Effective Kan fibrations for simplicial groupoids, semisimplicial sets and Ex^∞

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Homotopy type theory has a model inside the category of simplicial sets which is based on Kan fibrations, but the proof of this fact can demonstrably not be made constructive. Effective Kan fibrations were introduced as an alternative to Kan fibrations in hopes of acquiring a constructive model. We contribute various results to the theory of effective Kan fibrations. Firstly, three constructions of Kan fibrations based on simplicial groupoids are redone for effective Kan fibrations. Secondly, we show that all the information about degeneracy maps is stored in the lifting structure of an effective Kan complex, but argue that this has no practical application. Finally, we prove that the Ex^∞ functor does not automatically produce an effective Kan complex, and argue that it is most likely not usable for fibrant replacement in the context of effective Kan fibrations. The positive results encourage a continued study of effective Kan fibrations, while the negative results teach us about its possible obstacles.

Contents

Chapter 1

Introduction

Simplicial sets can be thought of as combinatorial approximations to topological spaces that preserve homotopical information. Together with an auxiliary notion of *fibration*, specifically the *Kan fibration*, they provide a useful approach to homotopy theory. In addition, it is possible to define a model of *homotopy type theory*, an extension of Martin-Löf's dependent type theory, in the category of simplicial sets, in which dependent types are interpreted by Kan fibrations, proving that homotopy type theory is consistent (relative to the axioms of set theory, at least) [KL12]. However, the proof that this model satisfies the axioms of homotopy type theory cannot be made constructive [BCP15].

This is unfortunate, because homotopy type theory was proposed as a foundation of constructive mathematics, with possible applications to computer proof assistants. Admittedly, there already exists a constructive model based on *cubical sets*, which are built out of (hyper)cubes instead of simplices. But given the preference of algebraic topologists for simplicial sets, a simplicial model continues to be desirable. It is also simply puzzling that simplicial sets should differ from cubical sets in any essential way.

In an attempt at a remedy, Benno van den Berg and Eric Faber have suggested to replace ordinary Kan fibrations by their *effective Kan fibrations* [vdBF22]. These still bring forth the same homotopy theory, but come with additional structure that facilitates constructive proofs. This thesis is intended as a small contribution to the theory of effective Kan fibrations.

Chapter 3 contains the main positive results. Many specific examples of Kan fibrations spring from *simplicial group(oid)s*, which are to topological group(oid)s as simplicial sets are to topological spaces. Three such constructions, appearing in [vdBM18], are successfully redone for effective Kan fibrations in Theorems 3.9, 3.11 and 3.22. The third of these theorems is the most important: it generalizes an existing result, namely that a simplicial group is an effective Kan complex (a special kind of effective Kan fibration), to simplicial groupoids.

In Chapter 4, it is investigated whether *semi*simplicial sets, a generalization of simplicial sets, can be of any use in the study of effective Kan fibrations. A semisimplicial set can sometimes be turned into a Kan complex. Proposition 4.6 tells us when this conversion also yields an effective Kan complex, and a subsequent counterexample tells us when it does not. It is argued that these results mean that semisimplicial sets carry little relevance for effective Kan fibrations.

Chapter 5 treats *fibrant replacement*, through which a generic simplicial set can be substituted by a *fibrant* simplicial set, for which the homotopy groups can be defined. It is found in Section 5.3 that the most useful implementation of fibrant replacement, the Ex^∞ functor, problematically does not give rise to a simplicial set that is by default fibrant in terms of effective Kan fibrations.

These three chapters are preceded by an introduction to simplicial sets and the various fibrations based on them. The primary sources I have drawn from are [Cis19], [GJ09], [Lur23], [vdBF22] and the master's thesis of Freek Geerligs [Gee23].

The context of this thesis lies at the intersection of algebraic topology, (higher) category theory, and type theory. I have chosen a topological narrative whenever possible, using light category theory mainly for its notational benefits, and eschewing type-theoretic language altogether. Nonetheless, whenever the reader finds the topological analogies unenlightening, they can be skipped without consequence.

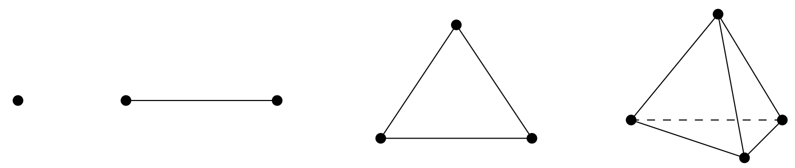
Chapter 2

Simplicial fibrations

This chapter covers most of the prerequisite knowledge for the later chapters. We assume decent familiarity with category theory, but we will use it in a very practical way, so that a thorough understanding of the technical definitions is not necessary. Other than this, we will introduce everything from simplices to the various kinds of fibrations we use.

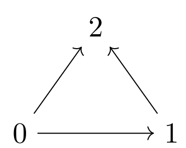
2.1 The category of simplicial sets

A *geometric simplex* is an element of the continuation of the following sequence:



More formally, we can define a geometric n-simplex as the convex closure of $n+1$ affinely independent vertices in n-dimensional Euclidean space. The boundary of a geometric n-simplex consists of $n+1$ geometric $(n-1)$-simplices, called its *faces*.

We want to introduce an *orientation* for the geometric simplices, so that for every way of rotating and reflecting an unoriented geometric simplex onto itself, there exists a different oriented simplex sharing the same vertices. One way of recording the orientation of a simplex is by ordering its vertices. We can represent this order by assigning increasing numbers to the vertices, or by drawing arrows that depict the *less-than* relation:



Thus we can abstractly regard an n-simplex as the totally ordered set of $n + 1$ elements (which is unique up to isomorphism), represented by the set $[n] = \{0, 1, \ldots, n\}$. Next, we consider morphisms of simplices, sending each vertex of one simplex to a vertex of another simplex. We postulate that such a map preserves the order of the vertices, so a morphism of simplices can be represented by an order-preserving function $[m] \to [n]$. These morphisms form a category:

Definition 2.1. The *simplex category* $\mathbf{\Delta}$ is the category with finite non-empty initial segments of $\mathbb{N}$ as its objects and order-preserving functions as its morphisms.

The category $\mathbf{\Delta}$ is generated by the minimally non-surjective and non-injective maps $\partial_j : [n] \to [n+1]$ and $\sigma_l : [n+1] \to [n]$, defined for each $n \in \mathbb{N}$, $j \leq n$ and $l \leq n + 1$ as follows:

$$\partial_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \geq j \end{cases}$$

$$\sigma_l(i) = \begin{cases} i & \text{if } i \leq l \\ i-1 & \text{if } i > l \end{cases}$$

In geometric language, the former includes the abstract simplex $[n]$ into the abstract simplex $[n + 1]$ as the face opposite vertex j. The latter projects $[n + 1]$ onto $[n]$ by identifying vertices l and $l + 1$.

A simplicial complex is a system of unoriented geometric simplices glued together along their faces. We want to define a version with oriented simplices, in such a way that if two simplices share a face, then the orderings of the vertices on the common face match up. This is achieved by defining a *simplicial set* as a presheaf on $\mathbf{\Delta}$, i.e. a (contravariant) functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ from the opposite category of $\mathbf{\Delta}$ to the category of sets. Taking a *morphism of simplicial sets* (also called a *simplicial map*) to be a natural transformation between two such functors, we obtain a category:

Definition 2.2. The *category of simplicial sets* $\widehat{\mathbf{\Delta}}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.

Given a simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and a number n, we denote $X([n])$ by X_n, which we interpret as the set of all n-simplices in the simplicial set, also called the *simplices of degree n*. By the Yoneda embedding, there is a simplicial set Δ^n which represents the abstract n-simplex from the simplex category $\mathbf{\Delta}$. By the Yoneda lemma, we can conceptually identify elements of X_n with simplicial maps $\Delta^n \to X$. Thus we can apply our terminology for the simplex $[n]$ also to the object Δ^n, as well as to any simplex in X. In particular, for any $k \in [n]$ we may speak of "the vertex with index k", or simply of "vertex k" in all three cases. We write the image of $\partial_j\colon [n] \to [n+1]$ under the functor X as $d_j\colon X_{n+1} \to X_n$, and call it a *face map*. It is a function that we interpret as taking a simplex in the simplicial set and returning the face of the simplex opposite the vertex with index j. Two simplices are "glued together" if they share a face in this manner. We write the image of $\sigma_l\colon [n+1] \to [n]$ as $s_l\colon X_n \to X_{n+1}$, and call it a *degeneracy map*. The maps d_j and s_l satisfy the *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ s_k s_l &= s_l s_{k-1} && \text{if } k > l \end{aligned}$$

$$d_j s_l = \begin{cases} s_{l-1} d_j & \text{if } j < l \\ \text{the identity map} & \text{if } j \in \{l, l+1\} \\ s_l d_{j-1} & \text{if } j > l+1 \end{cases}$$

where we omitted the composition symbol "$\circ$", writing e.g. $d_j s_l$ for $d_j \circ s_l$. We will also omit brackets, so that $d_j s_l x$ stands for $d_j(s_l(x))$. We call a simplex *degenerate* if it is in the image of some degeneracy map. From the last simplicial identity, we see that a degenerate simplex $s_l x$ has two identical faces $d_l s_l x = d_{l+1} s_l x = x$, while its other faces are degenerate themselves. In particular, vertices l and $l+1$ of $s_l x$ must be identical. Be mindful that an arbitrary simplex with two identical faces need not be a degenerate simplex. This is because two simplices can share all their faces without being equal. For example, a simplicial set might contain multiple 1-simplices, i.e. "edges" or (since we have defined simplices with a notion of orientation) "arrows", having the same, single vertex as their endpoints, only one of which can be degenerate.

Under the Yoneda embedding, to each map $[m] \to [n]$ in the simplex category $\mathbf{\Delta}$ corresponds a simplicial map $\Delta^m \to \Delta^n$. We we will denote the representatives of the maps $\partial_j\colon [n] \to [n+1]$ and $\sigma_l\colon [n+1] \to [n]$ again by $d_j\colon \Delta^n \to \Delta^{n+1}$ and $s_l\colon \Delta^{n+1} \to \Delta^n$. The initial confusion has a payoff: when using the Yoneda lemma to alternatingly represent an n-simplex x of a simplicial set X as an element $x \in X_n$ and as a simplicial map $x\colon \Delta^n \to X$, its faces are accordingly denoted by $d_j x \in X_{n-1}$ and $x \circ d_j\colon \Delta^{n-1} \to X$, and its degenerate versions by $s_l x$ and $x \circ s_l$.

The category of simplicial sets, being a presheaf category, is a topos. For us this means that we have claim to limits and co-limits, with products, pullbacks and pushouts as special cases. These are all constructed "object-wise/pointwise" in $\mathbf{\Delta}$. For example, the product of two simplicial sets $X \times Y$ is given by $(X \times Y)_n = X_n \times Y_n$ for every $n \in \mathbb{N}$. The simplicial set Δ^0, consisting of only a single vertex, is the terminal object of the category; for any simplicial set X, there exists exactly one map $X \to \Delta^0$, sending every vertex to the unique vertex in Δ^0 and every n-simplex to the unique, fully degenerate n-simplex in Δ^0.

In the introduction, we motivated the study of simplicial sets by claiming that they approximate topological spaces. Informally, we can recover a topological space from a simplicial set by replacing each abstract simplex by a topological simplex, and gluing them together appropriately. This operation takes the form of a functor from the category of simplicial sets to the category of topological spaces, and comes with a right adjoint, establishing a strong similarity between both categories — at least regarding homotopy.

However, simplicial sets, having the potential to form a foundation of mathematics, come in many forms, and they do not always have a relevant geometric interpretation. The following construction shows how to turn an arbitrary category into a simplicial set.

Definition 2.3. Given a category $\mathcal{C}$, we define the *nerve* $N(\mathcal{C})$ of $\mathcal{C}$ as follows. Regarding the ordered set $[n]$ as a category, for every n we let $N(\mathcal{C})_n$ be the set of all functors $[n] \to \mathcal{C}$. Each map $[m] \to [n]$ from the simplex category induces a function $N(\mathcal{C})_n \to N(\mathcal{C})_m$ by precomposition. These data make up a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, i.e. a simplicial set which we denote by $N(\mathcal{C})$.

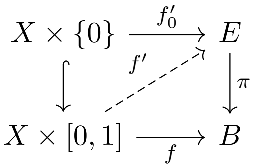
We can think of the category $[n]$ as a string of $n+1$ objects connected by n arrows. A functor $[n] \to \mathcal{C}$ highlights a string of the same shape in the category $\mathcal{C}$. So a vertex of $N(\mathcal{C})$, i.e. an element of $N(\mathcal{C})_0$, is essentially an object of $\mathcal{C}$, while an arrow in $N(\mathcal{C})$ is a morphism of $\mathcal{C}$. A degeneracy map $s_l \colon N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ takes a string in $\mathcal{C}$ and inserts an identity morphism into the string, duplicating the l-th object. For $j \notin \{0, n\}$, the face map $d_j \colon N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ takes a string and removes the j-th object in the string, composing the two morphisms that shared this object. If $j = 0$ (if $j = n$), then d_j removes the first (last) object of the string, together with the single arrow starting (ending) at this object.

2.2 Kan fibrations

The definition of a Kan fibration is quite technical, and we will first try to motivate it through a comparison with fiber bundles in topology. A useful property of a fiber bundle $\pi \colon E \to B$ is that, given a path $f \colon [0,1] \to B$ in the base space and a point f_0' in the fiber $\pi^{-1}(f(0))$, we can *lift* f to a path $f' \colon [0,1] \to E$ in the total space that starts at f_0' and gets projected back to f by π. Trying to find f' is called a *lifting problem*, and it can be summarized as finding a diagonal for a commutative square

$$\begin{array}{ccc} \{0\} & \xrightarrow{f_0'} & E \\ \big\downarrow & \overset{f'}{\nearrow} & \big\downarrow \pi \\ [0,1] & \xrightarrow[f]{} & B \end{array}$$

that makes the whole diagram commute. If π is a fiber bundle, then a solution f' exists for every choice of f and f_0' that makes the square commute: we say that π has the *right lifting property* with respect to the map on the left. We may think of f here as a homotopy between the points $f(0)\colon \{*\} \to B$ and $f(1)\colon \{*\} \to B$. By generalization, we can also try to lift homotopies of arbitrary maps $X \to B$:



Ignoring certain restrictions, this problem can be solved for any X, f_0' and f, as long as π is a fiber bundle. Now we take this as the defining property of π: a fibration is a map that has the right lifting property with respect to inclusions $X \times \{0\} \hookrightarrow X \times [0,1]$ for every space X.[1] In the background of this thesis lies an approach to homotopy theory that lends a foundational role to fibrations, called *Quillen model structure*, although we will not delve into this.

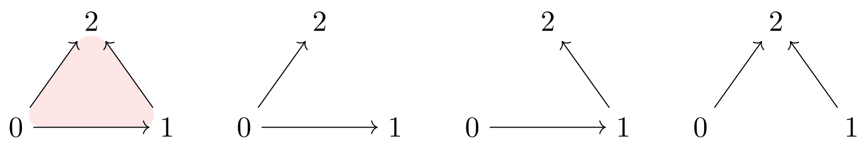
If we replace the inclusion $\{0\} \hookrightarrow [0,1]$ by the map $d_1\colon \Delta^0 \hookrightarrow \Delta^1$ (the face opposite vertex 1 is vertex 0), then this definition of fibration readily carries over to simplicial sets. However, it turns out that it is better to define a fibration of simplicial sets as possessing the right lifting property with respect to a larger class of morphisms, generated by the *horn inclusions*.

Definition 2.4. For any $n \geq 1$ and $k \leq n$, the *horn* Λ^n_k is the simplicial set given by

$$(\Lambda^n_k)_m = \{\alpha \in \mathrm{Hom}([m],[n]) \mid [n] \not\subseteq \alpha([m]) \cup \{k\}\}$$

for every m.

Here $\mathrm{Hom}([m],[n])$ is the set of all morphisms $[m] \to [n]$ in the simplex category $\mathbf{\Delta}$. Since Δ^n is defined by $(\Delta^n)_m = \mathrm{Hom}([m],[n])$, the horn Λ^n_k can be thought of as a subobject of Δ^n, and we write $\Lambda^n_k \hookrightarrow \Delta^n$ for its inclusion. Λ^n_k is the boundary of the simplex Δ^n with the face opposite vertex k removed. As an illustration, the simplicial set Δ^2 and each of its horns Λ^2_0, Λ^2_1 and Λ^2_2 can be drawn respectively as follows:



[1]This notion of fibration is the *Hurewicz fibration*. In practice, it is more useful to only allow X to be a CW-complex, defining a *Serre fibration*. We will not distinguish the two variants in the main text.

Given a simplicial set X, we will refer to a morphism $\phi\colon \Lambda^n_k \to X$ as a *horn in* X, similar to how we think of a map $\Delta^n \to X$ as a simplex in X. Since Λ^n_k is a subobject of Δ^n, we can use the notation and terminology for simplices also for horns, such as "the vertex with index i". The horn Λ^n_k consists of n of the faces of the simplex Δ^n, so a map $\phi\colon \Lambda^n_k \to X$ defines a set of $(n-1)$-simplices of X, which we write as $\{d_i\phi\}_{i\neq k}$. Conversely, for any $n \geq 1$ and $k \leq n$, an arbitrary set of $(n-1)$-simplices $\{\phi_i\}_{i\neq k}$ satisfying $d_i\phi_j = d_{j-1}\phi_i$ whenever $i, j \in [n] \setminus \{k\}$ and $i < j$ determines a horn $\phi\colon \Lambda^n_k \to X$ in X (the condition $d_i\phi_j = d_{j-1}\phi_i$ can be recognized as the first simplicial identity). Returning to the topic of fibrations, we make the following definition.

Definition 2.5. A morphism of simplicial sets $f\colon X \to Y$ is a *Kan fibration* if it has the right lifting property with respect to all horn inclusions.

We shall also more succinctly say that "f has lifts against horn inclusions". How does this relate to the fibrations along which homotopies can be lifted? The answer is that a Kan fibration automatically has the right lifting property with respect to a much larger class of "unproblematic inclusions":

Definition 2.6. A morphism is *anodyne* if it is an element of the smallest set that contains the horn inclusions and is closed under composition, pushouts and retractions.

The definition of a retraction of a morphism is irrelevant for our purposes. Although it is nontrivial to prove, the inclusions of the form $X \times \Delta^0 \hookrightarrow X \times \Delta^1$ are anodyne (see Proposition 3.1.2 of [Cis19]), so that a Kan fibration is an instance of the kind of fibrations we defined earlier. As an exercise, we will prove one part of the claim that a Kan fibration has lifts against all anodyne morphisms:

Proposition 2.7. *If* $p\colon X \to Y$ *has the right lifting property with respect to* $i\colon A \to B$*, then it has the right lifting property with respect to any pushout of* i*.*

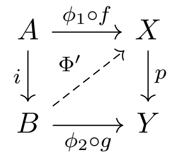
Proof. Let i' be the pushout of i along a map $f\colon A \to A'$, i.e. the right map in the pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle i'} \\ B & \xrightarrow[g]{} & B' \end{array}$$

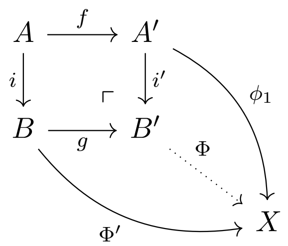
We need to show that any lifting problem of the form

$$\begin{array}{ccc} A' & \xrightarrow{\phi_1} & X \\ {\scriptstyle i'}\big\downarrow & \overset{\Phi}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow[\phi_2]{} & Y \end{array} \tag{2.1}$$

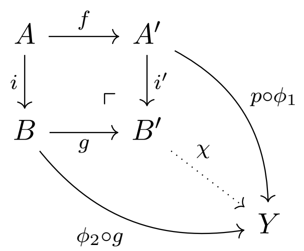
admits a solution Φ. Since p has the right lifting property with respect to i, we can find a solution Φ' for the lifting problem



Using the universal property of the pushout square, we can find a map Φ that makes the following diagram commute:



We need to check that this choice of Φ solves the lifting problem 2.1. By definition, $\Phi \circ i' = \phi_1$ so that the upper triangle commutes. To prove that $p \circ \Phi = \phi_2$, consider the following diagram:

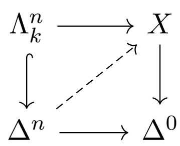


This diagram commutes when we set $\chi = \phi_2$ and also when we set $\chi = p \circ \Phi$. But by the universal property of the pushout square, the map χ must be unique, so $p \circ \Phi = \phi_2$, making the lower triangle of diagram 2.1 commute. Therefore we have found a solution to the lifting problem, completing the proof. □

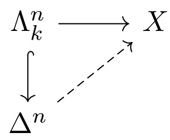
Together with the Kan fibration, the *Kan complex* is the most important notion in traditional simplicial homotopy theory.

Definition 2.8. A simplicial set X is a *Kan complex* if the unique map from X to the terminal object is a Kan fibration.

In a lifting problem of the form



the map on the bottom is trivial and the square is commutative automatically, so this problem is equivalent to the problem given by the diagram



which consequently always has a solution, provided that X is a Kan complex. We say that every horn in a Kan complex can be *filled* (extended to a simplex), and that a solution to such a *filling problem* is a *filler* for the horn. We will often use this alternative characterization of Kan complexes instead.

2.3 Fibration as structure

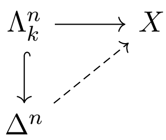
Simplicial sets can be made into a model of homotopy type theory, in which the Kan fibrations play the role of dependent types [KL12]. However, it has been shown by [BCP15] that a constructive proof of this fact is impossible. This is disappointing, because homotopy type theory has specifically been put forward as a possible foundation of constructive mathematics, under the name *Univalent Foundations*. To make a model out of simplicial sets that does facilitate a constructive proof, we can try to replace the interpretation of dependent types by a different kind of fibration. The issue with ordinary Kan fibrations is that they only tell us that there exists *some* solution to a given lifting problem, but they do not point out one *particular* solution. A generic lifting problem has multiple solutions, so in case we want an explicit solution, we would have to use the axiom of choice to pick one ourselves. Thus arises the idea to define a fibration as having extra structure that chooses the solutions for us:

Definition 2.9. An *algebraic Kan fibration* is a Kan fibration together with a *lifting function* that assigns a particular solution to every lifting problem against a horn inclusion.

This definition does not choose lifts against anodyne morphisms other than the horn inclusions. We obtain a lift against an arbitrary anodyne morphism by (roughly speaking) decomposing it into horn inclusions. In most cases, this can be done in various inequivalent ways. We can enforce some compatibility among these different ways of lifting against an anodyne map by imposing the *uniformity condition* on the lifting function of an algebraic Kan fibration. It turns out that the compatibility gained in this way allows for at least some progress in constructing a model of homotopy type theory based on simplicial sets, in which dependent types are algebraic Kan fibrations satisfying the uniformity condition. It is this notion of fibration, which we will call *effective Kan fibration*, that is the object of our study. So let us find out what the uniformity condition is. It relies entirely on the following fact:

Proposition 2.10 (Proposition C.1. of [vdBF22])**.** *A horn in a simplicial set has at most one degenerate filler.*

By a degenerate filler we mean a degenerate simplex that solves the filling problem

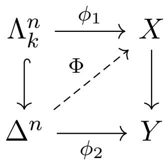


We define a *degenerate horn* as a horn in a simplicial set that has a degenerate filler. Of course, the proposition implies that a lifting problem against a horn inclusion also has at most one degenerate solution, and we call it a *degenerate lifting problem* if the degenerate solution exists. A special case of the uniformity condition is the following:

Definition 2.11. An algebraic Kan fibration is *degenerate-preferring* if its lifting function assigns the unique degenerate solution to every degenerate lifting problem.

Note that *any* Kan fibration can be given the structure of a degenerate-preferring algebraic Kan fibration, by assigning the degenerate solution to a lifting problem if it is degenerate, and an arbitrary solution if it is not. The point is that this cannot be done constructively, because we need the principle of excluded middle to make the case distinction, and the axiom of choice to pick the arbitrary solutions.

Although the definition is elegant, it is more restrictive than necessary. We motivated the uniformity condition as a means to harmonize the different ways of solving lifting problems. Accordingly, it does not concern the chosen solutions per se, but only the way in which they relate to each other. Therefore, consider how an arbitrary lift, as in the following diagram, can give rise to a degenerate lifting problem:



Fix any $l \leq n$, and choose a number k^* satisfying $\sigma_l(k^*) = k$ (recall the non-injective map σ_l from the simplex category $\mathbf{\Delta}$). The meaning of k^* is that it is the index of the vertex in Δ^{n+1} that is sent to vertex k of Δ^n under the degeneracy map s_l; when $l = k$, there are two such vertices. We define a degenerate simplex $\psi_2 = \phi_2 \circ s_l$ and a degenerate horn $\psi_1 = \Phi \circ s_l \circ c$, where c is the horn inclusion $\Lambda^{n+1}_{k^*} \hookrightarrow \Delta^{n+1}$. This horn is missing the face opposite vertex k^*, and therefore corresponds to the horn ψ_1, which misses the face opposite vertex k. It is degenerate because it is filled by the degenerate simplex $\Phi \circ s_l$, which is also the (unique) degenerate solution to the lifting problem posed by ψ_1 and ψ_2:

$$\begin{array}{ccc} \Lambda^{n+1}_{k^*} & \xrightarrow{\psi_1} & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \xrightarrow[\psi_2]{} & Y \end{array}$$

We refer to this as a degenerate lifting problem *based on the lifting problem for* Φ. Note that it depends both on the *formulation* of that lifting problem, as well as on the fact that Φ was the *chosen* solution. The uniformity condition simply states that the degenerate solution is chosen at least cases like this:

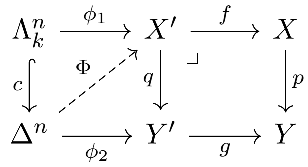
Definition 2.12. An *effective Kan fibration* is an algebraic Kan fibration whose lifting function assigns the degenerate solution to every degenerate lifting problem that is based on some other lifting problem.

This definition, as well as the degenerate-preferring version, was discovered by [Gee23], who calls it a *symmetric* effective Kan fibration. It is a simplified and more restrictive variant of the *effective Kan fibration* introduced by [vdBF22]. Since we will not use their original version, and since we already have more than enough qualifiers to deal with, we have dropped the adjective "symmetric". Degenerate-preferring *algebraic* Kan fibrations are always effective Kan fibrations, so we will occasionally refer to them as degenerate-preferring *effective* Kan fibrations. A *(degenerate-preferring) algebraic/effective Kan complex* is defined by analogy with the ordinary Kan complex in an obvious fashion.

This completes the exposition of the most important concepts for this thesis. Additional definitions will be given in the chapters for which they are relevant. We end this preliminary chapter by proving the following basic property of fibrations, in order to have some practice with the new definitions.

Proposition 2.13. *The class of (degenerate-preferring) effective Kan fibrations is closed under pullback.*

Proof. In a sense, this proposition is dual to Proposition 2.7, and the proof is reminiscent of it. We first prove the variant without degenerate-preference. So given an effective Kan fibration, we need to show that its pullback along an arbitrary map is a Kan fibration. Moreover, we need to assign it the structure of an effective Kan fibration by constructive means. Therefore, consider a commutative diagram



in which the right square is a pullback diagram for an effective Kan fibration p, and the left square is a lifting problem for q. The outer square poses a lifting problem for p; call the chosen solution $\Phi' \colon \Delta^n \to X$. Using the universal property of the pullback square, find Φ from the unique factorization $\phi_2 = q \circ \Phi$ and $\Phi' = f \circ \Phi$. We must have $\Phi \circ c = \phi_1$, since $f \circ \phi_1 = f \circ \Phi \circ c$ and $q \circ \phi_1 = q \circ \Phi \circ c$, while this factorization must be unique. Hence Φ solves the original lifting problem.

To prove that this choice of lifts satisfies the uniformity condition, consider an arbitrary degenerate lifting problem based on the lifting problem for Φ, given by $\psi_1 = \Phi \circ s_l \circ c$ and $\psi_2 = \phi_2 \circ s_l$, where $c \colon \Lambda^{n+1}_{k^*} \to \Delta^{n+1}$ is a horn inclusion. Let Ψ be the solution to this problem obtained in the same way as Φ. We need to show that this chosen solution Ψ is equal to the degenerate solution $\Phi \circ s_l$. The lifting problem posed by $f \circ \psi_1$ and $g \circ \psi_2$ is a degenerate lifting problem based on the lifting problem for Φ', so its chosen solution Ψ' is the degenerate solution $\Phi' \circ s_l$, since we assumed that p satisfies the uniformity condition. The map Ψ was defined from the factorization $\psi_2 = q \circ \Psi$ and $\Psi' = f \circ \Psi$. But we also have the factorization $\psi_2 = q \circ \Phi \circ s_l$ and $\Psi' = f \circ \Phi \circ s_l$. By the universal property of pullback squares, this factorization is unique, so it must be that $\Psi = \Phi \circ s_l$, which is what we wanted to show. Since the lifting problem for Φ was arbitrary, we conclude that the given lifting procedure satisfies the uniformity condition. We have therefore shown that any pullback of any effective Kan fibration can be made into an effective Kan fibration by constructive means, completing the proof.

To prove the degenerate-preferring variant of the proposition, we only have to slightly modify the second half of the proof: we assume that Ψ came from an arbitrary degenerate lifting problem, but now p is degenerate-preferring, so we get a degenerate solution for Ψ' regardless. The rest of the proof that Ψ is degenerate is the same, so that q chooses the degenerate solution whenever possible. □

Chapter 3

Simplicial groupoids

Simplicial groupoids are the simplicial equivalent of topological groupoids. They turn up in simplicial homotopy theory in various places, and Kan fibrations can be constructed from simplicial groupoids in several ways. For example, a classic result is that a simplicial group is always a Kan complex. Some of these constructions are given by [vdBM18] and subsequently used to generate *univalent* Kan fibrations; *univalence* is an important concept in homotopy type theory. We will revisit those constructions and prove that, under some additional assumptions, they also define effective Kan fibrations. More specifically, Theorem 3.8 is the original result from [vdBM18], consisting of four parts, and the "effective" versions of the first three parts are Theorem 3.9, 3.11 and 3.22 respectively, each of which is covered in a separate section. We first introduce simplicial groupoids and some definitions associated with them.

3.1 Description

A groupoid is a category in which every morphism has a two-sided inverse. The prime example is the fundamental groupoid of a topological space. The objects of the category are the points of the topological space and the morphisms are the homotopy equivalence classes of paths between two points. The composition of two such equivalence classes is obtained by concatenating their paths, and the inverse of an equivalence class by reversing the direction of all of its paths. A groupoid $\mathbb{G}$ consists of the following:

- a set of objects $\mathrm{ob}(\mathbb{G})$;
- a set of morphisms $\mathrm{ar}(\mathbb{G})$;
- source and target functions $s, t\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G})$;
- a function assigning identities $\mathrm{id}_{(-)}\colon \mathrm{ob}(\mathbb{G}) \to \mathrm{ar}(\mathbb{G})$;

- a function assigning inverses $(-)^{-1}\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ar}(\mathbb{G})$;
- a function for composition $(- \circ -)\colon \mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} \mathrm{ar}(\mathbb{G}) \to \mathrm{ar}(\mathbb{G})$;

where $\mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} \mathrm{ar}(\mathbb{G})$ contains all composable pairs of arrows, as in the following pullback square:

$$\begin{array}{ccc} \mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} \mathrm{ar}(\mathbb{G}) & \longrightarrow & \mathrm{ar}(\mathbb{G}) \\ \big\downarrow & \lrcorner & \big\downarrow s \\ \mathrm{ar}(\mathbb{G}) & \xrightarrow[t]{} & \mathrm{ob}(\mathbb{G}) \end{array}$$

The maps need to satisfy familiar relations (expressible by commuting diagrams) that enforce the properties of associativity, inverses and the identity. If, in the list above, we replace each instance of "set" by "object" and each instance of "function" by "morphism", we can evaluate the definition in *any* category that has the required pullbacks (rather than just the category **Set**), giving rise to the notion of an *internal groupoid.* In the category of topological spaces, we obtain topological groupoids, with topological groups and Lie groups as special instances. We will study the simplicial equivalents of these objects:

Definition 3.1. A *simplicial groupoid* is an internal groupoid in $\widehat{\mathbf{\Delta}}$.

In essence, a simplicial groupoid $\mathbb{G}$ provides an ordinary groupoid $\mathbb{G}_n$ for every n (similar to how a simplicial set X has a set X_n for every n). The groupoid morphisms being natural transformations means that they commute with the face and degeneracy maps. For example, $s_l \mathrm{id}_c = \mathrm{id}_{s_l c}$ for any $c \in \mathbb{G}_n$. By an object (arrow) of $\mathbb{G}$ we mean a simplex of $\mathrm{ob}(\mathbb{G})$ ($\mathrm{ar}(\mathbb{G})$), which can be incarnated either as an element of some $\mathrm{ob}(\mathbb{G})_n$ or as a simplicial map $\Delta^n \to \mathrm{ob}(\mathbb{G})$, according to the Yoneda lemma. For any simplicial groupoid, we continue using the notation from the list above for the source, target, identity, inverse and composition maps. As we will now see, some standard definitions regarding (topological) group(oid)s carry over to simplicial groupoids in a straightforward fashion.

Definition 3.2. Suppose we have a (simplicial) groupoid $\mathbb{G}$ and a map of (simplicial) sets $\pi\colon X \to \mathrm{ob}(\mathbb{G})$. First, write $\mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} X$ for the fiber product of the source map s with π. Then an *action of* $\mathbb{G}$ *on* π is a (simplicial) map $(- \cdot -)\colon \mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} X \to X$ such that $\pi(g \cdot x) = t(g)$, $h \cdot (g \cdot x) = (h \circ g) \cdot x$ and $\mathrm{id}_{\pi(x)} \cdot x = x$ for all compatible $x \in X$ and $g, h \in \mathrm{ar}(\mathbb{G})$.

Remark 3.3. Consider what an groupoid action does in **Set**. The map π provides a partition of X, namely by the preimages of the objects in $\mathbb{G}$. Each arrow of $\mathbb{G}$ becomes a function between the preimages of its source and target objects, and this representation preserves composition, identities and inverses. Accordingly, $\mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} X$ keeps track of which elements of X can be acted on by which arrows of $\mathbb{G}$ — it contains precisely the pairs $\langle g, x\rangle$ such that $x \in \pi^{-1}(s(g))$.

Remark 3.4. When working out the simplicial variants of this definition and the next one, one can interpret an element "$x \in X$" as a simplex $x\colon \Delta^n \to X$, and function evaluations "$f(x)$" as the composition $f \circ x$. The postulated relations can alternatively (but equivalently) be evaluated separately for each groupoid $\mathbb{G}_n$ in the simplicial groupoid; for example, $\pi_{[n]}(g \cdot x) = t_{[n]}(g)$ for every $n \in \mathbb{N}$, $x \in X_n$ and $g \in \mathrm{ar}(\mathbb{G})_n$.

Definition 3.5. Given a (simplicial) groupoid $\mathbb{G}$ and an action on a (simplicial) map $\pi\colon X \to \mathrm{ob}(\mathbb{G})$, we define the *(simplicial) action-groupoid* $X_{\mathbb{G}}$ as follows. Its simplicial set of objects is $\mathrm{ob}(X_{\mathbb{G}}) = X$ and its simplicial set of arrows is the pullback $\mathrm{ar}(X_{\mathbb{G}}) = \mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} X$ of the source map of $\mathbb{G}$ along π. The source map of $X_{\mathbb{G}}$ is the second projector of this product and its target map is the action of $\mathbb{G}$ on π. The remaining maps are given by $\mathrm{id}_x = \langle \mathrm{id}_{\pi(x)}, x\rangle$, $\langle g, x\rangle^{-1} = \langle g^{-1}, g \cdot x\rangle$ and $\langle h, g \cdot x\rangle \circ \langle g, x\rangle = \langle h \circ g, x\rangle$. It can indeed be verified that this constitutes a (simplicial) groupoid.

Remark 3.6. If two arrows $\langle g, x\rangle, \langle h, y\rangle \in \mathrm{ar}(X_{\mathbb{G}})$ are to be composable, we must have $y = s(\langle h, y\rangle) = t(\langle g, x\rangle) = g \cdot x$. This is why we wrote $g \cdot x$ instead of y in the composition law.

Since a groupoid $\mathbb{G}$ is a category, we can take its nerve (Definition 2.3) to get a simplicial set $N(\mathbb{G})$. An n-simplex of $N(\mathbb{G})$ is a string of composable arrows of $\mathbb{G}$:

$$c_0 \xrightarrow{h_1} c_1 \xrightarrow{h_2} \dots \xrightarrow{h_n} c_n$$

We would like to have a similar construction for a *simplicial* groupoid $\mathbb{G}$. Since taking the nerve turns a set-like object (a groupoid) into a simplicial set, we expect it to turn a simplicial object (a simplicial groupoid) into a *bisimplicial* set: a functor $N(\mathbb{G})\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. Its sets are now indexed by two natural numbers, and the set $N(\mathbb{G})_{(n,m)}$ contains all strings of n composable arrows in the groupoid $\mathbb{G}_m$. We can condense a bisimplicial set X into a simplicial set $\delta^* X$ by only looking at the sets along the diagonal of the double index: $(\delta^* X)_n = X_{(n,n)}$. In the case of the nerve of a simplicial groupoid $\mathbb{G}$, a degeneracy map of $\delta^* N(\mathbb{G})$ combines the effect of degeneracy in a nerve (repeating one of the objects) and degeneracy in a simplicial groupoid (making each object and arrow degenerate individually), and similarly for the face maps. So a face map d_j and a degeneracy map s_l respectively take a string of arrows, represented as above, to the following strings:

$$d_j c_0 \xrightarrow{d_j h_1} \dots \xrightarrow{d_j h_{j-1}} d_j c_{j-1} \xrightarrow{d_j(h_{j+1} \circ h_j)} d_j c_{j+1} \xrightarrow{d_j h_{j+2}} \dots \xrightarrow{d_j h_n} d_j c_n$$

$$s_l c_0 \xrightarrow{s_l h_1} \dots \xrightarrow{s_l h_l} s_l c_l \xrightarrow{\mathrm{id}_{s_l c_l}} s_l c_l \xrightarrow{s_l h_{l+1}} \dots \xrightarrow{s_l h_n} s_l c_n$$

To get the face d_j of a string in case $j \in \{0, n\}$, we need to remove the first or last arrow, instead of composing it. We have thus found a way to make a simplicial groupoid into a simplicial set:

Definition 3.7. The *classifying space* $B\mathbb{G}$ of a simplicial groupoid $\mathbb{G}$ is the simplicial set $\delta^* N(\mathbb{G})$.

We will not go into the significance of a classifying space, but just note that it can also be defined for a topological group G, for which it has purpose of establishing a bijection between principal G-bundles over a topological space X (up to isomorphism) and continuous functions $X \to BG$ (up to homotopy equivalence).

It follows immediately from the definition that $(B\mathbb{G})_n = N(\mathbb{G})_{(n,n)}$ for all n. We can also consider the classifying space of an action-groupoid, $BX_{\mathbb{G}}$. A simplex of $BX_{\mathbb{G}}$ is a string of arrows $\langle f_m, x_m \rangle$ with all the f_m and x_m simplices of $\mathrm{ar}(\mathbb{G})$ and X respectively. If we project out each first component, we get a string of arrows f_m, which is just a simplex of $B\mathbb{G}$. We can thus associate a morphism of classifying spaces $BX_{\mathbb{G}} \to B\mathbb{G}$ to every action of a simplicial groupoid.

With these definitions in hand, we can finally restate Theorem 3.1 of [vdBM18].

Theorem 3.8. *Let a simplicial groupoid* $\mathbb{G}$ *be given, together with an action on a map* $X \to \mathrm{ob}(\mathbb{G})$*.*

(i) If $\mathrm{ob}(\mathbb{G})$ *is a Kan complex and* $s\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G})$ *is a Kan fibration, then* $B\mathbb{G}$ *is a Kan complex.*

(ii) If $\pi\colon X \to \mathrm{ob}(\mathbb{G})$ *is a Kan fibration, then so is* $BX_{\mathbb{G}} \to B\mathbb{G}$*.*

(iii) If the pair of source and target maps $(s,t)\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G}) \times \mathrm{ob}(\mathbb{G})$ *is surjective, then it is a Kan fibration.*

(iv) If $(s,t)\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G}) \times \mathrm{ob}(\mathbb{G})$ *is a Kan fibration, then for any vertex* c *of* $\mathrm{ob}(\mathbb{G})$*, there is a natural weak equivalence* $\mathbb{G}(c,c) \to \Omega(B\mathbb{G}, c)$*.*

Note that we have not covered some of the definitions required to understand part *(iv)*. The reason is that it is essentially a corollary to part *(ii)* of the theorem, so that a constructive proof of part *(ii)* can immediately be transformed into a constructive proof for part *(iv)*, and therefore we will not cover it separately — we simply included it for completeness.

Thus our burden is to state and prove variants of parts *(i)* through *(iii)* that use effective Kan fibrations. Of course, we will use constructive proof methods only. We will treat each part in a separate section, and try to make clear in which ways the proofs presented here differ from those in [vdBM18].

3.2 Part *(i)*

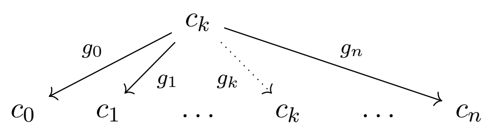
We will first prove a modified version of the first part of Theorem 3.8, and only afterwards discuss the choices we have made in its formulation.

Theorem 3.9. *Let* $\mathbb{G}$ *be a simplicial groupoid such that* $\mathrm{ob}(\mathbb{G})$ *is an effective Kan complex and* $s\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G})$ *is a degenerate-preferring algebraic Kan fibration. Suppose moreover that* s *chooses an identity arrow as the solution to lifting problems whenever this is possible. Then* $B\mathbb{G}$ *can be given the structure of an effective Kan complex by constructive means.*

Before we dive into the proof, we need to introduce a trick. If we have a string of n arrows in $\mathbb{G}_m$, called χ, as follows:

$$c_0 \xrightarrow{h_1} c_1 \xrightarrow{h_2} \dots \xrightarrow{h_n} c_n$$

then we can use inverses to alternatively represent it as a tree-like structure:



Here k can be chosen arbitrarily and the g_i are suitable compositions of the h_i or their inverses. Specifically:

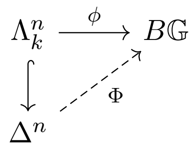
$$g_i = \begin{cases} (h_k \circ h_{k-1} \circ \dots \circ h_i)^{-1} & \text{if } i < k \\ \mathrm{id}_{c_k} & \text{if } i = k \\ h_i \circ h_{i-1} \circ \dots \circ h_{k+1} & \text{if } i > k \end{cases}$$

The arrow g_k has been distinguished in the diagram by a dotted arrow because even though it is always the identity arrow id_{c_k} and is in that sense redundant, it does record the position of c_k among the other objects and therefore tells us how the diagram should be put back together to form the string it derives from. We shall refer to this diagram as the *tree-representation* of χ with the k-th object of χ as its *trunk*. Instead of drawing such a diagram every time, we can also write it as $(k; g_0, \dots, g_n)$, where the number k on the left designates which arrow in the list is the dotted one. The objects c_i are not written explicitly but are still present as the sources and targets of the arrows. The advantage of this tree-representation is the simple effect of the face and degeneracy maps d_j and s_l of $B\mathbb{G}$:

$$\begin{aligned} d_j\chi &= (\sigma_j(k); d_jg_0, \dots, d_jg_{j-1}, \widehat{d_jg_j}, d_jg_{j+1}, \dots, d_jg_n) \\ s_l\chi &= (\partial_l(k); s_lg_0, \dots, s_lg_{l-1}, s_lg_l, s_lg_l, s_lg_{l+1}, \dots, s_lg_n) \end{aligned}$$

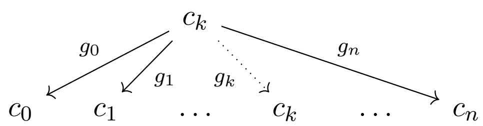
Here σ_j and ∂_l are the maps from the simplex category $\mathbf{\Delta}$, and the hat on $\widehat{d_jg_j}$ signifies that this arrow has been removed and does not actually appear in the list. Compare these tree-representations to the strings appearing just before Definition 3.7. In the string-representation of the face $d_j\chi$, we had to take a composition $h_{j+1} \circ h_j$. But since, for example, g_n is the composition of h_{k+1} through h_n anyway, it does not matter if two of its constituents are already composed "beforehand". Similarly, the identity arrow appearing in the degenerate string is assimilated into one of the $s_l g_l$ appearing in the tree-representation. Moreover, this representation of $d_j\chi$ is also valid when $j \in \{0, n\}$. We just need to be careful with the case $j = k$, because then the above representation is invalid (since in general the highlighted arrow, with index $\sigma_j(k)$, is then no longer the identity $\mathrm{id}_{d_j c_k}$).

Proof of Theorem 3.9. We are showing that $B\mathbb{G}$ is an effective Kan complex. This means that we need to solve filling problems



in such a way that the uniformity condition is satisfied. We first reproduce the filling procedure by [vdBM18], using a different notation but extending it only in one place to make it constructive. We then show that this procedure satisfies the uniformity condition.

Therefore consider the filling problem above, with ϕ now fixed. By the Yoneda lemma, a morphism $\Phi\colon \Delta^n \to B\mathbb{G}$ corresponds to an element of the set $(B\mathbb{G})_n$, i.e. a string of n composable arrows in the groupoid $\mathbb{G}_n$, which we represent as

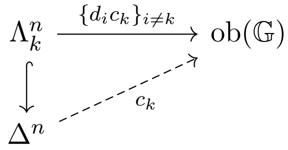


(3.1)

The horn ϕ captures all but one of the faces of this string, which we can represent suggestively as the trees given by

$$d_i\phi = (\sigma_i(k); d_i g_0, \dots, \widehat{d_i g_i}, \dots, d_i g_n)$$

for $i \neq k$. Note the abusive notation (which we will employ often): we write e.g. $d_i g_0$ even though g_0 has not been defined yet — they are instead provided by the horn ϕ. Our task is to extend these faces to a diagram 3.1, i.e. to construct all the g_j. Since $\mathrm{ob}(\mathbb{G})$ is an effective Kan complex, we obtain c_k by solving the filling problem



where each $d_i c_k$ can be found as the source of any arrow $d_i g_j$. Next, we define g_k as the identity arrow on c_k. To construct g_j for a fixed $j \neq k$, note that we are only given $d_i g_j$ for $i \notin \{k, j\}$. The union of these faces does not constitute an entire horn, but rather a simplicial set that we denote by $\Lambda^n_{k,j} \hookrightarrow \Delta^n$, since two faces are missing instead of just one. In the original proof, it is simply noted that this inclusion is anodyne, and therefore poses a lifting problem for the Kan fibration $s\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G})$ that can be solved to obtain g_j:

$$\begin{array}{ccc} \Lambda^n_{k,j} & \xrightarrow{\{d_i g_j\}_{i \neq k,j}} & \mathrm{ar}(\mathbb{G}) \\ \big\downarrow & \overset{g_j}{\nearrow} & \big\downarrow s \\ \Delta^n & \xrightarrow[c_k]{} & \mathrm{ob}(\mathbb{G}) \end{array}$$

For a constructive proof, we need to be more explicit about how we perform this lift. The interior of face j of $\Lambda^n_{k,j}$ is missing, but the remaining boundary of this face is a horn of one degree lower (this can be easily imagined in the case $n = 3$, when $\Lambda^n_{k,j}$ looks like two triangles sharing one edge). We can therefore first fill this horn to extend $\Lambda^n_{k,j}$ to a horn Λ^n_k, which we then extend to a simplex in the usual way. More precisely, we solve the lifting problem above by solving

$$\begin{array}{ccc} \Lambda^n_{k} & \xrightarrow{\{d_i g_j\}_{i \neq k}} & \mathrm{ar}(\mathbb{G}) \\ \big\downarrow & \overset{g_j}{\nearrow} & \big\downarrow s \\ \Delta^n & \xrightarrow[c_k]{} & \mathrm{ob}(\mathbb{G}) \end{array}$$

where we define $d_j g_j$ as the lift

$$\begin{array}{ccc} \Lambda^{n-1}_{\sigma_j(k)} & \xrightarrow{\{d_i d_j g_j\}_{i \neq \sigma_j(k)}} & \mathrm{ar}(\mathbb{G}) \\ \big\downarrow & \overset{d_j g_j}{\nearrow} & \big\downarrow s \\ \Delta^{n-1} & \xrightarrow[d_j c_k]{} & \mathrm{ob}(\mathbb{G}) \end{array}$$

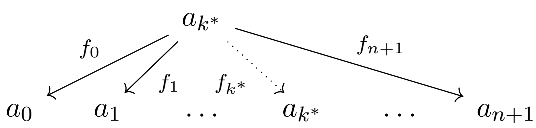
Here the $d_i d_j g_j$ can be drawn from existing faces by using simplicial identities:

$$d_i d_j g_j = \begin{cases} d_{j-1} d_i g_j & \text{if } 0 < i < j \\ d_j d_{i+1} g_j & \text{if } i \geq j \end{cases}$$

If $n = 1$, $d_j g_j$ cannot be defined as a lift, and we instead choose it to be the identity arrow on $d_j c_k$. Having found all the g_j, we can put them together to form the diagram 3.1. This defines a solution Φ to the original filling problem for ϕ, completing the first part of the proof.

We now turn to the uniformity condition of being an effective Kan complex (this aspect is, of course, absent from the original theorem in [vdBM18]).

Let ϕ and Φ be as before, fix l and k^* satisfying $\sigma_l(k^*) = k$, and construct the degenerate horn $\psi\colon \Lambda^{n+1}_{k^*} \to B\mathbb{G}$ as $\Phi \circ s_l \circ c$, with c an appropriate horn inclusion. We represent this horn as $\{d_i s_l \Phi\}_{i \neq k^*}$ — the degree of the faces in this set imply that i ranges from 0 to $n+1$, skipping k^*. Fill ψ using the construction we just defined, and call this solution $\Psi\colon \Delta^{n+1} \to B\mathbb{G}$. Since the filling problem posed by ψ is a degenerate problem based on the filling problem for Φ, the uniformity condition dictates that Ψ should be equal to the unique degenerate solution $s_l\Phi$. So let us try to show that this is indeed the case. We represent Φ as in diagram 3.1 and Ψ as



(3.2)

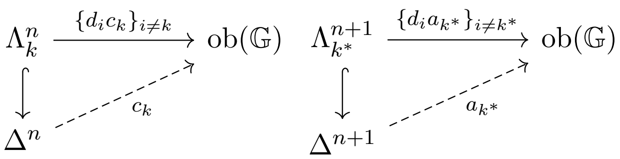
We also want to represent the string $s_l\Phi$ as a tree. The obvious choice is the tree given by

$$(\partial_l(k); s_l g_0, \ldots, s_l g_{l-1}, s_l g_l, s_l g_l, s_l g_{l+1}, \ldots, s_l g_n).$$

However, it is possible that $\partial_l(k) \neq k^*$, namely when $l = k = k^*$ so that $\partial_l(k) = k+1$, and in that case it is not meaningful to compare this tree with the one for Ψ. But this particular case also implies that the trunk of this tree, $s_l c_k$, is the object in the string $s_l\Phi$ that was duplicated by the degeneracy; $s_l c_l$ appears twice in this string, connected by an identity arrow. So we can instead consider the tree with the *first* appearance of $s_l c_l$ as its trunk (the k-th object of the string). Since the new definitions of the branches g_i differ from the old ones only by a composition with an identity arrow, they are identical. We can therefore draw the same tree but highlight arrow $\partial(k) - 1 = k^*$ instead of arrow $\partial_l(k)$. Since in all other cases we *do* have $\partial_l(k) = k^*$, we can in full generality represent $s_l\Phi$ by

$$(k'; g'_0, \ldots, g'_{n+1}) := (k^*; s_l g_0, \ldots, s_l g_{l-1}, s_l g_l, s_l g_l, s_l g_{l+1}, \ldots, s_l g_n)$$

To prove $\Psi = s_l\Phi$, we just need to show that their tree-representations are equal. First, let us prove that the trunks of the trees are equal: $a_{k^*} = s_l c_k$. The relevant definitions are



By construction, the faces of Ψ and $s_l\Phi$ with index $i \neq k^*$ are equal to each other. The horn given by $\{d_i a_{k^*}\}_{i \neq k^*}$ only comes from those faces of Ψ. Using these facts, we can rewrite $\{d_i a_{k^*}\}_{i \neq k^*} = \{d_i s_l c_k\}_{i \neq k^*}$. We see that the problem on the right is a degenerate problem based on the problem on the left. Since $\mathrm{ob}(\mathbb{G})$ is an effective Kan complex, the chosen solution a_{k^*} must be equal to the unique degenerate solution $s_l c_k$.

Next, we want to prove the equality of arrows $f_j = g'_j$ for all j. The two cases $g'_j = s_l g_j$ for $j \leq l$ and $g'_j = s_l g_{j-1}$ for $j \geq l+1$ are similar, and we will only treat the former; so fix $j \leq l$. We make a further distinction of cases:

- $j = k = k^*$.
- $j = k$ and $k^* = k+1$. This is only allowed if $k \geq l$. But we are assuming $j \leq l$, so $j = k = l$.
- $j \neq k$. It follows for similar reasons that $j \neq k^*$ necessarily.

First assume the third case holds. Then the definitions of g_j and f_j are

$$
\begin{array}{ccc}
\Lambda^n_k & \xrightarrow{\{d_i g_j\}_{i \neq k}} & \mathrm{ar}(\mathbb{G}) \\
\big\downarrow & \overset{g_j}{\nearrow} & \big\downarrow s \\
\Delta^n & \xrightarrow[c_k]{} & \mathrm{ob}(\mathbb{G})
\end{array}
\qquad
\begin{array}{ccc}
\Lambda^{n-1}_{\sigma_j(k)} & \xrightarrow{\{d_i d_j g_j\}_{i \neq \sigma_j(k)}} & \mathrm{ar}(\mathbb{G}) \\
\big\downarrow & \overset{d_j g_j}{\nearrow} & \big\downarrow s \\
\Delta^{n-1} & \xrightarrow[d_j c_k]{} & \mathrm{ob}(\mathbb{G})
\end{array}
$$

$$
\begin{array}{ccc}
\Lambda^{n+1}_{k^*} & \xrightarrow{\{d_i f_j\}_{i \neq k^*}} & \mathrm{ar}(\mathbb{G}) \\
\big\downarrow & \overset{f_j}{\nearrow} & \big\downarrow s \\
\Delta^{n+1} & \xrightarrow[s_l c_k]{} & \mathrm{ob}(\mathbb{G})
\end{array}
\qquad
\begin{array}{ccc}
\Lambda^n_{\sigma_j(k^*)} & \xrightarrow{\{d_i d_j f_j\}_{i \neq \sigma_j(k^*)}} & \mathrm{ar}(\mathbb{G}) \\
\big\downarrow & \overset{d_j f_j}{\nearrow} & \big\downarrow s \\
\Delta^n & \xrightarrow[d_j s_l c_k]{} & \mathrm{ob}(\mathbb{G})
\end{array}
$$

Here we have used the equality $a_{k^*} = s_l c_k$ for the arrows on the bottom. As before, since the $d_i f_j$ come from faces shared by Ψ and $s_l \Phi$, we have $d_i f_j = d_i s_l g_j$ for $i \neq k^*, j$. If $n = 1$, we have $d_j g_j = \mathrm{id}_{d_j c_k}$ instead of the lifting problem for $d_j g_j$. We are proving that f_j equals $g'_j = s_l g_j$. By the degenerate-preference of s, this is automatic if $s_l g_j$ is actually a solution for f_j. They agree on their faces $i \neq k^*, j$ by definition, so that we only need to check face j. We first show that the lifting problem for $d_j f_j$ is solved by $d_j s_l g_j$. For $i < j$, $d_i d_j f_j = d_{j-1} d_i f_j = d_{j-1} d_i s_l g_j = d_i d_j s_l g_j$, and similarly for $i \geq j$ (so that we can use $d_i d_j = d_j d_{i+1}$). We see that $d_j s_l g_j$ is indeed a solution. Distinguish two cases:

- $j = l$. Then $\sigma_j(k^*) = \sigma_l(k^*) = k$ and $d_j s_l$ is the identity map, so that the defining lifting problem for $d_j f_j$ is identical to the one for g_j.
- $j < l$. Then $d_j s_l g_j = s_{l-1} d_j g_j$ is degenerate and must be the chosen solution for $d_j f_j$, since s is degenerate-preferring.

In either case, $d_j f_j = d_j s_l g_j$ and hence $f_j = s_l g_j$, being again the degenerate solution to a degenerate lifting problem. Thus we have shown that $f_j = g'_j$ in case $j \neq k$.

Now assume $j = k$ (we are also still assuming $j \leq l$). If $k^* = k$, then both g_j and f_j are defined as identity arrows and not through lifting problems: $g'_j = g'_k = s_l g_k = s_l \mathrm{id}_{c_k} = \mathrm{id}_{s_l c_k} = \mathrm{id}_{a_{k^*}} = f_{k^*} = f_j$. The case $k^* \neq k$ is more complicated: g_j is still defined as the identity arrow on c_k, but f_j is now defined the same way as when we considered the case $j \neq k$. There we found that $d_j s_l g_j$ is a solution to the lifting problem for $d_j f_j$. This time, instead of appealing to degenerate-preference, we use the assumption that s chooses identity arrows as solutions whenever possible. Since $d_j s_l g_j$ is indeed an identity arrow, it is the chosen solution for $d_j f_j$ (of course, the identity on each object in a groupoid is unique). From this it follows that $g'_j = s_l g_j$ is a solution for f_j, and being an identity arrow, it is again the chosen solution.

We conclude that $f_j = g'_j$ whenever $j \leq l$. Since the case of $j \geq l + 1$ is similar, we have thus shown that the tree-representation of Ψ is identical to the one for $s_l \Phi$, meaning that Ψ and $s_l \Phi$ are equal as simplices of $B\mathbb{G}$. This is what we needed to show for the uniformity condition to apply, and therefore we have proved that $B\mathbb{G}$ can be equipped with the structure of an effective Kan complex in a constructive manner. $\square$

Let us discuss the assumptions required by Theorem 3.9 in its present form. The difficulties in the proof stem from the fact that a degeneracy map of $B\mathbb{G}$ combines two kinds of degeneracy: one in the "simplicial direction" (putting s_l before each object and arrow), and the other in the "nerve direction" (inserting an identity in the list of arrows, or duplicating an arrow in the tree-representation). Something similar holds for the face maps of $B\mathbb{G}$. A priori, we might expect that only demanding s to be an effective Kan fibration should be sufficient: when solving lifting problems, the effective structures on $\mathrm{ob}(\mathbb{G})$ and s regulate the first kind of degeneracy, and the second kind of degeneracy is constrained by properties of the nerve (the nerve of an ordinary groupoid is a Kan complex, and it can be given a unique algebraic structure that is necessarily degenerate-preferring). However, the two types of degeneracy (and the two types of face maps) are inextricably mixed, and to deal with this we need the additional assumptions on s.

Nonetheless, the current assumptions can be weakened slightly. When treating the case $k \neq j < l$, we wrote "$d_j s_l g_j = s_{l-1} d_j g_j$ is degenerate and must be the chosen solution for $d_j f_j$, since s is degenerate-preferring". In the cases where $d_j g_j$ is defined through a lifting problem (namely when $n > 1$), we recognize the definition of $d_j f_j$ as a degenerate problem based on this lifting problem, so that s being an effective Kan fibration would have sufficed. We only need unconditional degenerate-preference when filling horns of degree 1.

Conversely, we could wonder whether we can strengthen the theorem. If we are forced to demand degenerate-preference anyway, why not also let $\mathrm{ob}(\mathbb{G})$ be degenerate-preferring, in hopes of getting $B\mathbb{G}$ to be degenerate-preferring as well? Surprisingly enough, this seems to be impossible. We would have to redo the proof without the assumption that Φ came from a lift. The catch now lies in the case $k \neq j = l$, when we write "the defining lifting problem for $d_j f_j$ is identical to the one for g_j". Since Φ did not come from a lift, g_j might not be the chosen solution to the lifting problem, so that this step is no longer justified. We have thus discovered a rare example of a naturally arising effective Kan complex that is not necessarily degenerate-preferring, motivating us to continue the study of both structures in parallel.

It is also worth remarking that if we assume that $\mathrm{ob}(\mathbb{G})$ chooses degenerate fillers for horns of degree 1 (this is always possible, since such a horn is essentially just a vertex), the filling procedure defined for $B\mathbb{G}$ does the same. This is not as trivial as it may sound: if we blindly alter the filling function of an arbitrary Kan complex to choose degenerate fillers for horns of degree 1, we might violate the uniformity condition. We therefore claim without further proof:

Theorem 3.10 (Variant of Theorem 3.9)**.** *Let $\mathbb{G}$ be a simplicial groupoid such that $\mathrm{ob}(\mathbb{G})$ is an effective Kan complex and $s\colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G})$ is an effective Kan fibration, both being degenerate-preferring regarding horns of degree 1. Suppose moreover that s chooses an identity arrow as the solution to lifting problems whenever this is possible. Then $B\mathbb{G}$ can be given the structure of an effective Kan complex that is degenerate-preferring regarding horns of degree 1.*

3.3 Part *(ii)*

Adjusting the second part of Theorem 3.8 is more straightforward. We can add effective structures to its statement in the simplest way imaginable; the lifting procedure described by [vdBM18] translates without a problem.

Theorem 3.11. *Let a simplicial groupoid $\mathbb{G}$ be given, together with an action on a map $\pi\colon X \to \mathrm{ob}(\mathbb{G})$. If π is a (degenerate-preferring) effective Kan fibration, then the morphism of classifying spaces $BX_{\mathbb{G}} \to B\mathbb{G}$ induced by the action can be given the structure of a (degenerate-preferring) effective Kan fibration by constructive means.*

Proof. We first prove the variant without degenerate-preference, after describing the lifting procedure by [vdBM18]. Suppose we have a lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\phi_1} & BX_{\mathbb{G}} \\ \big\downarrow & \overset{\Phi}{\nearrow} & \big\downarrow \\ \Delta^n & \xrightarrow[\phi_2]{} & B\mathbb{G} \end{array}$$

Retracing definitions, we see that ϕ_2 points out a string of n arrows in $\mathbb{G}_n$:

$$c_0 \xrightarrow{h_1} c_1 \xrightarrow{h_2} \dots \xrightarrow{h_n} c_n = (h_1, \dots, h_n)$$

while Φ, once it is found, is a string of arrows in $(X_{\mathbb{G}})_n$ like this:

$$(\langle h_1, x_0\rangle, \ldots, \langle h_n, x_{n-1}\rangle)$$

satisfying $\pi(x_m) = s(h_{m+1}) = c_m$ and $x_{m+1} = h_{m+1} \cdot x_m$ for all m. Recall that the effect of the morphism $BX_{\mathbb{G}} \to B\mathbb{G}$ is only forgetting all of the x_m, which is why the same arrows h_m of ϕ_2 appear in Φ. We also see that any single x_j determines all the other x_m, so we could just as well represent Φ by the string of arrows h_m together with just one x_j. Before we make this choice, consider the information that ϕ_1 carries; namely a face $d_i\Phi$ of (the yet to be defined) Φ for each $i \neq k$. Such a face $d_i\Phi$ also contains faces of the x_m, but because the i-th object c_i is removed when we restrict to the face d_i, information about x_i is also lost. In short, $d_i\Phi$ captures only the faces $\{d_i x_m\}_{m\neq j}$. If we take together all of the faces $\{d_i\Phi\}_{i\neq k}$ provided by ϕ_1, we see that for $m \neq k$ we are missing two faces of x_m, and only for x_k do we have all but one of its faces. We can therefore try to obtain x_k through an extension problem for the horn $\{d_i x_k\}_{i\neq k}$, and since x_k fixes all the other x_m, this would define Φ uniquely. The compatibility condition $\pi(x_k) = s(h_{k+1}) = c_k$ gives us just the desired lifting problem:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\{d_i x_k\}_{i\neq k}} & X \\ \big\downarrow & \overset{x_k}{\nearrow} & \big\downarrow \pi \\ \Delta^n & \xrightarrow[c_k]{} & \mathrm{ob}(\mathbb{G}) \end{array} \tag{3.3}$$

Since π is assumed to be an algebraic Kan fibration, we obtain the simplex x_k that defines the solution Φ.

Note that we have been careless with the possible values of k: if $k = n$ then $x_k = x_n$ is not defined. This is not an actual problem, since x_n resides in Φ implicitly as $x_n = h_n \cdot x_{n-1}$. We construct its faces from ϕ_1 by $d_i x_n = d_i h_n \cdot d_i x_{n-1}$ for $i \notin \{n, n-1\}$ and $d_{n-1}x_n = d_{n-1}(h_n \circ h_{n-1}) \cdot d_{n-1}x_{n-2}$.

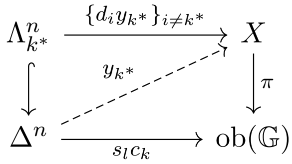
We will now show that the above lifting procedure satisfies the uniformity condition, provided that π does. With ϕ_1, ϕ_2 and Φ an arbitrary lifting problem and chosen solution as before, formulate a degenerate lifting problem $\psi_1 = \{d_i s_l \Phi\}_{i\neq k^*}$, $\psi_2 = s_l\phi_2$ with chosen solution Ψ. We need to show $\Psi = s_l\Phi$. We can represent Ψ and $s_l\Phi$ by

$$\begin{aligned} \Psi &= (\langle f_1, y_0\rangle, \ldots, \langle f_{n+1}, y_n\rangle) \\ s_l\Phi &= (\langle h'_1, x'_0\rangle, \ldots, \langle h'_{n+1}, x'_n\rangle) \\ &= (\langle s_l h_1, s_l x_0\rangle, \ldots, \langle s_l h_l, s_l x_{l-1}\rangle, \langle \mathrm{id}_{s_l c_l}, s_l x_l\rangle, \langle s_l h_{l+1}, s_l x_l\rangle, \ldots) \end{aligned}$$

We can again use that the arrows of $\psi_2 = s_l\phi_2$ must reappear in Ψ, so that $f_m = h'_m$ for all m. This implies in particular that $f_{l+1} = \mathrm{id}_{s_l c_l}$. But it must be that $f_{l+1} \cdot y_l = y_{l+1}$ (see Remark 3.6), so we get $y_l = y_{l+1}$. Defining $z_m = y_{\partial_l(m)}$, we can therefore rewrite Ψ as

$$\Psi = (\langle s_l h_1, z_0\rangle, \ldots, \langle s_l h_l, z_{l-1}\rangle, \langle \mathrm{id}_{s_l c_l}, z_l\rangle, \langle s_l h_{l+1}, z_l\rangle, \ldots)$$

Each y_m determines every other $y_{m'}$, and similarly for the x'_m. So if we can prove $z_m = s_l x_m$ for an arbitrary m, we have proved it for all m, and may conclude $\Psi = s_l \Phi$. We choose to prove $z_k = s_l x_k$. The definition of x_k is diagram 3.3, while the definition of y_{k^*} is



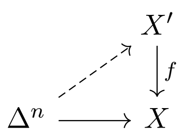
where we used $\pi(y_{k^*}) = s(f_{k^*+1}) = s(h'_{k^*+1}) = s_l c_{\sigma_l(k^*)} = s_l c_k$. By construction, Ψ and $s_l\Phi$ agree on the horn ψ_1, and therefore we may rewrite $\{d_i y_{k^*}\}_{i \neq k^*} = \{d_i s_l x_k\}_{i \neq k^*}$. This means that the lifting problem for y_{k^*} is a degenerate problem based on the lifting problem for x_k, so that $y_{k^*} = s_l x_k$ by virtue of the uniformity condition for π. Since $y_{l+1} = y_l$ and $\partial_l(k) \neq k^*$ only when $k = k^* = l$, in which case $\partial_l(k) = k^* + 1$, we can make the identification $y_{k^*} = y_{\partial_l(k)} = z_k$. We conclude that $z_k = s_l x_k$ and hence, as argued before, that $\Psi = s_l \Phi$, which is what we needed to show.

For the degenerate-preferring variant, we drop the assumption that Φ came from a lifting problem, we note that $s_l x_k$ is still a solution for y_{k^*}, and invoke the degenerate-preference of π. □

3.4 Part *(iii)*

Reworking part three of Theorem 3.8 will be the largest undertaking of this thesis, mostly because it relies on three lemmas that also require substantial modification in order to accommodate effective Kan fibrations. We will not state these lemmas in their original form; they can be retrieved by eliminating the effective structures that appear in our adapted versions of them. For the first lemma, we require a new definition.

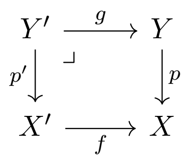
Definition 3.12. We call a simplicial map $f\colon X' \to X$ an *algebraic surjection* if it comes with a choice of lifts for all diagrams of the form



and we call it an *effective surjection* if this choice preserves degeneracy: if ϕ lifts to ϕ', then $s_l\phi$ lifts to $s_l\phi'$, for all $\phi\colon \Delta^n \to X$ and $l \leq n$.

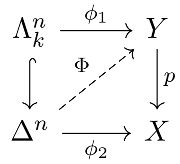
This definition is a thematic way of saying that there is a constructive proof of f being surjective, in the form of a right-inverse for each component function $f_n\colon X'_n \to X_n$. It is an effective surjection if these right-inverses commute with the degeneracy maps, and if they also commute with the face maps, then the choice function is a morphism of simplicial sets constituting a right-inverse, or section, of f. It can be shown non-constructively that any surjective simplicial map can be made into an effective surjection (see [Gee23], where they go under the moniker of "degeneracy sections").

Lemma 3.13 (Adapted from Lemma 3.3 of [vdBM18])**.** *Consider a pullback diagram of simplicial sets*

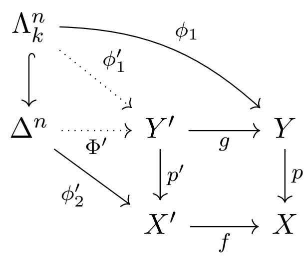


in which f is an effective surjection. If p' is a (degenerate-preferring) effective Kan fibration, then p can be given the structure of a (degenerate-preferring) effective Kan fibration by constructive means.

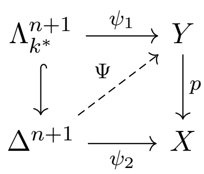
Proof. We continue following the pattern that should be familiar by now: exhibit the lifting procedure, prove the uniformity condition, prove degenerate-preference. Given a lifting problem

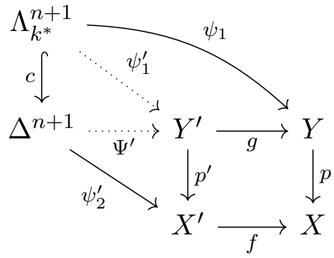


we can lift ϕ_2, using the effective surjection, to some $\phi_2' \colon \Delta^n \to X'$ and obtain a commuting diagram



where ϕ_1' comes from the universal property of the pullback and Φ' is a lift. We solve the original lifting problem by setting $\Phi = g \circ \Phi'$. Now consider a degenerate lifting problem based on it; $\psi_1 = \Phi \circ s_l \circ c$ and $\psi_2 = \phi_2 \circ s_l$, where by c we mean the horn inclusion $\Lambda^{n+1}_{k^*} \hookrightarrow \Delta^{n+1}$, and we have opted to not abuse the Yoneda lemma for once. Solve this problem in the same manner, and call the chosen solution Ψ:

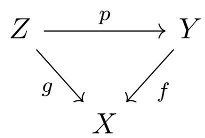




To satisfy the uniformity condition for p, we need to prove $\Psi = \Phi \circ s_l$. We can get there if we can show that the induced lifting problem for Ψ' is a degenerate problem based on the lifting problem for Φ'. Since lifting along the effective surjection f preserves degeneracy, we have $\psi'_2 = \phi'_2 \circ s_l$. Next, we want to show that $\psi'_1 = \Phi' \circ s_l \circ c$. The map ψ'_1 is unique, so in order to prove that ψ'_1 is equal to $\Phi' \circ s_l \circ c$, we only need to show that they solve the same problem, namely $g \circ \chi = \psi_1$ and $p' \circ \chi = \psi'_2 \circ c$. Substituting $\Phi' \circ s_l \circ c$ for χ we indeed get $g \circ \Phi' \circ s_l \circ c = \Phi \circ s_l \circ c = \psi_1$ and $p' \circ \Phi' \circ s_l \circ c = \phi'_2 \circ s_l \circ c = \psi'_2 \circ c$, so that it must be that $\psi'_1 = \Phi' \circ s_l \circ c$. This means that the defining lifting problem for Ψ' is a degenerate problem based on the one for Φ', so that $\Psi' = \Phi' \circ s_l$ by virtue of the uniformity condition for p'. Composing both sides with g, we derive $\Psi = \Phi \circ s_l$. This is what we needed to show, and we conclude that the given construction turns p into an effective Kan fibration.

To prove the degenerate-preferring version, we drop the assumption that Φ came from a lift, and in the penultimate step we get the degenerate solution regardless of whether the degenerate lifting problem for Ψ' is based on another lifting problem. □

Lemma 3.14 (Adapted from Lemma 3.4 of [vdBM18])**.** *Suppose we have a commuting diagram*



in which p and g are degenerate-preferring algebraic Kan fibrations, and suppose a right-inverse p_0^{-1} of $p_0 \colon Z_0 \to Y_0$ is also given. Then f can be given the structure of a degenerate-preferring algebraic Kan fibration by constructive means.

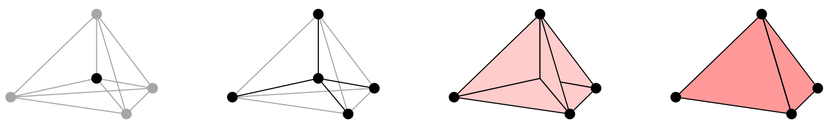
The proof of the original version of this lemma is quite simple, but relies on lifting against maps $\Delta^0 \to \Lambda^n_k$. Since such a map is anodyne, we know that this is possible, but an explicit construction is not given by [vdBM18]. We have to come up with one ourselves, and it has to mesh well with degeneracy.

Lemma 3.15. *Let $p\colon Z \to Y$ be a degenerate-preferring algebraic Kan fibration, and let p_0^{-1} be a right-inverse of $p_0\colon Z_0 \to Y_0$. Consider lifting problems of the form*

$$\begin{array}{ccc} & & Z \\ & \overset{\phi'}{\nearrow} & \big\downarrow p \\ \Lambda^n_k & \xrightarrow[\phi]{} & Y \end{array}$$

There exists a constructive lifting procedure solving these problems that preserves the degeneracy of a horn: if $\phi = \{d_i s_l \chi\}_{i \neq k}$ for some $\chi\colon \Delta^n \to Y$ and $l \leq n$, then its lift ϕ' can be written as $\{d_i s_l \chi'\}_{i \neq k}$ for some $\chi'\colon \Delta^n \to Z$.

Proof. We first describe the lifting procedure, and then prove that it preserves degeneracy. Intuitively, the lifting procedure goes as follows. We first lift the vertex opposite the missing face of ϕ using p_0^{-1}. A vertex looks like the horn of an edge; so we can now lift the edges that do *not* lie on the missing face. These edges define lifting problems for the 2-simplices that do not lie on the missing face; those 2-simplices in turn precisely define lifting problems for the 3-simplices not on the missing face; and so on. The following figure illustrates this process for the horn of a 4-simplex (projected stereographically so that the missing face is the volume on the outside, stretching to infinity).



More formally, we define the lifting procedure using recursion as follows. A Λ^1_k-horn is just a 0-simplex, and we can lift it using p_0^{-1}. Now suppose we have a horn $\phi\colon \Lambda^{n+1}_k \to Y$, which we represent as $\{d_i\phi\}_{i \neq k}$. We lift each face $d_j\phi$ separately, so let j be fixed. The vertex of $d_j\phi$ corresponding to vertex k of ϕ is the one with index $\sigma_j(k)$. Invoking recursion, we first lift the $\Lambda^n_{\sigma_j(k)}$-horn defined as $\phi_j = \{d_i d_j \phi\}_{i \neq \sigma_j(k)}$ to a horn $\phi'_j\colon \Lambda^n_{\sigma_j(k)} \to Z$. We then fill it using a lift against a horn inclusion:

$$\begin{array}{ccc} \Lambda^n_{\sigma_j(k)} & \xrightarrow{\phi'_j} & Z \\ \big\downarrow & \overset{d_j\phi'}{\nearrow} & \big\downarrow p \\ \Delta^n & \xrightarrow[d_j\phi]{} & Y \end{array}$$

Having thus lifted all the faces of ϕ, we need to verify that they match up to form a horn. The condition for this is $d_i\phi'_j = d_{j-1}\phi'_i$ whenever $i < j$. If $n = 1$, then ϕ_i and ϕ_j, regarded as vertices, must be identical and are lifted to the same vertex. If $n > 1$, we see from the recursive definition that $d_i\phi'_j$

and $d_{j-1}\phi'_i$ both are lifted from the face $d_i d_j \phi = d_{j-1} d_i \phi$ in the same way, and must therefore be equal. This completes the description of our lifting procedure for horns.

We now want to prove that it preserves degeneracy; supposing that ϕ was degenerate, we must show that ϕ' is similarly degenerate. If $\phi = \{d_i s_l \chi\}_{i \neq k}$, we can also represent it as $\{d_i s_l d_j \phi\}_{i \neq k}$ for an arbitrary $j \in \{l, l+1\} \setminus \{k\}$. With this l and j now fixed, we will show that ϕ' similarly satisfies $\phi' = \{d_i s_l d_j \phi'\}_{i \neq k}$. We use induction on n. First consider $n = 0$. Then $l = 0$ by necessity, and since $i \in \{0, 1\}$, we can use a simplicial identity to get $\{d_i s_l d_j \phi'\}_{i \neq k} = \{d_i \phi'\}_{i \neq k}$, which always represents ϕ' correctly.

Now let $n > 0$ and suppose that the lifting procedure preserves degeneracy of Λ^n_k-horns for any value of k. We want to prove that $d_i s_l d_j \phi' = d_i \phi'$ for all $i \neq k$. We distinguish the following cases.

- $i \in \{l, l+1\}$ and $i = j$. Then $d_i s_l$ is the identity map and the equality follows.

- $i \in \{l, l+1\}$ and $i \neq j$. We use the same simplicial identity, but we still need to show $d_j \phi' = d_i \phi'$. We do have $d_j \phi = d_i \phi$. Moreover, the condition for this case (together with the premise $j \in \{l, l+1\}$) implies that either $i, j < k$ or $i, j > k$, so that $\sigma_j(k) = \sigma_i(k)$. This means that the faces $d_j \phi$ and $d_i \phi$ are lifted in precisely the same way, and the desired equality follows.

- $i < l$. The face $d_i \phi$ is degenerate, and its $\Lambda^n_{\sigma_i(k)}$-horn ϕ_i can be represented as $\{d_m \phi_i\}_{m \neq \sigma_i(k)} = \{d_m d_i \phi\}_{m \neq \sigma_i(k)} = \{d_m d_i s_l d_j \phi\}_{m \neq \sigma_i(k)}$. Simplicial identities turn this into $\{d_m s_{l-1} d_{j-1} d_i \phi\}_{m \neq \sigma_i(k)}$. In order to apply the induction hypothesis, we need to justify that $d_{j-1} d_i \phi$ can be rewritten as $d_{j-1} \phi_i$. So d_{j-1} cannot be allowed to hit the missing face of ϕ_i. We know that $j \neq k$ (because we chose it that way). If $k > i$, then $\sigma_i(k) = k-1 \neq j-1$. If $k < i$, then $\sigma_i(k) = k < i < l \leq j$, so again $\sigma_i(k) \neq j-1$. We can therefore write $\phi_i = \{d_m s_{l-1} d_{j-1} \phi_i\}_{m \neq \sigma_i(k)}$ and apply the induction hypothesis to get $\phi'_i = \{d_m s_{l-1} d_{j-1} \phi'_i\}_{m \neq \sigma_i(k)}$. This, together with $d_i \phi$ being degenerate, implies that the lifting problem for $d_i \phi'$ has a degenerate solution $s_{l-1} d_{j-1} d_i \phi'$, which is also the chosen solution because p is degenerate-preferring. Simplicial identities finally give us $d_i \phi' = d_i s_l d_j \phi'$.

- $i > l + 1$. This is analogous to the previous case.

We conclude that the horn represented by $\{d_i s_l d_j \phi'\}_{i \neq k}$ is identical to ϕ'. By induction on n, our lifting procedure preserves the degeneracy of any horn. □

Proof of Lemma 3.14. We are showing that the morphism f appearing in the diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

is a degenerate-preferring algebraic Kan fibration. To solve an arbitrary lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\phi_1} & Y \\ \big\downarrow & \overset{\Phi}{\nearrow} & \big\downarrow f \\ \Delta^n & \xrightarrow[\phi_2]{} & X \end{array}$$

first lift the horn ϕ_1 to a horn ϕ_1' in Z using Lemma 3.15. Solve the lifting problem for g posed by ϕ_1' and ϕ_2, and call the solution Φ'. Then choose $p \circ \Phi'$ as the solution for Φ.

To prove the degenerate-preference of this lifting procedure, suppose that the lifting problem for Φ admits a degenerate solution, so that ϕ_1 and ϕ_2 are degenerate. We want to show that the chosen solution for Φ is the unique degenerate solution. By Lemma 3.15, ϕ_1' is degenerate in the same way that ϕ_1 and ϕ_2 are, so the lifting problem for Φ' admits a degenerate solution, which is also the chosen one, by virtue of g being degenerate-preferring. It follows that $\Phi = p \circ \Phi'$ is also degenerate. We conclude that f is a degenerate-preferring algebraic Kan fibration. □

We will need additional definitions to understand the next lemma, but it is instructive to first state it.

Lemma 3.16 (Adapted from Lemma 3.5 of [vdBM18]). *Suppose we are given a simplicial group H with a free action $\alpha\colon H \times E \to E$ and an effectively surjective co-equalizer*

$$H \times E \underset{\alpha}{\overset{\pi_2}{\rightrightarrows}} E \xrightarrow{q} X$$

where π_2 is the second projection map. Suppose moreover that the division map $E \times_X E \to H$ is also given. Then q can be given the structure of a degenerate-preferring algebraic Kan fibration by constructive means.

A group is just a groupoid with a single object, and analogously:

Definition 3.17. A *simplicial group* is a simplicial groupoid $\mathbb{G}$ such that $\mathrm{ob}(\mathbb{G}) \simeq \Delta^0$. We will often identify a simplicial group $\mathbb{G}$ with the simplicial set of its arrows $\mathrm{ar}(\mathbb{G})$.

Remark 3.18. A simplicial group is more commonly defined as a functor from the simplex category $\boldsymbol{\Delta}$ to the category of groups. In both cases we are dealing with a group for every natural number such that the group operations commute with the face and degeneracy maps, so the two definitions are equivalent.

Recall that the action of a simplicial groupoid $\mathbb{G}$ on a map $\pi\colon E \to \mathrm{ob}(\mathbb{G})$ is a certain map $\mathrm{ar}(\mathbb{G}) \times_{\mathrm{ob}(\mathbb{G})} E \to E$. If $H = \mathbb{G}$ is a simplicial group, then morphisms to the terminal object $\mathrm{ob}(H) \simeq \Delta^0$ are unique, so π is trivial and $\mathrm{ar}(H) \times_{\mathrm{ob}(H)} E = \mathrm{ar}(H) \times E$. We therefore choose to speak of actions on simplicial sets, rather than actions on simplicial maps, and write $H \times E \to E$ for such an action. An action is *free* when, for any simplices $g \in H_n$ and $x \in E_n$, $g \cdot x = x$ implies that g is the identity element of the group H_n.

In topology, a principal G-bundle could be defined (with some caveats) as the quotient map $q\colon P \to P/G$ for the quotient of P by a free topological group action $G \times P \to P$, collapsing G-orbits to a single point. The action of the group on a fiber $P_p = q^{-1}(p) \subseteq P$ leaves it invariant and is transitive by construction — every point in P_p can be sent to every other point in P_p by some element of G. We want to transport this construction to the realm of simplicial sets. The co-equalizer q in Lemma 3.16 serves just this purpose. As a refresher:

Definition 3.19. A *co-equalizer* of a pair of morphisms $f, g\colon A \to B$ is a morphism $q\colon B \to C$ satisfying $q \circ f = q \circ g$, such that any other morphism q' with this property factors uniquely through q.

The meaning of Lemma 3.16 is that this "quotient map" q is, in fact, the simplicial equivalent of a fiber bundle. In our category $\widehat{\boldsymbol{\Delta}}$, co-equalizers always exist, but we ask for one explicitly, one that is in addition an effective surjection, in order to make a constructive proof possible. Similarly, the division map provides, for every two elements x, y of the same H-orbit, the unique group element g that sends y to x, i.e. $g \cdot y = x$; it can be thought of as a witness for the fact that the action is transitive on fibers, and (depending on our meta-mathematics) we need it to be given explicitly for a constructive proof. To prove Lemma 3.16, we need the following crucial theorem due to Freek Geerligs [Gee23].

Theorem 3.20. *We can give the structure of a degenerate-preferring algebraic Kan complex to any simplicial group by constructive means.*

Proof of Lemma 3.16. Our proof is essentially identical to the original. Consider the diagram

$$\begin{array}{ccccccc}
H & \longleftarrow & H \times E & \xrightarrow{\;\theta\;} & E \times_X E & \xrightarrow{\;\pi_1\;} & E \\
\big\downarrow & \llcorner & \big\downarrow{\scriptstyle \pi_2} & & {\scriptstyle \pi_2}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle q} \\
\Delta^0 & \longleftarrow & E & \xrightarrow[\mathrm{id}_E]{} & E & \xrightarrow[q]{} & X
\end{array}$$

The squares on the left and on the right are pullback squares, viz. the definitions of the respective products. The map θ is defined by $\theta(g, x) = \langle \alpha(g, x), x \rangle$, so that the central square commutes. Moreover, θ is an isomorphism because the division map paired with id_E is its inverse. By Theorem 3.20, the map on the left is a degenerate-preferring algebraic Kan fibration. By Proposition 2.13, its pullback $\pi_2 \colon H \times E \to E$ is a degenerate-preferring algebraic Kan fibration, and so is the isomorphic $\pi_2 \colon E \times_X E \to E$ (in this step, we implicitly use the constructed inverse of θ). Finally, q is a degenerate preferring algebraic Kan fibration by Lemma 3.13. □

Remark 3.21. We have not used the universal property of the co-equalizer q explicitly. Co-equalizers provide the categorical definition of quotients, so we need it for the simplicial group action to be transitive on fibers, and hence for the existence of the division map. We will not prove this claim. We just want to remark that the lemma can (and will) be applied even when the universality of q has not been proved, as long as the division map is provided.

We can finally take on part *(iii)* of Theorem 3.8, bringing the current chapter to an end. The change in its statement is minimal, and the proof differs from the original only in the need to show that the modified lemmas still apply.

Theorem 3.22. *Given a non-empty simplicial groupoid $\mathbb{G}$, if the pair of maps $(s, t) \colon \mathrm{ar}(\mathbb{G}) \to \mathrm{ob}(\mathbb{G}) \times \mathrm{ob}(\mathbb{G})$ is an effective surjection, then it can be given the structure of a degenerate-preferring algebraic Kan fibration by constructive means.*

Remark 3.23. If a simplicial groupoid is empty, the statement is vacuously true. We take $\mathbb{G}$ being non-empty to mean that some element can actually be exhibited.

Remark 3.24. Theorem 3.22 can be considered a generalization of Theorem 3.20. If $\mathbb{G}$ is a simplicial group, then (s, t) is the unique map to the terminal object and can be made into an effective surjection by lifting with the identity-assigning map $\mathrm{ob}(\mathbb{G}) \to \mathrm{ar}(\mathbb{G})$. Since we identify a simplicial group $\mathbb{G}$ with the object of arrows $\mathrm{ar}(\mathbb{G})$, to say that (s, t) is a Kan fibration is to say that the simplicial group is a Kan complex.

Proof. Since $\mathbb{G}$ is non-empty, we can fix an object $c \colon \Delta^0 \to \mathrm{ob}(\mathbb{G})$ and write H for the simplicial group of arrows from c to c. More precisely, for every n define $H_n = \{h \in \mathrm{ar}(\mathbb{G})_n \mid s(h) = t(h) = c^n\}$, where $c^n := (s_0)^n(c)$ is the n-fold degenerate simplex based on c. Consider the pullback square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{ar}(\mathbb{G}) \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \mathrm{ob}(\mathbb{G}) & \xrightarrow{\mathrm{id}_{\mathrm{ob}(\mathbb{G})} \times c} & \mathrm{ob}(\mathbb{G}) \times \mathrm{ob}(\mathbb{G}) \end{array}$$

where we used $\mathrm{ob}(\mathbb{G}) \simeq \mathrm{ob}(\mathbb{G}) \times \Delta^0$ for the morphism on the bottom. E is the simplicial set of arrows into c: for every n, $E_n = \{f \in \mathrm{ar}(\mathbb{G})_n \mid t(f) = c^n\}$. We can compose each arrow in H_n with each arrow in E_n, defining an action $\alpha\colon H \times E \to E$ that is free by virtue of the ordinary properties of groupoids. We want to apply Lemma 3.16 with s as the quotient map. We evidently have $s \circ \alpha = s \circ \pi_2$, and we do not need to prove that s is universal as a co-equalizer because of what has been said in Remark 3.21. Since (s, t) is an effective surjection, s can be made into one as well. We still need to find the division map $E \times_{\mathrm{ob}(\mathbb{G})} E \to H$. A simplex of $E \times_{\mathrm{ob}(\mathbb{G})} E$ is a pair of arrows with a common source, so we find the division map as $(f, g) \mapsto f \circ g^{-1}$ since $(f \circ g^{-1}) \cdot g = f$. We conclude by Lemma 3.16 that s is a degenerate-preferring Kan fibration.

We can also get an action $\beta\colon H \times (E \times E) \to E \times E$ sending $\langle h, f, g\rangle$ to $\langle h \circ f, h \circ g\rangle$, which is again free. Now we want to apply Lemma 3.16 with the quotient map $p\colon E \times E \to \mathrm{ar}(\mathbb{G})$ defined by $p(f, g) = g^{-1} \circ f$. It satisfies $p \circ \beta = p \circ \pi_2$ (here π_2 is the projection $H \times (E \times E) \to E \times E$). We first show that p is an effective surjection. Given an arrow $h \in \mathrm{ar}(\mathbb{G})_n$, use the effective surjection (s, t) to find a lift $f_h \in \mathrm{ar}(\mathbb{G})_n$ of the pair $(s(h), c^n)$. Now we can choose $p_n^{-1}\colon h \mapsto (f_h, f_h \circ h^{-1})$. Clearly p_n^{-1} acts like a right-inverse of $p_n\colon (E \times E)_n \to \mathrm{ar}(\mathbb{G})_n$. We still need to show that it preserves degeneracy, so consider a degenerate arrow $s_l h$ of degree $n + 1$. Since lifting along (s, t) preserves degeneracy, $f_{s_l h} = s_l f_h$. Therefore $p_{n+1}^{-1}(s_l h) = (f_{s_l h}, f_{s_l h} \circ (s_l h)^{-1}) = s_l(f_h, f_h \circ h^{-1}) = s_l p_n^{-1}(h)$ so that lifting using the p_m^{-1} also preserves degeneracy, making p into an effective surjection.

Next, consider the division map $E^2 \times_{\mathrm{ar}(\mathbb{G})} E^2$ of β. The fiber product means that an element $\langle f_1, g_1; f_2, g_2\rangle$ satisfies $p(f_1, g_1) = p(f_2, g_2)$, i.e. $g_1^{-1} \circ f_1 = g_2^{-1} \circ f_2$ which implies $g_2 \circ g_1^{-1} = f_2 \circ f_1^{-1}$, so that we can define the division map as $\langle f_1, g_1; f_2, g_2\rangle \mapsto f_2 \circ f_1^{-1}$. Having satisfied the conditions of Lemma 3.16, we obtain that p is a degenerate-preferring algebraic Kan fibration. Since s in a degenerate-preferring algebraic Kan fibration, $s \times s$ can straightforwardly be made into one as well. This means that Lemma 3.14 applies to the diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{\quad p \quad} & \mathrm{ar}(\mathbb{G}) \\ & {\scriptstyle s \times s} \searrow \quad \swarrow {\scriptstyle (s,t)} & \\ & \mathrm{ob}(\mathbb{G}) \times \mathrm{ob}(\mathbb{G}) & \end{array}$$

and we conclude that (s, t) is a degenerate-preferring algebraic Kan fibration. □

Chapter 4

Semisimplicial sets

In this short chapter, we will explore semisimplicial sets. They are like simplicial sets without a notion of degeneracy. For example, since there no longer are degeneracy maps, a semisimplicial set X may contain a finite number of elements in total, having $X_n = \emptyset$ when n is larger than some fixed number. In contrast, each set of a simplicial set contains at least the degenerate versions of the simplices of lower degree.

Since the definition of an (algebraic) Kan fibration does not make use of the degeneracy maps, it readily carries over to semisimplicial sets. Rourke and Sanderson proved by a geometric argument that a semisimplicial Kan complex can be given a simplicial structure [RS71]. James McClure found a combinatorial proof of the same fact [McC12]: we can exploit the filling properties of a semisimplicial Kan complex to define degeneracy maps that turn it into a simplicial set. This simplicial set still has to be a Kan complex, because we did not alter its simplices.

Thus noting that in a certain sense, a filling structure stores information about degeneracy, we wonder how this relates to the effective versions of Kan complexes. We will find that the filling structure of an effective Kan complex perfectly records its degeneracy maps. On the other hand, contrary to what one might hope for, an arbitrary semisimplicial algebraic Kan complex is not turned into an effective Kan complex. We first review the results of [McC12] and then discuss our own.

4.1 Defining degeneracy maps

Definition 4.1. We define $\mathbf{\Delta}_{\text{inj}}$ as the subcategory of the simplex category $\mathbf{\Delta}$ containing only the injective functions. The category of *semisimplicial sets* is the presheaf category $\widehat{\mathbf{\Delta}_{\text{inj}}}$. There is an evident forgetful functor $\widehat{\mathbf{\Delta}} \to \widehat{\mathbf{\Delta}_{\text{inj}}}$ discarding all degeneracy maps.

Definition 4.2. A semisimplicial set X is a *semisimplicial Kan complex* if every horn $x_i, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$ of n-simplices in X, satisfying $d_i x_j =$

$d_{j-1}x_i$ whenever $i < j$ with $i, j \neq k$, has an extension $x \in X_{n+1}$ such that $d_i x = x_i$ for all $i \neq k$. We call X an *algebraic semisimplicial Kan complex* if it comes with a structure choosing a particular solution for every such extension problem.

As announced earlier, the definition is completely analogous to the simplicial variant. We chose not to define it in terms of horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ because we have not defined their semisimplicial versions yet; the forgetful functor does not help here because we would need to prune away all their degenerate simplices, not just the degeneracy maps. As a matter of fact, we will find it convenient to use set-theoretic notation, rather than semisimplicial morphisms, throughout this chapter.

The following theorem is Theorem 1.2 of [McC12], itself based on Theorem 5.7 of [RS71].

Theorem 4.3. *Let X be a semisimplicial Kan complex. Then there are functions $s_j \colon X_n \to X_{n+1}$, for $n \geq 0$ and $0 \leq j \leq n$, with the following properties.*

$$\begin{aligned} d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_i s_j x &= x && \text{if } i \in \{j, j+1\} \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\ s_j s_i &= s_i s_{j-1} && \text{if } i < j \end{aligned}$$

These are precisely the properties that a simplicial set requires of its degeneracy maps. Since a semisimplicial set already contains correct face maps, it follows from the theorem that there exists a simplicial Kan complex that is sent to X by the forgetful functor. We will now review the definitions of the s_j, omitting the proof that they satisfy the listed properties. Be aware that we work with an algebraic filling structure, while [McC12] implicitly uses the axiom of choice to pick solutions to filling problems (that is not to imply that the version presented here is constructive — we will return to this matter later).

Definition 4.4. Given a semisimplicial algebraic Kan complex X, define degeneracy maps s_j by double recursion as follows. Let $x \in X_n$ and $k \leq n$ be given and suppose that $s_j y$ has been defined when $\deg(y) < n$ and when $\deg(y) = n$ and $j < k$ ("deg" stands for the degree of a simplex). If x is in the image of s_j for some $j < k$, then for the smallest such j, find a simplex $w \in X_{n-1}$ satisfying $x = s_j w$, and define $s_k x = s_j s_{k-1} w$. Otherwise, define $s_k x = d_0 T_k x$, where $T_k x$ is the chosen filler of the horn

$$d_j T_k x = \begin{cases} T_{k-1} d_{j-1} x & \text{if } 0 < j < k+1 \\ y & \text{if } j \text{ is } k+1 \text{ or } k+2 \\ T_k d_{j-2} x & \text{if } j > k+2 \end{cases}$$

in which y is the chosen filler of the horn

$$d_j y = \begin{cases} x & \text{if } j = 0 \\ d_k T_{k-1} d_{j-1} x & \text{if } 0 < j < k+1 \\ d_{k+1} T_k d_{j-1} x & \text{if } j > k+1 \end{cases}$$

Note that this definition also works for the base case $n = 0 = k$, since in that case the definitions of faces depending on other $T_{k'}$ are not invoked. This completes the definition of $s_j x$. But x was arbitrary, so that we have defined all the s_j, by recursion in k and n.

$T_k x$ can be thought of as $s_0 s_k x$ — this identification holds after the degeneracy maps have been defined. It is shown in [McC12] that the T_k have the following useful properties:

(A)	$d_i T_j = T_{j-1} d_{i-1}$	if $0 < i < j+1$
(B)	$d_i T_j = T_j d_{i-2}$	if $i > j+2$
(C)	$d_{j+1} T_j = d_{j+2} T_j$	for all j
(D)	$d_0 d_{j+1} T_j x = x$	for all j and x

4.2 From an effective Kan complex

We will show that redefining the degeneracy maps of an effective Kan complex, viewed as an algebraic semisimplicial Kan complex, retrieves the original degeneracy maps. We need the lemma that in this case the fillers $T_k x$ as defined above are degenerate.

Lemma 4.5. *Let X be an effective Kan complex with degeneracy maps s_j. Then $T_k x = s_{k+1} d_{k+1} T_k x$ for all k and x, where $T_k x$ is defined as in Definition 4.4.*

Proof. We use induction on the degree of x. Suppose $T_k x = s_{k+1} d_{k+1} T_k x$ whenever $k \leq \deg(x) < n$. Now fix $x \in X_n$ and $k \leq n$. By definition, $T_k x$ is the chosen filler of a certain horn Λ, and its face $d_{k+1} T_k x = y$ is also a filler. If $s_{k+1} y$ fills Λ, then the filling problem for $T_k x$ is a degenerate filling problem based on the filling problem for y, and $s_{k+1} y$ must be the chosen solution for $T_k x$ by the uniformity condition of X. We will verify that $s_{k+1} y$ indeed fills Λ, meaning that they share their faces. So we must prove that $d_j T_k x$ is equal to $d_j s_{k+1} y = d_j s_{k+1} d_{k+1} T_k x$ for $j \neq 0$ (here we used $y = d_{k+1} T_k x$, which follows from the definition of $T_k x$). For $j \in \{k+1, k+2\}$, this is immediate from the simplicial identities and property (C) above. For $0 < j < k+1$, $d_j T_k x$ was defined as $T_{k-1} d_{j-1} x$. Using the inductive hypothesis, this becomes $s_k d_k T_{k-1} d_{j-1} x$, which equals $s_k d_k d_j T_k x$ by property (A) and simplifies to $d_j s_{k+1} d_{k+1} T_k x$ under the simplicial identities, as desired. Analogously, but using property (B) instead, for $j > k+2$ we get $d_jT_kx = T_kd_{j-2}x = s_{k+1}d_{k+1}T_kd_{j-2}x = s_{k+1}d_{k+1}d_jT_kx = s_{k+1}d_{j-1}d_{k+1}T_kx = d_js_{k+1}d_{k+1}T_kx$. In any case we have $d_jT_kx = d_js_{k+1}d_{k+1}T_kx$ (note that for the base case $n = 0 = k$, the only values allowed for j are $k+1$ and $k+2$, so we never actually ask for a face of a 0-simplex). This means that T_kx must be the chosen unique degenerate filler $s_{k+1}d_{k+1}T_kx$, since $d_{k+1}T_kx = y$ was also a filler. By induction, this holds for any n (k and x were already arbitrary), completing the proof. □

Proposition 4.6. *Given an effective Kan complex X, the degeneracy maps found through Definition 4.4 are identical to the existing degeneracy maps of X.*

Proof. Denote the newly defined degeneracy maps by $\tilde{s}_j : X_n \to X_{n+1}$, reserving the notation s_j for the original degeneracy maps of X. We need to show that $\tilde{s}_k = s_k$ for all n and $k \leq n$. We follow the double recursion of Definition 4.4: given $k \leq n$, suppose that $\tilde{s}_jx = s_jx$ for all j when $\deg(x) < n$, and for $j < k$ when $\deg(x) = n$. Fix $x \in X_n$. If $x = \tilde{s}_jw$ for some $w \in X_{n-1}$ and minimal $j < k$, then $\tilde{s}_kx = \tilde{s}_j\tilde{s}_{k-1}w = s_js_{k-1}w = s_ks_jw = s_k\tilde{s}_jw = s_kx$. Otherwise $\tilde{s}_kx = d_0T_kx = d_0s_{k+1}d_{k+1}T_kx = s_kd_0d_{k+1}T_kx = s_kx$, where the second equality follows from the foregoing lemma and the last equality from the property (D) from before. By induction, we have $\tilde{s}_kx = s_kx$ for all k and x. □

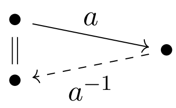
The proposition says that an algebraic semisimplicial Kan complex, obtained from an effective Kan complex by applying the forgetful functor, will be turned into that same effective Kan complex if we assign it the system of degeneracy maps from Definition 4.4. So the filling structure records all the information about degeneracy. However, a generic algebraic semisimplicial Kan complex X will not be turned into an effective Kan complex. For example, given a horn $y : \Lambda^2_2 \to X$ and writing F for the filler function, we would need $s_0F(y) = F(\{d_is_0F(y)\}_{i\neq 3})$. But the definition of s_0 only depends on filling horns that miss face 0 or 1, while $\{d_is_0F(y)\}_{i\neq 3}$ is a Λ^3_3 horn, so we have enough freedom to design F in such a way that the equation is not satisfied.

A more severe problem for our program is that the procedure of Definition 4.4 is not constructive. The transgression occurs when we write "if x is in the image of s_j, find a simplex $w \in X_{n-1}$ satisfying $x = s_jw$". First of all, the image of $s_j : X_n \to X_{n+1}$ is not a decidable subset of X_{n+1}, and second, selecting such a w requires the axiom of choice. It seems unlikely that there exists a constructive alternative. The takeaway of this chapter is that semisimplicial sets, though interesting in their own right, are probably not a useful tool in the study of effective Kan fibrations.

Chapter 5

Infinite extension

We motivated our study of simplicial sets by arguing that they are approximations to topological spaces, especially regarding homotopy. However, because we gave simplices an orientation, it might be that (roughly speaking) a homotopy between two simplicial morphisms f, g exists only in one direction, $f \Rightarrow g$, and not as $g \Rightarrow f$. The purpose of a Kan complex is to make sure that simplices come in every possible orientation. For example, if we have an arrow a, then we can append a degenerate arrow to get a horn whose filler defines an inverse for a:



This means that we can define homotopy groups for Kan complexes, but not for generic simplicial sets. The reason why we do not restrict ourselves to just Kan complexes is that the category of simplicial sets is much more convenient to work with than the category of Kan complexes. But the real justification of this choice is the fact that for every simplicial set, there exists a Kan complex of the same (weak) homotopy type. The definitions are technical, but intuitively this means that we can replace any simplicial set by a Kan complex that looks the same for the purposes of homotopy, but is no longer asymmetric in the sense we described above. This process is called *fibrant replacement*; a Kan complex is sometimes referred to more abstractly as a *fibrant object*, because its map to the terminal object is a fibration.

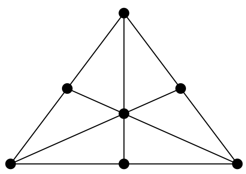
Infinite extension[1], the subject of this chapter, is a "brute force" approach to fibrant replacement. In essence, we extend a simplicial set X once by defining a new simplex for every horn in X with the purpose of filling it.

[1] "Infinite extension" is non-standard terminology; in the literature it is usually referred to as "(Kan's) ex-infinity functor" or "Ex^∞". In Kan's original paper [Kan57] we find that "Ex" stands for "extension".

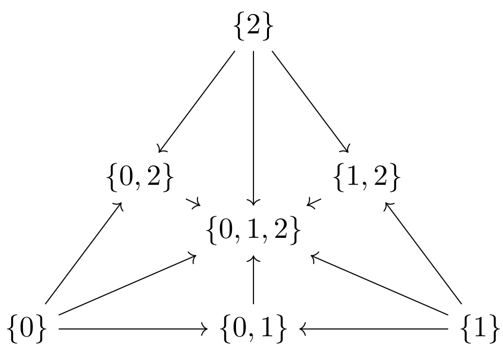
In doing so we may create new horns, which is why we have to repeat the procedure. Infinite extension is the limit of repeating this infinitely many times, filling all horns in the end. This implementation of fibrant replacement is favored because it extends to a functor $\mathrm{Ex}^\infty\colon \widehat{\mathbf{\Delta}} \to \widehat{\mathbf{\Delta}}$ with useful properties. However, it has a lot of redundancy: it always adds new solutions to filling problems, even when a solution already exists. In particular, it does not identify degenerate solutions. We will argue that it is implausible that this method could be converted into a constructive way of producing effective Kan complexes. But first we need to get familiar with Ex^∞. We have drawn techniques and notation from [Cis19], [GJ09] and [Lur23].

5.1 Definitions

The extension functor makes use of *barycentric subdivision.* The barycentric subdivision of the geometric 2-simplex looks as follows:



We can conveniently use the linear orders of the simplex category $\mathbf{\Delta}$ to define barycentric subdivision. First, for each n-simplex $[n] = \{0, \ldots, n\}$ we define $s([n])$ as the set of all non-empty chains (totally ordered subsets) of $[n]$, and we partially order this set by the subset relation. The chain $\{p_1, \ldots, p_k\} \subseteq [n]$ represents the barycenter of the vertices $p_1, \ldots, p_k$. We can draw $s([2])$ as follows:



(5.1)

where the arrows represent the ordering by inclusion. We can make s into a functor (from $\mathbf{\Delta}$ to the category of partial orders) by defining $s(f)\colon s([n]) \to s([m])$ as $s(f)(S) = f(S)$ for each $f\colon [n] \to [m]$ (the image of the set $S \subseteq [n]$). Since we can regard a partial order as a category, we define the barycentric subdivision of the n-simplex $[n]$ as the nerve $N(s([n]))$ (Definition 2.3. We also write $\mathrm{Sd}(\Delta^n)$ for this simplicial set. As an illustration, the triangle on the bottom right of diagram 5.1 above is now the 2-simplex given by the string of arrows $\{1\} \longrightarrow \{0,1\} \longrightarrow \{0,1,2\}$. We can also obtain the subdivision $\mathrm{Sd}(X)$ of an arbitrary simplicial set X by subdividing all of its simplices, which defines a functor $\mathrm{Sd}\colon \widehat{\mathbf{\Delta}} \to \widehat{\mathbf{\Delta}}$. An intuitive understanding is sufficient for our purposes; the formal definition writes X as a co-limit of representables and takes the co-limit of the corresponding diagram composed with Sd (which we have already defined on representables) [GJ09].

We can map the subdivided simplex $s([2])$ back to $[2]$ by retracting it to the face on the bottom left in diagram 5.1. This generalizes to other simplices: define the map $s([n]) \to [n]$ by $S \mapsto \max(S)$. Using these maps we can send each subdivided simplex in $\mathrm{Sd}(X)$ to the simplex it came from in X, defining a simplicial morphism $\lambda_X\colon \mathrm{Sd}(X) \to X$. It can be shown that the following diagram is commutative for all $f\colon X \to Y$:

$$\begin{array}{ccc} \mathrm{Sd}(X) & \xrightarrow{\lambda_X} & X \\ {\scriptstyle \mathrm{Sd}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Sd}(Y) & \xrightarrow[\lambda_Y]{} & Y \end{array}$$

so that $\lambda\colon \mathrm{Sd} \to 1_{\widehat{\mathbf{\Delta}}}$ is a natural transformation from the subdivision functor to the identity functor (Definition 3.1.17 of [Cis19]).

Given a simplicial set X, we define the simplicial set $\mathrm{Ex}(X)$ by $\mathrm{Ex}(X)_n = \mathrm{Hom}(\mathrm{Sd}(\Delta^n), X)$. Here $\mathrm{Hom}(\mathrm{Sd}(\Delta^n), X)$ is the set of simplicial morphisms from $\mathrm{Sd}(\Delta^n)$ to X, so $\mathrm{Ex}(X)$ represents all the ways in which a subdivided simplex can be embedded in X. This construction again extends to a functor $\mathrm{Ex}\colon \widehat{\mathbf{\Delta}} \to \widehat{\mathbf{\Delta}}$, which moreover is right-adjoint to Sd [GJ09]. This means that there is a natural bijection $m_{X,Y}$, called *transposition*, from maps of the form $f\colon X \to \mathrm{Ex}(Y)$ to maps of the form $g\colon \mathrm{Sd}(X) \to Y$. By naturality we mean that $m_{X,Y}$ sends the composite map

$$A \xrightarrow{a} B \xrightarrow{f} \mathrm{Ex}(C)$$

to

$$\mathrm{Sd}(A) \xrightarrow{\mathrm{Sd}(a)} \mathrm{Sd}(B) \xrightarrow{\overline{f}} C$$

where $\overline{f} = m_{X,Y}(f)$. A similar relation holds for $(m_{X,Y})^{-1}$, for which we write $(m_{X,Y})^{-1}(g) = \widehat{g}$. This can be summarized by the algebraic relations

$$\overline{f \circ a} = \overline{f} \circ \mathrm{Sd}(a) \tag{5.2}$$

$$\widehat{b \circ g} = \mathrm{Ex}(b) \circ \widehat{g} \tag{5.3}$$

For each simplicial set X, we define an embedding $\rho_X\colon X \to \mathrm{Ex}(X)$ by $\rho_X = \widehat{\lambda_X}$, resulting in a natural transformation $\rho\colon 1_{\widehat{\mathbf{\Delta}}} \to \mathrm{Ex}$. Of course, ρ_X satisfies the relation

$$\overline{\rho_X} = \lambda_X \tag{5.4}$$

The most important things of all of the foregoing are the conceptual interpretations of Sd (subdivision of all simplices in a simplicial set) and Ex (the ways to embed subdivided simplices in a simplicial set), and the abstract relations 5.2 and 5.4. We now finally define infinite extension.

Definition 5.1. Given a simplicial set X, $\mathrm{Ex}^\infty(X)$ is the co-limit of the diagram

$$X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{\rho_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \longrightarrow \dots$$

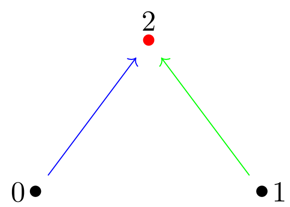
The co-limit means that we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\rho_X} & \mathrm{Ex}(X) & \xrightarrow{\rho_{\mathrm{Ex}(X)}} & \mathrm{Ex}^2(X) & \longrightarrow & \dots \\ & \searrow & & \searrow & & \searrow & \\ & & & & & & \mathrm{Ex}^\infty(X) \end{array} \tag{5.5}$$

The simplicial set $\mathrm{Ex}^\infty(X)$ can roughly be thought of as the union of all the $\mathrm{Ex}^m(X)$. Recall the significance of $\mathrm{Ex}^\infty(X)$: it is a functor sending a simplicial set to a Kan complex of the same weak homotopy type. We have not defined weak homotopy equivalence, but intuitively it is the simplicial equivalent of two topological spaces sharing all their homotopy groups, taking into account that a homotopy of simplicial morphism might come in only one direction. In the next section we will prove that $\mathrm{Ex}^\infty(X)$ is always a Kan complex.

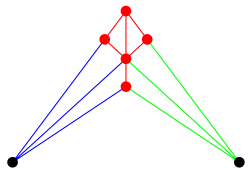
5.2 Fibrancy

First, let us consider pictorially how Ex fills horns. Suppose we have a horn in some simplicial set X, drawn as follows:



then the following are two of the ways to embed the subdivided 1-simplex $\bullet \longrightarrow \bullet \longleftarrow \bullet$ in this horn:

meaning that the first arrow of $\bullet \longrightarrow \bullet \longleftarrow \bullet$ gets sent to one of the legs of the horn, and the second arrow to the degenerate arrow on the red vertex. These two ways of embedding $\mathrm{Sd}(\Delta^1) \to \Lambda^2_2 \to X$ are elements of $\mathrm{Ex}(X)_1 = \mathrm{Hom}(\mathrm{Sd}(\Delta^1), X)$, i.e. 1-simplices of $\mathrm{Ex}(X)$, and this is how the legs of the horn are carried over to $\mathrm{Ex}(X)$. We call them a and b. The horn is filled by wrapping the subdivided 2-simplex around it:



and this embedding of $\mathrm{Sd}(\Delta^2)$ becomes a 2-simplex g of $\mathrm{Ex}(X)$ (again, the red lines are sent to the degenerate arrow). Since the left and right sides of this subdivided simplex are embedded in the same way as the legs a and b, we get $d_1 g = a$ and $d_0 g = b$, so that g fills the horn in $\mathrm{Ex}(X)$ spanned by a and b, which is the representative of the original horn in X.

To define this procedure rigorously and for arbitrary horns, we need to introduce some more notation. Recall the functor $s([n])$. It can straightforwardly be extended from the simplex category to the category of partial orders: $s(E)$ is the set of finite non-empty chains of the partial order E, ordered by inclusion. Since both s and Sd represent barycentric subdivision, the following property should be unsurprising:

Lemma 5.2 (Lemma 3.1.25 of [Cis19])**.** *For any partially ordered set E there is a canonical isomorphism*

$$\mathrm{Sd}(N(E)) \simeq N(s(E))$$

Let Φ^n_k be the set of non-empty subsets of $[n]$ that do not contain the complement of $\{k\}$, i.e. $\Phi^n_k = s([n]) \setminus \{[n], [n] \setminus \{k\}\}$. For example, Φ^2_2 is diagram 5.1 without the points $\{0, 1\}$ and $\{0, 1, 2\}$. It is true in general that Φ^n_k looks like the subdivision of a horn Λ^n_k:

Lemma 5.3 (Lemma 3.1.26 of [Cis19])**.** *For any n and $k \leq n$ there is a canonical isomorphism*

$$\mathrm{Sd}(\Lambda^n_k) \simeq N(\Phi^n_k)$$

We can now prove the fibrancy of $\mathrm{Ex}^\infty(X)$.

Theorem 5.4 (Kan)**.** *For any simplicial set X, we have a Kan complex $\mathrm{Ex}^\infty(X)$.*

Proof. We give a sketch of the proof, skimming over some unimportant details. Let a horn $y \colon \Lambda^n_k \to \mathrm{Ex}^\infty(X)$ be given. $\mathrm{Ex}^\infty(X)$ is the co-limit of the diagram

$$X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{\rho_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \longrightarrow \dots$$

Each face $d_i y$ of y must be added at a finite step $\mathrm{Ex}^{m_i}(X)$. Since y consists of finitely many faces, we can fix any $m \geq 1$ larger than all the m_i so that y has to factor through $\mathrm{Ex}^m(X)$:

$$\Lambda^n_k \xrightarrow{\phi} \mathrm{Ex}^m(X) \longrightarrow \mathrm{Ex}^\infty(X)$$

We will find a filler in the next extension:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\phi} & \mathrm{Ex}^m(X) \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{Ex^m(X)}} \\ \Delta^n & \underset{\Phi}{\dashrightarrow} & \mathrm{Ex}^{m+1}(X) \end{array} \tag{5.6}$$

We transpose this diagram twice. Using the relation 5.2, we can write

$$\overline{\overline{\Phi \circ c}} = \overline{\overline{\Phi}} \circ \mathrm{Sd}^2(c)$$

For the other path, we use $\overline{\rho} = \lambda$ and the fact that λ is a natural transformation:

$$\overline{\overline{\rho_{\mathrm{Ex}^m(X)} \circ \phi}} = \overline{\lambda_{\mathrm{Ex}^m(X)} \circ \mathrm{Sd}(\phi)} = \overline{\phi \circ \lambda_{\Lambda^n_k}} = \overline{\phi} \circ \mathrm{Sd}(\lambda_{\Lambda^n_k})$$

From now, we will suppress the subscripts on ρ and λ, which can always be recovered from context. These composite maps should be equal to each other:

$$\begin{array}{ccc} \mathrm{Sd}^2(\Lambda^n_k) & \xrightarrow{\mathrm{Sd}(\lambda)} & \mathrm{Sd}(\Lambda^n_k) \\ {\scriptstyle \mathrm{Sd}^2(c)}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\phi}} \\ \mathrm{Sd}^2(\Delta^n) & \underset{\overline{\overline{\Phi}}}{\dashrightarrow} & \mathrm{Ex}^{k-1}(X) \end{array}$$

Once this problem has been solved, we can use the inverse transposition to solve problem 5.6, so the two problems are equivalent to each other. Our solution factors through $\overline{\phi}$:

$$\begin{array}{ccc} \mathrm{Sd}^2(\Lambda^n_k) & \xrightarrow{\mathrm{Sd}(\lambda)} & \mathrm{Sd}(\Lambda^n_k) \\ \big\downarrow & \overset{u^n_k}{\nearrow} & \big\downarrow{\scriptstyle \overline{\phi}} \\ \mathrm{Sd}^2(\Delta^n) & \underset{\overline{\overline{\Phi}}}{\dashrightarrow} & Ex^{k-1}(X) \end{array} \tag{5.7}$$

So all that remains to do is to find u^n_k. Noticing that $\Delta^n \simeq N([n])$, we use Lemma 5.2 twice to get the isomorphism $\mathrm{Sd}^2(\Delta^n) \simeq N(s(s[n]))$. From Lemma 5.3 we have $\mathrm{Sd}(\Lambda^n_k) \simeq N(\Phi^n_k)$. So u^n_k is essentially a morphism $N(s(s[n])) \to N(\Phi^n_k)$, which we take to be the nerve of the map $\psi^n_k \colon s(s([n])) \to \Phi^n_k$ defined by

$$\psi^n_k(\mathcal{S}) = \{c^n_k(S) \mid S \in \mathcal{S}\}$$

where $c_k^n \colon s([n]) \to [n]$ is given by

$$c_k^n = \begin{cases} \max(S) & \text{if } S \in \Phi_k^n \\ k & \text{otherwise} \end{cases}$$

Remember, the 'otherwise' case applies when $S = [n]$ or $S = [n] \setminus \{k\}$. It can be checked that this choice of u_k^n makes the upper triangle in diagram 5.7 commute, defining a valid solution $\overline{\overline{\Phi}}$ which can be inverse transposed twice to obtain Φ. We use the co-cone for $\mathrm{Ex}^\infty(X)$ to transport this solution to $\mathrm{Ex}^\infty(X)$, where it has to be a filler for the original horn y, by definition of the co-cone. Since we have shown that any horn has a filler, $\mathrm{Ex}^\infty(X)$ is a Kan complex. □

5.3 Degenerate fillers

Recall that we filled a Λ_2^2-horn by wrapping a subdivided 2-simplex around it in figure 5.2. This is essentially what u_2^2 does, only using higher subdivision. Assuming that the legs of the horn are not degenerate, this embedding is evidently different from the embedding that constitutes a degenerate 2-simplex in $\mathrm{Ex}^\infty(X)$ based on the left leg of the horn:

But if the horn is degenerate, then this embedding constitutes the degenerate filler. In other words, the filling procedure defined in the proof does not choose degenerate solutions, and in particular does not define an effective Kan complex.

Before we derive a more rigorous counterexample, note that there is another problem with the proof. Each extension by Ex generates new fillers for every horn all over again. This was necessary because the previous extension might have introduced as yet unfilled horns, but it also means that the chosen filler depends on which extension $\mathrm{Ex}^m(X)$ we factor the horn through. To even get an *algebraic* Kan complex, we would have to be consistent in our choice of m; we would probably always choose the smallest possible value. In a constructive setting, a horn would be given from the start as a map $y \colon \Lambda_k^n \to \mathrm{Ex}^{m'}(X)$ (and not as a map to $\mathrm{Ex}^\infty(X)$; this is the analog of "$x \in \bigcup_i A_i$" meaning "$x \in A_m$ for some m"). But in general it is undecidable whether y factors through $\mathrm{Ex}^m(X)$ for any m smaller than

m'. So using the current approach we cannot even make $\mathrm{Ex}^\infty(X)$ into an algebraic Kan complex constructively, let alone an effective Kan complex.

We return to the issue of degenerate fillers. Let X be a simplicial set. Consider a horn $\phi\colon \Lambda^2_2 \to \mathrm{Ex}^{m-1}(X)$ for some $m \geq 2$, and let $\Phi\colon \Delta^2 \to \mathrm{Ex}^m(X)$ be the chosen filler of $\rho \circ \phi$. Define a degenerate horn $\psi\colon \Lambda^3_3 \to \mathrm{Ex}^m(X)$ as $\Phi \circ s_0 \circ c$ (where $c\colon \Lambda^3_3 \hookrightarrow \Delta^3$ is a horn inclusion), and let $\Psi\colon \Delta^3 \to \mathrm{Ex}^{m+1}(X)$ be the chosen filler of $\rho \circ \psi$. We compare Ψ to the degenerate solution $\rho \circ \Phi \circ s_0$, transposing both maps three times:

$$
\begin{aligned}
\overline{\overline{\overline{\Psi}}} &= \overline{\overline{\psi} \circ u^3_3} = \overline{\overline{\Phi \circ s_0 \circ c} \circ u^3_3} \\
&= \overline{\overline{\Phi}} \circ \mathrm{Sd}^2(s_0) \circ \mathrm{Sd}^2(c) \circ \mathrm{Sd}(u^3_3) \\
&= \overline{\phi} \circ u^2_2 \circ \mathrm{Sd}^2(s_0) \circ \mathrm{Sd}^2(c) \circ \mathrm{Sd}(u^3_3) \\
&\overset{?}{=} \overline{\overline{\overline{\rho \circ \Phi \circ s_0}}} \\
&= \overline{\overline{\lambda \circ \mathrm{Sd}(\Phi)}} \circ \mathrm{Sd}^3(s_0) \\
&= \overline{\overline{\Phi \circ \lambda}} \circ \mathrm{Sd}^3(s_0) \\
&= \overline{\overline{\Phi}} \circ \mathrm{Sd}^2(\lambda) \circ \mathrm{Sd}^3(s_0) \\
&= \overline{\phi} \circ u^2_2 \circ \mathrm{Sd}^2(\lambda) \circ \mathrm{Sd}^3(s_0)
\end{aligned}
$$

Suppose by way of contradiction that the equality holds. Since we are working towards a counterexample, we are free to assume X and ϕ are such that $\overline{\phi}$ is a monomorphism. This convenient assumption is stronger than necessary, but not unreasonable; we can e.g. take $\overline{\phi}$ to be the identity on $\mathrm{Sd}(\Lambda^2_2)$. Because $\overline{\phi}$ is a monomorphism, we can take it away from both sides of the equation:

$$
u^2_2 \circ \mathrm{Sd}^2(s_0) \circ \mathrm{Sd}^2(c) \circ \mathrm{Sd}(u^3_3) = u^2_2 \circ \mathrm{Sd}^2(\lambda) \circ \mathrm{Sd}^3(s_0) \tag{5.8}
$$

This is a morphism from $\mathrm{Sd}^3(\Delta^3) \simeq N(s(s(s([3]))))$ to $\mathrm{Sd}(\Lambda^3_3) \simeq N(\Phi^3_3)$. Now we really have to get our hands dirty. We will show that both sides of the equation map a particular vertex to different images. Since the nerve of a category sends objects to vertices bijectively, we can interpret the expressions in equation 5.8, on the level of vertices, as functions from $s(s(s([3])))$ to $\Phi^3_3 \subseteq s([3])$. The latter set contains chains of the simplex $[3] = \{0, 1, 2, 3\}$, whereas the former is the set of chains of chains of chains of [3] (all chains are non-empty). From its original definition, we find that λ acts like "max" on this level. The degeneracy map s_0 becomes the integer map σ_0 from the simplex category, subtracting by 1 from all but 0. The maps u^n_k are defined by ψ^n_k from the proof of Theorem 5.4. Finally, the inclusion map c of course still acts trivially. Also recall that the functor s was defined on morphisms as $s(f)\colon S \mapsto f(S)$, or in words: the subdivision of a map is the element-wise application of that map. Therefore, the effect of the right hand side of equation 5.8 on the element $\{\{\{0,1,2\}\}\} \in s(s(s([3])))$ is

$$\{\{\{0,1,2\}\}\} \xmapsto{\mathrm{Sd}^3(s_0)} \{\{\{0,1\}\}\} \xmapsto{\mathrm{Sd}^2(\lambda_{\Delta^2})} \{\{1\}\} \xmapsto{u_2^2} \{1\}$$

while the left hand side takes

$$\{\{\{0,1,2\}\}\} \xmapsto{\mathrm{Sd}(u_3^3)} \{\{3\}\} \xmapsto{\mathrm{Sd}^2(c)} \{\{3\}\} \xmapsto{\mathrm{Sd}^2(s_0)} \{\{2\}\} \xmapsto{u_2^2} \{2\}$$

We must conclude that equation 5.8 does not hold. But this is a contradiction, and therefore Ψ cannot be the degenerate filler of $\rho \circ \psi$. If we transported everything to $\mathrm{Ex}^\infty(X)$ using the co-cone, we would find that the filling problem for Ψ is effectively a degenerate problem based on the filling problem for Φ. Hence we see that the proof of Theorem 5.4 does not naturally generate an effective Kan complex.

5.4 Outlook

We have not proved rigorously that the infinite extension of a simplicial set cannot be given the structure of an effective Kan complex constructively in some other way. Non-constructively, when filling a horn we would simply check whether it is degenerate or not and act accordingly, but this requires the principle of excluded middle (more prosaically: we do not always have an algorithm that can decide whether a given horn is degenerate in a finite amount of time). Whenever we were successful at constructing an effective Kan fibration, we first defined a lifting procedure and afterwards found out that it "accidentally" chooses degenerate solutions when necessary. The conceptual design of infinite extension seems to rule out such a filling procedure; the way in which horns are provided with new fillers is inherently different from the way in which (degenerate) simplices are transported across extensions, and they do not coincide, except in trivial cases.

Another implementation of fibrant replacement uses the pair of adjoint functors between the category of simplicial sets and the category of topological spaces we referred to in Section 2.1. They are called *geometric realization* $|-|\colon \widehat{\mathbf{\Delta}} \to \mathbf{Top}$ and the *singular simplicial set* functor $\mathrm{Sing}\colon \mathbf{Top} \to \widehat{\mathbf{\Delta}}$. There exists a weak homotopy equivalence $X \simeq \mathrm{Sing}(|X|)$, and the singular simplicial set of any topological space is a Kan complex. However, it was found in [Gee23] that the traditional proof of the latter fact does not provide the structure of an effective Kan complex, similar to the issue with Ex^∞.

It seems likely that we will have to give up on these luxurious constructions, and rely on the most basic implementation of fibrant replacement, which is obtained from a so-called *weak factorization system* on simplicial sets. A weak factorization system allows us to factor any simplicial morphism as a weak homotopy equivalence followed by a fibration. We obtain a fibrant object Q from a simplicial set X by factorizing the map from X to the terminal object as $X \to Q \to \Delta^0$ in this way. Developing a weak factorization system is an essential part of the program of effective Kan fibrations anyway, and should therefore provide us with at least this version of "effectively fibrant" replacement.

Conclusion

We have seen both positive and negative results for effective Kan fibrations. Chapter 3 demonstrates that a proof of some simplicial map being a Kan fibration often induces the structure of an effective Kan fibration automatically, even though it takes considerable extra work to prove this. An exception to this tendency is the Ex^∞ construction, which is a true loss, because as a form of fibrant replacement it is much more useful than any other implementation. At the same time, we have developed a good understanding of *why* Ex^∞ fails to produce effective Kan complexes, so that we know what to look out for in the future. As a small excursion, we have found that semisimplicial sets are of limited utility in the theory of effective Kan fibrations.

Degenerate-preferring algebraic Kan fibrations are significantly easier to work with than generic effective Kan fibrations. Nonetheless, in Theorem 3.9 we had to work with both variants, so that it is advisable to continue studying them together. On top of this, one should keep in mind that we have used a simplified and more restrictive version of the original effective Kan fibration, and that the latter might remain the ultimate variant of Kan fibrations.

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