method and foundations, as witnessed by his early writings on the topic.

2.4.3 The definitional turn and Elucidation

The second period of Bolzano's mathematical writings starts in the 1830s and concludes only with the death of the man himself (Berg 1975, p. 9). This period sees him bring to completion the *WL* and *GL* (which is thought to have been written down already by 1835, cf. Berg 1975, p. 9), as well as the *PU*. These works have long been known for their foundational aspirations { Bolzano had meant for the *GL* to be a most precise rewriting of then-contemporary mathematics, presumably in accordance to the principles of good scientific treatises which he laid out in the *WL*. It also seems that he presented part of his work to the Bohemian Royal Society of the Sciences, and here is an excerpt from one of the summaries for these talks:

Mr Bolzano began to deliver 'an overview of the ideas in his system of pure mathematics' in a free-standing presentation; for this time however he only worked out the concept of mathematics (and while doing so he gave a more precise definition of the concept of quantity),

and then his view on its division [into subdisciplines].²¹ (Berg 1975, p. 13; my trans.)

And then, regarding the last of the three presentations:

Mr Bolzano continued the presentation, already begun in two previous sittings, of the ideas that he considers necessary for a system of mathematics that appeals to the strictest scientific standards, and this time he worked out the important concepts of finite and infinite, and the concept of a Reihe.²² (quoted in Berg 1975, pp. 13-14)

These summaries tell us two things: first, that Bolzano's contemporaries saw something of value in his 'more precise definition of the concept of quantity',²³ and second, that Bolzano himself considered his definitions of the finite/infinite dichotomy and of Reihe noteworthy enough to be presented in front of his intellectual peers at the time. Given therefore the pride of place conferred to these three notions, as well as their importance for the remaining chapters of this dissertation, I am going to examine how he defines each of them in turn. Clearly, I will only be using definitions that date from the 1830s onward.

Bolzano's new definition of quantity as given in the *GL* reads as follows:

§1. Definition. [...] [T]hings are quantities and moreover of a specific kind and quantities in the broadest sense of this word, whenever one of the following two relations must hold between any two M and N of said kind: either **a** that both are equal to one another, $M = N$, or **b** that one of the two, e.g. N , is constituted as a sum out of two summands, as $N = M^1 + n$, where $M^1 = M$, and n on the other hand is a thing precisely of this kind.²⁴ (*EB*, p. 220)

21. Hr. Bolzano begann in einem freien Vortrage 'eine Übersicht von dem Ideengange in seinem Systeme der reinen Mathematik' zu liefern; entwickelte aber für diesmal nur den Begriff der Mathematik (wobei er zugleich eine genauere Erklärung von dem Begriffe der Grösse gab), dann seine Ansicht von ihrer Eintheilung [...].

22. Hr. Bolzano setzte seine schon in zwei vorhergehenden Sitzungen begonnene Darstellung des Ideenganges fort, den er in einem die strengste Wissenschaftlichkeit ansprechenden Systeme der Mathematik für nöthig erachte, und entwickelte diesmal die wichtigen Begriffe des Endlichen und des Unendlichen, und den Begriff der Reihe.

23. We come back to Bolzano's definition in Chapter 4 and Chapter 5.

24. Translation mine. Here is the original German:

§. 1. Erklärung[.] [...] Dinge Gröen und zwar von einerlei Art und Gröen in der weitesten Bedeutung dieses Wortes sind, wenn zwischen je zweyen derselben M und N jedesmal eines von folgenden zwey Verhältnissen Statt finden muß. **a** entweder daß beyde einander gleich sind $M = N$; oder **b**, daß sich das Eine derselben z. B. N als eine Sum[m]e aus zwey Sum[m]en der $N = M^1 + n$ darstelle, denen die Eine $M^1 = M$, die andere n dagegen abermahl ein Ding von eben dieser Art ist.

The definitions one finds in the *WL* (§87) and the *PU* (§6) are small variations on the same (cf. also Section 2.2.1). Even though Bolzano does not define quantities as sums, it follows from all three definitions that something is a quantity only if it can be obtained as the sum of two other objects of the same kind. Bolzano seems to be saying as much, as he writes that 'Surely though I do require that a quantity which is composite, and is composite as quantity, may not be composed in any way other than as a Summe, that is, the way in which the parts are connected to one another is indifferent from the point of view of [it being a] quantity, and the parts of the parts can replace the parts themselves' (from *EB* §2, p. 220).²⁵ The fact that he concludes this remark on the importance of conceptualising a quantity that is not itself a unit as a sum, and not some other kind of composite object, by expressing hope that 'no one will object to this' ('Die wird man hoffentlich nicht mißbilligen', *ibid.*) suggests that Bolzano views this component of his definition as the most innovative and at the same time the one that makes the definition an improvement over other alternatives. Just as for the definition of quantity, the definition of *Reihe* (which we already discussed in Section 2.2.1) describes these as a specific kind of collection (*Inbegri*). What is essential to Bolzano's definition of *Reihen* is that there is an intrinsic rule of formation (*Bildungsgesetz*) which determines how to obtain one term of the *Reihe* from the preceding (or subsequent) one. From the remarks that follow the definition in the *GL* (*EGL* III §145, pp. 189-190) it seems that part of the target of this specification were definitions of continua (space and time) as *Reihen*, series, of points or instants. But this is false, Bolzano wants to say, because for any point in time or space it is impossible to determine an immediate predecessor or successor (because of density, we might add). Discussing the ramifications of this rejection would take us too far afield, but it is connected to Bolzano's rejection of appeals to spatio-temporal arguments in mathematics. In this sense then, Bolzano's definition of *Reihe* can be seen as part of his broader project of sanitising mathematics from conceptual confusions. I believe one could even argue that the definition of *Reihe* can constitute partial evidence of Bolzano's pursuit of something like **Elucidation**, but the cogency of the argument might depend on how much of an achievement one considers Bolzano's mathematical uses of *Reihen*, which I cannot get into here.

I will therefore now turn to what most clearly bears the hallmark of **Elucidation** in Bolzano's work: Bolzano's definition of infinity in the *PU*, his arguments for the need of a new, sharper definition (such as his), and his illustration of the mathematical advantages it unlocks. The simplest way to argue that Bolzano's is an **Elucidation** in Maddy's sense is to first show that Bolzano's elucidation

25. My translation. Original German: Wohl aber setze ich Voraus, daß eine Größe, die zusammengesetzt und zwar als Größe zusammengesetzt ist, auf keine andere als auf die Art einer Summe zusammengesetzt seyn dürfe, d.h. daß man die Art wie ihre Theile mit einander verbunden sind in Hinsicht auf die Größe als gleichgültig ansehe und die Theile der Theile statt ihrer selbst setzen könne.

of infinity fits the mould of Dedekind's set-theoretic **Elucidation** per Maddy's reconstruction, and then examine in detail how Bolzano's theory of collections resembles it.

Regarding Dedekind's 1874 construction of the continuum via cuts, Maddy writes:

[W]e have a vague picture of continuity that's served us well enough in many respects, well enough to generate and develop the calculus, but now isn't precise enough to do what it's being called upon to do: allow for rigorous proofs of the fundamental theorems. This isn't just a set-theoretic surrogate, designed to reflect the features of the pre-theoretic item; it's a set-theoretic improvement, a set-theoretic replacement of an imprecise notion with a precise one. So here's another foundational use of set theory: **Elucidation**. (Maddy 2017, p. 293)

Maddy's words suggest that **Elucidation** is called for whenever there is an important mathematical notion that neither can be given up nor can it continue to be used in its current understanding or formulation. Then the role of a foundational theory is to provide a new formulation that allows such a notion to continue to be useful. In Dedekind's case, the notion is that of the real numbers, and its use is being able to prove rigorously some fundamental theorems of analysis. In Bolzano's case, I want to argue, the notion is that of infinity, and its use is resolving some putative paradoxes in mathematics. Bolzano resolves to give a definition that properly separates the finite from the infinite, which we find in *PU* §9:

According to the different nature of the concept designated here by *A* there may sometimes be a greater and sometimes a smaller multitude [*Menge*, my addition] of objects which it comprehends, i.e. the units of the kind *A*. And therefore there is sometimes a greater and sometimes a smaller multitude of terms in the series [*Reihe*, my addition] being discussed. In particular there can even be so many of them that this series, to the extent that it is to exhaust all these units (taken in themselves), may have absolutely no last term. We shall prove this in more detail in what follows. Therefore assuming this for the time being I shall call a plurality [*Vielheit*, my addition] which is greater than every finite one, i.e. a plurality which has the property that every finite multitude represents only a part of it, an infinite plurality. (*PU* §9, translation in Russ 2004, p. 603)

In this definition Bolzano makes use of many of his collection notions: *Menge*, *Reihe*, *Vielheit*. This allows him to draw attention to the fact that the only kind of thing that can be infinite is something with several parts, a *Menge*, and more precisely one in which we treat each part as quantitatively interchangeable, so a

Vielheit (which by definition is just a Menge of units). Here we see the need to have more than one kind of collection for Bolzano: to each conceptual distinction he draws that is finer than simple or atomic versus composite he matches a collection notion, so that in the definition of infinity, for example, he is able to summon various necessary features for something to be called infinite just with one word, Vielheit. The choice of Vielheit specifically as the concept-kind that can be said to be infinite properly speaking implies that for something to be infinite, it has to allow for a conceptualisation such that:

1. It has parts (it is a collection);
2. It does not matter how the whole is structured from those parts;
3. In assessing how large the collection is as collection of a specific kind, each part counts as one object of that kind.

It is these three features, plus the fact that by definition each *Vielheit* of kind *A* is a quantity of kind *A* that allow for Bolzano's precise definition { nothing less would do. Bolzano argues for the superior clarity and usefulness of his definition initially by contrasting it with then-current definitions, to show their relative conceptual shortcomings (§§11, 12). Perhaps surprisingly, proponents of these definitions were sometimes mathematicians themselves. For example, according to Bolzano 'some mathematicians, among them even Cauchy (in his *Cours d'Analyse* and many other writings), and the author of the article 'Unendlich' in *Klugel's Wörterbuch*, have believed in infinity to be defined if they describe it as a variable quantity whose value increases *without bound* and which can be proved to become greater than *every given quantity however large*. The limit of this unbounded increase is the *ininitely large quantity*' (*PU* §12, Russ 2004, p. 604). The problem with this definition is that it fails to distinguish between the infinite range of values a variable can assume, and each single value being infinite. For this reason, Bolzano declares the definition as 'too wide': it counts as infinite even things which are not. By contrast, the definition given by Spinoza and others is 'too narrow', for it declares 'only that is infinite which is capable of no further increase, or to which nothing more can be attached (added)'. The problem is, '[t]he mathematician is allowed to add to every quantity, even infinitely large ones, other quantities, and not only infinite ones but even other quantities which are already infinite' (*PU* §12, p. 605). Then nothing counts as infinite according to this definition, and this is the proof of the inadequacy of such definition. Bolzano criticises a few other attempts at defining infinity, but the one he is the least impressed with is the 'qualitative infinity' of Hegel and his followers (§11). Bolzano rejects their claim that a qualitative infinity is 'a much higher one [than the infinity of mathematics], the true, the *qualitative infinity* which they find especially in *God* and generally only in the *absolute*.' (*PU* §11, Russ 2004, p. 603) Instead, if something like God can be said to be infinite at all, it is only to the extent that one of his attributes

may be attributed infinite magnitude. Bolzano's example is that of omniscience: to say that God has infinite knowledge simply means that God knows infinitely many propositions to be true, and this use of 'infinitely many' conforms to Bolzano's definition of infinity, no need to appeal to a sui generis infinitude.

Just as in the case of Dedekind and the continuum, then, Bolzano sees his definition as supplanting a scientifically inferior one. And regarding the applications of this improved definition, the first one of these can be found in *PU* §15, where Bolzano addresses the apparent paradox of how the natural numbers, which are all finite, can be infinite when taken collectively.

It might be said, 'If every number, as a concept, is a merely finite multitude, how can the multitude of all numbers be an infinite multitude? If we consider the series of natural numbers:

$$1;2;3;4;5;6;:::$$

then we notice that the multitude of numbers which this series contains, starting from the first (the unit) up to some other one, e.g. the number 6, is always represented by this latter one itself. Thus the multitude of all numbers must be as large as the last of them and thus itself be a number and therefore not infinite.' (*PU* §15)

Clearly, a definition like that attributed to Cauchy would be of no help, for what would be the quantity that admits of no bound in this context? Each natural number admits of a bound, namely, itself. Bolzano's definition of infinity, on the other hand, clearly allows for an infinite series of finite terms if the series has no last term. This is precisely the case for the series of the natural numbers, so the paradox has been defanged.

In short, Bolzano uses his theory of collections to give a more rigorous definition of infinity, when compared to pre-existing definitions; he argues by comparison for its rigour; and he illustrates its usefulness by using it to resolve apparent mathematical paradoxes involving the notion. Thus, when it comes to infinity, Bolzano's use of his theory of collection follows closely the template Maddy lays down for **Elucidation**.

2.4.4 Generous Arena

To sum up the work of this section so far: If we look at Bolzano's programmatic writings from the early years up to his death, it is undeniable that there was a strong foundational component to his investigations into mathematics and logic (the latter broadly construed). Moreover, if we agree with Maddy's characterisation of the ways in which set theory is a foundation, then there are meaningful overlaps with the ways in which Bolzano's theory of collection can be said to be a foundation. There are also ways in which set theory can be a foundation but Bolzano's theory

cannot, and I want to conclude this comparison between Bolzano's theory and set theory by highlighting the starkest difference between the two, namely that something like a **Generous Arena** for mathematics is both impossible and undesirable in Bolzano's framework. I want to argue that there are two reasons why Bolzano's theory of collections does not provide a **Generous Arena**. The first one is that Bolzano conceives of the conceptual organisation of mathematics as much more hierarchical than Maddy does, and this makes a **Generous Arena** unwanted. To wit, here is Maddy's description of contemporary mathematics which aims at motivating the need for a **Generous Arena**:

[T]he branches of modern mathematics are intricately and productively intertwined, from coordinate geometry, to analytic number theory, to algebraic geometry, to topology, to modern descriptive set theory (a confluence of point-set topology and recursion theory), to the kind of far-fung interconnections recently revealed in the proof of Fermat's Last Theorem. What's needed is a single arena where all the various structures studied in all the various branches can co-exist side-by-side, where their interrelations can be studied, shared fundamentals isolated and exploited, effective methods exported and imported from one to another, and so on. (Maddy 2017, p. 297)

Incidentally, Maddy herself refers back to Burgess's formulation of the interactions between different 'branches' of mathematics, which emphasises that 'the starting points of the branches being connected should . . . be compatible The only obvious way to ensure compatibility of the starting points . . . is ultimately to derive all branches from a common, unified starting point' (Burgess 2015b, pp. 60-62, as quoted in Maddy 2017, p. 297). Despite what Maddy and Burgess write, however, their picture is reminiscent of a bush more than a tree, because for them the interactions between different areas of mathematics do not have to follow necessarily one direction (from the most foundational to the least, but not vice versa) to be rigorous. For Bolzano,²⁶ by contrast, and especially the early

26. To the best of my knowledge, Bolzano does not explicitly invoke the no-kind-crossing rule in the *GL* or in the *PU*, however in both texts he is careful to follow a specific order of exposition that goes from the most general kind of quantity to the various specific kinds. Here is how he explains the notion of generality at stake (*EB* §2, pp. 38-39, my trans.):

The denomination of 'general' [in 'general science of quantity'] means that in this science we concern ourselves only with those kinds of quantities that are not *subordinated* [my emphasis] to any specific kind of quantity in the specialised theories of quantity such as geometry, mechanic etc., but rather appear in several of these sciences alike.[. . .] Thus the generality that pertains to the doctrine of the general theory of quantity is to be taken only relatively to those specific kinds of quantities on the basis of which the division of the specialised theories of quantities takes place. The essential difference though through which the general theory of quantities distinguishes itself from each applied one from a conceptual standpoint is merely that in the former quantities are studied only *in abstracto* (from the

Bolzano (see Section 2.4.2), mathematical results can translate only from a more to a less general area of mathematics { say, from the general theory of quantity to geometry, since geometry is the theory of extended (or spatial) quantity, but not the other way round. This is the famed kind-crossing prohibition that we already mentioned and it is one reason why I am sceptical that Bolzano would want a foundation of mathematics that can play the role of a **Generous Arena** the way Maddy explains it.

The second reason why Bolzano's framework cannot provide a **Generous Arena** is that it cannot accommodate a reduction of the kind that Maddy herself sees as necessary for **Generous Arena** (Maddy 2017, p. 197, footnote 16). Unlike what happens in set theory, in Bolzano's theory of collections the structure of a structured collection cannot be understood as something that is superimposed on a carrier the way that a group is just the result of superimposing the group-structure on to a set. As an example of Bolzano's understanding of structure, let us examine again Bolzano's clarification of what the structure of a Reihe is. Bolzano appeals to the example of a Reihe of propositions in themselves: 'a is b, b is c, c is d' form a Reihe, although, as propositions in themselves, they do not need to stand in a certain connection [*gar nicht in einer gewissen Verbindung zu stehen brauchen*], and their signs can be placed next to each other in whatever order [*in was immer für eine Ordnung*]. To me, then, the essence of a Reihe seems to consist merely in the relationship [*Verhältnis*] that must prevail between the terms of it.' (EGL III §146, p. 191) This passage draws a distinction between the intrinsic relationship between the terms of the Reihe, on the one hand, and the order in which the propositions are written. The use of 'essence' in this passage is technical: what Bolzano is saying is that what makes 'a is b, b is c, c is d' a Reihe is not that 'a is b' is written right before 'b is c', and so on, but that 'b is c' can be derived from 'a is b' according to a specific rule of formation. There is a conceptual structure linking the terms of the Reihe to one another, and this is independent of the manner of presentation of the Reihe. Interestingly, the order of a Reihe (where 'order' here is the same 'order' that is indifferent to the identity of a Menge) is viewed as part of this mutable manner of presentation of the Reihe, rather than as equivalent to the essential structure of the Reihe. At most, the relationship between the terms is what induces a canonical order, but the order itself is secondary. This is openly at odds with a contemporary understanding of order in a sequence, for example. If we let p : 'a is b', q : 'b is c', r : 'c is d' then a set theoretic representation of the sequence $pp; q; rq$ would be $R_1 = \{p1; pq; p2; qq; p3; rqu\}$. But if we were to change the order of p , q and r in the sequence, we would be changing the sequence into, say, $R_2 = \{p1; qq; p2; rq; p3; pqu\}$, and R_1 and R_2 are different sets. This is

point of view of their being quantifiable), in the latter instead they are always studied in connection with certain other properties, which have nothing to do with their being quantifiable. Hence the denomination 'pure theory of quantity' is more apt in principle.

possible because there is no way of capturing Bolzano's notion of a relationship between the terms of a sequence (Reihe) that is not just the position they occupy within the sequence. It should also be clear by now that the relationship between terms Bolzano talks about isn't quite the same as the structure of a structured set, either. For in the case of a structured set, say, the natural numbers \mathbb{N} , the structure is given by the tuple $\langle \mathbb{N}; \cdot; +; 0; 1 \rangle$, where each element of the tuple is a set and the tuple itself is also a set. This kind of construction also does not capture the way in which the propositions of Bolzano's example are related to one another. In other words, Bolzano does not achieve the extensionalisation of structure for mathematical objects, that is, he does not come to conceive of the structure of a structured collection (such as a Reihe) as a tuple of the relations defined on the domain of the object in question. For Bolzano, the structure of a structured collection remains a conceptual relation, and this constitutes a hard limit on the kind of reduction to collections one can perform within his theory of collections.

Bolzano's failure to see the possibility of presenting relations just as sets of tuples is all the more interesting, given how close he seems to come to it. In the *GL*, he exemplifies what a collection is by describing the relation obtaining between three objects (a theorem and two people who proved it) and declaring the three objects a collection, precisely in virtue of the relation that binds them:

Thus for example we can quite easily say that the famous theorem about the square on the hypotenuse stands in the relation to Pythagoras and Heinrich Boad: the one discovered it the other however provided the most straightforward proof. The notion of a relation between certain things is however nothing other than the notion of a property which we do not attribute to the one of them alone, but only to their union, to the whole. And we could not truthfully ascribe this property [*Eigenschaft*] to the whole if it did not possess it, even without our thinking of it. So must this whole also exist without our own conception of it, so that also between things as heterogenous as truth in itself, which has no existence [i.e. it is not spatiotemporally located], and two men, of which one lived a couple thousand years after the other, there occurs a conjoining in the sense given here; they form a whole to which certain unique qualities belong objectively, that is, regardless of whether anyone conceives of it.²⁷ (*EGL* III §6, p. 101, my trans.)

27. So können wir z.B. recht füglich sagen, daß der bekannte Lehrsatz vom Quadrate der Hypothenuse zu Pythagoras und zu Heinrich Boad in dem Verhältnisse stehe, daß jener ihn entdeckt, dieser aber den sinnfälligsten Beweis für ihn ersonnen. Der Gedanke eines Verhältnisses zwischen gewissen Dingen ist aber kein anderer als der Gedanke einer Beschaffenheit, welche wir nicht dem Einen derselben allein, sondern nur ihrer Vereinigung, dem Ganzen beilegen. Und diese Eigenschaft könnten wir dem Ganzen nicht mit Wahrheit beilegen, wenn es sie nicht besäße, auch ohne daß wir sie uns denken. Somit muß dieses Ganze auch bestehen, ohne daß wir es uns vorstellen, also auch zwischen so ungleichartigen Dingen, wie eine Wahrheit an sich, welche

What seems to be the natural further step, of taking the ternary relation of 'X and Y have proved theorem Z' as determining the collection of all the triples $\langle X; Y; Z \rangle$ that can satisfy the relation, is not taken by Bolzano. It is almost as if Bolzano failed to see how to properly generalise the operation 'collection of' beyond the concrete case of the three relata he is interested in. This is the deepest difference between Bolzano's theory of collections and modern set theory: the way structures are handled and put to fruition.

I have presented Bolzano's ordering of mathematical disciplines and anti-extensional understanding of structure as two distinct reasons why his theory of collections cannot play the role of a **Generous Arena** for mathematics in Maddy's sense. However, I would like to suggest that the two are related. I already mentioned that what allows **Generous Arena** is the set-theoretic reduction that underlies many of the foundational roles set theory plays. I also already mentioned that Bolzano's theory of collections cannot accommodate this reduction because of the way it understands structure. One consequence of this is that Bolzano cannot see a quantity in geometry, say, and one in algebra as 'essentially the same', and this is what enforces his mathematical hierarchy. By contrast, modern mathematics enjoys an interconnectedness which becomes fruitful precisely thanks to set theoretic representations, which allow us to recognise a structure of a certain kind within the context of a different structure, and exploit the interaction between the two. A typical example of this is the fundamental group of a topological space. Both a topological space and its fundamental group can be given a set theoretic representation, and this allows to exploit the group-theoretic properties of the fundamental group to gather information about the space *as a topological object*. But if the topological structure of the space and the group-theoretic structure of the fundamental group are seen as inseparable from the carrier sets, the way a Bolzanian understanding of structure seems to require, then it becomes impossible to exploit the properties of one type of structure to explore the other.

2.5 Conclusion

This chapter covers a lot of ground: Bolzano's theory of collections as a whole, its place relative to set theory and mereology as a metaphysics of collections, and its foundational role for mathematics according to Bolzano. I chose to focus on some issues, most notably, extensionality, that I thought would benefit from a more direct attention than what they have received so far in the literature. I also tried to show that how we answer the metaphysical question of what Bolzano's collections really are has some bearings on the foundational uses Bolzano's theory of collection

nichts Existirendes ist, und zwischen zwei Menschen, deren der Eine ein paar Jahrtausende später als der Andere gelebt, bestehet ein Zusammen in der hier angegebenen Bedeutung; sie bilden ein Ganzes, welchem gewisse eigenthümliche Beschaffenheiten objectiv zukommen[,] d.h. auch abgesehen davon, ob irgend Jemand sich dieselben vorstelle.

can fulfil $I \mid$ and these foundational uses are dictated not by metaphysical needs but by mathematical ones. Conversely, I have tried to show that if we want our investigations into the metaphysics of Bolzano's collections to bear on how we interpret Bolzano's mathematics, we need to pay attention to specific points of contact between collections and sets.

To be clear, I take what has been done here to be merely preliminary work to a more systematic investigation that puts less emphasis on the metaphysical comparison and more on the functional comparison between sets and collections not just in the work of Bolzano, but that of all those authors who are considered, to a greater or less extent, as the forefathers of set theory. For now, however, I hope to have convinced the reader that, yes, we can ask what Bolzano's collections are, *really*, and we can try to give identity criteria and wonder whether they are the same criteria as for mereological wholes or sets, but that kind of investigation cannot be carried out independently of a close inspection of how Bolzano's collections are supposed to work in his mathematics.

Chapter 3

Natural Number Concepts¹

3.1 Introduction

Bolzano's writings on infinite collections of natural numbers span about two decades, from the *WL* completed around 1830 (published in 1837), to the *PU*, finished in 1848 and published posthumously in 1851. His observations about infinite collections of natural numbers have enjoyed much popularity when compared to the rest of his contributions to philosophy, mostly because they seem to both anticipate and contradict Cantor's approach to measuring the size of infinite collections. The very comparison with Cantor's work, however, is made possible by neglecting the role which Bolzano's theory of ideas plays in his determination of the relative sizes of infinite collections of natural numbers. This omission unfortunately deprives us of an accurate account of how Bolzano's views on infinite collections of natural numbers changed over time.

In this chapter, we argue for an understanding of Bolzano's concept of natural number that allows us to account for the evolution of his views on the relative sizes of collections of natural numbers between the *WL* and the *PU*, all the while clarifying certain aspects of his writings on the topic. The chapter begins with an overview of the dilemma raised by Galileo's Paradox when it comes to choosing a criterion for comparing the size of infinite collections of natural numbers, and of how secondary literature rephrases Bolzano's contributions within that context. We stress that if one views Bolzano's writings on different infinite sizes only through the prism of Galileo's Paradox one is led to misunderstand the change of mind evidenced by a letter he wrote in 1848 to one of his former students, thus making Bolzano's position more obscure than it actually is. We then exploit Krickel's interpretation of Bolzano's theory of ideas (Krickel 1995) to argue that what determines the size relations between collections of natural numbers in his view is the relation of subordination between the concepts { a

1. This chapter is based on a paper currently under review (Bellomo and Ginammi 2021).

particular kind of ideas { of which said collections are the extensions, and we show how Bolzano's definition of natural numbers supports this interpretation. We conclude by showing that our interpretation explains Bolzano's later writings on infinite collections of natural numbers while preserving the continuity between them.

3.2 A misleading dilemma

3.2.1 Bolzano and Galileo's Paradox

Broadly speaking, when comparing two collections one can decide to follow either of the following principles:

one-one Two collections are equal in size if and only if there is a one-to-one correspondence between their elements.

part-whole If a collection B is a proper part of another collection A , then the size of B is strictly smaller than the size of A .

While **one-one** and **part-whole** are compatible in the finite case, this is no longer true in the infinite case. Take for example the collection of all natural numbers and then compare it to that of all squared natural numbers. If one applies **one-one** to the collection of natural numbers and that of their squares, since one can pair off each number with its square and each square with its (positive) root, then it appears that the two collections have equal size. If on the other hand one applies **part-whole**, one finds that the collection of *all* natural numbers surely contains all squares, plus some (infinitely many) other numbers. It then has to be greater. The equally strong intuitive pull of both answers to the question of whether there are strictly more natural numbers than squared natural numbers is famously noted in *Two New Sciences* (Galileo 1958), and in fact the whole puzzle is sometimes referred to as Galileo's Paradox (see Mancosu 2009, p. 621 and Parker 2008 where the expression is in the title). Mancosu and Parker use Galileo's paradox to rephrase the problem of how to compare the sizes of infinite countable collections as a dilemma: either one chooses to compare sizes according to **one-one**, like Cantor did, or one chooses to compare sizes according to **part-whole**. Within this context then Bolzano is mentioned as someone who, unlike Cantor, chose to compare sizes while preserving **part-whole**. It is relatively well known among Bolzano readers however that a letter he wrote in 1848, so only a few months before his death, seems to complicate this picture of Bolzano as adhering to **part-whole**. It is now to this letter, and to the passage from *WL* the letter objects to, that we turn our attention.

3.2.2 Bolzano's texts

The problematic passage from the letter to his former student Robert Zimmermann is the following (*BBGA* 2A/12.2, pp. 187-8, our translation):

Wissenschaftslehre vol. I, p. 473. The matter is not only obscurely presented, but also, as I just began to recognize, quite wrong. If one designates by n the concept of *every arbitrary whole number*, or to say it better, if by n every arbitrary whole number would be represented, then with this it is already decided *which* (in *nite*) set of objects this sign represents. This will not change the least, if we by means of addition of an exponent like n^2 , n^4 , n^8 , n^{16} , ... require that each of these numbers now must be raised to the second, now to the fourth, ... power. The *set* of these objects which is represented by n is still exactly the same as before, although the *objects themselves*, which are represented by n^2 are not the same as those represented by n . The wrong result was due to an unwarranted inference from a *nite* set of numbers, namely those not exceeding the number N , *to all of them*."²

It is clear from this passage that Bolzano is now convinced that somewhere in the *WL* he made a mistake, and it taints the argument allowing him to conclude that in the nested sequence of concepts $n; n^2; n^4; \dots$: every successor of the concept n is in *nite*ly many times smaller than n itself. To specify what portion of the argument does not convince Bolzano anymore, it is worth looking at the argument in full:

WL §102 [...] Let us abbreviate the concept of any arbitrary integer by the letter n . Then the numbers $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$ express concepts each of which includes in *nite*ly many objects (namely, in *nite*ly many numbers). Furthermore, it is clear that any object that stands under one of the concepts following n , e.g., n^{16} , also stands under the predecessor of that concept, n^8 . It is also clear that very many

2. Here is the original:

Wissenschaftslehre Bd I S 473. Die Sache is nicht nur unklar vorgetragen, sondern wie ich soeben zu erkennen anfangte, ganz falsch. Bezeichnet man durch n den Begriff *jeder beliebigen ganzen Zahl*, oder was besser gesagt wäre, soll durch das Zeichen n jede beliebige ganze Zahl vorgestellt werden, so ist damit schon entschieden, *welche* (unendliche) Menge von Gegenständen dieses Zeichen vorstelle. An dieser ändert sich nicht das geringste dadurch daß wir durch Zusatz eines Exponenten wie n^2 , n^4 , n^8 , n^{16} ... verlangen, daß jeder dieser Zahlen jetzt auf zweite, jetzt auf die vierte ... Potenz erhoben werden soll. Die *Menge* dieser Gegenstände welche das n vorstellt, ist genau immer noch dieselbe wie vorhin, obgleich die *Gegenstände selbst*, die n^2 vorstellt, nicht eben die nemlichen sind, welche n vorstellt. Das falsche Ergebnis wurde nur durch den unberechtigten Schluß von einer *endlichen* Menge Zahlen, nemlich der die Zahl N nicht übersteigenden, *auf alle* herbeigeführt.

objects that stand under the preceding (n^8) do not stand under the following (n^{16}). Thus of the concepts $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$, each is subordinated to its predecessor. It is, furthermore, undeniable that the width of any of these concepts is *in nitely larger* than the width of the concept immediately following it. (And this holds even more for concepts that follow later in the sequence.) For, if we assume that the largest of all numbers to which we want to extend our computation is N , then the largest number that can be represented by the concept n^{16} is N and thus the number of objects that it includes equals or is smaller than $N^{\frac{1}{16}}$ and likewise the number of objects that stand under the concept n^8 equals or is smaller than $N^{\frac{1}{8}}$. Hence the relation between the width of the concept n^8 and that of the concept n^{16} is $N^{\frac{1}{8}} : N^{\frac{1}{16}} = N^{\frac{1}{16}} : 1$. Since $N^{\frac{1}{16}}$ can become larger than any given quantity, if N is large enough, and since we can take N as large as we please, and since we can come closer and closer to the true relation between the widths of the concepts n^8 and n^{16} , the larger we take N , it follows that the width of the concept n^8 surpasses in nitely many times that of the concept n^{16} . Since the sequence $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$ can be continued indefinitely, this sequence itself gives us an example of an infinite sequence of concepts each of which is of infinitely greater width than the following.³

This passage comes from a section of the *WL* that aims to establish that there is no finite set of units to measure and compare the sizes of the extensions of infinite concepts (i.e. concepts under which infinitely many objects stand). The

3. Original German: Wenn wir zur Abkürzung den Begriff jeder beliebigen ganzen Zahl durch den Buchstaben n bezeichnen: so drücken die Zahlen $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$ Begriffe aus, deren jeder ohne Zweifel unendlich viele Gegenstände (nämlich unendlich viele Zahlen) umfaßt. Eben so offenbar ist ferner, daß jeder Gegenstand, der unter einem der auf n folgenden Begriffe, z. B. n^{16} steht, auch unter dem nächstvorhergehenden n^8 stehe, daß aber umgekehrt sehr viele Gegenstände, die unter dem vorhergehenden Begriffe n^8 stehen, nicht unter dem folgenden n^{16} enthalten sind. Von den Begriffen $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$ ist also jeder folgende immer den vorhergehenden untergeordnet. Eben so unläugbar ist ferner auch, daß die Weite jedes von diesen Begriffen die Weite des nächstfolgenden (um so mehr die eines späteren) unendliche Male übertrifft. Denn setzen wir, daß die größte aller Zahlen, bis zu der wir unsere Berechnungen ausdehnen wollten, N wäre, so wäre die größte Zahl, die der Begriff n^{16} vorstellen kann, N , und folglich die Zahl der Gegenstände, die er umfaßt, = oder $N^{\frac{1}{16}}$. Und ebenso die Zahl der Gegenstände, die der Begriff n^8 umfaßt, = oder $N^{\frac{1}{8}}$. Das Verhältnis der Weite des Begriffes n^8 zu jener des Begriffes n^{16} wäre daher $N^{\frac{1}{8}} : N^{\frac{1}{16}} = N^{\frac{1}{16}} : 1$. Da aber $N^{\frac{1}{16}}$ größer als jede gegebene Größe zu werden vermag, wenn man N groß genug annehmen darf; und da wir N so groß annehmen dürfen, als wir nur immer wollen; ja da wir dem wahren Verhältnisse, das zwischen den Weiten der Begriffe n^8 und n^{16} obwaltet, nur näher kommen, je größer wir N nehmen: so folgt, daß die Weite des Begriffes n^8 jene des Begriffes n^{16} unendliche Male übertrifft. Da nun die Reihe $n; n^2; n^4; n^8; n^{16}; n^{32}; \dots$ sich so weit fortsetzen läßt, als man nur immer will: so haben wir an ihr selbst ein Beispiel einer unendlichen Reihe von Begriffen, deren ein jeder unendliche Male weiter als der nächstfolgende ist.

sequence of concepts $n; n^2; n^4; \dots$ is used to provide a witness for said statement: here is a sequence of concepts, each with an infinite extension and each with a strictly smaller extension than the one immediately preceding it, and such that each predecessor concept has an infinite extension that is infinitely greater than the extension of the concept which immediately succeeds it. There are potentially three aspects of the quoted passage that Bolzano might not endorse anymore in 1848: the conclusion the argument establishes; the principle **part-whole** that seems to underpin the argument; or finally, some step or other of the argument, but not the conclusion by itself.

The interpretation of the letter as a rejection of **part-whole** in favour of **one-one** is an important one, given that it is enshrined in the editorial footnotes of the one and only publication of said letter, and it is due to Jan Berg, a very influential Bolzano scholar. In an editorial footnote to the letter Berg comments that (*BBGA 2A/12.2*, p. 188, n. 451):⁴

It seems as if in the end Bolzano restricted the proposition that the whole is greater than any of its parts to the finite case and came to recognise isomorphism as a sufficient condition for the equality of power of infinite sets.

And in *Bolzano's Logic*, Berg writes:

In the *PU*, and implicitly in the *WL* too, Bolzano repudiates the notion of equivalence as a sufficient condition for the identity of powers of infinite sets. [...] it seems that at the last Bolzano confined the doctrine that the whole is greater than its parts to the finite case and accepted equivalence as a sufficient condition for the identity of powers of infinite sets. (Berg 1962, pp. 176-177)

In other words, according to Berg's interpretation, in the *WL* and *PU* Bolzano based his notion of size on the part-whole principle, but in the letter he argues that it should be based on one-to-one correspondence instead.

As Mancosu 2009 wryly notes, Berg was arguing for Bolzano's glory as presciently anticipating **one-one** and it is this that makes him read too much into the letter. Mancosu (2009) as well as Rusnock (2000, pp. 194-196) and Parker (2008, p. 94) doubt that what Bolzano is objecting to in the letter is indeed **part-whole**, but they all refrain from offering a competing explanation of what changes between *WL* and the letter. Van Wierst et al. (2018) add that the object of Bolzano's rebuttal is not so much the conclusion of the argument from *WL* §102, as the argument itself. To date, no interpretation exists that both analyses the letter faithfully and accounts for a substantive change between *WL* §102 and the letter when it comes to Bolzano's views on infinite collections. We think that what

4. Berg seems to use 'isomorphism' and 'equivalence' in these quotes as meaning one-to-one correspondence.

stands in the way of actually doing so is the very framing of Bolzano's writings as having to belong to one of the two horns of what we call Galileo's dilemma. Let us clarify how.

To frame Bolzano's contributions on the infinite as belonging to either horn of Galileo's dilemma, one has to assume that his infinite collections can be regarded as *sets* (in the sense of modern set theory) without interpretive costs. While it is now widely acknowledged that such a move leads to an interpretive overreach in the case of Berg, many of Berg's critics themselves paraphrase Bolzano's claims as claims about sets and their sizes. For example, Mancosu 2016 writes:

(Bolzano 1837, 1973, 2014) offers an example constructed by a nested sequence of infinite countable sets and states that each one of the sets in the sequence is infinitely smaller than the preceding one. (Mancosu 2016, p. 130)

And a few pages later:

While the criterion for equality of 'pluralities' [*Vielheiten*, our note] is left vague in the infinite case, there is no doubt that at the time of *WL* Bolzano thought that many infinite countable sets have different 'pluralities' (he called them 'Weite' in §102 of the *Wissenschaftslehre* (Mancosu 2016, p. 168)⁵

At the very least, then, Mancosu takes Bolzano's pluralities and other kinds of collections to be interpretable as sets, when one's goals are not of exegesis of Bolzano alone, but of comparison among Bolzano and others (as is the case for Mancosu). Similarly in (Rusnock 2000, pp. 195-196) one reads:

[referring to the letter to Zimmermann] [...] Thus the sets designated by $n; n^2; n^4; n^8; n^{16}; \dots$ may be said to be equinumerous. This is certainly a change in Bolzano's opinion; but not, I think, in the direction suggested by Berg. For Bolzano makes no mention here of the circumstance that the sets $n; n^2; n^4; n^8; n^{16}; \dots$ can be mapped one to one onto each other. Rather, his point seems to be that considered *as sets*, $n; n^2; n^4; n^8; n^{16}; \dots$ are identical, in that they only differ by

5. The following shows that Mancosu is perhaps more sensitive to Bolzano's conceptual architecture than these isolated quotes would otherwise suggest (Mancosu 2016, pp. 168-169):

While in the *WL* he was obviously ready to accept differences in *Vielheiten* between what we would now call infinite-countable sets, in his later works the situation is crystal clear only for uncountable sets such as intervals of points on the real line but it is less clear for countable sets. Regardless, Bolzano did not accept one-to-one correlation as a sufficient criterion for equality of 'plurality' and wanted to preserve the part-whole principle at least for some classes of infinite sets.

uniform transformations of their members: all the sets, in other words, are generated in the same way.⁶

And finally, one reads in (Parker 2008):

Bolzano boldly claimed that infinite sets differed in numerosity [Parker's word for 'number of elements'], and that transfinite numerosity did not satisfy both Euclid's Principle and Hume's [Hume's Principle is what we call **one-one**] (Parker 2008, p. 94)

This is not an exhaustive list of interpretations of Bolzano's writings as being about *sets* (in the modern sense), but it suffices as an illustration of how commonplace they are, even among critics of Berg's interpretation of the letter.

Allow us then to explicitly mention what we take to be the two fundamental { and often implicit { steps of any set-theoretic interpretation of Bolzano's writings on infinity. The first step is to tacitly suppose that, if Bolzano is comfortable saying that the extension of the concept n of all natural numbers is (in finitely) greater than the extension of the concept of all natural numbers squared, he must be endorsing the view that the collection of all natural numbers is in finitely greater than the collection of all squares. The second step is to assume that, based on Bolzano's commitment to the Euclidean principle that the whole is greater than any of its proper parts, his handling of infinite collections of natural numbers is governed by the principle:

PW_{set}: If $B \hat{=} A$ then $sizepBq \leq sizepAq$

where A and B { since mathematical collections according to the interpretations in question are just sets { stand for sets of natural numbers. This move from Bolzano's talk of extensions to talk of collections (step 1) that are then interpreted as sets (step 2) is precisely what we detect in all the interpretations we encountered (see paragraphs above). We should emphasise that such implicit assumptions { which come down to the assumption that modern mathematical tools can unproblematically be used to formalise (philosophy of) mathematics that predates them { are rather commonplace and in many cases justified by the ends of the enquiry at hand. It is however our contention that, in the specific instance of accurately portraying Bolzano's evolving thought on infinity, it is misleading to interpret his talk of collections as talk of sets (we will come back to this point in Section 3.3.1).

In sum, we believe that interpretations of Bolzano's writings on the infinite generally try to home Bolzano's position within one of the horns of Galileo's

6. It is true that Rusnock distinguishes between the sets of set theory, and Bolzano's *Mengen*, which he nevertheless opts to translate as 'sets'. However this passage we quote only makes sense if we read Rusnock's use of 'sets' as systematically ambiguous between Bolzano's sets and the sets of set theory.

dilemma; these interpretations perform both step 1 and step 2 above and are in that sense set-theoretic. Such set-theoretic interpretations are however problematic. For, to start, in one of our key texts (*WL* §102) Bolzano never even mentions the collection (his term is *Inbegri*) { let alone the set (*Menge*) { of all natural numbers (or of all squares, or fourth, eighth, . . . powers); rather, he phrases his argument in terms of concepts (*Begri e*) and their objects (*Gegenstände*). Further, Bolzano bases his argument in that text on the relation of subordination (*Unterordnung*) between the concepts which constitute the sequence he considers. The principle he abides by there is not \mathbf{PW}_{set} but rather a part-whole principle in terms of the extensions of ideas (here concepts), which we will discuss in the next section.

3.3 Ideas and the notion of size

In this section, we appeal to the same interpretation of Bolzano's theory of collections as from Chapter 2. This will form the basis of our alternative understanding of the part-whole reasoning present in Bolzano's determination of the relative sizes of collections of natural numbers. As already mentioned in the previous chapter, our interpretation is based on Krickel's (1995) interpretation of Bolzano's theory of collections. On the basis of our interpretation, we show in Section 3.3.2 how this allows us to reframe Bolzano's use of part-whole reasoning in comparing sizes of different infinite collections of natural numbers.

3.3.1 A relative interpretation of Bolzano's collections

There are two key aspects of Krickel's (1995) interpretation of Bolzano's theory of collections which are relevant for our purposes.⁷ First: Bolzano's theory of collections is primarily a theory of ideas (*Vorstellungen*) of collections, and only derivatively a theory of collections themselves;⁸ an adequate analysis of Bolzano's theory of collections should thus start from his theory of ideas, rather than from the collections themselves (Krickel 1995, p. 105). Second: Bolzano's theory of collections is relative (rather than absolute), which means that it takes a collection to have properties { such as what counts as a part of it,⁹ and whether or not there is an order between the parts { only with respect to some idea which represents it (Krickel 1995, pp. 64, 122-3, 126). We will discuss both aspects here in turn, together with their bearing on Bolzano's views on the sizes of collections of natural numbers.

7. We do not embrace Krickel's interpretation in all detail, but merely work with these two key aspects as hypotheses.

8. Cf. **Idea** from Chapter 2.

9. This claim is similar to **Part** from Chapter 2, but in Chapter 2 we distinguish between relative interpretations and our **Part** so as to make the comparison with Simons and Behboud easier to express.

We start with the first relevant aspect of Krickel's interpretation: Bolzano's theory of collections is primarily a theory of ideas of collections, and only derivatively a theory of collections themselves. Ideas, in the logic which Bolzano develops in the *WL*, are abstract, mind-independent entities { which is why Bolzano calls them also ideas 'in themselves' (*an sich*) { which can have (*haben*) or represent (*vorstellen*) objects, or, which means the same, can have objects standing under (*stehen unter*) them (*WL* §§49, 66). They ultimately make up propositions (*Sätze*) { also called propositions 'in themselves' {, the mind-independent abstract entities which are either true or false (*WL* §48). For every object in Bolzano's universe there is in his view an idea which represents exactly it (*WL* §101.2, cf. Krickel 1995, p. 46). Krickel's (1995) thesis that Bolzano's theory of collections is primarily a theory of ideas of collections, and only derivatively a theory of collections themselves, means that instead of developing a theory about collections directly, Bolzano develops a theory about ideas which have collections as objects.¹⁰

Indicative of the view that Bolzano's theory of collections is primarily a theory of ideas which have collections as objects, is, as Krickel (1995) points out, that Bolzano answers the question of the existence¹¹ conditions for collections by means of answering the question of which ideas of collections have objects (Krickel 1995, p. 78). Not all ideas have objects or are, as he calls it, objectual (*gegenständlich*) in Bolzano's view: some of them are objectless (*gegenstandslos*; *WL* §67). The question of which ideas of collections are objectual is again a question which Bolzano answers at the level of ideas, as Krickel (1995) explains: to answer this question Bolzano investigates the conditions which follow from the concept of collection (Krickel 1995, p. 78).¹² Krickel does not explain what he takes such conditions following from the concept of collection to be for Bolzano, but we propose to understand it as follows.

The relationship between an idea and its objects remains unde ned in the *WL*. Bolzano simply writes that the object of an idea is 'that (existing or not existing) something, of which we say that it is *represented* by it, or that it is the idea *of it*' (*WL* §49, cf. Krickel 1995, p. 45). Nonetheless, Bolzano holds that ideas have

10. Approaching issues which are not restricted to logic or linguistics in terms of the theory of ideas is not unusual for Bolzano. For example, he addresses the issue of necessary existence also by means of his theory of ideas (see e.g. Roski and Rusnock 2014, cf. Krickel 1995, p. 17).

11. Strictly speaking, 'existence' is not the right word here, for some collections might be abstract, and abstract objects do not exist but 'are there' (*gibt es*) in Bolzano's view (*WL* §30).

12. Krickel (1995) does not mention that answering the question of which collection ideas are objectual by means of investigating the properties implied by the concept of collection only works, in Bolzano's framework, for collections of *abstract* objects. For collections of concrete objects, to the contrary, other factors such as contingent circumstances should be taken into account. We follow Krickel (1995) in limiting the discussion to collections of abstract objects, because that is in line with the focus of this chapter. Furthermore, we believe that Bolzano himself was primarily interested in collections of abstract objects as well, since the theories of ideas and of collections which he develops in the *WL* was to serve as a foundation for mathematics (cf. Sebestik 2017, §§1, 2).

a property (*Beschaenheit*) which determines which objects it represents (*WL* §§66.2, 66.4). Bolzano calls this property the extension (*Umfang, Gebiet, Sphaere*) of that idea, although sometimes he uses these terms not to indicate this property, but rather the collection of objects which the idea represents (e.g. *WL* §70).¹³ The extension of an idea (in both senses) is in Bolzano's view a product of its content (*Inhalt*), that is, the simple ideas which make up that idea, and their arrangement, that is, the way in which these simple ideas are combined (*WL* §94.n, cf. *WL* §§56, 70). Roughly, one could say that as a property, an idea's extension is the having of a certain content and arrangement such that the idea attributes certain properties to its objects; as a collection of objects, an idea's extension consists of all and only those objects which have these properties.¹⁴ Now, it may be clear that the properties which an idea attributes to its objects are properties which objects *must* have in order to be represented by that idea. But these are not the only properties which objects thus must have in order to be represented by that idea: besides the properties which an idea attributes to its objects, the objects of that idea must also have the properties which *follow from* (in the sense of Bolzano's notion of logical consequence, i.e. *Ableitbarkeit*; *WL* §155) the properties which it attributes to its objects (*WL* §113). These two kinds of properties taken together, i.e. all properties which objects must have in order to be represented by a given idea, Bolzano calls 'essential' (*wesentliche*) properties of the objects of that idea (*WL* §111, Ginammi et al. 2021). When Krickel (1995) writes that Bolzano, in order to answer the question of which ideas of collections are objectual, investigates the conditions which follow from the concept of collection, we take this to mean that Bolzano investigates the essential properties of the objects of the idea of collection, that is, the properties which objects must have in order to be a collection.

Let us continue with the second relevant aspect of Krickel's (1995) interpretation: Bolzano's theory of collections is relative, which means that it takes collections to have properties always with respect to some idea which represents the collection.¹⁵ Krickel (1995) holds this relativity to be inherent to Bolzano's theory of ideas and to stem from Bolzano's view that in general objects are given always only mediated by ideas (Krickel 1995, pp. 63-64, cf. *WL* §111.n3). Again, Krickel

13. This ambiguity in Bolzano's use of the term 'extension' is often pointed out in the literature (e.g. in Claas 2018, Rusnock and Sebestik 2019, Textor 1996, Centrone 2010, Casari 2016). For a thorough analysis of Bolzano's notion of extension, see (Ginammi 2021). In this chapter we will follow Bolzano in his ambiguous use of the term 'extension'.

14. Consider for example the idea [*Something which has complexity*] (where appropriate, we follow the practice common to Bolzanian scholarship to refer to Bolzanian ideas and propositions using square brackets): as a property, its extension consists in the property of attributing complexity to its objects, as a collection of objects it consists in all and only those objects which have complexity, i.e. are complex.

15. More precisely, Krickel (1995) writes that this relative interpretation of Bolzano's theory of collections is not forced by Bolzano's writings, has however many benefits, especially when it comes to Bolzano's use of his theory of collections in mathematics (Krickel 1995, pp. 123, 148).

does not clarify how this mediation works. We think it should be understood as follows. Earlier in this section we saw that Bolzano took there to be for every object an idea which represents exactly it; since properties are objects in Bolzano's view (*WL* §80), he must have thought that for each property too there is an idea representing exactly it. We also saw that he took ideas to represent all and only those objects which have the properties which they attribute to their objects. On the natural assumption that (most) objects have multiple properties, it follows that for (most) objects there are multiple ideas which represent them.¹⁶ Importantly, the different ideas which represent a given object may be (*WL* §101.5), but need not be equivalent (in the sense of *Wechselvorstellungen*; *WL* §96): the latter case is illustrated by Bolzano's example that we may consider a square both as represented by the concept [*square*] and by the concept [*quadrangle*] (*WL* §§110, 111.3). But Bolzano takes it that depending on which concept represents the object, the essential properties of the object differ (*WL* §111.3, cf. Krickel 1995, pp. 63-4). For example, he writes, equilaterality is an essential property of some object *o* represented by the concept [*square*], but it is not an essential property of that same object *o* represented by the concept [*quadrangle*] (*WL* §111.3). Accordingly, we take Bolzano's view that objects are given always only mediated by ideas to mean that one cannot attribute properties to objects directly, but only with respect to some idea which one takes to represent it.

The view that one cannot attribute properties to objects directly, but only with respect to some idea which one takes to represent the object in question, has the following consequences in the context of Bolzano's theory of collections. Bolzano distinguishes different kinds of collections and accordingly different concepts representing collections of these different kinds. (We will refer to concepts which represent collections as 'collection concepts'.)¹⁷ The most general and indeterminate of these collection concepts { Bolzano's term is *Inbegri*, translated here as 'collection' { is defined as something which has compositeness (*Zusammengesetztheit*; *WL* §82). This general collection concept does not involve any restriction on what counts as a 'part' of the collection, and thus any object which has two or more proper parts of any kind is represented by Bolzano's general concept of collection. More specific collection concepts which Bolzano distinguishes { clearly inspired by Bolzano's mathematical work { are that of a *Menge*, *Summe*, and sequence (*Reihe*; cf. Chapter 2 for an overview of each). As we saw in Section 2.2 and Section 2.3.3 *Menge* is a *Summe* if some parts of the parts are parts of the whole (*WL* §84.3). Following Lapointe (2011, pp. 118-23), we think that this should be interpreted as meaning that in Bolzanian *Summen* { but not

16. In fact, Bolzano writes that for every property b, b^1, b^2, \dots which object A has, there is a corresponding truth [A has b], [A has b^1], [A has b^2], \dots (*WL* §§109, 110.1, 360) and for every truth [A has b], [A has b^1], [A has b^2], \dots , the object A is represented by the corresponding ideas [*Something which has b*], [*Something which has b^1*], [*Something which has b^2*], \dots (*WL* §§66.1, 95.n, 196.4).

17. To each collection concept there corresponds a collection kind of the ones listed in Section 2.2.

in Bolzanian Mengen in general { parthood is transitive up to homogeneity, which is to say: the parts of the (more proximate) parts which are of the same kind as these (more proximate) parts are also parts of the Summe.

Bolzano's view that one cannot attribute properties to objects directly, but only with respect to some idea which one takes to represent it, means that the same object, i.e. the same collection, can be represented by different collection concepts. Moreover, depending on which idea represents it, the collection is considered to have certain properties such as an order among its parts or transitivity up to homogeneity of parthood.¹⁸

Now, the two discussed aspects of Krickel's interpretation of Bolzano's theory of collections are relevant for the following reason. The first aspect, i.e. that Bolzano's theory of collections is primarily a theory of ideas of collections, suggests that the issue of size might be determined at the level of collection ideas (or concepts), rather than at the level of collections themselves. The second aspect, i.e. the relativity of Bolzano's theory of collections, implies that there are different concepts which Bolzano might have taken to represent the natural numbers. For instance, Bolzano might have taken the natural numbers to be represented by a collection concept of any of the different kinds discussed in this section { Menge, Summe, sequence, or collection in general {, and there seems to be no privileged way among these to conceive of the natural numbers. Now, we have seen that in Bolzano's view objects have properties relative to an idea which represents them. Possibly, the size of a collection is in Bolzano's view a property which changes depending on which idea represents the collection. In the next section we will show that to a significant extent, size is in Bolzano's view indeed determined at the level of ideas. In Section 3.4 we will then investigate Bolzano's writings on the natural numbers, in order to determine which collection concept he took to represent them.

3.3.2 Part-whole for extensions

As we have seen in Section 3.2.2, in *WL* §102 Bolzano discusses the sequence $n; n^2; n^4; n^8; n^{16}; \dots$. Here, the letter 'n' denotes the concept of any arbitrary whole number (*jeder beliebigen ganzen Zahl*), and thus, as Bolzano explains, each of the terms of this sequence has infinitely many objects (namely, he adds, infinitely many numbers) standing under it. The issue which Bolzano addresses in the section of *WL* §102 is that of how to measure the sizes (*Größen*) of the respective collections of objects standing under the terms of this sequence, that is,

18. Evidence for Krickel's relative interpretation of Bolzano's theory of collections is that Bolzano on some occasions when discussing collection notions mentions essential properties, and, as we have discussed above, essential properties are relative in his view (Krickel 1995, pp. 140, 168-9). For example, in *PU* §6 Bolzano argues that the same object grasped under [*Drinking glass*] and [*drinking glass which is broken*] respectively appears *essentially* different with regard to the ordering of its parts.

as he calls it, the terms' widths (*Weiten*; cf. *WL* §66). It is thus clear that, in accordance with what we called in *crefsumkri* the first relevant aspect of Krickel's (1995) interpretation, Bolzano conceives here of the collections of natural numbers, squares, and so on, as extensions of ideas { namely, of the concepts denoted by $n; n^2; n^4; n^8; n^{16}; \dots$ } and furthermore that he conceives of the sizes of these collections as the widths of the ideas representing the collections.

In *WL* §102, Bolzano goes about addressing the issue of the respective widths of the terms of the sequence $n; n^2; n^4; n^8; n^{16}; \dots$ by pointing out that for any non-initial term in this sequence, the objects standing under it also stand under its predecessor, but the converse is not true. For example, he writes, all objects standing under n^{16} also stand under n^8 , but there are many objects standing under n^8 which do not stand under n^{16} . Thus, as he puts it, in this sequence every non-initial term is subordinated (*untergeordnet*) to its predecessor.

Subordination (*Unterordnung*) is a notion from Bolzano's theory of ideas and is defined as follows (*WL* §97):

(Sub): An idea $[B]$ is subordinated to an idea $[A]$ *i* all objects which stand under $[B]$ also stand under $[A]$ and there are objects which stand under $[A]$ but do not stand under $[B]$.

It seems natural to interpret **(Sub)** as stating that an idea $[B]$ is subordinated to an idea $[A]$ *i* the extension of (meaning here: the collection of objects represented by) $[B]$ is a *part* of the extension of $[A]$, and this is indeed how Bolzano expresses it himself on some occasions (*WL* §§66.3, 97.n). Accordingly, we can rephrase **(Sub)** as a part-whole principle concerning extensions:¹⁹

Sub_{part}: For $[A]$, $[B]$ objectual ideas, $[B]$ is subordinated to $[A]$ *i* $[B]$'s extension is a part of $[A]$'s extension.

Now, if an objectual idea $[B]$ is subordinated to an idea $[A]$ and thus $[B]$'s extension is a part of $[A]$'s (**Sub_{part}**), then Bolzano says that $[A]$ is *wider* (*weiter*) than $[B]$ (*WL* §93, cf. *WL* §§97.n, 102). Since width is, as we have seen above, Bolzano's notion of size of extension, this means that if an objectual idea $[B]$ is subordinated to an idea $[A]$, then $[A]$'s extension is *greater than* $[B]$'s. Accordingly, we can formulate Bolzano's notion of size based on the part-whole principle for extensions as follows:

PW_{ext}: If $[B]$'s extension is a part of $[A]$'s extension, then $[A]$'s extension is *greater than* $[B]$'s extension.

19. This principle must however be restricted to objectual ideas, since in Bolzano's view also objectless ideas can be subordinated to one another (*WL* §108), but it seems unlikely that he took there to be part-whole relations among 'things' which are not there or exist in any way, i.e. among 'collections' of 'objects' represented by objectless ideas.

(Note however that $[A]$ can be wider than $[B]$, and thus $[A]$'s extension greater than $[B]$'s extension, even if $[B]$ is not subordinated to $[A]$ (*WL* §97.n.) \mathbf{PW}_{ext} says, in other words: if $[B]$'s extension is a part of $[A]$'s extension, then there are *more A's than B's* (cf. *WL* §93).

Since, as Bolzano argued in *WL* §102, in the sequence $n; n^2; n^4; n^8; n^{16}; \dots$ each non-initial term is subordinated to its predecessor, his theory of ideas implies that each term in this sequence is wider than all its successors and thus that there are more natural numbers than squares, more squares than fourth powers, and so on. In *WL* §102, Bolzano however goes on to argue that the width of each term in this sequence is *in nitely* greater than that of all its successors by putting an upper bound N on the extensions of the terms $n; n^2; n^4; n^8; n^{16}; \dots$ of this sequence, and showing that in this way one can calculate the ratios between the widths of these terms. It seems that by the time of the letter to Zimmermann he had come to consider it an error to limit the extensions of the terms $n; n^2; n^4; n^8; n^{16}; \dots$ in order to determine these ratios, for, as we have seen in Section 3.2.2, he writes that 'the false result was due to an unjust inference from a *nite* amount of numbers, namely which do not succeed the number N , to all of them' (cf. van Wierst et al. 2018, §5).

But why did Bolzano argue in the letter to Zimmermann that there are as many natural numbers as squares? That there are as many natural numbers as squares does not seem to be a direct consequence of rejecting the calculation of the ratios between the widths of the terms of the sequence by means of putting an upper limit on them. It might seem that Bolzano could have arrived at this view only by giving up his notion of size based on **part-whole**, and this is indeed Berg's conclusion. However, we would like to point out that there is another possibility to arrive at the view that there are as many natural numbers as squares which does not require giving up part-whole reasoning. This possibility builds on Krickel's (1995) views that Bolzano's theory of collections is primarily a theory of ideas of collections, and that Bolzano's theory of collections is relative (see Section 3.3.1). For, as we have seen in that section, in Bolzano's view objects $\{$ and thus the collections of natural numbers, squares, and so on $\{$ can be represented by multiple different ideas, and these objects have properties always only relative to the ideas by which they are represented. Furthermore, as we have seen in the present section, subordination is in Bolzano's view a relation which obtains between *ideas* $\{$ for example, between ideas of collections $\{$ and thus not between objects $\{$ collections $\{$ directly. Accordingly, the following seems possible in Bolzano's framework: there are ideas $[N]$, $[S]$ such that $[N]$ represents the natural numbers and $[S]$ represents the squares and $[S]$ is subordinated to $[N]$ *and* there are ideas $[N']$, $[S']$ such that $[N']$ represents the natural numbers and $[S']$ represents the squares and $[S']$ is *not* subordinated to $[N']$. In other words, it seems possible in Bolzano's framework that there are more natural numbers than squares if they are taken to be represented by certain ideas, and there are as many natural numbers as squares if they are taken to be represented by certain other ideas instead. According to

this line of reasoning, the change between *WL* §102 and the letter to Zimmermann is not that $\{$ as is Berg's view $\{$ Bolzano came to give up part-whole reasoning concerning the size of collections of natural numbers, but rather that he came to conceive of the ideas which represent collections of natural numbers and of squares and so on differently.

In order to determine the viability of this explanation of the change between *WL* §102 and the letter to Zimmermann, in the next section we will discuss Bolzano's writings on (the ideas representing) the natural numbers.

3.4 Bolzano's concept(s) of natural number

In the previous section we presented an interpretation of Bolzano's theory of ideas and in particular of collection concepts to argue for a version of part-whole reasoning in *WL* §102 that is not contradicted by the letter to Zimmermann the way Berg advocated. In this section we apply this interpretation to Bolzano's definition of the natural numbers in his later mathematical works (*Reine Zahlenlehre* (henceforth *RZ*) and *PU*) and show that (at least the late) Bolzano conceived of collections of natural numbers primarily as sequences (Section 3.4.1). In Section 3.4.2 we then argue that Bolzano conceived of the size of collections of natural numbers as determined by the relation of subordination between the relevant sequence concepts. On this basis, we arrive at an interpretation of the Zimmermann letter which is compatible with **part-whole**.

3.4.1 Bolzano's definition(s) of the natural numbers

In order to determine which concept Bolzano took to represent the natural numbers, let us look at how he defines them. In *PU* §8 he gives the following definition:

Let us imagine a *sequence* [*Reihe*] of which the *first term* is a unit of the kind *A*, but every succeeding term is derived from its predecessor by our taking an object equal to it and combining it with a new unit of kind *A* into a *Summe*. Then obviously all the terms appearing in this sequence $\{$ with the exception of the *first* which is a *mere unit* of the kind *A* $\{$ are *Vielheiten* of the kind *A* and in fact these are such as I call *finite* or *countable Vielheiten*, indeed I call them straightforwardly (and even including the *first term*) *numbers* [*Zahlen*], and more definitely, *whole numbers*.²⁰

20. Denken wir uns eine *Reihe*, deren erstes Glied eine *Einheit* von der Art *A* ist, jedes nachfolgende aber aus seinem vorhergehenden auf die Weise abgeleitet wird, daß wir einen ihm gleichen Gegenstand nehmend, denselben mit einer neuen Einheit von der Art *A* zu einer *Summe* verbinden: so werden offenbar alle in dieser Reihe vorkommenden Glieder | mit Ausnahme des ersten, das eine bloße Einheit von der Art *A* darbietet | *Vielheiten von der Art A* sein und dies zwar solche, die ich *endliche* oder *zählbare Vielheiten*, auch wohl geradezu (und selbst mit

Thus, Bolzano defines the natural numbers here as terms of a sequence such that the first term is a unit of some kind A ,²¹ and every subsequent term of the sequence is a Summe obtained by adding a new unit to an object which is similar to its predecessor in the sequence. Similarly, also in *RZ I* §1, he defines the natural numbers as terms of such a sequence, and remarkably { favouring an interpretation that stresses the continuity between the Bolzano of *WL* §102 and the Bolzano of later years { also in *WL* §87.4.²²

The upshot of this definition of the sequence of natural numbers is the following. It is clear that already in the *WL*, and importantly later on in the *RZ* and the *PU*, Bolzano conceived of the natural numbers in the first place as a sequence. Accordingly, it seems that with the expression 'natural number' (*ganze Zahl*) and the sign n used in *WL* §102 and in the letter he associated a sequence concept. This is a pregnant choice for comparing sizes of different collections of natural numbers, because the treatment of sequences in the *PU* suggests that the amount of terms (*Gliedermenge*) of sequences is one of the two quantity notions along which one can compare the size of infinite collections in Bolzano's view (see Bellomo and Massas 2021). Furthermore, as we will show in the following, Bolzano's reasoning in *PU* §33 is strikingly similar to that in the letter to Zimmermann, which suggests that the size comparison between n and n^2 in the letter should { similar to that in *PU* §33 { be understood as a comparison of the amount of terms of n and n^2 as sequences.

In *PU* §33 Bolzano compares two infinite sums, $\overset{1}{S}$ and $\overset{2}{S}$, respectively the sum of all natural numbers and the sum of all squares of natural numbers, and in this comparison he claims that considered as sequences $\overset{1}{S}$ and $\overset{2}{S}$ have the same amount of terms (*Gliedermenge*).²³ He then adds that the difference between $\overset{1}{S}$ and $\overset{2}{S}$ is that in $\overset{2}{S}$ one raises each individual term of $\overset{1}{S}$ to its second power. Importantly, as Bolzano writes here, this induces a change in the quality (*Beschaenheit*) of the terms, not in their plurality (*Vielheit*).²⁴ Compare this with what one reads

Inbegriff des ersten Gliedes) *Zahlen*, bestimmter: *ganze Zahlen* nenne.

21. A can be any kind, and any object standing under the concept which represents A { i.e. $[A]$ { is a unit of kind A . If, for example, $[A]$ is the concept [*pair of cherries*], then every two cherries will count as a unit of kind A ; if $[A]$ is instead the concept [*cherries*], then every *single* cherry will count as a unit of kind A . Bolzano calls numbers whose kind is given 'named' numbers (*benannte Zahlen*; *RZ I* §10). Since in mathematics one does not specify the kind A when dealing with numbers, the numbers of mathematics and hence the ones we are concerned with in this chapter are in Bolzano's terminology 'unnamed' numbers (*unbenannte Zahlen*; idem, cf. Simons 2003, p. 131).

22. Given that, in the *WL*, Bolzano defines the natural numbers as terms of a sequence, it is puzzling that he does not treat the terms of the sequence n, n^2, n^4, \dots as sequences themselves in §102. We assume that this is part of what Bolzano later on came to conceive of as problematic about his argument in that passage.

23. For a much more detailed treatment of *PU* §33, see Chapter 5.

24. Dadurch, daß wir jedes einzelne Glied der Reihe $\overset{1}{S}$ in der $\overset{2}{S}$ auf das Quadrat erheben,

in the letter to Zimmermann:

The Menge of these objects which n represents is still exactly the same as before, even though the objects themselves, which n^2 represents, are not quite the same as the ones n represents.²⁵

Both passages seem to say that, despite there being some qualitative differences between the objects of the concept of natural number and those of the concept of square of natural number, there is something that they have in common, namely, how many they are. One aspect of our proposal is therefore to interpret the letter to Zimmermann as another version of the argument in the passage from *PU* §33, on account of their thematic and temporal proximity.

If we read the letter as another version of the argument in *PU* §33, then there is enough circumstantial evidence to suggest that Bolzano is not rejecting **part-whole** in the letter. First, there is no rejection of **part-whole** in *PU* §33, nor elsewhere in the *PU*, and several of Bolzano's arguments in *PU* rest on a notion of size based on the part-whole principle (see for example §§19-20). Second, §29 of the *PU*, Bolzano claims that the Menge of all natural numbers ($\overset{0}{N}$) is greater than the Menge of all natural numbers greater than a certain finite number n ($\overset{n}{N}$), because their difference is exactly n ; if these two Mengen are not the same, it means that they are not compared by using **one-one**. Third, Bolzano's statement in *PU* §33 that $\overset{1}{S}$ and $\overset{2}{S}$ have the same amount of terms $\{$ as well as his statement in the letter that n and n^2 represent the same Menge $\{$ does not imply that Bolzano accepted a notion of size based on **one-one**. For, as we will show in the next section, in Bolzano's framework $\{$ particularly, on the basis of the theory of ideas presented in Section 3.3.1 and the notion of size based on subordination (\mathbf{PW}_{ext}) presented in Section 3.3.2 $\{$ it is possible to argue that the sequence of natural numbers and that of squares of natural numbers have the same amount of terms while at the same time adhering to **part-whole**.

In summary, we have established two things. First, Bolzano defines the natural numbers as terms of a sequence. This makes it less likely that the sizes of n , n^2 , ... should be determined $\{$ as is usually done in the secondary literature, as we have shown in Section 3.2.2 $\{$ by means of set-inclusion, and suggests that instead they should be evaluated as the size of a sequence is evaluated in Bolzano's view. Second, if n and n^2 in the letter are to be considered as sequences in the way that infinite collections of natural numbers are in the *PU*, then Berg's argument for **one-one** and against **part-whole** is not borne out by the textual evidence. In the next section, we will therefore look at the different options for subordination

ändern wir bloß die Beschaffenheit (die Größe) dieser Glieder, nicht ihre Vielheit.

25. Die Menge dieser Gegenstände welche das n vorstellt, ist genau immer noch dieselbe wie vorhin, obgleich die Gegenstände selbst, die n^2 vorstellt, nicht eben die nemlichen sind, welche n vorstellt.

relations between the relevant ideas, in order to determine how this bears on their respective sizes.

3.4.2 Subordination between number concepts in the letter

In the previous section we have seen that Bolzano conceived of the collection of natural numbers primarily as a sequence and we propose that this is the case also in the letter to Zimmermann. Accordingly, when Bolzano writes in the letter that n and n^2 are the same *Menge*, we take it that this should be interpreted as saying that n and n^2 are sequences which have the same amount of terms (Bolzano's expression: *Gliedermenge*). How can we explain that n and n^2 have the same amount of terms in Bolzano's view, if not on the basis of **one-one**? In this section we aim to show that Bolzano's reasoning in the letter is consistent with the part-whole principle as based on subordination as we proposed in Section 3.3.2.

In order to arrive at an explanation of the view that n and n^2 have the same amount of terms, which does not amount to an acceptance of **one-one**, we propose to look at a passage from Bolzano's philosophical diaries (1827-1844, henceforth *PT*). As part of the preparatory notes that Bolzano took for the *PU* in the 1840s, there is a passage that reads (*PT*, p. 87; our translation):

Belonging to the general theory. Every infinite Menge is similar to some other infinite Menge; if it is considered just by itself, without another infinite Menge given through intuition, it cannot exhibit any property that can be grasped through pure concepts, which would not be exhibited also by any other [infinite Menge]. It is only through intuitions that a difference can be perceived, that for example the one Menge is a part of the other (?) that the Menge 1;2;4;8;16;... is a part of the Menge 1;2;3;4;5;6;7;8;... can be recognised through concept(s). An example of two infinite Mengen that are to be considered as *halves* of another infinite Menge can be found in the general theory of quantity through the sequences 1 2 3 ... in *inf* and 1 2 3 ... in *inf* when they are thought under [the concept of?] the whole infinite sequence of natural numbers, the one time with the determination [*Bestimmung*] of 'positive', the other time with the determination of 'negative'. Are there not infinite Mengen of any ratio e.g. $\sqrt{2} : \sqrt{9}$?²⁶

26. Zu der allgemeinen Lehre gehört. Jede unendliche Menge ist einer anderen unendlichen Menge ähnlich; sie für sich allein betrachtet, ohne eine durch Anschauung gegebene andere unendliche Menge kann keine durch reine Begriffe erfasslichen Beschaffenheiten darbieten, die nicht auch eine jede andere darbiete. Nur durch Anschauungen kann ein Unterschied wahrgenommen werden, da z.B. die eine Menge ein Theil der anderen ist (?) da die Menge 1;2;4;8;16;... ein Theil der Menge 1;2;3;4;5;6;7;8;... ist, kann durch Begriffe erkannt werden, { Ein Beispiel

This passage is difficult to interpret given that it appears without context in Bolzano's diaries, and that it contains terminology which appears unusual for Bolzano's mature mathematical writings (in particular, the appeal to 'intuitions'). Furthermore, it is not clear whether this passage reflects Bolzano's own view or that of someone else. Nonetheless, we think it worth considering the passage, because it presents an alternative picture to comparing collections of natural numbers which Bolzano certainly { in one way or another { has considered.

On our reading, the passage states the following. Infinite Mengen considered just as such are indistinguishable from one another; for example, one cannot tell whether they are related as parts and wholes. Mengen can appear in ratios to one another *only* when they are considered, not as Mengen as such, but as Mengen with some specific determinations. Such determinations are, for example, that one Menge consists of natural numbers and the other of squares, or that one consists of the terms of the positive sequence of natural numbers and the other of those of the negative sequence of natural numbers.

For our purposes, the most notable aspect of this passage of *PT* is that Bolzano writes that the sequence of positive integers and the sequence of negative integers are to be considered as *halves* of the whole infinite sequence of natural numbers. Bolzano is not explicit about the reasoning here, but it seems that, because the attributes of positive and negative as applied to numbers are mutually exclusive, once they are applied to the sequence of natural numbers as a whole they induce two copies of the same sequence which are identical except for the one attribute. Thus, it seems that Bolzano considers the sequence of positive integers and the sequence of negative integers as halves of the whole infinite sequence of integers *not* because half of the terms of the whole infinite sequence of integers are positive and the other half are negative (this is not the case), but rather because these sequences are the result of adding a determination and its opposite (namely, positive and negative) to the concept of infinite sequence of natural numbers. That is, in other words, Bolzano seems to consider the sequence of positive integers and the sequence of negative integers as halves of the whole infinite sequence of natural numbers because of relations of subordination between the respective *sequence concepts* rather than because of relations of inclusion between the terms of the respective sequences.

Now, importantly, Bolzano's reasoning about the sequences of even powers of natural numbers { and thus in particular his reasoning in the Zimmermann letter about the terms of the sequence $n, n^2, n^4, n^8, n^{16}, \dots$ { can very well be understood as similar to the reasoning about the sequences of positive and

von 2 unendlichen Mengen, die als *Halften* einer anderen unendliche Menge zu betrachten sind, haben wir in der allgemeinen Grössenlehre an den Reihen $1\ 2\ 3\ \dots :: in\ inf$ und $1\ 2\ 3\ \dots :: in\ inf$ wenn beide unter die ganze unendliche Reihe der natürlichen Zahlen, das einmal mit der Bestimmung der Positiven, das andere Mal mit der Bestimmung der Negativen gedacht werden sollen. Gibt es nicht unendliche Mengen von jedem beliebigen Verhältnis z.B. $\frac{1}{2} : \frac{1}{9}$!

negative natural numbers in the *PT*; it is in fact our proposal to understand it as such. That is to say, we propose to understand Bolzano's reasoning in the Zimmermann letter in parallel with the reasoning in the *PT* in two respects. First, where in the *PT* the attributes of positivity and negativity added to the concept of the whole infinite sequence of natural numbers generate two otherwise identical copies of the natural number sequence, the positive power attributes added to the concept of the natural number sequence in the letter do the same. Thus, in the letter, the sequences of even powers of the natural numbers $n^2, n^4, n^8, n^{16}, \dots$ are obtained from n by adding an attribute to the concept of n { as we have seen in Section 3.4.1, similar to Bolzano's reasoning in *PU* §33, this induces a change in the quality in the terms of the sequence but not in their plurality { which explains why in the letter Bolzano writes that the objects represented by n and those represented by n^2 are the same Menge. Second, similar to the relations between n and the positive and negative natural numbers in the *PT*, Bolzano conceived of the relations between n and $n^2, n^4, n^8, n^{16}, \dots$ in the letter as subordination relations between the *concepts* representing these sequences, rather than as inclusion relations between the terms of these sequences. That is to say, where in the *PT* the positive and negative natural numbers are halves of n , in the letter $n^2, n^4, n^8, n^{16}, \dots$ are *parts* of n { as well as n^4, n^8, n^{16}, \dots parts of n^2 , and so forth { on account of subordination between these concepts.

Thus, the view that the sequences of even powers of natural numbers are obtained from adding attributes to the concept of natural number sequence explains Bolzano's view that all these sequences have the same amount of terms. Importantly, however, Bolzano's calculations in *PU* §29 and §33 (see Section 3.4.1) show that in his view it is *not* the case that *all* infinite sequences of natural numbers have the same amount of terms: the amount of terms of for example tails of number sequences, such as the sequence of all natural numbers greater than 5, is in his view strictly smaller than that of the whole infinite number sequence. How can it be explained that in Bolzano's view some infinite sequences of natural numbers do and others do not have the same amount of terms?

As we see it, on the view that some sequences like that of the squares of natural numbers and that of all natural numbers greater than 5 are obtained by adding determinations to the general concept of the whole sequence of natural numbers, it is plausible that the added determinations can be of different kinds { specifically, that whereas some such determinations, as Bolzano puts it (Section 3.4.1), induce a change in quality but not in plurality of the terms of the sequence, other determinations do (also) induce a change in the plurality of the terms. In the cases of sequences of even powers of the natural numbers on the one hand, and tail sequences of the natural numbers on the other, we think it plausible that the determinations added to the concept of n in order to get the sequences of these respective kinds are of a completely different nature: sequences of even powers seem to be obtained from n by a modification of the determination rule in the sequence concept, whereas tail sequences of the natural numbers seem to be

obtained from n by a modification on its terms. As such, it is plausible that the latter, but not the former determination implies a change in amount of terms of the sequence.

Since, as we argued, Bolzano conceived of both sequences of even powers of natural numbers and tail sequences of natural numbers as obtained from the whole sequence of natural numbers by adding a determination of some kind, we maintain that Bolzano conceived of concepts of both kinds of sequences as subordinated to the concept of the sequence of natural numbers. Sequences of both kinds thus qualify as a 'part' of n on the basis of \mathbf{PW}_{sub} , though the way each relates to the whole of n is different. Since Bolzano's theory of collections is primarily a theory of ideas of collections and the same object can in Bolzano's view be represented by different ideas (see Section 3.3.1), part-whole relations among objects can take many different shapes in Bolzano's framework (cf. Krickel 1995, part C IV). Some of these part-whole relations { such as that between the sequence of natural numbers greater than 5 and the sequence of natural numbers as a whole } are similar to set-inclusion, whereas others { such as that between n and n^2 } are not. As we see it, the letter to Zimmermann is compatible with Bolzano's part-whole reasoning, because, as we argued, Bolzano conceived of n and n^2 , and so on as sequences such that the latter is obtained from the former by means of a modification of the determination rule (as opposed to a modification of its terms), and *as such* whether or not there is inclusion of the terms of the respective sequences is irrelevant to the question of (relative) size. In other cases, however, whether or not there is inclusion of terms *is* relevant to the question of (relative) size, namely, where one is concerned with two sequences such that one is obtained from the other by modification of its amount of terms (as opposed to modification of its determination rule), such as for example in the case of n and the sequence of natural numbers greater than 5. Crucially, whether or not there are part-whole relations between objects is in Bolzano's view, we argue, determined *not* by the objects themselves, but by the ideas which represent these objects.

To summarise, then, this is how we suggest one should understand the relations between number sequence concepts according to Bolzano in the 1840s:

1. There is no fact of the matter as per whether, say, the collection of natural numbers is smaller, greater, or equal to the collection of natural number squares unless one first specifies under which collection concept these objects are to be grasped.
2. Some sequences obtained from the general sequence of natural numbers have the same amount of terms as it, some do not.
3. This depends on exactly how the concepts representing the sequences are related to each other in terms of their attributes.

Bolzano argued in the letter to Zimmermann that n and n^2 are the same Menge, because at that time he conceived of these Mengen as sequences, and *as sequences*,

n, n^2, \dots have the same amount of terms (*Gliedermenge*), that is, as sequences, there are as many natural numbers as squares of natural numbers, and so on (see also Chapter 5). As we have shown, Bolzano can argue for the sequences having the same amount of terms without thus giving up part-whole reasoning altogether.

3.5 Conclusion

In this chapter we showed that the identification of Bolzano's collections with sets in the modern sense, as well as of Bolzano's part-whole principle with \mathbf{PW}_{set} comes with interpretative shortcuts which distort and obscure the part-whole reasoning present in Bolzano's writings on infinite collections, his letter to Zimmermann included. We argued with Krickel (1995) that Bolzano's collections are relative to an idea which represents them, and we argued that as a consequence, Bolzano's commitment to the part-whole principle (\mathbf{PW}_{ext}) goes as deep as his commitment to his definition of concept subordination. Thus, we emphasise that according to Bolzano collections have a size only with respect to an idea which represents it. Our interpretation has the benefit over Berg's that it preserves the continuity between *PU* and the letter to Zimmermann and that it is in accordance with the chronology of the writing of these works as reconstructed by Steele (Bolzano 1950, p. 54).

Chapter 4

Measurable Numbers

4.1 Introduction

In 1962, Karel Rychlík published a manuscript from the first half of the 1830s containing Bolzano's theory of the real numbers, which he calls *measurable* numbers (Rychlík 1962).¹ The portion of Bolzano's *Nachlass* published by Rychlík was reprinted in Bolzano's *Gesamtausgabe (BBGA)* in 1976 as the seventh section of Bolzano's *Pure Theory of Numbers (Reine Zahlenlehre, henceforth RZ VII)*.²

In much of the historical literature on the real numbers, Bolzano's presentation has gone largely unnoticed. For instance, there is no mention of Bolzano in Epple's (2003) extensive list of the numerous 19th-century attempts at a theory of the real numbers (Epple 2003, p. 292), nor in Gray's roundup of constructions of the real numbers in (Gray 2015). The lack of recognition for Bolzano's work on the reals is unexpected, two things considered.

One, Bolzano was one of the first³ to present the real numbers as obtained out of the natural numbers by performing addition, multiplication, subtraction or division on them (cf. Sebestík 2017).

In addition, constructions of the reals within the 'pure theory of numbers' like Bolzano's are generally considered of special interest because they are what gives

1. Summaries of Rychlík's findings had already appeared in 1956 in Czech, followed by a 1957 German (Rychlík 1957) and a 1961 French translation (Rychlík 1961). The dating of Bolzano's presentation of the reals to the 1830s is due to Winter (Rychlík 1957, p. 553).

2. The *BBGA* contains significant alterations with respect to Rychlík's text. According to Jan Berg (*RZ*, pp. 7-8) Rychlík only included the easily readable portions of the seventh section of the *RZ*. The text Rychlík omitted contained 'replacements and improvements that are of fundamental significance (*Bedeutung*) for Bolzano's theory' (Berg, *ibid.*). We follow the *BBGA* in this chapter.

3. We say this on the basis of (Sebestík 1992; Ewald 1996; Crossley 1987), from which we conclude that the only mathematician regarded as attempting to define the real numbers earlier than Bolzano is Hamilton (1837) { see Ewald (1996, p. 764), and Crossley (1987, pp. 143-144) { who, however, did not try to establish completeness of the reals.

the name to Felix Klein's 'arithmetization of mathematics' (Klein 1896). Bolzano's contributions to analysis, in particular his arithmetical proof of the intermediate value theorem, are nothing short of extraordinary (Lützen 2003, pp. 174-176). When Klein called Bolzano 'one of the fathers of arithmetization' (Klein 1926), he did so without consideration for Bolzano's work on the reals, as the latter was to appear many decades later. But if Bolzano's attempt is so remarkable, why is it neglected?

One of the reasons is arguably that *RZ VII* is a text of difficult interpretation { an unfinished manuscript never intended for publication, for which we depend on punctilious and complicated contemporary editorial work, and which was presented from the outset as containing a theory that 'left a lot to be desired' Rychlík (1961, p. 323). The work ends abruptly with an incomplete sentence, and on more than one occasion Bolzano offers multiple, contrasting definitions of key concepts. The difficulty of the text has motivated a number of interpreters over the past fifty years to offer commentaries and reconstructions of Bolzano's definitions and arguments in *RZ VII*. Most of these interpretations, following an approach inaugurated by Rychlík himself (Rychlík 1961, pp. 323-324), typically focus on repairing the construction (so that the theory would not have 'a lot to be desired' anymore), and do so by identifying Bolzano's reals with sequences (see e.g. van Rootselaar 1964, Laugwitz 1965, Sebestík 1992, Rusnock 2000). The problem with sequence interpretations, as Russ and Trlifajová (2016) call them, is that if on the one hand they make Bolzano's reals appear closer to those of Cantor (and, to a lesser extent, Dedekind's), on the other they do not preserve Bolzano's arguments nor, in some cases, his results.

Our aim in this chapter is to offer a close reading of some key passages from *RZ VII* that clarifies Bolzano's construction on its own terms { and this can only be done by resisting anachronistic readings of Bolzano's attempts.⁴ When a proper account is at hand, the broader issue of the significance of Bolzano's presentation within the arithmetization of analysis can be tackled. Consider Epple's characterization of the attempts to define the reals as situated within the conceptual space determined by three poles or 'centers of attraction' (Epple 2003, pp. 292, 303): the attempt to hold on to traditional views, which Epple associates with Hankel and Frege, according to whom analysis still has to be founded on the concept of continuous quantity (Epple 2003, p. 292); the *arithmetizing* strategy Epple associates with Dedekind, Weierstrass and Cantor, according to whom a construction of the real numbers requires an arithmetic foundation { that is, must be attained via operations on the natural or rational numbers; finally the *formalist* view Epple ascribes to Heine, Thomae and Hilbert (Epple 2003,

4. In this we side with Spalt (1991), who instead analyses Bolzano's key definitions and computations in *RZ VII* on the basis of the work carried out in the preceding portions of the *GL*, and with Fuentes Guillen (2021).

p. 292). Where would Bolzano's approach to the reals belong then? We want to argue that it belongs somewhere in the middle between the arithmetizing pole and the traditional pole, but in order to make our case, we need to clarify what we mean by 'arithmetization' first, and then give a close-reading analysis of *RZ VII*.

This chapter is organised as follows. In Section 4.2, we build on Gandon's 2008 analysis of the arithmetization programme in mathematics to articulate our view on Bolzano's measurable numbers. In Section 4.3 we summarise Bolzano's *RZ VII* and the interpretations thereof that we want to engage. We end the section with a list of the problems they raise as to the success of Bolzano's efforts, namely: his contrasting definitions of equality, his proof that the sum of two measurable numbers is still a measurable number, and his proof of the sufficiency of Cauchy's convergence criterion. In Section 4.4 we examine each of these three issues more closely and find that they mostly do not arise in Bolzano's text, but they are largely due to the sequence interpreters' efforts to align Bolzano's presentation with a specific kind of arithmetizing constructions. Our conclusion is that instead Bolzano's measurable numbers achieve a different kind of arithmetization, one that is also compatible with his adherence to the traditional definition of mathematics as the science of quantity.

4.2 Real numbers and the arithmetization of analysis

Roughly, real numbers are the numbers we use when we want to study continuous, as opposed to discrete, quantities. The reals cannot be obtained by simply adding, subtracting, multiplying or dividing any two integers, and this aspect sets them apart from the rationals in a fundamental way. The key property of the reals considered as a system is a property that used to be called 'continuity', and that modern textbooks usually call 'completeness'. Indeed, in modern algebraic terms, the reals as a system form a complete ordered field. Coming to a precise formulation of this property has been the goal of all 19th century attempts at defining the real numbers, which culminated in 1872 in Dedekind's formulation (Dedekind 1872).

At the time Bolzano published his celebrated arithmetical proof of the intermediate value theorem (1817), analysis was still seen as founded on geometry rather than arithmetic, in the sense that reasoning about properties of the real quantities of analysis was founded or justified by reasoning about properties of intuitively given continuous quantities { i.e. geometrical ones, such as line segments. Bolzano (*RAB*) himself cites the already existing proofs of the intermediate value theorem as examples of an improper use of truths of geometry in analysis (*RAB*, pp. 3-5).⁵

5. Here is the relevant passage:

The most common kind of proof [of the intermediate value theorem] depends on a

Bolzano's work is considered a typical example of a well-attested trend reversal in 19th century analysis that shows dissatisfaction with proofs relying on properties of continuous quantities, and preference for arithmetical proofs instead (see e.g. Detlefsen 2008, p. 182).

Now, one might wonder why the arithmetization of analysis emerges in the first place. This, however, is possibly even more complicated to clarify, and again, it will not be our main concern in this chapter. Suffices to say that in this development intrinsic technical questions in mathematics are inextricably mixed with traditional methodological standards on proper science that at least in Bolzano's case were of paramount importance.⁶

Somewhat as a generalisation of this specific situation in analysis, a tendency exists among historians of mathematics to see the 19th century as witnessing a progressive development from mathematics as the science or theory of quantity (*Größe*)⁷ { meaning by 'quantities' typically *continuous* ones } to the science of

truth borrowed from geometry, namely: *that every continuous [continuirlich] line of simple curvature of which the ordinates are first positive and then negative or conversely, must necessarily intersect the abscissae-line somewhere at a point lying between those ordinates.* There is certainly nothing to be said against *correctness*, [...] [b]ut it is also equally clear that it is an unacceptable breach of *good method* to try to derive truths of *pure* (or general) mathematics [...] from [...] *geometry*. (RAB, trans. in Russ 2004, p. 254)

6. According to the historical reconstructions of Schubring (2005), Lützen (2003), Epple (2003) and Grattan-Guinness (1970) between the second half of the 19th century and the early 20th century, several concurrent developments in mathematics between modern-day Germany, France and Great Britain force mathematicians to gradually modify their notion of quantity so as to accommodate new entities as legitimate mathematical objects. Interpreters have variably stressed at times more the technical, at times more the generally methodological aspects. Epple stresses technical difficulties. He writes one particularly important propeller for changes in the notion of quantity and mathematics more broadly are the technical difficulties raised by the more advanced analysis of the time:

The traditional relation between the real quantities of analysis and intuitively given magnitudes such as line segments lost its supposed self-evidence and these intuitive ideas ceased to be viewed as a sufficient basis for technical arguments. (Epple 2003, p. 291)

Under Epple's reconstruction of events, this is why the definition of the real numbers is so important for mathematicians at the turn of the century, and how it connects to the broader issue of mathematics ceasing to be the theory of (continuous) quantities.

Pierpont (1899) sees the difficulties with the notion of quantity as having to do explicitly with the confusion in its understanding, namely a lack of explicit definition of the notion of quantity (pp. 395-96). Moreover, Lützen (2003, p. 155) writes, 'the developments of new technical theorems of mathematics provided one of the main backgrounds for the growing interest in foundations questions'. It is in this context that mathematicians start to look for a precise definition and theory of the real numbers { i.e. of the domain of the argument of functions which are studied in analysis at the time (Lützen *op. cit.*).

7. We follow Russ (2004) and Cantu (2010, 2014) in translating *Größe* as 'quantities' rather

numbers and sets.

As we saw, Epple (2003) characterises the various 19th century presentations of the reals as traditional, arithmetizing and formalistic. While Epple does not consider these labels to be mutually exclusive, the overall tone of his account reflects the general tendency just mentioned. And so the fact that we use Epple's labels { for want of better ones { might give the impression that we commit to a view according to which whoever champions arithmetization rejects the view of mathematics as a science of quantity. This is not our position. The Bolzano of *RZ VII*⁸ sees mathematics as science of quantity but also champions, as we will see, a certain form of arithmetization.

What exactly is arithmetization?⁹ As the issue is quite complex (cf. Ferreiros 2007a, p. 155), we are not going to address it in this chapter in any satisfactory way, but we still feel some terminological clarity is called for. Our starting point is Gandon's (2008) analysis of Klein's (1896) description of the arithmetizing trend in mathematics of the late 19th century.

According to Gandon, Klein's examples of arithmetization of mathematics can be categorised as achieving or trying to achieve at least one of the following:

- (AM1) A mathematics purged of all intuition;
- (AM2) Gapless proofs in mathematics;
- (AM3) The reduction of mathematics to arithmetic, that is, the theory of the integers.

Note that AM1-AM3 can be seen as an explication of tenets involving three rich notions often associated with arithmetization: AM1 concerns the rejection of visual or diagrammatic intuition as a *source* of mathematical *knowledge*, in favour of pure concepts as a source for mathematical knowledge; AM2: the notion of *rigour*; AM3 the notion of *purity*. If making AM3 only one of three possible ways of pursuing arithmetization strikes one as unexpected, it is nevertheless intentional on Gandon's part, seeing that he considers Hilbert's axiomatisation of geometry as an example of arithmetization of the AM1 and AM2 kind, but not of the AM3 kind (Gandon 2008, p. 3).

Klein (1896) does not draw a sharp distinction between the arithmetization of analysis and that of mathematics. Some remarks surrounding his introduction of the expression 'arithmetizing of mathematics' nevertheless suggest that one distinction could be that arithmetization of analysis consists of the innovations of analysis ushered in by Weierstrass, Cauchy, Dedekind and Cantor among others,

than as Epple's 'magnitudes' as the best translation for how Bolzano uses the term.

8. Bolzano's definition of mathematics changed back and forth between the traditional definition and a more modern-sounding, mathematics-as-science-of-forms definitions. An excellent treatment of the topic can be found in (Cantu 2014).

9. Some have 'arithmetisation', cf. eg. Grattan-Guinness (1970).

and the arithmetization of mathematics consists of the broader changes that came about in mathematics as a whole because of the arithmetization of analysis (Klein 1896, p. 242).

Gandon follows Klein's lead when he does not draw an explicit distinction between the two scopes of arithmetization, however in Gandon's case the two would seem to boil down to the same phenomenon, as the 'reduction schema' he has in mind when talking about the arithmetization of mathematics is actually just the reduction of 'analysis and the real numbers' to 'arithmetic and whole numbers' (Gandon 2008, p. 3). Given that our focus in this chapter is strictly on the real numbers and analysis, we phrase Gandon's components of arithmetization in terms of *analysis* only, and we replace the three components of arithmetization of mathematics above with their analysis versions:

- (AA1) An analysis purged of all intuition;¹⁰
- (AA2) Gapless proofs in analysis (pursuit of rigour);
- (AA3) The reduction of analysis to arithmetic, that is, the theory of the integers (pursuit of purity).

We now have three different senses of 'arithmetization of analysis'.¹¹ It is important to clearly distinguish at least these senses, because in the remainder of this chapter we are going to argue that it is not possible for Bolzano to pursue all of them at the same time. More precisely, a close reading of some of Bolzano's arguments regarding the measurable numbers will show that previous interpretations have been too quick to presume that Bolzano's construction is meant to be an attempt at achieving AA3, while sacrificing AA2.

As to AA1, we will not address the intuition/concept distinction in Bolzano and his issues with Kant's position in this respect, but it is important to fix right from the start a key underlying assumption about Bolzano's conception of mathematics as a science in this context: for Bolzano mathematics is the science of quantity, but by this he means both *discrete* quantities (that is, numerical ones) and *continuous* quantities (geometrical ones). The crucial Bolzanian innovation here is that Bolzano comes to his definition of quantity by trying to eliminate

10. For Frege as adhering to AA1-AA3, see Tappenden (2013, pp. 128-129), Demopoulos (1994, pp. 230-231).

11. Tappenden (2006, p. 109) points out that AM2 is routinely identified with the arithmetization of analysis, in particular Weierstrass-like arithmetization, i.e. 'Weierstrass' broadly computational program of extending the techniques of algebraic analysis by exploiting power series'. See also Demopoulos (1994, p. 231), who runs together AA1-AA3, in fact, all under 'rigour' { while Letzen (2003) seems to run AA2, AA3 together under rigour. For one thing, Kronecker, who is indicated by Pringsheim (1898, p. 58) as the first to have spoken of arithmetizing (*arithmetisieren*) (Kronecker 1887, p. 338), explicitly explains the term as grounding the content of the mathematical disciplines of analysis and algebra uniquely and just (*einzig und allein*) on the concept of number taken in its narrowest meaning (*im engsten Sinne genommen*) (our AA3).

ambiguities and shortcomings from the definitions of his contemporaries, thus achieving a noticeably sharper one. Bolzano's position is that mathematics is a purely conceptual science (AM1 and, *a fortiori*, AA1), so geometry, as a part of mathematics, is as much a purely conceptual science as analysis is, that is, Bolzano rejects that intuitions play a role in geometry.

The upshot of this section is this: the finer conceptual distinctions we propose with respect to the notion of arithmetization help us thematise to what extent Bolzano's measurable numbers can be said to be part of an arithmetizing trend, and to what extent Bolzano is still methodologically committed to a more traditional view of mathematics as the science of quantity. In particular, they clarify that our characterisation of Bolzano as between Eppe's arithmetic and traditional poles hides no obvious conceptual tensions.

For now, we will proceed to summarise Bolzano's presentation of the measurable numbers in the next section.

4.3 RZ VII: Text and interpretations

4.3.1 Numbers, quantities or expressions?

Before we begin with our summary, a couple of terminological notes. First, on 'number' and 'quantity'. Jan Berg, editor of RZ VII in its *BBGA* edition, notes that while the title we now see for the text is 'Infinite quantity concepts (quantity expressions)' (*Unendliche Grö̃enbegriffe (Grö̃enausdrücke)*), an earlier title was 'Infinite number concepts' (*unendliche Zahlenbegriffe*), without the parenthetical which Berg also hypothesises is coeval to the switch from 'number' to 'quantity' in the title. Fuentes Guillen (2021) shows that Bolzano's hesitation between the use of 'quantity' and that of 'number' for anything other than the natural numbers dates back to the 1810s at least, and while there are passages (both from the mathematical diaries and from works such as the *PU*, cf. Fuentes Guillen 2021, p. 10) where numbers are spoken of as specific quantities, the use of 'quantity' and 'number' in RZ VII does not always respect the distinction.

Another terminological matter worth discussing is the distinction between infinite number (quantity) *expressions* (*Ausdrücke*) and infinite number *concepts* (*Begriffe*). Bolzano mentions the distinction already in RZ I and V (*RZ*, pp. 21-22, 78), and accordingly every expression denotes (*bezeichnet*) a concept, so that, for infinite numbers, say, there can be several infinite number expressions that denote one and the same infinite number concept. By his own admission, though, Bolzano does not follow the distinction rigorously but rather talks simply of number concepts or numbers *tout court* even in cases where he means expressions, to make the text easier to follow (*RZ*, pp. 21-22).

In the current chapter we follow Fuentes Guillen (2021) and Rusnock (2000) and use 'number' throughout, resorting to 'number concept' when a contrast with

number expressions is needed. We are now ready to sketch the contents of *RZ VII*.

4.3.2 Summary and segmentation

The definition of measurable number presupposes a number of other notions, namely that of infinite number expression, 'purely positive' number expression, and others. We will therefore first illustrate these definitions and then move on to the definition of measurable numbers proper.

RZ VII concerns itself at least on paper with infinite number concepts and the infinite number expressions which designate them, first and foremost. These are defined in contrast with the finite number concepts and expressions which are the main object of investigation of the previous sections of *RZ*. The infinite number concepts are infinite in the sense that the operations of addition, multiplication, subtraction and division required to fully describe the concept are iterated infinitely many times (§2). Bolzano offers as examples of infinite number expressions $1\ 2\ 3\ \dots\ \text{in inf.}$ and $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}\ \dots\ \text{in inf.}$, and also the infinite product $p\ 1\ \frac{1}{2}qp\ \frac{1}{4}qp\ \frac{1}{8}qp\ \frac{1}{16}q\ \dots\ \text{in inf.}$ Note how, although the definition does not exclude that the operation to be iterated infinitely many times be division, Bolzano does not use such an example.

Next, a purely positive (or 'strictly' positive) number expression is an expression that does not contain subtraction.¹² Since '0' does not count as a positive number for Bolzano, note that the definition of purely positive number concept (or expression) does not encompass '0' or expressions that end up designating 0.

Now that we know what an infinite number concept is for Bolzano, we are also in a position to explain when such a number concept is said to be measurable. In §5, Bolzano writes:

Among the infinite number concepts there are some which are of such a kind that, to every arbitrary actual number [*wirklichen Zahlen*] q which we want to consider as the denominator of a fraction, a numerator p can be found which is again a positive or negative actual number, or even sometimes zero, with the property that we obtain the two equations $S = \frac{p}{q} = P$ and $S = \frac{p-1}{q} = P^1$. (*RZ VII* §5)

In the equations above, S is the infinite number expression designating a measurable number concept, and P^1 is a strictly positive number expression, whereas P

12. Note that here the difference between a purely positive number concept and a purely positive number expression matters { a purely positive number concept may be designated by a number expression that does in fact contain a minus sign. So sometimes the definition of purely positive number expression is weakened to an expression that does not contain subtraction signs, or is equivalent to one with no subtraction signs (presumably the equivalence being that it designates a purely positive number concept).

is either a strictly positive number expression, or zero. $\frac{p}{q}$ is called the measuring fraction of S (with respect to denominator q).

Bolzano offers as an example of a measurable number expression the schematic expression $a \frac{b}{1 \ 1 \ 1 \ \dots \text{in inf.}}$, for $a; b \in \mathbb{N}$ (that is, for $a; b$ 'actual numbers'). He also proves that, for an arbitrary denominator q , adequate $p; P$ and P^1 can be found for this expression.

The notion of measurability plays a key role throughout RZ VII, and not simply for the study of measurable numbers, that is, the number concepts that are measurable, but also for the study of those infinite number concepts which fail to be measurable. Measurability allows Bolzano to neatly categorise infinite number concepts into measurable, semi-measurable, and non-measurable. Among the semi-measurable, there are the infinitely great numbers (number concepts), and among the measurable numbers there are the infinitely small numbers. Thanks to measurability, both the infinitely small and the infinitely large can be given precise definitions. Infinitely small numbers are those measurable numbers such that for any given q the numerator of the attending measuring fraction is always zero, 'but without our being justified in calling the number concept concerned itself zero' (§21). Thus the equations for an infinitely small (positive) number are always of the form $S = P^1 \frac{1}{q} = P^2$ (§22).

Infinitely large numbers can also be characterised with respect to their behaviour when one tries to determine their measurement via equations in the way Bolzano shows in §§5-6. An infinite number concept such that for every q there is a p such that either $S = \frac{p}{q} = P^1$ or $S = \frac{p-1}{q} = P^2$, but not both, is called an infinitely large number (§§26-27). This makes sense if one thinks of the two equations as approximating the desired number 'from above' and 'from below', so to speak. Then an infinitely large number can be approximated at most from one of these two directions, but not both, because it is always going to be greater (or smaller, if negative) than any quantity we can express via measuring fractions and P 's.

After a few sections proving results about what happens when one adds, subtracts or multiplies any combination of infinite measurable numbers, infinite measurable numbers, and infinitely large numbers, thus giving the reader a feel for what it is like to work with these mathematical objects, Bolzano proves that any measurable number can be given the appropriate representation through the measuring equations (§48). Then the problem arises of when two expressions which seem to designate two different measurable numbers actually designate one and the same $\{$ that is to say, the problem of when two measurable number expressions can be said to be equal. The discussion of equality runs from §53 to §56 and, as we shall see in subsequent sections, is very consequential for one's interpretation and appraisal of Bolzano's theory of measurable numbers.

§53 begins with a remark that considers how the comparison between infinite number expressions A and $A \frac{1}{1 \ 1 \ 1 \ \dots \text{in inf.}}$ cannot be carried out with the

extant definitions of greater-than and equality, because for $B \dot{=} B^1$ to hold, say, the difference $B - B^1$ needs to be a positive rational number $\frac{m}{n}$ (cf. *RZ IV*, §5), whereas the difference between the two numbers at hand, that is $\frac{1}{\underbrace{1 \ 1 \ 1 \ \dots}_{\text{in inf.}}}$, is not a positive rational number. At the same time, $A \dot{=} A - \frac{1}{\underbrace{1 \ 1 \ 1 \ \dots}_{\text{in inf.}}}$ because $A - \frac{1}{\underbrace{1 \ 1 \ 1 \ \dots}_{\text{in inf.}}} \dot{=} A - 0$. Bolzano's conclusion from this is that he needs to define a relation that is something like a weaker version of equality for measurable numbers, so that cases like the above can be unambiguously be described as either two quantities such that one is greater than the other, or the two are in fact "equal". The following two paragraphs §54 and §55 contain in fact two proposals on how to define an equality-like relation that does the job for measurable numbers.

One proposal, that is to be found in §§54-55, is to say that $A \dot{=} B$ ¹³ whenever for any q the respective measuring fractions of A and B coincide. This is what Bolzano calls 'being equal in the process of measuring' (§54). The other proposal, that Berg and Russ say comes after chronologically but in the text appears before the one just given (in the same paragraph, §54), is to say that two measurable numbers count as equal or equivalent whenever the absolute value of their difference 'behaves as zero in the process of measuring'. This definition is motivated by the fact that pairs such as 1 and $1 - \frac{1}{\underbrace{1 \ 1 \ 1 \ \dots}_{\text{in inf.}}}$ may differ from one another in the process of measuring, and yet Bolzano would want to consider them as equivalent. Let us call this definition the 'small difference' definition of equality, and the other the 'equal measure' definition of equality. Two measurable numbers that are equivalent with respect to the equal measure definition are also equivalent with respect to the small difference definition, but not the other way around, hence it is important to keep the distinction between these two notions of equality in mind when examining results about measurable numbers that Bolzano proves from §56 onwards in *RZ VII* (we come back to this in Section 4.3.4). It has already been noted several times by Bolzano commentators (Berg in *RZ*, pp. 134-135, Rusnock 2000, p. 184, Russ and Trlifajova 2016, p. 44) that even though Bolzano does not explicitly prove that either of these two notions of equality are equivalence relations in the modern sense (namely, that they are reflexive, transitive and symmetric relations and therefore induce equivalence classes of the old measurable numbers) it is helpful to think about equality from §55 onwards as an equivalence relation between measurable numbers, and about measurable numbers from §55 onwards as classes of those measurable numbers from the first half of *RZ VII*.

If this suggestion feels like interpretive overreach, consider the corollary expressed in §59, that if $A; B; C; \dots$ are finitely many measurable numbers and each of them has a unique value, then so does their total sum (and product, and difference). This sounds a lot like the claim that the sum defined among the new measurable numbers is well-defined in the standard algebraic sense, that is, if

13. We are introducing a different symbol from the equality sign to signal that this is not straightforward equality, but Bolzano uses always the same symbol, ' $\dot{=}$ ', thus introducing some ambiguity.

$A < B$ and $C < D$ then the sum $A + C < B + D$ is uniquely determined up to the equivalence relation \sim .

The definition of equality from these paragraphs is what allows Bolzano to examine relations of order between measurable numbers (§§60-74), which in turn forms the basis for a basic theory of intervals of measurable numbers (§§75-90).

From §91 onwards, Bolzano starts making use more systematically of what he calls variable measurable numbers which can decrease indefinitely { where variable numbers are numbers that can assume several values, as defined in RZ I (RZ, p. 25). Variable numbers *that can decrease indefinitely* are introduced in RZ VI, §1 (RZ, pp. 94-96), and they are variable numbers that can always assume a smaller value than any positive nonzero number. We will come back to Bolzano's use of variable numbers in Section 4.4, but for now it is worth mentioning that, while the notion of variable quantity (or variable number) was not Bolzano's invention and was in relatively widespread use in the 18th and early 19th centuries (Rusnock and Kerr-Lawson 2005, p. 396), Bolzano's deliberate use to prove auxiliary results concerning single-valued (as opposed to variable) measurable numbers can be considered one of the most prominent differences between Bolzano's approach to measurable numbers and later attempts at defining the real numbers that came from Dedekind and others.

Bolzano's work on measurable numbers, including the variable measurable numbers, and the attending theory of intervals, allows him to give an updated proof of the Bolzano-Cauchy theorem (§107) and of the Intermediate Value Theorem (§109) that had already been proved in 1817 (RAB, §7 and §12, respectively). In Section 4.4 we will analyse §107 in more detail. In the final paragraphs (§§110-122) Bolzano starts sketching results about performing division between measurable numbers, but the text ends abruptly with an unfinished suggestion of amendment to the definition of measurable number, thus casting doubt over the status of the manuscript as a whole.

4.3.3 Interpretations and interpreters

Ever since Rychlík's first edition of Bolzano's theory of measurable numbers (Rychlík 1962), scholars have attempted to give a rational reconstruction of Bolzano's measurable number expressions in terms of sequences and series. We follow Russ and Trlifajova (2016) in calling all interpretations that reconstruct Bolzano's theory of measurable numbers as a theory about sequences of one kind or another *sequence interpretations*. To clarify what such interpretations have in common, in Section 4.3.3 we briefly go over the most notable ones and in Section 4.3.4 consider a few difficulties most interpreters raise in relation to RZ VII.

The dominant view: sequence interpretation

Already Rychlik's edition of Bolzano's text contained a sketched proposal of how to reinterpret Bolzano's theory into a sequence-based construction of the reals 'so that it may be corrected' (Rychlik 1962, pp. 96-97). The first extensive appraisal and rational reconstruction of Bolzano's theory however is due to van Rootselaar (1964), and it proceeds as follows. First, and following a suggestion of Riediger's (Rychlik 1962, p. 8) van Rootselaar interprets Bolzano's infinite number expressions as sequences of partial computations suggested by the expression used by Bolzano. For example, if Bolzano's infinite number expression is $1 \ 2 \ 3 \ 4 \ \dots \text{ in inf.}$ then van Rootselaar's proposal interprets it as equivalent to the sequence $\{t_{np} = \frac{np}{2}\}_{n \in \mathbb{N}}$. Among sequences though van Rootselaar needs to identify some that can capture the idea of 'purely positive' number expressions. He achieves this by defining a purely positive sequence to be a sequence such that all but finitely many of its terms are positive. More precisely, $\{a_n\}_{n \in \mathbb{N}}$ is purely positive if there is some $n \in \mathbb{N}$ such that for all $m \geq n$; $a_m \geq 0$ (van Rootselaar 1964, p. 171).

In particular, then, the purely positive expressions P^1 and P^2 that Bolzano uses to define a measurable number will be sequences that have co-finitely many positive terms (to mirror Bolzano's weaker condition that P^1 may also be zero, van Rootselaar (1964, p. 173) also allows that for a fixed but arbitrary q , P^1 may either be always zero, or there may be a term P_n^1 such that for all $m \geq n$, $P_m^1 \neq 0$). P^1 and P^2 though are not computed independently of the relevant measuring fraction. Rather, for each q and attending $p \in \mathbb{Q}$ (that is, for each measuring fraction $\frac{p}{q}$) there is a corresponding purely positive sequence P^1 , therefore indicated as P_q^1 . The 'measuring equations' given by Bolzano (two for each choice of q):

$$S = \frac{p}{q} P^1 + \frac{p-1}{q} P^2 \quad (4.1)$$

are to be replaced by infinitely many equations indexed by \mathbb{N} , one for each term of the sequence $s = \{s_n\}_{n \in \mathbb{N}}$ corresponding to Bolzano's infinite number expression S :

$$s_i = \frac{p}{q} P_{q,i}^1 + \frac{p-1}{q} P_{q,i}^2 \quad (4.2)$$

The operations are then defined the usual termwise way on the sequences. For example, the sum of two measurable number expressions $A = \{a_n\}$ and $B = \{b_n\}$ is the resulting sum $C = \{c_n\}$ where $c_i = a_i + b_i$ for all $i \in \mathbb{N}$. Van Rootselaar himself recognises that his interpretation is 'inadequate to recover several theorems of Bolzano's theory [...]' (van Rootselaar 1964, p. 176). This is not a problem however if the original proofs are themselves thought to be flawed, which is indeed van Rootselaar's line. Among the results his interpretation cannot recover are the closure under addition of measurable numbers (more on this in Section 4.3.4), and the result that the sum of two infinitely small numbers is still an infinitely

small number. Because of this, van Rootselaar considers Bolzano's theory of measurable numbers fundamentally flawed. Despite van Rootselaar's overall negative assessment of Bolzano's theory, his reconstruction of Bolzano's theory as a theory about sequences of rational numbers forms the backbone of much of the subsequent literature, even for authors who otherwise distance themselves from van Rootselaar's evaluation of Bolzano's contributions. It is to some of these further versions of the sequence interpretation that we now turn.

Laugwitz and Rusnock Two interpreters that accept van Rootselaar's interpretation as correct or plausible are Laugwitz (1965) and Rusnock (2000). Both Laugwitz and Rusnock depart from van Rootselaar's overall negative assessment of Bolzano's work and try to offer what they think are amendments to the definition of measurability that Bolzano himself could have come up with.

Laugwitz (1965) writes just after Rychlik's edition of Bolzano's writings on measurable numbers is published. He takes van Rootselaar's sequence interpretation as fundamentally correct, but remarks that, first of all, it is reductive to present Bolzano's text merely as a rudimentary attempt at defining the real numbers, because the scope of the booklet actually encompasses much more (namely, all those Bolzano calls in finite number expressions, including the ones that give rise to infinitely small and infinitely large numbers. Second, van Rootselaar is too fast in labelling Bolzano's a doomed attempt because, even though the definition of measurable number he gives (per Rychlik edition) cannot sustain the proof of a number of key results, it can be amended so that those results go through. By the time Rusnock contributes to the debate, Bolzano's text has been given a more careful edition thanks to the editors of the *BBGA*, edition which includes, unlike Rychlik's, a note in Bolzano's handwriting that proposes a different definition of measurability that is quite close to Laugwitz's.¹⁴ Rusnock's preferred fix, like Laugwitz's, consists in amending the definition of measurable number, but it also requires a modification of the definition of *infinitely small*. Rusnock's (2000, p. 185) proposal is to, first, define the absolute value of a (finite) number expression $A = \sum a_n u$ as the sequence $\{ |a_n| u \}$, and then to use this in an updated definition of infinitely small number concepts as

$$|A| = \frac{0}{q} \quad P^1 \quad \frac{1}{q} \quad P^2 \quad (4.3)$$

With $P^1 \neq 0; P^2 \neq 0$ as usual. This allows Rusnock to write $A = 0$ for $A = 0$ or A infinitesimal, and $A \neq 0$ if $A = 0$ or $A \neq 0$. Then the definition of a

14. Bolzano's amendment to the original definition of measurability is that instead of letting S be measurable only if for any q there is a p such that $\frac{p}{q} \in S \subseteq \frac{p+1}{q}$, the second inequation can be weakened to $S \subseteq \frac{p+n}{q}$, for $n \neq 1$. Laugwitz's proposal is to let $n = 2$ (Laugwitz 1965, p. 407).

measurable number (concept) becomes

$$S = \left\{ \frac{p}{q} \mid P^1 \leq \frac{p}{q} \leq P^2 \right\} \quad (4.4)$$

for $P^1 \geq 0$ (unlike in Bolzano's original definition) and $P^2 > 0$ (like in Bolzano's original definition). This succeeds to maintain the 'old definition of equivalence', that is, that two measurable numbers are equivalent iff their measuring fractions coincide for all denominators q , as well as Bolzano's theorem that $A \approx B$ for ϵ infinitely small, A measurable (RZ VII §57).

Interval Interpretation

The last interpretation we want to consider here is due to Russ and Trlifajova (2016). While the authors seem to stress the continuity of their proposal with the sequence interpretation, we think it may be helpful to actually pay attention to the ways in which their interpretation actually differs from the sequence interpretation. As Russ and Trlifajova themselves write:

The common approach is that of *partial computation* [...]. Another approach is to begin from a concept like $\sqrt{2}$, or a rational like $\frac{2}{3}$, for either of which we may derive an algorithm, or a decimal expansion, which will allow us to generate approximating intervals.

To carry out the example of $\sqrt{2}$, if we want to show that $\sqrt{2}$ is a measurable number we need to show that for any q there is a p such that $\frac{p}{q} < \sqrt{2} < \frac{p+1}{q}$. So for $q = 1$, $1 < \sqrt{2} < 2$, for $q = 2$, $\frac{2}{2} < \sqrt{2} < \frac{3}{2}$, for $q = 3$, $\frac{4}{3} < \sqrt{2} < \frac{5}{3}$, and so on. Notice that each choice of q determines an interval that approximates the value we want to characterise (in this case, $\sqrt{2}$), even though the intervals themselves are not nested. The interval for $q = 3$, for instance, is not included in the interval for $q = 2$. The interval sequence however displays a property similar to that of directed sets in topology (which Russ and Trlifajova (2016, p. 43) call being 'dually directed'), namely for any two intervals I_i, I_j such that $i < j$ there is a third interval I_k such that $I_k \subset I_i \times I_j$; $i, j < k$. A 'dually directed' sequence of intervals can then be reduced to a subsequence that is actually nested, thus creating a representation of Bolzano's measurable numbers into nested interval sequences. Russ and Trlifajova (2016, pp. 53-54) appeal to the work of Mainzer (1991) to claim that such interval sequences can be used to retrieve a construction of the real numbers as equivalence classes of nested intervals, thus showing how an 'interval interpretation' of Bolzano's measurable numbers can still prove them to be anticipating a rigorous (in the AA2 sense) construction of the reals after all.

4.3.4 Three problems

Despite the decades of scholarship poured over Bolzano's measurable numbers, there remain still quite a few points where it is unclear whether Bolzano's attempt

at a rigorous treatment of irrational numbers has been successful. We will focus now on three of these that we think are the most significant for our purposes to establishing how successful an arithmetization Bolzano's measurable numbers exemplify.

Addition One of the problems first highlighted by Bolzano scholars, starting with van Rootselaar and Berg (in van Rootselaar 1964, pp. 175-176 and RZ, p. 122, respectively) is that Bolzano's claim that the sum of any two measurable numbers is still measurable seems to be false. To illustrate this, van Rootselaar gives the example of two measurable numbers (in his sequence interpretation) A and B defined as the sequences of partial computations $A = \{a_n\}_n, B = \{b_n\}_n$ where $a_n = \frac{1}{n}, b_{2n-1} = \frac{1}{2n}; b_{2n} = \frac{1}{2n-1}$. Then the sum $C = \{c_n\}_n$ is the result of adding A and B termwise, that is C is defined by cases as: $c_{2n-1} = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)}$, $c_{2n} = \frac{1}{2n} + \frac{1}{2n-1} = \frac{1}{2n(2n-1)}$.

It should be clear then that the sequence $\{c_n\}_n$ oscillates around 0, that is, it converges to zero but no two consecutive terms of the sequence have the same sign (+ or -). This means that, for any given $q \in \mathbb{N}$, it is impossible to find a pair of a strictly positive or zero P_q^1 and a strictly positive P_q^2 such that $C = \frac{p}{q} P_q^1 + \frac{p-1}{q} P_q^2$. Let us illustrate with a toy computation. First, let us compute c_1 and c_2 :

$$c_1 = \frac{1}{2-1} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

$$c_2 = \frac{1}{2} + \frac{1}{2-1} = \frac{1}{2} + 1 = \frac{3}{2}$$

Now let us $x = q = 2$ and write the measuring equations for the $i = 1; 2$ for $C; P_2^1$ and P_2^2 :

$$c_1 = \frac{1}{2} - \frac{p}{2} = P_{2,1}^1 - \frac{p-1}{2} = P_{2,1}^2 \tag{4.5}$$

$$c_2 = \frac{1}{2} + \frac{p}{2} = P_{2,2}^1 + \frac{p-1}{2} = P_{2,2}^2 \tag{4.6}$$

By definition of measurability, the p in Eq. (4.5) and Eq. (4.6) needs to be the same. Then the only two options are $p = 0$ or $p = 1$ (this is true for all terms of C and for all choices of q , because C converges to 0). If $p = 0$ then we can make Eq. (4.5) true with $P_{2,1}^1 = \frac{1}{2}; P_{2,1}^2 = 0$, but Eq. (4.6) is going to require $P_{2,2}^1 = \frac{1}{2}$, which is negative, so P_2^1 cannot be strictly positive. Similarly for $p = 1$ we obtain $P_{2,1}^1 = 1$ and $P_{2,1}^2 = \frac{1}{2}$ from Eq. (4.5). This pattern repeats itself, which means that regardless of whether we choose $p = 0$ or $p = 1$ we will always have infinitely many negative terms for P_2^1 or P_2^2 , which means that at least one of them cannot be strictly positive (for a full proof to complement our illustration, see van Rootselaar 1964, p. 175). So, C cannot be measurable, hence there seems to be a problem with Bolzano's claim that the sum of any two measurable numbers is still measurable.

Equality A second significant issue is Bolzano's two definitions of equality for the measurable numbers. As we have already mentioned, the two definitions are not incompatible since 'equal measure' implies 'small difference', but they are also not equivalent. There can be measurable numbers that differ only by an infinitely small number, so they are equal according to the 'small difference' definition of equality, and yet they behave differently in the process of measuring, so they are not equal according to the 'equal measure' definition. Bolzano's own example is that of $\frac{3}{5}$ and $\frac{3}{5} - \frac{1}{1 \cdot 1 \cdot 1 \cdots \text{in inf.}}$. For, whenever we take $q = 5n$, $\frac{3}{5}$ is perfectly measurable (in Bolzano's technical sense) with respect to that q , but $\frac{3}{5} - \frac{1}{1 \cdot 1 \cdot 1 \cdots \text{in inf.}}$ is not. Notably, it is only under the 'small difference' definition of equality that all infinitely small numbers turn out to be equal to zero, and this is because the 'small difference' definition, but not the 'equal measure' definition, does not discriminate between positive and negative infinitely small numbers, whereas the 'equal measure' definition does. Positive infinitely small numbers do always have the same measuring fractions as zero, but negative ones do not. As a consequence of this discrepancy between the two definitions of equality, Bolzano's corollary §57 that for J infinitely small, $J = 0$ is true only with the 'small difference' definition of equality, but not with the equal measure one because if J is a negative infinitely small number then its measuring fraction is not the same as that of zero, hence $J \neq 0$ (cf. Berg in *RZ*, p. 136). In a sense then depending on which definition of equality we take to be Bolzano's authentic one we may or may not believe that Bolzano's measurable numbers also comprise something akin to infinitesimals. Bolzano's stance on infinitesimals is however crucial if we want to evaluate to what extent his measurable numbers constitute an anticipation of standard real numbers.

§§107,109 Still regarding the extent to which Bolzano's measurable numbers can be said to anticipate later attempts at arithmetical constructions of the real numbers, the last interpretive issue we want to raise concerns §§107 and 109 of *RZ* VII. It is known that these are paragraphs that restate and reprove results which Bolzano first proved in (*RAB*, §§7, 12). The first question this raises is why Bolzano would feel the need to reproduce the same results twice, and why in this text about infinite number expressions. The early proofs contained in *RAB* are quite universally recognised to have flaws, if of varying degrees (cf. Rusnock 2000, pp. 69-84). Spalt (1991, p. 66) is, to our knowledge, the first commentator to explicitly hypothesize that the whole point of *RZ* VII was to fix those flaws. Rusnock (2000, p. 70) shares a similar view of Bolzano's intentions when he writes:

Bolzano ran into difficulties in his attempt to prove the sufficiency of the "Cauchy" criterion for the convergence of a sequence (the Bolzano-Cauchy Theorem). Only later, in the 1830's, would he come to grips with the problems involved, developing a theory of real (or, as he called them, "measurable") quantities and sorting out some of the conceptual

problems in which his first proof had become tangled.

And then on the same topic on p. 186:

The proof of the Bolzano-Cauchy theorem offered in the theory of measurable numbers indicates that Bolzano had a very clear idea of how his definition of measurability supported completeness.

Rusnock also writes that, while he cannot fully agree with the stronger views expressed by others that Bolzano's problems in *RAB* are due to a lack of a theory of real numbers at the time of writing, and that *RZ VII* was written 'primarily to fill this gap', as (Spalt 1991, p. 66) suggests, he nevertheless agrees that the proofs of the Cauchy-Bolzano criterion of convergence and the attending least upper bound property for the measurable numbers represent a significant improvement on the 1817 version of the proofs of the same results (Rusnock 2000, p. 188).¹⁵ To sum up then, the question of why Bolzano reproves his main theorems from *RAB* has been partially addressed in the literature already. What we think deserves further attention however is whether Bolzano's reproving of those results in *RZ VII* supports or weakens the case that Bolzano's measurable numbers represent a further step towards the arithmetization of analysis that we discussed at length in Section 4.2. It is to this question, namely to what extent Bolzano's theory can be said to be an instance of arithmetization of analysis, that we now turn to.

4.4 Measurable numbers and arithmetization

Having summarised Bolzano's *RZ VII* and the usual ways it is interpreted we are in a position to address the question we raised in Section 4.1, namely in which ways Bolzano's theory of measurable numbers is an instance of arithmetization of analysis. We will use the Gandon-inspired framework of Section 4.2 to argue that it is the very wish to read Bolzano's theory as an instance of arithmetization in the sense of AA3 that motivates the sequence interpretation. As a consequence, Bolzano's text appears to come short time and again of proving the theorems it is supposed to prove or of providing the conceptual clarity it is meant to provide on what measurability is and similar important notions. On the other hand, one can choose to take Bolzano's framework at face value, without trying to interpret his measurable numbers as infinite (converging) sequences of rational numbers or intervals. This allows for a more charitable and more insightful reading of Bolzano's text, a reading which allows us to pinpoint more precisely where Bolzano's approach resembles an arithmetization, and where it does not. We will focus on three topics that have already been mentioned in Section 4.3.4: Bolzano's claim that the sum of two measurable numbers is still measurable, the two different generalised definitions of equality, and Bolzano's theorem in §107.

15. In Rusnock's words: '[O]n the essential point of conceptual structure, Bolzano was almost entirely successful in characterizing the reals'.

4.4.1 Closure under addition

Van Rootselaar argued that the sum of two measurable numbers is not always itself a measurable number because, by taking two sequences which, in van Rootselaar's formalisation of Bolzano's work, model measurable numbers, the result of termwise adding those two sequences is not a measurable number. We now face a dilemma. Either van Rootselaar's interpretation is misleading, and the sequences he has chosen to model Bolzano's measurable numbers are inadequate, or Bolzano's proof does not go through. To solve this dilemma, let us look at Bolzano's proof closely.

He considers two measurable numbers A , B with measuring equations

$$A \quad \frac{p_1}{q} \quad P^1 \quad \frac{p_1 - 1}{q} \quad P^2 \quad (4.7a)$$

$$B \quad \frac{p_2}{q} \quad P^3 \quad \frac{p_2 - 1}{q} \quad P^4 \quad (4.7b)$$

From a previous corollary we know that besides A and B themselves, also $P^1; P^2; P^3$ and P^4 are measurable numbers. This means that for any $n \in \mathbb{N}$ one can write the equations

$$P^1 \quad \frac{r_1}{nq} \quad P^5 \quad \frac{r_1 - 1}{nq} \quad P^6 \quad (4.8a)$$

$$P^3 \quad \frac{r_2}{nq} \quad P^7 \quad \frac{r_2 - 1}{nq} \quad P^8 \quad (4.8b)$$

$$P^2 \quad \frac{s_1}{nq} \quad P^9 \quad \frac{s_1 - 1}{nq} \quad P^{10} \quad (4.8c)$$

$$P^4 \quad \frac{s_2}{nq} \quad P^{11} \quad \frac{s_2 - 1}{nq} \quad P^{12} \quad (4.8d)$$

Where $r_1; r_2; s_1; s_2$ are parametrized by n .

If we let $P^{13} : P^5 - P^7$ and $P^{14} : P^9 - P^{11}$ then we can write $A - B$ as

$$A - B \quad \frac{p_1 - p_2}{q} \quad \frac{r_1 - r_2}{nq} \quad P^{13} \quad \frac{p_1 - p_2 - 2}{q} \quad \frac{s_1 - s_2}{nq} \quad P^{14} \quad (4.9)$$

From these equations it follows that $A - B$ lies between $\frac{p_1 - p_2}{q}$ and $\frac{p_1 - p_2 - 2}{q}$. In the remainder of §45 then Bolzano argues that there can be two cases: either $A - B$ is measurable but not rational, in which case its measuring fraction is $\frac{p_1 - p_2}{q}$ or $\frac{p_1 - p_2 - 1}{q}$, or it is rational (i.e. it is perfectly measurable) and it is actually equal to $\frac{p_1 - p_2 - 1}{q}$ itself.

The first option is easily established once one examines the case in which $\frac{r_1 - r_2}{nq} \neq \frac{1}{q}$ or $\frac{s_1 - s_2}{nq} \neq \frac{1}{q}$. The second option requires a more convoluted argument to be proved, therefore this is the one we will focus on here.

By adding up Eq. (4.8a)-Eq. (4.8d), Bolzano obtains

$$P^1 \quad P^2 \quad P^3 \quad P^4 \quad \frac{r_1 \quad r_2 \quad s_1 \quad s_2}{nq} \quad P^5 \quad P^7 \quad P^9 \quad P^{11} \quad (4.10)$$

$$\frac{r_1 \quad r_2 \quad s_1 \quad s_2 \quad 4}{nq} \quad P^6 \quad P^8 \quad P^{10} \quad P^{12} \quad q \quad (4.11)$$

(He shortens $P^5 \quad P^7 \quad P^9 \quad P^{11}$ to P^{19} and $P^6 \quad P^8 \quad P^{10} \quad P^{12}$ to P^{20}). By solving Eq. (4.8b) and Eq. (4.8d) for $P^1 \quad P^2$ and $P^3 \quad P^4$ he obtains that

$$P^1 \quad P^2 \quad P^3 \quad P^4 \quad \frac{2}{q} \quad (4.12)$$

Thus by substituting (4.12) into (4.10) and (4.11) in turn, he obtains eventually that:

$$P^{19} \quad P^{20} \quad \frac{4}{nq} \quad (4.13)$$

Since $\frac{4}{nq}$ decreases as n increases, this allows Bolzano to conclude that P^{19} and P^{20} { themselves parametrised by n { 'decrease indefinitely with the infinite increase of n '. Therefore also $\frac{r_1 \quad r_2 \quad s_1 \quad s_2}{nq}$, which previous steps in the proof have shown to be equal to $\frac{2}{q} \quad P^{19}$, 'approaches $\frac{2}{q}$ indefinitely' the more n increases.

Now recall that we are working under the assumption that $\frac{r_1 \quad r_2}{nq} \quad \frac{1}{q}$ and $\frac{s_1 \quad s_2}{nq} \quad \frac{1}{q}$. Because neither $\frac{r_1 \quad r_2}{nq}$ nor $\frac{s_1 \quad s_2}{nq}$ can get $\neq \frac{1}{q}$, each of them must approach $\frac{1}{q}$ indefinitely as n increases:

$$\frac{r_1 \quad r_2}{nq} \quad \frac{1}{q} \quad \epsilon_1 \quad (4.14a)$$

$$\frac{s_1 \quad s_2}{nq} \quad \frac{1}{q} \quad \epsilon_2 \quad (4.14b)$$

Where ϵ_1 and ϵ_2 are, as usual, indefinitely decreasing variable rational numbers (cf. *RZ* VI, §7). Recall now the equations we could write for $A \quad B$ (Eq. (4.9)). They can now be rewritten as

$$A \quad B \quad \frac{p_1 \quad p_2}{q} \quad \frac{1}{q} \quad \epsilon_1 \quad P^{13} \quad \frac{p_1 \quad p_2 \quad 2}{q} \quad \frac{1}{q} \quad \epsilon_2 \quad P^{14} \quad (4.15)$$

At this point Bolzano performs what may look like a slight of hand: he argues that, because ϵ_1 and ϵ_2 decrease indefinitely, their sum is actually 0. Since from the equation above also $P^{13} \quad P^{14} \quad \epsilon_1 \quad \epsilon_2$, $P^{13} \quad P^{14} \quad 0$ as well. This allows him to conclude that, for $\frac{r_1 \quad r_2}{nq} \quad \frac{1}{q}$ and $\frac{s_1 \quad s_2}{nq} \quad \frac{1}{q}$, $A \quad B \quad \frac{p_1 \quad p_2 \quad 1}{q}$.

The result that $A \quad B \quad \frac{p_1 \quad p_2 \quad 1}{q}$ might seem suspicious, firstly because of the auxiliary claim that $\epsilon_1 \quad \epsilon_2 \quad 0$. Bolzano however does not just state the equality

as a matter of course, but appeals to a theorem from *RZ VI*, §8 according to which the algebraic sum of finitely many variable *rational* numbers that decrease indefinitely either is itself a variable number that decreases indefinitely, or it is zero. It is true that Bolzano has not given a full argument for why, if $M; N$ are rational numbers such that $M > N$, $!$, $!$ an indefinitely decreasing rational number, then $M > N$. Such a result though seems consistent with Bolzano's thinking in §§53-55, so its lack of a proof seems more of an oversight than an argumentative gap in the theorem of §45. The problem is though that in the case at hand our M is $A - B$, and $A - B$ is not assumed to be rational. Thus our Bolzano-inspired principle that if $M > N$, $!$ then $M > N$ as rational numbers does not seem to help the proof. Without that, the proof does seem to be incomplete.

Let us be clear though that this is a much lighter fault than what the literature has otherwise hinted at. Virtually no commentators engage directly with Bolzano's proof.¹⁶ The closest we come to an investigation and evaluation of Bolzano's proof is in (Russ and Trlifajova 2016) and (Spalt 1991). Spalt notes that the counterexample provided by the likes of van Rootselaar (1964) depend on the sequence interpretation itself, so they do not provide a direct rejection of Bolzano's own argument for the closure of addition. Surprisingly though even Spalt does not engage with Bolzano's own proof from §45, preferring to provide a Bolzanian argument for the closure of addition, argument that Spalt himself then deems circular. So if we want an analysis of Bolzano's proof we need to look elsewhere.

Russ and Trlifajova (2016) note the problem that is the source of the claim that Bolzano's measurable numbers are not closed under addition, namely that sequences whose partial sums are non-monotonic cannot be considered measurable even when they are converging to a real or rational number. If one holds the view that non-monotonic converging series are not measurable, then it seems that it is possible to produce counterexamples to the closure of addition (within the sequence interpretation). A second option suggested by Russ and Trlifajova (2016, p. 50) is that such series are indeed measurable { but then a modification of the definition of measurability is required, possibly along the lines of Laugwitz's suggestion (Laugwitz 1965). The third and last option they entertain is that Bolzano 'had a different concept of an infinite calculation' { different, that is, from the one that is implicit in the sequence interpretation. We agree with Russ and Trlifajova on this point, but we do consider it possible to make sense of the concluding steps of Bolzano's proof (the ones we just commented on in this section) if one is willing to accept his appeal to variable quantities as a coherent argumentative move.¹⁷

In conclusion, then, the challenge posed by the sequence interpreters to the soundness of §45 appears to be unfounded. First, the counterexamples to closure

16. van Rootselaar (1964) might be considered an exception in principle, but the proof he discusses is not the proof we see in the *BBGA*, rather Rychlík's incomplete rendition of the passage in question (Rychlík 1962).

17. We say something more on the legitimacy of Bolzano's variable quantities in footnote 21.

of addition rely on a specific way of understanding Bolzano's infinite number expressions, as Russ and Trlifajova (2016) also seem to acknowledge. Second, once one considers Bolzano's proof on its own merits, the overall argumentative structure turns out to be solid { for someone who accepts variable quantities as legitimate mathematical objects (we will come back to this in Section 4.4.4).

4.4.2 Equality and equivalence

In this section we move on to the next interpretive challenge, namely, Bolzano's definition of equality for measurable numbers. In §53 Bolzano presents the work on equality (*Gleichheit*) he is about to carry out in §§54-55 by noting that '[i]t is nevertheless desirable that we obtain concepts in several of the cases (if not all) where our previous concepts have not been adequate.' He sets himself the goal of 'extending' the concepts of equality and order so that they can also be applied to infinite number expressions.

We already mentioned that the text we have access to thanks to the efforts of the *BBGA* editors contains two different definitions of this generalisation of equality for infinite number expressions, one which we called the 'small difference' definition, the other 'equal measure'. Since our primary goal is to assess whether Bolzano's theory of measurable numbers represents some kind of arithmetization of analysis, more than to establish which of his theorems are right and which ones are wrong, we will not examine which of Bolzano's theorems hold only with one but not the other definition of equality. Rather, we intend to investigate whether either of Bolzano's definitions can support the thesis that Bolzano ultimately constructs the measurable numbers as equivalence classes induced by either of the equivalence relations defined in §§54-55.

It has in fact been suggested (Russ 2004, p. 349 and Rusnock 2000, p. 184, for example) that Bolzano's generalisation of equality is motivated by an attempt at defining equivalence classes of measurable numbers, in the naïve sense of defining a relation of partial identity between two numbers such that, under certain conditions, they can be considered as of the same value and thus identified. Indeed, some of what he writes supports this reading:

A and *B* are here called equal to one another in the sense that both have the same properties, and that their difference [...] has equal characteristics in the process of measuring to those of zero. (Russ 2004, p. 391, §54)

Is what Bolzano writes here enough to claim that his construction *reduces* real numbers to equivalence classes of measurable numbers? Before we answer this question, we have to clarify the connection between this question about reduction and question of when an AA3-arithmetization occurs. To do that, it is helpful to revisit the connection between Cantor's (and, to a lesser extent, Dedekind's) real numbers and arithmetization.

Cantor-Dedekind real numbers Hallett (1984, p. 30) mentions that Cantor's and Dedekind's presentations of real numbers are both meant to 'free analysis from formal reliance on geometrical or spatial intuition' (thus, theirs is an arithmetization in the AA1 sense), and the main conceptual tool that allows them to do so is to think of the real numbers essentially as collections (more specifically, equivalence classes) obtained from rational numbers: 'whichever definition one follows, each real number is either *itself* a completed infinite domain of a certain kind (an equivalence class of sequences of rational numbers or a segment of the rational numbers), or is a primitive term defined by reference to such a domain' (Hallett 1984, *ibid.*). Hallett also argues that part of the Cantor-Dedekind approach to real numbers is a certain *reduction* of the real numbers to sets (that is, equivalence classes, intervals of rational numbers, or sequences of rationals), though this reduction does not necessarily imply an ontological reduction { all it requires is that the real numbers be 'something defined in terms of rationals by appeal to the notion of collection' (Hallett 1984, p. 31). Hallett's reduction corresponds in our framework to AA3. To sum up then, Hallett is claiming that Cantor and Dedekind's arithmetization is primarily of the AA1 kind, and the way they achieve it is by an arithmetization in the sense of AA3. The AA3-arithmetization is in turn obtained by (a) providing equivalence classes of objects built out of rational numbers which (b) are essentially identified with real numbers, that is, to provide an equivalence class is to provide a real number (hence the reduction Hallett mentions).

We can now return to Bolzano and ask whether his generalisation(s) of equality in §§54-55 actually mean that Bolzano also carries out an arithmetization in the AA3-sense through the (a)-(b) route just mentioned. We are going to argue that, if Cantor can be said to have introduced equivalence classes, then so can Bolzano, so (a) is fulfilled, but the case for (b) is trickier and will be left for Section 4.4.3.

The argument for (a) is an indirect argument that goes through the comparison of Bolzano's measurable numbers with Cantor's introduction of the B -domain in his (Cantor 1872, pp. 92-93). Cantor (1872) associates a 'symbol' a to each Cauchy sequence $t_{a,n}$ of his construction ostensibly for the purpose of showing that he is treating each (equivalence class of) sequence(s) as one object in the number domain he is defining (Dauben 1990, pp. 37-38). Bolzano's generalised definitions of equality both yield equivalence relations (cf. Berg in *RZ*, p. 135), and while it is true that he does not explicitly mention that all measurable number expressions that are equal under either of his generalised definitions actually represent/are one and the same measurable number, neither does Cantor (1872). Same for checking that the relation in question is an equivalence relation (that is, reflexive, symmetric and transitive): Bolzano does not prove that, but neither does Cantor. Yet, in the case of Cantor, it is widely accepted to present his approach as involving an identification of real numbers with equivalence classes of Cauchy sequences, even though Cantor does not explicitly present his own work that way. So if we are justified in saying that Cantor's approach relies on equivalence classes, we are

justified in saying the same about Bolzano's.

This is not enough to conclude that Bolzano identifies or defines the measurable numbers as equivalence classes, because we have not shown that (b) Bolzano's approach consists in constructing the measurable numbers or, as Dedekind would have it, creating them, from the rational numbers. Whether Bolzano's approach amounts to a construction of the real numbers from the natural numbers is not an easy issue to solve, and as already mentioned, it is the pivot question for how we interpret Bolzano's proof in *RZ* VII §107. In the next section, we illustrate what options there are for an answer to it.

4.4.3 Cauchy convergence criterion and *RZ* VII §107

RZ VII §107 presents us with a second proof of the sufficiency of a criterion for convergence of infinite sequences, result which Bolzano already attempted to prove in (*RAB*, §5). This result is sometimes called the Bolzano-Cauchy theorem (Rusnock 2000; Russ 2004; Russ and Trlifajova 2016) because Bolzano's criterion can be interpreted as an early formulation of the Cauchy criterion for convergent sequences, though recent work has put into question such an identification between Bolzano's criterion and Cauchy's (Fuentes Guillen 2021). Here we will not enter the fray of the debate, instead we will focus on the stated result and the way Bolzano proves it.

The statement of the theorem reads as follows (in Russ's (2004) translation):

Suppose the infinitely many measurable numbers $X^1; X^2; X^3; \dots; X^n; \dots; X^{n+r}; \dots$, which we can consider as the terms of an infinitely continuing series distinguished by the indices $1; 2; 3; \dots; n; \dots; n+r; \dots$, proceed according to such a rule that the difference between the n th term and the $n+r$ th term of the series [*Reihe*], i.e. $(X^{n+r} - X^n)$, considered in its absolute value, always remains, however large the number r is taken, smaller than a certain fraction $\frac{1}{N}$ which itself can become as small as we please, providing the number n has first been taken large enough. Then I claim that there is always one and only one single measurable number A , of which it can be said that the terms of our series approach it indefinitely, i.e. that the difference $A - X^n$ or $A - X^{n+r}$ decreases indefinitely in its absolute value merely through the increase of n or r .

Let us analyse the statement of the theorem. First we are given what the theorem is about { a series of infinitely many measurable numbers that satisfies a certain property, namely, that for any positive integer N there is an n such that for all r , $|X^{n+r} - X^n| < \frac{1}{N}$ ($n; r$ also positive integers). This is the property that is usually identified with the Cauchy property of infinite series. What Bolzano wants to prove is that a series that is such-and-such always has a measurable number A that it approaches indefinitely. The reader's expectation is then that

Bolzano's proof needs to first establish the existence of A , and then prove that A is measurable. To test whether this expectation is met, let us go over Bolzano's proof.

Proof

The proof is carried out in detail only for the case where $X^1; X^2; X^3; \dots$ is strictly increasing { the cases of strictly decreasing and non-monotonic series are briefly sketched at the end of the proof and we do not need to worry about those for our purposes. Bolzano distinguishes several cases depending on whether, for q arbitrary but fixed, the equation $X^n = \frac{p}{q}$ holds for some p (where n is such that, given any r , $|X^{n+r} - X^n| < \frac{1}{q}$). If yes then **Case 1** 'there is no doubt that the equations $A = \frac{p}{q} = P^1 = \frac{p-1}{q} = P^2$ can also be asserted'.

If no (**Case 2**), since X^n is measurable by assumption we know that there must still be such that $\frac{p}{q} < X^n < \frac{p+1}{q}$. Since the series is monotonic and increasing by assumption, the question then becomes whether $X^{n+r} < \frac{p+1}{q}$ for any $n+r > n$ (**Case 2.1**) or not (**Case 2.2**). If it is the case that $X^{n+r} < \frac{p+1}{q}$ for any $n+r > n$, there can be two scenarios: either the difference $\frac{p+1}{q} - X^{n+r}$ decreases indefinitely as X^{n+r} increases (**Case 2.1.1**), or not.

If it does not decrease (**Case 2.1.2**) then Bolzano says that the difference remains always greater than a certain number, in which case it is still true that $\frac{p}{q}$ is the measuring fraction for A and $A = \frac{p}{q} = P^3 = \frac{p-1}{q} = P^4$. If instead we are in **Case 2.1.1** then the series actually approaches indefinitely $\frac{p+1}{q}$ itself, so $A = \frac{p+1}{q}$.

What happens though in **Case 2.2**, when there is some r such that $X^{n+r} \geq \frac{p+1}{q}$? If X^{n+r} is equal to $\frac{p+1}{q}$ then, since the series is strictly increasing, there will be $n+r > n$ such that $X^{n+r} > \frac{p+1}{q}$. Because of the property the series has by assumption, for all $n+r > n$ we have $X^{n+r} > X^n + \frac{1}{q}$. Therefore, $X^{n+r} > \frac{p}{q} + \frac{1}{q} = X^n$. Moreover, since $X^n < \frac{p+1}{q}$, a fortiori $X^{n+r} < \frac{p+2}{q}$. Then the measuring fraction for A is $\frac{p+1}{q}$. In all possible cases then A is measurable. Can there be though another measurable number B such that $X^1; X^2; X^3; \dots$ approaches it indefinitely, and $A \neq B$? Bolzano does not think so, and here is his proof. Suppose there is such a B . Then, in virtue of what it means to approach a number indefinitely, for a certain n we would have the following two equations: $A = X^n + !_1$ and $B = X^n + !_2$, where $!_1$ and $!_2$ are two indefinitely decreasing variable measurable numbers. Then, by subtracting one equation from the other we obtain $A - B = !_1 - !_2$. By RZ VII §91, $!_1 - !_2$ is itself either an indefinitely decreasing number or zero, and by §92¹⁸ we can conclude that $A = B$ after all. So A is indeed unique.

18. The theorem in §92 reads as follows:

If A and B denote a pair of measurable numbers which remain unchanged, while the measurable numbers $!_1$ and $!_2$ decrease indefinitely and the equation $A = !_1 + B = !_2 + B$ is always to hold, then it must be $A = B$.

Commentary

Rusnock (2000) mentions two putative faults with this proof. First, that Bolzano never proves that A exists, but actually his proof presumes the existence of A and this makes it circular. Second, that Bolzano does not check that the measuring fractions for A are 'compatible' (Rusnock 2000, p. 188), that is to say, Bolzano does not check that the method he outlines to determine a numerator p for any given q determines a sequence of approximating intervals S_i such that the intersection

S_i is non-empty. While we believe that it is possible to show that Bolzano's method does yield 'compatible' measuring fractions, the fact that Bolzano did not spot the need to verify what Rusnock calls the compatibility of the measuring fractions for A does nevertheless constitute a flaw.

As for the charge that Bolzano's proof does not establish the existence of A though it ought to, and otherwise the proof of §107 is circular (the first putative fault), Rusnock himself does not think it to be a convincing charge because 'a careful study of his [Bolzano's] language shows that he uses A throughout in a hypothetical sense { i.e., he says in effect, that if there is a limit A , then A will be such and such.' (Rusnock 2000, p. 188) In other words, Rusnock believes that Bolzano's proof establishes the existence of A in a non-circular way, because the proof has this 'hypothetical' structure he mentions. We agree with Rusnock that a careful read of the text shows that Bolzano uses the fact that A should be the limit of the sequence to argue that it should be measured by his proposed measuring fraction in several instances, hence it is not circular, but we disagree on whether Bolzano should have established the existence of A , not just its measurability. In short, there seem to be two possible readings for §107: either the theorem of §107 makes an existence claim, and therefore requires an existence proof (Rusnock's reading), or §107 is merely saying that if a series exists such that it approximates something indefinitely, that something has to be a measurable number (our reading). Let us now explore each option in turn.

Suppose the first reading is correct. Then the goal of Bolzano's proof is indeed to construct or create a measurable number from its measuring fractions, and §107 supports the hypothesis first raised in Section 4.4.2 that Bolzano does try to reduce certain quantities (the ones we would call real numbers) to his measurable numbers. Under this reading, Bolzano would be trying to achieve an arithmetization in the sense of AA3. The proponent of this reading, such as a sequence interpreter, is then forced to bite the bullet and admit that Bolzano's proof, while ambitious in its reductionist aims, is not rigorous, because it does not successfully prove the existence of a limit A for any sequence of measurable numbers satisfying the Cauchy-Bolzano criterion (Rusnock 2000, p. 188 does admit as much: 'Bolzano's proof lacks a demonstration [...] that the number A [...] is indeed the limit of the sequence'). Bolzano's proof successfully provides an algorithm

to compute a measuring fraction for A for any q . But, as Rusnock already noticed, if A is not assumed to exist already, then providing a measuring fraction for each q is not enough to warrant A 's existence. The fractions need to be compatible.

The second reading, that Bolzano never intended for §107 to comprise an existence proof, rather to establish that, if a series of measurable numbers approximates something indefinitely, then it has to be a measurable number, proves more charitable. This way we are not charging Bolzano with having missed a key portion of his own argumentative goal, and we honour the fact that the text itself shows no trace of an attempt at an existence proof. But if the proof was never meant to show that the limit of a sequence of measurable numbers can itself be reduced to something that is constructed out of natural numbers and arithmetical operations, to an arithmetical object, then Bolzano's proof cannot be said to be an instance of the arithmetization of analysis in the sense of AA3. At the same time, Bolzano's proof still relies on algebraic arguments about equations, and without the expectation that the proof should establish the existence of something out of natural numbers and arithmetical operations, Bolzano's proof can be regarded as rigorous again, thus fulfilling the description of arithmetization in the sense of AA2.

4.4.4 Bolzano and the arithmetization of analysis

We began this section wanting to investigate whether Bolzano exhibited some of the hallmarks of arithmetization, per Section 4.2, and our analysis seems to diverge somewhat from the consensus in the literature. Regarding the closure of the measurable numbers under addition, we have argued that, taken on its own terms, it does not show the problems previous scholarship suggests. Of course it is the 'taken on its own terms' that does the heavy lifting here, because it means not to presume that there is a correspondence between Bolzano's infinite number expressions { or at least measurable number expressions { and infinite (converging) sequences. If we limit ourselves to seeing measurable numbers as those objects that follow all and only those rules that Bolzano outlines for them, both explicitly through definitions and theorems, and implicitly through his proof and argument techniques, then Bolzano's proof in §45 is correct. Part of accepting Bolzano's proof and argument techniques though is also to accept his appeal to 'variable quantities/numbers', and this is perhaps where some want to argue that such a notion is itself a hallmark of a non-rigorous proof. Here is where our AA-distinctions about arithmetization bear fruit, though, because they allow us to argue that Bolzano's appeal to variable quantities does not endanger rigour per se (there are no gaps in the proof), but it is a clear violation of AA3, because the notion of a variable quantity is not defined merely as the result of applying standard arithmetical operations to the natural numbers. Something else also

needs to be in place to account for the 'variability' of variable quantities.¹⁹

We reach the same conclusion { Bolzano does not achieve arithmetization of the AA3 kind { by analysing Bolzano's (putative) use of equivalence classes to (re)define the measurable numbers from §54 onwards (Section 4.4.2) and the two horns of the dilemma regarding §107 (Section 4.4.3). This indicates that Bolzano's arithmetization is not of the same kind as what the sequence and interval interpreters seem to have presumed so far. Although van Rootselaar's interpretation of Bolzano's measurable numbers via sequences is what has become the blueprint for subsequent sequence interpretations, Rychlík had sketched it first (Rychlík 1962, pp. 96-99). Rychlík's explicit goal was to thus use the Cantorian theory of the real numbers to give a corrected version of Bolzano's own. The difference between the two would just be terminological.²⁰ It should be no surprise then that interpretations which use a technique meant to highlight the similarities between Bolzano's and Cantor's (and also Dedekind, if one believes Dedekind's reals to be relevantly similar to Cantor's) ends up making Bolzano's approach also conceptually close to Cantor! Going back to the text, however, as we have striven to do, shows that this is too close for comfort. The reduction that philosophers such as Hallett (1984) attribute to Cantor cannot be attributed to Bolzano, but this is what a sequence interpretation encourages us to do, unwittingly. Ultimately, though, we have hopefully corralled enough evidence to show why seeing an AA3-style arithmetization in Bolzano is perhaps misguided.

4.5 Conclusion

Part of our claim from Section 4.4.4 is that Bolzano's arithmetization is not an AA3 one, but it is an AA2-arithmetization. What this means in terms of our original question of housing Bolzano within Epple's framework is that it can be housed within the arithmetizing camp, because Bolzano's arguments do have a claim to be considered, for the most part, gapless. However, Bolzano's is not 'a strictly *arithmetical* construction of real numbers' (Epple 2003, p. 292), if such a construction requires a full reduction of traditional notions such as variable quantities and limits of converging sequences to arithmetical or set-theoretic ones.²¹

19. See footnote 21.

20. 'Es würde sich also darum handeln, für die Bolzanosche TRZ [Theorie der reellen Zahlen], eine solche Deutung zu suchen, durch die soviel als möglich von ihr gerettet werden könnte. Diese Möglichkeit bietet die Cantorsche (Cantor-Meraysche) TRZ. Im Grunde genommen entsteht die so korrigierte Bolzanosche TRZ aus der Cantorschen nur durch eine Abänderung der Terminologie.' (Rychlík 1962, p. 96).

21. One could make the case that, in keeping with his foundational project (cf. Section 2.4, and the literature quoted in footnote 19 on page 36 in that section), Bolzano attempts to achieve a reduction of sorts to his logic or at least 'general mathematics' (*allgemeine Grössenlehre*), because it should be possible to give a rigorous definition of variable quantity within Bolzano's logic by appealing to his notion of variation. Since both logic and general mathematics are more

Bolzano's measurable numbers are thus an attempt at a rigorous treatment of the real numbers that falls in between Epple's arithmetization pole and the traditional one: on the one hand, they achieve an AA2-arithmetization, but on the other, Bolzano does not do away with variable quantities and his proof of the completeness of the system of measurable numbers requires the existence of limits as quantities. We feel that this analysis is the one that does the most justice both to Bolzano's conceptual contributions { the measurable numbers do constitute an intriguing historical attempt at making certain quantities treatable within the confines of mathematics as Bolzano knew it { and to his mathematical skills, because our face-value reading does not lead us to see mistakes where there aren't any.

We began this chapter with the goal of locating Bolzano's rightful place within the 'conceptual triangle' delineated by Epple. We have argued that, if one distinguishes different strains of arithmetization as we do in Section 4.2, Bolzano can be shown to be arithmetizer without sacrificing his strong commitment to rigour, the way other interpretations do. We have also argued that it is plausibly due to an attempt to show Bolzano as a contributor to the arithmetization of analysis in the sense of AA3 that the sequence interpretation, despite the rather uncharitable reading it entails, has been so influential for so long in studies of Bolzano's *RZ VII*. Our approach may depart somewhat from the usual, but we think it actually opens up the opportunity of truly understanding and appreciating Bolzano's measurable numbers as part of his broader attitudes towards mathematics and the place in finite quantities occupy therein, and it helps us assess more accurately the way Bolzano's measurable numbers contribute to the arithmetization of analysis.

general than just arithmetic, from Bolzano's point of view this would be an even more desirable reduction. This would also dovetail well with claims already made in the literature on Bolzano's interest in purity as an ideal of proof (see e.g. Detlefsen 2008; Centrone 2016).

Chapter 5

The Mathematical Infinite¹

5.1 Introduction

One of Bolzano's more famous writings is a booklet his pupil Pr honsky published under the title *Paradoxien des Unendlichen* (from now on *PU* for short), *Paradoxes of the Infinite*. Likely contributing to its fame, this booklet was read and referred to by both Cantor and Dedekind. Perhaps because of this association, the booklet is also routinely interpreted as a text anticipating several ideas of Cantor's transfinite set theory (cf. Berg 1962, 1992; Sebestik 1992; Rusnock 2000), especially in sections §§29-33, in which Bolzano sketches a 'calculation of the infinite'. As a consequence, appraisal of the *PU* is almost exclusively conducted in terms of how much Bolzano's work on the infinite agrees with later developments in set theory. In particular, many shortcomings of Bolzano's calculation of the infinite are attributed to his adherence to the *part-whole principle*:

PW1 For any sets $A; B$, if $A \subset B$, then $\text{size}(A) < \text{size}(B)$.²

In the case of infinite sets, it is well-known that this principle contradicts the *bijection principle*, according to which the existence of a one-to-one correspondence between two sets is a necessary and sufficient condition for the equality of their sizes. One *locus classicus* for the tension between these two principles is the seventeenth-century dialogue of Galileo's *Discourses and Mathematical Demonstrations Relating to Two New Sciences* (Galileo 1958, pp. 44-45), which we have already encountered in Chapter 3 but is worth summarising again here. The characters debate among themselves the example of the set of natural numbers \mathbb{N} , which can be put into one-to-one correspondence with the proper subset \mathbb{N}^{p2q} of square natural numbers. Thus \mathbb{N} and \mathbb{N}^{p2q} have the same size according to the bijection principle, while the size of \mathbb{N}^{p2q} is strictly smaller than that of \mathbb{N}

1. This chapter has been published as (Bellomo and Massas 2021).

2. This is the same principle as **PW_{sets}** from Chapter 3.

according to **PW1**. While Bolzano is commonly taken to have adopted **PW1** in the *PU*, Cantor successfully founded his theory of powers and cardinal numbers on the bijection principle. Thus, as long as Cantor's way out of Galileo's Paradox is perceived as the 'right' way to compute the size of infinite collections, Bolzano's alternative can only be seen as an intriguing yet fundamentally flawed attempt.

This privileged status of the bijection principle however has started to be scrutinised in recent years thanks to a renewed interest in potential alternatives to Cantor's theory of the mathematical infinite. In particular, Mancosu (2009) shows that there is a long historical tradition of thinkers and mathematicians who favoured **PW1** over the bijection principle, and that recent mathematical developments in (Benci and Di Nasso 2003) establish that a consistent theory of the sizes of infinite collections can be founded on **PW1** rather than on the bijection principle. This theory, called the theory of *numerosities*, is a refinement of the Cantorian theory of cardinals that allows for two sets A and B to be considered of different sizes even in the presence of a bijection between the two (Benci and Di Nasso 2003, p. 51). This directly contradicts the claim that Cantor's theory is the only viable theory of the infinite, and thus calls for a reappraisal of alternative theories that until recently had been dismissed as essentially misguided or inconsistent.

Our main goal is to offer such a reappraisal of Bolzano's mature theory of the mathematical infinite. In particular, we propose an interpretation of Bolzano's calculation of the infinite in §§29-33 of the *PU* which stresses its conceptual and mathematical independence from set theory proper, and argue that Bolzano is more interested in developing a theory of infinite sums rather than a way of measuring the sizes of infinite collections. This leads us to reassess the role that part-whole reasoning plays in Bolzano's computations and to provide a formal reconstruction of his position that underscores its coherence and originality, and is overall a more charitable appreciation of Bolzano's ideas on the infinite. In particular, we show that Bolzanian sums in our interpretation form a non-commutative ordered ring, a well-behaved algebraic structure that nonetheless vastly differs from Cantorian cardinalities.

We proceed as follows. In Section 5.2 we discuss several sources of what we call the received view of the *PU*, and introduce enough background to set the stage for our novel interpretation. In Section 5.3 we focus on Bolzano's calculation of the infinite and argue that his work is best understood as a theory of infinite sums. This leads in Sections 5.4 and 5.5 to a formal reconstruction of Bolzano's computations with infinite quantities, which aims to establish both the consistency and the originality of his position. Finally, in Section 5.6, we recap the main points of our formalisation and discuss its implications for the interpretation of the *PU*. To improve readability, in this chapter we have opted to translate Bolzano's Menge as 'multitude', Vielheit as 'plurality', Summe as 'sum' and Reihe as 'series'.

5.2 The received view on the *PU*

Bolzano's *PU* is a short yet ambitious booklet in which the author aims to show that, when properly defined and handled, the concept of the infinite is not intrinsically contradictory, and many paradoxes having to do with the infinite in mathematics (but also in physics and metaphysics) can actually be solved. In the course of addressing the paradoxes of the infinite in mathematics, Bolzano develops what looks like a theory of transfinite quantities (§§28-29, 32-33), which is what commentators tend to focus on when appraising the contents of the *PU*.

One such commentator is, as is known (Sebestik 1992; Rusnock 2000; Ferreiros 2007b), Cantor (1883). He introduces Bolzano as a proponent of actual infinity, and specifically actually infinite numbers in mathematics, in contrast to Leibniz's arguments against infinite numbers:

Still, the actual infinite such as we confront for example in the well-defined point sets or in the constitution of bodies out of point-like atoms [...] has found its most authoritative defender in Bernard Bolzano, one of the most perceptive philosophers and mathematicians of our century, who has developed his views on the topic in the beautiful and rich script *Paradoxes of the Infinite*, Leipzig 1851. The aim is to prove how the contradictions of the infinite sought for by the sceptics and peripatetics of all times do not exist at all, as soon as one makes the not always quite easy effort of taking into account the concepts of the infinite according to their true content. (Cantor 1883 in Cantor 1932, p. 179)³

And still:

Bolzano is perhaps the only one who confers a certain status to actually infinite numbers, or at least they are often mentioned [by him]; nevertheless I completely and wholly disagree with the way in which he handles them, not being able to formulate a proper definition thereof,

3. In this and all other cases for which a published English translation is not cited, the translations are ours. Original German:

Doch den entschiedensten Verteidiger hat das Eigentlich-unendliche, wie es uns beispielsweise in den wohldefinierten Punktmengen oder in der Konstitution der Körper aus punktuellen Atomen [...] entgegentritt, in einem höchst scharfsinnigen Philosophen und Mathematiker unseres Jahrhunderts, in Bernard Bolzano gefunden, der seine betrefsenden Ansichten namentlich in der schönen und gehaltreichen Schrift: „Paradoxien des Unendlichen, Leipzig 1851“ entwickelt hat, deren Zweck es ist, nachzuweisen, wie die von Skeptikern und Peripatetikern *aller Zeiten* im Unendlichen gesuchten Widersprüche gar nicht vorhanden sind, sobald man sich nur die freilich nicht immer ganz leichte Mühe nimmt, die Unendlichkeitsbegriffe allen Ernstes ihrem wahren Inhalte nach in sich aufzunehmen.

and I consider for instance §§29-33 of that book as untenable and wrong. For a genuine definition of actually infinite numbers, the author is lacking both the general concept of *power*, and the accurate concept of *number*. It is true that the seeds of both notions appear in a few places in the form of special cases, but it seems to me he does not work his way through to full clarity and distinction, and this explains several contradictions and even a few mistakes of this worthwhile script. (ibid., p. 180)⁴

Cantor's comments in many ways set the tone of how the *PU* are mainly perceived even today, namely as a rich and interesting essay that nevertheless displays some serious shortcomings. Cantor diagnoses Bolzano's mistakes as being fundamentally due to an imprecise characterization of *power* and *number*. Without entering a discussion on Cantorian powers, it is useful for us to notice how Cantor is readily reinterpreting Bolzano's text in the light of his own research. The concept and terminology of *powers* was Cantor's own, which he introduced starting from 1878 in his papers. What Cantor means is that Bolzano did not have the right notion of size for infinite sets, the right notion being Cantor's own powers, and this shortcoming causes Bolzano to go astray in §§29-33. Another aspect of Cantor's comments on the *PU* which we want to stress is that Cantor straightforwardly presents Bolzano's 'calculation of the infinite' (*Rechnung des Unendlichen*, §28) as a version of his own transfinite arithmetic, albeit imprecise and imperfect.

All commentaries on the *PU* we were able to find seem to follow suit from Cantor in that they evaluate and interpret the *PU*, and §§29-33 in particular, against the backdrop of the development of set theory. Thus Bolzano's *PU* are about infinite sets according to editors and translators of Bolzano's text (e.g. Hans Hahn in Bolzano 1920, Donald Steele in Bolzano 1950), as well as scholars such as Berg (1992, 1962), Sebestik (1992, 2017), Lapointe (2011), Ferreiros (2007b) and Rusnock (2000). We now examine the most informative of these interpretations in some detail.

Among Bolzano scholars, Jan Berg is perhaps the one that embraces a set theoretic reading of Bolzano with the most conviction. Berg (1962, p. 176) writes:

4. Bolzano ist vielleicht der einzige, bei dem die eigentlich-unendlichen Zahlen zu einem gewissen Rechte kommen, wenigstens ist von ihnen vielfach die Rede; doch stimme ich gerade in der Art, wie er mit ihnen umgeht, ohne eine rechte Definition von ihnen aufstellen zu können, ganz und gar *nicht* mit ihm überein und sehe beispielsweise die §§29-33 jenes Buches als haltlos und irrig an. Es fehlt dem Autor zur wirklichen Begriffsfassung bestimmt-unendlicher Zahlen sowohl der allgemeine *Mächtigkeit*begriff, wie auch der präzise *Anzahl*begriff. Beide treten zwar an einzelnen Stellen ihrem Keime nach in Form von Spezialitäten bei ihm auf, er arbeitet sich aber dabei zu der vollen Klarheit und Bestimmtheit, wie mir scheint, *nicht* durch, und daraus erklären sich viele Inkonsistenzen und selbst manche Irrtümer dieser wertvollen Schrift.

In *PU* [...] Bolzano repudiates the notion of equivalence as sufficient condition for the identity of powers of infinite sets. [...] As a result, a number of statements follow which do not correspond to Cantor's view on this subject. E.g. if ' N_0 ' denotes the number of natural numbers (PU 45) [§29; Berg refers to the page of the 1851 edition], then in the series: $N_0; N_0^2; N_0^3; \dots$ each N_0^m is said to 'exceed infinitely' the preceding term N_0^{m-1} (PU 46) [§29]. But Bolzano's comparison of the powers of infinite sets is impossible to understand, since nowhere does he offer any clear sufficient condition for the equinumerousness of infinite sets.

Berg makes the same points as Cantor, namely that Bolzano's writings in *PU* are about the *powers* of infinite sets, and that his reasoning is impossible to follow as he does not offer sufficient conditions for the equality of size of sets. However Berg (see, for instance, his 1962, p. 177) remains convinced that a letter⁵ written by Bolzano in the last year of his life witnesses a change of heart regarding how infinite sets should be compared, moving from his rejection of one-to-one correspondence to an acceptance of it as a sufficient criterion for size equality.

On the heels of this interpretation, Berg (1992, pp. 42-43) sketches what he takes to be Bolzano's theory of the infinite. In a nutshell, Berg believes that any two infinite sets of natural numbers are of the same size according to Bolzano just in case 'the members are related to each other by infinitely many rational operations (addition, multiplication and their inverses)' (Berg 1992). Even though Berg does not use this terminology, his interpretation seems to suggest that \mathbb{N} is equinumerous with an infinite subset $S \subset \mathbb{N}$ whenever the bijection $f: S \rightarrow \mathbb{N}$ is primitive recursive. This is an interesting suggestion, but it would imply that, for example, $\mathbb{N} \setminus \{1\}$ and \mathbb{N} are equinumerous, while this seems to contradict Bolzano's reasoning in *PU* §29 (see Section 5.3 below). Moreover, Berg's interpretation of the letter is far from uncontroversial (see Rusnock 2000, pp. 194-195, Sebestik 1992, pp. 469-470, to be discussed below, and Mancosu 2009, 2016), and elsewhere in this dissertation we have already offered an alternative reading (Chapter 3).

A more nuanced view is offered by Sebestik (1992, pp. 435-473). When presenting the contribution of Bolzano's *PU*, Sebestik summarises it thus:

For the first time, the actual infinite, whose properties cease to be contradictory to simply become paradoxical, is admitted in mathematics as a well-defined concept, having a referent and only attaching to those objects capable of enumeration or measurement, that is, to sets and quantities.⁶ (Sebestik 1992, p. 435)

5. This letter, dated 9 March 1848 and intended for Bolzano's former pupil Robert Zimmermann, has been published in (*BBGA* 2A/12.2, pp. 187-189). This is the letter we discuss in Chapter 3.

6. Original French: Pour la première fois, l'infini actuel dont les propriétés cessent d'être

Sebestik also interprets the *PU* as about sets and their being infinite. Even though at p. 445 he more faithfully writes that 'the infinite is first and foremost a property of pluralities [our emphasis]',⁷ on p. 462 he then reverts to set talk at a crucial point, namely when giving his interpretation of *PU* §33:

[Referring to §33] It is the first and last time within the *Paradoxes of the Infinite* that Bolzano deduces from the reflexivity of the set of natural numbers to the equality of number between a set and one of its proper subsets.⁸

According to Sebestik's interpretation then, and unlike Berg's, it is not quite the case that Bolzano changed his mind regarding what criterion to use to compare the size of infinite sets *after* the *PU* and just before his death. Rather, Bolzano's views in the *PU* itself are already inconsistent, because at various points in the text Bolzano either implicitly or explicitly endorses the following views:

1. The part-whole principle, that is, the whole is greater than any of its proper parts.
2. All infinite sets can be put in one-to-one correspondence with any of their infinite subsets.
3. Every set has a definite size.
4. If two sets are in one-to-one correspondence then they have the same plurality.

It is quite telling that for 1, 3, and 4 Sebestik (1992, pp. 463-464) feels the need to add set theoretic glosses, so that 1 becomes ' $\text{card}A < \text{card}B$ if A is equivalent to a proper part of B ' (' $\text{card}A < \text{card}B$ si et seulement si A est equivalent a une partie propre de B '), 3 is 'Every set has a unique cardinal number' ('nombre cardinal unique') and 4 'If two sets are in one-to-one correspondence then they have the same cardinal number' ('ont le même nombre cardinal').

Thus formulated, 1-4 do indeed yield a contradiction. Consider any two infinite sets A and B such that A is a proper part of B . By 3, they each have a unique cardinality, and by 1 $\text{card}A < \text{card}B$. But also, since A and B can be put into one-to-one correspondence (by 2), they have the same cardinality, by 4, so $\text{card}A = \text{card}B$, contradicting our earlier deduction that $\text{card}A < \text{card}B$. We will give our argument as per why Sebestik's contradiction does not go through

cotradictories pour devenir simplement paradoxales, est admis en mathématiques a titre de concept de *ni*, ayant une référence et attache aux seuls objets susceptibles de dénombrement ou de mesure, c'est-à-dire aux ensembles et aux grandeurs.

7. Original: 'L'infini est d'abord et avant tout une propriété des multitudes'.

8. C'est pour la première et dernière fois que, dans les *Paradoxes de l'Infini*, Bolzano conclut de la réflexivité de l'ensemble des nombres naturels à l'égalité numérique entre un ensemble et l'un de ses sous-ensembles propres.

in Section 5.3, where we highlight that a crucial ingredient in this family of counterexamples to Bolzano's claim to internal consistency in the *PU* is largely due to the set theoretic interpretation of 4.

The last interpretation we want to consider in detail is Rusnock's (2000). Rusnock (2000, p. 193) writes that in §§21-22 Bolzano 'apparently based this opinion [of the insufficiency of one-to-one correspondence for equality of size] on considerations involving parts and wholes, assuming perhaps that the multiplicity of the whole must be greater than those of its parts.' (Rusnock translates with 'multiplicity' what we, following (Russ 2004), translate as 'plurality', namely *Vielheit*.) Rusnock then continues:

But this seems to be a mistake, even in Bolzano's own terms. For his sets (*Mengen*) are by definition invariant under rearrangements of their members, and thus the appeal to the "mode of determination" seems to be illegitimate in this context. (Rusnock 2000, *ibid.*)

Rusnock then produces an example to show why Bolzano is mistaken by his own lights when embracing 'considerations of parts and whole'. Consider the straight line abc , where a is to the left of b and b is to the left of c ; call A the set of points between a and b , B the set of points between a and c . Then it is possible to map each point of A to a point of B via a translation map that is also a one-to-one correspondence. Since a translation map only 'rearranges' points from one region of space to another, then B is just a rearrangement of A . Thus, A and B should be the same 'set', since Bolzano's definition of 'set' (*Menge*) entails that something considered as a 'set' is invariant under rearrangement of parts. Yet, because A is a proper part of B , A should be strictly smaller than B , in virtue of what from now on we call 'the part-whole principle': The whole is greater than any of its proper parts. This principle then is inconsistent with Bolzano's own definition of multitude.

It is not warranted however that an example such as Rusnock's really counts as a rearrangement of parts on Bolzano's terms, essentially because it relies on a metaphorical use of the term 'rearrangement' in a geometric context. This metaphorical use in turn suggests conceiving of geometric figures (points and lines) as objects that move through the two-dimensional (Euclidean) space. Yet Bolzano famously rejected metaphorical talk of motion in mathematical contexts (*RAB*, Introduction), and lacking that, we are not sure there is a way of rephrasing Rusnock's example so that it really counts as a rearrangement of parts on Bolzano's terms.

On the basis of our overview, we can now distil the received view about the *PU* into two theses:

(Sets) In §§29-33, Bolzano is concerned with determining size relationships between infinite sets.

(Set-PW) Bolzano's computations in §§29-33 are, at least partially, motivated by the part-whole principle for sets.

As we have seen above, the combination of these two theses motivates a reading of Bolzano's calculation of the infinite as a pre-Cantorian transfinite arithmetic that is either mistaken or downright inconsistent because of its adherence to the part-whole principle. As it will soon become apparent, we believe however that both theses incorrectly describe §§29-33 of the *PU*. Our main claim is that the standard view's identification of Bolzanian collections with the modern notion of set, and of all instances of part-whole reasoning in the *PU* to **PW1**, is too quick. Discussing the standard interpretation of Bolzano's calculation of the infinite therefore requires a clarification of the status of collections in the *PU*, and an assessment of the role that part-whole reasoning plays in Bolzano's arguments. We will take those two issues in turn. First, we briefly recap once again (cf. Section 2.2), the various notions of collections that Bolzano introduces at the beginning of the *PU*, and explain the role they play in his definition of the infinite. Second, we review sections §§20-24 of the *PU*, in which Bolzano is usually interpreted as rejecting the bijection principle in favour of something like **PW1**. We believe this will provide the reader with the necessary background for our in-depth discussion of §§29-33 in Section 5.3.

5.2.1 Bolzano's collections, multitudes, and sums

Bolzano's first goal in the *PU* is to arrive at a rigorous definition of the infinite. To that end, he relies on his logical system first developed in his *Wissenschaftslehre* (*Theory of Science*, *WL* for short). In particular, Bolzano devotes the first section of the *PU* to defining several distinct notions of collection. Without going into too much detail, we summarise here the most important definitions.

Collection The concept of collection (*Inbegri*) applies to any and all objects which are made of parts, i.e. that are not simple. In that sense, [collection] is the most general concept as it applies to any composite object. Collections, as opposed to units (*Einheiten*, sometimes also translated as unity/unities), can be decomposed into simpler parts. Anything that is made of at least two parts is a collection. (see *PU* §3)

Multitude The concept of multitude (*Menge*) is best illustrated with a slight modification of Bolzano's own example of a drinking glass (*PU* §6). Consider the glass as intact, and then as shattered into pieces. What changes between these two states of the glass is the arrangement (*Anordnung*) of the pieces, although the amount of glass is the same before and after. When we consider the glass as that which remains unchanged before and after the breakage, we are considering it as a multitude. A collection which we put under a concept so that the arrangement of its parts is unimportant (in which therefore

nothing essential changes for us if we merely change this arrangement) I call a multitude.' (PU §4)⁹

Plurality When the parts of a multitude all fall under the same concept A and are therefore considered as units of kind A (i.e. simple objects of kind A), that multitude is called a plurality (*Vielheit*) of kind A . (ibid.)

Sum A sum (*Summe*) is a collection such that (a) its parts can also be collections, and (b) the parts of its parts can be considered as parts of the whole sum, without the sum itself having changed (PU §5). Consider the glass example again. Suppose we break our glass G and it shatters in exactly three pieces, $a; b$ and c . Then suppose a breaks also into two pieces a_1 and a_2 . Then our glass G , considered as a sum, is still the same: $G = a + b + c = a_1 + a_2 + b + c$.

Quantity Bolzano defines a quantity (*Größe*) as an object that can be considered of a kind A such that any two objects $M; N$ of kind A satisfy a certain law of trichotomy (not Bolzano's expression): either they are equal to one another ($M = N$) or 'one of them presents itself as a sum which includes a part equal to the other one' (PU §6), that is to say, $M = N + X$ or $N = M + X$. The remaining parts X ; themselves also need to satisfy the condition that, for any other X of kind A , either $X = Y$ ($X = Y$, respectively) or one of them can be presented as a sum of which the other is just a part.

To avoid any confusion, it should be noted that the concepts of multitudes, pluralities, sums and quantities are specifications of the concept of collections, and the same object can be conceptualized as more than one kind of collection at once, as we have argued in Chapters 2 and 3 following Krickel (1995). Quantities are a great example. From their definition, it is clear that anything that is a quantity is also a plurality, because a quantity is a multitude (of a certain kind, say A) whose parts are also objects of kind A . At the same time, the way Bolzano expresses the trichotomy law holding of relationships between quantities suggests that a quantity is also a sum, namely, an object such that the parts of its immediate parts are also parts of the object itself, and nothing about the object changes if we consider it as made of the parts of its parts, instead of just of its own immediate parts.

Moreover, the existence of various notions of collections in Bolzano's framework is at odds with the thesis (**Sets**) of the received view, according to which Bolzano tries to develop an arithmetic of infinite sets. Indeed, in Chapter 2 we have already defended the view that neither Bolzano's multitudes, nor his sums or his pluralities can be identified with sets. Since multitudes, pluralities and sums are the infinite collections Bolzano concerns himself with, the identification of his infinite collections with Cantorian infinite sets is unwarranted and far from obvious.

9. Translations of Bolzano's *PU* are always from (Russ 2004).

Nevertheless, (**Sets**) might gain some traction from the fact that Bolzano's definition of the infinite only applies to collections, or, more precisely, to pluralities, and per our argument in Section 2.3, these are the Bolzanian collections that come closest to being just sets:

[...] I shall call a plurality which is greater than every finite one, i.e. a plurality which has the property that every finite multitude represents only a part of it, an *infinite plurality*. (*PU* §9)

However, the choice of defining an infinite *plurality* as opposed to simply infinity is justified in §10, where Bolzano argues that in the use made by mathematicians, 'the infinite' is always an infinite plurality:

Therefore it [is] only a question of whether through a mere definition of what is called an infinite plurality we are in a position to determine what is [the nature of] the infinite in general. This would be the case if it should prove that, strictly speaking, there is nothing other than pluralities to which the concept of infinity may be applied in its true meaning, i.e. if it should prove that infinity is really only a property of a plurality or that everything which we have defined as infinite is only called so because, and in so far as, we discover a property in it which can be regarded as an infinite plurality. Now it seems to me that is really the case. The mathematician obviously never uses this word in any other sense. For generally it is nearly always quantities with whose determination he is occupied and for which he makes use of the assumption of one of those of the same kind for the unit, and then of the concept of a number. (*PU* §10)

Bolzano's target when defining infinity solely as the attribute of certain collections are the imprecise definitions of infinity given by some philosophers (Hegel and his followers are cited explicitly here) who consider the mathematical infinity Bolzano talks about to be the 'bad' kind (*PU* §11), while the one true infinity is God's absolute infinity. The strategy to push against this *qualitative infinite* of the philosophers is to show that, even in the case of God, who is the unity *par excellence*, when we assign infinity to Him as one of His attributes, what we are really saying is that some other attribute of His has an infinite multitude as a component.

What I do not concede is merely that the philosopher may know an object on which he is justified in conferring the predicate of being infinite without first having identified in some respect an infinite magnitude [*Größe*] or plurality in this object. If I can prove that even in God as that being which we consider as the most perfect unity, viewpoints can be identified from which we see in him an infinite

plurality, and that it is only from these viewpoints that we attribute infinity to him, then it will hardly be necessary to demonstrate further that similar considerations underlie all other cases where the concept of infinity is well justified. Now I say we call God infinite because we concede to him powers of more than one kind that have an infinite magnitude. Thus we must attribute to him a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths because all truths in general etc. (*PU* §11)

With that, Bolzano considers himself to have exhaustively argued for his definition of mathematical infinity as being inextricable from the concepts of plurality and quantity and inapplicable to the one-ness of any unity, even God. Thus, we conclude that Bolzano's insistence on defining only an infinite plurality does not lend particular credence to (**Sets**) after all. Bolzano's definition unequivocally makes of infinity a quantifying attribute which, as such, can only apply to pluralities and quantities. But his insistence on discussing only infinite pluralities should be understood as in contrast with the Hegelian infinite as an attribute of a single infinite being. Talking about infinite collections, for Bolzano, is a way of clearly setting apart the *quantitative* infinite he is interested in from the qualitative infinite of the Hegelians.¹⁰

5.2.2 Bolzano's commitment to part-whole in the *PU*

As the discussion of the received view on the *PU* made clear, one point of contention in interpreting Bolzano's work on the infinite is whether (and to what extent) the principles that guide his computations with infinite quantities mirror those later used by Cantor. While part-whole considerations play an important role in Bolzano's *WL* (in particular, §102 therein; cf. Mancosu 2016, pp. 130-131, Mancosu 2009, pp. 624-625, see also Chapter 3), the discussion in Berg and Sebestik's interpretations has brought to light the issue of whether, on the whole, Bolzano's treatment of infinite quantities in the *PU* obeys the part-whole principle or not. Setting aside the issue of whether an adoption of one-to-one correspondence is implicit in Bolzano's §33 (something we will come back to in Section 5.3), here we review §§20-24, which are usually taken to be Bolzano's discussion of one-to-one correspondence as an insufficient criterion for size equality of infinite collections on the grounds of part-whole considerations.

Let us note first that some form of part-whole reasoning seems to be present in the very notion of 'being greater/smaller than' employed in the *PU*, as this passage from §19 witnesses:

10. Cf. the discussion of Bolzano's definition of infinity in Section 2.4.3, which emphasises how Bolzano's collections are what enables him to give a definition that is better suited to mathematicians' goals.

Even with the examples of the infinite considered so far it could not escape our notice that not all infinite multitudes are to be regarded as equal to one another in respect of their plurality, but that some of them are greater (or smaller) than others, i.e. another multitude is contained as a part in one multitude (or on the contrary one multitude occurs in another as a mere part). (*PU* §19)

Here, Bolzano glosses the claim that some multitudes are greater than others as some containing others as a part. A similar use of the part-whole principle is to be found in §20, when Bolzano compares the size of the collection of quantities smaller than 5 and the size of the collection of those smaller than 12:

If we take two arbitrary (abstract) quantities, e.g. 5 and 12, then it is clear that the multitude of quantities which there are between zero and 5 (or which are smaller than 5) is infinite, likewise also the multitude of quantities which are smaller than 12 is infinite. And equally certainly the latter multitude is greater since the former is indisputably only a part of it. (*PU* §20)

This suggests that Bolzano's writings commit him to upholding the part-whole principle even when it comes to the comparison of infinite quantities, because the principle is part and parcel of the definition of the order relation among quantities.

Having thus established Bolzano's commitment to part-whole, let us also show his explicit rejection of what nowadays we call one-to-one correspondence as a sufficient criterion for equality of size for infinite collections:

I claim that two multitudes, that are both infinite, can stand in such a relationship to each other that, on the one hand, it is possible to combine each thing belonging to one multitude, with a thing of the other multitude, into a pair, with the result that no single thing in both multitudes remains without connection to a pair, and no single thing appears in two or more pairs, and also, on the other hand it is possible that one of these multitudes contains the other in itself as a mere part, so that the pluralities which they represent if we consider the members of them all as equal, i.e. as units, have the most varied relationships to one another. (*PU* §20)

In the quote above, Bolzano remarks that it is possible for two infinite multitudes to both be in a one-to-one correspondence with each other and be related as a part to its whole. This state of affairs can have the appearance of a paradox, because in the infinite case checking whether two multitudes can be put into one-to-one correspondence suffices to determine whether they have the same number of terms, whereas the part-whole relation implies that one multitude must be greater than the other. Bolzano insists that the part-whole relation is what determines the greater-than relation, too:

Therefore merely for the reason that two multitudes A and B stand in such a relation to one another that to every part a occurring in one of them A , we can seek out according to a certain rule, a part b occurring in B , with the result that all the pairs $pa - bq$ which we form in this way contain everything which occurs in A or B and contains each thing only once | merely from this circumstance we can | as we see | in no way conclude that these two multitudes are equal to one another if they are infinite with respect to the plurality of their parts (i.e. if we disregard all differences between them). But rather they are able, in spite of that relationship between them that is the same for both of them, to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. (*PU* §21)

This consideration is illustrated in the preceding §20 by way of two examples, or, two versions of the same example, which considers the two intervals $p0;5q$ and $p0;12q$ on the real line and concludes that, since $p0;5q$ is only a part of $p0;12q$, $p0;12q$ contains more quantities (or more points) than $p0;5q$.

The reason why one has to drop the apparently successful one-to-one correspondence criterion when considering infinite quantities is that what makes one-to-one correspondence work in the finite case is precisely that one has to do with finite collections; hence at some point the process of pairing off each element from the collection with a natural number stops, whereas in the infinite case there is no last element, so the pairing-off never ends. Hence the need for a different criterion for size comparison (*PU* §22). Bolzano gives a brief explanation of how one-to-one correspondence does not suffice to reach conclusions regarding comparisons of infinite sums in §24:

[From the proposition of §20] follows as the next consequence of it that we may not immediately put equal to one another, two sums of quantities which are equal to one another pair-wise (i.e. every one from one with every one from the other), if their multitude is infinite, unless we have convinced ourselves that the infinite plurality of these quantities in both sums is the same. That the summands determine their sums, and that therefore equal summands also give equal sums, is indeed completely indisputable, and holds not only if the multitude of these summands is finite but also if it is infinite. But because there are different infinite multitudes, in the latter case it must also be proved that the infinite multitude of these summands in the one sum is exactly the same as in the other. But by our proposition it is in no way sufficient, to be able to conclude this, if in some way one can discover for every term occurring in one sum, another equal to it in the other sum. Instead this can only be concluded with certainty if both multitudes have the same basis for their determination. (*PU* §24)

Bolzano considers here the case of a one-to-one correspondence between the terms of two infinite sums S_1 and S_2 that would map each term in S_1 to an equal term in S_2 . Since the existence of a one-to-one correspondence is not enough to guarantee that S_1 and S_2 have the same number of terms, one cannot conclude that S_1 and S_2 are equal, unless the two sums also have the same 'basis for their determination'. This phrase does not have, to our knowledge, a standard interpretation in Bolzanian scholarship. Sebestik (1992, p. 460) does attempt an explanation of what the 'determining elements' (*bestimmende Stücke*) of an object can be, according to Bolzano. However, we are not convinced that the explanation offered there extends to a notion of determination for mathematical entities. For now, we simply draw the reader's attention to the fact that Bolzano concludes his discussion of the one-to-one correspondence criterion with a methodological point about infinite sums which plays a crucial role in §32 and §33 (see Section 5.3.2 and Section 5.3.3 below).

To sum up, in this section we have presented what we take to be the received view on Bolzano's calculation of the infinite, and shown that it relies on the two theses (**Sets**) and (**Set-PW**). We have argued that the existence of various notions of collections in Bolzano's framework puts some pressure on (**Sets**), as it does not seem obvious that any of Bolzano's notions closely matches our modern notion of set. Regarding (**Set-PW**) we have shown how Bolzano appeals in §§20-24 to part-whole reasoning in the context of determining size relationships between certain infinite collections. However, we also noted that, by §24, Bolzano has pivoted from discussing sufficient criteria for the equality of size of two infinite collections to discussing sufficient criteria for the equality of two infinite sums. As we will argue in the next section, this is a crucial shift in perspective that is missed by the standard interpretation of Bolzano's calculation of the infinite. We now turn to a close analysis of the text and to our arguments in favour of a different reading of *PU* §§29-33.

5.3 Bolzano's calculation of the infinite

As discussed in the previous section, up to §24 Bolzano has established the following facts about infinite multitudes and pluralities:

1. Some infinite multitudes are greater than others 'with respect to their plurality' (§19).
2. Two infinite multitudes can both be related as part and whole and be in a one-to-one correspondence (§20).
3. One-to-one correspondence is not sufficient to determine equality of infinite multitudes (§§21-22).

4. In the case of comparing two infinite sums, if one wants to conclude that they are equal, one needs to make sure both that there are as many summands in one as there are in the other and that each term from one sum is equal to the corresponding one in the other sum (§24).

These are the 'basic rules' (*Grundregeln*, *PU* §28) which govern a proper handling of the infinite in mathematics. Bolzano is aware however that his readers might still be skeptical towards the possibility of computing with the infinite, so he explains what he means by 'calculation of the infinite' in the following passage:

Even the *concept* of a *calculation of the infinite* has, I admit, the appearance of being self-contradictory. To want to *calculate* something means to attempt a *determination of something* through numbers. But how can one determine the infinite through numbers| that infinite which according to our own definition must always be something which we can consider as a multitude consisting of infinitely many parts, i.e. as a multitude which is greater than every number, which therefore cannot possibly be determined by the statement of a mere number? But this doubtfulness disappears if we take into account that a calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the *relationship* of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples. (*PU* §28)

Bolzano's calculation of the infinite is minimal. He does not purport to have extended the concept of number so as to introduce infinite *numbers* (pace Cantor { see Section 5.2 above),¹¹ but he aims to study the relationship { that is, the ratios as well as the 'greater than' relation { between two infinities whenever this can be done in a sound way, that is, in accordance with the principles he has argued for in the preceding portion of the *PU*. Armed with such principles, Bolzano can show his reader how to properly handle some apparently paradoxical results in mathematics, starting from the general theory of quantity.

5.3.1 Computing with infinite sums

The first computations with infinite quantities are found in earnest in §29; as we will see, these quantities are always introduced and treated as *sums*.

Bolzano introduces the symbol $\overset{0}{N}$ through a symbolic equation { that is, an equation which establishes that the reference of two signs is the same (cf. definition in *Grö enlehre*, *GL* I, pp. 131-132) { to stand for the *Menge* of all natural numbers.

11. As Mancosu (2016, p. 163) notes, this refusal to admit infinite numbers was not unique to Bolzano's position but was shared also by Dedekind (1888) and perhaps Schröder (1873).

He then introduces $\overset{n}{N}$ to stand for the *Menge* of all natural numbers strictly greater than $n \in \mathbb{N}$. $\overset{1}{S}$, on the other hand (which is first introduced as $\overset{0}{S}$), is the symbol for the sum of all natural numbers.

In Bolzano's words:

[...] if we denote the series of natural numbers by

$$1; 2; 3; 4; \dots; n; n+1; \dots \text{ in inf.}$$

then the expression

$$1^0 + 2^0 + 3^0 + 4^0 + \dots + n^0 + (n+1)^0 + \dots \text{ in inf.}$$

will be the sum of these natural numbers, and the following expression

$$1^0 + 2^0 + 3^0 + 4^0 + \dots + n^0 + (n+1)^0 + \dots \text{ in inf.}$$

in which the single summands, $1^0; 2^0; 3^0; \dots$ all represent mere units, represents just the number [*Menge*] of all natural numbers. If we designate this by $\overset{0}{N}$ and therefore form the merely symbolic equation

$$1^0 + 2^0 + 3^0 + 4^0 + \dots + n^0 + (n+1)^0 + \dots \text{ in inf. } \overset{0}{N} \quad (1)$$

and in the same way we designate the number [*Menge*] of natural numbers from $n \in \mathbb{N}$ by $\overset{n}{N}$, and therefore form the equation

$$(n+1)^0 + (n+2)^0 + (n+3)^0 + \dots \text{ in inf. } \overset{n}{N} \quad (2)$$

Then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \dots + n^0 = \overset{0}{N} - \overset{n}{N} \quad (3)$$

This passage mentions several notions that will be central to the remainder of our analysis of Bolzano's *PU*, hence we will briefly go over them now.

First is the notion of 'series' (*Reihe*), which Bolzano defines (*PU* §7) as a collection of 'terms' (*Glieder*) $a; b; c; d; \dots$ such that for each term c there is exactly another term d such that, by using the same rule for any pair $c; d$ we can obtain (determine, *bestimmen*) c by applying said rule to d , or the inverse rule to c to obtain d instead.¹² The natural numbers, that is, the 'whole numbers' (*ganze Zahlen*) are defined as a series of objects of a certain kind A where the first term

12. See also the discussion in Chapter 2.

is a unit of kind A and the subsequent terms are sums obtained by adding one unit to their immediate predecessor.¹³

The second concept we want to introduce is that of *Gliedermenge* (alternatively expressed by Bolzano as *Gliedermenge*, *Menge von Gliedern* or *Menge der Glieder*). As one can infer from *PU* §9, Bolzano considers any number series to have a *Gliedermenge*. Because a *Gliedermenge* is said to be sometimes greater, sometimes smaller, it seems reasonable to assume that this *Gliedermenge* is, if not a quantity properly said, at least something that can be quantified, i.e. treated as a quantity. In the passage we quote from §29, Bolzano introduces first the series of all natural numbers, then their sum and the *Menge* of such a sum. Given what was just said about series and *Gliedermenge* thereof, this occurrence of the word *Menge* should be read as a shorthand for *Gliedermenge* or one of its synonyms.

This occurrence of *Menge* is therefore at odds with any interpretation of Bolzano's definition of 'multitude' (*Menge*) that sees it as (almost) synonymous with 'set' in the modern sense. If the concept of multitude is virtually identical with that of set, then the multitude of $1\ 2\ 3\ 4\ \dots\ in\ inf.$ should be just $1;2;3;4;\dots\ in\ inf.$ and not $1\ 1\ 1\ 1\ \dots\ in\ inf.$ For the sake of preserving coherence in Bolzano's work in *PU* §§29-33 it is therefore sensible to insist that '*Gliedermenge*' is a quantitative concept. As a consequence, since we believe that translating *Menge* here as 'set', like Steele (Bolzano 1950), or 'multitude', as we would have to if we were to translate *Menge* rigidly, obfuscates this quantitative aspect of the concept of '*Gliedermenge*', we prefer to respect Russ's (2004) choice and translate *Menge* as 'number' when it seems to be short for *Menge der Glieder* or similar. As long as it is clear that we do not think Bolzano is introducing here genuine infinite numbers (in the sense of the German *Zahlen*), we will translate *Menge* as 'number' in these contexts.

$\overset{0}{N}$ thus denotes the number (*Menge*) of all natural numbers, and for any natural number n , $\overset{n}{N}$ represents the size of the collection of all natural numbers strictly greater than n . This is all written as follows:

$$1^0\ 2^0\ 3^0\ 4^0\ \dots\ n^0\ p_n\ 1q^0\ \overset{0}{N} \tag{4}$$

$$p_n\ 1q^0\ p_n\ 2q^0\ \dots\ \overset{n}{N} \tag{5}$$

The 0^{th} power works in the standard way here, meaning $n^0 = 1$ for any natural number n . So for instance the size of the set of all natural numbers up to n is $1^0\ 2^0\ 3^0\ 4^0\ \dots\ n^0\ 1\ 1\ 1\ 1\ \dots\ 1\ n$.

Having defined $\overset{0}{N}$ and $\overset{n}{N}$, Bolzano proceeds to show how they can be added or multiplied with one another thanks to distributivity. One then obtains a hierarchy of infinite quantities of ever-increasing order:

13. We already discussed at length the importance of this definition of the natural or whole numbers in Bolzano in Chapter 3.

$$\begin{array}{ccccccc}
 1^0:\overset{0}{N} & 2^0:\overset{0}{N} & 3^0:\overset{0}{N} & \dots & \text{in inf.} & \overset{0}{pNq^2} \\
 1^0.\overset{0}{pNq^2} & 2^0.\overset{0}{pNq^2} & 3^0.\overset{0}{pNq^2} & \dots & \text{in inf.} & \overset{0}{pNq^3} \\
 & & & & & \text{etc:}
 \end{array}$$

The notion of quantities being of different orders of infinity does not start with Bolzano and already existed in the context of infinitesimal calculus.¹⁴ However, we will argue in Section 5.5 that Bolzano's computation of the product of infinite quantities is in fact very original and hence very significant for a comparison with Cantor's theory of the infinite (which we carry out in Section 5.6).

Having looked carefully at Bolzano's first computations with infinite sums, we now proceed to our next piece of evidence for interpreting Bolzano as primarily interested in infinite sums, namely, §32 of the *PU*.

5.3.2 Grandi's series

In *PU* §32, Bolzano criticizes a report by a certain M.R.S. in *Gergonne's Annales* (M.R.S. 1830) which purports to prove that the infinite sum

$$a + a + a + a + a \dots (1)$$

has value $\frac{a}{2}$.

The series Bolzano focuses on is sometimes called Grandi's series after the Italian 18th century monk who first tried to compute a value for this infinite sum. Kline (1983) reports that this series was an object of great interest for mathematicians throughout the 19th century, that 'caused endless dispute' (Kline 1983, pp. 307-308). It is not necessary for our summary of Bolzano's views to rehash the whole debate surrounding Grandi's series (and other divergent series) in great detail, though it is perhaps worth mentioning that Grandi's opinion, that the value of this series should be $\frac{a}{2}$, was shared also by Leibniz (Kline 1983, p. 307). Kline also reports that Leibniz's argument { which differed from Grandi's } was accepted by the Bernoulli brothers. This acceptance notwithstanding, by the time Bolzano is active there is still no clear consensus on how to treat what we would now consider divergent series. For Bolzano and his contemporaries, the question of how to assign a value to infinite sums such as Grandi's series was still a live question, one which would later lead some mathematicians (e.g. the Italian Cesaro) to define different sorts of summation.

It is therefore not surprising that one should come across a piece of writing such as M.R.S.'s. M.R.S. purports to prove that the value of Grandi's series is

14. See for example the debate between Leibniz and Nieuwentijt on the existence of such higher-order infinitesimal, as presented in (Mancosu 1996, pp. 160-164).

$\frac{a}{2}$ via an algebraic reasoning, as opposed to Leibniz's more 'probabilistic' (per Kline) approach { and presumably, as opposed to Grandi's geometric approach, too. Here we quote M.R.S.'s own exposition of his proof:

The summation of the terms of a geometric progression decreasing into the infinite can be easily deduced from the above; in fact, if one has

$$x = a + aq + aq^2 + aq^3 + aq^4 + \dots;$$

one can then write

$$x = a + qa + aq^2 + aq^3 + \dots; q;$$

then $x = a + qx$ or $(1 - q)x = a$, hence $x = \frac{a}{1 - q}$. As per the remarks in (5), the equation

$$x = a + a + a + a + a + a + \dots$$

could not help in the approximation of x , as it successively gives the approximate values $a; 0; a; 0; a; 0; \dots$ among which the differences are constant; but, without resorting to Leibniz's subtle reasoning, one can immediately see that this equation comes to

$$x = a + x;$$

hence $x = \frac{1}{2}a$.¹⁵ (M.R.S. 1830, pp. 363-364)

As the text shows, M.R.S.'s treatment of Grandi's series has the virtue of treating it uniformly with other (converging) geometric series. Bolzano however is not impressed with M.R.S.'s algebraic manipulations and sees two mistakes in them. Bolzano spells out M.R.S.'s argument as follows. First, he sets

$$x = a + a + a + a + a + a + \dots \text{ in inf.} \tag{1}$$

Then, one can rewrite (1) as

$$a + qa + a + a + a + a + \dots \text{ in inf.} \tag{2}$$

15. Original French: La sommation des termes d'une progression geometrique decroissante a l'infini se deduit bien simplement de ce qui precede; si en e et on a

$$x = a + aq + aq^2 + aq^3 + aq^4 + \dots;$$

on pourra d'abord ecrire

$$x = a + qa + aq^2 + aq^3 + \dots; q;$$

puis $x = a + qx$ ou $(1 - q)x = a$ d'ou $x = \frac{a}{1 - q}$.

This yields $x = a + x$ and therefore $x = \frac{a}{2}$. Bolzano points out that while x is defined as $a + a + a + a + a + \dots$ in *inf.*, the expression within brackets in (2) is not identical with it, because it does not have the same *Gliedermenge* as $a + a + a + a + a + \dots$ in *inf.* in (1). The first a is missing so that the correct substitution ought to be the tautological $x = a + px = aq$.

Even though Bolzano does not pause to point this out to the reader, M.R.S. is making exactly one of those mistakes Bolzano was cautioning against in §24: he has assumed equality of two quantities arising from summing up two series without checking that the two series have the same *Gliedermenge*. Note that here again Bolzano seems to be using *Menge* in a way that is closer to the meaning of 'number' than to that of 'set', and Russ's (2004) translation accordingly translates the term as 'number'. While again one should not take the translation literally, we agree with the attempt to capture a more quantitative use of *Menge* in this kind of context.

The second criticism Bolzano levels at M.R.S.'s argument is that it presupposes that $a + a + a + a + a + \dots$ refers to an actual quantity, whereas Bolzano argues that it does not. The argument Bolzano gives for this position is an example of Bolzano putting to (mathematical) use his logico-philosophical apparatus: Grandi's infinite sum is a spurious one because it does not display the sum property (*PU* §31)

$$p(A + B) = C \quad A + p(B + C) = B:$$

If one tries to rewrite Grandi's sum according to Bolzano's equations, the left-hand side becomes $pa + aq + pa + aq + \dots$ in *inf.*, which according to Bolzano equals 0, whereas if one rearranges the parentheses as $a + p(a + aq) + p(a + aq) + p(a + aq) + \dots$ in *inf.*, one obtains a as a result. Thus indeed Grandi's expression does not satisfy Bolzano's definition of sum. Tapp (Bolzano 2012, p. 193) notes here that Bolzano's criterion is quite similar to Riemann's result (Apostol 1974, p. 197) which states that every infinite series is absolutely convergent if and only if it is preserved under permutation (an absolutely convergent series is one in which the series of the absolute values of its terms also converges). It is unfortunate though that Bolzano's criterion taken literally is too strong, as it seems to be also implying that $\sum_0^{\infty} N$ does not designate an actual quantity (see Section 5.4 below).

We take this section of the *PU* as helping our case that Bolzano's work in §§29-33 should not be read as an imperfect set theory. Indeed, §32 is an example of Bolzano's principles for the computations of the infinite at work: a result published by a fellow mathematician about the computation of infinite sums is rejected on the basis of a violation of one of these principles. However most other commentators do not devote particular attention to §32. One notable exception is Steele, who thus summarises §32: 'Some errors in the pretended summation of $\sum_{n=0}^{\infty} p^{-n} a$, which is a symbol not expressing any true quantity at all' (Bolzano 1950, p. 66). Even more intriguingly, he mentions Grandi's series and the whole

controversy surrounding it when introducing the historical context of the *PU* (Bolzano 1950, pp. 3-4). Yet it is as if this does not leave a trace when giving an overall appraisal of the contributions of the *PU*, or of Bolzano's contributions to mathematics and its philosophy. Bolzano is still presented as someone who *almost* anticipated Cantorian set theory, except he did not.

5.3.3 The sum of all squares

In the previous section, we argued that some passages of the *PU* offer textual evidence for the claim that Bolzano's work on the sizes of infinite collections should be understood as about sizes of infinite sums, that is, infinite series in modern terminology, rather than as about sizes of infinite countable sets. We now make a theoretical case as per why this interpretation is also the most charitable one.

Just following the discussion of §32, Bolzano writes that

[...] if we wish to avoid getting onto the wrong track in our calculations with the infinite then we may never allow ourselves to declare two infinitely large quantities, which originated from the summation of the terms of two infinite series, as equal, or one to be greater or smaller than the other, because every term in the one is either equal to one in the other series, or greater or smaller than it. (*PU* §33)

So, for two infinite sums $\sum a_i$ and $\sum b_i$, it is not the case that, say, $\sum a_i > \sum b_i$ if for every term of $\sum b_i$ there is one in $\sum a_i$ that is strictly smaller.

He then continues:

We may just as little declare such a sum as the greater just because it includes all the terms of the other and in addition many, even infinitely many, terms (which are all positive), which are absent in the other.

As an example of this principle in action, Bolzano asks us to consider the two series

$$1 \quad 4 \quad 9 \quad 16 \quad \dots \quad \text{in inf.} \quad \sum^2$$

and

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad \text{in inf.} \quad \sum^1$$

According to Bolzano, 'no one can deny that every term of the series of all *squares*' { that is, \sum^2 { 'because it is also a natural number, also appears in the series of first powers of the natural numbers and likewise in the latter series \sum^1 , together with all the terms of \sum^2 there appear many (even infinitely many) terms which are missing from \sum^2 because they are not square numbers.' (*PU*, §33)¹⁶ So, the series

16. This passage was already mentioned in Chapter 3.

S^1 and S^2 are such that the terms of the latter all appear in the former, and the former also includes infinitely many terms that the second series does not include. The next step in Bolzano's argument is to claim the following:

Nevertheless S^2 , the sum of all square numbers, is not smaller but is indisputably greater than S^1 , the sum of the first powers of all numbers. (PU, §33)

Bolzano argues for this point by claiming two things: first, that 'in spite of all appearance to the contrary, the *multitude of terms* [*Gliedermenge*] in both series (not considered as sums, and therefore not divisible into arbitrary multitudes of parts) is certainly the same.' Second, that with the exclusion of the first term, all terms of S^2 are greater than the corresponding term in S^1 . Since then the two series have the same amount of terms, but the terms of S^2 are greater than all but one of the terms in S^1 , Bolzano concludes that S^2 is greater than S^1 , because it is possible to termwise subtract S^1 from S^2 and one would still have a positive remainder as a result:

But if the multitude of terms [*Menge der Glieder*] in S^1 and S^2 is the same, then it is clear that S^2 must be much greater than S^1 , since, with the exception of the first term, each of the remaining terms in S^2 is definitely greater than the corresponding one in S^1 . So in fact S^2 may be considered as a quantity which contains the whole of S^1 as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as S^1 , [...] (PU, §33)

As we can see, in §33 Bolzano repeats twice the idea that S^1 and S^2 have the same *Gliedermenge* (translated by Russ as 'multitude of terms'). He is committed then to the claim

(Terms) The *Gliedermenge* in series S^1 and S^2 is the same.

This is often (see e.g. Berg 1962, 1992; Sebestik 1992) interpreted as a sign that Bolzano was using here one-to-one correspondence to compare the size of the sets corresponding to S^1 and S^2 , namely \mathbb{N} , the set of all natural numbers, and \mathbb{N}^{p^2} , the set of all squares, respectively. But if this is the case, then Bolzano is essentially violating part-whole as applied to sets, the way Sebestik suggests (cf. Section 5.2).

§29 and §33 taken together raise the question of how, if at all, Bolzano envisioned to generalise his notion of *Gliedermenge* from the collection of all natural numbers to any infinite subcollection thereof (or what would be a 'Bolzanian enough' way of doing this).

Let us take a step back and reconsider what Bolzano does in §29. Recall that $N = 1^0 + 2^0 + 3^0 + 4^0 + \dots$ in *inf.*, where each n^0 is one unit, as Bolzano reminds us. Assuming that N is what Bolzano intended to be the *size* of N just in the same way as cardinals are considered to capture set size in modern set theory, the question is how to extend Bolzano's notion of size of N to infinite (proper) subsets of N . Given the importance that the example of squares has in Bolzano scholarship (see our Section 5.2 and Chapter 3), let us try to answer the question for N^{sq} , specifically.

Per §29, the procedure to obtain the *Menge* (of terms, *von Gliedern*) of a series $1 + 2 + 3 + \dots$ is to first consider it as a sum

$$1 + 2 + 3 + \dots \text{ in inf.};$$

and then raise each term to the power of 0. The *number* of terms in $1 + 2 + 3 + \dots$ is then identified with the value of the infinite sum $N = 1^0 + 2^0 + 3^0 + \dots$ in *inf.* This means that if we list all square numbers as $sq: 1; 4; 9; 16; 25; 36; \dots$, the number of terms (hence the number of square numbers) should be identified with

$$N^{sq} = 1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \dots \text{ in inf.}$$

Now notice that if we apply the same procedure to the series of terms of S^2 , we obtain exactly the same. Since S^2 as a sum is S itself, i.e.

$$1 + 4 + 9 + 16 + 25 + 36 + \dots \text{ in inf.};$$

raising each term to the power of 0 yields

$$1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \dots \text{ in inf. } N^{sq}$$

Thus the number of square numbers is the same as the number of terms in S^2 . But since Bolzano endorses **(Terms)**, the number of terms in S^2 is equal to the number of terms in S , which is itself computed as $1^0 + 2^0 + 3^0 + \dots$ in *inf.* N . From this it immediately follows that N^{sq} and N have the same *Gliedermenge*. Moreover, since any term in each sum is regarded as a unit, both sums also have equal terms. Now by Bolzano's remark (*PU* §24) that 'equal summands also give equal sums', we must therefore conclude that $N^{sq} = N$. But if the first one

is the number of squares and the second one is the number of natural numbers, then under the standard (set theoretic) interpretation those two *sets* have the same *size*, which directly contradicts the part-whole principle. So it seems that we have reached a contradiction similar to the one highlighted by Sebestik (1992, pp. 463-464).

The first reaction would be of course to bite the bullet and accept that perhaps Bolzano did not realize that §29 and §33 would lead to a contradiction, and what is more, to a violation of part-whole. This seems to be the line that a set theoretic interpretation forces upon the reader. For, if N^0 , being the *Gliedermenge* of S^1 , is somehow also the size of N , and the *Gliedermenge* of S^2 is also the size of N^{p^2q} , then of course Bolzano's remark in §33 that S^1 and S^2 have the same *Gliedermenge* cannot be reconciled with part-whole as applied to sets (**PW1**).

A second option would be to reject the generalisation of the procedure of §29 to arrive at N^0 and argue that there is no analogue to N^0 for S^2 . One could defend this position by pointing out that, in §28, Bolzano only commits to be able to *sometimes* compute with the infinite { not always. In particular, he does not commit to be able to determine the size of every subset of N . We believe however that this answer is not entirely satisfactory. For one, this solution might feel *ad hoc*, because even though Bolzano may have not intended for the procedure of §29 to be applied indiscriminately to any set composed only of natural numbers, there is nothing intrinsic to the procedure itself that bars such a generalisation from being carried out. Moreover, while §29 does not explicitly mention a general procedure for determining the *Gliedermenge* of an infinite sum, determining when two sums have the same *Gliedermenge* is necessary to determine whether one is greater than another, as Bolzano himself notes (see xx24 and 32). Since *Gliedermengen* are *Mengen*, multitudes, it is natural to ask whether part-whole reasoning applies to, or is even compatible with, the procedure of determining when the *Gliedermengen* of two sums are equal. In a way, then, this second option does not solve the theoretical problem raised by Bolzano's work so much as skirt around it via a 'monster-barring' move.

There is a third option though, which hinges upon a closer reading of §29. Indeed, when computing quantities of the form N^n , which for him corresponds to the number of natural numbers greater than n , Bolzano does seem to apply the procedure sketched above, namely writing down the sum $p_n - 1q - p_n - 2q - p_n - 3q \dots$ in *inf.*, and then raising each term to the 0^{th} power, thus obtaining the sum $p_n - 1q^0 - p_n - 2q^0 - p_n - 3q^0 \dots$ in *inf.*. However, if, as evidenced again in §33, the difference of two infinite sums is computed termwise, $N^0 - N^n$ should be computed as:

$$p_1^0 - p_n - 1q^0q - p_2^0 - p_n - 2q^0q \dots$$

But each term in this sum is the difference of a unit and a unit, so it equals 0.

Hence Bolzano should conclude $\overset{0}{N} = \overset{n}{N} = 0$. Instead, Bolzano writes that

$$\overset{0}{N} = \overset{n}{N} = 1^0 + 2^0 + \dots + n^0;$$

which strongly suggests that Bolzano thinks that $\overset{0}{N} = \overset{n}{N}$ is equal to the infinite sum

$$1^0 + 2^0 + \dots + n^0 + (n+1)^0 + (n+2)^0 + \dots + (2n)^0 + (2n+1)^0 + (2n+2)^0 + \dots \text{ in inf.}$$

But this in turn suggests that a more accurate way of representing $\overset{n}{N}$ is in fact as

$$\underbrace{1^0 + 2^0 + \dots + n^0}_{n \text{ times}} + (n+1)^0 + (n+2)^0 + \dots \text{ in inf.}$$

In other words, $\overset{n}{N}$ is not obtained by listing all the numbers above n in an infinite sum and raising each of them to the power of 0, but is instead obtained by erasing the first n terms from the sum corresponding to $\overset{0}{N}$. This procedure clearly changes the number of terms in the resulting sum. In order to compare $\overset{n}{N}$ to $\overset{0}{N}$, we must therefore make sure first that the two sums have the same *Gliedermenge*, which implies adding n terms to $\overset{n}{N}$ which act, quite literally, as the 'ghosts of departed quantities'.

This reading of Bolzano's text now gives a way out of the problem of the sum of all squares presented above. Let us consider again the example of \mathbb{N}^{p^2q} . If we want to compute its size as a subset of \mathbb{N} , the way to obtain said size is first to compute that of \mathbb{N} , namely, $\overset{0}{N}$. We then remove from $\overset{0}{N}$ the elements whose base is not an element of \mathbb{N}^{p^2q} , thus obtaining

$$\overset{SQ}{N} = 1^0 + 4^0 + 9^0 + 16^0 + \dots \text{ in inf.}$$

The difference between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is that, in the former, 4^0 is the second term of the sum, while it is the fourth term in $\overset{SQ}{N}$ { and so on. The idea would be then that such an erasure procedure *does* change the number of elements from one set to the other, because \mathbb{N}^{p^2q} considered as a subset of \mathbb{N} has a different size from when considered as the set underlying the sum $\overset{2}{S}$. Note that this distinction between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is not available to a proponent of the received view: if $\overset{sq}{N}$ and $\overset{SQ}{N}$ are *sets*, i.e. entirely determined by their elements, then as the two sums clearly have the same terms, they should also be equal to one another. By contrast, the difference between the two sums is easy to express in our interpretation of Bolzano's computations (see next section), because $\overset{sq}{N}$ would

correspond to a countable sequence with graph $\langle x_1; 1^0y; x_2; 4^0y; x_3; 9^0y; \dots \rangle$ whereas N has graph $\langle x_1; 1^0y; x_2; 0y; x_3; 0y; x_4; 4^0y; \dots \rangle$. Incidentally, Tapp (Bolzano 2012, p. 191) suggests a similar idea for the interpretation of §29, raising the question whether such an interpretation can actually lead to a fully-edged coherent reading of the *PU*. Our next two sections address that question.

5.4 Modelling Bolzano's arithmetic of the infinite

Our goal in this section is to offer a model of Bolzano's computations with infinite sums. More precisely, we interpret Bolzano's talk of infinite sums and operations between them as statements about a certain model and show that all of Bolzano's positive results as summarised in the previous section also hold in our model. Additionally, we argue that our model accurately represents Bolzano's *reasoning*, in that several of the proofs we provide closely match Bolzano's own arguments in the *PU*.

Our main idea is to associate to each infinite sum a corresponding infinite quantity. Our proposal here is closely related to the theory of numerosities (Benci and Di Nasso 2003; more recently Benci and Di Nasso 2019, Ch. 17), in which the numerosity of a set of natural numbers is defined as an element in an ultrapower of \mathbb{N} . However, since our focus is on assigning infinite quantities to certain infinite sums of integers, and not on assigning numerosities to sets of natural numbers, our proposal will be slightly different. Part of our model is in fact closer to the construction presented by Trlifajova (2018, pp. 20-24), which we will discuss in Section 5.4.4. In order to do that, we first need to outline our own proposal.

5.4.1 The basic framework

We start by representing Bolzano's infinite sums of integers as countable sequences of integers. Formally, we write \mathbb{N} for the set of positive natural numbers and \mathbb{Z} for the set of all integers, and we consider functions from $\mathbb{N} \rightarrow \mathbb{Z}$. To any infinite sum $a_1 + a_2 + a_3 + \dots$ in *inf.*, we associate the function $f : \mathbb{N} \rightarrow \mathbb{Z}$, i.e. the function that maps each positive natural number i to the i^{th} summand of the infinite sum. As is customary, we will often identify a function $f : \mathbb{N} \rightarrow \mathbb{Z}$ with the countable sequence of integers $\langle f(1); f(2); f(3); \dots \rangle$. In the case of a Bolzanian sum which has a different *Gliedermenge* because it has been obtained from another sum by erasing certain terms, we treat the erased terms as 0 and obtain the function associated to it accordingly. For example, since the sequence associated to $\overset{0}{N}$ is $\langle p_1; 1; 1; \dots \rangle$, the sequence associated to $\overset{2}{N}$ is $\langle p_0; 0; 1; 1; \dots \rangle$.

We consider the structure $Z : \langle \mathbb{Z}; +; \cdot; 0; 1; < \rangle$ of integers with their usual ordering and addition operation, and take an ultrapower Z_U of this structure by a non-principal ultrafilter on I (i.e. a non-empty collection U of infinite subsets of I closed under supersets and finite intersections and such that for any $A \subseteq I$, precisely one of $A; I \setminus A$ belongs to U). Ultrapowers are standard constructions in mathematical logic, and a detailed presentation of their theory is beyond the scope of this chapter. Instead, we refer the reader to Bell and Slomson (1974, Chs. 5, 6) for a standard introduction to ultrapowers and ultraproducts, and simply list some crucial facts below:

Lemma 5.4.1.

1. Elements in the ultrapower Z_U are equivalence classes of functions from I to Z . For any $f : I \rightarrow Z$, we write its corresponding equivalence class as f . For any $f; g : I \rightarrow Z$, $g \sim f$ if and only if f and g are equal for U -many elements in I , i.e. $\{i \in I : f(i) = g(i)\} \in U$.
2. There is a canonical elementary embedding of Z into Z_U , obtained by mapping any integer z to the equivalence class of the constant function $e_z : I \rightarrow Z$ sending any $i \in I$ to z . It is customary to identify z with e_z and to view Z as an elementary substructure of Z_U .
3. Addition and subtraction are defined in Z_U . Given $f; g : I \rightarrow Z$, $f + g$ is the equivalence class of the function $h : I \rightarrow Z$ such that $h(i) = f(i) + g(i)$ for any $i \in I$. Similarly, $f - g$ is the equivalence class of the function $h : I \rightarrow Z$ such that $h(i) = f(i) - g(i)$ for any $i \in I$.
4. Elements in Z_U are linearly ordered. More precisely, given any $f; g : I \rightarrow Z$, we have that $Z_U \models f < g$ if and only if $\{i \in I : f(i) < g(i)\} \in U$.
5. Given any first-order formula $\varphi(x_1; \dots; x_n)$ and any functions $f_1; \dots; f_n : I \rightarrow Z$, we write $\|\varphi(f_1; \dots; f_n)\|$ for the set $\{i \in I : Z \models \varphi(f_1(i); \dots; f_n(i))\}$. Los's Theorem states that for any $\varphi(x_1; \dots; x_n)$ and any functions $f_1; \dots; f_n$,

$$Z_U \models \varphi(f_1; \dots; f_n) \text{ iff } \|\varphi(f_1; \dots; f_n)\| \in U$$

6. As a direct consequence of Los's Theorem, Z and Z_U are elementarily equivalent.

An intuitive motivation for our use of an ultrapower of Z can be provided along the following lines. As we have argued, we take Bolzanian infinite quantities to be infinite sums. Given an infinite sum $\sum_{i \in I} a_i$, we may decompose it into a sequence of partial sums $\langle \sum_{i \in U_n} a_i \rangle_{n \in \mathbb{N}}$, where, for any positive integer n , U_n is the sum of the first n terms in $\sum_{i \in I} a_i$. Any such sum can be seen as providing some partial information

about \sum , and if \sum were a finite sum with n terms, then \sum_n would be \sum itself. However, since \sum is infinite, there is no last term of \sum and no partial sum that would give us total information about \sum . In order to overcome this difficulty, we must try to organize the partial information given by each partial sum of the first n terms of \sum into a coherent whole. This is precisely the role that a non-principal ultrafilter U on \mathbb{N} will play for us. One may think of U as a collection of properties of positive integers that describe a natural number 'at infinity', distinct from all finite numbers, and providing a vantage point from which all the partial sums of \sum form a coherent picture. We therefore encourage the reader who may not be familiar with ultrapowers to keep the following two principles in mind:

- Properties of an infinite sum \sum are those that are shared by 'most' partial sums of the form \sum_n ;
- What 'most' partial sums means is determined by U . Given a set of positive integers A , the set $\{n : n \in A\}$ contains 'most' partial sums of \sum if and only if $A \in U$.

Given a function $f : \mathbb{N} \rightarrow \mathbb{Z}$, we define the *approximating sequence* of f to be the function $\langle pfq \rangle : \mathbb{N} \rightarrow \mathbb{Z}$ defined by $\langle pfq \rangle_i = f(p_i)$ for any $i \in \mathbb{N}$. In the case of a function f representing a Bolzanian sum \sum , the approximating sequence of f is simply the sequence of partial sums $\langle p_1; p_2; \dots; p_n \rangle$ mentioned above. Our proposal consists in identifying the (possibly infinite) quantity designated by a Bolzanian sum \sum with $\langle pfq \rangle$, i.e. with the equivalence class of its approximating sequence. To simplify notation, we will write \mathbf{f} for the element $\langle pfq \rangle$ in Z_U , but we will sometimes abuse notation and write \mathbf{f}/q for $\langle pfq \rangle/q$.

We are now able to represent all infinite sums and infinite quantities discussed by Bolzano, except products of infinite quantities, which we will discuss in Section 5.5. As outlined above, the procedure consists in turning a Bolzanian infinite sum into a countable sequence of integers, to which (the equivalence class of) an approximating sequence is then associated. Additions and order relations between infinite sums are then determined by the ultrapower. As an example, the infinite sum $1^0 + 2^0 + 3^0 + \dots$ in *inf.* is represented by the sequence $\overset{0}{N} : \langle p_1; 1; 1; \dots; q \rangle$, since, according to Bolzano, each summand of this sum is a unit. Consequently, the approximating sequence of $\overset{0}{N}$ is the sequence $\overset{0}{pNq} = \langle p_1; 2; 3; \dots; q \rangle$, which corresponds to the identity function on \mathbb{N} , and $\overset{0}{N}$ is the equivalence class of the sequence $\langle p_1; 2; 3; \dots; q \rangle$. Similarly, infinite sums of the form $\langle p_n - 1; p_n - 2; \dots; p_n - n \rangle$ in *inf.*, which Bolzano writes as $\overset{n}{N}$, are sums that according to him have n fewer terms than $\overset{0}{N}$. We therefore propose to model $\overset{n}{N}$ as a countable sequence in which the first n summands are 0, i.e. by the sequence $\overset{n}{pNq} = \langle \underbrace{0; \dots; 0}_n; 1; 1; \dots; q \rangle$. The corresponding

n times

approximating sequence $\{p^i N^i / q^i\}_{i \in \mathbb{N}}$ is $\{p^i / q^i\}_{i \in \mathbb{N}}$. Equivalently, for any $i \in \mathbb{N}$, $p^i N^i / q^i < i$, where $i < n$ if $i \neq n$ and $i = n$ otherwise.

A similar approach can be applied to represent the sums $\sum_{i=0}^n 1$ and $\sum_{i=1}^n i$, as well as Grandi's series of the form $G_a = a - a + a - a + \dots$ in inf . For clarity's sake, we have collected the representation of $\sum_{i=0}^n 1$; $\sum_{i=1}^n i$; $\sum_{i=0}^n a^i$ and G_a in the table below:

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$1^0 - 2^0 + \dots$ in inf .	$\sum_{i=0}^n 1$ $\{p^i / 1; 1; 1; \dots; q\}$	$\{p^i N^i / q^i\}_{i=0}^n$ $\{p^i / 2; 3; 4; \dots; q\}$	$\{p^i N^i / q^i\}_{i \in \mathbb{N}}$	$\sum_{i=0}^n 1$ $\sum_{i=0}^n p^i N^i / q^i$
$p^n - 1q^0 - p^n - 2q^0 + \dots$ in inf .	$\sum_{i=0}^n 1$ $\{p^i / 1; 1; 1; \dots; q\}$	$\{p^i N^i / q^i\}_{i=0}^n$ $\{p^i / 1; 2; 3; \dots; q\}$	$\{p^i N^i / q^i\}_{i \in \mathbb{N}}$	$\sum_{i=0}^n 1$ $\sum_{i=0}^n p^i N^i / q^i$
$1 - 2 + 3 - \dots$ in inf .	$\sum_{i=0}^n 1$ $\{p^i / 1; 2; 3; 4; \dots; q\}$	$\{p^i S^i / q^i\}_{i=0}^n$ $\{p^i / 3; 6; 10; \dots; q\}$	$\{p^i S^i / q^i\}_{i \in \mathbb{N}}$	$\sum_{i=0}^n 1$ $\sum_{i=0}^n p^i S^i / q^i$
$1^n - 2^n + 3^n - \dots$ in inf .	$\sum_{i=0}^n 1$ $\{p^i / 1^n; 2^n; 3^n; 4^n; \dots; q\}$	$\{p^i S^i / q^i\}_{i=0}^n$ $\{p^i / 1^n; 2^n; 3^n; \dots; q\}$	$\{p^i S^i / q^i\}_{i \in \mathbb{N}}$	$\sum_{i=0}^n 1$ $\sum_{i=0}^n p^i S^i / q^i$
$a - a + a - a + \dots$ in inf .	G_a $\{p^i / a; a; a; a; \dots; q\}$	$\{p^i G_a^i / q^i\}_{i=0}^n$ $\{p^i / 0; a; 0; \dots; q\}$	$\{p^i G_a^i / q^i\}_{i \in \mathbb{N}}$	G_a $\sum_{i=0}^n p^i G_a^i / q^i$

Table 5.1: Representation of Bolzanian sums in Z_U

5.4.2 Modelling Bolzano's results about infinite sums

We now establish some results that echo Bolzano's own computations. We will first give proofs in our framework, then argue that those proofs are very close in spirit to Bolzano's arguments. We start with results about infinite sums of the form $\sum_{i=0}^n 1$ and $\sum_{i=1}^n i$:

Lemma 5.4.2.

1. For any natural numbers $i; n$, $Z_U \models i < \sum_{i=0}^n 1$.
2. For any natural number n , $Z_U \models \sum_{i=0}^n 1 < \sum_{i=0}^n n$.
3. For any natural number i , $Z_U \models i < \sum_{i=0}^1 1$.
4. For any natural numbers $i; n$, $Z_U \models \sum_{i=0}^n i < \sum_{i=0}^n n + 1$.

The first result asserts that all sums of the form $\sum_{i=0}^n 1$ are infinite, in the sense that they are greater than any finite number. The second shows that our model preserves Bolzano's part-whole intuition that certain infinite sums might have fewer terms than some others and that, as a consequence, two infinite quantities might differ by a finite quantity. Finally, the last two correspond to Bolzano's claim that some infinite quantities might be infinitely greater than some others. Note that we write n as a shorthand for the sum of n with itself n times, which

is defined in the ultrapower.

The proofs for all four items are all similar and can be thought of as 'arguments by co-finiteness'. In all cases, we show that Z_U satisfies a formula by showing that $\{j \in P : \dots\}$ is a co-finite subset of P and must therefore belong to U (since U is non-principal, it contains no finite set, so it must contain all co-finite sets).

Proof.

1. Recall that, in Z_U , the natural number i corresponds to (the equivalence class of) the function $e_i : m \rightarrow \mathbb{N} : i$. Moreover, for any natural number n , $\{j \in P : i < j < n\}$ is a co-finite subset of P . Thus $\{j \in P : i < j < n\} \in U$. Hence $\{j \in P : i < j < n\}$ is a co-finite subset of P and belongs to U , from which it follows that $Z_U \models i < n$.

2. Again, in Z_U , n is (the equivalence class of) the function $e_n : m \rightarrow \mathbb{N} : n$. Moreover, $\{j \in P : n < j < 2n\}$ is (the equivalence class of) the function $f : m \rightarrow \mathbb{N} : 2m - n$ such that

$$f(p/q) = \{j \in P : n < j < 2n\} \cap \{i - p, nq\}$$

for any $i \in P$. Hence $\{j \in P : n < j < 2n\} \in U$. Hence $\{j \in P : n < j < 2n\}$ is a co-finite subset of P , and $Z_U \models n < 2n$.

3. Since $\{j \in P : n < j < 2n\} \in U$, we have that $\{j \in P : i < j < 2n\} \in U$. On the other hand, $\sum_{k=1}^j k$ which, by Gauss's summation theorem, is equal to $\frac{j(j+1)}{2}$. Hence

$$\begin{aligned} \{j \in P : i < j < 2n\} &= \{j \in P : i < j < \sum_{k=1}^j k\} \\ &= \{j \in P : i < j < \frac{j(j+1)}{2}\} \\ &= \{j \in P : i < \frac{j^2}{2}\}. \end{aligned}$$

Hence $\{j \in P : i < \frac{j^2}{2}\}$ is co-finite, and $Z_U \models i < \frac{j^2}{2}$.

4. The argument is a simple generalisation of the one above. Fix some natural numbers i and n . Then for any $k \in P$, we have $\{j \in P : i < j < k\} \in U$, and

$\sum_{j=1}^{n-1} p^k j^{n-1}$. This means that

$$\sum_{j=1}^{n-1} p^k j^{n-1} = \sum_{j=1}^{n-1} p^k j^{n-1} - \sum_{j=1}^{n-1} p^k j^{n-1}$$

for any $k \in \mathbb{N}$. Now since $p^k j^{n-1} - ij^{n-1}$ is positive for any $j > i$ and in fact assumes arbitrarily large positive values, it follows that $\sum_{j=1}^{n-1} p^k j^{n-1} - \sum_{j=1}^{n-1} ij^{n-1}$ is positive for any large enough k . Thus $\{ \sum_{j=1}^{n-1} p^k j^{n-1} - \sum_{j=1}^{n-1} ij^{n-1} \mid k \in \mathbb{N} \}$ is a cofinite subset of \mathbb{N} . Now since $\sum_{j=1}^{n-1} p^k j^{n-1} - \sum_{j=1}^{n-1} ij^{n-1} > 0 \iff \sum_{j=1}^{n-1} p^k j^{n-1} > \sum_{j=1}^{n-1} ij^{n-1}$, by Los's Theorem we have that $\sum_{j=1}^{n-1} p^k j^{n-1} \succ \sum_{j=1}^{n-1} ij^{n-1}$. Hence $\sum_{j=1}^{n-1} p^k j^{n-1} \succ \sum_{j=1}^{n-1} ij^{n-1}$ for any natural numbers i and n .¹⁷ \square

Let us now compare the proofs above with Bolzano's arguments in sections 29 and 33 of *PU*. Bolzano does not explicitly argue for results p1q and p3q: in §29, he seems to take for granted that sums of the form $\sum_{j=1}^n j$ and $\sum_{j=1}^n j^2$ designate infinite quantities, and he simply writes that $\sum_{j=1}^n j$ is 'far greater than $\sum_{j=1}^n j^2$ '. However, the same section contains the following argument for p2q:

If we designate [the number of all natural numbers] by $\sum_{j=1}^{\infty} j$ and therefore form the merely symbolic equation

$$1^0 + 2^0 + 3^0 + \dots + n^0 + (n+1)^0 + \dots = \text{in inf. } \sum_{j=1}^{\infty} j \quad (1)$$

and in the same way we designate the number of natural numbers from $(n+1)$ by $\sum_{j=1}^n j$, and therefore form the equation

$$(n+1)^0 + (n+2)^0 + (n+3)^0 + \dots = \text{in inf. } \sum_{j=1}^n j; \quad (2)$$

then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \dots + n^0 = n \sum_{j=1}^n j \quad (3)$$

17. A more direct proof of this result can also be given using more advanced resources from number theory. It is a standard number-theoretic fact (using for example Faulhaber's formula) that for any natural numbers k, n , $\sum_{j=1}^k j^n$ is a polynomial of degree $n+1$ in k , with leading term $\frac{1}{n+1} k^{n+1}$. Thus $\sum_{j=1}^n p^k j^{n-1}$ is a polynomial in k of degree $n+1$ with leading term $\frac{1}{n+1} k^{n+1}$, while $\sum_{j=1}^{n-1} ij^{n-1}$ is a polynomial in k of degree n with leading term $\frac{1}{n} k^n$. This means that $\sum_{j=1}^n p^k j^{n-1} \succ \sum_{j=1}^{n-1} ij^{n-1}$ for k sufficiently large, and thus $\{ \sum_{j=1}^n p^k j^{n-1} - \sum_{j=1}^{n-1} ij^{n-1} \mid k \in \mathbb{N} \}$ is a cofinite subset of \mathbb{N} .

from which we therefore see how two infinite quantities $\overset{0}{N}$ and $\overset{n}{N}$ sometimes have a completely definite difference.

As mentioned in Section 5.3, we read Bolzano as arguing that subtracting $\overset{n}{N}$ from $\overset{0}{N}$ amounts to subtracting from each term i^0 after the n^{th} summand in $\overset{0}{N}$ the corresponding term i^n in $\overset{n}{N}$. The only terms left in $\overset{0}{N} - \overset{n}{N}$ after this procedure are the first n summands in $\overset{0}{N}$, from which it follows that $\overset{0}{N} - \overset{n}{N} = n$. In our setting, $\overset{0}{N}$ is represented by (the equivalence class of) the sequence $\langle 1; 2; 3; \dots; q \rangle$, while $\overset{n}{N}$ is represented by the sequence $\langle \underbrace{0; \dots; 0}_n; 1; 2; 3; \dots; q \rangle$, and $\overset{0}{N} - \overset{n}{N}$ is the

sequence obtained by subtracting $\overset{0}{N}$ from $\overset{n}{N}$ n times componentwise, i.e. the sequence $\langle 1; 2; 3; \dots; n; n; n; \dots; q \rangle$, which over U is equivalent to n . Similarly to Bolzano's argument, the difference between the two infinite sums $\overset{0}{N}$ and $\overset{n}{N}$ is determined by the difference between matching summands (i.e. the difference is computed componentwise) and is precisely n .

Finally, Bolzano does not explicitly argue for p4q in its full generality. In a very revealing passage in §33, however, he gives a detailed argument for the $n = 1$ instance of p4q when arguing that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$:

But if the multitude of terms [*Menge der Glieder*] in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the first term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms [*Gliederzahl*] as $\overset{1}{S}$, namely:

$$0; 2; 6; 12; 20; 30; 42; 56; \dots; npn - 1q; \dots \text{ in inf.};$$

in which, with the exception of the first two terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. If we therefore subtract from this remainder the series $\overset{1}{S}$ for the second time, then we obtain as the second remainder a series of the same number of terms [*Gliedermenge*]

$$1; 0; 3; 8; 15; 24; 35; 48; \dots; npn - 2q; \dots \text{ in inf.}$$

in which, with the exception of the first three terms all the following terms are greater than the corresponding ones in S^1 , so that also this third remainder is without contradiction greater than S^1 . Now since these arguments can be continued without end it is clear that the sum S^2 is infinitely greater than the sum S^1 , while in general we have

$$S^2 = mS^1 = p_1 - mq + p_2^2 - 2mq + p_3^3 - 3mq + p_4^4 - 4mq \\ \dots pm^2 - m^2q \dots n^n - nq \dots \text{in inf.}$$

In this series only a finite multitude of terms [*Menge von Gliedern*], namely the first $m - 1$ are negative and the m^{th} is 0, but all succeeding ones are positive and increase indefinitely.

Let us note two features of Bolzano's argument that are shared by our interpretation. First, when determining whether one infinite sum is greater than another one, Bolzano considers which terms in the first sum are greater than the corresponding terms in the second one: this is reminiscent of the way relations between (equivalence classes of) functions are determined in an ultrapower. Moreover, Bolzano's reason to claim that S^2 is greater than $S^1, 2S^1, 3S^1$, and so on, is that in all such cases, all but finitely many terms in S^2 are strictly greater than the corresponding terms in any finite multiple of S^1 . This seems very similar to the 'argument by co-finiteness' that we presented above: even though the first terms of the sum mS^1 might be greater than the first terms of the sum S^2 , the terms in the second sum become greater than the corresponding terms in the first one from some point onwards. In our setting, we prove that $Z_U \models S^2_i > mS^1_i$ by showing that $S^2_i - mS^1_i > 0$ for co-finitely many natural numbers i . To establish this, it is enough to observe, like Bolzano, that $i^2 - mi$ is positive for any $i > m$, as this implies that the sum $\sum_{j=1}^i pj^2 - mj$ must be positive for i large enough. It is worth mentioning that, unlike in Bolzano's argument, our 'tipping point', i.e. the value i at which S^2_i becomes strictly greater than mS^1_i is not $m - 1$. This is because S^2 and S^1 are the approximating sequences of the sequences $p_1; 2; 3; \dots; q$ and $p_1; 4; 9; \dots; q$ respectively, while Bolzano is reasoning with the sequences of terms themselves. We therefore conclude that the general proof given for 4 closely matches Bolzano's own reasoning. In particular, our use of an ultrapower construction enables us to lift the following criterion for the inequality of two integers:

$$m > n \iff \exists n \in \mathbb{N} \exists m_i \in \mathbb{Q} \tag{5.1}$$

to a criterion for the inequality of two infinite sums:

$$\sum_{i \in P} p_i > \sum_{i \in Q} q_i \iff \exists U \forall i \in U (p_i > q_i) \tag{5.2}$$

In other words, in our formalism, in order to determine whether an infinite sum is greater than another infinite sum, it is enough to compute their difference, which is defined termwise, and then determine whether the sum of the first i terms of the difference is positive for U -many i . Our claim is that this reasoning is very close to the one displayed by Bolzano in §33. Moreover, let us note that when he argues that S^2 is greater than mS^1 for any m , because all but finitely many terms in the infinite sum $S^2 - mS^1$ are positive, Bolzano can be seen as implicitly displaying a form of part-whole reasoning about *sums*, rather than sets: mS^1 is smaller than S^2 because it is contained 'as a part'. This is established by showing that the difference $S^2 - mS^1$ is positive, and this latter fact is established in turn by noticing that all but finitely many terms in $S^2 - mS^1$ are positive. Thus Bolzano can be read here as providing a criterion for when the quantity designated by a sum is a proper part of the quantity designated by another sum. We will come back to this point in Section 5.6, and we will discuss its implication for the role that part-whole reasoning plays in Bolzano's computations with the infinitely large.

5.4.3 Grandi's series

Finally, let us address some of Bolzano's remarks on Grandi's series. As noted above, Bolzano disagrees with the claim (attributed to M.R.S.) that the infinite sum

$$x = a - a + a - a + \dots \text{ is } \textit{in inf.}$$

designates the quantity $\frac{a}{2}$. In particular, Bolzano claims that the mistake in M.R.S.'s proof is to treat the sum obtained by discarding the first term of x as x . In our setting, x designates the quantity G_a , i.e. the equivalence class of the sequence $\langle p; 0; a; 0; \dots; q \rangle$. On the other hand, following the strategy adopted for 'truncated' infinite sums like $\sum_{i \in \mathbb{N}} p_i$, it seems that the infinite sum obtained by discarding the first term in x should be interpreted as the countable sequence $\langle 0; a; a; a; \dots; q \rangle$. If we write this sequence as G_a^1 , we then have that G_a is the equivalence class of the sequence $\langle 0; a; 0; a; \dots; q \rangle$. But then, it follows that

$$\sum_{i \in U} G_a = G_a^1 - a$$

Indeed, for any $i \in \mathbb{N}$, $G_a p/q = a$ if i is even and 0 if i is odd, while $G_a^1 p/q = 0$ if i is even and a if i is odd. Thus $G_a p/q = G_a^1 p/q - a$ for any i . Hence our interpretation agrees with Bolzano's diagnostic of the fallacy in M.R.S.'s proof:

The series in the brackets obviously does not have the same multitude of terms [*Gliedermenge*] as the one put x at first, rather it is lacking the first a . Therefore its value, supposing it could actually be stated, would have to be denoted by $x - a$. But this would have given the identical equation

$$x - a = x - a:$$

Moreover, recall that Bolzano raises a second, deeper argument against M.R.S.'s conclusion: the infinite sum x cannot designate an 'actual quantity', since different ways of parsing this infinite sum yield different conclusions regarding which quantity it allegedly designates. According to Bolzano, the infinite sum

$$a - a + a - a + \dots \text{ in inf.}$$

represents the same quantity as the sums

$$(a - a) + (a - a) + (a - a) + \dots \text{ in inf.}$$

and

$$a - (a - a) + (a - a) - (a - a) + (a - a) - (a - a) + \dots \text{ in inf.}$$

But the first expression simplifies as

$$0 - 0 + 0 - 0 + \dots \text{ in inf.};$$

while the second one simplifies as

$$a - 0 + 0 - 0 + \dots \text{ in inf.};$$

Therefore, if it were a real quantity, x should be equal to both 0 and a , which is a contradiction.

What does this argument become in our interpretation? At first sight, it seems that we cannot make sense of Bolzano's claim that Grandi's series does not represent any actual quantity, since we attributed to this series the element \mathbf{G}_a in Z_U . However, it is straightforward to verify that, depending on which subsets of I are in U , \mathbf{G}_a is computed differently in the ultrapower. Indeed, since \mathbf{G}_a is the (equivalence class of) the sequence $\langle a; 0; a; 0; \dots \rangle$, we have that $\|\mathbf{G}_a = a\| = \{i \in I : a \in u_i\}$, while $\|\mathbf{G}_a = 0\| = \{i \in I : 0 \in u_i\}$. Now since U is an ultrafilter, exactly one of $\|\mathbf{G}_a = a\|$ or $\|\mathbf{G}_a = 0\|$ belongs to U . This implies that $Z_U \models \mathbf{G}_a = a \vee \mathbf{G}_a = 0$ regardless of our choice of ultrafilter, but the choice of U determines whether $Z_U \models \mathbf{G}_a = a$ or $Z_U \models \mathbf{G}_a = 0$. Thus we seem to recover at last part of Bolzano's intuition that the quantity designated by the sum $a - a + a - a + \dots \text{ in inf.}$ is indeterminate, as it can be computed to be equal to 0 or to a .

Bolzano also argues that the sum $a + a + a + a + \dots$ in *inf.* should represent the same quantity as the sum

$$a + pa + aq + pa + aq + \dots \text{ in } \textit{inf.};$$

which simplifies to

$$a + 0 + 0 + \dots \text{ in } \textit{inf.};$$

and should therefore designate the quantity a . His argument is that one may first compute Grandi's series as

$$pa + aq + pa + aq + \dots \text{ in } \textit{inf.}$$

Using commutativity of addition an infinite number of times, swap each pair of terms in order to obtain the series

$$p + a + aq + p + a + aq + \dots \text{ in } \textit{inf.};$$

which, by associativity is then equivalent to

$$a + pa + aq + pa + aq + \dots \text{ in } \textit{inf.}$$

In our setting, the infinite sum $a + a + a + a + \dots$ in *inf.* is represented by its approximating sequence $\langle p + a; 0; a; 0; \dots; q \rangle$. As a consequence, the infinite sums $a + a + a + a + \dots$ in *inf.* and $a + a + a + a + \dots$ in *inf.* will be identified in Z_U precisely if $\exists i: i \in P \wedge \exists u \in P \cup U$. In fact, as shown above, in such a case both series will be identified with 0.

In light of the remarks above, it might be tempting to conclude that Bolzano's criterion for an infinite sum to represent an actual quantity, namely that the order in which the terms are summed do not change the result of the summation, could be interpreted in our framework as some kind of absoluteness of the corresponding sequences under the choice of a non-principal ultrafilter U . However, it is straightforward to observe that Bolzano's own criterion is too strong for his purposes. Indeed, let us consider again the infinite sum $\sum_{n=0}^{\infty} 1^0 2^0 3^0 + \dots$ in *inf.* If we interpret, as we have done so far, n^0 as equal to 1 for any natural number n , then this infinite sum may actually be written as $1 + 1 + 1 + \dots$ in *inf.*, which is a special case of a geometric series of the form $\sum_{n=0}^{\infty} ar^n$ where $a = r = 1$. Similarly to Bolzano's argument for Grandi's series, we may now rewrite $\sum_{n=0}^{\infty} 2^n$ as

$$\begin{aligned} p + 1 + q + p + 1 + q + p + 1 + q + \dots \text{ in } \textit{inf.} &= 2 + 2 + 2 + \dots \text{ in } \textit{inf.} \\ &= 2p + 1 + 1 + \dots \text{ in } \textit{inf.} \\ &= \sum_{n=0}^{\infty} 2^n; \end{aligned}$$

from which we would be forced to conclude that $\overset{0}{N} = \overset{0}{N} = \overset{0}{N}$, implying that $\overset{0}{N} = 0$. Thus $\overset{0}{N}$ does not designate any infinite quantity after all, since it is equal to 0. This means that the order in which the terms in $\overset{0}{N}$ are summed determine which quantity the sum designates, which, by Bolzano's own criterion, is impossible. Of course, a Bolzanian could reply to that argument that there is a fallacy in deriving this equality, because the sum between parenthesis on the second line above does not have the same *Gliedermenge* as the original $1 + 1 + 1 + \dots$ in *inf.* Note that this response implies that changing the order in which terms are summed together, although it does not change the quantity designated by the sum, does change its *Gliedermenge*. Moreover, this answer is not entirely satisfactory. Indeed, if we assume that the right-hand side of the first equation above does not have the same *Gliedermenge* as $\overset{0}{N}$, we may therefore represent the two sums $p_1 + 1q + p_1 + 1q + p_1 + 1q + \dots$ in *inf.* and $1 + p_1 + 1q + p_1 + 1q + \dots$ in *inf.* by the sequences $A_1 : p_0; 2; 0; 2; 0; 2; \dots; q$ and $A_2 : p_1; 0; 2; 0; 2; 0; \dots; q$ respectively. Since both sums correspond to different ways of writing $\overset{0}{N}$, we should expect that $A_1 = A_2 = \overset{0}{N}$. However, one quickly notices that $p_{A_1} q / i = p_{A_2} q / i$ whenever i is odd, and $p_{A_2} q / i = p_{A_1} q / i$ whenever i is even. But this immediately implies that $Z_U \not\subseteq A_1 = A_2$. In other words, if we interpret the two infinite sums $p_1 + 1q + p_1 + 1q + p_1 + 1q + \dots$ in *inf.* and $1 + p_1 + 1q + p_1 + 1q + \dots$ in *inf.* by A_1 and A_2 , then in order to satisfy Bolzano's requirement that infinite associativity holds, we would need both the set of even numbers and the set of odd numbers to be in U , which is not possible. Note however that this has little to do with our formalisation: Bolzano himself seems committed to the following equalities:

$$p_1 + 1q + p_1 + 1q + p_1 + 1q + \dots \text{ in inf. } \overset{0}{N} = 1 + p_1 + 1q + p_1 + 1q + \dots \text{ in inf.}$$

$$2 + 2 + 2 + \dots \text{ in inf. } = 1 + 2 + 2 + \dots \text{ in inf.};$$

but there does not seem to be any reasonable way of establishing directly the latter equality. Let us conclude this section by noting that a weaker requirement could be imposed on infinite sums which designate actual quantities, namely that any finite permutation of the terms or of the order in which such terms are summed does not change the value of the sum. However, it is straightforward to verify that all sums in our formalisation satisfy this criterion: any two infinite sums that differ from one another only by a finite permutation of their terms or by finitely many rearrangements of the order in which those terms are summed are represented by approximating sequences which agree on a cofinite set and are therefore identified in Z_U . Thus this alternative criterion is too weak to rule out Grandi's series. In short, while Bolzano's first argument against M.R.S. can easily be translated in our framework, his second argument seems to prove either too much, or too little, for his purposes.

5.4.4 Comparisons with related work

Our central proposal is to model Bolzano's computations inside an ultrapower of the integers, and to identify the quantities designated by Bolzano in infinite sums with equivalence classes of functions from the positive integers to the integers. This idea is very close to a proposal made by Trlifajova (2018), although there are a few important differences that we must remark on. First, Trlifajova seems to be primarily interested in connecting Bolzano's ideas with some modern approaches to non-standard analysis, while we are more interested in a close reading of Bolzano's arguments and in establishing the consistency of our interpretation. Second, Trlifajova works mainly with equivalence classes of functions from \mathbb{N} to the real numbers. By contrast, we work with countable sequences of integers. Indeed, we believe that determining whether Bolzano's notion of a real number corresponds to our modern notion is a difficult problem. Bolzano, of course, made some significant contributions to the foundations of analysis. In particular, he developed a theory of *measurable* numbers (*RZ*, Part VII) which is often seen as an attempt to define the real numbers (see e.g. van Rootselaar 1964; Spalt 1991; Russ and Trlifajova 2016, and our discussion in Chapter 4). Trying to model Bolzano's computations with real numbers would require us to provide a detailed technical discussion of Bolzano's theory of measurable numbers, and it should be clear from Chapter 4 that it would be a highly nontrivial task. Since we are primarily interested in challenging the received view according to which Bolzano's computations should be read as a flawed attempt to develop an arithmetic of the transfinite, we believe that addressing this issue would take us too far astray. Just as Bolzano's measurable numbers are beyond the scope of our goals for this chapter, so are Bolzano's arguments in *PU* involving infinitely small quantities or infinitesimal calculus. Third, let us note that, in Trlifajova's framework, two sequences are identified if they agree on a cofinite set of natural numbers. Formally, this means that she works with a reduced power of \mathbb{R} rather than an ultrapower. While we do see the appeal of using only the Frechet filter on the natural numbers instead of a non-principal ultrafilter, we have several reasons to believe that our framework is more suitable to our purposes.

For one, only a weaker version of Los's theorem holds for reduced powers (see Hodges 1993, p. 445), which means that the resulting structure will not be as well-behaved as the ultrapower construction we are using. While this does not create significant technical issues at this stage, we will argue in the next section that the most accurate way of modelling Bolzano's views on the product of infinite sums is to conceive of it as an iterated infinite summation. This means that one will have to work with either iterated ultrapowers, or iterated reduced powers, the general theory of which is much less developed.

Moreover, we believe that the use of a non-principal ultrafilter rather than the Frechet filter can also be justified on interpretive grounds. Indeed, it is straightforward to verify that the reduced power Z_F , in which two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are

identified only if $\| \cdot \|$ is countable, does not satisfy trichotomy. For example, for the sequence $\langle p_1; 0; 1; 0; \dots; q \rangle$, we have that none of $\| \cdot \| = 0$, $\| \cdot \| > 0$ or $\| \cdot \| = \infty$ is a countable set, and thus $Z_F \not\models p = 0q \wedge p < 0q \wedge p = \infty q$. One might argue that this is a desirable feature of a formal reconstruction of Bolzano's ideas about the infinite, since, in §28, Bolzano writes that 'a determination of the relationship of one infinity to one another [...] is feasible, in certain cases at any rate [...]'. Nonetheless, we think that Bolzano should not be read in this passage as claiming that trichotomy may not hold in the case of infinite quantities. Indeed, as mentioned in Section 5.2.1, it is part of Bolzano's very definition of a quantity that it must obey the law of trichotomy. All things considered, then, we believe that a formalisation that preserves trichotomy (such as ours, using ultrapowers) is more faithful to the text than a formalisation that preempts the very possibility of trichotomy for infinite quantities, such as one using the Frechet filter.

A second related work is the recently proposed theory of numerosities (Benci and Di Nasso 2003), which we have already mentioned in the introduction to this chapter. Numerosities form a positive semi-ring that is meant to capture an intuitive notion of the size of sets of natural numbers. Benci and di Nasso introduce the technical notion of a labelled set of natural numbers, i.e. a set $A \subseteq \mathbb{N}$ with an associated labelling function $\lambda_A : A \rightarrow \mathbb{N}$ which is finite-to-one and represents a certain way of counting the elements of the set. One can then define the sum and product of two labelled sets in a natural way. Labelling functions allow for the representation of (disjoint unions and finite products of) subsets of \mathbb{N} as *approximating sequences*, which are non-decreasing functions from $\mathbb{N} \rightarrow \mathbb{N}$. The numerosity of a set A can then be defined as the equivalence class of its approximating sequence in an ultrapower N^U of \mathbb{N} by a Ramsey ultrafilter U . Benci and di Nasso show that the requirement that U be Ramsey guarantees that any element of N^U is the numerosity of some subset of \mathbb{N} . They also show that for any $A, B \subseteq \mathbb{N}$, if $A \sim B$ then $N^U \models \text{num}A = \text{num}B$, and that the numerosity of a disjoint sum (respectively, product) of two labelled sets is equal to the sum (respectively, product) of the numerosities of the labelled sets as computed by the ultrapower.

Numerosities share some features with our interpretation of Bolzano's computations, in particular regarding the way sums of infinite quantities are defined. However, a central motivation for the numerosity framework is to develop a theory of the size of sets of natural numbers that is consistent with what we called the set-theoretic part-whole principle **PW1**. As we will argue in Section 5.6, we take Bolzano's arithmetic of the infinite to be compatible with the set-theoretic part-whole principle but not motivated by it, as we do not believe that Bolzano is primarily concerned with counting sets of natural numbers but rather with developing a theory of infinite sums.

5.5 Higher-order infinities

5.5.1 The product of two infinite quantities

So far, we have shown how to interpret Bolzano's computations regarding infinite sums of the form $\sum_{i \in \mathbb{N}} a_i$ and $\sum_{i \in \mathbb{N}} b_i$, as well as Grandi's series. We have, however, refrained from giving an interpretation of Bolzano's computations involving products of two infinite quantities. Although our treatment of Bolzano's computations so far closely matches Trlifajova's and is consistent with numerosities, our account of Bolzanian products of infinite quantities will be quite different. Indeed, it seems at first sight that there is a natural way to define the product of two quantities in Z_U . Similarly to the way addition is defined, we could define the product componentwise. Formally, for any $f, g: \mathbb{N} \rightarrow Z$, letting $f \cdot g: \mathbb{N} \rightarrow Z$ be the function mapping any $i \in \mathbb{N}$ to $f(i) \cdot g(i)$, we may define $f \cdot g$ as $\langle f \cdot g \rangle$. This is the definition adopted by Benci and Di Nasso (2003) and Trlifajova (2018), and it is straightforward to check that, under this definition, the structure $\langle Z_U; +; \cdot; 0; 1; < \rangle$ is an ordered commutative ring. However, we believe that this definition of the product does not satisfactorily account for Bolzano's ideas as exposed in *PU*. We will first lay out our textual evidence for this claim and then explain how our interpretation works.

Bolzano gives explicit computations of the product of two infinite quantities in only one passage towards the end of §29:

The purely symbolic equation [(1)]¹⁸ underlying all this will surely allow the derivation, through successive multiplication of both sides by $\frac{0}{N}$, of the following equations:

$$\begin{aligned} 1^0 : \frac{0}{N} \quad 2^0 : \frac{0}{N} \quad 3^0 : \frac{0}{N} \quad \dots \quad \text{in inf.} \quad \langle \frac{0}{N} q^2 \rangle \\ 1^0 : \frac{0^2}{N} \quad 2^0 : \frac{0^2}{N} \quad 3^0 : \frac{0^2}{N} \quad \dots \quad \text{in inf.} \quad \langle \frac{0}{N} q^3 \rangle \quad \text{etc:} \end{aligned}$$

from which we are convinced that there [are] also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times. But it also certainly follows from this [that] there are infinite quantities which have every arbitrary rational, as well as irrational, ratio $\frac{0}{N}$ to one another, because, as long as $\frac{0}{N}$ denotes some infinite

18. The German version of the text reads (4) here, but the context clearly suggests that this is a mistake.

quantity which always remains the same, $1^0:N^0$ and $2^0:N^0$ are likewise a pair of infinite quantities which are in the ratio $1:2$.

Bolzano defines the product of the quantity N^0 with itself, noted pN^0q^2 , as the result of summing N^0 with itself N^0 many times. The equation

$$1^0:N^0 \quad 2^0:N^0 \quad 3^0:N^0 \quad \dots \text{ in inf. } pN^0q^2$$

is obtained from the equation

$$1^0 \quad 2^0 \quad 3^0 \quad \dots : n^0 \quad pn \quad 1q^0 \quad \dots \text{ in inf. } N^0$$

by multiplying by N^0 on both sides. This seems to suggest that Bolzano assumes some form of distributivity of multiplication over infinite summation, which allows him to equate $p1^0 \quad 2^0 \quad 3^0 \quad \dots \text{ in inf. } q:N^0$ with $1^0:N^0 \quad 2^0:N^0 \quad 3^0:N^0 \quad \dots \text{ in inf.}$ on the left-hand side of the equality symbol. Understood as such, pN^0q^2 is an infinite sum in which all terms are infinite quantities. Quantities of the form pN^0q^n are the only example in Bolzano's text of quantities defined explicitly as infinite sums of infinite quantities. It is also worth mentioning that, even though Bolzano discusses other examples of infinite quantities being infinitely smaller or larger than one another, this is the only case in §§29-33 where some infinite quantities are explicitly referred to as being 'of higher order' than some others.¹⁹

If we were to interpret pN^0q^2 in a similar fashion as Trlifajova and Benci and Di Nasso, we would have to define the quantity pN^0q^2 in such a way that

19. The authors thank an anonymous referee for noting that an alternative interpretation of §29 is also plausible. When introducing pN^0q^2 and pN^0q^3 , Bolzano writes that this 'convinc[es us] that there are also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times.' This can be read as meaning that whenever an infinite quantity A exceeds an infinite quantity B infinitely many times, then A is an infinite of higher order with respect to B . In other words, the definition of infinities of higher order is infinities that exceed smaller infinities by an infinitely large factor. This understanding of 'higher order' is problematic, however, for at least two reasons. First, if 'of higher order' simply meant 'infinitely larger or smaller', then the introduction of S^1 in §29 should have succeeded to establish the existence of infinite quantities of higher-order, since Bolzano has already noted by that point that S^1 is 'far greater than' N^0 . Second, in the definition of 'infinite' (§10), Bolzano presents the concept of infinitely smaller and infinitely greater quantities of higher order as quantities derived from, but not identical with, infinitely small and infinitely large quantities. The referee's interpretation, by contrast, would collapse the notion of infinities of higher order into that of infinities *simpliciter*, per Bolzano's definition.

$\sum_{i=1}^n p/q^2 = \sum_{i=1}^n p/q + \sum_{i=1}^n p/q$ for all $n \in \mathbb{N}$. However, due to the well-known fact that the sum of the first n odd numbers is always equal to n^2 , the infinite sum $\sum_{i=1}^{\infty} (1 + 3 + 5 + 7 + \dots)$ is also represented by (the equivalence class of) the sequence $\{1; 4; 9; 16; \dots\}$. It would therefore follow that $\sum_{i=1}^{\infty} (1 + 3 + 5 + \dots) = \sum_{i=1}^{\infty} i^2$. We should conclude that the two infinite sums $\sum_{i=1}^{\infty} (1 + 3 + 5 + \dots)$ and $\sum_{i=1}^{\infty} i^2$ actually designate the same quantity. But this seems a clear violation of Bolzano's treatment of order relationships between infinite sums. Indeed, we saw above that, in showing that $\sum_{i=1}^{\infty} i^2$ was infinitely greater than $\sum_{i=1}^{\infty} i$, Bolzano reached his conclusion by showing that the difference between matching summands in $\sum_{i=1}^{\infty} i^2$ and in any finite multiple of $\sum_{i=1}^{\infty} i$ is always positive for all but finitely many summands. In this case too, since $\sum_{i=1}^{\infty} i^2$ and $\sum_{i=1}^{\infty} i$ have the same number of terms, we could also argue along Bolzanian lines that, for any natural number n , the difference $\sum_{i=1}^{\infty} i^2 - n \sum_{i=1}^{\infty} i$ is given by the sum $\sum_{i=1}^{\infty} (i^2 - ni) = \sum_{i=1}^{\infty} i(i-n)$, in which all summands are positive (and in fact infinite). As we have argued in Section 5.4.2, one can extract from Bolzano's writings a sufficient criterion for one sum to be strictly greater than another sum, namely when all but finitely many terms in the sum are positive. We will come back to this issue at greater length in Section 5.6. For now, let us note that, if our interpretation is correct, we must conclude in the present case that $\sum_{i=1}^{\infty} i^2$ is greater than any finite multiple of $\sum_{i=1}^{\infty} i$, and thus that $\sum_{i=1}^{\infty} i^2 > \sum_{i=1}^{\infty} i$. The componentwise definition of the product of two quantities is therefore incompatible with Bolzano's own criterion for comparing infinite sums.

Moreover, another passage from §29 seems to explicitly contradict the 'componentwise' interpretation of the product of two infinite quantities. Indeed, when introducing the sum of all natural numbers $\sum_{i=1}^{\infty} i$, Bolzano writes:

On the other hand if we designate the quantity which represents the sum of all natural numbers by $\sum_{i=1}^{\infty} i$, or assert the merely symbolic equation

$$1 + 2 + 3 + \dots + n + pn + 1q + \dots = r \sum_{i=1}^{\infty} i \quad (4)$$

then we will certainly realize that $\sum_{i=1}^{\infty} i$ must be far greater than $\sum_{i=1}^{\infty} i$. But it is not so easy to determine precisely the difference between these two infinite quantities or even their (geometrical) ratio to one another.

For if, as some people have done, we wanted to form the equation

$$rS = \frac{N + pN + 1q}{2}$$

then we could hardly justify it on any other ground than that for every finite multitude of terms [*Menge von Gliedern*] the equation

$$1 + 2 + 3 + \dots + n = \frac{n \cdot pn + 1q}{2}$$

holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes N . However it is in fact not so, because with an infinite series it is absurd to speak of a last term which has the value N .

Bolzano's point here seems to be that one cannot infer from the validity of Gauss's summation theorem for finite numbers that an 'infinite' version of the summation theorem also holds for infinite quantities. His rejection of the infinite summation theorem can be given two readings, one stronger, and one weaker. On the stronger reading, Bolzano is arguing that the infinite summation theorem is false, because the only way of justifying it, namely, through an inference from the finite to the infinite, leads to a false consequence.²⁰ On the weaker reading, by contrast, Bolzano is not asserting the falsity of the infinite summation theorem, but he is merely refraining from asserting its truth, because what is ostensibly the only argument to prove its truth is a defective argument.

Under the first reading, which we tend to find more natural, the componentwise definition of the product *a la* Trlifajova (2018) and Benci and Di Nasso (2003) is simply inconsistent with Bolzano's own views, as the infinite summation theorem is true in the structure $\mathcal{Z}_U; \leq; 0; 1; q$:

Lemma 5.5.1. *Let $N = pN + 1q$ be such that $N = pN + 1qp/q = Np/q + pN + 1qp/q$ for any $i \in \mathbb{P} \setminus \emptyset$. Then $\mathcal{Z}_U \models \exists N = pN + 1q \exists S$.*

Proof. By definition, $\|\exists N = pN + 1q \exists S\| = \{i \in \mathbb{P} \setminus \emptyset : pN + pN + 1qp/q = 2Sp/q\}$. Now for any $i \in \mathbb{P} \setminus \emptyset$, $2Sp/q = 2 \cdot \frac{ip + 1q}{2} = ip + 1q$ by Gauss's summation theorem. On the other hand, $pN + pN + 1qp/q = Np/q + pN + 1qp/q = i + pi + 1q$. Thus $\|\exists N = pN + 1q \exists S\| = \emptyset$, and therefore is contained in U . \square

²⁰ Compare with Bolzano's rejection of a similar argument from the *WL*, which we discussed in depth in Chapter 3.

Since we are interested in establishing at least the consistency of Bolzano's calculation of the infinite, the stronger reading of this passage of the infinite summation theorem compels us to provide an alternative definition of the product of two Bolzanian quantities.

Moreover, we find that this conclusion also follows from the second, weaker reading mentioned above. Indeed, even if Bolzano is merely punting here on the truth of the infinite summation theorem, we find it quite revealing that he would object to the infinite summation theorem being a direct consequence of Gauss's summation theorem. Indeed, this passing from the finite to the infinite is very similar to the various 'arguments by co-finiteness' that Bolzano appeals to in *§29* and *§32*, and which we discussed at length in the previous section. As we have noticed above, the formal setting of ultrapowers, in which operations can be defined componentwise, allows for a straightforward reconstruction of such arguments by co-finiteness, with the help of Los's theorem. In fact, the proof of Lemma 5.5.1 above proceeds precisely in the same way as the inference rejected by Bolzano: since the summation theorem holds for any $i \in \mathbb{N}$, it transfers to the infinite quantities $\overset{1}{S}$ and $\overset{0}{N}$. Bolzano therefore seems to have two distinct attitudes with regard to these inferences from the finite to the infinite: while he uses arguments by co-finiteness when establishing results about sums and differences of infinite sums, he explicitly rejects this style of reasoning when discussing ratios of infinite sums, i.e. results about products of infinite sums. If we were to model such products componentwise, we would be allowing in our formal setting precisely the type of inference that Bolzano objects to. This seems cause enough to us to propose an alternative definition of the products of two Bolzanian sums.

5.5.2 Second-order infinities via an iterated ultrapower

As shown above, the componentwise interpretation of the product adopted both by Trlifajova and Benci and Di Nasso has unfortunate consequences for our project. If we want to model Bolzanian computations with the infinite as accurately as possible, we must therefore propose an alternative interpretation. Our solution springs from the observation above that the product $\overset{0}{p/q^2}$ is written by Bolzano as an infinite sum in which the summands themselves are infinite quantities. Since we decided to model infinite sums of integers as functions from an index set I into the integers, we should therefore model infinite sums of possibly infinite quantities as functions from I into a structure that contains those infinite quantities, i.e. into Z_U .

Formally, this means that we should now work in an ultrapower of Z_U , i.e. in an iterated ultrapower. Letting $\overset{0}{pZ_Uq^2}$ denote this ultrapower, we have a straightforward embedding $\iota : Z_U \rightarrow \overset{0}{pZ_Uq^2}$, induced by the map sending any $f : I \rightarrow Z$ to the map $i \in I \mapsto \overline{f(i)}$, where $\overline{f(i)}$ is the constant function returning $f(i)$ for

any $j \in \mathbb{P} \setminus \{0\}$. Given an infinite sum of (possibly infinite) quantities in Z_U , say $x_1 + x_2 + x_3 + \dots$ in *inf.*, we proceed as before by identifying this sum with the countable sequence $\langle x_1, x_2, x_3, \dots \rangle$, and determining its quantity as the equivalence class in the iterated ultrapower ${}^p Z_U q^2$ of the sequence $\langle x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots \rangle$, where the partial sums of the first n terms in $\langle x_i \rangle$ are computed inside Z_U . In the case of ${}^p \mathbb{N} q^2$, this means that we identify the infinite sum with the sequence ${}^p \mathbb{N} q^2 : \langle {}^0 p\mathbb{N}; {}^0 \mathbb{N}; {}^0 \mathbb{N}; \dots \rangle$. The corresponding approximating sequence is then $\langle {}^0 p\mathbb{N}; {}^0 2\mathbb{N}; {}^0 3\mathbb{N}; \dots \rangle$, which means that ${}^p \mathbb{N} q^2$ is the equivalence class of the function assigning to each $i \in \mathbb{P} \setminus \{0\}$ the quantity $i^0 \mathbb{N}$. Similarly, we could form the infinite sum $S + S + S + \dots$ in *inf.*, which corresponds to summing the quantity S \mathbb{N} -many times to itself. This sum is interpreted as the series $\mathbb{N} : S : \langle {}^1 pS; {}^1 S; {}^1 S; \dots \rangle$, with approximating sequence $\langle {}^1 pS; {}^1 2S; {}^1 3S; \dots \rangle$, so $\mathbb{N} : S$ is the equivalence class of the function assigning $i^1 S$ to each $i \in \mathbb{P} \setminus \{0\}$.

Going one step further, we could also wonder how the product $S : \mathbb{N}$, i.e. summing S -many times the quantity \mathbb{N} , should be interpreted. Just as we computed ${}^p \mathbb{N} q^2$ by taking \mathbb{N} as a unit in our summation instead of 1, it seems that, in computing $S : \mathbb{N}$, we should take \mathbb{N} as a unit in the summation $x_1 + x_2 + x_3 + \dots$ in *inf.* which yields S . This suggests that summing \mathbb{N} S -many times with itself yields the infinite sum

$$\mathbb{N} : 2\mathbb{N} : 3\mathbb{N} : \dots \text{ in inf.}$$

According to our interpretation, this sum is represented by the sequence

$$S : \mathbb{N} : \langle {}^0 p\mathbb{N}; {}^0 2\mathbb{N}; {}^0 3\mathbb{N}; \dots \rangle;$$

whose approximating sequence is $\langle {}^0 p\mathbb{N}; {}^0 3\mathbb{N}; {}^0 6\mathbb{N}; \dots \rangle$. Hence $S : \mathbb{N}$ is the equivalence class of the function that assigns ${}^1 S p/q \mathbb{N} = \frac{ip}{2} \frac{1q}{2} \mathbb{N}$ to any $i \in \mathbb{P} \setminus \{0\}$. More generally, given any two infinite quantities x and y in Z_U , we may define the product $x : y \in \mathbb{P} \setminus \{0\}$ ${}^p Z_U q^2$ as the equivalence class of the function mapping any $i \in \mathbb{P} \setminus \{0\}$ to ${}^i p/q$, where ${}^i p/q = \underbrace{p/q}_{i \text{ times}}$. The relevant definitions are

summarised in the table below:

This definition of the product of two infinite quantities has three important consequences. First, as evidenced already by the examples of $\mathbb{N} : S$ and $S : \mathbb{N}$ above, the product operation will in general not be commutative. Although this might seem as a highly non-Bolzanian feature of our setup, we remark that this does not

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$\overset{0}{N} \overset{0}{N} \overset{0}{N} \dots \text{in inf.}$	$\overset{0}{pNq^2} \overset{0}{pN:N} \dots q$	$\overset{0}{pNq^2q} \overset{0}{pN:2N:3N} \dots q$	$\overset{0}{ppNq^2qp/q} \overset{0}{iN}$	$\overset{0}{pNq^2} \overset{0}{ppNq^2q}$
$\overset{1}{S} \overset{1}{S} \overset{1}{S} \dots \text{in inf.}$	$\overset{0}{N:S} \overset{1}{pS:S} \dots q$	$\overset{0}{pN:Sq} \overset{1}{pS:2S:3S} \dots q$	$\overset{0}{pN:Sqp/q} \overset{1}{iS}$	$\overset{0}{N:S} \overset{1}{pN:Sq}$
$\overset{1}{1N} \overset{0}{2N} \overset{0}{3N} \dots \text{in inf.}$	$\overset{1}{S:N} \overset{0}{p1N:2N:3N:4N} \dots q$	$\overset{1}{pS:Nq} \overset{0}{p1N:3N:6N:10N} \dots q$	$\overset{1}{pS:Nqp/q} \overset{0}{j-1N}$	$\overset{1}{S:N} \overset{0}{pS:Nq}$
$\overset{1}{p1q} \overset{0}{p2q} \dots \text{in inf.}$	$\overset{1}{p} \overset{0}{p1q} \overset{0}{p2q} \dots q$	$\overset{1}{p} \overset{0}{p} \overset{0}{qp1q} \overset{0}{p} \overset{0}{qp2q} \dots q$	$\overset{1}{p} \overset{0}{qp/q} \overset{0}{pqp/q}$	$\overset{1}{p} \overset{0}{p} \overset{0}{q}$

Table 5.2: Representation of Bolzanian products in pZ_Uq^2

directly contradict any of Bolzano's computations in *PU*. Moreover, contrary to the associativity and commutativity of addition, which he sees as rooted in the concept of sum and therefore a feature of the general theory of quantity, associativity and commutativity of multiplication of integers are introduced as theorems in Bolzano (*RZ*, §§19-20 part III, pp. 62-63), instead of being part of the definition of a product. Moreover, we think that the non-commutativity of the product of two infinite quantities is itself motivated by Bolzanian considerations. Indeed, if one agrees that the correct interpretation for $\overset{0}{N:S}$ and $\overset{1}{S:N}$ are the infinite sums $\overset{1}{S} \overset{1}{S} \overset{1}{S} \dots \text{in inf.}$ and $\overset{0}{N} \overset{0}{2N} \overset{0}{3N} \dots \text{in inf.}$ respectively, then the Bolzanian strategy for comparing two infinite sums, namely computing their difference term by term, yields that $\overset{0}{N:S} \overset{1}{S:N} \overset{1}{pS} \overset{0}{Nq} \overset{1}{pS} \overset{0}{2Nq} \overset{1}{pS} \overset{0}{3Nq} \dots \text{in inf.}$ is itself an infinite sum of positive quantities. It is therefore positive, which means that $\overset{0}{N:S}$ should be strictly greater than $\overset{1}{S:N}$.

Second, it is easy to verify that, under this definition of the product, the summation theorem does not hold in the infinite case. Indeed, in our interpretation, $\overset{0}{N:pN} \overset{0}{1q}$ is the function mapping any $i \in P \setminus \{0\}$ to $\overset{0}{Np/q} \overset{0}{pN} \overset{0}{1q}$. Now since $\overset{0}{N} \overset{0}{1}$ is (the equivalence class of) the function mapping any $j \in P \setminus \{0\}$ to $j-1$, it follows that $\overset{0}{Np/q} \overset{0}{pN} \overset{0}{1q} \overset{0}{i} \overset{0}{pN} \overset{0}{1q}$ maps any $j \in P \setminus \{0\}$ to $\overset{0}{ipj} \overset{0}{1q}$. On the other hand, in pZ_Uq^2 , $\overset{1}{2S}$ maps any $i \in P \setminus \{0\}$ to $\overset{1}{2Sp/q} \overset{0}{ipj} \overset{0}{1q}$. Hence $\|\overset{1}{2S} \overset{0}{N:pN} \overset{0}{1q}\| \overset{1}{t} \overset{0}{i} \overset{0}{P} \setminus \{0\} : Z_U \overset{1}{i} \overset{0}{ipj} \overset{0}{1q} \overset{0}{i} \overset{0}{pN} \overset{0}{1q}$. Now for any $i, j \in P \setminus \{0\}$, $\overset{0}{ipj} \overset{0}{1qp/j} \overset{0}{ipj} \overset{0}{1q}$, while $\overset{0}{pi} \overset{0}{pN} \overset{0}{1qp/j} \overset{0}{ipj} \overset{0}{1q}$, hence $Z_U \overset{1}{i} \overset{0}{ipj} \overset{0}{1q} \overset{0}{i} \overset{0}{pN} \overset{0}{1q}$ for all $i \in P \setminus \{0\}$. Therefore $pZ_Uq^2 \not\subseteq \overset{1}{2S} \overset{0}{N:pN} \overset{0}{1q}$.

Finally, we argue that this definition of the product gives a better interpretation of Bolzano's remark that quantities like $\overset{0}{pNq^2}$ are infinities of a 'higher order'. Indeed, our construction introduces a clear stratification between integers, infinite quantities of the first order (i.e. elements introduced in the first ultrapower Z_U), and infinite quantities of the second order (i.e. elements introduced in the second ultrapower pZ_Uq^2). In fact, in our interpretation, genuine second-order infinite

positive quantities are always larger than any first-order infinite quantity:

Lemma 5.5.2. *Suppose $\langle \mathbb{P}, \mathbb{Z}_U \rangle$ are such that $\mathbb{Z}_U \models \exists i \ m \wedge i < m$ for any integer m . Then $\mathbb{P} \models \exists i \ i < i$.*

Proof. We claim that $\mathbb{P} \models \exists i \ i < i$. This amounts to showing that, for U -many $j \in \mathbb{P}$, $\mathbb{P} \models \exists i \ i < j$. Now suppose $\mathbb{P} \models \exists i \ i < j$ (which is true for U -many $j \in \mathbb{P}$). Then $k \in \mathbb{P} \models \exists i \ i < j$ if and only if $k \in \mathbb{P} \models \exists i \ i < \frac{j}{q}$, which is true for U -many k since, letting m be the smallest integer greater than $\frac{j}{q}$, we have that $\mathbb{Z}_U \models \exists i \ i < m$. \square

However, an obvious drawback of modelling second order infinite quantities by iterating the ultrapower construction is that we must repeat this procedure again in order to account for third-order infinite quantities, and so on. In fact, provided we want to make sense of quantities of the form $\mathbb{P} \models \exists i \ i < i^n$ for any natural number n , we must iterate our ultrapower construction countably many times. This requires us to construct models of the form $\mathbb{P} \models \exists i \ i < i^n$ for any n , with embeddings from each $\mathbb{P} \models \exists i \ i < i^n$ into $\mathbb{P} \models \exists i \ i < i^{n-1}$:

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}_U \xrightarrow{1} \mathbb{P} \models \exists i \ i < i^2 \xrightarrow{2} \mathbb{P} \models \exists i \ i < i^3 \xrightarrow{3} \dots$$

Limits of iterated ultrapowers are a standard tool in mathematical logic. The direct limit B of this chain of ultrapowers contains quantities of arbitrarily large orders of infinity, and allows for a rigorous definition of the product \cdot of two infinite quantities. In fact, we obtain a particularly well-behaved structure:

Theorem 5.5.3. *The structure $B = \langle \mathbb{P}; \cdot; 0; 1; \cdot; \cdot \rangle$ is a non-commutative ordered ring.*

We refer the interested reader to the Appendix for a proof of this theorem as well as details about the structure B . For now, let us simply conclude that this formal result establishes that our interpretation of Bolzani sums yields a rich and original structure which nonetheless shares many properties with the integers.

5.6 Reassessing the PU

In the preceding sections we have touched on the following three issues:

1. Whether Bolzano's work truly was about (something like) the sets of set theory, or not. We argued that Bolzano's work in *PU* §§29-33 is best understood as being an attempt at giving solid foundations to the handling of infinite series (which correspond to Bolzano's infinite sums).

2. Whether part-whole reasoning plays an important role or not in Bolzano's computations. We argue that a form of part-whole reasoning about infinite sums, not about infinite sets, plays a central role in Bolzano's argument, even though Bolzano's argument does not contradict set-theoretic part-whole (PW1 in the Introduction).
3. Whether Bolzano's relation to what we may call the 'first generation' of set theorists (specifically Cantor) needs to be reassessed. We think it does.

In this section, we discuss in detail where we stand on each point in turn, making use of the formalisation from Section 5.4 and Section 5.5 whenever necessary.

5.6.1 A theory of infinite sums

We have argued that Bolzano's primary interest in *PU* §§29-33 is in infinite sums of integers, rather than sets and their sizes. To be more specific, we wanted to illustrate that by interpreting these sections as trying (and largely failing) to anticipate Cantorian inventions, one would fundamentally misrepresent Bolzano's work. Instead of there being one notion that, like Cantor's cardinals (or powers) captures the quantitative aspect of a collection, Bolzano has rather two quantity notions associated to each of his infinite sums: the *Gliedermenge* of the corresponding series of summands, and the sum itself (which for us would be the *value*, or the result of performing the infinite addition { Bolzano's notion of sum does not allow for a distinction between a sum and its value).

These infinite sums (or the underlying series) can undergo certain transformations, which may induce a change in the *Gliedermenge*, a change in the value of the sum, or both. We saw in Bolzano's work three examples of such operations:

1. raising all the terms in a sum to the same power;
2. 'erasing' some of the terms in a sum;
3. permuting terms in a sum or computing summands in a different order.

§§29 and 33 suggest that raising all natural numbers at once to the same power does not change the *Gliedermenge* of an infinite sum, but it does change its value. Indeed, N^0 and S^2 are obtained from the infinite sum $1 + 2 + 3 + \dots$ in *inf.* (i.e. S^1) by raising all terms in this sum to the 0^{th} and 2^{nd} power, respectively. Bolzano explicitly states in §33 that this operation does not change the *Gliedermenge* of the corresponding sum, which is why he is able to determine that $N^0 = S^1 = S^2$. On the other hand, the second operation, which consists in erasing some of the terms in an infinite sum, does change the *Gliedermenge* of the infinite sum in such a way that also induces a change in the overall value of the sum. Bolzano's clearest examples of this are quantities of the form N^n , which vary from N^0 only in that the

first n terms of the sum are removed. Nonetheless, as we have seen above, this reasoning also appears in §32, where it plays a crucial role in Bolzano's rejection of M.R.S.'s identification of the infinite sum $a + a + a + a + \dots$ in *inf.* with the sum within brackets in $a + pa + a + a + a + \dots$ in *inf.*. Finally, regarding the third operation, Bolzano seems to adhere to the idea that because the laws of commutativity and associativity should always hold for addition, this operation should not change the value of the sum if the sum designates any value at all. As we have shown above, Bolzano uses this criterion to argue that Grandi's series does not designate any actual quantity, but seems unaware of the fact that his argument also creates difficulties for infinite sums like $1 - 1 + 1 - 1 + \dots$ in *inf.* We have also argued that those issues should commit Bolzano to the thesis that changing the order in which terms are summed in an infinite sum also changes its *Gliedermenge*, although he does not explicitly make this point.

In our formalisation of Bolzano's computations, we treat all infinite sums as countable sequences of integers, to which we associate a countable sequence of partial sums. For infinite sums which have the same *Gliedermenge* as \mathbb{N} , this can be done in a straightforward way by identifying an infinite sum with its sequence of partial sums, and our ultrapower construction allows us to assign different values to such sums. For infinite sums which have a different *Gliedermenge*, like \mathbb{N} or \mathbb{G}_a , we only need to make some natural choices in the way we represent them to retrieve Bolzano's results. We therefore believe to have established that Bolzano's computations in *PU* form a consistent theory of divergent infinite sums, which paint a picture of the arithmetic of the infinite largely different from our modern, set-theoretic, conception. In particular, interpreting Bolzano as developing a theory of infinite sums allows us to reassess the role that part-whole considerations play in his theory.

5.6.2 Part-whole reasoning in Bolzano's computations

As we have mentioned above, we do not think, *pace* Berg and Sebestik, that Bolzano's computations in *PU* §§29-33 are incompatible with his use of part-whole reasoning in §§17-24. In fact, we argue that part-whole reasoning plays a central role in Bolzano's determination of the relationship between infinite quantities. However, since, as we have argued, Bolzano is developing in §§29-33 a theory of infinite sums and not a theory of infinite (set-like) collections, we must exert caution in determining how we should understand the principle that 'the whole is always greater than its proper parts'. The more common interpretation of this principle (see e.g. Mancosu 2009) is set-theoretic:

PW1 For any sets $A; B$, if $A \subset B$, then $\text{size}pAq < \text{size}pBq$.

This formulation of the part-whole principle is, by and large, the one satisfied for labelled sets of natural numbers by numerosities as defined by Benardete and

Di Nasso (2003). In particular, in the numerosity structure $\langle \mathbb{N}; \preceq \rangle$ constructed by Benci and di Nasso, the following holds:

Num For any (labelled) set of natural numbers A and any numerosity ν , $\nu(A) > \nu(B)$ if and only if there is a (labelled) set $B \subset A$ such that $\nu(B) = \nu(A)$.

However, a more general version of the part-whole principle, which avoids set-theoretic parlance entirely, is given by Bolzano in his *GL*. This is to be found in the definition of 'greater than', which we transcribe here together with the immediately following remark, which shows that Bolzano is aware of the difficulty his definition of 'less/greater than' creates for determining relationships between quantities which may be infinitely large or infinitely small, but adopts it nonetheless:

§27 Def. If the quantity N lets itself be considered as a whole, which includes in itself the quantity M or one that is equivalent to it as part, then we say that N is greater than M , and M is smaller than N and we write it as $N \succ M$ or $M \prec N$. Should this much be established, that M is not greater or not smaller than N ; then we write in the first case $M \sim N$ and in the second case $M \cdot N$.

§28 Remark. What I here pick as definition, that each whole must be greater than its part, and the part smaller than the whole (as long as they are both quantities) some, namely already Gregory of St. Vincent and in more recent times also Schultz (in his Foundations of the pure Mathesis), do not want to concede, because of quantities which are infinitely large or infinitely small. If M is infinitely large, but m is finite, or M is finite, but m infinitely small, then people say that the whole $pM \sim mq$ composed from the parts m and M isn't to be truly called greater than the part M . [...] ²¹ (*EB*, p. 237)

The quote above clearly indicates both that Bolzano sees himself as employing some version of the part-whole principle as the criterion for size comparison between quantities, and that two quantities A and B are related as whole and part, respectively, if and only if there is a positive (non-negative, non-zero) quantity

21. §.27 Erkl. Wenn sich die Grö ß e N als ein Ganzes ansehen lä ß t, welches die Grö ß e M oder eine ihr gleichkommende als ein Theil in sich schlie ß t; so sagen wir, N sey grö ß er als M , M aber kleiner als N und schreiben die $N \succ M$ oder $M \prec N$. Wenn um so viel bestimmt werden soll, da ß M nicht grö ß er oder nicht, kleiner als N sey; so schreiben wir im ersten Falle $M \sim N$ oder im zweyten $M \cdot N$.

§28 Anm. Was ich hier als Erklärung annehme, da ß jedes Ganze grö ß er als sein Theil, und der Theil kleiner als das Ganze seyn müsse, (so fern beyde Grö ß en sind), haben Einige, namentlich schon Gregor v. St.Vincenz und in neuerer Zeit auch wieder Schultz (in seinen Anfangsgr. d. rei. Mathesis) in Hinsicht solcher Grö ß en, die unendlich grö ß oder klein sind, nicht zugestehen wollen. Wenn M unendlich grö ß, m aber endlich ist, oder wenn M endlich, m aber unendlich klein ist; so behauptet man da ß aus den Theilen M und m zusammengesetzte Ganze $pM \sim mq$ sey nicht wirklich grö ß er als der Theil M zu nennen. [...]

C such that $A + B = C$. Then Bolzano's definition of less-than ($<$) can be formulated as follows:

PW2 For any two quantities $A; B$, $A < B$ if and only if there is some positive quantity C such that $A + C = B$.

This latter principle can indeed be seen as preserving the part-whole intuition: if A is a proper part of B , then the part C of B obtained by removing A from B is non-null, and clearly its sum with A yields back B . In particular, if the operation of taking the sum of two quantities has an inverse (removing a part from a whole), then **PW2** can be rephrased as follows:

PW3 For any two quantities $A; B$, $A < B$ if and only if $B - A$ is positive.

Our claim is that Bolzano is endorsing **PW3** when determining order relations between infinite sums. Note that for **PW3** to apply to infinite sums, one needs first to define two things:

- a) the difference of two infinite sums and ;
- b) when an infinite sum is positive.

As we have argued above, Bolzano solves those two issues in his calculation of the infinite as follows:

- a) For two infinite sums and having the same *Gliedermenge*, their difference is computed termwise: is the infinite sum in which the i^{th} term is $a_i - b_i$, i.e. the difference of the i^{th} terms of and respectively;
- b) An infinite sum is positive if all but finitely many of its terms are positive.

Bolzano is thus able to derive from **PW3** a *sufficient* criterion for order relationships between infinite sums:

PW4 For any two infinite sums , , if all but finitely many terms in are positive.

It is worth noting once again that this criterion is exactly the version of **PW3** at play in Bolzano's proof that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$ in §33. Moreover, Bolzano explains his reasoning in terms of part-whole relationships between sums:

So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, namely:

$$0; 2; 6; 12; 20; 30; 42; 56; \dots; npn - 1q; \dots \text{ in inf.};$$

in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. (*PU*, §33)

We therefore conclude that the part-whole principle plays an important role in Bolzano's computations, but also that, in his calculation of the infinite, Bolzano's text should not be interpreted as displaying some instances of part-whole reasoning about sets and their proper subsets. Rather, in deriving those results, part-whole reasoning is applied to infinite sums in the precise sense of **PW4**.²² In our formalisation of Bolzano's computations, we have shown that computations with infinite sums based on **PW4** could be carried out in a consistent fashion. In fact, as a simple consequence of the fact that our structure B is elementarily equivalent to the integers, we have that $B \models \forall p \in \mathbb{N} \exists q \in \mathbb{N} p < q$. Moreover, we have also argued that Bolzano's criterion could also be applied in a productive way to determine order relations between infinities of higher order. As a consequence, we showed how a Bolzanian product of infinite quantities could be interpreted as a non-commutative monoidal operation, i.e. a well-behaved operation which is nonetheless considerably different from the product of Cantorian cardinalities or even the product of numerosities.

Finally, let us note that, although we have argued that the correct way to interpret Bolzano's part-whole reasoning does not commit him to the set-theoretic part-whole principle (**PW1**), we nonetheless believe that **PW1** is *compatible* with Bolzano's arguments. In fact, we are now in a position to fully describe a way out for Bolzano from the apparent contradiction of §33 (cf. Section 5.3) that we believe is satisfactory even from a modern standpoint. Indeed, following the position sketched in Section 5.3, we may argue that the number (*Menge*) of natural squares is not equal to the *Gliedermenge* of the infinite sum $\overset{2}{S}$ but that it must be computed, in relation with $\overset{0}{N}$, as the *value* of the sum $\overset{SQ}{N} = 1^0 + 2^0 + \dots + q^0 \dots$ *in inf.* The approximating sequence of this sum is $p; 1; 1; 2; 2; \dots; q$, and it is therefore

22. This does not mean that **PW4** is the correct interpretation of Bolzano's part-whole reasoning throughout the *PU*. As we have noted in Section 5.2, Bolzano is clearly committed to a form of part-whole reasoning about collections in $\text{xx}19 - 24$. We also thank an anonymous referee for pointing out that in the following passage from $\text{x}29$, Bolzano seems to endorse a form of set-theoretic part-whole principle about continuous magnitudes:

the whole *multitude* (plurality) of quantities which lie between two given quantities, e.g. 7 and 8, although it is equal to an *infinite* [multitude] and therefore cannot be determined by any number however great, depends solely on the magnitude of the distance of those two boundary quantities from one another, i.e. on the quantity $8 - 7$, and therefore must be an equal [multitude] whenever this distance is equal.

This suggests that a more fine-grained analysis might be required in order to fully assess the role that part-whole reasoning plays in the *PU* as a whole.

straightforward to verify that, in our model, $B \cup \overset{0}{\mathbb{N}} \overset{\text{SQ}}{\mathbb{N}} \neq \emptyset$. In other words, this interpretation avoids making Bolzano's computations inconsistent with his adherence to the principle that the whole is always greater than its proper parts. The price to pay is to argue that the existence of a one-to-one correspondence between natural numbers and squares does not imply that the two sets have the same size, even though, in the specific case of $\overset{1}{S}$ and $\overset{2}{S}$, it is instrumental in establishing that the two sums have the same *Gliedermenge*. In fact, this strategy can be generalised to any set of natural numbers. Indeed, if $A \subseteq \mathbb{N}$, let $\chi_A : \mathbb{N} \rightarrow \{0,1\}$ be the characteristic function of A , i.e. for any $n \in \mathbb{N}$, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$. We may then consider the infinite sum $\sum_{i=1}^{\infty} \chi_A(i)$ and identify the *number* of elements in A with $\sum_{i=1}^{\infty} \chi_A(i)$. It is then straightforward to verify the following fact:

PW5 For any two $A, B \subseteq \mathbb{N}$, if $A \sim B$, then $\sum_{i=1}^{\infty} \chi_A(i) = \sum_{i=1}^{\infty} \chi_B(i)$.

Indeed, if $A \sim B$, let f be the smallest number in $B \setminus A$, and observe that, for any $j \neq f$, $\chi_A(j) = \chi_B(j)$. Thus $\sum_{i=1}^j \chi_A(i) = \sum_{i=1}^j \chi_B(i)$. This is finite, so $\sum_{i=1}^{\infty} \chi_A(i) = \sum_{i=1}^{\infty} \chi_B(i)$. In fact, this 'Bolzanian' way of assigning quantities to sets of natural numbers completely coincides with how a set of natural numbers is assigned a numerosity when the structure is constructed out of an ultrapower of the natural numbers, as in Benci and Di Nasso (2003).

We stop short of arguing that this was Bolzano's position, as we do not believe that there is enough evidence in the text of *PU* to make this claim; nor are we convinced that Bolzano had a notion of sets of natural numbers and of their sizes that would allow him to conceive of the problem in those terms. Our point, however, is that Bolzano's computations with infinite sums, and his attempts to develop a general theory of a calculation of the infinite, do not, as our formalisation makes clear, commit him to a rejection of the part-whole principle for sets of natural numbers.

5.6.3 Bolzano and early set theory

Even though our interpretation sees Bolzano as not necessarily concerned with sets and their cardinalities, this should not be seen as a claim that Bolzano's work is completely separate from, and irrelevant for, the historical development of set theory. We believe that ours is just a more cautious evaluation of the interactions between the *PU* and the early development of set theory as seen mainly in Cantor's work.

What follows is not an exhaustive comparison between Bolzano's §§29-33 and Cantorian set theory but a selective comparison on just a couple of points: the status of infinite quantities in Bolzano's and Cantor's work and the arithmetic of the infinite, respectively.

Insofar as the actual infinite in mathematics is concerned, Bolzano and Cantor are both advocates for its existence. In addition to defending the existence of the actual infinite, Bolzano provides specific examples of infinite multitudes of mathematical objects such as the multitude of all natural numbers, which is an infinitely large quantity (*PU*, §16). Infinitely large quantities exist, and they are fully legitimate objects for mathematics, meaning their relationships to one another can be computed. Although Bolzano asserts this in *PU* §28, he also makes it clear that he is not claiming to be able to express the infinite quantities themselves through numbers. The symbols $\overset{0}{N}; \overset{n}{N}; \overset{1}{S}; \overset{2}{S}$ are just shorthand for the infinite sum expressions Bolzano concludes with ‘::: in inf.’ { they are not separate entities, like cardinals (and ordinals) with respect to sets.²³

Indeed, in modern set theory, ordinals are defined as canonical representatives of order types of well-ordered sets, while cardinals are canonical representatives of equivalence classes of equipollent sets (i.e. sets that can be bijected with one another). Thus, while cardinals are sets and each cardinal is the cardinal of itself, in general a set and its cardinal are two distinct entities. Whether or not Cantor himself held precisely such a view at some point during his lifetime is a complex issue that depends on how one understands the role that Cantor assigns to abstraction in his original construction of the transfinite numbers. Cantor defines the cardinal number or power of a set M to be the result of a ‘double act of abstraction’ performed on M : first, to abstract from the nature of each individual element of M , and second, to abstract from the order of the elements relative to one another. A detailed discussion of the correct interpretation of Cantor’s abstraction is beyond the scope of this chapter, and we therefore refer the interested reader to Hallett (1984, pp. 119-128) and Mancosu (2016, pp. 52-59).

For our purposes, it suffices to stress that the definition of cardinal Cantor gives is such that any set, in principle, can be abstracted from twice and hence give rise to its own cardinal. Thus for instance the cardinal \aleph_0 can be obtained from the set of natural numbers \mathbb{N} by abstracting first from the nature of each single natural number and then from the order of \mathbb{N} as a whole. But one fundamental consequence of Cantor’s double abstraction definition is that any set has a cardinal.²⁴ For Bolzano instead not all infinite strings of integers can give rise

23. Florio and Leach-Krouse (2017) have recently proposed a non-objectual interpretation of ordinals. Provided an analogous treatment can be extended to cardinals, the objectuality of cardinals as a *conceptual* difference between contemporary set theory and Bolzano’s approach to infinite collections might appear less significant than what it seems to be right now. However, it would still be the case that a Cantorian definition by abstraction for cardinals certainly lends itself to a straightforward objectual interpretation, and thus our point regarding the difference in conception between Cantor and Bolzano would still hold true.

24. Note however that if one reads Cantor as associating to any set not only its equipollence class but also a canonical well-ordered representative for it, this is actually equivalent to the Well-Ordering Principle according to which any set can be well-ordered. Therefore, if one were to reject the Well-Ordering Principle, not all sets would have a Cantorian cardinal.

to a sum, as the case of Grandi's series witnesses, and determining which such expressions do correspond to a sum is one of the problems he tries to solve.

A second point of comparison between Cantor's and Bolzano's treatments of the infinite is the computations they perform with infinite quantities. They both strive to give a meaningful account of arithmetical operations (addition and multiplication, but also subtraction and division, or 'ratios' in Bolzano's case) between transfinite cardinals and infinite sums. What this means and how they achieve it is however very different for each of them.

Cardinal multiplication is defined as taking the cardinal of the product of two sets $A; B$, and addition is defined as the cardinality of the disjoint union of two sets (according to Hallett (1984, p. 82) this was already Cantor's own definition). In the presence of the axiom of choice, it is an elementary fact of cardinal arithmetic that for any two infinite cardinals $\kappa; \lambda$, $\kappa \cdot \lambda = \max\{\kappa; \lambda\}$. This was already proved in the early 20th century by Hessenberg and Jourdain, who were able to generalise Cantor's result that $\aleph_0^2 = \aleph_0$ to $\aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha; \aleph_\beta\}$ (cf. Hallett 1984, pp. 79, 82). They were also able to show that for addition the same holds, namely $\aleph_\alpha + \aleph_\beta = \max\{\aleph_\alpha; \aleph_\beta\}$. This collapse of addition and multiplication into taking the greatest of the addends in the addition case, or factors in the multiplication case, is very far from Bolzano's approach to computing with the infinite.

One important similarity between Cantor and Bolzano is that, for both of them, an actually infinite quantity, like \aleph_0 for Bolzano or ω for Cantor, can be obtained by iterating a finite operation (adding units for Bolzano, taking successor ordinals for Cantor) on finite quantities. But they seem to conceive of this process of inductive addition in different terms, as evidenced by the role subtraction plays in their respective systems. Cantor does not define subtraction of infinite cardinals, while, as we have seen, for Bolzano the ability to compute the difference between two infinite sums is an essential tool in determining order relationships between infinite quantities. Moreover, no two infinite cardinals can have a finite difference, in the sense that for any two infinite cardinals $\kappa; \lambda$, if $\kappa < \lambda$ and there is a cardinal μ such that $\lambda = \kappa + \mu$, then μ must be finite (in fact $\mu = \lambda - \kappa$). Here again Bolzano's infinities behave vastly differently, since one of his most basic results is that two infinite sums such as $\sum_{n=0}^{\infty} n$ and $\sum_{n=1}^{\infty} n$ have a strictly finite difference, namely n .

Similarly, in §29 we see Bolzano generate new infinities, infinities of higher order, as he claims, simply by multiplying \aleph_0 by itself, so that $\aleph_0 \cdot \aleph_0 = \aleph_0^2$, $\aleph_0 \cdot \aleph_0^2 = \aleph_0^3$. This is in stark contrast with Cantor's result that $\aleph_0^n = \aleph_0$, mentioned above. Moreover, we have argued that a faithful interpretation of Bolzano's criterion for inequality between infinite sums implies that the Bolzanian product of two infinite sums should be non-commutative. In fact, according to us, Bolzanian products are significantly different from products of cardinals. Bolzano does not conceive

of multiplying quantities as akin to taking Cartesian products of sets. He rather seems to be extending the definition of multiplication of natural numbers that he had in his Bolzano (*RZ*, p. 57), without introducing infinite numbers. Just like the product of two finite numbers $m \cdot n$ is defined as $\underbrace{100001100001}_{m \text{ times}}$, i.e. as obtained from the sum $m \cdot \underbrace{100001100001}_{n \text{ times}}$ by replacing each unit by n , the product of two infinite quantities $\underbrace{100001100001}_{m \text{ times}}$ may be obtained by writing the corresponding infinite sum for $\underbrace{100001100001}_{n \text{ times}}$ and replacing each unit by n , as in the case of $p \sqrt[n]{q^2}$. Perhaps surprisingly, this latter feature of Bolzano's computation may in fact be seen as the most modern one, especially under our interpretation of the Bolzanian product. Indeed, by allowing not only his finitary operations, but also his infinitary operations (like infinite summation) to range over both finite and infinite quantities, Bolzano, just as Cantor, is able to generate a hierarchy of infinities of ever increasing order.

5.7 Conclusion

Our goal was to provide a faithful interpretation of the *PU* and especially of Bolzano's calculation of the infinite as presented in §§29-33. We argued that Bolzano's computations should not be judged as failed attempts at anticipating Cantor's transfinite arithmetic, and that Bolzano's primary interest was not in measuring the sizes of infinite collections of natural numbers, but in developing an arithmetic of infinite sums of integers. As a consequence, one should not read Bolzano as failing to anticipate Cantor's work because of his commitment to a set-theoretic version of the part-whole principle but rather as developing from part-whole considerations an original and productive way of reasoning about infinite sums. Moreover, far from shutting Bolzano out of future historiographies of set theory, this new interpretation clarifies where Bolzano's approach to the infinite stands within that history. The intentions and methods of Bolzano when computing with the infinitely large are radically different from Cantor's, yet, as we have shown, amenable to a consistent mathematical interpretation. We hope that the present work may mark only the beginning of deeper scholarly engagement with Bolzano's mathematical infinite.

5.8 Appendix

In this appendix, we describe in more detail the ring of Bolzanian quantities B mentioned in Section 5.5.2. In particular, we show how to construct B as a direct limit of iterated ultrapowers, define rigorously the product of two infinite quantities, and prove Theorem 5.5.3.

Let us first note that a standard presentation of our construction would require us to take a direct limit of the structures:

$$Z \xrightarrow{0} Z_U \xrightarrow{1} pZ_U q^2 \xrightarrow{2} pZ_U q^3 \xrightarrow{3} \dots$$

where for any natural number n , $pZ_U q^{n+1}$ is the ultrapower of $pZ_U q^n$ by U , and each $\pi_{n+1} : pZ_U q^{n+1} \rightarrow pZ_U q^n$ maps (any equivalence class of) a function $f : I \rightarrow Z_U^n$ to the function mapping any i to $\overline{f \upharpoonright i}$. The inconvenience of this approach is that it requires us to introduce elements of increasing complexity in our structure, i.e. functions from I into the integers, functions from I into functions from I into the integers, and so on. However, we may present our construction differently, by drawing on the well known fact that for any sets A, B and C , there is a canonical bijection between functions from A into C^B and functions from $A \times B$ into C : given any $f : A \rightarrow C^B$, the function $\overline{p f q} : A \times B \rightarrow C$ is such that $\overline{p f q}(a, b) = \overline{f(a)(b)}$ for any $a \in A$ and $b \in B$. Instead of working with functions of higher and higher complexity, we may therefore simply work with functions of finite arity, or, equivalently functions from finite sequences of elements in I into Z . However, since we still need to identify functions using an ultrafilter U , we also need to generalise our definition of when two n -ary functions are equivalent according to U . This requires the following definition.

Definition 5.8.1. Let U be a non-principal ultrafilter on I . For any natural number n , we define U^n by induction as follows:

- $U^0 = \{ \langle \rangle \}$
- U^{n+1} is a collection of subsets of $pI^n q$ such that for any $X \subseteq pI^n q$, $X \in U^{n+1}$ if and only if $\{ i \in I : X \upharpoonright i \in U^n \} \in U$, where for any $i \in I$, $X \upharpoonright i$ is the set of n -tuples \vec{j} in $pI^n q$ such that the $n+1$ -tuple $\langle i, \vec{j} \rangle \in X$.

Note that $pI^0 q$ is the set of all 0-ary sequences of elements of I , i.e. contains only the empty sequence. It is also straightforward to see that, given the previous definition, $U^{n+1} \subseteq U^n$. The following lemma will be useful later on, and is established by a straightforward induction on the natural numbers.

Lemma 5.8.2.

- For any natural number n , U^n is an ultrafilter on $pI^n q$ which is non-principal if $n \geq 0$.
- Let m, n be two natural numbers and $X \subseteq pI^{m+n} q$. Then $X \in U^{m+n}$ if and only if

$$\{ \vec{i} \in pI^m q : \{ \vec{j} \in pI^n q : \langle \vec{i}, \vec{j} \rangle \in X \} \in U^n \} \in U^m;$$

We can then define the following structures:

Definition 5.8.3. Let n be a natural number. We let $Z_{U^n} : \mathcal{P}Z_{U^n}; \mathcal{P}; 0; 1; q$ be the ultrapower of Z by U^n . More precisely, elements in Z_{U^n} are equivalence classes of functions from $\mathcal{P}! q^n$ to Z , where for any two functions $f; g : \mathcal{P}! \mathcal{N} Z$:

- $f \sim g \iff \{i \in \mathcal{P}! q^n : f(i) = g(i)\} \in U^n$;
- $\mathcal{P}f \sim \mathcal{P}g \iff f \sim g, \mathcal{P}f \sim \mathcal{P}g \iff f \sim g$;
- $f \sim g \iff \{i \in \mathcal{P}! q^n : f(i) = g(i)\} \in U^n$.

In particular, it is straightforward to verify that Z_{U^0} is isomorphic to Z .

Since U^n is an ultrafilter on $\mathcal{P}! q^n$ for any natural number n , the previous definition is a generalisation of the original construction of Z_U . Moreover, we have natural embeddings $\pi_n : Z_{U^n} \rightarrow Z_{U^{n+1}}$. In fact, those embeddings are always elementary:

Lemma 5.8.4. For any $f : \mathcal{P}! q^n \rightarrow Z$, let $\pi_n \mathcal{P}f : \mathcal{P}! q^{n+1} \rightarrow Z$ be such that for any n -tuple \vec{i} and any $j \in \mathcal{P}!$, $\pi_n \mathcal{P}f(\vec{i}j) = f(\vec{i})$. Then the function $\pi_n : Z_{U^n} \rightarrow Z_{U^{n+1}}$ defined by $\pi_n \mathcal{P}f \sim \pi_n \mathcal{P}g$ is an elementary embedding.

The proof of this lemma is a simple application of the Tarski-Vaught test of elementary substructures. For any natural numbers $m \leq n$, we let $\pi_{m,n}$ be the composition of the embeddings $\pi_{m-1, m}, \pi_{m-2, m-1}, \dots, \pi_{m, m-1}$. We can then define the structure $\mathcal{P}B; \mathcal{P}; 0; 1; q$ as the direct limit of the system

$$Z_{U^0} \xrightarrow{\pi_0} Z_{U^1} \xrightarrow{\pi_1} Z_{U^2} \xrightarrow{\pi_2} \dots$$

We will refer to elements in B as *quantities*. By definition of the direct limit of a directed system, quantities are equivalence classes of elements in some Z_{U^n} , where for any $m \leq n$ and any two equivalence classes $f \in \mathcal{P}Z_{U^n}; g \in \mathcal{P}Z_{U^m}$, f and g are identified if and only if $Z_{U^n} \cap \pi_{m,n} \mathcal{P}f \sim g$. For any quantity α , we let the *order* of α be the smallest natural number n such that there is some $f \in \mathcal{P}$ such that $f \sim \alpha$. Clearly, any $\alpha \in B$ has a finite order n , and moreover, if α has order n witnessed by some f , then for any natural number m , any $g \in \mathcal{P}Z_{U^m}$, and any tuples \vec{i} and \vec{j} of length n and m respectively, $f(\vec{i}) = g(\vec{j})$. We may therefore abuse notation and view α as a function from m -tuples of elements in $\mathcal{P}!$ into Z for any $m \geq n$.

Let α and β be two quantities of order m and n respectively, represented by the functions f and g of arity m and n respectively. We define the *product* $\alpha \cdot \beta$ as (the equivalence class of) the function $f \cdot g : \mathcal{P}! q^{m+n} \rightarrow Z$ such that for any tuples \vec{i} and \vec{j} of length m and n respectively, $f \cdot g(\vec{i}\vec{j}) = f(\vec{i}) \cdot g(\vec{j})$, i.e. $f(\vec{i}) \cdot g(\vec{j})$. It is straightforward to verify that this operation is well-

defined. Indeed, suppose $g \in \mathcal{P}$ and $g' \in \mathcal{P}$ are functions of arity m and n

respectively. Clearly for any m -tuple \bar{i} and any n -tuple \bar{j} , if $f : \bar{i}\bar{q} \rightarrow g : \bar{i}\bar{q}$ and $f : \bar{j}\bar{q} \rightarrow g : \bar{j}\bar{q}$, then $g : \bar{i}\bar{j}\bar{q} \rightarrow f : \bar{i}\bar{j}\bar{q}$. Moreover, since f and g are U^m equivalent, and f and g are U^n equivalent, it follows that for U^m -many \bar{i} there are U^n -many \bar{j} such that $f : \bar{i}\bar{j}\bar{q} \rightarrow g : \bar{i}\bar{j}\bar{q}$. Equivalently,

$$t\bar{i} P p! \cdot q^m : t\bar{j} P p! \cdot q^n : f : \bar{i}\bar{j}\bar{q} \rightarrow g : \bar{i}\bar{j}\bar{q} u P U^n u P U^m;$$

which by Lemma 5.8.2 implies that $t\bar{i}\bar{j} P p! \cdot q^{m+n} : f : \bar{i}\bar{j}\bar{q} \rightarrow g : \bar{i}\bar{j}\bar{q} u P U^{m+n}$, and therefore $f : g$.

The next lemma establishes that the product of two quantities of order m and n is of order $m+n$. The proof is a simple application of Los's theorem.

Lemma 5.8.5. *Let f and g be two quantities of order m and n respectively, and let h be a quantity of order $l = m+n$. Then $B \models (f \cdot g) : h$.*

Finally, we can now prove Theorem 5.5.3 and establish that Bolzanian sums and products form a non-commutative ordered ring.

Theorem 5.8.6. *The structure $B = \langle pB; \cdot; +; 0; 1; \cdot; q \rangle$ is a non-commutative ordered ring.*

Proof. Note first that by construction, we have an elementary embedding from Z into the reduct $\langle pB; \cdot; +; 0; 1; q \rangle$, which immediately implies that B is an ordered additive group. We therefore only need to verify the following properties:

- **Associativity:** Let f , g , and h be three quantities of order $l; m$ and n respectively. Then for any tuples \bar{i}, \bar{j} and \bar{k} of arity $l; m$ and n respectively, we have that:

$$\begin{aligned} p : q \bar{i}\bar{j}\bar{k}\bar{q} & \rightarrow p : \bar{i}\bar{q} \cdot p : \bar{j}\bar{k}\bar{q}\bar{q} \\ & \rightarrow p : \bar{i}\bar{q} \cdot p : \bar{j}\bar{q} \cdot p : \bar{k}\bar{q}\bar{q} \\ & \rightarrow p : \bar{i}\bar{q} \cdot p : \bar{j}\bar{q}\bar{q} \cdot p : \bar{k}\bar{q} \quad \text{pby associativity of } \cdot \text{ in } Zq \\ & \rightarrow p : \bar{i}\bar{j}\bar{q}\bar{q} \cdot p : \bar{k}\bar{q} \\ & \rightarrow p : q : \bar{i}\bar{j}\bar{k}\bar{q} \end{aligned}$$

- **Multiplicative identity:** Note that any integer z is represented in B by a quantity z of order 0, which corresponds to the set of all constant functions from finite sequences of elements in I into Z with range $\{z\}$. For any quantity f of order l , we therefore have that $f : z$ and $z : f$ are quantities of order n such that for any l -tuple \bar{i} , $f : z \bar{i}\bar{q} \rightarrow p : \bar{i}\bar{q} \cdot z$ and $z : f \bar{i}\bar{q} \rightarrow z \cdot p : \bar{i}\bar{q}$. Thus $f : z \cdot z : f = \underbrace{1 \dots 1}_{z \text{ times}}$. Hence in particular $1 : f = f : 1$.

z times

- **Left-distributivity:** Let $\mathcal{A}; \mathcal{B}; \mathcal{C}$ be as above. Without loss of generality, assume that the order m of \mathcal{A} is greater than or equal to the order n of \mathcal{B} , which implies that \mathcal{C} is also of order m . Fix an l -tuple \vec{i} and an n -tuple \vec{j} . Note that even though \mathcal{A} is of lower order, we may still write $p\vec{j}q$. Then:

$$\begin{aligned} p \quad q &: p\vec{i}\vec{j}q \quad p\vec{i}q \quad p \quad p\vec{j}qq \\ & \quad p\vec{i}q \quad p \quad p\vec{j}q \quad p\vec{j}qq \\ p \quad p\vec{i}q \quad p\vec{j}qq \quad p \quad p\vec{i}q \quad p\vec{j}qq & \text{ pby left-distributivity of } \mathcal{A} \text{ over } \mathcal{B} \text{ in } \mathbb{Z}q \\ p &: p\vec{i}\vec{j}qq \quad p &: p\vec{i}\vec{j}qq \\ p &: q \quad p &: qp\vec{i}\vec{j}q: \end{aligned}$$

- **Right-distributivity:** Let $\mathcal{A}; \mathcal{B}; \mathcal{C}$ as above, and assume the order l of \mathcal{A} is greater than or equal to the order m of \mathcal{B} . Let \vec{i} be an l -tuple and \vec{k} a n -tuple. Then:

$$\begin{aligned} p \quad q &: p\vec{i}\vec{k}q \quad p \quad p\vec{i}qq \quad p\vec{k}q \\ & \quad p \quad p\vec{i}q \quad p\vec{k}qq \quad p\vec{k}q \\ p \quad p\vec{i}q \quad p\vec{k}qq \quad p \quad p\vec{i}q \quad p\vec{k}qq & \text{ pby right-distributivity of } \mathcal{A} \text{ over } \mathcal{B} \text{ in } \mathbb{Z}q \\ p &: p\vec{i}\vec{k}qq \quad p &: p\vec{i}\vec{k}qq \\ p &: q \quad p &: qp\vec{i}\vec{k}q: \end{aligned}$$

- **Order axiom:** Suppose \mathcal{A} and \mathcal{B} are two quantities of order l and m respectively and are such that $\mathcal{B} \leq 0$ and $\mathcal{B} \leq 0$. We claim that $\mathcal{B} \leq 0$. Indeed, since $\mathcal{B} \leq 0$, we have that $t\vec{i} P p! q^l : 0$ and $p\vec{i}qu P U^l$, while it follows from $\mathcal{B} \leq 0$ that $t\vec{j} P p! q^m : 0$ and $p\vec{j}qu P U^m$. Now clearly for any l -tuple \vec{i} such that $0 \leq p\vec{i}q$, if \vec{j} is an m -tuple such that $0 \leq p\vec{j}q$, then $0 \leq p\vec{i}q \quad p\vec{j}q$, i.e. $0 \leq p\vec{i}\vec{j}q$. Thus

$$t\vec{i} P p! q^l : t\vec{j} P p! q^m : 0 \quad : p\vec{i}\vec{j}qu P U^m \cup P U^l;$$

which by Lemma 5.8.2 implies that $t\vec{i}\vec{j} P p! q^{l+m} : 0$ and $p\vec{i}\vec{j}qu P U^{l+m}$, and hence $\mathcal{B} \leq 0$. □

Let us conclude this appendix with a few remarks regarding the Bolzanian ring of infinite quantities \mathcal{B} . First, our formalisation only allows us to represent infinite quantities of a finite order, i.e. infinite sums of the form $p_1q + p_2q + p_3q$ for which there is an $n \in \mathbb{N}$ such that for all $m \neq n$, the order of p_mq is less than or equal to the order of p_nq . For example, the following infinite sum is not represented by any element in \mathcal{B} :

$$\sum_{n \in \mathbb{N}} p_n q^n \quad \text{::: in } \text{inf.}$$

Of course, if we wanted to include this sum in our model, we would have to take an ultrapower of \mathcal{B} by U and construct another countable sequence of

ultrapowers. In fact, if we wanted to close our domain of infinite quantities under taking infinite sums, we would need to keep iterating the ultrapower until the first ordinal with uncountable cofinality, i.e. until \aleph_1 . Our structure B , however, is more than enough to account for Bolzano's examples, and we certainly do not want to claim that the consistency of Bolzano's system requires anything like uncountable ordinals.

Second, it is quite straightforward to observe that the situation described in Lemma 5.5.2 generalises to the full structure B . Indeed, for any n , the product of any n^{th} order quantity with at least a first-order infinite quantity is always greater than or smaller than any quantity of strictly lower order. Thus, in accordance with Bolzano's original claims, multiplying infinite quantities together yields new quantities that are infinitely larger or infinitely smaller than the previous ones in a very strong sense.

Chapter 6

Domain Extension and Ideal Elements in Mathematics¹

In Chapters 4 and 5 we have seen Bolzano contending with the problem of how to extend the notion of size and size measurement from finite to infinite collections (Chapter 5), and with the problem of extending the notion of number and arithmetical operations when one allows for infinitely many iterations of them (Chapter 4). In the current chapter we move away from Bolzano and treat the germane topic of domain extension, and specifically domain extension via ideal elements, from the model-theoretic perspective first suggested by Manders (1989).

6.1 Introduction

In field theory, algebraic number theory and Galois theory, one often studies number domains of the form $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Q}[i]$, $\mathbb{R}[i]$, etc. These are number domains which are obtained from \mathbb{Z} , \mathbb{Q} and \mathbb{R} , respectively, by *adjoining* new elements. This means the new elements are added to the old structure and then the mathematician works in the structure that results when the expanded domain is closed under the operations that were already defined on the old domain. A similar procedure can be carried out in geometry. There one can view the projective plane as obtained by adding points and lines at infinity to the standard Euclidean plane and then closing the structure under e.g. linear transformations. Historically, certain successful cases of such domain extensions have come to be referred to as extensions via *ideal elements* (see the discussion on ideal elements in the writings of Dedekind, but also Gauss, Veronese and others, contained in Cantu 2013, summarised below in Section 6.2).

The philosophical significance of ideal elements and of the method of ideal elements has mainly been discussed in the context of Hilbert's philosophy of mathematics (cf. Detlefsen 1993; Hallett 1990; Stillwell 2014). In his 1919 lecture

1. This chapter has been published as (Bellomo 2021).

'The role of ideal entities' (Hilbert 1992, pp. 90-101), Hilbert characterises the method of ideal elements as consisting in moving from a given 'system' in which the handling of certain questions is complicated to one where such questions become simple to handle (pp. 90-91). In addition, the new system contains a subsystem isomorphic to the old system. Thus, at least according to Hilbert, ideal elements are introduced to simplify certain mathematical problems, while preserving the old setting in which the problems arose.

Besides Hilbert, though, other mathematicians such as Poncelet (see for example Chemla 2016), Kummer, and Dedekind (Cantu 2013) talk of ideal elements; this suggests that domain extension via ideal elements was perhaps understood as a mathematical technique even before Hilbert. Despite the existence of many treatments of ideal elements in the context of Hilbert's philosophy, formal investigations of what makes domain extensions successful are rare. One such investigation focusing on domain extensions via ideal elements in particular can be found though in (Manders 1989), where Manders sketches an account for domain extension. Manders argues that extended domains are productive to work with, because they are the existential closure of the original domain. In other words, for an extended domain to count as a good domain extension it is sufficient that it be the existential closure of the domain it extends.

In this chapter, however, I will argue that if we understand ideal elements as heuristic tools ordering the mathematician certain pragmatic or epistemic advantages, Manders's proposed explanation of the fruitfulness of domain extensions can only be a partial one, since it cannot explain some historically important cases of domain extension via ideal elements. I will then turn to a different approach to domain extension inspired by Dedekind (1854) and defend the view that, if interpreted correctly, it can provide a framework for the domain extensions motivated by closure under properties and operations. Given the historical context in which (Dedekind 1854) was written, in Section 6.6 I explore the question of how this second criterion fares with respect to concurrent developments in number theory. I conclude (Sections 6.7 and 6.8) that the comparison between Manders's framework and mine leaves us with three distinct options concerning the philosophical treatment of domain extension via ideal elements in mathematics.

6.2 Ideal elements

Cantu (2013) offers a historically informed reconstruction of the role ideal elements play in a mathematician's toolbox. She argues that 'ideal', 'imaginary' mathematical entities have been used by mathematicians in their proofs or theory building whenever the accepted mathematical domain would not warrant them in pursuing a certain simplification or generalisation of mathematics. Thus, the introduction of ideal elements is justified, in the eyes of the mathematician, on the basis of the following argument:

Premise (1) I, as a mathematician, have the goal (G'') of removing exceptions, allowing direct and inverse operations to satisfy closure properties, and dual transformations between models to be introduced, whenever possible.

Premise (2) The goal (G') is supported by the set of values (V) and (V').

Premise (3) The method of introduction of ideal elements is a means for me, as a mathematician, to bring about (G'').

Conclusion (4) Therefore, I should (practically ought to) introduce ideal element. (Cantu 2013, pp. 86, 88)

The values Cantu recognises as supporting the mathematician's goal are the following:

(V) Value V. The generality of a theory, i.e. its being without exceptions, is a desirable value in mathematics. (Cantu 2013, p. 83)

(V') Value V' as a warrant for value V. Generality is desirable because it increases simplicity. (Cantu 2013, p. 84)

Cantu reconstructs this argument on the basis of writings by Hilbert, Dedekind, Gauss, Veronese. The new elements are *ideal*, or imaginary, etc. because they might enjoy a different ontological, epistemic or pragmatic status from 'real' elements. In other words, they might exist in a different sense, they might be less epistemically secure, or they might be used differently than real elements (Cantu 2013, pp. 79-80).

The argument above is supposed to offer a defence of the use of ideal elements in these mathematicians' work, based on their own writings on the matter. Cantu however is not arguing that this argument alone warrants the individual mathematician to use ideal elements { she is noting though that several mathematicians use the above argument to justify the adoption of ideal elements in their practice. This argument cannot justify, for example, why a mathematician subscribes to (G''), or what happens when (G'') conflicts with another mathematical goal. Depending on the mathematician, these issues are fended off by different arguments.²

Having thus settled on a working notion of ideal elements as heuristic tools having epistemic and/ or pragmatic advantages, I now introduce the first of the two accounts for domain extension via ideal elements this chapter considers.

2. For a more thorough treatment of objections to the argument (1)-(4), see (Cantu 2013, p. 89) onwards.

6.3 Manders's framework

Manders (1989) proposes to use the notions of existential closure and model completion from model theory to explain why certain historical cases of domain extensions, including some important cases of extension via ideal elements, are mathematically fruitful. Before sketching Manders's proposal, a few terminological clarifications are in order. For the remainder of the chapter, a *structure* A is an ordered pair where the first element is a set of individuals, which is what we call a *domain* A , and the second element is the *interpretation* of all symbols of a given language L in A . For each symbol I of L , if I is a constant symbol then its interpretation is an element of A , if I is an n -ary relation symbol then its interpretation is a set of n -tuples of elements of A , and if I is an n -ary function symbol then its interpretation is an n -ary function on A , that is a function from A^n to A (see e.g. Tent and Ziegler 2012, p. 2). Now, let a theory T be a set of sentences in L . If A makes those sentences true, we say that A is a *model of* T (Tent and Ziegler 2012, p. 10). We can now say what existential closure consists in. Roughly, existential closure is the property exhibited by a structure A , considered as the model of a given theory T , or equivalently, as a member of a class K of structures (the class of all and only those structures which are models of T), whenever A contains in its domain all the solutions to equations and inequations which can be expressed in the language of A . This language needs to be a first-order language with no relation symbols.

According to Manders, when performing domain extension via existential closure, the mathematician is trying to preserve three things: the original domain of objects, which we want to extend without modifying the objects we started with; conditions on said objects which we do not want to give up on, which he dubs 'invariant conditions' ('invariants' for short), indicated as ' pq '; and the properties these conditions give rise to, sentences of the form $\exists x' p x q$, where ' pq ' is itself an invariant. While the first one, namely the objects, are almost always preserved, invariants and the properties they give rise to sometimes have to be given up in order for the desirable extension to take place. Manders claims that this (informal) process has a formal counterpart in the notion of existential closure:

Definition (Existential closure). *Let L be a first-order language with no relation symbols (but possibly function symbols), and K be the class of L -structures. Call a formula ' φ ' primitive if and only if ' $\varphi = \exists y_{i \in I} \psi$ ', where each ψ is either an atomic formula or a negated atomic formula.*

Then a structure A from class K is existentially closed in K (e.c.) if and only if for every primitive formula ' φ ' of L , and every tuple \bar{a} in A , whenever there is a structure B in K such that $A \vDash B$ (A is a substructure of B) and $B \vDash \varphi$ then already $A \vDash \varphi$.³

3. Existential closure in this formulation is due to (Hodges 1993, p. 361).

Manders's goal is to convince his reader that by using existential closure (and model completion, where applicable) of contemporary model theory to conceptualise historical cases of domain extension in mathematics, one can achieve an analysis of what guides choices of fruitful theories in mathematics.

Manders's further claim is that, if we understand good domain extensions in terms of existential closure then we have accounted for the conceptual unification such extensions afford (p. 554). This is how conceptual unification follows from existential closure. Once a given domain is existentially closed, the new structure, considered as a model of the old theory, will be such that for certain propositions, they will either hold universally or not at all (Manders calls this 'squeezing out the middle case'). Manders's example is that equations of second degree only have a solution in some cases over the real numbers, but once this is extended to the complex numbers, *every* second-degree equation has a solution in the extended domain.

The notion of existential closure is quite common in algebra: we can talk of an existentially closed (e.c.) lattice, an e.c. group, an e.c. field. One needs to use some caution, though, when talking about e.c. structures, for the notion itself is always relative to a class of structures. In the case of fields, for example, if K is the class of models of the theory of fields then the e.c. structures are exactly the algebraically closed fields (see e.g. Hodges 1993, p. 362). If K on the other hand is the class of models for the theory of ordered fields, then the e.c. structures are the real closed fields (where algebraically closed and real closed fields are not extensionally the same class of structures).

If existential closure is, in a sense, quite common, what makes it noteworthy for the purposes of explaining the advantages of domain extensions? In short, existential closure can be a stepping stone towards an important model-theoretic feature of certain theories, *quantifier elimination* (or properties which can approximate the advantages brought about by quantifier elimination proper). A theory T is said to have quantifier elimination whenever, for any formula ϕ in the language of T , T proves that ϕ is equivalent to a quantifier free formula. Quantifier elimination is an important model-theoretic feature for algebraic theories, because it enables the proof of mathematically rich results such as the Nullstellensatz.⁴

As it turns out, one can strengthen the definition of existential closure so that the formula ϕ can be any existential formula, instead of just an existential quantifier followed by atomic or negated atomic formulas. This is just a consequence of the disjunctive normal form theorem for D_1 formulas (Hodges 1993).

4. *Nullstellensatz* actually is the name given to several theorems of modern algebra, which however are generalisations of Hilbert's Nullstellensatz. One standard formulation of Hilbert's Nullstellensatz is as follows:

Suppose A is an algebraically closed field, I is an ideal in the polynomial ring $A[x_0, \dots, x_n]$ and $\mathfrak{p} = \langle p \rangle$ is a prime ideal in $A[x_0, \dots, x_n]$ such that for all $a \in A$ if $ap \in \mathfrak{p}$ then $a \in \mathfrak{p}$. Then for some positive integer k , $p^k \in I$. (Hodges 1993, p. 366.)

The Nullstellensatz is proved via quantifier elimination, and its generalisation called the Strong Nullstellensatz is used to establish certain results in duality theory. (nLab 2019)

6.4 Domain extensions and ideal elements

Manders's goal is to use cases of historical domain extensions which turn out to be existential closures of preexisting models⁵ as evidence against the claim that fruitfulness of mathematical theories is an empirical, historical fact. Manders also suggests that existential closure is the formal model-theoretic notion that captures (Hilbert's) method of ideal elements. On the face of it, Manders sees existential closure as a sufficient condition for the success of certain domain extensions { in particular, successful domain extensions that Hilbert would consider as extensions via ideal elements. It is the scope of application of this explanation that I am interested in probing.

One of Hilbert's examples for ideal elements are lines and points at infinity. Manders (1984) shows how, under certain conditions, the models of projective geometry are existential closures of the Euclidean plane. So in that sense, Manders's account is correct in the case of ideal elements in geometry.

What about arithmetic and algebra? Let me start by the easiest case, namely the complex numbers. If we consider the field of complex numbers \mathbb{C} as a structure in the class of models of the theory of fields, then, since it is an algebraically closed field, it is actually existentially closed (this follows almost immediately from the definitions). Moreover, the theory of algebraically closed fields is model-complete. So Manders's framework works well for this case { and indeed, if we look back at how he introduced the notion of existential closure, he generically spoke of all those cases in which one 'rounds off' a mathematical domain by adjoining roots. That is exactly one way of constructing the complex numbers, as $\mathbb{R}[i]$. Moreover, his historical discussion in the (1989) paper can be seen as a way of demonstrating how the extension of the reals into the complex number system is one of those instances of domain extension which does deliver conceptual unity; one can treat equations which used to be analysed separately as members of one and the same class of equations.

6.4.1 Infinitesimals as ideal elements

The next case one may want to consider is that of infinitesimals. Although infinitesimals are not explicitly listed by Hilbert as one of the canonical cases of ideal elements in his 1984, nor do they appear to be considered as such by the other authors Cantu considers,⁶ I will briefly illustrate how modern authors such as Robinson (1996) and Goldblatt (1998) present the advantages of working in nonstandard analysis.

In his (1996, pp. 1-3), Robinson writes that the 'meaning' of a limit is more

5. The term is used in a non-technical sense, since these examples predate model theory by some time.

6. With the exception perhaps of Veronese (cf. Cantu 2013, pp. 94-95).

appealing if given in terms of infinitesimals { it is simpler.⁷ Moreover, 'Leibniz's ideas [that is, in infinitesimal calculus] can lead to a *fruitful* [emphasis mine] approach to classical Analysis and to many other branches of mathematics. [...] In infinitesimals have generalisations in topology which lead to fruitful applications' (Robinson 1996, p. 2). Thus, in infinitesimals are fruitful, they lead to simplifications and generalisations in mathematics.

Similarly, one reads in the preface to (Goldblatt 1998):

What does nonstandard analysis offer to our understanding of mathematics? [...] New definitions of familiar concepts, often simpler [...] New and insightful (often simpler) proofs of familiar theorems'. (Goldblatt 1998, p. vii.)

Thus, at least some mathematicians seem to argue in favour of infinitesimals because they allow for more perspicuous proofs, clearer expression of foundational concepts, and novel results. Working with infinitesimals, they claim, presents some epistemic advantages. Although the quotes above do not constitute conclusive evidence in that respect, it seems reasonable to allow infinitesimals under the umbrella of ideal elements as understood by Cantu.⁸

7. Here is the full quote:

Underlying the fundamental notions of the branch of mathematics known as Analysis is the concept of a limit. Derivatives and integrals, the sum of an infinite series and the continuity of a function all are defined in terms of limits. For example, let $f(x)$ be a real-valued function which is defined for all x in the open interval $(0; 1)$ and let x_0 be a number which belongs to that interval. Then the real number a is the *derivative of $f(x)$ at x_0* , in symbols 1.1.1 $f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = a$ if 1.1.2 $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = a$. Suppose we ask a well-trained mathematician for the meaning of 1.1.2. Then we may rely on it that, except for inessential variations and terminological differences (such as the use of certain topological notions), his explanation will be thus:

For any positive number ϵ there exists a positive number δ such that $|\frac{f(x) - f(x_0)}{x - x_0} - a| < \epsilon$ for all x in $(0; 1)$ for which $0 < |x - x_0| < \delta$.

Let us now ask our mathematician whether he would not accept the following more direct interpretation of 1.1.1 and 1.1.2.

For any x in the interval of definition of $f(x)$ such that $dx = x - x_0$ is *infinitely close to 0* but not equal to 0, the ratio $\frac{df}{dx}$, where $df = f(x) - f(x_0)$, is *infinitely close to a* . To this question we may expect the answer that our definition might be *simpler* [my emphasis] in appearance, but totally wrong. [...] (Robinson 1996, pp. 1-2.)

8. Even though I believe there is little doubt that infinitesimals are ideal elements at least in the epistemic sense, I should note that one would have to be pretty liberal with what counts as 'removing exceptions', if one wanted to argue that Cantu's argument (1)-(4) can be effortlessly read off of Robinson and Goldblatt's quotes. Here is one possible modification of Cantu's argument: We replace goal (G'') with goal (G''') of making formal mathematics easier

Let me now turn to the question of whether the reals, augmented by infinitesimals, constitute a good domain extension for Manders (and therefore whether infinitesimals count as ideal elements for him). If one considers the reals extended by infinitesimals (from now on denoted by \mathbb{R}), then the model one obtains is not the existential closure of \mathbb{R} over the theory of the reals. Adjunctions that are conservative over the theory one is considering are not going to be existential closures, hence they cannot be good cases of domain extension according to Manders's framework. In the specific case of the real numbers, any nonstandard model for the theory is going to be conservative over the theory of the reals. Hence, the theory of the original model, namely, \mathbb{R} , does not undergo the simplification that Manders is after { i.e. there is no 'squeezing out the middle case', nor any quantifier-elimination kind of simplification occurring.

Thus Manders's proposal seems to work well in several cases of adjunction of ideal elements, but not all.⁹ While this does not undermine his proposal of existential closure as one sufficient condition for deeming a domain extension good or successful, it does seem to suggest that his explication of traditional theoretical virtues via model-theoretical ones is more limited than it might seem at first sight. If the adjunction of infinitesimals is not a case of existential closure, the 'fruitfulness' and 'simplification' afforded by infinitesimals remains unexplained on Manders's framework.

In the next section, I introduce an alternative conceptualisation of domain extensions and consider whether it can account for the status of infinitesimals as ideal elements.

to understand and as close as possible to naive intuitions, and the supporting values (V) and (V') with values (V'') Ease of comprehension of a mathematical theory is a desirable value in mathematics, and (V''') Ease of comprehension is desirable because it increases fruitfulness. Historical proponents of infinitesimal calculus however may have appealed to the argument precisely as it is in (Cantu 2013) though.

9. Here the reader might wonder what happens if instead of considering \mathbb{R} as the starting point of an extension, as I just did, we consider cases where \mathbb{R} is the extended domain { for example, with respect to \mathbb{Q} . It is indeed true that there is a tradition regarding irrational numbers as ideal elements with respect to \mathbb{Q} , and the case of \mathbb{R} as an extension of \mathbb{Q} could potentially be problematic for Manders's account; \mathbb{R} is not the existential closure of \mathbb{Q} as a field. Since algebraic closure and existential closure collapse into the same notion for fields, this means that \mathbb{R} is not the existential closure of \mathbb{Q} , so it is problematic to accommodate on Manders's framework. To this, the adopter of Manders's framework for extensions via ideal elements could give two replies. One is that indeed, \mathbb{R} can be regarded as an extension of \mathbb{Q} via ideal elements, but only if one departs from the classical mathematician's viewpoint. The second is that this is only to be expected, since what makes the real numbers worthy of the mathematician's attention is their completeness, and completeness is not expressible as a first-order formula, while existential closure only deals with preservation of first-order formulas. One may accept these two replies as satisfactory, but note that they seem to have the consequence of making Manders's account of domain extension more restrictive.

6.5 Domain extension according to Dedekind

In a footnote in his paper, Manders refers in passing to an alternative way of conceiving of domain extensions for number domains called the Law of Permanence of Forms (Manders 1989, p. 555). There he summarises the content of the law of permanence as requiring that certain universal properties about basic arithmetical operations be preserved in an extension of a mathematical domain. Manders seems to quickly dismiss the law of permanence as not being specific enough in determining what properties are worth preserving in a domain extension. In order to assess the limits of the law of permanence as an alternative to Manders's notion of successful domain extension, in this section I will (i) briefly discuss the origin of this law, and then (ii) introduce what seems to be Dedekind's take on the law of permanence. This will then form the basis for an alternative (semi-)formal criterion for good domain extension, against which I will compare Manders's own.

6.5.1 The law of permanence of forms

The law of permanence, first introduced by British algebraist George Peacock (1791-1858), states that the only algebraic laws the mathematician should accept are those that { in modern terms { are conservative over certain¹⁰ results of elementary arithmetic. Peacock introduces said 'law' or 'principle' in the context of justifying formal algebra as a generalisation of arithmetic, where 'formal' algebra stands for the part of algebra that studies forms (of equations). A much more detailed discussion of Peacock's views on mathematics and the precise role the principle was meant to fulfil in his philosophy of mathematics can be found in Detlefsen (2005, pp. 271-277). Here I merely explain the principle to the extent that is needed to give some context to Dedekind's views (to be examined in the next subsection).

First let us consider one of Peacock's own formulations of the law of permanence:

Let us again recur to this principle or law of the permanence of equivalent forms [...]. "Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote." Conversely, if we discover an equivalent form in Arithmetical Algebra or any other subordinate science, when the symbols are general in form though specific in their nature, the same must be an equivalent form, when the symbols are general in their nature as well as in their form. (Peacock 1830, §132 p. 104)

'Arithmetical Algebra' in the passage above just is arithmetic, and 'Symbolic Algebra' is algebra. Peacock's claim is that expressions of elementary arithmetic

10. As we will also see for Dedekind, this restriction of which laws of arithmetic are the ones to preserve under extensions is doing some rather non-trivial work in these criteria for extensions.

such as $5 = 5$, or $5 = 5 + 2 - 5$, which are valid only for arithmetical quantities, become laws of symbolic algebra when expressed via symbols that are 'general in their form' (i.e. variables) and 'in nature' (i.e. they are allowed to range over any kind of quantity, not just arithmetical quantities). As the quote below will clarify, Peacock sees arithmetic and algebra as being connected as a more specific and a more general formulation of the same science, the difference being in the semantic value of the symbols deployed by each in the statement of its propositions:

But though the science of arithmetic, or of arithmetical algebra, does not furnish an adequate foundation for the science of symbolical algebra, it necessarily *suggests* its principles, or rather its law of combination; for in as much as symbolical algebra, though arbitrary in the authority of its principles, is not arbitrary in their application, being required to include arithmetical algebra as well as other sciences, it is evident that their rules must be identical with each other, as far as those sciences proceed together in common: the real distinction between them will arise from the *supposition or assumption that the symbols in symbolical algebra are perfectly general and unlimited both in value and representation, and that the operations to which they are subject are equally general likewise.* (Peacock 1834, p. 195)

The principle roughly prescribes that 'symbolic algebra' is, for the most part, a recasting in variables of the already well known truths of 'arithmetical algebra'. Thus, for example, if in arithmetic(al algebra) one finds that $1 + 1 = 0$; $2 + 2 = 0$; $3 + 3 = 0$; in symbolic algebra one can simply assert the general symbolic principle $a + a = 0$.

Peacock however recognises that some of the laws (statements) of his symbolic algebra may not be so 'derived' (or to use Peacock's own terminology, 'suggested') from arithmetic. It is therefore necessary to offer a principled way of guiding formation of new principles in symbolic algebra, and what Peacock offers is more or less a conservativity criterion. If a certain statement is true in arithmetic, then one cannot accept into symbolic algebra another statement that would contradict the arithmetical one.

Peacock's law, as Detlefsen (2005, p. 272) also points out, was already somehow foreshadowed by other writers, and it is also quoted almost verbatim in the German speaking context by Hankel (1867, pp. 11, 15).¹¹ Thus, even though I could not find direct evidence of Dedekind having read Peacock's writings, there does seem

11. On p. 11 one reads:

Der hierin enthaltene hodegetische Grundsatz kann als das Prinzip der Permanenz der formalen Gesetzen bezeichnet werden und besteht darin: Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedrückte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfachen Größen zu bezeichnen, und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen.

to be a similarity in the mathematicians' ideas about generalisation of arithmetic via algebra, and extension of functions and domains in mathematics, respectively. Dedekind (1854) can be read as offering a criterion for fruitful domain extension which is strongly reminiscent of Peacock's principle. This is also noted by Ferreiros (2007b, p. 219), who writes:

'This principle [Dedekind's, *my note*] is analogous to Ohm's ideas on how to generalize arithmetical operations, and to the famous "principle of permanence" formulated by Peacock around 1830 (still found in [Hankel 1867])'.

In the next subsection I thus present Dedekind's analogous ideas on domain extension as expressed in (Dedekind 1854).

6.5.2 Early Dedekind on domain extension

Dedekind's main claim in his *Habilitationsrede* is that, just as in the other sciences,

In mathematics too, the definitions necessarily appear at the outset in a restricted form, and their generalisation emerges only in the course of further development. (Dedekind 1854, §6)¹²

He then follows immediately with a remark that is both puzzling to the modern reader, and familiar to someone acquainted with Peacock's principle:

But [...] these extensions of definitions no longer allow scope for arbitrariness; on the contrary, they follow with compelling necessity from the earlier restricted definitions, provided one applies the following principle: Laws which emerge from the initial definitions and which are characteristic for the concepts that they designate are to be considered as *of general validity*. (Dedekind 1854, *ibid.*)

Note how, just as Peacock rushes to defend algebra as a non-arbitrary generalisation of arithmetic, so does Dedekind not just for algebra, but for any extended mathematical definition (or function). How the extension happens is however somewhat different: for Peacock, the extension concerns the range of validity of certain algebraic propositions; for Dedekind, the extension seems to consist in augmenting the domain of objects that fall under a certain concept (for example, number). Dedekind's understanding of extension however can be seen as equivalent to Peacock's; for, concepts are determined by 'characteristic' laws which 'emerge

My translation: The hodegetic base principle therein contained can be dubbed as the Principle of Permanence of formal Laws and consists in the following: whenever two forms expressed in general signs of *arithmetica universalis* are equal to one another, they should also remain equal to one another when the signs cease to designate simple quantities and therefore also the operations acquire some other content [i.e. meaning].

12. Throughout, I am quoting from the English translation in (Ewald 1996, pp. 755-762).

from the initial definitions' of said concepts. So in the end to expand a concept in Dedekind's sense (at least in arithmetic) is the same as interpreting certain special arithmetic statements as being not just about a restricted domain, but a wider, richer one. This is Peacock's law for the permanence of forms { the law guiding the generalisation of arithmetical results to algebra.

There is also a difference in scope between Dedekind's criterion and Peacock's law; for, Dedekind seems to be offering a (prescriptive, as well as descriptive) criterion for *all* conceptual expansions in mathematics, while Peacock seems to be focused on the generalisation (where generalisation consists in expanding the domain of application of a statement) of arithmetic only. Having thus substantiated the claim of similarity between Dedekind and Peacock, there is still another aspect of Dedekind's reflections that is worth mentioning, namely, his focus on functions, i.e. operations. That is, Dedekind's criterion seemingly applies to more than just the extended domain and codomain of functions. His interest is particularly clear in the following passage concerning numbers and basic arithmetical operations:

[7] Elementary arithmetic is based upon the formation of ordinal and cardinal numbers; the successive progress from one member of the sequence of positive integers to the next is the first and simplest operation of arithmetic; all other operations rest on it. If one collects into a single act the multiply repeated performance of this elementary operation, one arrives at the concept of addition. From this concept that of multiplication is formed in a similar manner, and from multiplication that of exponentiation. But the definitions we thereby obtain for these fundamental operations no longer suffice for the further development of arithmetic, and that is because it assumes that the numbers with which it teaches us to operate are restricted to a very narrow domain. That is, arithmetic requires us, upon the introduction of each of these operations, to create the entire existing domain of numbers anew; or, more precisely, it demands that the indirect, inverse operations of subtraction, division, and the like be unconditionally applicable. And this requirement makes it necessary to create new classes of numbers, since with the original sequence of positive integers the requirement cannot be satisfied. Thus one obtains the negative, rational, irrational, and finally also the so-called imaginary numbers. Now, after the number domain has been extended in this manner it becomes necessary to define the operations anew [...]. (Dedekind 1854, §7)

This passage lays bare how domain extension and operation expansion relate for Dedekind, at least in the case of numbers: the given domain is that of the natural numbers, and the given operation just successor. From the successor function one obtains the other direct operations of addition and multiplication, each defined as iterations of the previously defined function. Once all the 'direct' operations are

defined, one may want to introduce the inverses. For addition, this is subtraction. However for subtraction to be defined between two arbitrary elements of the domain, the domain has to be extended (i.e. the concept of number is expanded) to include also negative numbers. Similarly, introducing the inverse operation for multiplication, namely, division, together with a closure requirement for the domain under the new operation, leads to the introduction of rational numbers. This iterative construction (introduce a new operation, then new numbers so that the domain is closed under said operation) goes all the way up to the imaginary numbers. But with each round of extension of the number domain, old operations also need to be defined anew.¹³ Dedekind is not explicit about this, but it seems that what allows the process to stop is the achievement of a sufficiently rich (number) domain that is also closed under all the defined operations, taken in their most general form. To see how one can adapt the 'definition' of an operation to an extended domain, consider Dedekind's example of multiplication:

We already have a definite example in multiplication. This operation arose from the requirement that a multiply-repeated performance of an operation of the next lower rank [*Ordnung*] | namely the addition of a fixed positive or negative addend (the so-called multiplicand) | be collected together into a single act. The multiplier | that is, the number which states how often the addition of the multiplicand is to be thought of as repeated | is therefore at the outset necessarily a positive integer; a negative multiplier would, under this first definition of multiplication, make absolutely no sense. A special definition is therefore needed in order to admit negative multipliers as well, and thereby to liberate the operation from the initial constraint; but such a definition involves a priori complete arbitrariness, and it would only later be decided whether then this arbitrarily chosen definition would bring any real use to arithmetic; and even if the definition succeeded, one could only call it a lucky guess, a happy coincidence | the sort of thing a scientific method ought to avoid. So let us instead apply our general principle. We must investigate which laws govern the product if the multiplier undergoes in succession the same general alterations which led to the

13. Note that Dedekind seems to say that at each round of extension, strictly speaking one is not simply adding new elements to the number domain or redefining operations, but the whole number domain is 'creat[ed] [...] anew'. I find it plausible that here Dedekind is merely recognising that adding numbers to the old domain is effectively a change in the concept of number. Consequently, adding new numbers yields a rewriting of the definition of the concept of number altogether, and in that sense, the previously existing numbers are also recreated once the new numbers are in place. This reading is admittedly weaker than other readings of Dedekind's 'creationism' about numbers especially in *Was sind und was sollen die Zahlen* and *Stetigkeit und irrationale Zahlen* (Dedekind 1888, 1872) as presented e.g. in (Tait 1996; Hallett 2019). A careful discussion of the relationship between definitions and creation in Dedekind's writings goes beyond the scope of the present chapter.

creation of the sequence of negative integers out of the sequence of positive integers. For this it suffices if we determine the alteration which the product undergoes if one makes the simplest numerical operation with the multiplier, namely, allowing it to go over into the next-following number. By successive repetition of this operation we obtain the familiar addition theorem for the multiplier: in order to multiply a number by a sum, one multiplies it by each summand and then adds these partial products together. From this theorem a subtraction theorem immediately follows for the case where the minuend is greater than the subtrahend. If one now declares this law to be valid in general (that is, to hold also when the difference which the multiplier represents is negative) then one obtains the definition of multiplication with negative multipliers; and it is then of course no accident that the general law which multiplication obeys is exactly the same for both cases. (Dedekind 1854 §8)

The 'original definition' of multiplication as iterated addition has to be amended so that it may also be defined for negative factors, because one cannot repeat an action a negative number of times. Instead, left distributivity is considered as the 'general law' that is to be preserved even in the extended domain.

At this point it is important to notice an element of imprecision in Dedekind's discussion, namely that he seems to be considering simultaneously two types of what one may call conceptual extensions in mathematics. On the one hand there is the introduction of new operations (or functions, as per his lecture title) alongside 'the chain of previous ones'. This is akin to an expansion of the language which one uses to 'talk about' the domain, and here is an example of how that is supposed to work. If we keep the domain of a structure A fixed, we can add, say, relation symbols to the language so as to obtain a new structure A^1 that also interprets these new symbols as well as the old ones. If we let N be the set of all natural numbers, then we can consider both the structure N of the natural numbers in the language $L = \{0; u\}$ and the structure N^1 of the natural numbers in the language $L^1 = \{0; 1; ; u\}$. The domain underlying both N and N^1 is the same, no new elements have been added to N . Yet there is an expansion occurring between the two, which involves operations and constants only. The second type of extension consists in adding elements to the domain of functions, or the introduction of new objects under an old concept. For example, while multiplication as originally defined can only be performed among two positive integers, it can be later redefined so as to allow also negative integers among its domain (and range). This extension can be exemplified as follows. The L -structure one starts with is N , where the domain is just N , and the language L comprises a symbol for addition, '+', that N interprets as the function $\text{ppm}; \text{mq}; m \text{ nq} : m; n \in N \rightarrow u$. One then adds the negative integers to N , thus using Z as the domain of the new L -structure Z , and moreover '-' is now interpreted as

$\text{tpm}; \text{nq}; m \text{ nq} : m; n \text{ P Zu} \dots \text{tpm}; \text{nq}; m \text{ nq} : m; n \text{ P Nu}$. This second change is more straightforwardly a case of adding elements to the total domain of the model as well as to the domains of the individual functions. These two (expansion of the language versus extension of the domain) are, in principle, two distinct kinds of extension, yet Dedekind does not seem to note this. I believe the reason why Dedekind does not examine the two cases of extension separately is because he does not believe one can occur in the absence of the other: if the mathematician introduces new elements to the domain in question, then she needs to be able to determine how the old operations or functions apply to the new objects.

6.5.3 Formalising Dedekind's proposal

As the quote illustrates, there is a lot happening in Dedekind's text. Hence, in order to bring out the points of comparison with Manders's notion of domain extension, it might be helpful to give a model-theoretic characterisation of Dedekind's views. I propose the following:

Definition (Dedekind-extension). *Let L, L^1 be two first-order languages without relation symbols such that $L \subseteq L^1$. Let A be an L -structure. Then a Dedekind-extension of A consists in finding a class of L^1 -structures K such that for all $B \in K$:*

- (i) $A \subseteq B \restriction L$
- (ii) $B \models \phi$ whenever $A \models \phi$, ϕ a quantifier-free, positive formula in L^1

Condition (i) of the definition asks that A be embedded in B . This ensures the preservation of functions among the individuals of the original model A , if the languages L, L^1 include function symbols interpreted in A and B .

Condition (ii) aims to capture Dedekind's rule about certain laws that are to be considered 'as of general validity'. The positivity restriction on ϕ is motivated by technical issues one would otherwise encounter,¹⁴ and also by the fact that equations seem to have a privileged status over inequations. Preservation of equations is an important theme of results in universal algebra, as witnessed by the stream of research in universal algebra consisting in generalisations and applications of Birkhoff's theorem.¹⁵ Moreover, other 19th century mathematicians

14. Consider for example Z as A , Q as B . If the formalisation is to capture Dedekind's notion of good domain extension, then Q should turn out to be one such for Z . In order to do that, my definition needs to rule out e.g. $\exists x \exists x \dots$ from the class of sentences which one wants to preserve between Z and Q , and one way of doing that is by excluding order relation(s) from appearing in ϕ .

15. The theorem states that a class of algebras K is equationally axiomatisable if and only if it is a variety (i.e. if and only if it satisfies certain closure properties). Establishing that a class of algebras is a variety is easier than having to explicitly give an axiomatisation of a class of algebras, and knowing that a certain structure is equationally definable is extremely valuable.

such as Hankel (1867, pp. 26, 40-41), and Peacock (1834) implicitly recognise the importance of preserving equations when extending the number domain and arithmetical operations. Thus the restrictions on \mathcal{L} in the formalisation do not have to be seen as arbitrary.

To be clear, the definition of Dedekind-extension alone does not answer the question of whether a given mathematical domain is or is not a good domain extension of another domain, according to the view I ascribe to Dedekind. The choice of languages $L; L^1$ also plays a non-trivial role in that sense. Consider for instance the following example: let $N; Z$ be the models under considerations, with $\exists x P(x) \in L$. Then Z cannot be a Dedekind extension of N , because $N \models \exists x P(x) \wedge \exists y Q(y)$, which is false in Z . If we exclude \exists from our language, however, the problem does not arise and Z can be considered a Dedekind extension of N . This means that the notion of Dedekind extension is still, to some extent, context dependent. This however is also true of Manders's notion, for existential closure and model completion are also language- and class-sensitive.

6.6 Extending the concept of number

In the previous section, I presented Dedekind's 1854 reflections as suggesting a conception of domain extension akin to that underlying the principle of permanence of equivalent forms, and I proposed a model-theoretic semi-formalisation of the criterion. This was done in an attempt to make progress on the *normative* question of what makes certain domain extensions 'good'. At the same time, work in the previous section might leave the reader wondering about the *historical* question of whether the criterion thus formalised truly does justice to Dedekind's attitude towards new number systems appearing on the mathematical fore in the mid 1850s. To answer this question, I briefly recall in this section two such number theoretic developments and argue that in both cases it appears unlikely that Dedekind would regard them as extensions in the sense of his (1854).

6.6.1 Quaternions, octonions and other hypercomplex numbers

The first case in consideration is that of quaternions, or so-called hypercomplex numbers more generally.

A hypercomplex number is traditionally any number belonging to a (unital) algebra constructed on top of the real numbers. There are several distinct hypercomplex number systems that can be defined as vector spaces over the real numbers; quaternions and octonions are the number systems of dimensions 4 and 8, respectively as their names suggest. This means that each quaternion can be represented by a quadruplet of real numbers, while each octonion can be represented by an octuplet (or 8-tuple). The problem is that multiplication in

quaternions is not commutative, and in the octonions it even fails to be associative. But commutativity and associativity of multiplication are expressible as universal positive statements of the kind a Dedekind-extension should preserve, by definition. This makes clear that the proposal at hand, although inspired by Dedekind, cannot account for these extensions as good extensions.

Dedekind writes on the hypercomplex numbers in two papers (1885; 1887), and in both papers his presentation consists in letting the hypercomplex numbers be expressible as finite sums of the form $\sum e_i$, where e_i is a 'principal unit' (*Haupteinheit*) of the hypercomplex numbers (think $i; j; k$ for quaternions). Then operations between any two hypercomplex numbers can be defined as operations on the units, which taken together form a *basis*. These operations can be expressed as linear transformations, that is, matrices, and some of their crucial properties are therefore determined by the value of the determinant of the corresponding matrix. This is what Dedekind is concerned with in these writings.

By way of conclusion in the 1885 paper, Dedekind writes (my translation):

[...] every system of n principal units, as it happens in Mr Weierstrass' investigation, may always be understood as an n -valued system from n ordinary numbers, in this way, that each rational equality between the n principal units is true if and only if, it holds for each of the special systems $e_1^{p_{sq}}, e_2^{p_{sq}}, \dots, e_n^{p_{sq}}$ derived by us. So if we want to speak of such complex quantities as new numbers (which to me is inexpedient, because in our higher algebra there always emerge multi-value quantity systems in the manner here described), this can only be done though in a completely different, and indeed in a infinitely weaker sense, than in the introduction of imaginary numbers by hefty enrichment of the real numbers field, or also in the introduction of Hamilton's quaternions, which although their usefulness seems to be limited to a very small field, make an unconditional claim to the character of novelty against the other numbers. (Dedekind 1885, in Dedekind 1931, p. 16)

If on the one hand this supports my interpretation that hypercomplex numbers are not genuine new numbers, it also undermines the idea that quaternions count as a special case of hypercomplex numbers in that respect. Dedekind considers them 'new enough' to count as genuine new numbers. So we are left with the following: the Dedekind-inspired account correctly aligns with a distinction between two cases of extension { namely, a domain extension due to expanding the very concept of number (such would be the quaternions) } and number domains obtained as unique extensions (up to isomorphism) of the natural numbers. My definition of Dedekind extension adequately captures the latter kind of domain extension as good, but it leaves out hypercomplex numbers, quaternions included, despite what Dedekind writes in the excerpt above.

So, when it comes to numbers obtained by adjoining new imaginary units, it seems that Dedekind draws the line at quaternions in terms of what counts

as genuine new numbers. For, if on the one hand it already seemed suggested in his (1854) that both imaginaries and quaternions are numbers, only not yet equipped with a satisfactory account of how they are obtained, on the other hand these number systems can be obtained in roughly the same way as hypercomplex numbers, so one would expect Dedekind to regard all these as either uniformly in or uniformly out of the category 'genuine domain extensions' (something needs to be a genuine domain extension to be a good one, needless to say). By looking closely at (Dedekind 1854) one plausible suggestion is that, although both complex numbers and quaternions are genuine domain extensions (because in both cases genuinely new numbers are introduced), only the complex numbers are obtained as a closure of an already accepted number domain (the real numbers) under a certain inverse operation, namely, the inverse of exponentiation. Since the definition of Dedekind-extension strives to capture the idea of good domain extension expressed in (Dedekind 1854), and that idea is that one extends domains to close them under operations, it is a positive feature of the definition of Dedekind-extension that it is satisfied by the complex numbers as an extension of the real numbers, but not by quaternions, since it is only complex numbers that are introduced as closure of the real numbers under square roots.

6.6.2 Dedekind's ideals and ideal elements

A second case of what one might want and expect to turn out a case of 'good extension' for Dedekind is Dedekind's own ideals, or ideal numbers more generally. Ideal numbers were first introduced by Kummer in 1846 (Bordogna 1996, p. 6; Edwards 1980, p. 322) to solve the specific problem of uniqueness of factorisation for certain number domains. Unique factorisation in the case of natural (or even integer) numbers is pretty straightforward: for any non-prime natural number n , there is a unique decomposition of n into its prime factors, that is, into numbers that themselves cannot be written as the product of anything but themselves and the unit. While Kummer first introduced talk of 'ideal numbers' or 'ideal divisors' as numbers that exist beyond (outside) the domain of real (i.e. existing in reality) numbers, and seemed to consider these as additional numbers to be added to the already existing ones, Dedekind's position on the status of his ideals (and the corresponding ideal numbers) is not as clear. In the upcoming subsection, I will sketch Dedekind's second version of the theory of ideals. On the basis of this sketch I will then be able to address the question of what kind of domain extension it is, if it is at all one. For interested readers (White 2004) contains a detailed discussion of the differences between these two versions.

The class of ideals and Dedekind's 'rigorous definition of ideal numbers'

Dedekind presented his theory of ideals first in his supplements to Dirichlet's *Lectures on Number Theory* (Dirichlet and Dedekind 1999), and then in a series of

papers in the *Bulletin des Sciences Mathematiques et Astronomiques* (Dedekind 1876, 1877a, 1877b, 1877c, 1877d). In his *Bulletin* formulation of the theory, Dedekind comes to a 'precise' definition of *ideal number* in the following way.

The starting point is a finite degree extension (in the technical sense of a field, which can be seen as a one- or two-dimensional vector space over \mathbb{Q}) of \mathbb{Q} . In this field one identifies a subring of elements that for the purposes of divisibility behave similarly to the integers. This is the ring of integers \mathfrak{o} of the field \mathbb{K} . The problem is that, in general, unique factorisation will fail in \mathfrak{o} . The point of introducing ideal divisors is to partially retrieve some of the advantages of unique factorisation even in the cases where it strictly speaking fails. Viewed as a set, \mathfrak{o} is not just a subring of \mathbb{K} ; it is also an *ideal*, where an ideal I is a set of elements that is also an additive subring of the original ring R , and for any element $r \in R$ and $p \in I$, $rp \in I$.

Throughout, Dedekind is actually considering ideals of the ring \mathfrak{o} . I can now sketch Dedekind's definition of an ideal *number* (or *divisor*). Dedekind shows that for any ideal \mathfrak{a} , there is a positive integer h such that $\mathfrak{a}^h = \mathfrak{t}^h$; $\mathfrak{t} \in \mathfrak{o}$ is a principal ideal, i.e. $\mathfrak{a}^h = \mathfrak{t}b^{-1}$; $b \in \mathfrak{o}$ for some $b \in \mathfrak{o}$. From this it immediately follows that any $\alpha \in \mathfrak{a}^h$ is of the form $\alpha = b^{-1}t^h$ for some b , and thus that α itself is divisible by b^{-1} , and α is an 'algebraic integer' that does not belong to the field one started with, \mathbb{K} . Dedekind thus writes:

Thus the ideal \mathfrak{a} is composed of all the integer numbers contained in \mathfrak{o} and divisible for the integer b ; for this reason we will say that the number α , although not contained in \mathfrak{o} , is an *ideal number of the field* \mathbb{K} , and that it corresponds to the ideal \mathfrak{a} .¹⁶ (Dedekind 1877d, p. 246)

Dedekind stops short of identifying an ideal containing all the numbers divided by a certain ideal divisor with the ideal divisor itself. This distinction might seem analogous to that which Dedekind draws in the case of real numbers and cuts, where Dedekind says that to each cut that is not generated by a rational number there corresponds an irrational number, without saying that the cut and the number are *one and the same*.¹⁷ One might then consider it a shallow distinction that should not be taken at face value. However there is a substantial difference between the way Dedekind then handles the real numbers versus the number domain he defines ideals over, once the ideals have been defined. In the first case, Dedekind tries to establish that the cuts, *taken collectively as one domain*, satisfy certain arithmetical and order properties. He thus establishes some continuity between the cuts and the irrational numbers they determine, and the arithmetic of natural numbers. If one looks at how Dedekind treats also the

16. Donc l'ideal \mathfrak{a} est compose de tous les nombres entiers contenus dans \mathfrak{o} et divisibles par le nombre entier b ; pour cette raison nous dirons que le nombre α , lors même qu'il n'est pas contenu dans \mathfrak{o} , est un *nombre ideal du corps* \mathbb{K} , et qu'il correspond a l'ideal \mathfrak{a} .

17. For a discussion of Dedekind's views on cuts and real numbers, and attending difficulties, see (Reck 2020).

integers, and the rational numbers¹⁸ each of these steps have in common that the new numbers are defined as (ordered) pairs of the old numbers, and the arithmetic operations on the new numbers are defined in terms of the operations on the old numbers. Moreover, Dedekind seems to have a sense of the newly defined numbers as forming a new whole, a new structure (system), having certain arithmetical properties (for example, in the case of the integers, commutativity of addition and distributivity laws for addition and multiplication) that also hold for the natural numbers. It seems to me that no analogous interest can be detected in Dedekind's work on ideals. He is not trying to show that there is some deep continuity between the arithmetical properties of the natural numbers and the arithmetical properties of these putative new numbers (even though one might say that they are generated because of an investigation of divisibility and the fundamental theorem of arithmetic, and are therefore what one obtains when trying to define 'divisibility' in its most general form, i.e. extend divisibility, in some sense).

6.6.3 Good Dedekind-extensions?

In the presentation of ideals at hand, Dedekind explicitly distances himself from Kummer's approach to the ideals, which renders them as non-existing numbers which are only individuated by divisibility rules given through cumbersome equations (Bordogna 1996; Edwards 1983).

Dedekind by contrast defines ideals as classes of (complex) numbers. He claims that these equivalence classes are *not* to be seen as additions to the number domain. In other words, he does not see himself as having expanded the domain.

This is in accordance with the way extensions of the number concept are presented in the *Habilitation*. For Dedekind's ideals to count as new numbers, one should expect Dedekind to try and prove that the number domain extended to include the ideals still preserves certain 'laws' that were true of the same domain without the ideals. But Dedekind does not do that. Specifically, he does not try to prove that the fundamental arithmetical operations are preserved in the extension.

Similarly, such a concern seems to be absent from his treatment of hypercomplex numbers. Given that still in his (1872) and (1888) Dedekind proudly refers to (Dedekind 1854) as a script the aim of which Gauss himself approved, it seems unlikely that he would not take notice of an extension of the concept of number, namely, the hypercomplex numbers, which does not tally with his description of what constrains such extensions.

I therefore favour a different position when it comes to ideals and hypercomplex numbers, namely, these are not meant to be extensions in the sense of (1854).

18. Dedekind offers a construction for each of these in the *Nachlass*. I was able to gain access to his notes on the integers thanks to Emmylou Haener, but not to those on the 'analogous construction' for the rationals, and am therefore relying on Sieg and Schlimm's (2005) account on the matter.

There are two reasons for this position. First, as already mentioned, both cases present us with a conundrum: a (putative) case of extension that does not seem to satisfy Dedekind's criterion for a good extension. Second, I believe there is enough textual evidence to suggest that Dedekind treats these two cases differently from the way he treats the integers (as extension of \mathbb{N}), the rationals, and the reals. (The case of complex numbers is not an issue, for it comes out as a good domain extension on my semi-formal rendition of Dedekind's 1854 criterion, and Dedekind himself does consider those as extending the real numbers). I believe that the latter (integers, rationals, reals, complexes) are genuine extensions of the number concept for Dedekind in a way that ideals and hypercomplex numbers are not, and this much is also what my semi-formalisation of Dedekind's criterion suggests.

6.7 Comparison

If one considers Dedekind's criterion for extension, then the number domain cases which Dedekind is interested in (extensions from \mathbb{N} to \mathbb{Z} all the way up to \mathbb{C}) come out as good cases of domain extension { if one limits the signatures so as to exclude order; otherwise, already \mathbb{Z} as an extension of \mathbb{N} would not satisfy condition (ii) in Dedekind's definition.

$\mathbb{R}[i]$, for example, would be a structure obtained as completion of another one, namely \mathbb{R} . One starts with domain \mathbb{R} , adds one new element, i , and then adds also all the appropriate algebraic combinations of i with all the elements of \mathbb{R} . It also seems that in the process we have been conservative over \mathbb{R} as a field (not as an ordered field though, given that \mathbb{R} is linearly ordered whereas $\mathbb{R}[i]$ is not). Thus this particular example is a 'good case' extension both for Manders and for Dedekind.

The extension from \mathbb{R} to \mathbb{C} also counts as a good domain extension under Dedekind's framework, unlike under Manders's. This is a significant difference which can be explained in terms of what the two different frameworks are trying to capture. Manders's use of existential closure is meant to capture cases of domain extension that aimed at gaining simplification in terms of reduced quantifier complexity of the theory. Dedekind's extension, on the other hand, is meant to capture cases of domain extension that aim at extending a given concept (e.g. that of addition, or of number) as much as the *essence* of the concept allows.¹⁹

19. This remark might spur some readers to think that the case of forcing extensions in set theory are the kind of extensions that Dedekind should be able to account for. I have two replies to this issue. First, I am interested in Dedekind's notion of domain extension primarily in the instance where the elements added are 'ideal elements'. To the best of my knowledge, forcing extensions are not discussed in those terms in the literature. Secondly, a more appropriate condition (ii) for a notion of extension trying to capture good extensions of theories in the language of set theory would be one requiring preservation of *absolute*, that is, \in_0 notions between the original structure and the extension.

In the previous section I touched upon two prominent cases of putative domain extension and concluded that they seem not to qualify as such under my interpretation of Dedekind. We now turn briefly to the question of whether quaternions and ideals are well handled by Manders's notion. Quaternions (H) are not obtained as an existential closure of C , given that C is already existentially closed and not isomorphic to H . At the same time, it is not straightforward that there should be some class of structures K over which H is existentially closed. Similarly for ideals defined over some field. Without such results then, one cannot definitively rule whether quaternions and ideals count as good domain extensions for Manders.²⁰

In Section 6.3 I explained how Manders argues that for any theory such that each solvability condition has one weak complement, existential closure yields simplification and conceptual unification (Manders 1989, pp. 554-556). Manders spells out simplification and unification in terms of formal properties of the theories of the existentially closed models one obtains. In other words, Manders suggests that existential closure is a sufficient condition for considering a domain extension as a good, fruitful one,²¹ and he points out that a few historically important cases (complex numbers, points and lines at infinity) are indeed cases of existential closure. As such, they really are a means of partially pursuing goal (G''): via results like the Nullstellensatz, they allow 'dual transformations between models to be introduced', and in virtue of what Manders calls the 'squeezing out the middle case' property, they remove exceptions.

Dedekind's notion, meanwhile, focuses on the preservation of certain features (theorems) of a theory which are considered to be essential to the concepts involved (of addition, for example). Because of this, a Dedekind extension pursues the

20. Given that on the face of it hypercomplex numbers and ideals do not seem to straightforwardly fall in the category of good domain extensions under either of the frameworks considered here, one might wonder whether a satisfactory account of domain extension is one that validates hypercomplex numbers and ideals as good domain extensions.

My semi-formalisation of Dedekind's proposal allows considering commutativity of addition and multiplication as some of the laws any extension of the number concept (or of a number domain) should preserve. This has as a straightforward consequence that the quaternions therefore cannot count as a case of Dedekind-extension. Moreover, it is consistent with my explanation of what Manders's criterion is supposed to capture and what Dedekind's is supposed to capture that neither would then consider ideals and hypercomplex numbers as good extensions. For Manders's sufficient criterion, I believe, captures the cases of domain extension that are motivated by adjoining solutions to equations that are expressible, though unsolvable, in the original domain. Clearly, hypercomplex numbers and ideals are no such things. Dedekind's criterion on the other hand ought to capture the cases of domain extension that originate from expanding the domain of well-definedness of algebraic operations *as much as possible*. I want to also argue that introduction of ideals is not brought about by wanting to 'close' a domain under some operations { that is, functions { on the original, restricted, domain, which is the kind of domain extension I take Dedekind's notion to capture. The problem of course is that this makes both accounts (Manders's and mine, based on Dedekind) somewhat normative, instead of merely descriptive.

21. More precisely, Manders suggests that for theories satisfying certain properties, existential closure is a sufficient condition for a good domain extension.

goal (G'') by allowing direct and inverse operations to satisfy closure properties. This splitting of goal (G'') suggests the possibility of using both Dedekind's and Manders's proposal to develop a disjunctive characterisation of historical cases of ideal elements.

Nevertheless, there is a fundamental conceptual difference between Dedekind on one hand, and Manders on the other. Manders insists that, after the fact of the extension, we might find ourselves in a position to reject properties or facts which, before the extension, had been considered essential to the concept that the structure in question was meant to represent or model (in a loose sense of the terms). As already noted in Section 6.5, he even refers to Peacock's principle of permanence of equivalent forms while remarking that, despite its *prima facie* plausibility, it cannot be held true at all times. This seems to be an irreconcilable difference in the way the two opposing camps { Manders on one hand, Dedekind on the other } conceive of the goals and benefits of domain extensions. Preservation of the essence of a function is the criterion, for Dedekind, that guides the mathematician to extend her functions and concepts in one way rather than another.

6.8 Conclusion

In this chapter I started by giving an overview of ideal elements in mathematics, seen as an example of good extensions of mathematical domains. I considered Manders (1989) as a candidate for a model-theoretic explication of Hilbert's method of ideal elements and its role in the advancement of pure mathematics.

Manders's conception of domain extension however seemed to be ill-equipped to explain the 'ideality' of domain extensions which occur when the mathematician pursues closure under operations, or simplification and fruitfulness of a different sort than that granted by quantifier elimination. While it is true that Manders only aims at offering a sufficient condition for successful or good domain extensions, the number and kind of cases which do not exhibit the model-theoretic characteristics he focuses on suggest that Manders's explanation of the fruitfulness of domain extensions is, at best, partial.

In an attempt to shed light on the related questions of how one should understand attributions of theoretical virtues like simplicity and fruitfulness to extended mathematical domains (or attending theories), and of whether such virtues can be reduced to model-theoretic traits of the structures or theories in question the way Manders suggests we should do, I used (Dedekind 1854) as a basis for an alternative model-theoretic criterion of good domain extensions. The upshot of the comparison between Dedekind and Manders is that they both consider the complex numbers as a fruitful case of domain extension, but then seem to disagree on most other cases. Quaternions and ideals are not straightforward to adjudicate on Manders's framework, but also they do not seem like the kind of extension his

criterion is intended to capture as a good case of extension; they also do not satisfy the definition of Dedekind-extension. The case of the reals with infinitesimals, on the other hand, constitutes a good extension for Dedekind, although in a way that it does not enlighten us of the (epistemological) advantages of working with infinitesimals. It cannot for Manders. For infinitesimals then one is left with the following two options: either the understanding of ideal elements offered by Manders's formalisation is too restrictive, because it does not account for the role of ideal elements as 'proof simplifiers'; or, if one takes Manders's proposal as normative, infinitesimals are not ideal elements after all. However, there might be a third option if one looks more carefully at the discussion of ideal elements and extensions at the beginning of the chapter, namely one might want to distinguish between ideal elements which are introduced to round off a domain in Manders's sense or to simplify the mathematics in Manders's way, and ideal elements which are introduced to satisfy closure under certain operations. Under this suggestion, Dedekind-extensions are the extensions that involve a genuine enlargement of the domain of objects in the domain, and an enlargement that achieves closure under certain operations. This solution would do justice to the historical discussion brought forward by Cantor, while highlighting both the strengths and potential limitations of the use of model-theoretic concepts to understand domain extension in mathematics.

Chapter 7

Concluding Remarks

This dissertation makes two main contributions, one of content, and one of methodology. The contribution as of philosophical content is the cumulative argument developed in Chapters 2, 3 and 5 regarding Bolzano's theory of collections. Even though no single argument is enough to completely disavow the identification of sets and Mengen within Bolzano's work, the chapters taken together shift the burden of proof on to whoever wants to defend the view that Bolzano's Mengen are sets. First, there are the metaphysical argument from extensionality and the related functional argument that show that Bolzano's collections are not sets (Chapter 2). Second, there is the argument from Bolzano's theory of concepts (or ideas, to use Bolzano's more general term), namely that because Bolzano conceives of infinite collections of integers as concept extensions, these cannot be treated as sets when computing their size (Chapter 3). Third, there is the historical and mathematical argument from the *PU* (Chapter 5), namely that Bolzano's calculation of the infinite is a theory for how to compare infinite sums rather than a theory of size for infinite sets.

The metaphysical argument from extensionality in Chapter 2 rests on the assumption that extensionality { as a trait and as an axiom { is the most significant feature of sets and of a set-theoretic representation of mathematical structure. Of Bolzano's collection notions, only *Vielheit* satisfies the extensionality axiom or suitable forms of the extensionality property. Mengen are also extensional, but only in a mereological sense. Moreover, Bolzano's theory of collections plays a different foundational role than the sets of set theory (more precisely, than ZFC) because of the anti-extensional approach to structure it seems to presume.

The argument from Bolzano's theory of concepts (Chapter 3) consists in pointing out that Bolzano's adoption of the part-whole principle to adjudicate size comparisons for infinite collections is rooted in his understanding of the hierarchy of concepts, not the other way around. It therefore seems more plausible that our understanding of how infinite collection concepts are ordered needs to be adjusted so as to preserve the principle, as we suggest, than to conclude from a few difficult

passages that the late Bolzano abandoned part-whole.

Finally, the argument from the *PU* (Chapter 5) offers a mathematical counterpart to the argument from Chapter 3. Again we argue that Bolzano's comparisons are not quite size comparisons in the sense of Galileo's Paradox, only this time we defend this point by appealing to the historical context and to Bolzano's computations themselves to argue that Bolzano must have been concerned not quite with sizes but with how to handle infinite sums without appealing to notions of convergence and divergence. This paints a radically different picture than what is usually offered in at least two ways: one, Bolzano's mathematical programme is really not the same as the one of Cantorian transfinite arithmetic. Two, Bolzano's mathematical results, once interpreted as something other than an anticipation of Cantorian sizes, can be shown to be consistent.

Chapters 4 to 6 show the advantages and the disadvantages of formal reconstructions of historical mathematical results. In all three chapters we have considered different historical positions (Bolzano on the measurable numbers and on infinity, and Dedekind on domain extensions) and how they can be formalised using formal tools that were only developed much later than the texts at hand. Chapter 4 shows in detail how formalising Bolzano's theory of measurable numbers without first ascertaining to what extent the formalisation can be faithful to the original theory can make us forget the original in favour of just commenting on the modernised rendition of the theory. Chapter 5 by contrast provides a formalisation of Bolzano's finitary computations that preserves both the results and the arguments of the historical text in question (that is, *PU* 29-33), and that also helps us in understanding the original, rather than trying to supplant it with something 'better'. Finally, Chapter 6 uses Manders's formal approach to the historically important phenomenon of domain extension in mathematics to explore both the gains { which I would say are mostly in clarity and conceptual sharpness { and the losses of using formal approaches to explicate past mathematical developments.

What then do I take to be the methodological lesson from these chapters? Clearly, it is not that any kind of present-day formalisation is best avoided, also because the kind of exegetical work carried out in this dissertation is meant for an audience of present-day readers anyway. The point is to be explicit in both the goals and in the presuppositions one is starting from when approaching an historical text. This is not an original point by any means, seeing that it has been argued for at length by several scholars, but especially Betti and van den Berg (2014) for the history of ideas, and Chang (2021) for the history of science in recent work. Insofar as historically-oriented philosophy of mathematics can be considered as sharing in the challenges of history of ideas and history of science, respectively, it also warrants a conscious examination of its own methods. On the basis of the work in Chapters 4 to 6 I would formulate what I learned from it as follows: first, it can be helpful and insightful to try and compare historical mathematics with contemporary formal frameworks and concepts { to some extent, it is downright

unavoidable { but the line between what actually is in the texts and what is being added by appeal to contemporary tools needs to be explicitly acknowledged. Second, the most helpful formalisations are those that do not merely preserve the results of the historical source, but also its argumentative structure.

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Sums, Numbers and Infinity

Collections in Bolzano's Mathematics and Philosophy

This dissertation contains a series of studies on 19th century philosophy of mathematics. The essays are linked together by two common threads: Bolzano's theory of collections on the one hand, and the emergence of modern sets and analysis in the 19th century on the other. Bolzano is often mentioned as an important figure for both developments, but sometimes the apparent similarity between his contributions and those of other thinkers is not sufficiently probed. Much of the work of this dissertation offers new interpretations of key aspects of Bolzano's writings to give a historically accurate and technically sound reassessment of Bolzano's contributions to mathematics and its philosophy in its own terms, obtained by careful textual analysis, and mathematical probing. This applies to the key notions of collections, natural numbers, measurable (i.e. real) numbers, and infinity.

Chapter 2 is the one that focuses the most on Bolzano's theory of collections. Bolzano has different notions of collection: collection in general (*Inbegriff*), *Reihe*, *Menge*, *Vielheit* among others. Generally speaking, in his mathematics he usually appeals to *Mengen* in a way that has encouraged other scholars to interpret these as completely equivalent to the sets of set theory. Chapter 2 argues that Bolzano's *Mengen* though are markedly different from sets, for two reasons: first, they are not extensional in the sense of the extensionality axiom (although *Vielheiten*, a special kind of *Mengen*, are); second, they do not play the same foundational role as sets do. Ultimately this difference in function is insurmountable, because it stems from the fact that Bolzano does not extensionalise the notion of structure, whereas that is precisely the conceptual gain granted by set theory that allows for sets' foundational applications.

Chapter 3 is the first of the chapters that deal with Bolzano's mathematical objects, and we start with the most basic of those objects, namely, the natural numbers. The main focus of the chapter is to explain why, within Bolzano's

conceptual approach to natural numbers, the question of how he measures the sizes of infinite collections needs to be rephrased. We argue that it is not possible to determine *the* size of a given infinite collection of natural numbers, because this collection will always be given through a certain concept, and it is this concept that determines how the size of the collection is to be computed. This approach has the advantage of explaining how Bolzano's views on infinite collections of natural numbers evolve between the *Wissenschaftslehre* (1837) and the *Paradoxien des Unendlichen* (1851).

Chapter 4 considers Bolzano's most sophisticated number system, that of the measurable numbers. Ever since the relevant text first came to light, Bolzano's measurable numbers have been read as his attempt at giving a presentation of the real numbers. This argument has been made mostly by showing that Bolzano's presentation can be translated into a sequence-based presentation of the real numbers that strongly resembles Cantor's or Dedekind's (depending on how it is carried out). Whilst agreeing that Bolzano's measurable numbers ought to be seen as Bolzano's attempt at a rigorous presentation of the real numbers, we argue that, for it to be rigorous, it cannot be the sort of presentation that the sequence interpretations make it out to be. We also argue that sequence interpretations are motivated precisely by an effort to show that somehow Bolzano was 'right all along', where being right boils down to anticipating a Cantor-style approach, even though such a reading introduces a host of mistakes in Bolzano's presentation that are simply not there.

With Chapter 5, the last one on Bolzano's mathematics, we shift our attention beyond number systems and on to Bolzano's 'calculations of the infinite', as they appear in *Paradoxien* §§29-33. Here we argue against what has been one of the mainstays of Bolzanian interpretations, namely the thought that the *Paradoxien* contain an anticipation of Cantor's transfinite arithmetic. It was never Bolzano's intention to measure the size of infinite set-like collections, all he wanted was a principled way to compute with infinite sums. This new interpretation sheds light on passages from the *Paradoxien* that are otherwise hard to make sense of, and it also allows us to defend Bolzano's arithmetic of the infinite as coherent.

Finally, Chapter 6 contains a comparison between a common 19th century understanding of how to extend mathematical concepts and domains, as exemplified by Dedekind, and a recent attempt at using model-theoretic notions to explain what domain extensions, and especially domain extensions via ideal elements, are supposed to do. I test each proposal against an array of prototypical cases of domain extension, including some from Dedekind's own mathematical work, and conclude that neither the modern proposal by Ken Manders (1989) nor the Dedekind-inspired proposal can offer a complete characterisation of domain extensions via ideal elements, although this negative result is insightful: it makes us realise that each criterion is meant to capture extensions that are meant to preserve different features of the domain we start from.

Samenvatting

Sommen, getallen en oneindigheid

Verzamelingen in Bolzano's Wiskunde en Filosofie

Dit proefschrift bevat een reeks studies over de 19e eeuwse filosofie van de wiskunde. De essays zijn met elkaar verbonden door twee rode draden: Bolzano's theorie van collecties enerzijds en de opkomst van moderne verzamelingen en analyse in de 19e eeuw anderzijds. Bolzano wordt vaak genoemd als een belangrijke voorloper voor beide ontwikkelingen, maar soms wordt de schijnbare overeenkomst tussen zijn bijdragen en die van andere denkers niet voldoende onderzocht. Een groot deel van dit proefschrift biedt nieuwe interpretaties van belangrijke aspecten van Bolzano's geschriften om een historisch accurate en technisch verantwoorde herwaardering te geven van Bolzano's bijdrage aan de wiskunde en haar filosofie. Dit gebeurt in Bolzano's eigen termen, verkregen door zorgvuldige tekstuele analyse en wiskundig peilen. Dit is van toepassing op de kernbegrippen van collecties, natuurlijke getallen, meetbare (d.w.z. reële) getallen en oneindigheid.

In hoofdstuk 2 wordt het meest ingegaan op Bolzano's theorie van collecties. Bolzano gebruikt verschillende begrippen van collectie: onder andere collectie in het algemeen (Inbegri), Reihe, Menge en Vielheit. In het algemeen heeft de manier waarop hij zich in zijn wiskunde op Mengen beroept andere geleerden ertoe aangezet deze te interpreteren als volledig gelijkwaardig aan de verzamelingen van de verzamelingenleer. In hoofdstuk 2 wordt betoogd dat Bolzano's Mengen echter duidelijk verschillen van verzamelingen. Dit om twee redenen: ten eerste zijn ze niet extensioneel in de zin van het extensionaliteitsaxioma (hoewel Vielheiten, een speciaal soort Mengen, dat wel zijn) en ten tweede spelen ze niet dezelfde funderende rol als verzamelingen. Uiteindelijk is dit verschil in functie onoverkomelijk, want het vloeit voort uit het feit dat Bolzano het begrip structuur niet extensionaliseert, terwijl dat nu juist de conceptuele winst is die de verzamelingenleer toekent en die de funderende toepassingen van verzamelingen mogelijk maakt.

Hoofdstuk 3 is het eerste van de hoofdstukken die handelen over de wiskundige objecten van Bolzano. We beginnen met het meest elementaire van die objecten,

namelijk de natuurlijke getallen. Het belangrijkste doel van dit hoofdstuk is uit te leggen waarom, binnen Bolzano's conceptuele benadering van de natuurlijke getallen, de vraag hoe hij de grootte van oneindige collecties meet, geherformuleerd moet worden. Wij betogen dat het niet mogelijk is om de grootte van een gegeven oneindige collectie van natuurlijke getallen te bepalen, omdat deze collectie altijd door middel van een bepaald concept zal worden gegeven, en het is dit concept dat bepaalt hoe de grootte van de collectie moet worden berekend. Deze benadering heeft het voordeel dat ze verklaart hoe Bolzano's opvattingen over oneindige collecties van natuurlijke getallen evolueren tussen de *Wissenschaftslehre* (1837) en de tekst *Paradoxien des Unendlichen* (1851).

Hoofdstuk 4 behandelt Bolzano's meest geavanceerde getallenstelsel, dat van de meetbare getallen. Vanaf het moment dat de betrouwbare tekst voor het eerst aan het licht kwam, zijn Bolzano's meetbare getallen gelezen als zijn poging de reële getallen te presenteren. Dit argument is vooral naar voren gebracht door aan te tonen dat Bolzano's presentatie kan worden vertaald in een op reeksen gebaseerde presentatie van de reële getallen die sterk lijkt op die van Cantor of Dedekind (afhankelijk van hoe het wordt uitgevoerd). Hoewel we het ermee eens zijn dat Bolzano's meetbare getallen gezien moeten worden als Bolzano's poging tot een rigoureuze presentatie van de reële getallen, betogen we dat, wil het rigoureuze zijn, het niet het soort presentatie kan zijn dat de sequentie-interpretaties ervan maken. We betogen ook dat sequentie-interpretaties juist gemotiveerd zijn als poging aan te tonen dat Bolzano op de een of andere manier 'al die tijd gelijk had', waarbij gelijk hebben neerkomt op het anticiperen op een Cantor-achtige benadering. Bovendien introduceert een dergelijke interpretatie een groot aantal fouten in Bolzano's presentatie die er niet zijn.

Met hoofdstuk 5, het laatste over Bolzano's wiskunde, verleggen we onze aandacht van getallenstelsels naar Bolzano's 'berekeningen van het oneindige', zoals die voorkomen in *Paradoxien* §§29-33. Hier wordt ingegaan tegen wat een van de pijlers is geweest van Bolzano's interpretaties, namelijk de gedachte dat de *Paradoxien* een anticipatie bevatten op Cantor's transfinite rekenkunde. Het is nooit Bolzano's bedoeling geweest om de grootte van oneindige (op verzamelingen lijkende) collecties te meten, hij wilde alleen maar een principiële manier om met oneindige sommen te rekenen. Deze nieuwe interpretatie werpt licht op passages uit de *Paradoxien* die anders moeilijk te begrijpen zijn en stelt ons ook in staat Bolzano's rekenkunde van het oneindige als coherent te verdedigen.

Hoofdstuk 6 tenslotte bevat een vergelijking tussen een gangbaar 19e eeuws begrip van hoe wiskundige concepten en domeinen uit te breiden, zoals dat door Dedekind wordt geïllustreerd, en een recente poging om met behulp van modeltheoretische noties uit te leggen wat domeinuitbreidingen, en vooral domeinuitbreidingen via ideale elementen, geacht worden te doen. Ik toets elk voorstel aan een reeks prototypische gevallen van domeinuitbreiding, waaronder enkele uit Dedekind's eigen wiskundige werk, en concludeer dat noch het moderne voorstel van Ken Manders (1989) noch het op de 19e eeuw geïnspireerde voorstel een

volledige karakterisering kan bieden van domeinuitbreidingen via ideale elementen. Desondanks geeft dit negatieve resultaat inzicht: het doet ons bese en dat elk criterium bedoeld is om uitbreidingen te vatten die bedoeld zijn om verschillende kenmerken te behouden van het domein van waaruit we vertrekken.

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