Validity, Logic, and Models

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is an investigation into the nature of logic and validity. The motivating intuition is that we could understand why the intuitionist should come to their conclusions, even if we were Platonists ourselves. According to the standard account, an argument is valid only if it preserves truth in all Tarskian models due to logical form. Although information necessarily has logical structure, we argue, with Szabó [2012] and Brandom [1994], that the restriction of validity to "formal" arguments is hard to defend. Moreover, existing proposals to demarcate the logical constants by means of *invariance* are uncomfortably circular.

Furthermore, the Tarskian tradition focuses exclusively on particular set-theoretic structures. However, many reasoning problems require other models and therefore alternative logics, such as intuitionistic and relevance logic, modal logics, closed-world reasoning, and finite logics for computer science. Besides, Stenning and Van Lambalgen [2012] have shown that although people often don't conform to the standards of *classical* logic, they turn out quite consistent if we model their reasoning using other systems.

The core of this thesis is that an argument is valid if it necessarily preserves truth *on the model of interest*. I will argue that models are indispensable for thought and that necessity can be explained by the stability of models. The most basic models (of the "ordinary world") determine the construction of scientific and mathematical models and vice versa in a dialectical process. *Logics* are systems that capture patterns of valid arguments on types of models by focusing only on expressions that have a central role in logical structure. Normativity comes in at the level of constructing the right model and at the level of evaluating arguments on the model. The resulting position is a task-relative logical pluralism.

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Chapter 1

Introduction

1.1 Motivating intuitions

This thesis is an investigation into the nature of logic and validity, with the purpose of making sense of the different ways we use these concepts, in academics and everyday life. The motivating intuition behind this work is this: even if there are many different logics that logicians study and philosophers or mathematicians endorse, one can often admit that their opponent is right about *their* logic, *if* the world (or the task) is as they assume it is. In practice, adherents of incompatible logics are, in general, able to understand each other: they alter their assumptions, for the sake of argument, to match those of their opponent, and recognize why their opponent arrives at those conclusions. For instance, the real-life relevantist probably *gets* what the classical mathematician means by 'truth' and why ex falso quodlibet is a classically valid inference pattern — they just disagree with that.

These ideas are present in the literature. For instance, consider this remark from Barwise:

> "What is the logic of specific mathematical concepts?" [G]iven a particular mathematical property (...), what is the logic implicit in the mathematician's use of the property? What sorts of mathematical structures isolate the property most naturally? What sorts of languages best mirror the mathematician's talk about the property? (...) If you and I are discussing some topic, (...) and I say "The logic of that escapes me", what I mean is that I do not see how the conclusion you have come to follows from our shared assumptions and concepts, *including the conception of the task at hand* [my italics]. [Barwise and Feferman, 2017, p.3]

For another example of diverging judgments of validity, both of which make sense, suppose there is a person, Eve, who is trying to get from Amsterdam to Berlin. She wonders whether a train is a viable option. Let us assume the railway schedule says (schematically) that if train A leaves at midnight, it arrives in Berlin in the early morning. Eve takes train A, which leaves at midnight. Is Eve reasoning validly if she concludes that she will arrive in Berlin in the early morning? On the one hand, one might say she is not: there is enough that could go wrong — the train could have a breakdown, Germany could close its borders, there could be a train traffic jam. When one considers straightforward reality, the conclusion seems invalid. But that would never get anyone anywhere. In real life, we often reason under so-called closed-world assumptions; for instance, that we only take possible sabotaging exceptions into account if we have any evidence that they might be realized. Naturally, Eve planned the trip considering such a closed world version of reality, and could validly infer that the train would arrive.

1.2 Arguing about arguments

The multitude of logical systems and the diverging judgments about validity stand in contrast to the modern, dominant philosophical view on logic. This *model-theoretic* account, which originates with Tarski [2002] and was further developed by Sher [2008], equates validity with logical consequence and defines this as formal necessary truth-preservation over particular mathematical structures. Formality means that the necessary truth-preservation must be due to the form of the argument. In the contemporary view, this means that truth-preservation must be due to the *logical constants*.

For example, there can be little doubt about the truth of the conclusion of the following arguments, if one assumes that the premises are true:

- x is an even number and x is strictly greater than 100; therefore, x is an even number.
- All humans must die. Harrison is a human. Therefore, Harrison must die.
- Most bananas are fruit. Most bananas are yellow. Therefore, there are bananas that are yellow and fruit.

Terms like 'all', 'and', 'or', 'if ... then' are standardly considered to be logical constants; 'most' a bit less so, but it qualifies according to some authors. However, other arguments that seem fine, like

- Joe is taller than Jack. Wilma is taller than Joe. Therefore, Wilma is taller than Jack.
- Joe is running fast; therefore, Joe is running.
- The chair is red; therefore, the chair is colored.
- Sample x is water; therefore, the chemical composition of x is H_2O.

where the crucial terms are 'taller', '... fast', 'red', and 'water', are not traditionally seen as valid. But why? They certainly seem better than these:

- All humans must die. Therefore, there are only finitely many animals.
- x is an even number <u>or</u> x is strictly greater than 100; therefore, x is an even number.

This latter difference in "goodness" can be seen as a preamble for the first half of the critique against the Tarskian view.

The standard model-theoretic account displays a *monist* perspective on logic, which consists of two beliefs. One is that there is a fixed set of logical constants.

The other is that genuine logical consequence has to be defined on a particular kind of *structures*, usually Tarskian, set-theoretic models with a base domain from which higher-order functions and relations are defined.

A natural question is how one should distinguish validity based on form from validity based on definition or fact. Szabó [2012] argues that this cannot be done in a principled manner. Nevertheless, mathematical characterizations of the (exclusive) logical constants have been proposed by Tarski, Sher, and others. These revolve around *invariance* of the desired logical operators. When discussing this, we will notice that the authors make sure to get exactly the constants they believed to be logical, as Van Benthem [2002] points out. As a result, it will be apt to disentangle the concepts of validity and logical consequence. In other words, the arguments in the second set are eligible for validity.

Even if validity is not all about form, we can admit that it's *often* about form, in the sense of logical structure of propositions (the information that a sentence conveys). MacFarlane [2000] identifies several readings of the standard criterion of formality. We will observe that one of these — logical structure as constitutive of thought and (semantic) information — should be endorsed, even if it is probably impossible to determine what exactly counts as logical structure and what does not.

The other (objectionable) half of the Tarskian view is the focus on one type of structures. This idea neglects the fact that those set-theoretic models are also just *models*: tools to analyze propositions about abstract (mathematical) relationships between objects. The Tarskian models might be able to represent a lot, but still not everything there is. Many logics are defined on other kinds of models: intuitionist logic on *stages*, relevance logic on (incomplete) *situations*, modal logics on Kripke frames, nonmonotonic logics on closed worlds. These logics, given their clear merits, give a first rationale for a pluralist account of logic.

Another rationale can be found in Stenning and Van Lambalgen [2012] and has to do with the example above, of Eve planning a train trip. In a few paradigmatic psychological experiments on human reasoning, like Wason's selection task, it has been concluded that ordinary subjects are not very good at logic. However, although most of these subjects did admittedly not conform to the principles of *classical logic*, it is possible to investigate how these subjects understood the task. It turns out that subjects, given *their* interpretations, reasoned quite consistently — just like how Eve could make a valid argument given that she would not consider all and any possible exceptions to the train schedule.

The main point of this thesis will be developing a view on validity and logic that is in line with these intuitions about form and the logical constants, and the rationales for logical pluralism. I will argue that the crux of valid arguments, in many different logics, is necessary truth-preservation on the model(s) of interest. We will investigate what models are, what is meant by necessity on models, and why necessity of truth-preservation occurs.

Once validity is characterized, we will say more about logic. A logic is a "collection of mathematical structures, a collection of formal expressions and a satisfaction relation between the two. (...) [A] logic is something we construct to study the logic of a part of mathematics [Barwise and Feferman, 2017, p. 4-5]." I will add to this that logics are systematizations of *some* valid arguments, given a *type* of models (complete or incomplete, consistent or inconsistent, static or dynamic), by focusing on only certain expressions of the language — those expressions that most clearly play a role in structuring information (or thought).

One might wonder whether this undercuts the supposed normativity and universality of validity and logic. I will argue that it does not, by pointing out that the (albeit dialectical) process of determining the right model can have an objectively correct outcome. Thereafter, premises and conclusions that have a sufficiently determinate meaning are either valid or not.

1.3 Outline

The outline of this work is as follows.

In **chapter 2**, we look at the dominant account of validity. We start out with some preliminaries in §2.1. After this, we discuss the model-theoretic account and the criterion of formality in §2.2. The Tarski-Sher thesis and proposals of invariance criteria are presented in §2.3. In §2.4, I discuss the encountered flaws: the problems of the invariance criteria and the focus on Tarskian structures.

In **chapter 3**, we will discuss the rationales for logical pluralism. In §3.1, we examine a multitude of logics that are defined by using other structures than the standard Tarskian models. In §3.2, we look at Stenning and Van Lambalgen [2012]'s analysis of what subjects are doing in paradigmatic psychological experiments on reasoning. I will consider the consequences for an account of logic — that it has to be pluralist — in §3.3.

In **chapter 4**, I attempt to give an account of validity and logical pluralism. First, in §4.1, we discuss the model-theoretic pluralism that was proposed by Beall and Restall [2000], which has two major flaws: it fails to satisfy the requirements of normativity and necessity. §4.2 purports to give an explanation of validity and logic that does justice to the ideas developed in the rest of this thesis.

The conclusion follows in **chapter 5**.

Chapter 2

Validity via Tarskian models

In this chapter, I will present the dominant modern account of validity and discuss some of its flaws.

In §2.1, we will take note of some necessary preliminary remarks about (the logical structure of) language and information.

In §2.2, we will consider the model-theoretic account of validity, including the crucial criterion of *formality*. We will discuss MacFarlane [2000]'s analysis of the senses of formality discernible in the literature. We will also look at the historical roots of the "doctrine of logical form", following Dutilh Novaes [2012]. I will then evaluate the several criteria that fall under the header 'formality', and argue against form as the hallmark of *validity*, following insights from Szabó [2012] and Brandom [1994].

In §2.3, we will discuss the Tarski-Sher proposal for characterizing the logical constants by means of invariance under permutations or isomorphisms. We will also look at invariance proposals by Feferman et al. [2010] and Bonnay [2008].

After that, in §2.4, I discuss the encountered flaws of the monist model-theoretic account: the problems of the invariance criteria and the focus on Tarskian structures.

2.1 Preliminaries

Let us start with a few presuppositions, that need little justification.

First of all, *all* (semantic) information is by definition structured.[1] Information is formed by means of predication, conjunction, disjunction, implication, negation, quantification, et cetera. Whatever we mean exactly by 'That apple

[1] There are several things one can mean by 'information' [cf. Adriaans, 2020, Floridi, 2019]. Here we refer to the idea of (semantic) information as meaning-full, well-formed data, or that which (formal and natural) language utterances convey.

is red,' it encodes some kind of connection between two concepts (in our conceptual scheme)[2]. Several, say two, atomic judgments can only be connected in certain ways: by asserting both, by asserting that at least one is true, by refuting at least one, by asserting both and expressing some temporal order, by describing causal or other dependencies, and so on. And we can say things about multiple objects at a time: that all, most, some of them are red.

Judging from the academic practice that studies logic, (at least some of) this structure of information is what we call *logical*. This is where the phrase *logical form* comes from.

This concept of logical structure cannot be identified with 'logical form' as it is sometimes used in linguistics: a level of representation of a sentence where ambiguities have been resolved. As Szabó [2012] points out,

> Thus conceived, logical form encompasses all and only information required for interpretation. But semantic and logical information do not fully overlap. The connectives "and" and "but" are surely not synonyms, but the difference in meaning probably does not concern logic. (...) Logical form in a broadly Chomskyan sense would be more appropriately called "semantic form." [Szabó, 2012, p.105]

Rather, logical structural aspects are those upon which validity — which we will discuss extensively later — *often* depends. This is something that becomes clear from investigating patterns of valid arguments; thus it follows that 'and' and 'but' don't normally imply a difference in *logical* structure.

Let me already say that I don't think that we can exactly delimit what should count as logical structure and what does not. Logical structure or form should be more viewed as a label for a collection of (intensional and extensional) operators and ways of connecting concepts that induce (patterns of) valid arguments. More on this later.

So a proposition (that which is expressed by an uttered sentence) has logical structure, even though we cannot demarcate what counts as structure (but at least things like predication, quantification and conjunction/negation will count).

It is outside the scope of this work to take a well-argued stance on whether the proposition is fully determined by the semantic content of an uttered sentence (if that exists), and whether the exact logical operators in a proposition are completely determined by the semantic content of their natural language counterparts if that semantic content were considered to be unambiguous. Instead, I will posit the following points as justifiable by common sense, however they might be explained by a satisfying linguistic theory.

First, I will take it as a given that, all too often, we *can* understand what exactly was expressed by a sentence. Maybe sometimes extensive communication is needed, but the amount to which humans communicate and work together and achieve things is evidence for this supposition. If we truly believed we could not generally understand each other to this exact extent, it would make no sense

[2]The term 'conceptual scheme' comes from Quine [2013], and can be understood as the system of concepts that constitute the categorization or model (including mathematical laws and laws of nature) we have of the world.

to engage in the kind of philosophical and mathematical and other scientific debates that we do. It is good to acknowledge that we do not always know how understanding is possible, nor that we *certainly* know we're talking about the same things; but we have to realize that, mostly, we understand each other, to a degree that might be called stunning if that wasn't so self-congratulatory.

I will also assume that the standard natural language logical terms, like 'and' or 'if ... then,' can be used, and are often used, to express logical structure that is not traditionally seen as their "logical" meaning, like the temporally ordered conjunction ('and then') or a *default conditional* (one leaves room for exceptions: "if A *and nothing weird is going on*, then B") [cf. Carston et al., 1993].[3] In the philosophy of language debate on semantic versus pragmatic content, there are two broad positions on this topic [cf. Grice, 1975, Speaks, 2019]. One is that the proposition that an uttered sentence expresses is determined completely by the semantic content, which, then, also takes in all relevant contextual factors (for instance, via hidden indexicals). The other is that the proposition is not exhaustively determined by semantic content, but by pragmatic factors as well. The verdict is still out on whether these non-classical meanings of logical terminology are part of semantic content or just pragmatic implicatures. However, we can note that even if the correct linguistic position were that the *semantic content* of 'if ... then' is the classical material implication,[4] then, in colloquial speech, people regularly intend to express another operator, like the default conditional.

In practice it is often unclear what the proposition (or even intended interpretation of a sentence) is, so that investigating how the listener interprets an uttered sentence — including what logical form they assign to it — is interesting and important for understanding how people reason. We will see later, in chapter 3, that a number of famous psychological experiments, such as the Wason selection task, neglected this simple linguistic fact and therefore prematurely concluded that people are not good at logical reasoning, even though subjects often reasoned coherently from their interpretation of the instructions, which one might say they arrived at because there were not enough clues for the intended interpretation.

Given that we can express thoughts precisely, it is also evident that we can *draw conclusions* from premises: that the apple is red, that the apple is colored, that the apple is an apple; that from a conjunction we can jump to the truth of the conjuncts, that we can do modus ponens, that we can say that my apple is red if all apples are red.

The intuition here is that valid arguments are arguments where the conclusion *follows* from the premises, where the conclusion and premises are all pieces of (semantic) information, such that the premises are assumed to be true and the conclusion is judged to follow. What does this following or validity mean?

[3] Default conditionals: "If I flip the switch, the light will turn on," "If it rains, the streets get wet." Temporal conjunction (from [Carston et al., 1993]): "He handed her the scalpel and she made the incision," "She fed him poisoned stew and he died."

[4] Which can be defined as: "if A then B" is only false if A is true and B is false.

2.2 The model-theoretic account

Shapiro [2005a, p.655-659] gives a few definitions that aim at capturing our pre-theoretical intuitions about validity. He talks about *logical consequence* — these two terms are used interchangeably in the literature but I will later argue that we should disentangle them.

> Φ is a logical consequence of Γ if it is not possible for the members of Γ to be true and Φ false.
>
> Φ is a logical consequence of Γ if the truth of the sentences of Γ guarantees the truth of Φ in virtue of the meaning of the terms in those sentences.
>
> Φ is a logical consequence of Γ if it is irrational to maintain that every member of Σ is true but Φ is false. The premises alone justify the conclusion.

There are two general ways to flesh out these first intuitions. One way is via proofs. This position, the *proof-theoretic account*, claims that an argument is valid iff there is a proof from the premises to the conclusion. We will come back to this position in chapter 4.

The standard way is via truth in models or interpretations: the *model-theoretic account*. This view says that an argument is valid iff the conclusion is true in all models that make all the premises true.

The contemporary model-theoretic account takes models to be set-theoretic, Tarskian models. These mathematical structures are made up from a base domain of individuals and higher-order type relations, properties and functions. There is a specified formal language with a privileged collection of logical terms. For example, the logical constants of first-order logic are the standard propositional connectives, the existential and universal quantifier (and often equality).

Valid inferences under the model-theoretic account are all valid due to the meaning of those few logical terms. This property is seen as the formality of logic.

2.2.1 Formality

There is a long evolution of ideas, starting all the way from Antiquity, through Immanuel Kant, that lead up to the criterion of formality. The origin of the criterion lies in the recognition that some conclusions seem to follow from a set of premises because of the *form* of the sentences involved, and not the particularities of the objects that the argument refers to [Dutilh Novaes, 2012]. Many arguments are intuitively valid because they follow the same patterns.

Here is an example. We can suppose that *"the chemical composition of sample x is H_2O"* follows necessarily from *"sample x is water"*.[5] But then there is an even clearer sense in which *"x is alive"* follows necessarily from *"x is alive and x is young"*. We need to know some chemical facts to make the former inference, but with the latter, we need only understand the meaning of 'and'. It's apparent that, intuitively, the latter inference has nothing to do with either being alive or being young.

[5] Assuming that the famous analysis of metaphysical modality by Kripke [1972] was right.

MacFarlane's identifications of formality

The question "What does it mean to say that logic is formal?" is discussed in [MacFarlane, 2000]. He says that if logic is to play such an essential role in philosophy of science projects like logicism (basing all mathematics on a foundation of logic) or structuralism (explaining the ontological status of mathematics), a characterization of logic is needed. Because of these motivations, the characterization cannot be pragmatic. So formality is evoked; or what is sometimes called *generality* or *topic-neutrality* — these terms are understood in the same confused ways.

First of all, there are three properties that are often seen as the formality of logic. MacFarlane calls these "decoy" concepts of formality. They are all conceptually unproblematic, but none of them can do the job of demarcation, since they all have gaps: implicitly or explicitly, some further property of logic is presupposed.

The decoy senses of formality are the following three.

- Syntactic formality refers to axiomatic systems and effective rules for symbol manipulation. If one only pays attention to the syntactic elements of a language, the *meaning* of the terms is neglected.

  But "[s]yntactic-formal logics stand in need of application just as syntactic-formal systems of physics do [MacFarlane, 2000]." For example, a syntactic rule saying that $A \vee B$ is a direct consequence of A does not tell us anything until we understand what 'direct consequence' means.

  Moreover, to see that not just any syntactic rule can be logical, consider Prior's famous runabout inference ticket [Prior, 1960]. The constant "tonk" is governed by the following introduction and elimination rule. *For any statement B: A* tonk *B is a direct consequence of A. For any A, B: B is a direct consequence of A* tonk *B*. The result is: any sentence follows from any sentence. No one could want this rule in their logic. Some semantic concepts will be needed.

- Schematic formality refers to *patterns* in arguments. An example is

  All As are Bs.
  All Bs are Cs.
  Therefore, all As are Cs.

  To say that logic is formal in this sense is to say that if one keeps some terms in a valid argument fixed and replaces some terms with other ones of the same semantic category, the argument will stay valid.

  But schematic formality cannot *characterize* logic: a criterion is needed to decide what terms are logical and we need to decide upfront what the semantic categories are.

- Logical form is grammatical form. MacFarlane argues that this just shifts the problem to deciding what counts as grammatical *form*. For example, two competing grammatical analyses of the adjective "taller" will yield different logics: if "-er" is categorized as a particle, "taller" is not an item in the lexicon; if "-er" is not, "taller" is. On one analysis *Taller(Joe, Jack), Taller(Wilma, Joe); therefore, (Taller(Wilma, Jack)* is valid but not on the other. Again, a *prior* distinction between the logical and the non-logical is necessary.

When philosophers use 'formality' as the *characterizing* property of logic, then, they must be talking about something else. This is the idea of logic as being general or independent of content or subject matter. Throughout history, there are three ways people have conceptualized this idea:

1. Logical norms are constitutive of thought as such (1-formality).

   > [T]here are the rules of the game. These norms apply to chess playing as such, because they are constitutive of chess playing. One might violate them (...) and still count as playing chess. One might even be ignorant of some of them and still count as playing chess. But unless these norms are binding on one's moves, one is not playing chess, but some other game [MacFarlane, 2000, p. 52].

2. Logic is indifferent to the particular identities of different objects (2-formality). This is cashing out formality as not being concerned with the peculiarities of individuals.

3. Logic abstracts entirely from the semantic content of concepts (3-formality). This refers to the idea that logic abstracts entirely from the semantic side of thoughts: that logic considers only the *form* of arguments.

Historically grown metaphysical assumptions

Dutilh Novaes [2012] discusses the historical evolvement of the idea that arguments exist of two parts, their matter and their form. MacFarlane [2000] calls this metaphysical view *hylomorphism*. It has been so dominant that almost all currently pervasive views on logical consequence are influenced by it.

Despite what is often thought, formality as the distinctive characteristic of logicality is not an idea that is thousands of years old; it originates with Kant. Aristotle, the discoverer of syllogisms, and because of that widely considered the father of formal logic, did not apply the form-matter distinction to arguments. (We will discuss syllogisms more extensively later. For now, here is an example of a valid one: *Some A are B. All B are C. Therefore, some A are C.* The pattern formed by "some ... all ... some" is called the mood, the pattern "AB/BC" the figure of the argument.) But this is not so surprising: Aristotle only ever applied the distinction between matter and form to special kinds of entities, mostly primary substances, like ordinary objects such as the reader of this thesis.[6]

[6] Moreover, for Aristotle, form in general is not a *part* of the whole; it's the *principle of unity* articulating the different parts of the whole, which constitute its matter [Dutilh Novaes, 2012].

Nonetheless, it is evident that the Aristotelian form-matter dichotomy has been the key inspiration for the modern doctrine [Dutilh Novaes, 2012, p. 400]. The first steps were set in the tradition of the Ancient Commentators. In many of these early texts, there is an ambiguity: for some authors, the figure constitutes the form of an argument, for others, the mood. Along the way, the form of arguments becomes increasingly associated with moods. This is a crucial point, as only this association presupposes the partition of the vocabulary into two sorts. By the Latin fourteenth century, this step is fully taken.

This brings us to the modern doctrine. Here is a description from a well-known introductory textbook on logic:

> First, on the classical view, validity is a matter of form. Individual arguments are valid only in virtue of instantiating valid logical forms; one proposition is a logical consequence of others only if there is a valid pattern which the propositions together match. (...)
>
> [T]he task of logic is to provide techniques for identifying and discerning the logical form of various arguments, and for determining whether the forms discovered in this way are indeed valid. [Read, 1994, p.36-37]

Dutilh Novaes identifies three, historically grown, often tacit presuppositions about the metaphysical nature of arguments that the modern doctrine is dependent on. The first is *mereological hylomorphism*. The idea is that every argument is a whole made up of a matter-part and a form-part.The second is that there is one single form per argument. And thirdly, that the partition between form and matter is meant to be principled and sharp.

Dutilh Novaes' analysis is that the ideas about the metaphysical nature of form and matter in arguments lead to the following assumptions. Every whole argument has as parts form and matter. Form is determined by a proper subset of the vocabulary of the language, in a given disposition; matter is related to the complement of that subset. The form of an argument can be rendered by means of a schema. The form is what makes a valid argument valid. Logic is concerned with validity, and thus with forms of arguments. The technique for studying logical form is substitutivity. The terms that remain fixed are the logical terms — or *constants*. The rest are extra-logical. Logic can be demarcated by demarcating the logical constants.

She discusses three conceptual reasons to not take logical hylomorphism for granted. (1) The justification for the application of the form-matter distinction to arguments is usually not given. (2) It is questionable whether we can just import ideas and assumptions from metaphysical hylomorphism to logic. Can arguments sensibly be said to have two parts, a form and a matter, that are sharply divided? Is there one unique form to each argument? (3) Even if the division of form and matter makes sense for arguments, it's not clear why form should demarcate logic.

Dutilh Novaes' diagnosis of the situation is that the demarcation of the nature of logic by the criterion of formality has been caused by a confusion of a method used in logic with its subject-matter.

2.2.2 Remarks on the criterion of formality

It is not as clear to me as it is to Dutilh Novaes that the substantive metaphysical assumptions that could be observed in ancient, medieval, and early-modern philosophy are still fully present in the modern accounts of logical consequence. The conceptual lacunae of schematic formality in modern model-theoretic accounts are filled up mostly by a kind of 2-formality: indifference to particular identities of objects. It is evident that the idea of 2-formality has its roots in the hylomorphism that dates back to antiquity, but in itself, it does not necessarily imply acceptance of a metaphysical hylomorphism.

If the modern accounts are indeed not committed to these metaphysical assumptions, then that is a good thing. I find it hard to see what *the* logical form of an argument should be: information is structured, yes, but it is not conceivable that we could define that some of this structure is ontologically different from the matter, whatever that might be.

Let us evaluate the three conceptions of formality that MacFarlane identified. First of all, the criterion of 3-formality — that logic abstracts away from all semantic content — is ill-fated. Structural aspects of a proposition are part of the semantic content of the sentence. If something else than the standard linguistic concept is meant by semantic content — like *matter* — it needs to be explained what the form and matter of an argument exactly are: if matter is intended to mean the particular objects that the proposition refers to, the criterion is actually the criterion of 2-formality in disguise; if matter is intended to mean something else — well, then the burden is on the proponent to explain what it could be.

The supporter of a criterion of 2-formality might not have to presuppose the separation of the vocabulary but usually does just that. We will see in the next section that it's quite impossible to make sense of this criterion: because "logical" valid arguments actually do depend on some (numerical) qualities of individual objects and definitely depend on the particular identities of the models on which the arguments are evaluated — and the choices for these models are again dependent on assumptions about (mathematical) objects. We will discuss one suggestion for the formality criterion, by Sher, more extensively in the next section.

The idea of 1-formality — logical norms are constitutive of thought — is better: information necessarily has structure. But that does not mean that validity can only come from (a fixed category of) logical structure. To get an idea of why it is fruitless to try and develop a fixed set of constants that induce *the* logical form of a proposition and make this form the crux of validity, we will look at an argument by Szabó [2012]. Note first that Szabó remarks that the Quinean idea of logical form — language (and thereby thought) clearly has some structure, and when we regiment natural language into more *precise* formalized language, we can *choose* which structural aspects we encode, dependent on our interests and aims — is not problematic. But the traditional view on formality is. Szabó argues that the principled distinction between logical and extra-logical terms can only be made by *fiat*.

The argument is simple. Tradition says that valid inferences come in three (collectively exhaustive, mutually exclusive, nonempty) categories: factual, lexical and formal (non-formal ones are sometimes called "material", non-factual ones "analytic"). Here are three examples:

1. Alex is a father; therefore Alex has a Y-chromosome.
2. Alex is a father; therefore Alex has a child.
3. Alex is a father; therefore Alex is a father or a mother.

Traditionally, (1) is explained to be valid due to fact, definition, and form, (2) due to definition and form, (3) due to form. But Szabó argues that this is not correct. The validity of (2) and (3) depend *also* on the facts that all fathers have children and that everyone who is a father is a father or a mother. Similarly, the validity of (3) depends also on the definition of 'or'. Szabó argues that it would be absurd to reject the existence of those universal or necessary facts.[7] Nor does it make sense to deny that 'or' has a definition (even though that definition is of another kind than for content words such as 'father').

Now perhaps the proponent of the doctrine can claim that non-factual arguments might depend on facts but *hold in virtue of* definitions and form. But then the *facts* that are explanatorily *relevant* should themselves depend on definitions and form — otherwise *form* cannot be explanatorily *sufficient* for the validity of the argument. However, for this line of response to work, it is crucial that definitions and forms should not be facts. But again, whenever something is a definition or form, there are facts correlated with it, which justify the correctness of the definitions and forms.[8] (And the same argument, with the right adjustments, can made for the distinction between formal and non-formal validities).

The conclusion that Szabó draws seems to be that validities can have different kinds of explanations, of which the standard "logical" validities, based on "form", are the ones that stand least in need of justification.

It is interesting here to include some points made by Brandom [1994]. According to him, validity is indeed not a property only of "formal" arguments; rather, formal arguments are ways to make the material inferences (validities) (and how they relate) that can be seen as the network, or texture, of our conceptual scheme *explicit*:

> The "dogma" expresses a commitment to an order of explanation that treats all inferences as good or bad solely in virtue of their form (...). It trades primitive goodnesses of inference for the truth of conditionals. [p.98]
>
> But this substitutional conception of what it is for an inference to be good in virtue of its form is not essentially restricted to a notion of logical form. [p.105]
>
> Vocabulary deserves the appellation 'logical' just in case it serves to make explicit, as the content of a claim, proprieties concerning the use of the expression

[7] It would be absurd to reject the existence of facts alltogether; likewise, to reject the existence of necessary facts (making mathematics impossible); and rejecting the existence of lexical or formal facts would *presuppose* the classification into factual, lexical, and formal validities (since truths are valid consequences of the empty set of premises) [Szabó, 2012, p.114].

[8] Again, rejecting existence only of lexical or formal facts would presuppose the distinction.

> that otherwise remain implicit in practice, specifically the proprieties in virtue of which it has the conceptual content that it does. [p.114]
>
> The point of using these sorts of expressions [including (...) 'claims that', 'believes that', 'intends that', and normative talk of commitments and entitlements] is to make explicit as the contents of claims some of the pragmatic elements of the practices of talking, believing, and acting that confer propositional contents. [p.116]

This is a good way to think about logical notions, because it makes intelligible why so many notions can be seen as logical. Propositional connectives are the obvious case of making our inferential commitments explicit, by conjoining or disjoining or negating them; the quantifiers are another unmistakable example. Similarly, modal operators (which we will discuss in §3.1.2) make explicit, for instance, the temporal universality or particularity of our inferential commitments, or whether we judge things to be necessary or only possible. Even adverbs such as 'very' or adjectives such as 'more' can be said to have an explicitating and therefore sometimes logical role.

One might object that one *can* distinguish the logical constants by means of a *mathematical* characterization, which would show that these logical constants have a role in the mechanics of thought that make logical consequence a special kind of consequence — in terms of Szabó's argument, that validities due to the logical constants are justified by (besides, obviously, their definitions) a *special kind of necessary facts*: that logic deals with the highest generality or structurality. To assess that objection, we'll take a look at the proposed attempts at principled mathematical characterizations of the logical constants in the literature.

2.3 Logical constants: invariance criteria

2.3.1 The Tarski-Sher thesis

Tarski

The author with whom the contemporary model-theoretic account originates is Alfred Tarski. In [Tarski, 2002] he articulates for the first time the model-theoretic definition of logical consequence (i.e. validity): all models of the premises $\mathfrak{K}$ have to be models of the conclusion X, where the premises and conclusion are of a formalized language.

The question is where to draw the dividing lines between logical and extra-logical terms. Some terms must clearly be regarded as logical; other, unusual ones could be classified as logical without violating our intuitions. In the extreme, Tarski says, if all terms are seen as logical, formal consequence will coincide with material consequence. This point might be a bit confusing. It is true when one employs the kind of models that Tarski does: the domains are fixed; they are sequences of objects from a universe on which the discourse at hand is interpreted. So if *all* terms are classified as logical, and thus held fixed, conclusion X just follows from premises $\mathfrak{K}$ in case it *materially* follows. That is, whenever there is truth preservation. For example, "3 and 4 are natural numbers; therefore, a triangle has three sides" will be logically true. These fixed domains are a flaw in Tarski's account [cf. the translators comments in Tarski, 2002]. Suppose the domain contains infinite objects. This is not a bold supposition if our goal is to represent the mathematical universe. With a fixed, infinite domain, a sentence like *There are at least 2 objects* comes out as logically true. But that is undesirable. What seems to be missing is variation of the domain [Tarski, 2002, p. 171]. Tarski himself avoided this, probably to not make things unnecessarily complicated for the intended, philosophical audience. We will see in 2.3.1 how Sher accommodates modal variation.

If a division of the language cannot be justified objectively, Tarski says, we will have to treat logical consequence as relative to an arbitrary division. As he already mentioned in the introduction: "every precise definition (...) will to a greater or lesser degree bear the mark of arbitrariness" [Tarski, 2002, p. 176].

In his (posthumously published) [Tarski and Corcoran, 1986], Tarski returns to the question of demarcating logic by means of defining the logical constants. The criterion that he proposes is inspired by Klein's Erlanger Programm for classifying geometrical notions. The notions of metric geometry were discovered to be the notions that were invariant under similarity transformations. The notions of descriptive geometry were those that are invariant under the affine transformations. And topological notions are invariant under continuous transformations. As the number of admissible sorts of transformations grows, there are fewer invariant notions; and what invariant notions remain, are more general than the ones that disappear. Following this idea, Tarski considers the most general transformation of the space: that of every point onto another point. Instead of only geometrical spaces, he considers transformations of arbitrary universes of discourse. These are functions from the universe to itself: *permutations*. Consequently, the logical notions are defined to be the notions that stay invariant under these functions.

Sher

In [Sher, 2008], Sher adopts and adapts Tarski's proposal to provide a philosophically informative and mathematically precise characterization of the logical constants.[9] She explains that for the purpose of her contribution, we will think about "worlds" or "models" as set-theoretic structures, and objects as operators: characteristic functions representing the objects of interest. Moreover, the use of *permutations* of the domain is changed to *isomorphisms* between structures.

> **Invariance under Isomorphism** An operator O is logical iff it is invariant under all isomorphisms of its argument-structures
>
> where
>
> 1. A *structure* is an m-tuple, $m \geq 1$, whose first element is a universe, A (i.e., a nonempty set of objects treated as individuals, that is, as objects

[9] Sher distinguishes logical operators from logical constants. The former are the actual logical notions, the latter the linguistic entities that describe these.

lacking inner structure), and whose other elements (if any) are set-theoretic constructs of elements of A.

2. Two structures, $\langle A, \beta_1, ..., \beta_n \rangle$ and $\langle A', \beta'_1, ..., \beta'_k \rangle$ are *isomorphic* (...) iff $n = k$ and there is a bijection f from A to A' such that for every $1 \leq i \leq n$, β'_i is the image of β_i under f.

3. An operator O represents an *object of a given type*—an individual, a property of individuals (...), etc—and specifies its extension (...) in each universe.

   Specifically:

   - An operator representing an individual a assigns to each universe A a 0-place function whose fixed value is a if $a \in A$, and which is treated in some conventional manner otherwise.
   - An operator representing a first-order property assigns to each universe A a function from all members of A to a truth-value (which, provisionally, we assume is T or F). (...)
   - An operator representing a first-order monadic quantifier assigns to each universe A a function from all subsets of A to $\{T, F\}$.

   (...) Etc.

4. If O is an operator whose arguments are of types $t_1, ..., t_n$, A is a universe and β_1, β_n are constructs of elements of A of types $t_1, ..., t_n$ respectively, then β_1, β_n are arguments of O in A (or $\langle \beta_1, ..., \beta_n \rangle$ is an argument of O in A) and $\langle A, \beta_1, \beta_n \rangle$ is an argument-structure of O.

(...)
We now define:

> *An n-place operator O is invariant under all isomorphisms of its argument-structures*
> iff
> for any of its argument-structures, $\langle A, \beta_1, ..., \beta_n \rangle$ and $\langle A', \beta'_1, ..., \beta'_n \rangle$: if $\langle A, \beta_1, ..., \beta_n \rangle \cong \langle A', \beta'_1, ..., \beta'_n \rangle$, then $O_A(\beta_1, ..., \beta_n) = O_{A'}(\beta'_1, ..., \beta'_n)$.

[Sher, 2008, p. 302-304]

Let's explain this informally. To say that two structures are isomorphic is essentially to say that the *structure* of the structures is exactly the same. *Invariance* (under all isomorphims of its argument-structures) of an operator can be explained as it being not affected by changing only the *labels* of its input. If two structures A, A' are isomorphic, for all nonempty subsets B in domain A, their images $f(B)$ are also nonempty; so the operator $\exists$ gives the same output on A, B and $A', f(B)$.

Sher says that it is easy to see that the standard logical operators — the existential quantifier, the universal quantifier, the propositional connectives — are classified as logical under this criterion.[10] At the same time, "blatantly non-logical" notions like 'is red' are not.

[10] To see that the propositional connectives are logical, consider, for example, ordinary conjunction $\wedge$. $O_\wedge$, given any structure M, takes two arguments and outputs T iff both arguments are T. So a standard first-order sentence like $B(a) \wedge C(a)$, true on a given structure M with interpretation function I such that $I(B) = B, I(C) = C, I(a) = a$ (for simplicity), would in this proposal be formalized as $\wedge_M(B_M(a_M), C_M(a_M)) = \wedge_M(1,1) = 1$ (since $a \in M$ and operators B, C are applied to structure M and object a). This operator $\wedge$ is invariant under isomorphisms.

Some logical operators under this criterion are not usually considered as such. Examples are infinitistic (cardinality) quantifiers, the uncountability quantifier, generalized quantifiers such as 'most', and second-order set-theoretical membership.

Tarski's independent, conceptual justification was based on *generality*. But the transformations that he had in mind were not the most general transformations one can think of: even more general are any functions between structures whatsoever. Then all standard operators would fail the criterion and we would be left with operators that we could not reasonably classify as logical, such as 'is an individual'. So we have to conclude, says Sher, that generality cannot be the mark of logic. Instead, it should be *formality*.

Sher thinks logic and mathematics approach the same topic, the laws of the *structural*, from different perspectives: mathematics investigates these laws, logic applies them in general reasoning. And it is well known that isomorphisms capture *structurality* or *formality*. By characterizing logic as formal in this way, we explain why it is so universally applicable: whatever the topic of discourse, the formal laws are the same.

Now, logicism aimed to use logic as a foundation of mathematics, which lead nowhere [Sher, 2008, p. 318]. But on Sher's view, one can again reduce the two fields, and the two mysteries, to one, but now it is logic that is explained in terms of mathematics (or, alternatively, logic and mathematics are both explained in terms of the formal). Sher dubs this *mathematicism*.

Pros and cons of Sher's criterion of formality

Much of the standard logical validities are indeed valid precisely because of mathematical necessities. This is most clearly the case with arguments involving quantifiers.

However, as I have already mentioned, I doubt whether validity should be explained in terms of mathematicality (structurality) only. I do not think necessary truth-preservation is only induced from mathematical properties unless those are taken to mean something that they don't traditionally signify. For instance, validity arguments can depend on the relations between necessity and possibility, or the properties of time and space, or connections between "material" concepts (e.g., from *red* to *colored*), which are all not traditionally seen as part of mathematics.[11]

Also, the propositional connectives should not be seen as mathematical operators. Evidently, they can be characterized as such, but that's putting the

[11] Of course, if Sher allowed other interpretational models, such as standard Kripke frames that model phenomena such as alethic and temporal modality as a mathematical accessibility relation between (possible) worlds, then modal terms could also be characterized as mathematical operators. But first, Sher does not seem to allow this. Second, it's reasonable to say that such structures would only *model* the phenomena involved; characterizing the modal terms as mathematical operators then cannot lead one to the conclusion that valid arguments due to modal operators are actually valid due to *mathematics* — rather, the mathematical characterization is accurate given the nature of modality and this nature of modality is the reason for validity.

cart before the horse. The propositional connectives are, fundamentally, the means by which to assert, negate and connect propositions, not the mathematical (Boolean) functions. Admittedly, this is much based on intuition. Moreover, this distinction is based on what we *traditionally* call logic and not mathematics; but given that the boundary between these (traditional disciplines) cannot be sharply made, we could of course expand what we mean by *mathematics*. (That the boundary between logical and mathematical operators cannot be sharply made, will become clear in §2.3.2.)

In short, it is not sufficiently motivated why and undesirable that formality interpreted as *(mathematical) structurality* should be the defining criterion of validity. Rather, it makes sense to conceptualize *logical consequence* as one end of a spectrum of validity, based on structural aspects of information (1-formality). A strict boundary between valid logical arguments and valid non-logical arguments is not feasible.

Besides, Sher is not consistent: if mathematicality is what (logical) validity is all about, why are we not able, in her logic, to express as that an object is a circle, or that a property of individuals *is* a set, such that arguments like

- a is a circle; therefore, a is a round plane figure whose boundary consists of points equidistant from a fixed point
- B is a set; therefore, B is an unordered collection of objects

would count as valid? (The *property* of being a circle or a set would be clearly invariant under appropriate transformations of structures that model mathematical objects; so given that one can express these properties in the logic, all models of the premises should be models of the conclusion, hence validity.)

2.3.2 More invariance criteria

Let us take a look at two authors who have suggested other invariance criteria: Feferman et al. [1999, 2010] and Bonnay [2008].

Feferman: invariance under strong homomorphism

McGee [1996] showed that the notions that are invariant under the Tarski-Sher proposal are exactly the operators that are definable in the language $\mathcal{L}_{\infty,\infty}$.[12] This infinitary logic allows sets of arbitrarily size: arbitrary sized conjunctions, quantification over arbitrary sized sets of variables.

This result is unacceptable to Feferman et al. [1999]. He *feels* that first-order logic has a special role in our thought, and formulates three basic criticisms of the Tarski-Sher proposal.

[12] The standard notation here is $L_{\kappa,\lambda}$, where κ and λ are cardinals, and all the logical terms are just the same as in FOL, but where conjunctions of up to κ sentences are allowed, and quantification over up to λ variables. For example, $L_{\omega,\omega}$ is just FOL.

1. The thesis assimilates logic to mathematics, more specifically to set theory.
2. The set-theoretical notions involved in explaining the semantics of $L_{\infty\infty}$ are not robust.
3. No natural explanation is given by it of what constitutes the same logical operations over arbitrary domains. [Feferman et al., 1999, p. 37]

Let's start with the third point. It is no mere theoretical possibility that operators could behave differently on different domains; for example, Lindström's definition of generalized quantifiers leaves open the possibility of such non-uniformly behaving quantifiers.

It's good to know that one could define many reasonable restrictions on these quantifiers. For an oversight of such natural requirements, such as *uniform* behavior on argument tuples of the same cardinalities, or *continuous* semantic behavior of basic operators, [cf. Van Benthem, 1984].[13] However, in the light of the pluralist position that I wish to defend, these reasonable intuitions can come in handy for *pragmatic* reasons ("what kind of operators will we allow for the present inquiry?"), but will not be employed for a principled distinction between logic and non-logic. I disagree with Feferman that this phenomenon of unnatural logical operators is a problem for the Sher-Tarski proposal. Odd mathematical definitions can also describe straightforward mathematical operators — that does not make them less mathematical, only less useful.

The first critique depends only on intuitions, but those *clearly* conflict with the possibility to express something like the Continuum Hypothesis as logically determinate statements, says Feferman.[14] This possibility does exist under the Tarski-Sher thesis. The problem is that, if the Continuum Hypothesis is true, the corresponding logical expression would be true in all models, and would thus be a *logical truth*. But we do not think that such substantial mathematics are *logically* true if they are true, since we take logic to be "independent of "what there is" [Feferman et al., 1999, p. 38]."

The second critique is related to the first one. If logical notions should be explained by means of set theory, it should not matter what the exact extent of the set-theoretical universe is. There are many versions of set-theory, and we should find something that they all have in common to define the logic. For this, Feferman argues that one can use Gödel's concept of absoluteness.

Feferman's own proposal is as follows. Instead of only allowing isomorphic transformations between models, Feferman et al. [1999] considers homomorphisms, which are defined as follows: a structure $\langle A, \beta_1, ..., \beta_n \rangle$ is *homomorphic* to $\langle A', \beta'_1, ..., \beta'_n \rangle$ iff $n = k$ and there is a surjection f from A to A' such that for every $1 \leq i \leq n$, β'_i is the image of β_i under f.

[13] One formulation of continuity: $QAB, QAC, B \subseteq D \subseteq C$ implies QAD (with Q any quantifier and A, B, C arbitrary sets).

[14] The Continuum Hypothesis says that there is no set between the set of the natural numbers and the set of the real numbers [Koreň et al., 2014]. If we let $x > \mathbb{N}$ and $\mathbb{R} \leq x$ be two second-order definable properties, the logical formula $\forall X(X > \mathbb{N} \rightarrow \mathbb{R} \leq X)$ expresses the Continuum Hypothesis.

Now, consider two set-theoretic models: $\mathcal{M}$ with a basic domain of size $\aleph_1$, $\mathcal{N}$ with a basic domain of size $\aleph_0$. Assume for simplicity that on these domains, no relations or properties or anything of a higher type is defined (so that the requirement that, for each $0 \leq i \leq n$, B'_i should be the image of B_i is vacuously satisfied). There can be no isomorphism between these two models — they have a different cardinality, which means that there exists no bijection between them. However, one can define a homomorphic function $\{$ from $\mathcal{M}$ to $\mathcal{N}$ (which means that all $x \in \mathcal{N}$ are $\{(y)$ for one or more $y \in \mathcal{M}$). Evidently, a quantifier like "there exist uncountably many" is not invariant under these transformations. For instance, $\exists!\aleph_1 x(x = x)$ is true on $\mathcal{M}$, but false on $\mathcal{N}$.

With this criterion, Feferman avoids the operators that behave differently on domains of different sizes. The logical constants under this criterion are the propositional connectives, the existential and universal quantifier, and the quantifier 'is well-founded'. Non-logical are: the identity relation, cardinality quantifiers like 'there are exactly 5' and 'there are uncountably many', the monadic quantifier 'most', and the aforementioned operators that behave differently on different sized domains.

Some of the operators that pass the test are still not acceptable to Feferman [Feferman et al., 1999, p. 43]. He sets a type-restriction: in principle, only *monadic* first-order quantifiers are logical, which rules out relational quantifiers like 'most' and polyadic quantifiers like 'is a well-ordering'. Polyadic operators are only allowed when they are constructible, using the rules of the λ-calculus, from monadic operators.[15]

Feferman bases the restriction to monadic type quantifiers on linguistic data, claiming that (only) these play an essential role in human thought. Polyadic quantifiers *are* used in everyday communication, but they are constructed from combinations of monadic operators.

That the (infinite) cardinality quantifiers are excluded is good news too: the higher-infinity quantifiers clearly belong to mathematics, says Feferman; the quantifier 'there exists infinitely many' is "a borderline case to which intuition and experience do not provide a clear-cut answer (...)." It's not a problem if such a borderline case falls by the wayside. Something similar goes for identity, which has a "universal, accepted and stable logic," but does not qualify.

Sher's reply to Feferman

Sher has discussed Feferman's critiques and has attacked his solutions in return [Sher, 2008, p. 327]. First of all, in Sher's view, mathematicians *do* study non-logical notions. *Logic* is about applying these laws in discourse and reasoning. A mathematician might be studying the laws of arithmetic by looking at the system of numbers, as non-logical (non-invariant) objects, while the logician studies these numbers by defining them as (invariant) *places in a structure.*

[15] Feferman does not give an explicit explanation, but the main reason that he allows λ-definable polyadic operators must be that such a polyadic operator is actually the exact same operation as the successive application of the monadic quantifiers from which the polyadic one can be defined — and such successive application should naturally be allowed.

Moreover, Sher just disagrees that intuition should play such a role in science. Her invariance-under-isomorphism criterion

> (...) offers an informative and systematic account of the concept of logical operator, solves serious conceptual problems, explains the relation between logic and truth, elucidates the role of logic in our system of knowledge, critically establishes many of the intuitive attributes of logic, and offers a substantive and methodologically economical account of the relation between logic and mathematics. [Sher, 2008, p. 328]

Sher admits that her logic carries ontological commitments to sets. But that criticism applies to standard first-order logic too, as well as to the logics of the other invariance proposals. And, Sher claims, there is a difference between logical and mathematical commitments. Consider the following sentence.

$$(\exists! 2^{\aleph_0} x)x = x \equiv (\exists! \aleph_1 x)x = x \qquad (CH_L)$$

The *negation* of CH_L would not be logically true if the negation of the CH were added to the background set theory. This illustrates that mathematical commitment *is* just different than the indirect ontological commitments of logic. Aside from a few technical commitments, such as to the existence of at least one object, logic only has commitments through its background theory of formal structure. And, Sher argues, these are not even existential per se: they are commitments to the *formal possibility* of existence.

Meanwhile, Feferman's proposal has problems of its own. Yes, his criterion satisfies a higher *generality*. But these transformations are still not as general as possible. So his criterion cannot be *only* justified in terms of generality. But Feferman cannot resort to *formality*, since his criterion does not respect all structure of the models. Finally, Feferman falls prey to his own criticisms. Van Benthem and Bonnay have pointed out that his first point — assimilation of logic to mathematics and operators that are sensitive to high-content set theory — can be applied to his own proposal directly [Sher, 2008]. Bonnay criticized restricting the logical operators to those that are lambda-definable from monadic operators in two ways: (1) Feferman gave no justification as to why natural language evidence is relevant to the project of characterizing logicality. (2) *If* natural language evidence is relevant, logic should not be restricted in the way that Feferman suggests: in natural language, polyadic quantifiers not λ-definable from monadic ones are often employed, like some that involve branching quantification. Bonnay's example: "Quite a few of the boys in my class and most of the girls in your class have dated each other [Bonnay, 2008]."

Bonnay: invariance under potential isomorphism

Bonnay [2008] thinks, like Feferman, that the Sher-Tarski criterion overgenerates: logic had better be defined as something more basic than the whole of mathematics.

Bonnay also thinks the criterion is not conceptually well-motivated. Do permutations get us the biggest group of transformations possible? Tarski looked at *automorphisms*[16], Sher at *isomorphisms*; but both are still quite demanding. Instead, Bonnay investigates arbitrary *similarity relations* S between structures. The point is to find what similarity relation S does the best job, both from the prior view of the justification and from the posterior view of the resulting logical operators.

A *partial isomorphism* f holds between two structures $\mathcal{M}, \mathcal{M}'$ if there are two substructures A and A' such that f is an isomorphism between A and A'. This relation captures the idea of *local resemblance*. Maybe we can define a relation that requires that the partial isomorphism between two structures can always be extended, in any direction, for an *infinite number of times*. This is captured by *potential isomorphisms*.

Bonnay claims that Iso_p (invariance under potential isomorphisms) is the most suitable candidate for a proper criterion. It is the most general similarity relation that satisfies both closure under definability and absoluteness.

Bonnay distinguishes two parameters that determine how general a given similarity relation is. (1) the amount of extra-structure which has to be preserved. (2) the degree to which structure is to be preserved. By extra-structure is meant, for example, the structural aspects that a bisimulation preserves: the objects can be shuffled around, but the accessibility-relation between objects keeps the same structure. On the other hand, by (2) is meant that, for instance, an isomorphism might not keep extra-structure, but does preserve *all* explicit structure, whereas potential isomorphisms do not.

Now, we cannot want full generality. Invariant operators under all transformations would be those that contain either all or none of the structures of a given signature. Logic cannot abstract away from *all* features of objects. Nor is logic about full formality. However, if *no* mathematical content is allowed, we cannot even have operators that say that a set is nonempty. So we wish that logical notions are deprived only of *problematic* set-theoretic content. This gets us the second requirement on the desired similarity relation: the notion of absoluteness (which Bonnay borrows from Gödel). An explanation of this idea is outside of the scope of this work. Instead, it's enough to note that typically, formulas such as "x is transitive/an ordinal/a limit ordinal" are absolute with respect to ZFC, but "x is a cardinal/of size $\aleph_1$" are not. It turns out that Iso_p is absolute with respect to ZFC, where the isomorphism criterion is not.

'There exactly countably many' and 'there are at least uncountably many' do not come out as logical under Iso_p. 'There are infinitely many', 'there is (at least one)' and 'is well-founded' do. Cardinality quantifiers are not all in the trash: only quantifiers that distinguish among infinite cardinals. All arithmetical truths are logical; not all mathematical truths. Bonnay thinks this is reasonable: the language of arithmetic seems to belong to natural reasoning. The same holds for the difference between finite and infinite. Countability and uncountability are not intuitive in this sense — so it's good they do not qualify.

Further overgeneration with respect to standard logic is justified as well — just consider the following example.

[16] An automorphism is an isomorphism from an object to itself.

> *Most French movies favor introspection.*
>
> *Most French movies are commercial failures.*
>
> *Therefore, there are French movies which favor introspection and which are commercial failures.* [Bonnay, 2008, p. 65]

Arithmetical truths are treated as logical. This is a merit of the proposal: "arithmetical notions have in common with the more elementary logical notions a number of properties that grant them a special place in our conceptual scheme [Bonnay, 2008, p. 65]"

Feferman's critique of Bonnay

Feferman discusses a few proposals that satisfy intuitive requirements and capture a *smaller* logic than Bonnay's logic under Iso_p invariance. For example, Bonnay characterizes logicality by use of a single, global similarity relation; in combination with the requirement that the set of invariant operators are closed under definability, the resulting logic becomes quite big. But one could think that an operator should not be counted as logical just because it could be defined from operators that were already invariant under some global similarity relation; that, instead, an individual operator is only logical if it passes some more refined, local invariance condition. For instance, the following theorem tells us that for FOL, every (definable) operator is Iso_n-invariant for some n:

Theorem 1. *Q is definable in FOL if and only if there exists $n < \omega$ such that Q is Iso_n-invariant.*

Moreover, Bonnay could have used a more conservative notion of absoluteness, to encapsulate even less problematic set-theoretical content [Feferman et al., 2010]. Feferman estimates that the result would be closer to FOL. That we should require from a logical operator that it is absolute relative to a weak set theory, without the axiom of infinity, can be seen as only relying on mathematics that are "needed for a theory of the syntax of any humanly manageable system of logical reasoning" [Feferman et al., 2010, p. 17].

2.4 Remarks on model-theoretic monism

2.4.1 The circularity of the invariance approach

The first problem with the standard monist account is the attempt to demarcate the logical constants. Invariance proposals tried to characterize some operators as logical because they were invariant under the right kind of transformations. But as Van Benthem remarks,

> Permutation invariance is blatantly circular as a criterion for logicality! [Van Benthem, 2002, p. 428]

And Dutilh Novaes [2014] notes, similarly:

> The gist of the [proposals] seems to be: 'what invariance criterion will demarcate the realm of logic as (roughly) coinciding with first-order predicate logic?', but not 'let us take an independently motivated criterion and see where it draws the boundaries of logic.' [p.87]

It is a remarkable phenomenon: these authors have a set of operators in mind and adjust the criterion that is supposed to independently characterize the logical constants as soon as it doesn't suit their presuppositions.

It is important to note that the invariance approach is not without its use [Van Benthem, 2002]. For one, it can be very informative to see what operators are invariant under a natural class of transformations. The different results in the previous section give us good ideas about the nature of the different operators; to what extent the first-order quantifiers are the same as generalized quantifiers like 'most', how logicality of (higher-order) infinity quantifiers is dependent on the background set theory of the underlying structures, in what way propositional connectives differ from the rest. But invariance cannot serve as a philosophical demarcation of the logical constants.

Let's look at the problem using an example: models for syllogistic tasks. These models can be used to show that one always needs a tailor-made definition of transformations to get the right operators as invariant.

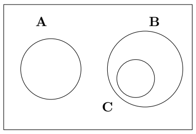
We briefly saw syllogisms already, in §2.2.1. A syllogism consists of two premises and a conclusion, each formed by an application of a binary quantifier ('all', 'some', 'not all' or 'no') to two terms representing sets. At least, this is the modern interpretation: a syllogistic premise like *All A are B* is interpreted to mean set-theoretic inclusion ($A \subseteq B$). [17]

Diagrams for illustrating and supporting syllogistic reasoning have been used for centuries [Mineshima et al., 2012]. We can distinguish three important representations. Traditional Euler diagrams represent logical relations among the terms of a syllogism by topological relations among regions of circles. In these diagrams, every region that contains no other region represents a nonempty set (see 2.1(a)). Venn diagrams remove this existential import from regions (2.1(c)). Instead, shading is used to express that a region is empty. As a result, *All A are B*, since it is logically equivalent to *There is nothing that is A but not B*, can be represented by a diagram like figure 2.1(b). Note that this diagram does not tell us whether the two parts of B are nonempty or empty. This information can be represented by use of the syntactic device $\times$, as shown in 2.1(d). The third kind is Euler diagrams in system EUL. In contrast to the traditional Euler diagrams, these do not make every minimal region nonempty. Because of this indeterminacy, all situations that make *All A are B* true are represented in just one diagram (like figure 2.1(e)). Nonemptyness of a region is again represented by $\times$ (see 2.1(f) for disjunction of nonemptyness).

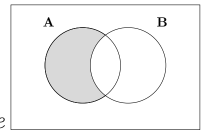
Just consider how one would define invariant operators on the different diagrams for syllogistic reasoning. For each kind of diagram invariance has to be

[17] The previously dominant *Fregean* formalization of syllogisms used predicate logic, such that, for example, *All A are B* became $\forall x(Ax \to Bx)$. However, although the Fregean formalization is useful for *solving* some of the syllogisms, it is an unnatural formalization of the standard natural language task, and therefore unintuitive and arguably undesirable.

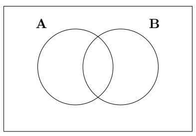
Figure 2.1: Diagrams for syllogisms



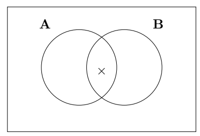
(a) Traditional Euler diagram for *No A are B*, *No A are C*, *Some C are B*, *All C are B*, *Some B are not C*.



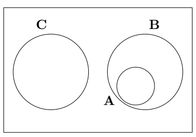
(b) Venn diagram for *All A are B*.



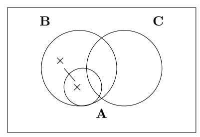
(c) Venn diagram that conveys no information on the relation between A and B.



(d) Venn diagram for *Some A are B*.



(e) EUL system diagram for *All A are B* and *No C are B*.



(f) EUL system diagram for *All A are B* and *Some B are not C*.

defined on transformations that will respect the relevant syntactic properties of the models. The topological relations between the circles have to stay the same for all three types of models; for Venn diagrams, images of shaded minimal regions need to be shaded; the same for images of regions with $\times$ or $\times - \times$, for Venn and EUL system diagrams. So how your invariance criterion must be defined is entirely dependent on the kind of diagrammatic representation of the relational information one chooses (or prefers).

One might say that the authors in the previous section had conceptual justifications for the transformations they allowed. But we observed that, actually, they mostly had strong intuitions about what operators should be included in the logic. The conceptual motivations, such as generality/2-formality (indifference to identities of particular objects) could never be fully met because the resulting logic would be close to trivial. For Sher, the resulting logic of meeting her criterion (structurality) was not trivial but I argued that narrowing down logical consequence to necessary truth-preservation on Tarskian structures was undesirable, insufficiently motivated, *and* inconsistent.

That the independent motivations could not be fulfilled is also pointed out by Dutilh Novaes [2014]. She argues that the resulting logics do depend on a number of substantive characteristics of objects: for example, that they be discrete, perdure, not merge, and not multiply spontaneously. These logics are not as ontologically neutral after all, says Dutilh Novaes; obviously, such ontological neutrality is never possible anyway.

There is another problem with the invariance approach, says Van Benthem:

> Should truly logical notions not be *independent* from particular choices of objects over which they are supposed to work? [Van Benthem, 2002, p. 429]

In natural languages, the same logical operators can be seen to work on different structures. 'Every' as in "every girl" and 'all' as in "all wine" use the same logical operator, but it is hard to imagine how to get this operator out of invariance for transformations of models that contain both kinds of objects. We will discuss the idea that logical operators have a meaning independent of the structures on which they are interpreted later, in chapter 4.

Other proposals to demarcate a set of exclusive logical constants are doomed to fail too. As we saw in Szabó's argument, it's implausible that one could principally distinguish validity based on form, definitions, or facts, except by fiat.

2.4.2 The focus on Tarskian structures

The syllogism example can also be employed as an argument against the idea that we should evaluate (logical) validity on just one type of structures. The different diagrammatic models for syllogistic reasoning have different mathematical properties, but no one of them has clear precedence: some are more expressive, others are more intuitive [cf. Mineshima et al., 2012].

One might object that there is a difference between those diagrammatic models and the Tarskian structures; that the first are tools for solving a syllogistic task, while the last are mathematical models of precisely the relationships that validity should care about. But both the diagrammatic models and the Tarskian models are tools to analyze propositions about abstract (mathematical) relationships; the Tarskian models might be able to represent more, but still not everything. Therefore, it's strange that the model-theoretic monist takes these abstract structures as the hallmark of validity without leaving space for considering other operations and concepts as logical notions. A principled justification for the essential role of Tarskian models is hard to find (except with Sher, who based it on the idea that logic should be about mathematical truth; but then it was strange that some mathematical properties were not at all represented). We will see in the next chapter how other types of models can be appropriate for many areas of reasoning.

One can find criticisms of the focus on Tarskian models in the literature as well. Here is a quote from Bueno and Shalkowski [2013]:

> That 'model' is a technical term when used in the philosophy of logic obscures the fact that models are models. They are not the genuine article; they are not the subject matter. They are the illustrations, the exhibits that illuminate the mind regarding the phenomenon of interest. They do so by making salient poorly understood features of that phenomenon. (...)
>
> Granting the existence of one or more languages, sets, relations, and functions defined over the languages and the sets (...), mathematical models can be used to model interesting features of many different things, logical consequence and the logical constants among them. [p.13-14]

2.5 Summary

In this chapter, we took a look at the model-theoretic definition of validity or logical consequence: an argument is valid iff it preserves truth in all models due to its logical *form*. The contemporary account, which originates with Tarski [2002] and Sher [2008], takes cases to be set-theoretic, Tarskian models.

Dutilh Novaes [2012] explained the historical roots of the modern doctrine of logical form, MacFarlane [2000] identified the philosophical conceptions of formality. With Szabó [2012] and Brandom [1994], I argued that restricting validity to "formal" arguments is hard to defend in a principled manner, and that "analytic" and "material" arguments can also be truly valid. I concluded that arguments that are traditionally seen as analytic or material can also be truly valid. On the other hand, we noted that information necessarily has (logical) structure, and agreed with the idea of formality that considers logical structure constitutive of thought (or semantic information). However, I held that formality should not be employed to demarcate the set of valid arguments.

After this, we saw the efforts by Tarski, Sher, Feferman et al. [2010] and Bonnay [2008] to characterize the logical constants by means of invariance criteria. We examined the debate on which transformations to employ and concluded that these authors more or less *picked* the type of transformations that resulted in the desired logical operators. With Van Benthem [2002], we concluded that invariance criteria could not serve as a foundation of logic. In fact, I pointed out that the Tarskian tradition suffered from another flaw: that it focused exclusively on set-theoretic structures, without strong independent justification. For instance, Sher claimed that logic should only be about what follows due to mathematical structure — but the Tarskian structures cannot model all of mathematics.

Chapter 3

Considerations for Logical Pluralism

In this chapter, we will discuss some facts about logical systems and human reasoning that evoke pluralist intuitions about the nature of validity and logic. First, there is the multitude of formally specified systems for reasoning about varying topics that are developed in academic practice and are called "logics". What these logics have in common is that they are defined by using other *structures* than the standard Tarskian structures from the model-theoretic account. Illustrating this will be the point of §3.1.

We look at Stenning and Van Lambalgen [2012]'s analysis of what subjects are doing — usually not classical logic, generally still *logical reasoning* — in paradigmatic psychological experiments on reasoning in §3.2.

We will discuss the consequences for our account of logic and validity — that it has to be a form of logical pluralism — in §3.3.

3.1 Validity defined on other structures

In the previous chapter, we saw attempts to fix a set of logical constants by means of invariance under suitable transformations on a particular set of structures. Not only were the choices for what transformations were allowed a little arbitrary; the idea that one kind of *model* should serve as the evaluative basis for validity is highly dubious.

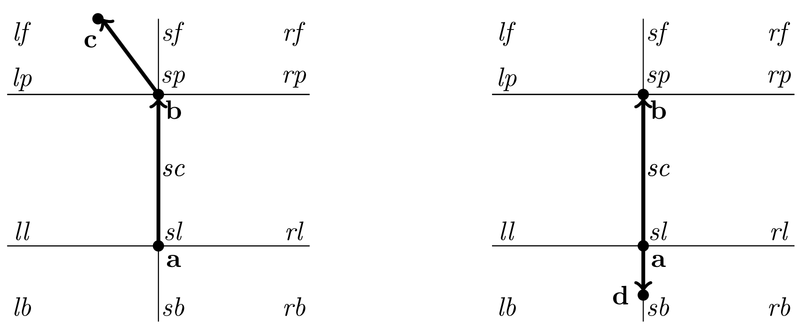
There are many kinds of structures (and to a lesser degree, semantics) that one can interpret (logical) language on, lots of which are thought to capture interesting aspects of the things we wish to reason about. In §2.4, we already considered diagrams for syllogistic reasoning: models that captured exactly the set-theoretic relations that were of interest for that task. Another example of models for specific tasks are those that (i.a.) humans employ for navigation:

> Although a few researchers remain skeptical (...), there is now a broad consensus that mammals (and possibly even some insects) navigate using mental

representations of spatial layout. [Rescorla, 2019]

A formal characterization of one type of such navigational model and calculus for these are discussed in Scivos and Nebel [2001]. The Double Cross calculus is a qualitative spatial calculus, which means it incorporates no measurements on the distance between points, or coordinates of a point in a 2D plane; the natural qualitative spatial dimensions are a relative spatial orientation and a front/back dichotomy. This means the models only represent the relations between three points by specifying an oriented line fragment relative to two points and giving each area that is a possible location for the third point a label. See figure 3.1 for an example. So one can express that point c is in front and to the left of (a, b), or that d is on the straight line through (a, b) in the back of the perpendicular line through a. In §4.2, when it is even more illuminating, we will discuss an example of an argument on this model.

Figure 3.1: The 15 base relations of the Double Cross calculus, with two examples: $lf(a, b : c)$ and $sb(a, b : d)$.



In this section, we will look at several other logical systems: formal languages for particular kinds of structures. Some of these logics are intended to be "all-purpose" logics [Field, 2009]. These were developed as systems for evaluating arguments that are about truth directly. The examples I will mention are intuitionistic logic and relevant logic. In this context, we will examine an exposition of ideas from Husserl and Brouwer by Van Atten [2006]. This will suggest that we view these logics as systems of validity for reasoning about *certain* objects or *regions* of objects in our conceptual scheme. For instance, if we are reasoning about mathematical objects that are not atemporal or omnitemporal, but get constructed *through time*, the definition of validity will not bring about the law of the excluded middle, since the model for these objects will not make every statement either true or false.

Other logics are systems for reasoning with other goals than truth. There are the modal logics, which can be used for questions about truth concerned with alethic or temporal modality, but also for knowledge and belief, or deontic reasoning.[18] And there are nonmonotonic logics for defeasible reasoning and sys-

[18]As regards logics for knowledge, belief, and deontic reasoning, one might argue that they

tems for evaluating arguments on finite structures, which represent databases, bit strings and graphs.

3.1.1 Intuitionism and relevant logic

First, some logics are thought of as fundamentally better at conceptualizing validity for arguments about (mathematical) reality. Intuitionists, for example, think that (in the context of mathematics, usually) truth should be defined as *constructibility* (or: provability). *Relevantists* believe that *truth conditions* should be spelled out in terms of *situations*, not whole worlds: it is not total reality that makes some thought or sentence true, it is only the relevant part thereof.

Intuitionistic logic is a logic for constructivist mathematics [Iemhoff, 2020]. The logic famously invalidates the law of excluded middle. Intuitionism originates with famous mathematicians such as Brouwer and Heyting. Unlike classical mathematicians that study intuitionistic logical systems for their interesting formal structure, or because those systems capture aspects of provability and constructibility in mathematics, ideological intuitionists *always* deny the validity of the law of excluded middle, because they believe it does not hold on all domains — in particular, they believe it does not hold on infinite domains [Williamson, 2014]. The motivation for this is the conviction that mathematical objects are constructed by the human mind: so as long as there is not a proof for P or a construction that transforms every proof for P into a proof for $\perp$ (falsum), P is neither true nor false.

The standard structures on which intuitionists interpret their logical language are *stages*, which can be thought of as steps in a process of construction or verification [Beall and Restall, 2000, p. 62]. There are a few important features about stages: they are potentially incomplete (some claims might be neither verified nor falsified), and they can be followed by a more complete stage, perhaps even an *end-stage* where everything (of interest) has been proven or disproved.

One way to model this idea mathematically was proposed by Beth and Kripke in the 1950s [Van Benthem, 2019, p. 575]. They suggested models over trees of finite or infinite sequences, where a node of a tree is an (incomplete) set-theoretic model like the previous (classical) Tarskian ones, and a formula is true at a node when they are 'verified' at that stage — which also means they will be true at any successors. The standard semantics here are almost the same as for classical models, but not entirely. The crucial example is negation, which works differently: $\neg p$ can only hold at a model $\mathcal{M}$ and stage s if p will not hold at s or any later stage — so it must really have been *falsified.*

are also about truth. However, this is less obvious, in any case for reasoning about moral judgments.

Brouwer's choice sequences

In Van Atten [2003, 2006] we find a more detailed account of why an intuitionist like Brouwer denies the law of the excluded middle. Van Atten investigates the *choice sequences* that Brouwer suggested as the objects by which the mathematical continuum (the "straight line" of the real numbers) can be defined, and attempts to give a philosophical foundation for Brouwer's proposal by means of Husserl's phenomenology.

Husserl is interesting on his own for his ideas on logic, as [Stenning and Van Lambalgen, 2012] point out:

> Husserl's Logische Untersuchungen brings the important innovation that logic must be viewed, not as a normative, but as a theoretical, or as we would now say, mathematical, discipline. [p.12]
>
> (...) Normativity comes in only via a principle of the form "in this particular field of knowledge, truth of such-and-such a form is good, therefore only such-and-such arguments are good." This means that logical laws are unassailable in the sense that they are mathematical consequences of the structure of the domain studied, but by the same token these laws are relative to that domain. [p.14]

If we connect this with Husserl's standpoint that different ontological regions of the world each have their appropriate logic, we obtain a view on logic that has many similarities with the view we are developing in this work.[19]

In particular, intra-temporal objects like choice sequences would form a region for which the law of the excluded middle is not fit [Van Atten, 2006, p.17]. A choice sequence is a sequence of (natural) numbers, started by a subject at a particular moment in time, obeying restrictions such as laws for generating the numbers *or* being a lawless sequence generated by free choice, which is always unfinished, or potentially infinite. For instance, $12, 3, 81, 12, 221, ...$ and $5, 10, 1003, 6, ...$ In other words, they are made up of *finite* initial segments [Van Atten, 2006, p.1]. Even though many mathematicians refuse Brouwer's choice sequences as mathematical objects, there are good arguments for accepting them. Least of all, one should be inclined to accept them as *objects* — for instance, when one accepts infinite objects (that can be described in a finite manner) and objects that grow in time like *melodies*[20], it would be odd to deny the existence of choice sequences [cf. Van Atten, 2006, p.26, p.93]. And, as remarked, Brouwer thinks the intuitive continuum can be defined in terms of choice sequences.[21]

[19] In fact, Husserl's view on logic has, mostly via other authors, obviously influenced the position defended in this thesis.

[20] "An ongoing melody is experienced as an identity even though it may not have been completed yet [Van Atten, 2006, p.93]."

[21] "Brouwer defines a point of the continuum (or real number) P as a choice sequence of nested rational intervals. (...) [W]e use choice sequences to analyze the continuum, but how we define choice sequences in turn depends on a prior understanding of the structure of the intuitive continuum. (...) The problem at hand is how choice sequences are constituted, not how the intuitive continuum is. The intuitive continuum itself is constituted in (...) our everyday world of pre-theoretical experience [p.86]. (...) [T]he homogeneity of the continuum, it is a whole of which the parts are fused with it (...). The definition of choice sequences is motivated by this inexhaustibility and non-discreteness of the (intuitive) continuum. (...)

It's good to note that according to Husserl, the mathematical universe is static: its objects are finished. However, Van Atten convincingly argues that one can reconstruct Brouwer's arguments for the existence of choice sequences from Husserl's philosophy. We will not go into his argument. Let me just point out that the choice sequences form a good example for the argument that one's logic will vary with the objects one is concerned with — more precisely, I will argue that logic and validity vary with the *model* or *structure* one employs for representing (regions of) reality.

Relevance logic

Yet another definition of validity comes from *relevance logic* [Beall and Restall, 2000, Read, 2006]. Relevance logicians believe that B should not follow from $A \wedge \neg A$, since the truth of the latter does not in any way seem relevant to the truth of the former. The models with which they define validity (in the Tarskian sense of all models of the premises are models of the conclusion), therefore, are incomplete *situations*. These are, informally, parts of the world that do not make everything either true or false. An example of a situation could be one where I am petting my cat, where my cat is purring, and where I'm making tea; where it is false that I am sleeping, and false that I am drinking tea; but undetermined whether I am a policeman or not, and whether I have any parrots.

Semantics for relevance logics are different from classical semantics, in particular, again, negation. If A is not true at a situation s, it doesn't follow that $\neg A$ is true there. Instead, a relevantist clause for negation could be:

> ¬A is true in s if and only if A is not true in s' for any s' where sCs' [Beall and Restall, 2000, p. 52].

C here is a compatibility relation; informally, this clause says that $\neg A$ is only true if A is not true in all situations that are compatible with s. For example, $\neg A$ where A means "it is raining in my living room" is true in the situation s where I am sitting in a sun-kissed living room since no situation compatible with this one could make A true.

Note that from these two short expositions, intuitionistic logic and relevant logic might seem a lot alike, but the differences in underlying structures actually make for very diverging logics.

3.1.2 More logics: modal operators and nonmonotonic systems

Modal operators

One important set of operators that are seen as "logical" in scientific (and philosophical) practice are modal operators. Examples are 'necessarily' ,'possibly', 'it has always been the case that', 'I know that', 'I believe that', 'it is morally

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The possibility of non-lawlike sequences represents the inexhaustibility of the continuum. [T]he identification of points with unfinished sequences of nested intervals expresses the non-discreteness of the continuum (...) [Van Atten, 2006, p.85-87]."

obligatory that'. These operators are used in logics that analyze modalities of metaphysical, temporal, and deontic nature and phenomena of knowledge and belief. They are *intensional* operators, as opposed to *extensional*: intuitively, the output of the application of a modal operator to a proposition (letter) depends not on the truth value, but the *meaning* of the proposition. For example, consider 'ought': perhaps I ought to be nice, but it's not true that I ought to be talented (only hard-working).

It is common practice to analyze modal logics by means of non-modal classical logic. This is traditionally done by a possible world semantics. A model for modal formulas and sentences is then a set W of worlds w, that are each standard first-order models: sets d of individuals. A function D assigns a domain d to every world w. Global relations Z and functions F specify an extension for every world w. @ can be seen as a designated member of W (the 'actual world'). An accessibility relation R specifies how the worlds are modally connected: if a world w 'sees' another world x, that world x is 'possible' as viewed from w. Besides the standard semantics for the propositional connectives and $\forall$ and $\exists$, the satisfaction rule for $\Diamond$ is: $\Diamond P$ is true at a world w iff w sees some world v where P is true. And for $\Box$, the rule is: $\Box P$ is true at w iff P is true at all worlds v that w sees.

For temporal systems, the underlying structures are slightly different: they are, depending on the formalization of one's ideas about the structural aspects of time, one or more (time-)lines or perhaps a branching structure, with more nodes in the future (usually, to the right) than in the past (usually, the left). A logic for knowledge, so-called *epistemic logics*, one might use a set of worlds to model someone's doxastic state: everything that someone knows must be true in all worlds in that set; what someone believes true in some of those worlds; when a person *learns* a new fact, the worlds where that fact is not true are removed. Deontic logics use similar mathematical structures, where an accessible world can for example mean something like "morally permitted".

Modal systems provide a striking case of interesting and — for many — intuitively *logical* operators that have been neglected in the main literature debating the boundaries of logicality.

Of course, one can (easily) extend classical (first-order) logic to include some modal operators (resulting in modal predicate logic). The point is that the standard attempts to characterize logic do typically not take the modal operators into account. But the modal operators should definitely be considered as logical, as pointed out in §2.4, and as argued by Dutilh Novaes [2014]:

> The failure of the permutation invariance criterion to count these modal operators as logical should make us reconsider the whole idea of permutation invariance as a criterion for logicality. After all, modal logics and their descendants are currently among the most widely studies logical systems; they are highly influential both for the interface of logic with computer science and for philosophical discussions on modalities and related topics. [Dutilh Novaes, 2014, p.95]

Even if we observed that invariance criteria cannot fundamentally demarcate the logical constants, it is informative to briefly analyze why the modal operators are not ruled in as logical under the permutation invariance criterion.

The standard models for interpreting first-order modal logics are Kripke frames, where every world is a Tarskian model on its own. One natural way that a permutation invariance criterion for these models might work, is by permuting every *world* w into another world that is *isomorphic* to w: otherwise none of the standard (first-order) operators will come out as invariant anyway. Consider such a first-order Kripke model $\mathcal{M}$. Suppose for simplicity that there are only two worlds, w and v, with each one object. Suppose furthermore that w is blind: that is, no world is accessible from w. Suppose that v is reflexive, and that there is only one predicate in the language, P, which has an empty extension in both w and v. It follows that the $\Box$-operator is not logical, since it is not invariant if we permute w into v or the other way around: $\Box\exists xPx$ is true at w but not at v.[22]

It is not only counter-intuitive that the modal operators are not seen as logical operators; it is also obvious that the modal operators are logical iff the Kripke models have empty, identity, or universal accessibility relations. It would be unlikely that we think that these accessibility relations capture the aspects of necessity and possibility in reality, so if that is the goal, Kripke models like that hardly seem relevant. But the point here is:

> what independent motivations would justify that the S4 modal operators do not count as logical, whereas their counterparts interpreted on universal frames do. What is the fundamental difference between these two cases besides the fact that they are interpreted on different structures? [Dutilh Novaes, 2014, p. 94]

By the way, we saw that invariance criteria were adjusted to get the desired operators, and we can also do that for modal operators. As Dutilh Novaes notes, this can be done by the notion of bisimulation as used in computer science.[23]

Nonmonotonic logics

Some reasoning is less concerned with *truth* and more with other goals, such as communicating or action. To model the sort of reasoning that human beings perform when they try to interpret each other's words or when they plan action in the world and to construct logical systems by use of which *computer programs* can solve similar tasks, so-called nonmonotonic logics were developed. The crucial feature of these logics is that they are defeasible: reasoners reserve the right to retract their conclusions in the light of new information. With these logics, the context often plays a role in how information should be linked together; so

---

[22] We are assuming standard semantics for $\Box$ here: $\Box\phi$ is true at w iff ϕ is true at all worlds v that w sees. w in our example sees no worlds, so $\Box\phi$ is true for any formula ϕ.

[23] A *bisimulation* between two Kripke τ-frames $M = \langle W\{R_\alpha\}_{\alpha\in\tau}, V\rangle$ and $M' = \langle W'\{R'_\alpha\}_{\alpha\in\tau}, V'\rangle$ is a nonempty relation $\rho \subseteq W \times W'$ satisfying the following conditions for any $w\rho w'$:

- **Atom equivalence:** w and w' satisfy the same atomic propositions;
- **Forth:** For any $\alpha \in \tau$, if $wR_\alpha u$ for some $u \in W$, then there is some $u' \in W'$ such that $w'R'u'$ and $u\rho u'$.
- **Back:** For any $\alpha \in \tau$, if $w'R'u'$ for some $u\prime \in W\prime$, then there is some $u \in W$ such that wRu and $u\rho u\prime$. [cf Blackburn et al., 2006, p.257]

that even though there might be a valid argument from a set of premises Γ to a conclusion ϕ, this conclusion might not follow anymore when Γ is extended with an extra premise γ — for instance, because this γ is incorporated as an exception to a conditional premise in Γ. To get an idea, I will discuss, in an informal and intuitive manner, one nonmonotonic system: that of closed-world reasoning.

In Human Reasoning and Cognitive Science, Stenning and Van Lambalgen [2012] show that much of *human* reasoning (in laboratory experiments and real life) can be seen and modeled as *logical* reasoning, even though often it does not conform to the standards of classical logic. We will discuss some of their insights more extensively in the next section. For now, we consider their exposition of closed-world reasoning [Stenning and Van Lambalgen, 2012, p. 33].

Consider again the example of Eve who wants to go to Berlin by train. Eve will plausibly use a conditional like "if I take the 10:00 AM train from Amsterdam, I will get to Berlin at 8 PM". She will assume that she cannot get to Berlin by trains that are not on the schedule. She will also *not* consider all possible exceptions to the conditional, like natural disasters or technical defects. However, if Eve knows that trains to Berlin are often canceled, she will perhaps consider this an exception that makes her suppress the conclusion — and try a bus instead.

Now, closed-world reasoning (using S&vL's preferred formalization) is a logical system that does a good job at modeling just this kind of defeasible reasoning. Syntactically, it allows for clauses of the form $p_1 \wedge ... \wedge p_n \to q$. Iteration of implication is not allowed, nor occurrences of negation in antecedent and consequent. Semantically, $\vee, \wedge$ get the normal truth rules. $\to$, however, has a special closed-world interpretation:

1. If all $p_1, ..., p_n$ are true, then so is q.

2. If one of $p_1, ..., p_n$ is false and no other implication has q as the consequent, then q is false.

3. If there are multiple formulas with q as the consequent, all of which have a false conjunct in the antecedent (one of $p_1, ..., p_n$), then q is false.

The most important feature of this logic is that validity is *nonmonotonic*: q might follow from $\Gamma = p_1 \wedge ... \wedge p_n$, but not from $\Gamma \cup \{u \to q\}$, if u is false. This is different from the classical account of logical consequence: there, $\Gamma \vDash q$ meant that all models that made all premises true make the conclusion true, which implies, in particular, also $\Gamma \cup \{\theta\} \vDash q$ for any formula θ. Intuitively, this property models that adding extra information can destroy earlier inferences.

Conditionals in closed-world reasoning are given an *abnormality/exception parameter*: $p_1 \wedge ... \wedge p_n \to q$ becomes $p_1 \wedge ... \wedge p_n \wedge \neg ab \to q$. Known exceptions w_i are incorporated as $w_i \to ab$. Also included is $\bot \to ab$, since it's trivially true. The information about ab can be collected by $ab \leftrightarrow w_1 \vee ... \vee w_k \vee \bot$. (This encodes the closed-world assumptions that are sensible for a task: take (only) *known* preconditions into account, assume that an event is caused by one of its *known* causes, et cetera.) This implies that if there is no information about ab, the bi-implication becomes $ab \leftrightarrow \perp$, which reduces to $\neg ab$.

Areas where closed-world reason fits perfectly are causal and counterfactual reasoning. Mostly, though, it is appropriate *planning*. Humans are distinctly better at offline planning than other species. What is involved in planning is mentally constructing a model, which represents relevant parts of the world, and computing the effects of action in these models over time. For this, various closed-world assumptions are necessary. Systems like classical logic just could not help us with these tasks. Considering all models of the available information is not feasible — not for humans, nor computer programs. For some tasks, like discourse comprehension or planning, searching through all models is computationally *intractable* (in AI, this is known as the frame problem).

... and more

There are many more logical systems that induce a non-classical concept of validity or are interpreted on completely incomparable (mathematical) structures, and that were essentially ignored by the predominant philosophical literature on logicality. I will give three examples.

Logics for topological structures are tailored for expressing interesting aspects of topological structures, and can thus only be interpreted on such structures. Topology is, informally characterized, originally the mathematical study of properties of objects that are invariant under deformations of space. According to Vickers [1996], one can call this also *rubber sheet geometry*, in that spatial objects are analyzed whereby we do not mind stretching our space. A *doughnut*, then, is not different from a *cup*. A second step in the domain was the abstraction to studying *open* and *closed* sets: the latter include their boundary points, the former do not. Here the underlying structures are an abstract set of points. In computer science, topology is used to explain *approximate* states of information. The idea here is that many concepts (properties) are semi-decidable: one can sometimes affirm or refute that an object satisfies the property, but as often one can do neither. As such, these structures capture interesting aspects about observations, knowledge, and information, and give rise to natural logical operators and semantics, which are quite distant from, to name a thing, those of first-order logic.

Probability logics are appropriate for first-order structures endowed with a probability measure on the universe [Keisler et al., 1985]. Usually, it is assumed that every definable set in a structure is measurable. Such a logic, for example, knows not the usual quantifiers $\forall, \exists$ but revolves around the probability quantifier $(Px \geq r)$, such that $(Px \geq r)\phi(x)$ is true if the set $\{x|\phi(x)\}$ has probability at least r. Furthermore, a completely natural syntax and semantics can be defined, as well as a proof theory. Also, there exists soundness and completeness theorems for (some of) these logics, and many other model-theoretic results (such as compactness and interpolation theorems).

Logics in finite model theory originated in computer science, where the objects of interest are finite mathematical structures, such as graphs, strings, and databases Libkin [2013]. This is in contrast with classical logics, which are mostly concerned with infinite objects and structures, for example, the natural or real numbers. The areas in computer science where finite model theory has a role to play are databases, complexity theory, and formal languages. These objects require the expressiveness of a finite logic.

From the early 1970's, database systems were structured as relational models. Databases could then be queried by a logic-based declarative language, of which the most standard, relational calculus, has exactly the expressive power of first-order predicate logic. However, the expressiveness of first-order logic is not enough to ask certain relevant questions to the database. For example, if the database stores a partial order on individuals (e.g. it stores the '"reports-to" relation between employees and their direct superior), a first-order logical formula cannot query the database on the transitive closure of this relation. Similarly, no first-order formula can express that there are any cycles in the "reports-to"-relation; but presumably, a company would want to prevent these. There are more of these inexpressibility proofs for FOL; but, on the other hand, second-order logic is way too expressive. Some logics in this area, therefore, are designed to be just expressive enough to be able to formulate interesting (read: *tractable*) queries but not more. For example, Immerman and Vardi have proved that, in the presence of linear order, the least-fixed-point extension of first-order logic captures polynomial time.

Another important limitation of classical logic (FOL): many famous results collapse when only finite structures are allowed; among which Gödel's completeness theorem, Craig's interpolation theorem, Bern's definability theorem, and the substructure preservation theorem [Gurevich, 1985, Libkin, 2013]. But it turns out that when the logical language (FOL) allows for a relation $<$ that defines a linear order on the universe, the expressive power grows substantially. For this, it does not even matter how $<$ is exactly interpreted, as long as $<$ is a linear ordering on the basic universe.

Other interesting logical systems formalize the **dynamic** character of some interesting phenomena; such as, again, objects that are studied in computer science, but also more theoretical models of information updating and exchange in (human) communicative interaction. For examples of these, [cf. i.e. Gurevich, 1985, Van Benthem, 2011].

3.2 Logic in human reasoning

Besides the fact that there are many logics other than classical (first-order) logic, that are defined on other structures than Tarskian models, a crucial observation for the position defended in this thesis comes from Stenning and Van Lambalgen [2012, 2019] (S&vL). Essentially, they argue that ordinary people often do not conform to the standards of classical logic, but are not reasoning at random either. We need other logical systems, like closed-world reasoning, to model their thinking.

S&vL take a number of paradigmatic results from the field of experimental psychology, as well as other examples and insights from evolutionary biology, to argue that analyzing tasks of reasoning that human beings concern themselves with, both in everyday life and in classrooms and academia, should not only be done against the standard of classical logic. Different logics are appropriate for different kinds of tasks and discourse, for which kinds S&vL use the term *domain*: a piece of text that can be seen as originating in communicative interaction. A domain can be more or less characterized by the class of mathematical structures on which the discourse is to be interpreted. A discourse admits of several domains; what the right one should be is, in a given context, a pragmatic matter.

The establishment of the domain is done in the stage of *parameter setting*. Human reasoning can be seen as consisting of trying to understand a task or piece of discourse (reasoning *to* an interpretation) and then solving that task or drawing one's conclusions from that piece of discourse (reasoning *from* an interpretation).

S&vL argue that one cannot read "the logical form" of a sentence off of the surface grammatical form: they criticize the idea that there is a literal meaning of the premises. Discourse interpretation is "not at all exhausted by composing the meanings of the lexical items in the way dictated by the syntax of the sentences [Stenning and Van Lambalgen, 2012, p. 21]." For example, consider the following text that consists of two sentences "Max fell. John pushed him." One natural way of processing this mini-discourse would lead to the interpretation that first, John pushed Max, and after and because of this, Max fell. Here the expressed events are related in a temporal and causal language. Moreover, if an utterance like "... into the hole expressly dug for the purpose' is added, the interpretation could change drastically (Max falls; after this, John pushes him into a hole). The situation model for the discourse is constructed incrementally [Stenning and Van Lambalgen, 2019].

The underdeterminacy of sentences in natural language can also be found in the context of syllogistic tasks. It is known that untrained subjects often encounter problems in getting to the intended interpretation of the (sentences that explain the) syllogistic task, as well as, once they get there, in making the right inferences *from* that intended interpretation [Stenning and Van Lambalgen, 2012, Sato and Mineshima, 2015, cf]. Perhaps some of these problems are due to *pragmatic* inferences rather than *semantic* ambiguity. One might think that *Some A are B* cannot be reasonably taken to have as its semantic content *Some A are not B*, but subjects do often infer this. The likely reason for this is that people assume that if the speaker knew that *All A are B*, they would have said so.[24]

But subjects also exhibit interpretations that point to more straightforward indeterminacies of the language. For instance, there has been some historical discontinuity in the intended meaning of the syllogism; the modern set-theoretic interpretation is not in accord with the original, Aristotelian interpretation,

[24]This was pointed out by Grice [1975]. Based on cooperative or adversarial contexts of communication, uttered sentences do not only express their normal semantic content but also convey *pragmatic implicatures*.

where existential import is assumed, such that *All A are B* implies *Some A are B* [Sato and Mineshima, 2015, p. 412]. This Aristotelian interpretation is intended in most psychological literature [Stenning and Van Lambalgen, 2012, p.300].

S&vL argue that it is evident that people have an advanced capacity for selecting the appropriate logics for the domains to which *their* constructed interpretations belong to. The (psychologist) reader might be inclined to think that formalisms are maybe useful and successful at solving these tasks, but that subjects do not *know* them, that they are just the theorist's tools. The critical point here is that subjects might not *know* these logics, like they do not know the grammar of English, but that they still use these logical systems, that they *follow the same rules.* Of course, the exact formal systems are not present in the mind, but S&vL do decidedly think that something computationally equivalently *is* what is going on inside of there [Stenning and Van Lambalgen, 2012, p. 41].

Of course, human reasoners make mistakes, and their reasoning does not always live up to the standards of the formal systems that can describe their reasoning. They get tasks wrong or yo-yo between interpretations or simply make mistakes. Meanwhile, the formal systems are designed to model reasoning in ideal circumstances with clear facts and knowledge and determinate goals. One can compare these systems with a visual tutorial of how to run. Most people do not have a flawless technique nor the optimal body structure, but they do *run*, and by generally the same universal principles as that virtual avatar.

So it is clear that subjects are *thinking* [Stenning and Van Lambalgen, 2019, p. 101]; and that they are quite consistent [p. 212]. S&vL conclude this from replications of famous experiments, like the Wason selection task and the suppression task. They conducted Socratic dialogues with subjects to get an idea of what they are doing — I will not quote any excerpts but happily refer to Stenning and Van Lambalgen [2012, p. p.59 and further] for the rich and insightful data.

Let's take a short look at how S&vL's analyze the suppression task (original experiment by Byrne). Subjects are presented with two problems [Stenning and Van Lambalgen, 2019]. One is a conditional reasoning problem consisting of two premises and a conclusion. The subjects are asked if they agree with the conclusion. In the second task, a conditional premise is added to the problem. The two problems are investigated in a number of patterns: modus ponens (classically valid), denial of the antecedent (classically invalid), affirmation of the consequent (classically invalid), and modus tollens (classically valid).

The task for modus ponens is as follows. First, the subject is presented with the premises.

> *If she has an essay to write, she will study late in the library.*
> *She has an essay to write.*

Then, the subject is asked to consider whether the conclusion

> *She has an essay to write.*

is true. In the second problem, a premise is added.

If the library is open, she will study late in the library.

A majority will support the conclusion in the first problem, but that percentage drops significantly when the second conditional is added (96.1% to 51.1%) [Stenning and Van Lambalgen, 2019]. We call this behavioral pattern *suppression* of the conclusion. If this task is formalized in classical logic, suppression is an illogical action: if $A \vDash C$ then also $A, B \vDash C$ (monotonicity). On a discourse view of reasoning, the three premises are first interpreted coherently; only after this, the inference to the conclusion is considered. The coherence in subjects' interpretations can be seen in terms of degrees and conjunctive linking has lowest coherence. One plausible and more coherent interpretation might be:

If she has an essay to write ***and*** *the library is open, she will study late in the library.*

She has an essay to write.

Therefore, one ***cannot*** *conclude that she will study late in the library.*

S&vL show how the formal system of *logic programming* does an excellent job of modeling exactly the kind of step-by-step closed-world reasoning that goes into the discourse interpretation and conclusion derivation in this task [Stenning and Van Lambalgen, 2012, p. 185-195].

It is *logic* that should be used for modeling this human reasoning. Other, non-logical formal systems *presuppose* a logical form of the information. For example, probability theory has been proposed to analyze vague terms like "usually". "Usually(p)" might then be explained to mean that p has a probability greater than 60%. But probability theory is explained in terms of classical logic: $P(\phi) = 0$ if ϕ is a contradiction; $P(\phi) = P(\psi)$ if ϕ is logically equivalent to ψ; if ϕ logically implies $\neg\psi$, then $P(\phi \vee \psi) = P(\phi) + P(\psi)$. It is actually considerably difficult to define probability on non-classical logics [Stenning and Van Lambalgen, 2012, p. 31]. Additionally, it is unlikely that something equivalent to a probability system is going on in the human mind — it is implausible that people actually assign precise numbers to possibilities.

So to understand what people are doing in reasoning situations, we have to first understand what *logical form* they assigned to the data at hand: the specification of what is to be computed, the formulation of the input and output of the task. The essence of this constitutive normative aspect to logic is that it studies and describes the kinds of structure that have to be given to a body of data before information can be extracted from it at all [Stenning and Van Lambalgen, 2012, p. 348]:

> In Kantian terms, we may think of the activity of imposing logical form and integrating the premises in a single representation as synthesis; this synthesis is a priori since the logical form imposed is not determined by experience, but a constraint contributed by cognition. One needs logical form in order to be able to extract information, but it is as little given in the data as an edge is given in the retinal array. [Stenning and Van Lambalgen, 2012, p. 351]

3.3 Three reasons for pluralism

The observations in the previous sections give us three reasons for a pluralist conception of logic.

First of all, even if the goal is just *truth in the world*, to think about different objects or ontological regions we need different underlying structures: for instance, for reasoning about mathematical objects that are omnitemporal versus objects that are intratemporal. A second example were modal logics for studying alethic and temporal modality: the structures there would encode the intuitions or convictions we have about the nature of metaphysical modality and time. And for other arguments about the world relevance logics employ *situations*: partial structures that can be embedded in the (physical) world. This is how we reason: when the topic is the economical condition of the country, we abstract away from individual grocers and foreign criminal law; when the goal is navigation, we consider a spatial layout that captures all the relevant features of the environment we wish to traverse.

Second, we care not only about truth, but also action, plans, evidence, et cetera. We saw that there are many logical systems for tasks with those goals, like nonmonotonic systems for defeasible inference and exceptions to conditional rules, logics from finite model theory for reasoning with databases, graphs, and strings, probabilistic or topological logics, or logics for deontic (ethical) reasoning. These systems make use of models that are, for instance, closed-worlds (either set-theoretic or more resembling the physical world), Kripke frames for whatever modal operators, finite (Tarskian) frames (perhaps including a linear order on the basic elements) and topological or probabilistic structures.

We might wonder what valid arguments in different logics have in common. So this is a good moment to make one important intuition explicit: that validity is about what information follows from other information, given that the information is structured (and: has a precise meaning) in some way and is interpreted on a particular model.

This intuition leaves room for a third sense of pluralism. When people get a task or (a part of) the world wrong, they are either making no linguistic sense (they are interpreting or using words in a way that cannot reasonably be seen as the meaning of those words) or they are constructing an incorrect model. The person interpreting or using 'or' as a conjunction is not speaking English, but the subject in the suppression task interpreting the 'if ... then' as a default conditional might make linguistic sense but got the intended task wrong. That same subject constructing a closed world is not picking the "correct" model. These were the observations we made in §3.2.

When someone understands or uses the language in an admissible way, we could investigate what model they are constructing, to evaluate whether they are reasoning validly *given their choice of model*. Given the view of logic (the science) as concerned with structural aspects of information (comprehension) and its interaction with valid arguments, it makes sense that logic is to be the science that *models* how people reason (among other things, the *logical form* that they assign to the texts and tasks that they are confronted with), even if it's incorrect for the "objective" problem.

This pluralist view on logic also agrees with the reality that we are often in doubt as to how the world is, what model is right for the task. Both sides, then, could be making valid arguments, even if only one of them would be correct. Equivalently, this view makes it possible to judge that people are reasoning *invalidly*, are being inconsistent, given their own assumptions.

If validity is always defined as necessary truth-preservation on the model of interest, why does it make sense to speak about *logical pluralism*? Models just validate different argument patterns because of their different constitutions. We get logical pluralism because there is more than one formal *system* that captures *patterns* of validity that are due to logical structure. We will discuss this more extensively in chapter 4.

Finally, this pluralist view is in line with academic practice. This is something that Dutilh Novaes [2014] supports: the philosophy of logic needs to stay close to the practices of logicians. Similar points are made by Van Benthem [2002, 2019]: logic, the science, should be concerned with studying information, knowledge, belief, action, agency, and other key topics in philosophy or computer science.

3.4 Summary

In this chapter, we saw several considerations for adopting a pluralist view on logic. After Van Atten [2006]'s exposition of Brouwer's choice sequences (which require an intuitionistic logic) and Husserl's views on logic, we concluded that we need different logics to reason about different ontological regions of the (mathematical) world. If phenomena like time, knowledge, belief, or morality are of interest, Kripke frames are the go-to structures to define the logic. Supporters of relevance logic believe the right underlying structures for (everyday) arguments are *situations*: partial structures that can be embedded in the (physical) world. If the problem is about navigation, we need an abstract spatial layout.

Secondly, we saw logical systems that are not designed for reasoning about truth, but about action, plans, evidence. Nonmonotonic systems, for example, make use of models that are closed-worlds ("don't consider unknown preconditions"). Logics for computer science are defined on finite (Tarskian) models (since graphs, databases, strings are finite).

A third reason for logical pluralism came from Stenning and Van Lambalgen [2012]. Essentially, they argued that ordinary subjects (in paradigmatic psychological experiments like Wason's selection task) do not conform to the standards of classical logic, but are in general not reasoning at random either. We need other logical systems, like closed-world reasoning, to model their thinking.

To be more precise, S&vL distinguished two stages of problem-solving: reasoning to and reasoning from an interpretation. Giving attention to the first stage, of assigning a logical form to the task, does justice to the (experienced) indeterminacy of natural language.

How can we endorse logical pluralism but still think valid arguments are valid in the same way? Here I made the following intuition explicit: validity is about what information follows from other information, given that the semantics of the information is sufficiently specified and the information is interpreted on a particular model. Finally, I argued that this pluralist view on logic is more in line with academic practice.

Chapter 4

A Story on Validity and Logic

In this chapter, we will develop an account of logical pluralism, based on one idea of validity, as necessary truth-preservation on the model of interest.

In §4.1, we discuss the model-theoretic pluralism that was proposed by Beall and Restall [2000]. Their idea has received a lot of criticism, most of it directed at its failure to make logic normative since Beall & Restall claim that multiple logics are equally valid, whatever the problem or context. Another point of criticism, which we take from Bueno and Shalkowski [2009], argues against the explication of necessary truth-preservation as truth-preservation on all models. Finally, we briefly discuss whether logical operators have a meaning invariant of the structures on which they are interpreted.

In §4.2, I will try to develop a story on the nature of validity and logic that does justice to the intuitions and facts and problems we have identified in the rest of this thesis. Here we will discuss the role of models, as well as the idea of necessary truth-preservation of propositions interpreted on a model. I will attempt to describe what logics are, and why and how people can still reason incorrectly.

4.1 A pluralistic model-theoretic proposal

Beall and Restall [2000] (B&R) opt for a generalization of Tarski's model-theoretic definition of logical consequence and attempt to accommodate logical pluralism. Their proposal might be thought to account for the intuitions that were developed in the last chapter: that validity should be defined as necessary truth-preservation *given* a model.

We will see, however, that there are two problems with their proposal. First, it does not do justice to the normative role that validity and logic are supposed to fulfill. Second, we will discuss a problem with the Tarskian definition in general, which has to do with explicating *necessity* as *quantification over all* *models*. The notion of necessity will be of importance in §4.2. These two flaws in Beall & Restall's proposal will give us a better idea of what an account of logic and validity should look like.

4.1.1 Beall & Restall

The logical pluralism that B&R defend maintains that there are multiple genuine consequence relations, that hold at the same time, in the same contexts, all at once. As [Caret, 2017, p. 741] says, their pluralism is not attributable to flexibility in the demarcation of logical terms: "The logical pluralist envisions a far more radical stance toward distinct logics, viz. that even when our choice of terms is kept fixed, there are several equally good accounts of logical consequence over the same argument form."

Their account purports to fulfill the standard requirements they identify for a definition of logical consequence. Thus, it must satisfy that valid arguments are truth-preserving, and that they are necessarily truth-preserving. They also think that logical consequence is *normative*, in the sense that "if an argument is valid, then you somehow go *wrong* if you accept the premises but reject the conclusion [Beall and Restall, 2000, p. 16]." Logic is also formal, B&R say. But they do not pick any of John MacFarlane's three senses of formality as the right criterion; instead, they wish to show that their accounts of validity will all be formal to some degree.

B&R's proposal essentially consists of the following thesis, plus the stipulation that different interpretations of *case*$_x$ and *truth in a case* yield different but genuine accounts of validity.

> [Generalized Tarski Thesis (GTT):] An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion. [Beall and Restall, 2000, p. 29]

According to B&R, logical pluralism is the claim that at least two different instances of GTT provide admissible precisifications of logical consequence. There is no *correct* account; only accounts with different *uses*.

Admissible ways to specify cases$_x$ are, among others, possible worlds, Tarskian models, situations, and stages (of inquiry).

Rivalry between logics enters at the level of application [Beall and Restall, 2000, p. 44]. For example, for classical mathematics, classical (first-order) logic will be (most) useful. Relevance logics can help analyze fictional discourse. Intuitionistic logic is useful for constructivist mathematics. But these different logics do not give *rival* answers on whether an argument is valid. They only give different answers, say B&R, and that's all.

It's crucial to note that B&R deny they are relativists about truth in a case:

> If α is true in [case] s, and if s is a member of $K1$ [the class of complete cases], then, by the $K1$-validity of the inference from α to β, it follows that β is true in s. That is not at issue. The pluralism in our position comes from the plurality of relations of logical consequence, not any plurality about what is true in a case. [Beall and Restall, 2000, p.395]

In other words, different logics are about capturing necessary preservation of truth (in the case of interest) simpliciter. We will see next how this point breaks them up.

The virtues of this account are several, B&R argue. Plurality comes at little or no cost. Also, pluralism allows a charitable interpretation of many debates in philosophical logic. And it does more justice to the mix of perspectives in the various debates about logic.

4.1.2 The collapse problem

There have been multiple attacks on Beall & Restall's logical pluralism. Most of these revolve around the *collapse problem*. The problem is quite straightforward. Let's look at two formulations, by Stephen Read and Rosanna Keefe.

Read

Read [2006] explains the collapse problem as follows (following Graham Priest): we often reason about some situation or other, call it s. Suppose two accounts of deductive validity, L_1 and L_2, deliver different answers about whether to conclude that, in s, a proposition β follows from the premises α. What should we conclude? Suppose α is true in s. Then the question, "Is β true in s?" is a determinate one. If L_1 deems the argument valid and L_2 deems it invalid, one should support the conclusion, since the former yields the stronger demand: invalidity does not mean that $\neg\beta$ follows, just that β doesn't follow. B&R's response is that the inference to β is classically valid, but not relevantly. But this response is not open to them, argues Read. They said both logics preserved truth in a case simpliciter. Moreover, a true *relevantist* wants to maintain that the inference to β is not justified — but saying that one should not infer β not possible for the logical pluralist who also endorses relevance logic.

Keefe

Read's version of the collapse problem can be generalized.

B&R's claimed that the different logical consequence relations were like precisifications of a vague, pre-theoretic notion of validity. Keefe [2014], first of all, argues that the analogy with philosophical accounts of natural language vagueness is mistaken. The different precisifications of validity are not presented as possible candidates for the one relation of logical consequence: for one, *precisifications* of vague terms agree on all settled cases (a person of two meters is definitely tall). This is how supervaluationism works: several cutoff points might give the right boundary for "tall" and we don't know which one. B&R's consequence relations do not do this; otherwise, we could only count as valid those arguments that are valid according to all admissible consequence relations. *That* would result in a very weak logic.

However, B&R's account doesn't depend on the notion of precisification, so Keefe turns to the core of B&R's pluralism. What do they mean by saying that pluralism is about endorsing several logics as true consequence relations?

Maybe, Keefe says, the solution is on the level of *application*. Are there wrong and right ways to reason in a particular context? There the threat of a contextualist pluralism looms — not what B&R want. Instead, we have to interpret them as saying that any true consequence relation delivers a true verdict in any context and that some choices are sometimes pragmatically better. But then we bump into the collapse problem: one should always adopt the stronger consequence relation if they're considering two logics with one containing the other. If there's no strongest relation between multiple admissible logics, there will still be one right answer to any question of validity: "yes" iff at least one of the admissible systems deems the argument valid. It follows that the logic that the pluralist should follow is some kind of argument mix of the logics that the pluralist accepts.

Evidently, our pluralism cannot be a radical one like Beall & Restall's. Their position falls prey to the collapse problem: if logic is to have a normative role, two senses of validity cannot play the role of arbiter for the same problem.

The solution for this is simple: validity has to be topic-relative or contextualist pluralism (as, for instance, Caret [2017] proposes). The idea is then that in one context, there is one way of reasoning that is normatively correct.

4.1.3 Necessity by quantification over cases

Before we move on, there is another point of criticism against B&R that is relevant to discuss, aimed at their explication of the requirement of necessity.

First of all, Bueno and Shalkowski [2009] identify a problem with B&R's (claimed fulfillment of the) necessity constraint:

> The rub is the necessity constraint. For the premises to necessitate a conclusion is a matter of them doing the right thing in all cases. Having recognized that cases may or may not be complete, and that they may or may not be consistent, to do the right thing over all cases is to do the right thing regardless of whether a case is complete or consistent. (...) Recognizing this is exactly why constructive logicians reject some classically and paraconsistently valid inferences, such as double negation elimination. [p.299-300]

So the problem is that if Beall & Restall take different kinds of cases to be (or represent) *possibilities*, necessary truth-preservation has to be defined over *all* cases, not only cases of a certain kind.

There also exists a more general problem with the model-theoretic explication of necessity. Bueno & Shalkowski again:

> To meet the necessity requirement, it is not good enough for 'all cases' to be simply all cases; 'all cases' need to be all possible cases. If the space of cases is not shown to exhaust all possible cases, no claim to satisfying the necessity constraint is warranted for any proposed logic. To articulate what it is to satisfy the necessity requirement we need some background modal notion. [Bueno and Shalkowski, 2009, p.306]

So the model-theoretic account explains necessary truth-preservation by truth-preservation in all cases, but this presupposes that we know exactly what cases represent possibilities, such that we only consider all *those*.

A similar (but slightly different) point has also been made by other authors. For instance, Prawitz [2005], Etchemendy [1983, 1994] argue that the model-theoretic definition misses an *epistemological* necessity: under the Tarskian definition of validity, if validity is a matter of truth-preservation in all cases, then one has to already know that truth *is* preserved in every case that makes the premises true, before one can say that an argument is valid.

According to Priest [1995], Etchemendy (and Prawitz) confuse(s) definitional order with epistemological order. What *makes* an argument valid is that it preserves truth in all cases. How we *recognize* an argument to be valid is not that it preserves truth in all cases — it's that we have a proof from premises to conclusion. Of course, it is a good question what makes an inference a proof — but a proof is not what makes an argument valid.

Now, Priest's response is exactly the kind of response that can be used for criticism at the explication of the criterion of necessity in the model-theoretic account: what *makes* something necessary is not that it is true in all possible cases; rather, that is a consequence of the necessity. Considering "all (possible) cases" is perhaps easier for evaluating (*recognizing* the truth of) modal statements than considering necessity directly, but in the end, the *definition* of a valid argument should be that it *necessarily* preserves truth, not that it preserves truth in all cases: we don't care about arguments that happen to always preserve truth, we care about arguments where the premises *necessitate* the conclusion. The model-theoretic definition forgets that quantification over cases is all *about* fulfilling a criterion of necessity — aside from the fact that considering possible cases already presupposes an understanding of possibility and thus necessity.

4.1.4 Meaning (in)variance

Let us touch briefly upon another debate that relates to logical pluralism: whether supporters of alternative logics *mean* the same by the (standard) logical operators or whether they are actually just talking about different things.

Beall & Restall maintain that their pluralism is not a "Carnapian pluralism", by which they mean a pluralism about the *meaning of logical connectives*. Carnap held that one's logic is internal to the choice of language; and a language must be chosen based on pragmatic considerations about application:

> *Principle of Tolerance: It is not our business to set up prohibitions, but to arrive at conclusions.* (...)
>
> *In logic, there are no morals.* Everyone is at liberty to build his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. [Carnap, 2002, p.51-55] [original emphasis]

Carnap's principle of tolerance implied that alternative logics never genuinely conflict: every logic is valid relative to a language. A Carnapian pluralist, say B&R, holds that negation in intuitionistic logic *means* something else than negation in classical logic. However, B&R claim that on their account, differences judgments of validity are not due to different choices of languages; they occur *within* a language. The logical constants in one logic or another *mean* the same.

Field [2009] is skeptical about the viability of a radical logical pluralism based on Carnap's principles of tolerance. He thinks difference and sameness of meaning are too obscure to explain logical pluralist positions:

> On some readings of "differ in meaning", any big difference in theory generates a difference in meaning. On such readings, the connectives do indeed differ in meaning between advocates of the different all-purpose logics, just as 'electron' differs in meaning between Thomson's theory and Rutherford's; but Rutherford's theory disagrees with Thomson's despite this difference in meaning, and it is unclear why we shouldn't say the same thing about alternative all-purpose logics. [Field, 2009, p. 345]

What Field means by this, is that on a *maximalist* understanding of the meaning of logical operators, which means that everything — the semantics, the interpretational structure, the logical laws that are validated by the whole system — contributes to the meaning of the operator, obviously, supporters of differing logics do not speak the same language. But then any difference of opinion about the logic leads to a difference in meaning of operators.

However, another position in this debate holds that we should see meaning (of logical operators) in a *minimalist* way: there exists some core meaning of, for instance, negation, which changes in a maximalist meaning only *after* the operator is applied to a structure [cf. Estrada-González, 2011]. This would mean that the *semantics* determine this core meaning. For instance, negation defined as "$\neg p$ is true if and only if p is not true" would be a different operator from "$\neg p$ is true if and only if p is false"; the conditional defined as "$q \rightarrow p$ is true if q is not true or p is true" is a whole other operator than "$q \rightarrow p$ is true if $[(p \wedge \neg e) \rightarrow q$ and $\neg e$ are true and there is no $r \neq p$ such that $(r \wedge \neg e) \rightarrow q]$".

The minimalist position does justice to some strong intuitions, even if it has its own problems.[25] That different uses of conjunction, or negation as failure ("Assume p is false if you have no evidence for p"), on different objects have *something* in common seems quite probable to me. I can have a formal language, with a fixed semantics, in mind but switch the structures on which I interpret that language. Van Benthem [2002] also seemed to have something like this in mind when he asked, "Should truly logical notions not be *independent* from particular choices of objects over which they are supposed to work? [p.429]"

Of course, this is not to say that in natural language we do not use words to refer to *different* logical operators. That is evident, and I have given examples before. To see *what* precise operators (and other linguistic devices) people *mean*, we need only engage in (sometimes extensive) communication, plausibly performing the kind of regimentation that Quine had in mind, where we *ask* people about the logical structure (and thus, semantics) they intended.

One proposal for defining the meaning of the (standard) logical operators comes from the side of the proof-theoretic account. We have only touched upon this position briefly, when we discussed MacFarlane's "decoy" concepts of

[25] For instance, it might be criticized for implying a kind of essentialism, which is a notoriously tricky philosophical position.

formality in §2.2.1, one of which was *syntactic formality*, the idea that logic is about inference rules that work only on the syntax of language. The meaning of a logical constant c is traditionally given by the truth (understood constructively, so: assertibility) conditions of sentences with c as the main sign. One of the essential observations of the proof-theoretic account, originally made by Gerhard Gentzen, is that the meaning of the logical constants is *determined* by the introduction rules for when and how these constants may be used [Prawitz, 2005]. The rules for elimination inferences, then, are justified by these meanings. An argument from Γ to A is (logically) valid if any proof of Γ can be transformed into a proof of A.

Now, Prior's renowned operator, 'tonk', showed that good logical rules cannot be defined as *just* manipulation rules on the syntax of the language. As Brandom [1994] explains, *logical* operators have to play the role of making explicit what is otherwise implicit in our inferential practices — 'tonk' does not do this (it leads to triviality). In other words, we need a notion of validity of *material* inferences to evaluate what syntactical rules are good. Consider this quote from Bueno and Shalkowski [2013]:

> That does not change the fact that in any particular instance it is the impossibility of some worldly affairs without another that makes that instance valid, which in turn provides part of the reasoned grounding for the conclusion of that instance. Both syntax and semantics play roles in making languages better or worse tools for communication. If syntax and valid inference are correlated in interesting ways, the syntax is interesting only insofar as it tracks inference, not the other way around. [p.16]

However, proof-theoretic semantics *can* play a role in giving minimalist meanings, at least for the standard logical operators. An author who has suggested this, for instance, is Paoli [2007]. However, how one could characterize a minimalist meaning of operators that are not among the standard (first-order) logical constants, like the modal operators or the default conditional ('if ... and there are no exceptions, then ...'), is an open problem.

4.2 Models, necessity, logic, normativity

In contrast to Beall & Restall, I wish to define validity as necessary truth-preservation on the model that is appropriate for the problem at hand, which means I propose one definition of validity but make space for a multitude of *logics*, which capture *patterns* of valid arguments on *types* of structures.

4.2.1 Necessity on models

Even though it is clear that we use models for reasoning, thinking, and interpreting language, there is little consensus in the literature on what models *are* [Frigg and Hartmann, 2020]. Some models are physical objects, like maps, scale models (e.g. of buildings), model organisms in the life sciences. But many models are *not* physical objects: the idealized models (including, e.g., frictionless planes) that are used in physics, abstractions from the real world for economic reasoning, graphs for how information spreads through social networks.

It is outside the scope of this work to investigate the (ontological) properties and give a precise characterization of models.[26] But it is undeniable that they are indispensable for human thought. This indispensability can be supported by some intuitive remarks about the role that models play for understanding, and *stabilizing*, the world as we experience it; and about their role as representational tools for thinking and speaking about past, impossible, future, counterfactual, plausible, fictional, abstract and idealized situations and objects.

So here is a schema of the world, our experience of it, and the models on which we interpret language. In science, we postulate (with good reason) that the world reaches us through raw sense data. However, as pointed out by phenomenologists like Merleau-Ponty and Smith [1962] and Husserl [cf. Van Atten, 2006], our *experience* of the world is one already full of sense, of a world of *objects*. This immediate categorization can be seen as a model already: it classifies individual things *as* individual things, which might not always do justice to how the world *is*, or otherwise definitely disagrees with how someone else experiences the world.[27] But this object-filled world is still too chaotic, too ever-moving, so we construct models to stabilize it, to describe a temporal interval (and spatial configuration) as a situation, to put things in categories, to identify properties. We see a grey small thing shoot between the greenish things and construct a model plus communicate that there is a rabbit in the bushes — but the world is more complex: it might be wrong (it was a haze), outdated (it's moved on), imprecise (there are two).

One might wonder, can we not reason about the world directly? This question evokes two responses. One, "the world" is not immediately given: as mentioned above, (our idea of) "the world" can also be said to be a model (or just a collection of models).[28] Two, it's not feasible to reason about totality directly: it's too big, we always need to consider relevant aspects, abstract from details or entire regions of our conceptual scheme.

We sometimes interpret language on partial structures that can be embedded in the "ordinary" (physical) world. For instance, observation sentences, like 'The door is closed', can be true of a particular door during some particular time in the observable world. However, many utterances are not about this ordinary

[26] Perhaps a formal characterization of models can be given along the lines of Barwise and Perry [1981]. They define a *situation* $\mathbf{s} = \langle l, s \rangle$ as a type s of objects standing in certain relations at a space-time location l. We discussed situations briefly in §3.1.1. However, it is unclear whether these could accommodate all the models I have in mind. Moreover, note that this suggestion does not imply that mine is secretly an account of relevance logic: (1) I do not wish to define validity as truth-preservation over all models, even if those models could be explicated as situations; and (2) my account, of course, purports to defend a form of logical pluralism, where a logic is a systematization of valid arguments due to *particular* logical structure on a *type* of models, as I will explain in §4.2.2.

[27] That we cannot know anything about the nature of the world outside of our experience of it was of course observed by Kant [1908]. That our categorizations are somewhat arbitrary is common sense: for instance, we might often disagree about whether something is red or orange, and if science sets a border on the color palette, that's a choice, not a discovery.

[28] Cf. Sellars' [1956] renowned argument.

world: what those are interpreted on, then, are models (which one might, of course, see as abstract objects *in* the real world). For instance, for a counterfactual statement, we conceive of one or more ways the world could have been (where we don't specify all and any details of those realities). For statements about omnitemporal mathematical objects, the model will be a static one; for temporal mathematical objects, like the choice sequences discussed in §3.1.1, the model will take time into account. Clearly, *time* will be an important element of models for the arguments they make valid: under a standard understanding of truth, the truth values of many statements relativized to a particular moment in time can be indeterminate.

What kind of models are closed worlds? Of course, for a formal language like logic programming, the interpretational structure is an abstract, Tarskian-like model, that is constructed by integration of the premises. But logic programming is a formal model of the kind of reasoning people likely perform in, for instance, planning tasks and discourse comprehension. When we consider a person planning a train trip, we can imagine that the model employed is a kind of partial structure that takes relevant features from (our model of) the real (physical) world as described before, with the difference that the model incorporates closed-world assumptions: that we do not take exceptions into account that we don't know of, that there are no more trains than the schedule says, or that fact D is caused by the occurrence of one of the facts A, B, C that we know *can* cause D.

Note that language and models interact. We do not interpret linguistic utterances on models that were somehow already determined, but on models that were constructed *for* understanding language, which are then improved and corrected in communication. Also, it can of course be clear from the context what the model of interest is, such that some sentences don't prompt the construction of a new model or changes to a model, but are evaluated on the model of interest. When I say, "Look, a rabbit in the bushes and a bird in the sky," you will often interpret my utterance by constructing a model where both are true, but you might also be sure that the rabbit was a haze and the bird a kite, which means you evaluated my utterance on your model of the situation and concluded it was false.

Also, it is natural to assume that the models we reason with get a mental representation in our cognitive processes. But I do not wish to define validity on mental representations ('mental models'). *Those* might be too individual and unstable. Consider the quote from §3.1 again:

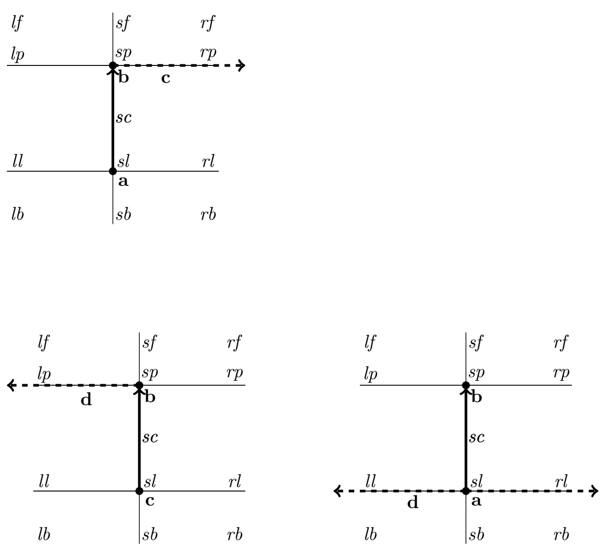
> Although a few researchers remain skeptical (...), there is now a broad consensus that mammals (and possibly even some insects) navigate using mental representations of spatial layout. [Rescorla, 2019]

Here the *spatial layout* that is mentioned is the kind of model I'm talking about: an abstraction from reality.

In fact, navigation makes for a good example of my theory. The quote above illustrates that we (both people and animals) use *models* to get around. Of course, for communicating and inter-subjective reasoning, such models need to be made very precise: it must be possible to determine most (simple) questions that might come up. In the most demanding reasoning context — scientific inquiry — models like these have to be characterized in a rigorous, mathematical manner.

This was done in the Double Cross calculus (see figure 4.1). Let us apply my point to this example: we might argue whether this model captures qualitative spatial reasoning best, and we might come to improve the model. But *if* we agree to use this model for our navigational problem, for instance, we decided to meet with someone at point d but we are wondering if point a, where we are standing, isn't just the same as point b, and we have some instructions about the spatial relations between a set of points a, b, c, d: $rp(a, b : c)$, $lp(b, c : d)$ and $\{ll, sl, rl\}(a, b : d)$[29] — then $a = d$ is necessarily true, and we have a valid argument.

Figure 4.1: Models for premises in the Double Cross calculus: $rp(a, b : c)$, $lp(c, b : d)$, $\{ll, sl, rl\}(a, b : d)$.



Similarly, $a = c$ is not a valid conclusion: it's even clearly false. Note that an argument can be truth-preserving (the premises and the conclusion are true) but not *valid* since it is not necessarily true. For example (uttered on the evening of the 3rd of February 2021): "It is raining in Amsterdam; therefore, my grandma is at home."

This brings us to necessity.

[29] $\{ll, sl, rl\}$ denotes the union of ll, sl and rl.

(Conditional) necessity

I have argued that validity is a property of arguments that depends on the structures they are interpreted on; that the way in which a intuitionists reasons validly is the same in which the Platonist reasons validly. What I tried to say is that once it has been made precise what the language is supposed to express and about what — on what model the sentences were intended to be interpreted — it is in some way inescapable whether a conclusion *follows* from premises or not.

It is of course mysterious why our world is as it is, it is interesting to think about why and how we experience and structure and classify the world, but there is nothing mysterious, *besides* our use and understanding of language, about the fact that I can conclude that there is no tea in my cup if I know that my cup is empty, or that I can conclude that 8 is an even number if I learned that 8 is bigger than 7 and an even number. The intuitions behind such simple (valid) arguments are precisely as strong as those behind simple arithmetic. (Which, of course, we do by simple valid arguments as well; I'm only mentioning arithmetic because our strong intuitions about it make it such an illustrating example.)

In this thesis, I will not give an account of (alethic) necessity — that is too big a topic on its own. One could take it to be a primitive notion, like Bueno and Shalkowski [2009, 2013]; however, I think necessity of truth-preservation can be explained in other terms. At the same time, we have to acknowledge that *explanation* can only get so far: attempts to justify each step in an explanation by an extra reason do of course lead to a (vicious) regress. As Szabó [2012] remarks, at some point, validity (necessity) just *shines* through.

Instead of giving a philosophical theory on necessity and possibility, I will make some remarks that should support our conviction that it's a proper concept to found validity on. First of all, it is evident that we do recognize necessities and possibilities in everyday life, in scientific research and philosophical debate. We know the rock will not fall up from the mountain to the sky, that I could become a grandpa but not a baby, that we can only see not hear color, that physical objects extend in space, that one and one people are two people.

This self-evidence is induced for a large part because of the *stability*, the enduring *identity*, of models and the objects in models. If it is true, on a model of the situation we are both witnessing, that "There is a rabbit in the bushes and a bird in the sky" (and it is clear enough what this *means*, then, because of the identity of the model, it is *necessarily true* that in my model there is a bird in the sky. Truth of a negated proposition $\neg P$ expresses that the model $\mathfrak{M}$ is different from all those that make P true (or leave it indeterminate). Conditional statements A implies B make explicit that there is a valid argument from A to B — so they are true when this is the case and similarly, if one assumes they are true (on the relevant model), this means B can be concluded if A is true: so we have modus ponens.

Similarly, quantifier arguments are (in)valid by the same necessity that makes arithmetic work. (Alethic) modal operators induce valid arguments on arbitrary models because they make our judgments of modality explicit. If they are explicated as mathematical operators on structures like Kripke frames, they induce necessary consequences because of their precise semantics and the constancy of the structure.

Some arguments are valid on models of the (concrete) world because concepts are part of the meanings of other concepts (typical "material" arguments). So again, via identity of the model, it is clear that *red* implies *colored*, *water* implies H_2O, and *Hesperus* implies *Phosphorus*).

Of course, what is necessary in models for specific problems is determined by what is necessary in the world (i.e. in our most basic models): partial structures of the concrete world inherit the laws of nature, mathematical structures (at least) the most basic laws of identity, difference, space — but since the latter is something like a model itself, this is not problematic; it is just the way that our idea of reality influences everything we can think. On the other hand, models could explicitly be constructed as to not obey some of these laws: then, if we make them precise enough, we will still be able to evaluate arguments on them.

In short, we might say that validity depends on a conditional necessity. What's necessary on a model is dependent on the identity and laws of the model. In other words, if the model is to represent parts of (concrete) reality, what's necessary on there will be determined by what we deem to be necessary in the world.

Often enough, we might not be able to evaluate the validity of an argument. On my account, Goldbach's conjecture is either a valid or an invalid statement on any classical model that represents the natural numbers, even if turns out that no one can prove or disprove it. When these are sufficiently (mathematically) specified, some arguments are clearly necessarily truth-preserving, some clearly not; for others, we might need intermediate steps (proofs).

What about the notion of logical truths? A logical truth is traditionally defined as a sentence that is true on all models (of interest) due to the logical structure of the sentence. Now if logical structure is not strictly demarcated, it follows that the notion of logical truth is not as strictly defined as well. However, it will still be reasonable to judge that some sentences are clearly *logical* truths and some are clearly not (given a set of models of interest). For instance, that a circle is round will be valid on any model that tries to capture ordinary mathematics, but it does not seem like this validity is due to any logical structure.

What is generally the method to obtain whether a conclusion follows from a set of premises, instead of being accidentally true? If the model is a partial structure of (mathematical) reality, considering all total models (assuming that such models exist) in which the partial structure can be embedded is probably the right strategy.[30] If the model is a total structure of concrete reality, the method of considering all other possible totalities that make the premises true is a good method. Either way, we have to presuppose a capacity for judging

[30] And, perhaps, varying the contingencies in the partial structure that do not contribute to the truth of the premises.

what is possible and what is necessary.

4.2.2 Logic and normativity

Logics: systems of validity inducing operators

What are logics? Logics are systematizations of valid arguments on particular types of models by focusing on only certain expressions of the (natural) language and, generally, by abstracting away from (concrete) aspects of models that do not influence the validity of those (expressible) arguments. The expressions that logic (the science) is interested in are those different phenomena in our language and thought that we discussed in chapter 2, which have a role in organizing and structuring our thoughts and information, like the propositional connectives, (generalized) quantifiers, modal operators, predication, prepositions, and so forth.

For instance, propositional logic captures argument patterns that are induced by the (standard) propositional connectives, on models that are just valuations of the proposition letters (and thus models of all the propositional formulas).

Like I have argued before, it is unlikely that we could principally characterize the logical constants of the language: those operators that play a structuring (explicitating) role are similar in this respect, but they all do it in a very different way. Logic should be thought of as a method rather than a science that can discover its exact subject matter. *Proof-theoretic semantics* can perhaps go a long way in characterizing logical structure.

My argument is that for varying models, particular *logics* have normative force since they systematize an important set of valid arguments on those models. This presupposes a few things. One: many alternative logics *truly* deserve the name of logic, because they capture logicality for some models. To put it otherwise, *logic* as such is not one system, nor is *a logic* some kind of metaphysical, mysterious entity.

The preoccupation of traditional accounts of validity with the connectives and quantifiers of first-order logic is probably caused by the crucial but not exclusive role these play in valid arguments. It's reasonable to suppose that the mathematical operators that are appropriate for reasoning are basic in some sense. Humans and computers alike have limitations on the amount of information that can be processed. There are operators that can express intractable queries, which could be undesirable [for computational properties of several logical systems, cf. Libkin, 2013]. Not only computational considerations could be relevant. For practical purposes, one might want to use a logic that is complete or that is not overly expressive. Very unnatural mathematical operators that function very differently on different (sized) domains are probably too impractical to play a big role in our thinking.[31]

[31] However, one can think of situations in which operators like these would be employed in the logic. For example, some generalized quantifiers, like "most", might be thought to change semantics given the domain on which they're interpreted: suppose "many people are

Even though a principled boundary between operators that are usually seen as logical operators and those that are not is not feasible, from the perspective of *cognitive science*, it is highly interesting to investigate what kind of operators are essential in human reasoning. Doubtlessly, those are the standard propositional connectives and several (generalized) quantifiers, as well as modal operators thinking about time, knowledge & belief, and deontic norms.

Given that equivalent models might be appropriate for a reasoning problem, it occurs that logical systems are equally fit for a problem. This does not have to mean that two logics make the same set of arguments valid — for if one system is much more expressive than the other, that set will be different. Instead, two systems are equally appropriate for a reasoning problem if they are both expressive enough for any of the questions that need to be answered in that context *and* give the same answers to those questions posed. As an example: if (classical) first-order logic is *exactly* expressive enough for a certain reasoning problem, for instance, syllogisms, then (classical) second-order logic will be expressive enough as well, and yield the same answers to any questions that can come up. Second-order is incomplete, so if this is a relevant consideration — for instance, it comes up while solving the syllogism that we need the logical truths to be provable — first-order logic will have normative precedence.

That several models and therefore logics can be fitting for the same problems should not be surprising. Van Benthem [2019] analyzes connections between two kinds of formal systems: on the one hand explicit extensions of classical logic, on the other hand implicit re-interpretations. In some newer formal systems, operators are added to the classical vocabulary, leaving the old notions intact. A typical case is modal logic, where the propositional base logic is extended by adding two modal operators. In other systems, we can see modifications of the meaning and use of the old language, to model new or other phenomena. For instance, we get new meanings for the logical constants, new semantics, new understanding of concepts such as "truth" and "validity".

For example, Van Benthem discusses formal systems that represent mechanisms of *knowledge*. In *epistemic logic*, we find all the classical operators (with standard semantics, usually interpreted on a basic Kripke frame) plus an operator K with the following truth definition: $M, s \vDash K\phi$ iff $M, t \vDash \phi$ for all t with $s \sim t$ (where $\sim$ is the epistemic accessibility relation). This says, intuitively, that I know ϕ iff ϕ is true in all worlds that I consider epistemic possibilities. Further intuitions about knowledge (such as $K\phi \to KK\phi$) are encoded on the semantic side in the structure of the models studied, and in the proof system as axioms — and different intuitions induce different systems; two well-known ones are $S5$ and $S4$). At the same time, these epistemic phenomena can be captured using *intuitionistic logic*, where "truth" of ϕ means evidence for or *knowledge* of ϕ. But in this system, these intuitions are not encoded in extra basic operators, but by redefining the meanings of the logical constants (for example, $\neg\phi$ is only true when we have evidence that ϕ is false). The interesting thing is that there

B" is already satisfied when 30% of the people are B, but we cannot say the same when we're talking about, say, ants.

exists a faithful translation, discovered by Gödel, from intuitionistic logic into $S4$; and, less-known, a converse translation as well.

Two such logics are not just the same system in different guises, says Van Benthem. Yes, we have faithful mutual embeddings, but one does not get a feeling of strong resemblance; and mutual translation does not imply system equivalence in all relevant aspects.

There are many more examples of connections between formal systems that are concerned with the same phenomena. Of course, in my proposal, if *all* considerations that come into play in formal system design are also relevant considerations for (constructing the model for) a given problem, it seems that there will actually be only *one* formal system that is the right logic for that problem. On the other hand, in real life, it won't usually be the case that every relevant consideration has a clear-cut answer:

> It is not always straightforward to come up with the best language to capture a given concept. For example, the "best" one for studying the concepts of finite and infinite is not at all the one that first came to mind (...). In other cases, even finding just the right collection of structures has been problematic. Finding natural logics takes trial, error and experience. [p. 6]

Normativity

The reader might be wondering whether validity and logic can still be about *judging* arguments to be good or bad, about evaluating whether we should draw a conclusion or not. The concern about Beall & Restall's pluralism was that it undercut the normative character of logic. However, on my view, normativity comes in at the level of choosing or constructing the right model, *and* at the level of evaluating arguments *given* the model.

This means that sometimes, there is an objectively correct model. It depends on what makes this model correct whether it is achievable to find out *what* the correct model is. If the goal is to reason about absolute mathematical truth, it's probable that one set of assumptions is right — but it is hard to imagine that we will find out, on short notice, which assumptions. However, if the goal is to reason *classically* about mathematics, as is often the default in classrooms, there is an objectively correct model. If we plan a military operation and agreed to only consider exceptions that we know *could* occur, there is a correct model. In other words, it will often be defensible that everyone was supposed to reason with this or that set of assumptions.

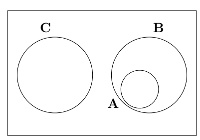
In real life, of course, the exact model (or relevant properties of the model) is not always explicitly decided upon. If there's disagreement, parties have to (1) either convince the other party, like the teacher telling his students how to interpret the problem; (2) find some kind of compromise, like two philosophers who agree for the present debate that they will understand truth in a Platonist manner; or (3) give up and cease the task at hand.

Here is an example of an underdetermined problem that can be specified such that, at some point, we can speak of (common-sense) objectivity. As we discussed before, untrained subjects often have problems interpreting the syllogistic task as intended, as well as deriving conclusions from the intended interpretation.

Sato and Mineshima [2015] conclude from experimental results that *diagrams* can significantly improve subjects' scoring on syllogistic tasks [cf. Mineshima et al., 2014, 2012]. Not all diagrams: we saw that traditional Euler diagrams hold the Existential Assumption for minimal regions (EA). In this system, categorical sentences like *All A are B* cannot be represented by just one diagram: the two cases $A \subset B$ and $A = B$ induce a separate one each. Checking validity of a syllogism via these diagrams leads to the problem of "combinatorial explosion."

However, Venn diagrams and especially EUL system Euler diagrams are very useful for (untrained) subjects. Any categorical premise, as well as any combination of consistent premises, can be represented in just one diagram, like figure 4.2. These diagrams improve task performance because they directly communicate the intended meaning of the sentences as relations between sets, and because they are useful for checking whether there exists a valid conclusion for the two premises (for the latter, EUL diagrams work significantly better than Venn diagrams — 4.1 is an example of a unification of two diagrams, from which the conclusion *No C are A* is easily read off).

Figure 4.2: EUL system diagram for *All A are B* and *No C are B*.



In the experiments that Sato & Mineshima conducted, subjects were told to not assume existential import for premises like *All A are B*. Those subjects that were also presented with the Venn or EUL diagrams received extra instructions, among which the EFA:

> Before the reasoning tasks, we provided the participants with instructions on the meanings of diagrams and sentences (...). More specifically, the point of the instruction is that a circle is used to denote a set of objects and point × is used to indicate the existence of an object for the Existence-Free Assumption for minimal regions (...). The convention of EFA seems technical so that some instructions are needed to understand it correctly. Accordingly, we provided participants with instructions on the meanings of diagrams to fix the intended interpretation of the diagrams used in the experiments. [p.430]
>
> (...) Here our instruction emphasized that the meaning of categorical sentences used in our experiment does not contain the existential import. Concretely, the following is given: *All A are B* does not imply that there are some objects which are *A*; thus, *All A are B* does not imply *Some A are B*. Similarly, *No A are B* does not imply that there are some objects which are *A*. Thus, *No A are B* does not imply *Some A are not B*. [Sato and Mineshima, 2015, p.432]

But it turns out that existential import "can be robust in novice learners' interpretation (...) [and] understanding the EFA requests learners' effort [Sato and Mineshima, 2015, p.434]." *After* it had been explained that one should not assume existential import for the universal quantifier nor the minimal regions in the diagrams, some reasoners still did just that. In this case, it is reasonable to say that there *were* objective models, which some reasoners failed to grasp.

So there are several ways one can go wrong. First, a person can be incorrect about what the right model is. For instance, it was not clear that we needed a logic that did not assume a background set theory that includes the axiom of choice; this parameter was not explicitly set, but now it turns up and warrants repair. However, it can also be a mistake in the reasoning *about* the background assumptions of the problem. Think about the philosopher that assumes introspection ($K\phi \rightarrow KK\phi$) for an epistemic phenomenon for which this is highly contentious.

One also reason invalidly given the right conception of the model. This is a clear mistake. For instance, the instructions of a mathematical problem *mentioned* that we needed, besides a proof of abstract existence of a kind of object, a method for constructing such an object; but the student employed a classical, bivalent logic (interpreted on a classical model for mathematics), which does not, contrary to constructivist (intuitionist) logic, have the property that existence proofs always deliver a way of actually constructing the desired object as well [cf. Stenning and Van Lambalgen, 2012, Bridges and Palmgren, 2018].

So there is a clear normative aspect to validity and logic: it provides standards against which to evaluate the arguments that people make in varying situations. However, precisely *because* real-life models are often underdetermined is it interesting to investigate what models people constructed and whether they reasoned validly from *those* or not.

Of course, it remains a question of how we *choose* the right models, what things (abstract objects or otherwise) could count as models, what could not count as models. Answering these questions is outside of the scope of this work, but here are some suggestions, inspired by Quine [2013]'s insights on semantic holism and the web of belief, and his evaluative standards for scientific theories, such as simplicity, elegance, utility (predictive accuracy).

Some models are of the more basic type, that classify the world as we know it in a common-sensical way, that account for the everyday experiences that humans share. These include our normal categories of standard objects: tables, rabbits, colors, currencies, countries, shapes, natural numbers, and temporal chronology. Some questions might not have a determinate answer on these: are there higher-order infinities, how does non-Euclidian geometry work, what are the laws of economics. Perhaps, therefore, these common-sense models might be best thought of as the *situations* that formed the basis of relevance logic — such that a claim of being the right logic for common-sensical arguments about the world could have some credibility. The more complex — scientific — questions, then, have to be answered by constructing (completing) these models, in a way that answers to the Quinean scientific standards. Two complex, scientific models can be compared (judged) by incorporating them into the more basic model of the world, and *observing* how well they fit. Now if there is strong evidence, for example, empirical (observational) data that suggest some boundaries of our common sense conceptual scheme need (slight) revision, this will be acceptable. So clearly, this is a dialectical process (in the ancient sense of discovering truth through debate); we can compare it with Rawls' idea of a reflective equilibrium [cf. Daniels, 2020]. We update our scientific models often, our commonsensical models sometimes. What was a valid argument before, might not be on the new model anymore.

4.3 Summary

In this chapter, I proposed a theory of validity and logic, taking into account the considerations developed in the rest of this thesis.

First, we looked at the model-theoretic logical pluralism by Beall and Restall [2000]. They proposed that different interpretations of "case" (as in: an argument is valid iff we have truth-preservation in every case) yield different but genuine consequence relations. Read [2006] and Keefe [2014] criticized this by means of the collapse problem: one needs exactly *one* notion of validity to determine whether a conclusion should be drawn or not. Bueno and Shalkowski [2009] pointed out that the *definition* of a valid argument should be that it *necessarily* preserves truth, not that it preserves truth in all possible cases: the latter is only a consequence of the former and presupposes an understanding of alethic modality anyway. Finally, I argued for a *minimalist* conception of the meaning of logical constants.

Next, I tried to explain the nature of validity and logic. I argued that *models* are indispensable for human thought, because of the role they play in understanding and *stabilizing* the world. Some models are partial structures that can be embedded in the (physical) world; some are abstract mathematical objects or idealized abstractions from reality (for instance, spatial layouts for navigational purposes). We saw that language and models interact: most of the time, models are constructed *for* discourse comprehension and are then improved and corrected in communication.

I pointed out that necessary truth-preservation was induced for a large part because of the *identity* of models. Of course, what is necessary in models is determined by what is necessary in the world: partial structures of the concrete world inherit the laws of nature, mathematical structures at least the most basic laws of identity and difference. Since the "world" is something like a model itself, this could just be seen as the way in which our idea of reality influences everything we can think. I also pointed out that often, determining validity of an argument might be hard, perhaps sometimes even impossible.

Logics were said to be systems that capture patterns of valid arguments on particular types of models by focusing on only certain expressions of the language: those which have a role in organizing and structuring our thoughts and information. I argued that the preoccupation of traditional accounts with the connectives and quantifiers of first-order logic is probably due to their crucial cognitive function.

Also, mathematical properties of logics as a whole, like completeness or expressiveness, can be relevant considerations for questions of application. We looked at an analysis of connections between formal systems by Van Benthem [2019]: this showed that often, different models might be appropriate for thinking about a problem, and therefore different (but connected) logics.

Furthermore, I argued that normativity comes in at the level of choosing or constructing the right model, *and* at the level of evaluating arguments *given* the model. This meant that sometimes we can speak of an objectively correct model. I concluded with some remarks about the dialectical nature of choosing the right models: that the most basic, common-sense models of everyday objects determine the construction of scientific (or mathematical) models and vice versa.

Chapter 5

Conclusion

The goal of this thesis was to provide a convincing story on the nature of validity and logic that explains why there can be so many valuable logical systems with such diverging verdicts on the validity of arguments. It tried to account for the commonplace experience that we can understand why our opponent *had to* come to their conclusion, given the assumptions that they made.

In chapter 2, we started out with the model-theoretic definition of validity (viewed as synonymous with logical consequence): an argument is valid iff it preserves truth in all models due to its logical *form*. The contemporary account, which originates with Tarski [2002] and Sher [2008], employed Tarskian, set-theoretic models.

With Szabó [2012] and Brandom [1994], I argued that restricting validity to formal arguments is hard to defend in a principled manner and that *analytic* and *material* arguments can also be truly valid. At the same time, we noted that information necessarily has (logical) structure, and agreed with the idea of formality that considers logical structure constitutive of thought; but formality should not be employed to demarcate the set of valid arguments.

After this, we looked at *invariance* criteria that tried to characterize the logical constants. We observed that these criteria were designed to obtain exactly the desired logical operators. In line with Van Benthem [2002], we concluded that invariance criteria as principled demarcations failed. Furthermore, the Tarskian tradition suffers from another flaw: that it focused exclusively on set-theoretic structures.

In chapter 3, we considered several reasons for logical pluralism. After Van Atten [2006]'s exposition of Brouwer's choice sequences and Husserl's views on logic, we concluded that we need different logics to reason about different ontological regions of the (mathematical) world. For phenomena like time, knowledge, belief or morality were of interest, we need modal logics, often interpreted on Kripke frames. Supporters of relevance logic believe the right underlying structures for arguments are *situations*: partial structures that can be embedded in the (physical) world. Besides, other non-classical logics are needed for reasoning about action, plans, evidence. For instance, closed-world reasoning ("don't consider unknown preconditions") is appropriate for tasks like planning a train trip. Logics for computer science are defined on finite models (since graphs, databases, and strings are finite).

A third reason came from Stenning and Van Lambalgen [2012]. They give an analysis of the performance of ordinary subjects on reasoning experiments like Wason's selection task. This performance usually does not live up to the standards of classical logic, but Stenning & van Lambalgen argue that people are in general not reasoning at random either — on the contrary, they are quite consistent if we use other logical systems to model what they are doing. Additionally, giving attention to the process of assigning a logical form to the task, does justice to the indeterminacy of natural language.

Finally, I stated the intuition that is the core of this thesis: that validity is about what information follows from other information, given that the meanings are sufficiently determined and the information is interpreted on a particular model.

In chapter 4, we first looked at the model-theoretic logical pluralism that Beall and Restall [2000] have proposed. They claimed that multiple logics are equally valid, whatever the problem or context. This idea crashed into the collapse problem: for every particular argument, we need one notion of validity to determine whether the conclusion should be drawn or not. Furthermore, Bueno and Shalkowski [2009] pointed out that the definition of a valid argument should be that it necessarily preserves truth, not that it preserves truth in all possible cases: the latter is only a consequence of the former and presupposes an understanding of alethic modality anyway. I also argued for a *minimalist* conception of the meaning of logical constants.

Next, I offered a story on the nature of validity and logic. I argued that *models* are indispensable for human thought, because of the role they play in understanding and stabilizing the world. Some models are partial structures that can be embedded in the (physical) world; some are abstract mathematical objects or idealized abstractions from reality (for instance, spatial layouts for navigational purposes). We saw that language and models interact: models are often constructed as a means to comprehend language; the models are then improved and corrected in communication.

I pointed out that necessary truth-preservation was induced for a large part because of the identity (stability) of models. Of course, what is necessary in models is determined by what is necessary in the world. Since "the world" was said to be a model itself, this could just be seen as how our idea of reality influences everything we can think: the most basic, common-sense models of everyday objects determine the construction of scientific (or mathematical) models and vice versa (which we called the dialectical process of constructing models).

Logics were said to be systems that capture patterns of valid arguments on particular types of models by focusing only on expressions of the language that have a role in structuring our thought (semantic information). The preoccupation of traditional accounts with the connectives and quantifiers of first-order logic is probably due to their crucial functions in cognition — but they are not the only operators to induce logical structure.

Finally, I argued that normativity comes in at the level of choosing (or constructing) the right model, *and* at the level of evaluating arguments *given* the model.

There are several natural directions for future research. One area is the philosophical characterization of models: what is their ontological status; and if they can be widely different, what do they have in common? How do we get from the most basic models of the ordinary (physical) world to scientific theories? A second topic is logical structure: what do we count as logical structure and why? Should sets of operators actually be arranged according to their different roles?

This ties into a third area, that of cognition. It is interesting to investigate what operators are most present or important in human thought. Likewise, we can examine how subjects understand tasks and whether they reason correctly if we specify the intended interpretation sufficiently. Finally, one evident and challenging task in cognitive science is to develop a detailed and accurate view (i.e. a precise model) of mental representation of models.

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