Stable Canonical Rules for Intuitionistic Modal Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **dr. Nick Bezhanishvili**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
June 23 2022

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Abstract

This thesis develops the theory of stable canonical formulas and rules for intuitionistic modal logics and Heyting-Lewis logics. We prove that every intuitionistic modal (or Heyting-Lewis) multi-conclusion consequence relation is axiomatizable by stable canonical rules. This allows us to assume without loss of generality that rules that we consider are stable canonical rules in many cases when we study intuitionistic modal logics and Heyting-Lewis logics, which turns out to be quite useful. In particular, our method gives an alternative proof of the Blok-Esakia theorem for intutionsitic modal logics, and helps us find an error in the proof of that theorem for Heyting-Lewis logics. Besides, using stable canonical rules, we also prove an analogue of the Dummett-Lemmon conjecture for intutionistic modal multi-conclusion consequence relations which states that an intuitionistic modal multi-conclusion consequence relation is Kripke complete if and only if its least modal companion is.

Acknowledgements

First, I am extremely grateful to my supervisor – Nick Bezhanishvili, for his guidance and support during the creation of this thesis. Nick aroused my interest in the topic about stable canonical rules even before I joined ILLC by his wonderful speech given at AiML 2020. After coming to ILLC, I did a research project under his supervision, which in the end grew up into this thesis. Nick assisted me with his useful comments and suggestions during the whole process. He sacrificed quite a lot of his precious time in order to discuss and exchange emails with me about the details of the project and the thesis (I cannot remember how many times he sent me his careful comments). I also owe him a lot for his help during PhD application season. It is no exaggeration to say that without Nick, many things I have achieved about logic would not have been possible.

I thank Lev Beklemishev, Malvin Gattinger and Tommaso Moraschini for joining my thesis defence committee, and for their stimulating questions about my work. I thank Antonio Cleani for his clarification (also correction) about his proof strategy for the Dummett-Lemmon conjecture for superintuitionisticlogics, which is quite helpful for me to get the final proof of the corresponding result for intuitionistic modal logics. I also thank Rodrigo N. Almeida for his useful comments on the first draft of this thesis and bringing the **Key Statement** into my attention.

I thank my academic mentor Yde Venema for his guidance about the master program and my bachelor supervisor Yanjing Wang for his help during my PhD application process. I also want to express my gratitude towards the whole ILLC community. It is a wonderful experience to work with so many friendly and excellent MoL students.

Finally, I want to thank my parents and other family members for their endless care and unconditional support. It was an unbearable loss that my grandfather passed away in the summer of my first year at ILLC. This thesis is also dedicated to him, as a compensation which is too late.

Contents

Chapter 1

Introduction

This thesis studies classical modal and intuitionistic modal logics, and connections between these systems. One important aspect of the study of modal logics is the investigation of lattices of logics. In this area, semantic methods in general and uniform axiomatization technique in particular, play a central role.

One such important method was developed by Zakharyaschev in 1980s. He introduced *canonical formulas* for superintuitionistic logics[1] and proved that every superintuitionistic logic can be axiomatized by canonical formulas [40]. The brief idea is as follows: for every formula, using a variant of selective filtrations, one can obtain a finite set of finite refutation patterns (i.e., a finite intuitionistic frame with a set of parameters). Canonical formulas syntactically encode these finite refutation patterns such that the conjunction of them is equivalent to the original formula. Zakharyaschev [43, 44] then also developed canonical formulas for transitive normal modal logics in a series of papers and proved that every normal extension of **K4** can be axiomatized by modal canonical formulas. Following the same idea, Jeřábek [27] generalized the result to multi-conclusion rules and showed that every normal modal multi-conclusion consequence relation over **K4** is axiomatizable by *canonical rules*. It turns out that canonical formulas and rules offer a uniform method to study superintuitionistic and modal logics, and are thus quite useful. For example, using canonical formulas, Zakharyaschev proved the Dummett-Lemmon conjecture stating that a superintuitionistic logic is Kripke complete iff its least modal companion is [41], and with canonical rules, Jeřábek gave an alternative proof of decidability of admissibility in the intuitionistic propositional calculus [27].

[1]These are extensions of the intuitionistic propositional calculus.

However, the mechanisms of developing canonical formulas and rules are model-theoretic and are quite involved. Bezhanishvili *et al.*[2, 3, 6] developed an algebraic approach to canonical formulas and rules via duality. They showed that from the algebraic perspective, Zakharyaschev's canonical formulas for superintuitionistic logics encode the $\vee$-free reducts of Heyting algebras which are locally finite, and the so-called *closed domain condition* encoded in the formulas corresponds to the preservation of joins of certain elements [2].

Besides, the algebraic perspective raises a natural question: can we develop canonical formulas and rules based on the $\to$-free reducts of Heyting algebras which are also locally finite? This idea led to the study of stable canonical formulas and rules [4, 5, 6, 7], which encode these $\to$-free reducts, and are alternatives to Zakharyaschev's canonical formulas and Jeřábek's canonical rules. Since stable canonical formulas and rules encode finite refutation patterns constructed by taking filtrations instead of selective filtrations, they can apply to non-transitive logics where Zakharyaschev's canonical formulas and Jeřábek's canonical rules do not apply[2]. In particular, it was proved in [6] that every normal modal multi-conclusion consequence relation is axiomatizable by stable canonical rules, which partially answered the question about developing canonical formulas for extensions of **K** proposed in [15, Problem 9.5].

Although the research on stable canonical formulas and rules is still in its infancy, quite some effort has already been put into the extension of stable canonical formulas and rules to different settings and development of related theories. For example, in her PhD thesis [26], Illn gave a thorough analysis of *stable logics* which are logics axiomatized by certain type of stable formulas and enjoy the finite model property. Melzer [30] developed the stable canonical formulas for the lax logic while Cleani [16] generalized stable canonical formulas and rules to the setting of bi-superintuitionistic logics, and also used them to prove the Blok-Esakia theorem and the Dummett-Lemmon conjecture in that setting.

On the other hand, compared to superintuitionisic logics and classical modal logics, the area of intuitionistic modal logics (i.e. superintuitionistic logics plus modal operators) is much more involved and less well-understood. Intuitionistic modal logics were introduced by Fisher Servi [33], whose basic aim was to define modalities from an intuitionistic point of view. These logics are applicable to a wide variety of situations, ranging from computer science [17] to epistemic logics [1]. However, the study of intuitionistic modal

[2]More details can be found in [26].

logics is far from an easy combination of the study of classical modal logics and of superintuitionistic logics. In fact, there still remain some fundamental philosophical and technical questions. For example, unlike in classical modal logics, $\Box$ and $\Diamond$ are not dual to each other in intuitionistic modal logics. A natural question is: what counts as a reasonable relation between these two operators? We still do not know much about the finite model property of intuitionistic modal logics[3] when they have both $\Box$ and $\Diamond$. Thus, this area may still benefit from some new uniform and effective methods.

Considering the above situation, in this thesis we will develop stable canonical rules and formulas for intuitionistic modal logics and related systems. These on the one hand will deepen our understanding of the theory of stable canonical formulas and rules and verify its wide applicability, and on the other hand may provide us with a uniform method to study intuitionistic modal logics and thus pave the way for further study.

In particular, we prove that every intuitionistic modal multi-conclusion consequence relation is axiomatizable by stable canonical rules, and every intuitionistic modal logic over $\mathbf{IntS4}_\Box$ is axiomatizable by stable canonical formulas. Following the main proof strategy of [16], we give an alternative proof of the Blok-Esakia theorem for intuitionistic modal logics [38] and generalize it to multi-conclusion consequence relations, which states that the lattice of intuitionistic modal multi-conclusion consequence relations is isomorphic to the lattice of extensions of the bimodal multi-conclusion consequence relation $\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R$. By adjusting the proof strategy of [16], we give a proof of the Dummett-Lemmon conjecture for intuitionistic modal multi-conclusion consequence relations stating that an intuitionistic modal multi-conclusion consequence relation is Kripke complete if and only if its least modal companion is, which, as far as we know, is a new result.[4]

Besides, we also develop stable canonical rules for Heyting-Lewis logics which are superintuitionistic logics with a weak implication and can be seen as an extension of intuitionistic modal logics. However, our strategy for proving the Blok-Esakia theorem does not go so smoothly in this setting. In particular, we identify a statement which is equivalent to the Blok-Esakia theorem, whose correctness is unclear.[5] Therefore, right now whether the Blok-Esakia theorem holds for intuitionistic modal logics remains an open

[3] See [34] for one example.

[4] The proof in this thesis is slightly different from that of [16], which contains a small gap. The restriction of our proof to the setting of superintuitionistic logics also fills this gap.

[5] In fact, we have reasons to believe that it may not hold.

problem.[6]

To summarise, our main contributions are as follows:

- Development of the theory of stable canonical rules and formulas for intuitionistic modal logics.
- Development of the theory of stable canonical rules for Heyting-Lewis logics.
- An alternative proof of the Blok-Esakia theorem for intuitionistic modal logics [38] and a generalization of it to multi-conclusion consequence relations.
- A proof of the Dummett-Lemmon conjecture for intuitionistic modal multi-conclusion consequence relations.
- Identification of a statement equivalent to the Blok-Esakia theorem for Heyting-Lewis logics, which leads to a gap in the proof of this theorem given in [23].

The thesis is structured as follows. Chapter 2 is devoted to general preliminaries which are needed thoroughout the thesis. In Chapter 3, we develop stable canonical formulas and rules for intuitionistic modal logics. Using duality theory, a dual description of stable canonical rules for intuitionistic modal logics is also given. Chapter 4 is about the application of stable canonical rules for intuitionistic modal logics. We first introduce the Gödel translation for intuitionistic modal logic, and then use our rules to prove the Blok-Esakia theorem. Then with the Blok-Esakia theorem, we prove the Dummett-Lemmon conjecture for intuitionistic modal multi-conclusion consequence relations. In Chapter 5, we develop stable canonical rules for Heyting-Lewis logics and then use them to point out an error in the proof of the corresponding Blok-Esakia theorem. In the last chapter, we conclude and discuss possible directions for future work.

[6] Although the Blok-Esakia theorem was claimed to hold for intuitionistic modal logic in [23], we found a gap in the proof of this theorem [23, Lem. 4.18]. So far, this gap remains unfilled, which has been confirmed by the authors in private communication.

Chapter 2

Preliminaries

In this chapter, we present basic notations, definitions and facts that will be used throughout the thesis. We assume familiarity with standard set-theoretic notations, elementary lattice theory and some fundamental concepts from topology and first-order logic. Familiarity with the categorical notion of duality is also expected but is not necessary.

2.1 Ordered sets

We begin with the following notations and definitions related to ordered sets.

Definition 2.1.1 (Maximal and passive elements)**.** Let X be a set, R be a transitive binary relation on X and $U \subseteq X$. We say that $x \in U$ is an *R-maximal* element of U if for any $y \in U$, Rxy implies that $x = y$. We define $\max_R(U)$ as the set of all R-maximal elements of U.

An element $x \in U$ is called *R-passive* in U if for all $y \in X \setminus U$(or $\bar{U}$), if Rxy, then there is no $z \in U$ such that Ryz. The set of all R-passive elements of U is denoted as $pas_R(U)$.

Definition 2.1.2. For any reflexive and transitive (binary) relation R on a set X, a subset $C \subseteq X$ is an *R-cluster* if it is an equivalence class under the relation $\backsim_R$ where $x \backsim_R y$ iff xRy and yRx.

An R-cluster is *proper* if it contains more than one element.

For any $U \subseteq X$, U is said to *cut an R-cluster* C if $U \cap C \neq \emptyset$ and $C \setminus U \neq \emptyset$.

Remark 2.1.3. As usual, for any equivalence relation R on a set X, we use $[x]$ (or $[x]_R$) to denote the equivalence class of x where $x \in X$.

By definition, U does not cut an R-cluster C if it either contains C or is disjoint from C.

Definition 2.1.4 (Upsets and downsets)**.** Let $(A, \leq)$ be a poset (partially ordered set) and let $B \subseteq A$. We call B *upwards closed* or an *upset* if $x \in B$, $y \in A$ and $x \leq y$ imply $y \in B$. If $C \subseteq A$, we write $\uparrow C$ for the least upset that contains C, namely $\{y \in A \mid \exists x \in C : x \leq y\}$. And we use $Up(A)$ to denote the set of all upsets of $(A, \leq)$.

Similarly, we call B *downwards closed* or a *downset* if $x \in B$, $y \in A$ and $y \leq x$ imply $y \in B$. If $C \subseteq A$, we define $\downarrow C = \{y \in A \mid \exists x \in C : y \leq x\}$.

2.2 Universal algebra

In this section we recall some basic definitions and results from universal algebra, all of which can be found in [12].

2.2.1 Algebras and operations

We first recall the definition of algebras and some operations on them.

Definition 2.2.1 (Signature)**.** Let $\mathcal{F}$ be a set and $\tau : \mathcal{F} \to \mathbb{N}$ be a map, we call τ a *signature* or *language*, and we call $\mathcal{F}$ the corresponding set of *function symbols*. For $f \in \mathcal{F}$, let $\tau(f)$ be the *arity* of f. We call f a *constant symbol*, if $\tau(f) = 0$.

Definition 2.2.2 (Algebra)**.** Let A be a non-empty set, $\tau : \mathcal{F} \to \mathbb{N}$ be a signature and $F = \{f^A \mid f \in \mathcal{F}, f^A : A^{\tau(f)} \to A\}$, we call (A, F) an *algebra* in the signature τ (or simply τ-*algebra*). If the context is clear, we will denote by A both the algebra and the underlying set (also called the *carrier*).

Remark 2.2.3. For convenience, we will use the same notations for function symbols and their corresponding interpretations.

Let (A, F) be a τ-algebra. Then (A, F') is called the *reduct* of (A, F) if $F' \subseteq F$. Because of the above remark, we may also write a reduct of (A, F) as $(A, F)|_{\tau'}$ or simply $A|_{\tau'}$ where $\tau' \subseteq \tau$.

Definition 2.2.4 (Homomorphism)**.** Let A and B be algebras in a signature $\tau : \mathcal{F} \to \mathbb{N}$. A function $g : A \to B$ is a *homomorphism* if for every $f \in \mathcal{F}$ with $n := \tau(f)$ and every $a_1, \ldots, a_n \in A$, we have:

$$g(f^A(a_1, \ldots, a_n)) = f^B(g(a_1), \ldots, g(a_n)).$$

B is a *homomorphic image* of A if there exists a surjective homomorphism from A to B.

Definition 2.2.5 (Embedding and isomorphism)**.** Let $f : A \to B$ be a homomorphism, f is called an *embedding* if f is injective. If, in addition, f is surjective, then it is an *isomorphism* and we say that A and B are *isomorphic*.

Definition 2.2.6 (Congruence)**.** Let A be a τ-algebra, an equivalence relation $\theta \subseteq A \times A$ is a *congruence* if for every $f \in \mathcal{F}$ with $n \coloneqq \tau(f)$, we have:

$$(a_1, b_1) \in \theta, \ldots, (a_n, b_n) \in \theta \implies (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta,$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$.

Definition 2.2.7 (Quotient)**.** Let A be a τ-algebra and let θ be a congruence on A. Let B be the τ-algebra with an underlying set $A/\theta \coloneqq \{[a]_\theta \mid a \in A\}$ such that for every $f \in \mathcal{F}$ with $n \coloneqq \tau(f)$ and every $a_1, \ldots, a_n \in A$:

$$f^{A/\theta}(a_1, \ldots, a_n) = [f^A(a_1, \ldots, a_n)]_\theta.$$

We denote B by A/θ and call it the *quotient algebra* of A by θ.

Remark 2.2.8. By the definition of congruence, the operations in the quotient are well defined.

Definition 2.2.9 (Subalgebra)**.** Let A and B be τ-algebras such that $A \subseteq B$. If the inclusion function of A into B is a homomorphism, then A is a *subalgebra* of B.

Definition 2.2.10 (Direct product)**.** Let $\{A_i \mid i \in I\}$ be a set of τ-algebras. Let B be the τ-algebra with an underlying set $\Pi_{i \in I} A_i$ such that for every $f \in \mathcal{F}$ with $n \coloneqq \tau(f)$, every $\overline{a}_1, \ldots, \overline{a}_n \in \Pi_{i \in I} A_i$ and $i \in I$, we have:

$$f^{\Pi_{i \in I} A_i}(\overline{a}_1, \ldots, \overline{a}_n)(i) = f^{A_i}(\overline{a}_1(i), \ldots, \overline{a}_n(i)).$$

We denote B by $\Pi_{i \in I} A_i$ and call it the *direct product* of $\{A_i \mid i \in I\}$.

In order to define the next operation on algebras, we need the notion of an ultrafilter.

Definition 2.2.11 (Filter and ultrafilter)**.** Let I be a set and let $F \subseteq \mathfrak{P}(I)$ be non-empty. F is called a *filter* on I if the following hold:

1. For any $A, B \in F$, we have that $A \cap B \in F$.
2. F is an upset in $(\mathfrak{P}(I), \subseteq)$.

If $F \neq \mathfrak{P}(I)$, it is *proper*. If F is maximal with this property, it is an *ultrafilter*.

Now we can define the last operation on algebras:

Definition 2.2.12 (Ultraproduct)**.** Let $\{A_i \mid i \in I\}$ be a set of τ-algebras and U be an ultrafilter on I. Let θ_U be the congruence on $\Pi_{i \in I} A_i$ defined by:

$$(\overline{a}, \overline{b}) \in \theta_U \iff \{i \in I \mid \overline{a}(i) = \overline{b}(i)\} \in U.$$

$\Pi_{i \in I} A_i / \theta_U$ is called the *ultraproduct* of $\{A_i \mid i \in I\}$ on U.

Let $\mathcal{K}$ be a class of τ-algebras, we then introduce the following class operators based on the operations we have defined:

$$\begin{aligned}
\mathbb{I}(\mathcal{K}) &\coloneqq \{A \mid A \text{ is isomorhpic to some } B \in \mathcal{K}\};\\
\mathbb{H}(\mathcal{K}) &\coloneqq \{A \mid A \text{ is a homomorphic image of some } B \in \mathcal{K}\};\\
\mathbb{S}(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is a subalgebra of some } B \in \mathcal{K}\});\\
\mathbb{P}(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is a direct product of some } \{B_i\}_{i \in I} \subseteq \mathcal{K}\});\\
\mathbb{P}_U(\mathcal{K}) &\coloneqq \mathbb{I}(\{A \mid A \text{ is an ultraproduct of some } \{B_i\}_{i \in I} \subseteq \mathcal{K}\}).
\end{aligned}$$

2.2.2 Varieties and universal classes

We have seen that an algebra is a set with some functions. Thus, it is also a model in first-order logic. In particular, we can evaluate formulas on an algebra in the obvious way. It turns out that there is a close connection between the syntactic types of the defining formulas and the closure conditions on the algebras validating them.

Definition 2.2.13 (Universal class)**.** Let τ be an arbitrary signature. A class of τ-algebras is a *universal class* if it is the class of models of some set of universal sentences[1].

If $\mathcal{K}$ is a class of τ-algebra, we write $Uni(\mathcal{K})$ for the least universal class that contains $\mathcal{K}$. $Uni(\mathcal{K})$ is also called the universal class *generated by* $\mathcal{K}$.

The following is a useful characterisation of Uni in terms of the operators from the previous section.

[1] "Universal" in the sense of the one defined in first-order logic. See [18, Def. 5.6].

Theorem 2.2.14. *[12, Thm. 2.20] Universal classes are closed under $\mathbb{S}$ and $\mathbb{P}_u$. Furthermore:*

$$Uni(\mathcal{K}) = \mathbb{S}\mathbb{P}_u(\mathcal{K}).$$

Another interesting kind of classes of algebras in this thesis is called *variety*.

Definition 2.2.15 (Variety)**.** Let τ be an arbitrary signature, a class of τ-algebras V is a *variety* if it is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$.

Let $\mathcal{K}$ be a class of τ-algebras, we write $Var(\mathcal{K})$ for the least variety which contains $\mathcal{K}$. $Var(\mathcal{K})$ is also called the variety *generated by* $\mathcal{K}$.

The characterisation of Var in terms of the operators from the previous section is given as follows:

Theorem 2.2.16. *[12, Thm. 9.5] Let τ be an arbitrary signature and Let $\mathcal{K}$ be a class of τ-algebras. Then $Var(\mathcal{K}) = \mathbb{H}\mathbb{S}\mathbb{P}(\mathcal{K})$.*

Similarly to universal classes, varieties are defined by special formulas called *equations*.

Definition 2.2.17 (Equation)**.** A first-order sentence φ is called an *equation* if it is of the form $\sigma = \tau$, where σ and τ are terms.

This following result is called Birkhoff's Theorem.

Theorem 2.2.18. *[12, Thm. 11.9] Let τ be an arbitrary signature and Let V be a class of τ-algebras. Then V is a variety if and only if V is definable by equations, namely there is a set of equations Φ such that:*

$$V = \{A \text{ is an algebra in } \tau \mid A \vDash \Phi\}.$$

Another useful result which is also referred to as Birkhoff's Theorem in many textbooks states that every variety is generated by a certain type of algebras called *subdirectly irreducible* algebras.

Definition 2.2.19 (Subdirect embedding and product)**.** An embedding $A \to \Pi_{i\in I} A_i$ is called *subdirect* if for every $i \in I$, we have $(\pi_i \circ f)[A] = A_i$. Here π_i denotes the projection onto the i-th coordinate.

If $A \leq \Pi_{i\in I} A_i$ and the inclusion function is a subdirect embedding, we call A a *subdirect product* of $\{A_i \mid i \in I\}$.

Definition 2.2.20 (Subdirect irreducibility)**.** An algebra A is called *subdirectly irreducible* (*s.i* for short) if for every subdirect embedding $f : A \to \Pi_{i\in I} A_i$, there exists $i \in I$ such that $(\pi_i \circ f) : A \to A_i$ is an isomorphism.

Theorem 2.2.21. *[12, Cor. 9.7] Every variety V is generated by its subdirectly irreducible members.*

Finally, we define the property called *locally finiteness* for varieties, which is closely related to the finite model property of logics.

Definition 2.2.22 (finitely generated)**.** Let A be an algebra and $X \subseteq A$, the least subalgebra of A containing X is called the subalgebra *generated by* X.

An algebra A is *n-generated*, where $n \in \mathbb{N}$, if there exists a set $B \subseteq A$ with $|B| \leq n$ that generates A. If A is n-generated for some $n \in \mathbb{N}$, A is said to be *finitely generated*.

Definition 2.2.23 (Locally finite)**.** A variety V is *locally finite* if every finitely generated algebra in V is finite.

2.3 Deductive systems

In this section, we present two types of deductive systems which will be explored throughout the thesis. The following presentation is mainly based on [25].

Definition 2.3.1. The *set of formulas in signature v over a set of variable X* (denoted by $Form_v(X)$) is the least set containing X such that for any $f \in v$, we have that $\varphi_1, ..., \varphi_n \in Form_v(X)$ implies that $f(\varphi_1, ..., \varphi_n) \in Form_v(X)$ where f is of arity n.

Let $Prop$ be a fixed countably infinite set of variables, we write $Form_v$ for $Form_v(Prop)$. A *substitution* s is map from $Prop$ to $Form_v$, which can be recursively extended to a map $\bar{s}$ from $Form_v$ to $Form_v$ in the obvious way.

First, we define what a *logic* is.

Definition 2.3.2. A *logic* over $Form_v$ is a set $L \subseteq Form_v$ such that if $\varphi \in L$, then $\bar{s}(\varphi) \in L$ for every substitution s.

Remark 2.3.3. By the above definition, classical propositional logic is a logic over $Form_{\wedge,\vee,\neg}$.

Definition 2.3.4 (Multi-conclusion rule)**.** A *multi-conclusion rule* in signature v over a set of variable X is a pair Γ/Δ of finite subsets of $Form_v(X)$.

Remark 2.3.5. In case $\Delta = \{\psi\}$, we simply write Γ/ψ for Γ/Δ, similarly if $\Gamma = \{\varphi\}$.

We write $Rul_v(X)$ for the *set of all multi-conclusion rules in signature v over the set of variable X*, and we let Rul_v stand for $Rul_v(Prop)$.

As logics are defined over formulas, *multi-conclusion consequence relations* are defined over multi-conclusion rules.

Definition 2.3.6 (Multi-conclusion consequence relation)**.** A *multi-conclusion consequence relation* over Rul_v is a set $S \subseteq Rul_v$ such that the following hold:

- If $\Gamma/\Delta \in S$, then $\bar{s}[\Gamma]/\bar{s}[\Delta] \in S$ for all substitutions s.
- $\varphi/\varphi \in S$.
- If $\Gamma/\Delta \in S$, then $\Gamma; \Gamma'/\Delta; \Delta' \in S$ for any finite sets of formulas Γ' and Δ'.
- $\Gamma/\Delta'; \varphi \in S$ and $\Gamma; \varphi/\Delta \in S$, then $\Gamma/\Delta \in S$ (Cut).

If L is a logic and Δ is a set of formulas, we write $L \oplus \Delta$ for the least logic extending L that contains Δ, and say that the logic is axiomatized over L by Δ. Similarly we define $S \oplus \Sigma$ where S is a multi-conclusion consequence relation and Σ is a set of multi-conclusion rules.

We then define the interpretation of formulas and rules over algebras in the same signature. Let $\mathfrak{A}$ be a v-algebra and A be its carrier, a *valuation* on $\mathfrak{A}$ is a map V from $Prop$ to A, which can be recursively extended to a map $\bar{V}$ from $Form_v$ to A in the most obvious way.

In the following, every algebra we consider is assumed to have the top element (or the largest element, denoted by 1). This will make the following definition of *validity* simpler, and will not cause any problem as we only consider such algebras in this thesis. For a more general definition of validity in algebraic semantics, one can consult [28].

Definition 2.3.7 (Validity)**.** A rule Γ/Δ in signature v is *valid* on a v-algebra $\mathfrak{A}$ if for any valuation V on $\mathfrak{A}$, if $\bar{V}(\gamma) = 1$ for any $\gamma \in \Gamma$, then $\bar{V}(\delta) = 1$ for some $\delta \in \Delta$ where 1 is the top element of $\mathfrak{A}$. We denote this as $\mathfrak{A} \vDash \Gamma/\Delta$.

A formula φ in signature v is *valid* on a v-algebra $\mathfrak{A}$ if the rule $/\varphi$ is valid on $\mathfrak{A}$, and we denote this by $\mathfrak{A} \vDash \varphi$.

Remark 2.3.8. With the above definition, the notion of validity of a rule Γ/Δ on a class $\mathcal{K}$ of v-algebras and the notion of validity of a set of rules S on a v-algebra $\mathfrak{A}$ are then defined as usual, and we denote them as $\mathcal{K} \vDash \Gamma/\Delta$ and $\mathfrak{A} \vDash S$ respectively.

Then for any logic L, we say that L is *complete* w.r.t a class of algebras $\mathcal{K}$ if $\mathcal{K} \vDash \varphi$ implies that $\varphi \in L$. Similarly we define the completeness of a multi-conclusion consequence relation.

Finally, we fix two useful notations. Let $\mathcal{A}_v$ be the class of all v-algebras and S be a logic or a multi-conclusion consequence relation, we write $Alg(S)$ for the set of all v-algebras which validate S, i.e., $Alg(S) = \{\mathfrak{A} \in \mathcal{A}_v \mid \mathfrak{A} \vDash S\}$. Conversely, if $\mathcal{K}$ is a set of v-algebras, we define $Ru(\mathcal{K}) = \{\Gamma/\Delta \in Rul_v \mid \mathcal{K} \vDash \Gamma/\Delta\}$ and $Th(\mathcal{K}) = \{\varphi \in Form_v \mid \mathcal{K} \vDash \varphi\}$.

2.4 Bimodal logics

2.4.1 Deductive systems for bimodal logics

We have presented the general theory of universal algebra and deductive systems. Now we can apply them in more concrete settings. In this section, we introduce bimodal deductive systems. For convenience, sometimes bimodal logics just mean bimodal deductive systems when whether they are logics or multi-conclusion consequence relations does not matter (the title of this section is an example).

The *bimodal signature* $bi = \{\wedge, \vee, \neg, \top, \bot, \Box_I, \Box_M\}$, and the set of *bimodal formulas* $Form_{bi}$ is then defined recursively as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box_I\varphi \mid \Box_M\varphi$$

As usual, $\varphi \to \psi$ stands for $\neg\varphi \vee \psi$ for any bimodal formulas φ and ψ.

Remark 2.4.1. The subscript I means "intuitionistic" while the subscript M means "modal". The reason for using these subscripts will only become clear in Chapter 4. Right now, we use them simply as a way to distinguish these two operators.

Definition 2.4.2. A logic L over $Form_{bi}$ is a *bimodal logic* if the following hold:

- **CPC** $\subseteq L$
- $\Box_I(\varphi \wedge \psi) \leftrightarrow (\Box_I\varphi \wedge \Box_I\psi) \in L$ and $\Box_M(\varphi \wedge \psi) \leftrightarrow (\Box_M\varphi \wedge \Box_M\psi) \in L$
- $\varphi \to \psi \in L$ implies $\Box_I\varphi \to \Box_I\psi \in L$ and $\varphi \to \psi \in L$ implies $\Box_M\varphi \to \Box_M\psi \in L$ (Reg)
- $\varphi \in L$ implies $\Box_I\varphi \in L$ and $\varphi \in L$ implies $\Box_M\varphi \in L$ (Nec)

- $\varphi \to \psi$ and $\varphi \in L$ implies $\psi \in L$ (MP)

We denote the least bimodal logic by $\mathbf{K} \otimes \mathbf{K}$. This notation is justified as if restricted to the signatures $\{\wedge, \vee, \neg, \top, \bot, \Box_I\}$ or $\{\wedge, \vee, \neg, \top, \bot, \Box_M\}$[2], the least bimodal logic is just the least normal modal logic $\mathbf{K}$. Then $\mathbf{S4} \otimes \mathbf{K}$ is $\mathbf{K} \otimes \mathbf{K} \oplus (\Box_I p \to p) \oplus (\Box_I p \to \Box_I \Box_I p)$ and $\mathbf{Grz} \otimes \mathbf{K}$ is $\mathbf{S4} \otimes \mathbf{K} \oplus \Box_I(\Box_I(p \to \Box_I p) \to p) \to p$[3].

Then we introduce *bimodal multi-conclusion consequence relations* as follows:

Definition 2.4.3. A *bimodal multi-conclusion consequence relation* is a multi-conclusion consequence relation M over Rul_{bi} satisfying the following conditions:

- $/\varphi \in M$ whenever $\varphi \in \mathbf{K} \otimes \mathbf{K}$
- $\varphi / \Box_I \varphi \in M$ and $\varphi / \Box_M \varphi \in M$
- $\varphi \to \psi, \varphi / \psi \in M$

Elements in Rul_{bi} are called *bimodal multi-conclusion rules.* If L is a bimodal logic, then $\mathbf{NExt}(L)$ is the lattice of all bimodal logics extending L with $\oplus$ as join and intersection as meet. Similarly we define $\mathbf{NExt}(M)$ where M is a bimodal multi-conclusion consequence relation. Clearly, for any $L \in \mathbf{NExt}(\mathbf{K} \otimes \mathbf{K})$, there is a least bimodal multi-conclusion consequence relation L^R containing all $/\varphi$ for $\varphi \in L$. In particular, we denote the one corresponding to $\mathbf{K} \otimes \mathbf{K}$ as $\mathbf{K} \otimes \mathbf{K}^R$ (the least bimodal multi-conclusion consequence relation) and the one corresponding to $\mathbf{S4} \otimes \mathbf{K}$ as $\mathbf{S4} \otimes \mathbf{K}^R$. Conversely, for any $M \in \mathbf{NExt}(\mathbf{K} \otimes \mathbf{K})$, $Taut(M) = \{\varphi \in Form_{bi} \mid /\varphi \in M\}$ is a bimodal logic.

The following proposition allows us to transfer results about multi-conclusion consequence relations to results about logics. The proof is routine.

Proposition 2.4.4. The mappings $(-)^R$ and $Taut(-)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathbf{K} \otimes \mathbf{K})$ and the sublattice of $\mathbf{NExt}(\mathbf{K} \otimes \mathbf{K}^R)$ consisting of all bimodal multi-conclusion consequence relations M such that $Taut(M)^R = M$.

[2] In either case, we would prefer to use simply $\Box$ instead of $\Box_I$ or $\Box_M$.

[3] When restricted to the signature $\{\wedge, \vee, \neg, \top, \bot, \Box_I\}$, $\mathbf{S4} \otimes \mathbf{K}$ is just $\mathbf{S4}$, and $\mathbf{Grz} \otimes \mathbf{K}$ is just $\mathbf{Grz}$. See [15] for more information about these normal modal logics.

2.4.2 Algebraic semantics for bimodal logics

Now we introduce the algebraic semantics for bimodal logics.

Definition 2.4.5. A *modal algebra* is a tuple $\mathfrak{A} = (A, \Box)$ where A is a Boolean algebra, $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $a, b \in A$.

A $K \otimes K$*-algebra* (or *bimodal algebra*) is a tuple $\mathfrak{A} = (A, \Box_I, \Box_M)$ where $(A, \Box_I)$ and $(A, \Box_M)$ are both modal algebras.

For any bimodal logic L, we call a bimodal algebra an L*-algebra* if it validates L.

Remark 2.4.6. In particular, a bimodal algebra $\mathfrak{A} = (A, \Box_I, \Box_M)$ is an $S4 \otimes K$*-algebra* if $\Box_I a \leq a$ and $\Box_I a \leq \Box_I \Box_I a$ for any $a \in A$ (or equivalently, $\Box_I a \to a = 1$ and $\Box_I a \to \Box_I \Box_I a = 1$). A bimodal algebra $\mathfrak{A} = (A, \Box_I, \Box_M)$ is a $Grz \otimes K$*-algebra* if $\Box_I(\Box_I(a \to \Box_I a) \to a) \leq a$ for any $a \in A$[4].

Let **BMA** be the class of all bimodal algebras, by Theorem 2.2.18, **BMA** is a variety. Let **Var**(**BMA**) and **Uni**(**BMA**) denote the lattice of subvarieties and the lattice of universal subclasses of **BMA** respectively, we have the following result as usual. It says that there is a correspondence between varieties and logics, and a correspondence between universal classes and multi-conclusion consequence relations. Proofs of similar results for modal algebras and (unary) normal modal logics can be found in [8, Thm. 2.5] and [15, Thm. 7.56].

Theorem 2.4.7. *The following maps form pairs of mutually inverse isomorphisms:*

- *Alg:* $\mathbf{NExt}(\boldsymbol{K} \otimes \boldsymbol{K}) \to \boldsymbol{Var}(\mathbf{BMA})$ *and Th:* $\boldsymbol{Var}(\mathbf{BMA}) \to \mathbf{NExt}(\boldsymbol{K} \otimes \boldsymbol{K})$
- *Alg:* $\mathbf{NExt}(\boldsymbol{K} \otimes \boldsymbol{K}^R) \to \boldsymbol{Uni}(\mathbf{BMA})$ *and Ru:* $\boldsymbol{Uni}(\mathbf{BMA}) \to \mathbf{NExt}(\boldsymbol{K} \otimes \boldsymbol{K}^R)$

Corollary 2.4.8. Every bimodal logic (resp. bimodal multi-conclusion consequence relation) is complete with respect to some variety (resp. universal class) of bimodal algebras.

[4] It is well known that every Grz-algebra is an $S4$-algebra as **Grz** is an extension of **S4**. See [15].

2.5 Intuitionistic modal logics

In this section, we introduce intuitionistic modal logics. Again, intuitionistic modal logics may sometimes just mean intuitionistic modal deductive systems when whether they are logics or multi-conclusion consequence relations does not matter (the title of this section is an example).

2.5.1 Deductive systems for intuitionistic modal logics

The *intuitionistic modal signature* $i_\Box = \{\wedge, \vee, \to, \top, \bot, \Box\}$, and the set of *intuitionistic modal formulas* $Form_{i_\Box}$ is defined recursively as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

$\varphi \leftrightarrow \psi$ stands for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Let **IPC** denote the intuitionistic propositional calculus, *intuitionistic modal logics* are defined as follows:

Definition 2.5.1. A logic L over $Form_{i_\Box}$ is an *intuitionistic modal logic* if the following hold:

- $\mathbf{IPC} \subseteq L$
- $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi) \in L$
- $\varphi \to \psi \in L$ implies $\Box\varphi \to \Box\psi \in L$ (Reg)
- $\varphi \in L$ implies $\Box\varphi \in L$ (Nec)
- $\varphi \to \psi, \varphi \in L$ implies $\psi \in L$ (MP)

We denote the least intuitionistic modal logic by $\mathbf{IntK}_\Box$, and $\mathbf{IntS4}_\Box$ is just $\mathbf{IntK}_\Box \oplus (\Box p \to p) \oplus (\Box p \to \Box\Box p)$.

Definition 2.5.2. An *intuitionistic modal multi-conclusion consequence relation* is a multi-conclusion consequence relation M over $Rul_{i_\Box}$ satisfying the following conditions:

- $/\varphi \in M$ whenever $\varphi \in \mathbf{IntK}_\Box$
- $\varphi/\Box\varphi \in M$
- $\varphi \to \psi, \varphi/\psi \in M$

Elements in $Rul_{i_\Box}$ are called *intuitionistic modal multi-conclusion rules.* We then define the notations $\mathbf{NExt}(L)$, $\mathbf{NExt}(M)$, $\mathbf{IntK}_\Box^R$ and $\mathbf{IntS4}_\Box^R$ where L is an intuitionistic logic and M is an intuitionistic multi-conclusion consequence relation in the same way as we did for bimodal logics and bimodal multi-conclusion consequence relations.

We have the following counterpart to Proposition 2.4.4 as well.

Proposition 2.5.3. The mappings $(-)^R$ and $Taut(-)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathbf{IntK}_\Box)$ and the sublattice of $\mathbf{NExt}(\mathbf{IntK}_\Box^R)$ consisting of all intuitionistic modal multi-conclusion consequence relations M such that $Taut(M)^R = M$.

2.5.2 Algebraic semantics for intuitionistic modal logics

The algebraic semantics for intuitionistic modal logics is given by so-called *modal Heyting algebras.* We start with the definition of *Heyting algebras.*

Definition 2.5.4. A tuple $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ is a *Heyting algebra* if $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice such that for any $a, b, c \in A$, $c \wedge a \leq b$ iff $c \leq a \to b$.

Definition 2.5.5. A *modal Heyting algebra* is a tuple $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$ where $(A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra such that $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $a, b \in A$.

A modal Heyting algebra is an *interior Heyting algebra* if $\Box a \leq a$ and $\Box a \leq \Box\Box a$ for any $a \in A$ (or equivalently, $\Box a \to a = 1$ and $\Box a \to \Box\Box a = 1$).

Obviously, for any modal Heyting algebra $\mathfrak{A}$, we have that $\mathfrak{A}$ validates $\mathbf{IntS4}_\Box$ iff $\mathfrak{A}$ is an interior Heyting algebra. For simplicity, we will write $\mathfrak{A} = (A, \Box)$ for $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$ where A is assumed to be a Heyting algebra.

Let **MHA** be the class of all modal Heyting algebras, by Theorem 2.2.18, **MHA** is a variety. Let **Var**(**MHA**) and **Uni**(**MHA**) denote the lattice of subvarieties and the lattice of universal subclasses of **MHA** respectively. Then we have the following results. Proofs of similar results for Heyting algebras and superintuitionistic logics can be found in [15, Thm. 7.56] and [27, Thm. 2.2].

Theorem 2.5.6. *The following maps form pairs of mutually inverse isomorphisms:*

- *Alg:* $\mathbf{NExt}(\boldsymbol{IntK}_\Box) \to \boldsymbol{Var}(\mathbf{MHA})$ *and Th:* $\boldsymbol{Var}(\mathbf{MHA}) \to \mathbf{NExt}(\boldsymbol{IntK}_\Box)$

- *Alg:* $\mathbf{NExt}(\boldsymbol{IntK}_{\Box}^{R}) \rightarrow \boldsymbol{Uni}(\mathbf{MHA})$ *and Ru:* $\boldsymbol{Uni}(\mathbf{MHA}) \rightarrow \mathbf{NExt}(\boldsymbol{IntK}_{\Box}^{R})$

Corollary 2.5.7. Every intuitionistic modal logic (resp. intuitionistic modal multi-conclusion consequence relation) is complete with respect to some variety (resp. universal class) of modal Heyting algebras.

2.6 Duality

In the last section of this chapter, we recall dual descriptions of bounded distributive lattices and Heyting algebras, which will be frequently used in this thesis.

We start with Priestley duality for bounded distributive lattices, for which we refer the reader to [32]. We first recall the definition of *prime filters*.

Definition 2.6.1. Let $\mathfrak{A}$ be a lattice, a non-empty subset $X \subseteq A$ is a *prime filter* if it satisfies the following conditions:

1. $X \neq A$.
2. For any $a \in A$, if $a \leq b$ and $a \in X$, then $b \in X$ (upward-closed).
3. For any $a, b \in X$, $a \wedge b \in X$.
4. For any $a, b \in A$, if $a \vee b \in X$, then either $a \in X$ or $b \in X$.

Definition 2.6.2 (Priestley space)**.** A tuple $(X, \leq)$ (where $\leq$ is a partial order on X) is a *Priestley space* if X is a compact space and for any $x, y \in X$, if $x \not\leq y$, then there is clopen (closed and open) upset U of X such that $x \in U$ while $y \notin U$.

Definition 2.6.3 (Priestley morphism)**.** For Priestley spaces $(X, \leq)$ and $(Y, \leq)$, a map $f : X \rightarrow Y$ is a *Priestley morphism* if f is continuous and order-preserving.

Let **BDL** be the category of bounded distributive lattices with bounded lattice homomorphisms and **PS** be the category of Priestley spaces with Priestley morphisms, the functors $(-)_* : \mathbf{BDL} \rightarrow \mathbf{PS}$ and $(-)^* : \mathbf{PS} \rightarrow \mathbf{BDL}$ that establish Priestley duality are constructed as follows. For a bounded distributive lattice $\mathfrak{A}$, its dual $\mathfrak{A}_*$ is the set of all prime filters

X_A of $\mathfrak{A}$ with $\subseteq$ as the order and $\{\beta(a) \mid a \in A\} \cup \{X_A \setminus \beta(a)^5 \mid a \in A\}$ where $\beta(a) = \{x \in X_A \mid a \in x\}$ as the basis. For a bounded lattice homomorphism $h : A \to B$, its dual h_* is given by h^{-1}. For a Priestley space $\mathcal{X} = (X, \leq)$, its dual $\mathcal{X}^*$ is the bounded distributive lattice of clopen upsets of $\mathcal{X}$ with intersection as meet and union as join. For a Priestley morphism $f : X \to Y$, its dual $f^* : Y^* \to X^*$ is given by f^{-1}.

We summarise some useful details about Priestley duality in the following theorem.

Theorem 2.6.4. **BDL** *is dually equivalent to* **PS***, which is witnessed by* $(-)^*$ *and* $(-)_*$*. In particular, for any bounded distributive lattice* $\mathfrak{A}$*,* $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ *witnessed by* β *where* $\beta(a) = \{x \in A_* \mid a \in x\}$*, and for any Priestley space* $\mathcal{X}$*, we have* $\mathcal{X} \cong (\mathcal{X}^*)_*$ *witnessed by* ϵ *where* $\epsilon(x) = \{U \in X^* \mid x \in U\}$*.*

The dual description of Heyting algebras is then given by Esakia duality. The reader may refer to [19] for more details.

Definition 2.6.5 (Esakia space)**.** A Priestley space $(X, \leq)$ is an *Esakia space* if for any clopen set U of X, we have that $\downarrow U$ is clopen.

Remark 2.6.6. Using an easy argument about general topology, one can easily check that every clopen subset of an Esakia space is of the form $\bigcup_{1 \leq i \leq n}(U_i \setminus V_i)$ where $n \in \mathbb{N}$ and U_i, V_i's are clopen upsets.

For any topological space X, we will write $Clop(X)$ for the set of all clopen subsets of X.

Definition 2.6.7. For Esakia spaces $(X, \leq)$ and $(Y, \leq)$, $f : X \to Y$ is an *Esakia morphism* if it is continuous, order-preserving and for any $x \in X$, $f(x) \leq z$ implies that there is $x \leq y$ such that $f(y) = z$.

Let **ES** be the category of Esakia spaces with Esakia morphisms and **HA** be the category of Heyting algebras with Heyting algebra homomorphisms, the functors $(-)_* : \mathbf{HA} \to \mathbf{ES}$ and $(-)^* : \mathbf{ES} \to \mathbf{HA}$ that establish Esakia duality are constructed as follows: $(-)_*$ is the same as above. For an Esakia space $\mathcal{X} = (X, \leq)$, its dual $\mathcal{X}^*$ is the Heyting algebra of clopen upsets of $\mathcal{X}$ where $U \to V = X \setminus \downarrow(U \setminus V)$. And $f^* = f^{-1}$ for any Esakia morphism f. In particular, we have the following theorem.

Theorem 2.6.8. **HA** *is dually equivalent to* **ES***, which is witnessed by* $(-)^*$ *and* $(-)_*$*. In particular, for any Heyting algebra* $\mathfrak{A}$*,* $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ *witnessed by* β *where* $\beta(a) = \{x \in A_* \mid a \in x\}$*, and for any Esakia space* $\mathcal{X}$*, we have* $\mathcal{X} \cong (\mathcal{X}^*)_*$ *witnessed by* ϵ *where* $\epsilon(x) = \{U \in X^* \mid x \in U\}$*.*

[5] We may also denote the set-theoretic complement as δ_a for convenience.

We finish this section with some useful results about *modal spaces*.

Definition 2.6.9 (Stone space)**.** A topological space is a *Stone space* if it is a compact Hausdorff space which has a basis of clopen sets.

Remark 2.6.10. The category of Boolean algebras with Boolean algebra homomorphisms is dually equivalent to the category of Stone spaces with continuous maps. This duality is called *Stone duality* which is not considered independently in this thesis. One only needs to note that for any Boolean algebra $\mathfrak{A}$, its dual Stone space is simply its dual Esakia space without the order.

Definition 2.6.11 (Modal space)**.** A *modal space* (X, R) consists of a Stone space X and a binary relation R on X such that the following two hold:

- For any $x \in X$, the set $R[x] = \{y \in X \mid xRy\}$ is closed.
- For any clopen subset U of X, $R^{-1}[U] = \{x \in X \mid xRy \text{ for some } y \in U\}$ is clopen.

Remark 2.6.12. The category of modal spaces with their "corresponding maps" is dually equivalent to the category of modal algebas with their homomorphisms. We will only spell out some details about this duality (in fact, for bimodal spaces and bimodal algebras) in Chapter 4 when we need it.

Because of the duality just mentioned, it is not surprising that we can define a *valuation* on a modal space for unary modal logic[6]: a *valuation* on a modal space (X, R) is a map $V : Prop \to Clop(X)$ which can be extended to all modal formulas in the standard way [7]. For any modal formula φ, we write $(X, R) \vDash \varphi$ if for any valuation V, $V(\varphi) = X$. Then for any unary modal logic L, we call a modal space (X, R) an *L-space* if for any $\varphi \in L$, we have that $(X, R) \vDash \varphi$. In particular, an *S*4*-space* is a modal space (X, R) where R is reflexive and transitive.

We recall the following useful results about *Grz*-spaces, whose proofs can be found in [21, Ch. 3].

Theorem 2.6.13. *For any Grz-space (X, R) and $U \in Clop(X)$, the following hold:*

- *$max_R(U)$ is closed.*

[6]They are simply bimodal logics restricted to one modal operator.

[7]Usually people still write V for the extended map.

- $max_R(U) \subseteq pas_R(U)$.
- $max_R(U)$ *does not cut any R-cluster.*

This concludes our general preliminaries. We can now start developing the theory of stable canonical rules for intuitionistic modal logics.

Chapter 3

Stable canonical rules for intuitionistic modal logics

The purpose of this chapter is to develop stable canonical rules and formulas for intuitionistic modal logics. We begin by introducing stable canonical rules and proving that every intuitionistic modal multi-conclusion consequence relation is axiomatizable by stable canonical rules. Next, focusing on intuitionistic modal logics over $\mathbf{IntS4}_{\Box}$, we proceed to showing how to turn stable canonical rules into stable canonical formulas. In particular, we prove that every one of such logics is axiomatizable by stable canonical formulas over $\mathbf{IntS4}_{\Box}$. Finally, we close this chapter by a dual (geometric) characterization of our stable canonical rules, which will be quite useful in their applications.

3.1 Stable canonical rules for intuitionistic modal multi-conclusion consequence relations

We start with the definition of *stable maps*.

Definition 3.1.1. Let $\mathfrak{A} = (A, \Box)$ and $\mathfrak{B} = (B, \Box)$ be modal Heyting algebras, and let $h : A \to B$ be a bounded lattice homomorphism. We say that h is *stable* if for any $a \in A$, we have $h(\Box a) \leq \Box h(a)$.

This definition is the analogue to the one given in [6, Def. 3.1] in the setting of classical modal logics.

Definition 3.1.2. Let $\mathfrak{A} = (A, \Box)$, $\mathfrak{B} = (B, \Box)$ be modal Heyting algebras, $D^{\to} \subseteq A^2$ and $D^{\Box} \subseteq A$. A bounded lattice embedding $h : A \to B$ satisfies

- the *closed domain condition* (CDC for short) for $D^{\to}$ if $h(a \to b) = h(a) \to h(b)$ for any $(a,b) \in D^{\to}$.
- the *closed domain condition* (CDC for short) for $D^{\Box}$ if $h(\Box a) = \Box h(a)$ for any $a \in D^{\Box}$.

The following proposition relates each intuitionistic modal multi-conclusion rule with finitely many finite refutation patterns by stable bounded lattice embeddings which satisfy CDC for some parameters.

Proposition 3.1.3. For each intuitionistic modal multi-conclusion rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box})$, ..., $(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that each $\mathfrak{A}_i$ is a finite interior Heyting algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{\Box} \subseteq A_i$, and for each interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$.

Proof. Let Γ/Δ be an arbitrary intuitionistic modal multi-conclusion rule. If $\Gamma/\Delta \in \mathbf{IntS4}_{\Box}^{R}$, take $n = 0$. Suppose $\Gamma/\Delta \notin \mathbf{IntS4}_{\Box}^{R}$, let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$. Clearly Θ is finite. Assume $|\Theta| = m$, since the variety of bounded distributive lattices is locally finite, there are only finitely many pairs $(\mathfrak{A}, D^{\to}, D^{\Box})$ satisfying the following two conditions up to isomorphism:

i) $\mathfrak{A} = (A, \Box)$ is a finite interior Heyting algebra such that $\mathfrak{A}|_{\{\wedge,\vee,1,0\}}$ is at most m-generated as a bounded distributive lattice and $\mathfrak{A} \not\models \Gamma/\Delta$.

ii) $D^{\to} = \{(V(\varphi), V(\psi)) \mid \varphi \to \psi \in \Theta\}$ and $D^{\Box} = \{V(\psi) \mid \Box\psi \in \Theta\}$ where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \Gamma/\Delta$.

Let $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ be the enumeration of such pairs. For any interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we prove that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$.

For the right-to-left direction, suppose there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$. Define a valuation V_B on $\mathfrak{B}$ by $V_B(p) = h(V_i(p))$ for any propositional letter p where V_i is the valuation on $\mathfrak{A}_i$ witnessing $\mathfrak{A}_i \not\models \Gamma/\Delta$. We then prove by induction that $V_B(\psi) = h(V_i(\psi))$ for any $\psi \in \Theta$. We only consider following two cases as other cases are trivial (h is a bounded lattice embedding):

If $\psi = \Box\varphi$, then as $\Box\varphi \in \Theta$, $V_i(\varphi) \in D_i^{\Box}$.

$$
\begin{aligned}
V_B(\Box\varphi) &= \Box V_B(\varphi)\\
&= \Box h(V_i(\varphi)) \quad \text{(IH)}\\
&= h(\Box V_i(\varphi)) \quad \text{(CDC)}\\
&= h(V_i(\Box\varphi)).
\end{aligned}
$$

If $\psi = \varphi \to \chi$, then as $\varphi \to \chi \in \Theta$, $(V_i(\varphi), V_i(\chi)) \in D_i^{\to}$.

$$
\begin{aligned}
V_B(\varphi \to \chi) &= V_B(\varphi) \to V_B(\chi)\\
&= h(V_i(\varphi)) \to h(V_i(\chi)) \quad \text{(IH)}\\
&= h(V_i(\varphi) \to V_i(\chi)) \quad \text{(CDC)}\\
&= h(V_i(\varphi \to \chi)).
\end{aligned}
$$

Since $V_i(\gamma) = 1_{A_i}$ for any $\gamma \in \Gamma$ and h is a bounded lattice embedding, $V_B(\gamma) = h(V_i(\gamma)) = h(1_{A_i}) = 1_B$ for any $\gamma \in \Gamma$. Since $V_i(\delta) \neq 1_{A_i}$ for any $\delta \in \Delta$ and h is a bounded lattice embedding, $V_B(\delta) = h(V_i(\delta)) \neq 1_B$ for any $\delta \in \Delta$. Thus $\mathfrak{B} \not\models \Gamma/\Delta$.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \Gamma/\Delta$. There exists a valuation V_B on B such that $V_B(\gamma) = 1_B$ for any $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for any $\delta \in \Delta$. Let B' be the bounded sublattice of B generated by $V_B(\Theta) = \{V_B(\varphi) \mid \varphi \in \Theta\}$. Note that B' is finite as the variety of bounded distributive lattices is locally finite. Clearly $|V_B(\Theta)| \leq |\Theta|$. Let $D^{\Box} = \{V_B(\psi) \mid \Box\psi \in \Theta\}$ and $D^{\to} = \{(V_B(\varphi), V_B(\psi)) \mid \varphi \to \psi \in \Theta\}$. We define $\to'$ and $\Box'$ on B' as follows: $a \to' b = \bigvee\{d \in B' \mid d \wedge a \leq b\}$ for any $a, b \in B'$; $\Box' a = \bigvee\{\Box b \mid \Box b \leq \Box a \text{ and } b, \Box b \in B'\}$ for any $a \in B'$.

We first check that $(B', \to', \Box')$ is an interior Heyting algebra. Clearly, $(B', \to')$ is a Heyting algebra by the definition of $\to'$. Since $\Box 1 = 1$ and $1 \in B'$, we have that $\Box' 1 = \Box 1 = 1$. Since B is an interior Heyting algebra, it follows that $\Box a \leq a$. Thus $\Box' a = \bigvee\{\Box b \mid \Box b \leq \Box a \text{ and } b, \Box b \in B'\} \leq \Box a \leq a$. Namely, $\Box' a \leq a$ for any $a \in B'$.

For any $a, b \in B'$, $\Box' a \wedge \Box' b$
$= \bigvee\{\Box x \leq \Box a \text{ and } x, \Box x \in B'\} \wedge \bigvee\{\Box y \mid \Box y \leq \Box b \text{ and } y, \Box y \in B'\}$
$= \bigvee\{\Box x \wedge \Box y \mid \Box x \leq \Box a, \Box y \leq \Box b \text{ where } x, y, \Box x, \Box y \in B'\}$(distributivity)
$= \bigvee\{\Box(x \wedge y) \mid \Box x \leq \Box a \text{ and } \Box y \leq \Box b \text{ where } x, y, \Box x, \Box y \in B'\}$
$= \bigvee\{\Box z \mid \Box z \leq \Box(a \wedge b) \text{ and } z, \Box z \in B'\} = \Box'(a \wedge b)$.

For any $a \in B'$, we have that $\Box'\Box' a = \bigvee\{\Box c \mid \Box c \leq \Box\Box' a \text{ and } c, \Box c \in B'\}$. For any $\Box x \leq \Box a$ where $x, \Box x \in B'$,
$\Box\Box' a = \Box \bigvee\{\Box b \mid \Box b \leq \Box a \text{ and } b, \Box b \in B'\}$
$\geq$ [1] $\bigvee\{\Box\Box b \mid \Box b \leq \Box a \text{ and } b, \Box b \in B'\}$
$\geq \bigvee\{\Box b \mid \Box b \leq \Box a \text{ and } b, \Box b \in B'\} \geq \Box x$ (note that $\Box\Box b \geq \Box b$).

[1] Here we use the fact that for any interior Heyting algebra $\mathfrak{A}$, $\Box a \vee \Box b \leq \Box(a \vee b)$ for any $a, b \in A$, which one can check easily.

Thus $\Box' a \leq \Box'\Box' a$ for any $a \in B'$. This proves that $(B', \rightarrow', \Box')$ is an interior Heyting algebra. Let $h : (B', \rightarrow', \Box') \to (B, \Box)$ be the inclusion map, h is clearly a bounded lattice embedding as B' is a bounded sublattice of B. h is stable as $\Box' a \leq \Box a$ for any $a \in B'$ by definition.

Then we check that h satisfies CDC for $D^{\rightarrow}$ and $D^{\Box}$. For any $a \in D^{\Box}$, $a = V_B(\psi)$ for some $\Box\psi \in \Theta$. And $V_B(\Box\psi) = \Box V_B(\psi) = \Box a \in B'$. Thus $\Box' a = \Box a$ by the definition of $\Box'$. For any $(a, b) \in D^{\rightarrow}$, $a = V_B(\varphi)$ and $b = V_B(\psi)$ for some $\varphi \to \psi \in \Theta$. Thus $V_B(\varphi \to \psi) = V_B(\varphi) \to V_B(\psi) = a \to b \in B'$. Then $a \to b' = a \to b$ by the definition of $\rightarrow'$. Therefore, the stable bounded lattice embedding h satisfies CDC for $D^{\rightarrow}$ and $D^{\Box}$.

Let V' be the valuation V_B restricted to B', we then prove that for any $\varphi \in \Theta$, $V'(\varphi) = V_B(\varphi)$ by induction on φ. We only consider the following two cases as others are trivial (B' is a bounded sublattice of B):

If $\varphi = \psi \to \chi$, as $\psi \to \chi \in \Theta$, we have that $(V_B(\psi), V_B(\chi)) \in D^{\rightarrow}$, and $V_B(\psi) \to V_B(\chi) \in B'$.

$$\begin{aligned} V'(\psi \to \chi) &= V'(\psi) \rightarrow' V'(\chi) \\ &= V_B(\psi) \rightarrow' V_B(\chi) \quad \text{(IH)} \\ &= V_B(\psi) \to V_B(\chi) \quad \text{(By the definition of } \rightarrow'\text{)} \\ &= V_B(\psi \to \chi). \end{aligned}$$

If $\varphi = \Box\psi$, as $\Box\psi \in \Theta$, we have that $V_B(\Box\psi) \in B'$, and $V_B(\psi), \Box V_B(\psi) \in B'$.

$$\begin{aligned} V'(\Box\psi) &= \Box' V'(\psi) \\ &= \Box' V_B(\psi) \quad \text{(IH)} \\ &= \Box V_B(\psi) \quad \text{(By the definition of } \Box'\text{)} \\ &= V_B(\Box\psi). \end{aligned}$$

Since V_B is a valuation which refutes Γ/Δ on $\mathfrak{B}$, V' is a valuation which refutes Γ/Δ on $(B', \rightarrow', \Box')$ by the above result. Thus $(B', \rightarrow', \Box') \not\models \Gamma/\Delta$. As for any $\varphi \in \Theta$, $V'(\varphi) = V_B(\varphi)$, we have that $D^{\Box} = \{V_B(\psi) \mid \Box\psi \in \Theta\} = \{V'(\psi) \mid \Box\psi \in \Theta\}$ and $D^{\rightarrow} = \{(V_B(\varphi), V_B(\psi)) \mid \varphi \to \psi \in \Theta\} = \{(V'(\varphi), V'(\psi)) \mid \varphi \to \psi \in \Theta\}$. As B' is generated by $V_B(\Theta)$ whose cardinality is no larger than that of Θ, $(B', \rightarrow', \Box', D^{\rightarrow}, D^{\Box})$ must be one of $(\mathfrak{A}_1, D_1^{\rightarrow}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\rightarrow}, D_n^{\Box})$. As h is a stable bounded lattice embedding from B' to B satisfying CDC for $D^{\rightarrow}$ and $D^{\Box}$, we get what we want.

■

It is not difficult to see that the above proposition still holds if we replace "interior Heyting algebra" by "modal Heyting algebra" in its statement. The reason why we consider interior Heyting algebras and $\mathbf{NExt}(\mathbf{IntS4}_{\Box}^{R})$ instead of modal Heyting algebras and $\mathbf{NExt}(\mathbf{IntK}_{\Box}^{R})$ is because the proof for the former requires more work as we have to make sure that the algebra $(B', \to', \Box')$ is not only a modal Heyting algebra but also an interior Heyting algebra[2]. Besides, when discussing stable canonical formulas for logics over $\mathbf{IntS4}_\Box$, we need to refer to the above proof. This way of organizing results exposes the key points while saves us from repetitions.

In the above proof, what we have done is a essentially transitive filtration in algebraic terms[3]: for the left-to-right direction, we begin with the assumption that $\mathfrak{B} \not\models \Gamma/\Delta$, and then use $\mathfrak{B}$ to construct a finite interior modal algebra $(B', \to', \Box')$ – a filtrated algebra – which still refutes Γ/Δ. And it turns out that the relation between the filtrated algebra $(B', \to', \Box')$ and the original algebra $\mathfrak{B}$ (i.e., a stable bounded lattice embedding satisfying CDC for some parameters) can be coded syntactically.

Definition 3.1.4. Let $\mathfrak{A} = (A, \Box)$ be a finite modal Heyting algebra, $D^\to \subseteq A^2$ and $D^\Box \subseteq A$. For each $a \in A$, we introduce a new propositional letter p_a and define the *stable canonical rule* $\rho(\mathfrak{A}, D^\to, D^\Box)$ based on $(\mathfrak{A}, D^\to, D^\Box)$ as follows:

$$\begin{aligned} \Gamma \quad = \quad & \{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \{p_0 \leftrightarrow \bot, p_1 \leftrightarrow \top\} \\ & \cup \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \{p_{\Box a} \to \Box p_a \mid a \in A\} \\ & \cup \{p_{a\to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D^\to\} \cup \{\Box p_a \to p_{\Box a} \mid a \in D^\Box\} \end{aligned}$$

$$\Delta = \{p_a \leftrightarrow p_b \mid a \neq b \in A\}$$

$\rho(\mathfrak{A}, D^\to, D^\Box) = \Gamma/\Delta$.

The above definition can be seen as a combination of [4, Def. 3.1] and [6, Def. 5.2]. It is easy to see that the following proposition holds:

Proposition 3.1.5. Let $\mathfrak{A} = (A, \Box)$ be a finite modal Heyting algebra, $D^\to \subseteq A^2$ and $D^\Box \subseteq A$, then $\mathfrak{A} \not\models \rho(\mathfrak{A}, D^\to, D^\Box)$.

Proof. Define a valuation V on A by $V(p_a) = a$ for any $a \in A$. It is then easy to check that V refutes $\rho(\mathfrak{A}, D^\to, D^\Box)$ on $\mathfrak{A}$. ■

The next result shows that the stable canonical rule does encode a stable bounded lattice embedding satisfying CDC for $D^\to$ and $D^\Box$.

Proposition 3.1.6. Let $\mathfrak{A} = (A, \Box)$ be a finite modal Heyting algebra, $D^\to \subseteq A^2$, $D^\Box \subseteq A$, and $\mathfrak{B} = (B, \Box)$ be a modal Heyting algebra. Then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D^\to, D^\Box)$ iff there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^\to$ and $D^\Box$.

[2] This is also the only place where we need the assumption that $\mathfrak{B}$ is an interior Heyting algebra.

[3] See [26] for more details.

Proof. For the right-to-left direction, suppose there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^{\to}$ and $D^{\Box}$. Define V_B on B by $V_B(p_a) = h(V(p_a)) = h(a)$ for any $a \in A$ where V is just the valuation in the proof of Proposition 3.1.5. As h is a bounded lattice embedding, for any $a, b \in A$, it follows that $h(a \wedge b) = h(a) \wedge h(b)$, $h(a \vee b) = h(a) \vee h(b)$, $h(0) = 0$ and $h(1) = 1$. Thus we have the following:

$$\begin{aligned} V_B(p_{a\vee b} \leftrightarrow p_a \vee p_b) &= V_B(p_{a\vee b}) \leftrightarrow V_B(p_a \vee p_b) \\ &= V_B(p_{a\vee b}) \leftrightarrow V_B(p_a) \vee V_B(p_b) \\ &= h(a \vee b) \leftrightarrow h(a) \vee h(b) \\ &= 1. \\ V_B(p_{a\wedge b} \leftrightarrow p_a \wedge p_b) &= V_B(p_{a\wedge b}) \leftrightarrow V_B(p_a \wedge p_b) \\ &= V_B(p_{a\wedge b}) \leftrightarrow V_B(p_a) \wedge V_B(p_b) \\ &= h(a \wedge b) \leftrightarrow h(a) \wedge h(b) \\ &= 1. \end{aligned}$$

Also $V_B(p_0) = h(0) = 0$ and $V_B(p_1) = h(1) = 1$.

As h is stable, $h(\Box a) \leq \Box h(a)$ for any $a \in A$. Thus $V_B(p_{\Box a}) = h(\Box a) \leq \Box h(a) = \Box V_B(p_a) = V_B(\Box p_a)$ for any $a \in A$. Therefore, $V_B(p_{\Box a} \to \Box p_a) = V_B(p_{\Box a}) \to V_B(\Box p_a) = 1$ for any $a \in A$.

As h satisfies CDC for $D^{\to}$ and $D^{\Box}$, for any $a \in D^{\Box}$, we have that $h(\Box a) = \Box h(a)$; for any $(a, b) \in D^{\to}$, we have that $h(a \to b) = h(a) \to h(b)$. Thus $V_B(p_{\Box a}) = h(\Box a) = \Box h(a) = \Box V_B(p_a) = V_B(\Box p_a)$, we get $V_B(\Box p_a \to p_{\Box a}) = 1$ for any $a \in D^{\Box}$.

For any $(a, b) \in D^{\to}$, we have that

$$\begin{aligned} V_B(p_{a\to b} \leftrightarrow p_a \to p_b) &= V_B(p_{a\to b}) \leftrightarrow V_B(p_a \to p_b) \\ &= V_B(p_{a\to b}) \leftrightarrow (V_B(p_a) \to V_B(p_b)) \\ &= h(a \to b) \leftrightarrow (h(a) \to h(b)) \\ &= 1. \end{aligned}$$

Since h is an embedding, for any $a \neq b \in A$, $h(a) \neq h(b)$. Thus $V_B(p_a) \neq V_B(p_b)$ and $V_B(p_a \leftrightarrow p_b) \neq 1$. Therefore, for any $\gamma \in \Gamma$, we have that $V_B(\gamma) = 1$ while for any $\delta \in \Delta$, $V_B(\delta) \neq 1$. Thus V_B refutes Γ/Δ on $\mathfrak{B}$, $\mathfrak{B} \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$. Then there exists a valuation V on B such that $V(\gamma) = 1$ for any $\gamma \in \Gamma$, and $V(\delta) \neq 1$ for any $\delta \in \Delta$. Define $h : A \to B$ by $h(a) = V(p_a)$ for any $a \in A$. For any $a, b \in A$, as $V(p_{a\vee b} \leftrightarrow p_a \vee p_b) = 1$, we have that $V(p_{a\vee b}) = V(p_a \vee p_b) = V(p_a) \vee V(p_b)$. Thus $h(a \vee b) = V(p_{a\vee b}) = V(p_a) \vee V(p_b) = h(a) \vee h(b)$. Similarly, we obtain $h(a \wedge b) = h(a) \wedge h(b)$ for any $a, b \in A$. And $h(0) = V(p_0) = 0$, $h(1) = V(p_1) = 1$. Thus h is a bounded lattice homormophism. As for any $a \in A$, $V(p_{\Box a} \to \Box p_a) = 1$, $V(p_{\Box a}) \leq \Box V(p_a)$, and thus $h(\Box a) \leq \Box h(a)$. h is stable.

For any $(a, b) \in D^{\to}$, as $V(p_{a \to b} \leftrightarrow p_a \to p_b) = 1$, we have $V(p_{a\to b}) = V(p_a) \to V(p_b)$, and thus $h(a \to b) = V(p_{a\to b}) = V(p_a) \to V(p_b) = h(a) \to h(b)$. For any $a \in D^{\Box}$, as $V(\Box p_a \to p_{\Box a}) = 1$, $V(\Box p_a) \leq V(p_{\Box a})$, $h(\Box a) = V(p_{\Box a}) \geq V(\Box p_a) = \Box V(p_a) = \Box h(a)$. Therefore, $h(\Box a) = \Box h(a)$ for any $a \in D^{\Box}$.

For any $a \neq b \in A$, as $V(p_a \leftrightarrow p_b) \neq 1$, it follows that $V(p_a) \neq V(p_b)$ and $h(a) = V(p_a) \neq V(p_b) = h(b)$. Therefore, h is a stable bounded lattice embedding satisfying CDC for for $D^{\to}$ and $D^{\Box}$. ■

Now, combining Propositions 3.1.3 and 3.1.6, we obtain immediately the following result:

Theorem 3.1.7. *For an intuitionistic modal multi-conclusion rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box})$, ..., $(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that each $\mathfrak{A}_i$ is a finite interior Heyting algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{\Box} \subseteq A_i$, and for each interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have:*

$$\mathfrak{B} \models \Gamma/\Delta \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., \rho(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box}).$$

As a corollary, we arrive at the main theorem of this section.

Theorem 3.1.8. *Every intuitionistic modal multi-conclusion consequence relation extending $\boldsymbol{IntS4}_{\Box}^{R}$ is axiomatizable by stable canonical rules over $\boldsymbol{IntS4}_{\Box}^{R}$.*

Proof. Let $M \in \mathbf{NExt}(\mathbf{IntS4}_{\Box}^{R})$. Then $M = \mathbf{IntS4}_{\Box}^{R} \oplus \{\Gamma_i/\Delta_i \mid i \in I\}$ where Γ_i/Δ_i is an intuitionistic modal multi-conclusion rule. For any $i \in I$, by Theorem 3.1.7, there exist stable canonical rules $\rho(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \rho(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box})$ such that for any interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have that $\mathfrak{B} \models \Gamma_i/\Delta_i$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \rho(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box})$. Therefore, for any interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have that $\mathfrak{B}$ validates M iff $\mathfrak{B}$ validates $\{\rho(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \rho(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box}) \mid i \in I\}$. By Corollary 2.5.7, this means that $M = \mathbf{IntS4}_{\Box}^{R} \oplus \{\rho(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \rho(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box}) \mid i \in I\}$. Therefore, M is axiomatized by stable canonical rules over $\mathbf{IntS4}_{\Box}^{R}$. This proves that every intuitionistic modal multi-conclusion consequence relation extending $\mathbf{IntS4}_{\Box}^{R}$ is axiomatizable by stable canonical rules over $\mathbf{IntS4}_{\Box}^{R}$. ■

It is easy to note that we do not need the assumption that the modal Heyting algebras we consider are interior Heyting algebras, except in Proposition 3.1.3, which, as mentioned above, still holds if we replace "interior Heyting algebras" by "modal Heyting algebras" in the statement. Thus the reader should find it not difficult to obtain the following result.

Theorem 3.1.9. *Every intuitionistic modal multi-conclusion consequence relation is axiomatizable by stable canonical rules.*

As every modal multi-conclusion consequence relation[4] is also an intuitionistic modal multi-conclusion consequence relation, Theorem 3.1.9 is a generalization of [6, Thm. 5.6]. When considering intuitionistic modal multi-conclusion rules, the above theorem allows us to assume that they are stable canonical rules in many cases. Since stable canonical rules are in a certain syntactical shape, it is more manageable to work with them instead of arbitrary intuitionistic modal multi-conclusion rules. This point may become more evident when we see the dual description of stable canonical rules, which gives us geometric intuitions about how these rules work.

3.2 Stable canonical formulas for intuitionistic modal logics over $\mathbf{IntS4}_\Box$

Now, similarly to [6], we may take a step further to transform stable canonical rules to stable canonical formulas for intuitionistic modal logics over $\mathbf{IntS4}_\Box$. Unlike the results of the first section of this chapter, in this section, it is quite crucial that we work with interior Heyting algebras and logics over $\mathbf{IntS4}_\Box$ instead of modal Heyting algebras and logics over $\mathbf{IntK}_\Box$.

First, we need a characterization of subdirectly irreducible interior Heyting algebras:

Proposition 3.2.1. A nontrivial interior Heyting algebra $\mathfrak{A}$ is subdirectly irreducible iff there exists an element $a \in A$ (called an *opremum*) with $a \neq 1$ such that for any $b \neq 1 \in A$, $\Box b \leq a$.

Proof. By [36, Prop. 1.6], a nontrivial interior Heyting algebra $\mathfrak{A}$ is subdirectly irreducible iff there exists an element $a \in A$ such that for any $b \neq 1$, there exists $n \in \mathbb{N}$ such that $b \wedge \Box b \wedge ... \wedge \Box^n b \leq a$. As $\mathfrak{A}$ is an interior algebra, it follows that $\Box c \leq c$ and $\Box c \leq \Box\Box c$ for any $c \in A$. Then the result follows immediately. ■

[4] $\Box$ is the primitive operator and $\Diamond$ is defined by $\Box$.

Definition 3.2.2. Let $\mathfrak{A}$ be an interior Heyting algebra, a filter F of $\mathfrak{A}$ is a $\Box$-*filter* if for any $a \in A$, we have that $a \in F$ implies that $\Box a \in F$.

It is well known that there is a one-to-one correspondence between congruences and filters of Heyting algebras [15, Chap. 3]. In the setting of interior Heyting algebras, it turns out that congruences correspond to $\Box$-filters.

Proposition 3.2.3. [24, Prop. 2.2] Let $\mathfrak{A}$ be an interior Heyting algebra, the map $f : F \mapsto \theta_F = \{(a, b) \mid a \leftrightarrow b \in F\}$ is an isomorphism from the complete lattice of $\Box$-filters of $\mathfrak{A}$ onto the complete lattice of congruences of $\mathfrak{A}$. The inverse map is given by $g : \theta \mapsto F_\theta = \{a \in A \mid (a, 1) \in \theta\}$.

Then we obtain the following technical lemma, which is the counterpart to that for **K4**-algebras in [3, Lem. 4.1].

Lemma 3.2.4. *Let $\mathfrak{A}$ be an interior Heyting algebra, $a, b \in A$ and $\Box a \not\leq b$. Then there exists a subdirectly irreducible interior Heyting algebra $\mathfrak{B}$ and an onto homomorphism $\eta : A \to B$ such that $\eta(\Box a) = 1$ and $\eta(b) \neq 1$.*

Proof. As $\mathfrak{A}$ is an interior Heyting algebra, for any $a \in A$, we have that $\Box a \leq a$. Thus $a \wedge \Box a = \Box a$ for any $a \in A$. By Proposition 3.2.3, there is a correspondence between $\Box$-filters and congruences of $\mathfrak{A}$. Then the proof is exactly the same as that of [3, Lem. 4.1]. For the reason of completeness, we sketch the main steps here.

First, consider the set Z of $\Box$-filters of $\mathfrak{A}$ containing $\Box a$ while missing b. It is easy to check that Z satisfies the assumptions of Zorn's Lemma, and thus Z has a maximal element M. We define $\mathfrak{B}$ as the quotient algebra $\mathfrak{A}/\backsim$ where $x \backsim y$ iff $x \leftrightarrow y \in M$. Then let $\eta : A \to B$ be the quotient map. One can check that $\mathfrak{B}$ is s.i (using the correspondence between $\Box$-filters and congruences) and η is an onto homomorphism such that $\eta(\Box a) = 1$ while $\eta(b) \neq 1$.

■

Now we can define stable canonical formulas as follows.

Definition 3.2.5. Let $\mathfrak{A}$ be a finite s.i interior Heyting algebra, $D^\to \subseteq A^2$ and $D^\Box \subseteq A$. For each $a \in A$, we introduce a new propositional letter p_a and define the *stable canonical formula* $\iota(\mathfrak{A}, D^\to, D^\Box)$ based on $(\mathfrak{A}, D^\to, D^\Box)$ as follows, in which Γ and Δ are just those in Definition 3.1.4:

$$\begin{aligned}\iota(\mathfrak{A}, D^\to, D^\Box) &= \bigwedge\{\Box\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box\delta \mid \delta \in \Delta\} \\ &= \Box\bigwedge\Gamma \to \bigvee\{\Box\delta \mid \delta \in \Delta\}\end{aligned}$$

The following proposition tells us what the stable canonical formulas encode:

Proposition 3.2.6. Let $\mathfrak{A}$ be a finite s.i interior Heyting algebra, $D^{\to} \subseteq A^2$ and $D^{\Box} \subseteq A$. For any interior Heyting algebra $\mathfrak{B}$, $\mathfrak{B} \not\models \iota(\mathfrak{A}, D^{\to}, D^{\Box})$ iff there is a s.i homomorphic image $\mathfrak{C} = (C, \Box)$ of $\mathfrak{B}$ and a stable bounded lattice embedding $h : A \to C$ satisfying CDC for $D^{\to}$ and $D^{\Box}$.

Proof. For the right-to-left direction, suppose there is a s.i homomorphic image $\mathfrak{C} = (C, \Box)$ of $\mathfrak{B}$ and a stable bounded lattice embedding $h : A \to C$ satisfying CDC for $D^{\to}$ and $D^{\Box}$. Define V on $\mathfrak{C}$ by $V(p_a) = h(a)$ for each $a \in A$. As h is stable bounded lattice embedding satisfying CDC for $D^{\to}$ and $D^{\Box}$, we can easily check that $V(\gamma) = 1$ for any $\gamma \in \Gamma$ while $V(\delta) \neq 1$ for any $\delta \in \Delta$ as we did in the proof of Proposition 3.1.6. For any $\delta \in \Delta$, $V(\Box\delta) = \Box V(\delta) \leq V(\delta)$. Thus $V(\Box\delta) \neq 1$ for any $\delta \in \Delta$. As $\mathfrak{C}$ is s.i, it has an opremum, say d. For any $\delta \in \Delta$, $V(\Box\delta) \leq d$, and thus $V(\bigvee\{\Box\delta \mid \delta \in \Delta\}) \leq d \neq 1$. V refutes $\iota(\mathfrak{A}, D^{\to}, D^{\Box})$ on $\mathfrak{C}$. Therefore, $\mathfrak{C} \not\models \iota(\mathfrak{A}, D^{\to}, D^{\Box})$. As $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, it follows that $\mathfrak{B} \not\models \iota(\mathfrak{A}, D^{\to}, D^{\Box})$.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \iota(\mathfrak{A}, D^{\to}, D^{\Box})$, then there exists a valuation V on $\mathfrak{B}$ refutes $\iota(\mathfrak{A}, D^{\to}, D^{\Box})$. Namely, $V(\Box \bigwedge \Gamma) \not\leq V(\bigvee\{\Box\delta \mid \delta \in \Delta\})$. By Lemma 3.2.4, there exists a s.i interior Heyting algebra $\mathfrak{C}$ and an onto homomorphism $\eta : B \to C$ such that $\eta(V(\Box \bigwedge \Gamma)) = 1$ and $\eta(V(\bigvee\{\Box\delta \mid \delta \in \Delta\})) \neq 1$. Define a valuation V_C on $\mathfrak{C}$ by $V_C(p_a) = \eta(V(p_a))$. Then $V_C(\Box \bigwedge \Gamma) = 1$ and $V_C(\bigvee\{\Box\delta \mid \delta \in \Delta\}) \neq 1$. Thus $V_C(\Box\delta) \neq 1$ for any $\delta \in \Delta$. As $V_C(\Box a) \leq V_C(a)$ for any $a \in \mathfrak{C}$ and $\Box 1 = 1$, we have that $V(\gamma) = 1$ for any $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for any $\delta \in \Delta$. Define $h : A \to C$ by $h(a) = V_C(p_a)$, it is then easy to check that h is a stable bounded lattice embedding satisfying CDC for for $D^{\to}$ and $D^{\Box}$ as we did in the proof of Proposition 3.1.6. ■

We next prove the following version of Proposition 3.1.6 for interior Heyting algebras, whose Heyting algebra analogue and **K4**-algebra analogue can be found in [39, Lem. 1] and [6, Thm. 6.5] respectively. Unlike the proof of [6, Thm. 6.5], we do some tricks directly in the process of taking filtrations.

Theorem 3.2.7. *For an intuitionistic modal formula φ, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that each $\mathfrak{A}_i$ is a finite s.i interior Heyting algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{\Box} \subseteq A_i$, and for each s.i interior Heyting algebra $\mathfrak{B} = (B, \Box)$, the following are equivalent:*

1) $\mathfrak{B} \not\models \varphi$.

2) There is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$.

3) There is a s.i homomorphic image $\mathfrak{C} = (C, \Box)$ of $\mathfrak{B}$, $1 \leq i \leq n$ and a stable bounded lattice embedding $h : A_i \to C$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$.

Proof. If $\varphi \in \mathbf{IntS4}_{\Box}$, then take $n = 0$. Suppose $\varphi \notin \mathbf{IntS4}_{\Box}$, let Θ be the set of all subformulas of φ, Θ is finite. Assume $|\Theta| = m$, since bounded distributive lattices are locally finite, there are only finitely many pairs $(\mathfrak{A}, D^{\to}, D^{\Box})$ satisfying the following two conditions up to isomorphism:

i) $\mathfrak{A} = (A, \Box)$ is a finite s.i interior Heyting algebra such that $\mathfrak{A}|_{\{\wedge,\vee,1,0\}}$ is at most $m+1$-generated as a bounded distributive lattice and $\mathfrak{A} \not\models \varphi$.

ii) $D^{\to} = \{(V(\varphi), V(\psi)) \mid \varphi \to \psi \in \Theta\}$ and $D^{\Box} = \{V(\psi) \mid \Box\psi \in \Theta\}$ where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \varphi$.

Let $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ be the enumeration of such pairs. Let $\mathfrak{B} = (B, \Box)$ be a s.i interior Heyting algebra.

1) $\Rightarrow$ 2): suppose $\mathfrak{B} \not\models \varphi$, there is a valuation V_B on $\mathfrak{B}$ which refutes φ. As B is s.i, it has an opremum, say c. Let B' be the bounded distributive sublattice of $\mathfrak{B}$ generated by $V(\Theta)\cup\{c\}$ where $|V(\Theta)\cup\{c\}| \leq |\Theta|+1 = m+1$. B' is finite, and we define $\to'$ and $\Box'$ on B' exactly the same way as we did in the proof of Proposition 3.1.3. Then $(B', \to', \Box')$ is a finite interior Heyting algebra which refutes φ by V' (the restriction of V_B to B'). By the definition of $\Box'$, for any $a \in B'$, $\Box' a \leq \Box a$. As c is an opremum of $\mathfrak{B}$, $\Box' a \leq \Box a \leq c \neq 1_B (= 1_{B'})$ for any $a \in B'$. Thus c is also an opremum of B', B' is s.i by Proposition 3.2.1. Therefore, $(B', \to', \Box', D^{\to}, D^{\Box})$ is one of $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ where $D^{\Box} = \{V'(\psi) \mid \Box\psi \in \Theta\}$ and $D^{\to} = \{(V'(\varphi), V'(\psi)) \mid \varphi \to \psi \in \Theta\}$, and there is a stable bounded lattice embedding $h : B' \to B$ satisfying CDC for $D^{\to}$ and $D^{\Box}$ as shown in the proof of Proposition 3.1.3.

2) $\Rightarrow$ 3) is obvious. 3) $\Rightarrow$ 1): suppose There is s.i homomorphic image $\mathfrak{C} = (C, \Box)$ of $\mathfrak{B}$, $1 \leq i \leq n$ and a stable bounded lattice embedding $h : A_i \to C$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$. Then it is easy to see that $\mathfrak{C} \not\models \varphi$ as we did in the proof of Proposition 3.1.3. As $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, $\mathfrak{B} \not\models \varphi$.

■

Combining Proposition 3.2.6 and Theorem 3.2.7 together, we obtain the following result:

Proposition 3.2.8. For an intuitionistic modal formula φ, there exist tuples $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box})$,..., $(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that each $\mathfrak{A}_i$ is a finite s.i interior Heyting algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{\Box} \subseteq A_i$, and for each s.i interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have that $\mathfrak{B} \vDash \varphi$ iff $\mathfrak{B} \vDash \bigwedge_{1 \leq i \leq n} \iota(\mathfrak{A}_i, D_i^{\to}, D_i^{\Box})$

Proof. By Theorem 3.2.7, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that each $\mathfrak{A}_i$ is a finite s.i interior Heyting algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{\Box} \subseteq A_i$, and for each s.i interior Heyting algebra $\mathfrak{B} = (B, \Box)$, $\mathfrak{B} \nvDash \varphi$ iff there is a s.i homomorphic image $\mathfrak{C} = (C, \Box)$ of $\mathfrak{B}$, $1 \leq i \leq n$ and a stable bounded lattice embedding $h : A_i \to C$ satisfying CDC for $D_i^{\to}$ and $D_i^{\Box}$. By Proposition 3.2.6, this is equivalent to the existence of $1 \leq i \leq n$ such that $\mathfrak{B} \nvDash \iota(\mathfrak{A}_i, D_i^{\to}, D_i^{\Box})$. Thus $\mathfrak{B} \vDash \varphi$ iff $\mathfrak{B} \vDash \bigwedge_{1 \leq i \leq n} \iota(\mathfrak{A}_i, D_i^{\to}, D_i^{\Box})$

■

The next theorem is the main result of this section.

Theorem 3.2.9. *Each intuitionistic modal logic over* $\textbf{IntS4}_\Box$ *is axiomatizable by stable canonical formulas over* $\textbf{IntS4}_\Box$.

Proof. Let $L \in \textbf{NExt}(\textbf{IntS4}_\Box)$. Then $L = \textbf{IntS4}_\Box \oplus \{\varphi_i \mid i \in I\}$ where φ_i is an intuitionistic modal formula. For any $i \in I$, by Proposition 3.2.8, there exist stable canonical formulas $\iota(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \iota(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box})$ such that for any s.i interior Heyting algebra $\mathfrak{B} = (B, \Box)$, we have that $\mathfrak{B} \vDash \varphi_i$ iff $\mathfrak{B} \vDash \bigwedge_{1 \leq i \leq n} \iota(\mathfrak{A}_i, D_i^{\to}, D_i^{\Box})$. Therefore, for any s.i interior Heyting algebra $\mathfrak{B} = (B, \Box)$, $\mathfrak{B}$ validates L iff $\mathfrak{B}$ validates $\{\iota(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \iota(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box}) \mid i \in I\}$.

By Corollary 2.5.7, every intuitionistic modal logic over $\textbf{IntS4}_\Box$ is determined by the variety of all interior Heyting algebras which validate it. Moreover, Theorem 2.2.21 tells us that every variety is determined by its s.i algebras.

Therefore, $L = \textbf{IntS4}_\Box \oplus \{\iota(\mathfrak{A}_{i1}, D_{i1}^{\to}, D_{i1}^{\Box}), ..., \iota(\mathfrak{A}_{in_i}, D_{in_i}^{\to}, D_{in_i}^{\Box}) \mid i \in I\}$, and L is axiomatizable by stable canonical formulas over $\textbf{IntS4}_\Box$.

■

Similarly to Theorem 3.1.9, the above result allows us to assume that all formulas are semantically equivalent to stable canonical ones when we consider intuitionistic modal logics over $\textbf{IntS4}_\Box$. For example, let $L \in \textbf{NExt}(\textbf{IntS4}_\Box)$, for any intuitionistic modal formula φ, we can effectively construct finitely many stable canonical formulas $\iota(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., \iota(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})$ such that $\textbf{IntS4}_\Box \oplus \varphi = \textbf{IntS4}_\Box \oplus \{\iota(\mathfrak{A}_1, D_1^{\to}, D_1^{\Box}), ..., \iota(\mathfrak{A}_n, D_n^{\to}, D_n^{\Box})\}$. Thus $\varphi \in L$ iff $\iota(\mathfrak{A}_i, D_i^{\to}, D_i^{\Box}) \in L$ for any $1 \leq i \leq n$. Therefore, L is decidable iff there is an algorithm which decides for each stable canonical formula ψ whether $\psi \in L$ or not. This is the key idea that leads to the applications of stable canonical formulas and rules, which we will see in the next chapter.

3.3 Dual descriptions of stable canonical rules

In the previous sections, we have introduced stable canonical formulas and stable canonical rules. Both of them are constructed based on a finite modal Heyting algebra together with some parameters (called a finite refutation pattern). In this section, we will provide a geometric description of stable canonical rules via duality.

We first recall the duality between the category of modal Heyting algebras and the category of modal Esakia spaces which is established in [31] and makes our dual description possible.

Definition 3.3.1. Let $(X, \leq, R)$ be a triple such that $(X, \leq)$ is an Esakia space and $R \subseteq X \times X$, then $(X, \leq, R)$ is a *modal Esakia space*[5] if the following two conditions hold:

- If U is a clopen upset of X, then $\Box_R U$ is a clopen upset as well where $\Box_R U = \{x \in X \mid R[x] \subseteq U\}$.
- For every $x \in X$, $R[x]$ is a closed upset.

The morphisms between modal Esakia spaces are given as follows:

Definition 3.3.2. Let $(X_1, \leq, R_1)$ and $(X_2, \leq, R_2)$ be modal Esakia spaces, a map $f : X_1 \to X_2$ is called a *p-morphism* if the following conditions are satisfied for any $x, x', y \in X_1$ and $z \in X_2$:

1. f is continuous.
2. If $x \leq y$, then $f(x) \leq f(y)$.
3. If $f(x) \leq z$, then $f(x') = z$ for some $x \leq x'$.
4. If xR_1y, then $f(x)R_2f(y)$.
5. If $f(x)R_2z$, then $f(x') \leq z$ for some xR_1x'.

[5]It is also called $I_\Box$-*space* in [31].

Let **MHA** be the category of modal Heyting algebras and modal Heyting algebra homomorphisms, **MES** be the category of modal Esakia spaces and p-morphisms, the functors $(-)_* : \mathbf{MHA} \to \mathbf{MES}$ and $(-)^* : \mathbf{MES} \to \mathbf{MHA}$ that establish the duality are constructed as follows. For a modal Heyting algebra $\mathfrak{A} = (A, \Box)$, let $\mathfrak{A}_* = (A_*, R)$ where A_* is the Esakia space of A and xRy iff $\forall \Box a \in A(\Box a \in x \implies a \in y)$. For a modal Esakia space $\mathcal{X} = (X, \leq, R)$, let $\mathcal{X}^* = (X^*, \Box_R)$ where X^* is the Heyting algebra of clopen upsets of X and $\Box_R U = \{x \in X \mid R[x] \subseteq U\}$. The duals of maps are exactly the same as that in Esakia duality. We spell out some useful details about the duality in the following theorem.

Theorem 3.3.3. *[31, Thm 6.12]* **MHA** *is dually equivalent to* **MES***, which is witnessed by $(-)^*$ and $(-)_*$. In particular, for any modal Heyting algebra $\mathfrak{A}$, $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ witnessed by β where $\beta(a) = \{x \in A_* \mid a \in x\}$, and for any modal Esakia space $\mathcal{X}$, $\mathcal{X} \cong (\mathcal{X}^*)_*$ witnessed by ϵ where $\epsilon(x) = \{U \in X^* \mid x \in U\}$.*

As an easy corollary, we obtain the following duality:

Theorem 3.3.4. *The category of interior Heyting algebras and their homomorphisms is dually equivalent to the category of S4 modal Esakia spaces (S4 means that R is reflexive and transitive) and p-morphisms.*

Proof. By Theorem 3.3.3, we only need to check that for any interior Heyting algebra, its dual is an S4 modal Esakia space, and for any S4 modal Esakia space, its dual is an interior Heyting algebra.

Let $\mathfrak{A} = (A, \Box)$ be an interior Heyting algebra, and $\mathfrak{A}_* = (A_*, R)$. For any $x \in A_*$ and any $\Box a \in x$, as $\Box a \leq a$ and x is a prime filter, it follows that $a \in x$. Thus xRx, and so R is reflexive. For any $x, y, z \in A_*$, suppose $xRyRz$, then for any $\Box a \in x$, as $\Box a \leq \Box\Box a$ and x is a prime filter, it follows that $\Box\Box a \in x$. As $xRyRz$, we have that $\Box a \in y$ and $a \in z$. Thus xRz, and R is transitive. Therefore, $\mathfrak{A}_*$ is an S4 modal Esakia space.

Let $\mathcal{X} = (X, \leq, R)$ be an S4 modal Esakia space, and $\mathcal{X}^* = (X^*, \Box_R)$. For any $y \in \Box_R U$ where U is a clopen upset of X, $R[y] \subseteq U$. As R is reflexive, $y \in R[y] \subseteq U$. Thus $\Box_R U \subseteq U$. As R is transitive, $R[R[y]] \subseteq U$. Thus $y \in \Box_R \Box_R U$, $U \subseteq \Box_R \Box_R U$. Therefore, $\mathcal{X}^*$ is an interior Heyting algebra.

■

By the above duality, we can now give a dual description of stable bounded lattice homomorphisms.

Definition 3.3.5. Let $(X, \leq, R)$ and $(Y, \leq, R)$ be modal Esakia spaces and $f : X \to Y$ be a Priestley morphism. The map f is called *stable* if for any $x, y \in X$, xRy implies $f(x)Rf(y)$.

Proposition 3.3.6. Let $\mathfrak{A} = (A, \Box)$ and $\mathfrak{B} = (B, \Box)$ be modal Heyting algebras. Let $(X_A, \leq, R)$ and $(X_B, \leq, R)$ be the dual of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. For a bounded lattice homomorphism $h : A \to B$, h is stable iff $h_* : X_B \to X_A$ is stable.

Proof. By Priestley duality, h_* is a Priestley morphism. Thus, it suffices to prove that $h(\Box a) \leq \Box h(a)$ for any $a \in A$ iff xRy implies $h_*(x)Rh_*(y)$ for any $x, y \in X_B$.

Suppose $h(\Box a) \leq \Box h(a)$ for any $a \in A$. Suppose xRy and $\Box a \in h_*(x)$, then $h(\Box a) \in x$. As $h(\Box a) \leq \Box h(a)$ and x is a prime filter, $\Box h(a) \in x$. As xRy, it follows that $h(a) \in y$ and $a \in h_*(y)$. Thus $h_*(x)Rh_*(y)$.

For the other direction, suppose xRy implies $h_*(x)Rh_*(y)$ for any $x, y \in X_B$. For any $x \in \beta(h(\Box a))$, we have that $h(\Box a) \in x$ and $\Box a \in h_*(x)$. Now for any xRy, by assumption, $h_*(x)Rh_*(y)$. As $\Box a \in h_*(x)$, it follows that $a \in h_*(y)$ and $h(a) \in y$. Thus $R[x] \subseteq \beta(h(a))$, and $x \in \Box_R(\beta(h(a))) = \beta(\Box h(a))$. As $x \in \beta(h(\Box a))$ is arbitrary, this proves that $\beta(h(\Box a)) \subseteq \beta(\Box h(a))$. As β is an isomorphism, $h(\Box a) \leq \Box h(a)$. ■

Similarly, we can use the duality to give a dual description of CDC for $D^{\to}$ and $D^{\Box}$.

Definition 3.3.7. Let $(X, \leq, R)$ and $(Y, \leq, R)$ be modal Esakia spaces, $f : X \to Y$ be a Priestley morphism, and D be a clopen subset of Y. We say that f satisfies the *implication closed domain condition ($CDC_{\to}$)* for D if the following holds:

$$\uparrow f(x) \cap D \neq \emptyset \text{ implies } f[\uparrow x] \cap D \neq \emptyset.$$

Furthermore, let $\mathcal{D}$ be a collection of clopen subsets of Y, f satisfies the *implication closed domain condition ($CDC_{\to}$)* for $\mathcal{D}$ if f satisfies ($CDC_{\to}$) for each $D \in \mathcal{D}$.

We then have the following proposition analogous to [4, Lem. 4.3], which connects the algebraic CDC for $D^{\to}$ with the geometric $CDC_{\to}$.

Proposition 3.3.8. [4, Lem. 4.3]

Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal Heyting algebras, $h : A \to B$ be a bounded lattice homomorphism, and $a, b \in A$, then the following two conditions are equivalent:

1. $h(a \to b) = h(a) \to h(b)$.
2. h_* satisfies $\mathrm{CDC}_\to$ for $\beta(a) \setminus \beta(b)$.

For the dual description of CDC for $D^\Box$, we have the following.

Definition 3.3.9. Let $(X, \leq, R)$ and $(Y, \leq, R)$ be modal Esakia spaces, $f : X \to Y$ be a Priestley morphism, and D be a clopen subset of Y. We say that f satisfies the *modal closed domain condition ($CDC_\Box$)* for D if the following holds:

$$f[R[x]] \subseteq D \text{ implies } R[f(x)] \subseteq D.$$ [6]

Furthermore, let $\mathcal{D}$ be a collection of clopen subsets of Y, f satisfies the *modal closed domain condition ($CDC_\Box$)* for $\mathcal{D}$ if f satisfies ($\mathrm{CDC}_\Box$) for each $D \in \mathcal{D}$.

Proposition 3.3.10. Let $\mathfrak{A} = (A, \Box)$ and $\mathfrak{B} = (B, \Box)$ be modal Heyting algebras, $h : A \to B$ be a stable bounded lattice homomorphism, and $a \in A$, then the following are equivalent:

1. $h(\Box a) = \Box h(a)$.
2. $h_* : X_B \to X_A$ satisfies $\mathrm{CDC}_\Box$ for $\beta(a)$.

Proof. As h is stable and β is an isomorphism, $h(\Box a) = \Box h(a)$ iff $\Box h(a) \leq h(\Box a)$ iff $\beta(\Box h(a)) \subseteq \beta(h(\Box a))$ iff $\Box_R \beta(h(a)) \subseteq \beta(h(\Box a))$.

Suppose $h(\Box a) = \Box h(a)$, thus $\Box_R \beta(h(a)) \subseteq \beta(h(\Box a))$. Suppose $h_*[R[x]] \subseteq \beta(a)$, then $R[x] \subseteq h_*^{-1}(\beta(a))$. For any $y \in R[x]$, we have that $h_*(y) \in \beta(a)$ and $a \in h_*(y)$, $h(a) \in y$. Thus $R[x] \subseteq \beta(h(a))$, and $x \in \Box_R(\beta(h(a))) \subseteq \beta(h(\Box a))$. Namely, $h(\Box a) \in x$ and $\Box a \in h_*(x)$. Thus, for any $y \in R[h_*(x)]$, we have that $a \in y$. Therefore, $R[h_*(x)] \subseteq \beta(a)$. This proves that h_* satisfies $\mathrm{CDC}_\Box$ for $\beta(a)$.

For the other direction, suppose h_* satisfies $\mathrm{CDC}_\Box$ for $\beta(a)$. Suppose $x \in \Box_R(\beta(h(a)))$, then $R[x] \subseteq \beta(h(a))$. For any xRy, we have that $h(a) \in y$, and thus $a \in h_*(y)$. As y is arbitrary, this means that $h_*[R[x]] \subseteq \beta(a)$. As h_* satisfies $\mathrm{CDC}_\Box$ for $\beta(a)$, it follows that $R[h_*(x)] \subseteq \beta(a)$. Then by [31, Cor. 5.6] which says that for any $w \in X_A$, $\Box a \notin w$ implies that $a \notin v$ for some wRv, we have that $\Box a \in h_*(x)$ and $h(\Box a) \in x$, namely $x \in \beta(h(\Box a))$. Thus, $\Box_R(\beta(h(a))) \subseteq \beta(h(\Box a))$. This proves that $h(\Box a) = \Box h(a)$. ∎

[6] By contraposition, this is equivalent to the condition that $R[f(x)] \cap \bar{D} \neq \emptyset$ implies $f[R[x]] \cap \bar{D} \neq \emptyset$. Sometimes it may be more convenient to use this one.

Combining the above results together, we obtain the following dual description of the algebraic CDC for $D^{\to}$ and $D^{\Box}$:

Proposition 3.3.11. Let $\mathfrak{A} = (A, \Box)$ and $\mathfrak{B} = (B, \Box)$ be modal Heyting algebras, $h : A \to B$ be a stable bounded lattice homomorphism, $D^{\Box} \subseteq A$ and $D^{\to} \subseteq A^2$, the following two are equivalent:

- h satisfies CDC for $D^{\Box}$ and $D^{\to}$.
- h_* satisfies $\mathrm{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and satisfies $\mathrm{CDC}_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$.

By Propositions 3.3.6 and 3.3.11, we now know what stable canonical rules code geometrically, which is a dual analogue of Proposition 3.1.6.

Proposition 3.3.12. Let $\mathfrak{A} = (A, \Box)$ be a finite interior Heyting algebra, $D^{\to} \subseteq A^2$, $D^{\Box} \subseteq A$, and let $\mathfrak{B} = (B, \Box)$ be an interior Heyting algebra. Then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$ iff there is a surjective stable Priestley morphism $f : X_B \to X_A$ satisfying $\mathrm{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and $\mathrm{CDC}_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$.

Proof. According to Proposition 3.1.6, $\mathfrak{B} \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$ iff there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^{\to}$ and $D^{\Box}$. Then by Proposition 3.3.6 and Proposition 3.3.11, the result follows immediately.

■

By the above proposition, we are then well-justified to write $\rho(\mathfrak{A}, D^{\to}, D^{\Box})$ as $\rho(\mathfrak{A}_*, \mathfrak{D}_{\to}, \mathfrak{D}_M))$ where $\mathfrak{D}_{\to} = \{\beta(a) \setminus \beta(b) \mid (a, b) \in D^{\to}\}$, $\mathfrak{D}_M = \{\beta(a) \mid a \in D^{\Box}\}$[7]. This notation will become quite useful in the next chapter when we need to operate on those parameters.

Besides, a dual description of stable canonical formulas can also be obtained similarly. One may note that we only lack the dual description of s.i interior Heyting algebras in order to state Proposition 3.2.6 in geometric terms. However, considering the complexity of Proposition 3.2.6, its direct dual translation will be quite lengthy and is not needed in this thesis.

[7] "M" stands for "modal".

Chapter 4

Applications of stable canonical rules for intuitionistic modal logics

In the previous chapter, we introduced stable canonical rules for intuitionistic modal logics and their dual characterization. This chapter will be devoted to the applications of these rules in establishing some intrinsic properties of intuitionistic modal logics.

First, using stable canonical rules for bimodal logics, we will give a new and self-contained proof of the Blok-Esakia theorem for intuitionistic modal logics which was first proved by Wolter and Zakharyaschev in [37], and generalize it naturally to intuitionistic modal multi-conclusion consequence relations. Then we will proceed to proving the Dummett-Lemmon conjecture in this setting by our stable canonical rules for intuitionistic modal logics, which, as far as we know, is a new result.

4.1 The Blok-Esakia theorem for intuitionistic modal logics

It is well-known that every superintuitionistic logic can be embedded into an extension of **S4** via the Gödel translation[1]. The embedding led to many important results, one of which is the Blok-Esakia theorem. It states that the lattice of superintuitionistic logics is isomorphic to the lattice of normal ex-

[1]It is the translation which transforms an intuitionistic propositional formula to a modal formula by putting $\Box$ in front of every subformula. See Definition 4.1.16 below.

tensions of **Grz**, via the mapping which sends each superintuitionistic logic L to the normal extension of **Grz** with the set of all Gödel translations of formulas in L. As its name suggests, the Blok-Esakia theorem was proved independently by Blok using algebraic methods [11] and by Esakia using duality theory [20]. It allows us to study superintuitionistic logics by methods and results from normal modal logics and vice versa[2]. In [37] and [38], Wolter and Zakharyaschev extended the Gödel translation to intuitionistic modal logics which embeds every intuitionistic modal logic to an extension of bimodal logic $\mathbf{S4} \otimes \mathbf{K}$. Besides, they proved a Blok-Esakia theorem for this translation, whose proof assumes that the reader is familiar with the so-called *selection procedure* and *cofinal subreductions* which are developed in [43] and are quite involved.

In this section, we will use stable canonical rules to prove the Blok-Esakia theorem for intuitionistic modal multi-conclusion consequence relations, which generalizes that for intuitionistic modal logics. The proof strategy was adopted from the one in [16] where Cleani proved the Blok-Esakia theorem for superintuitionistic logics using stable canonical rules for classical (unary) normal modal logics. Compared to the proof in [37], we believe that our alternative proof provides a new perspective and is arguably more self-contained.

4.1.1 Stable canonical rules for bimodal logics

We first introduce stable canonical rules for bimodal logics, which are simple and natural generalizations of those in [6] from the unimodal case (only one modal operator) to the bimodal case.

Definition 4.1.1. Let $\mathfrak{A} = (A, \Box_I, \Box_M)$ and $\mathfrak{B} = (B, \Box_I, \Box_M)$ be bimodal algebras, and $h : A \to B$ be a Boolean homomorphism, h is *stable* if for any $a \in A$, we have $h(\Box_I a) \leq \Box_I h(a)$ and $h(\Box_M a) \leq \Box_M h(a)$.

Definition 4.1.2. Let $\mathfrak{A} = (A, \Box_I, \Box_M)$, $\mathfrak{B} = (B, \Box_I, \Box_M)$ be bimodal algebras, $D^I \subseteq A$ and $D^M \subseteq A$. A Boolean embedding $h : A \to B$ is said to satisfy

- the *closed domain condition* (CDC for short) for D^I if $h(\Box_I a) = \Box_I h(a)$ for any $a \in D^I$.
- the *closed domain condition* (CDC for short) for D^M if $h(\Box_M b) = \Box_M h(b)$ for any $b \in D^M$.

[2] See [14] for more details.

It is natural to anticipate that we can encode a stable Boolean embedding which satisfies CDC for some parameters in a certain syntactic form in analogy with Definition 3.1.4. This is exactly what the following definition does.

Definition 4.1.3. Let $\mathfrak{A} = (A, \Box_I, \Box_M)$ be a finite bimodal algebra, $D^I \subseteq A$ and $D^M \subseteq A$. For each $a \in A$, we introduce a new propositional letter p_a and define the *stable canonical rule* $\mu(\mathfrak{A}, D^I, D^M)$ based on $(\mathfrak{A}, D^I, D^M)$ as follows:

$$\begin{array}{rcl}\Gamma & = & \{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \{p_0 \leftrightarrow \bot, p_1 \leftrightarrow \top\} \\ & & \cup\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \\ & & \cup\{p_{\Box_I a} \to \Box_I p_a, p_{\Box_M a} \to \Box_M p_a \mid a \in A\} \cup \{\Box_I p_a \to p_{\Box_I a} \mid a \in D^I\} \\ & & \cup\{\Box_M p_a \to p_{\Box_M a} \mid a \in D^M\}\end{array}$$

$$\Delta = \{p_a \leftrightarrow p_b \mid a \neq b \in A\}$$

$$\mu(\mathfrak{A}, D^I, D^M) = \Gamma/\Delta.$$

Using a proof analogous to that of Proposition 3.1.6, we have the following proposition. The proof for the unimodal case can be found in [6, Thm. 5.4].

Proposition 4.1.4. Let $\mathfrak{A} = (A, \Box_I, \Box_M)$ be a finite bimodal algebra, $D^I \subseteq A$, $D^M \subseteq A$, and $\mathfrak{B} = (B, \Box_I, \Box_M)$ be a bimodal algebra. Then $\mathfrak{B} \not\models \mu(\mathfrak{A}, D^I, D^M)$ iff there is a stable Boolean embedding $h : A \to B$ satisfying CDC for D^I and D^M.

The following result is essentially needed in our proof of the Blok-Esakia theorem. For convenience, let **Mix**[3] denote the formula $\Box_I \Box_M \Box_I p \leftrightarrow \Box_M p$.

Proposition 4.1.5. For any bimodal multi-conclusion rule Γ/Δ, there exist tuples $(\mathfrak{A}_1, D^I_1, D^M_1), \ldots, (\mathfrak{A}_n, D^I_n, D^M_n)$ such that each $\mathfrak{A}_i$ is a finite $S4 \otimes K \oplus Mix$-algebra, $D^I_i \subseteq A_i$ and $D^M_i \subseteq A_i$, and for each $S4 \otimes K \oplus Mix$-algebra $\mathfrak{B} = (B, \Box_I, \Box_M)$, we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is a stable embedding $h : A_i \to B$ satisfying CDC for D^I_i and D^M_i.

Proof. Let Γ/Δ be an arbitrary bimodal multi-conclusion rule. If $\Gamma/\Delta \in \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R$, then take $n = 0$. Suppose $\Gamma/\Delta \notin \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R$, let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$ which is clearly finite, define $\Theta' = \Theta \cup \{\Box_I \varphi \mid \Box_M \varphi \in \Theta\}$. Clearly Θ' is finite and closed under

[3]This was introduced by Wolter and Zakharyaschev in [37]. The reason why it is called "**Mix**" is that semantically the formula says that $R_I \circ R_M \circ R_I = R_M$, which means that R_M and R_I are "mixed" together in some sense.

subformulas, and for any formula $\Box_M\varphi$, we have that $\Box_M\varphi \in \Theta'$ implies $\Box_I\varphi \in \Theta'$. Assume $|\Theta'| = m$, there are only finitely many pairs $(\mathfrak{A}, D^I, D^M)$ satisfying the following two conditions up to isomorphism:

i) $\mathfrak{A} = (A, \Box_I, \Box_M)$ is a finite $S4 \otimes K \oplus Mix$-algebra such that A is an m-generated Boolean algebras and $\mathfrak{A} \not\models \Gamma/\Delta$.

ii) $D^I = \{V(\varphi) \mid \Box_I\varphi \in \Theta'\}$ and $D^M = \{V(\varphi) \mid \Box_M\varphi \in \Theta'\}$ where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \Gamma/\Delta$.

Let $(\mathfrak{A}_1, D_1^I, D_1^M), ..., (\mathfrak{A}_n, D_n^I, D_n^M)$ be an enumeration of such pairs. For any $S4 \otimes K \oplus Mix$-algebra $\mathfrak{B}$, we prove that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is a stable embedding $h : A_i \to B$ satisfying CDC for D_i^I and D_i^M.

For the right-to-left direction, suppose there is a stable embedding $h : A_i \to B$ satisfying CDC for D_i^I and D_i^M. Define a valuation V_B on $\mathfrak{B}$ by $V_B(p) = h(V_i(p))$ for any propositional letter p where V_i is the valuation on $\mathfrak{A}_i$ witnessing that $\mathfrak{A}_i \not\models \Gamma/\Delta$. We then can prove by induction that $V_B(\psi) = h(V_i(\psi))$ for any $\psi \in \Theta'$ in the same way as that in the proof of Proposition 3.1.3.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \Gamma/\Delta$. There exists a valuation V_B on B such that $V_B(\gamma) = 1_B$ for any $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for any $\delta \in \Delta$. Let B' be the Boolean subalgebra of B generated by $V_B(\Theta')$. Clearly $|V_B(\Theta')| \leq |\Theta'| = m$. Let $D^I = \{V_B(\varphi) \mid \Box_I\varphi \in \Theta'\}$ and $D^M = \{V_B(\varphi) \mid \Box_M\varphi \in \Theta'\}$, we define $\Box'_I$ and $\Box'_M$ on B' as follows: $\Box'_I a = \bigvee\{\Box_I b \mid \Box_I b \leq \Box_I a \text{ and } b, \Box_I b \in B'\}$ and $\Box'_M a = \bigvee\{\Box_M b \mid b \leq a \text{ and } b, \Box_M b, \Box_I b \in B'\}$ for any $a \in B'$. We first prove that $(B', \Box'_I, \Box'_M)$ is an $S4 \otimes K \oplus Mix$-algebra.

As $\mathfrak{B}$ is an $S4 \otimes K \oplus Mix$-algebra and $\Box'_I$ is defined the same way as that of $\Box'$ in the proof of Proposition 3.1.3, we can check that $(B', \Box'_I)$ is an $S4$-algebra in the same way as we checked that $(B', \to', \Box')$ is an interior Heyting algebra in that proof. As $1 = \Box_I 1 = \Box_M 1 \in B'$, by definition $\Box_M 1 \leq \Box'_M 1$ and thus $\Box'_M 1 = 1$.

For any $a, b \in B'$, $\Box'_M a \wedge \Box'_M b$
$= \bigvee\{\Box_M x \mid x \leq a \text{ and } x, \Box_I x, \Box_M x \in B'\} \vee \bigvee\{\Box_M y \mid y \leq b \text{ and } y, \Box_I y, \Box_M y \in B'\}$
$= \bigvee\{\Box_M x \wedge \Box_M y \mid x \leq a \text{ and } y \leq b \text{ where } x, \Box_I x, \Box_M x, y, \Box_I y, \Box_M y \in B'\}$(distributivity)
$= \bigvee\{\Box_M(a \wedge b) \mid x \leq a \text{ and } y \leq b \text{ where } x, \Box_I x, \Box_M x, y, \Box_I y, \Box_M y \in B'\}$
$= \bigvee\{\Box_M z \mid z \leq a \wedge b \text{ and } z, \Box_M z, \Box_I z \in B'\}$
$= \Box'_M(a \wedge b)$.

Therefore, $(B', \Box'_M)$ is a modal algebra. To check that $(B', \Box'_I, \Box'_M)$ is an $S4 \otimes K \oplus Mix$-algebra, it suffices to prove that for any $a \in B'$, $\Box'_I \Box'_M a = \Box'_M a$ and $\Box'_M \Box'_I a = \Box'_M a$[4].

For any $a \in B'$, as $(B', \Box'_I, \Box'_M)$ is an $S4 \otimes K$-algebra, $\Box'_I a \leq a$, and thus $\Box'_M \Box'_I a \leq \Box'_M a$. Suppose $b \leq a$ such that $b, \Box_I b, \Box_M b \in B'$, then $\Box_I b \leq \Box_I a$. Thus $\Box_I b \leq \bigvee\{\Box_I x \mid \Box_I x \leq \Box_I a \text{ where } x, \Box_I x \in B'\} = \Box'_I a$. As $\mathfrak{B}$ is an $S4 \otimes K \oplus Mix$-algebra, $\Box_I b = \Box_I \Box_I b \in B'$ and $\Box_M \Box_I b = \Box_M b \in B'$. Therefore, $\Box_M b \in \{\Box_M x \mid x \leq \Box'_I a \text{ where } x, \Box_I x, \Box_M x \in B'\}$. Thus $\Box_M b \leq \bigvee\{\Box_M x \mid x \leq \Box'_I a \text{ where } x, \Box_I x, \Box_M x \in B'\} = \Box'_M \Box'_I a$. By the definition of $\Box'_M$, we have that $\Box'_M a \leq \Box'_M \Box'_I a$. Therefore, $\Box'_M a = \Box'_M \Box'_I a$. For any $a \in B'$, as $(B', \Box'_I, \Box'_M)$ is an $S4 \otimes K$-algebra, $\Box'_I \Box'_M a \leq \Box'_M a$. Suppose $b \leq a$ such that $b, \Box_I b, \Box_M b \in B'$, then $\Box_M b \leq \Box'_M a$. As $\mathfrak{B}$ is an $S4 \otimes K \oplus Mix$-algebra, $\Box_I \Box_M b \leq \Box_I \Box'_M a$ and $\Box_I \Box_M b = \Box_M b \in B'$. Thus $\Box_M b = \Box_I \Box_M b \leq \bigvee\{\Box_I x \mid \Box_I x \leq \Box_I \Box'_M a \text{ where } x, \Box_I x \in B'\} = \Box'_I \Box'_M a$. By the definition of $\Box'_M$, we have that $\Box'_M a \leq \Box'_I \Box'_M a$, and thus $\Box'_M a = \Box'_I \Box'_M a$. Therefore, $(B', \Box'_I, \Box'_M)$ is an $S4 \otimes K \oplus Mix$-algebra.

Let $h : (B', \Box'_I, \Box'_M) \to (B, \Box_I, \Box_M)$ be the inclusion map, h is clearly an embedding as B' is a Boolean subalgebra of B. As $\Box'_I a \leq \Box_I a$ and $\Box'_M a \leq \Box_M a$ for any $a \in B'$ by definition, h is stable. Then we check that h satisfies CDC for D^I and D^M. For any $a \in D^I$, $a = V_B(\varphi)$ for some $\Box_I \varphi \in \Theta'$. Thus $V_B(\Box_I \varphi) = \Box_I a \in B'$. By the definition of $\Box'_I$, we then have that $\Box'_I a = \Box_I a$. For any $b \in D^M$, $b = V_B(\psi)$ for some $\Box_M \psi \in \Theta'$. Thus $\Box_I \psi \in \Theta'$, and $V_B(\Box_I \psi) = \Box_I V_B(\psi) = \Box_I b \in B'$. As $V_B(\Box_M \psi) = \Box_M V_B(\psi) = \Box_M b \in B'$, by the definition of $\Box'_M$, $\Box'_M b = \Box_M b$. This proves that h satisfies CDC for D^I and D^M.

Let V' be the valuation V_B restricted to B', we then prove that for any $\varphi \in \Theta'$, $V'(\varphi) = V_B(\varphi)$ by induction on φ. We only consider the modal cases as others are trivial:

If $\varphi = \Box_I \psi$, as $\Box_I \psi \in \Theta'$, we have that $\psi \in \Theta'$. Thus $V_B(\Box_I \psi) = \Box_I V_B(\psi) \in B'$ and $V_B(\psi) \in B'$.

$$\begin{aligned} V'(\Box_I \psi) &= \Box'_I V'(\psi) \\ &= \Box'_I V_B(\psi) && \text{(IH)} \\ &= \Box_I V_B(\psi) && \text{(By the definition of } \Box'_I) \\ &= V_B(\Box_I \psi). \end{aligned}$$

If $\varphi = \Box_M \psi$, as $\Box_M \psi \in \Theta'$, we have that $\Box_I \psi, \psi \in \Theta'$. Thus $V_B(\Box_M \psi) = \Box_M V_B(\psi) \in B'$, $V_B(\Box_I \psi) = \Box_I V_B(\psi) \in B'$ and $V_B(\psi) \in B'$.

[4] It is easy to check that for any $S4 \otimes K$-algebra, it validates **Mix** iff it validates $\Box_I \Box_M p \leftrightarrow \Box_M p$ and $\Box_M \Box_I p \leftrightarrow \Box_M p$. In fact, in [38], **Mix** was simply defined to be $(\Box_I \Box_M p \leftrightarrow \Box_M p) \wedge (\Box_M \Box_I p \leftrightarrow \Box_M p)$.

$$
\begin{array}{rcll}
V'(\Box_M\psi) & = & \Box'_M V'(\psi) & \\
& = & \Box'_M V_B(\psi) & \text{(IH)} \\
& = & \Box_M V_B(\psi) & \text{(By the definition of } \Box'_M) \\
& = & V_B(\Box_M\psi). &
\end{array}
$$

Since V_B is a valuation which refutes Γ/Δ on $\mathfrak{B}$, V' is a valuation on $(B', \Box_I, \Box_M)$ which refutes Γ/Δ. As for any $\varphi \in \Theta'$, $V'(\varphi) = V_B(\varphi)$, we have that $D^I = \{V_B(\varphi) \mid \Box_I\varphi \in \Theta'\} = \{V'(\varphi) \mid \Box_I\varphi \in \Theta'\}$ and $D^M = \{V'(\varphi) \mid \Box_M\varphi \in \Theta'\}$. As B' is generated by $V_B(\Theta')$ whose cardinality is no larger than m, $(B', \Box'_I, \Box'_M, D^I, D^M)$ must be one of $(\mathfrak{A}_1, D_1^I, D_1^M), ..., (\mathfrak{A}_n, D_n^I, D_n^M)$. Since $h : (B', \Box'_I, \Box'_M) \to (B, \Box_I, \Box_M)$ is a stable embedding satisfying CDC for D^I and D^M, we get what we want. ■

The above proposition is not a trivial generalization of [6, Thm. 5.1] to bimodal cases as we require that every A_i should validate **Mix**. This is quite crucial in the proof of Lemma 4.1.19 below. In fact, the above proof gives a concrete example of how to do filtrations in polymodal cases when there are interactions between different operators. It is far from clear how to do so in general, and an answer to it will shed light on some open problems about the finite model property of polymodal logics.

The above two propositions allow us to get the following result, which is analogous to Theorem 3.1.7.

Proposition 4.1.6. For any bimodal multi-conclusion rule Γ/Δ, there exist tuples $(\mathfrak{A}_1, D_1^I, D_1^M), \ldots, (\mathfrak{A}_n, D_n^I, D_n^M)$ such that each $\mathfrak{A}_i$ is a finite $S4 \otimes K \oplus Mix$-algebra, $D_i^I \subseteq A_i$ and $D_i^M \subseteq A_i$, and for each $S4 \otimes K \oplus Mix$-algebra $\mathfrak{B} = (B, \Box_I, \Box_M)$, we have that $\mathfrak{B} \vDash \Gamma/\Delta$ iff $\mathfrak{B} \vDash \mu(\mathfrak{A}_1, D_1^I, D_1^M), ...,$ $\mu(\mathfrak{A}_n, D_n^I, D_n^M)$.

Then we introduce the category of bimodal spaces, which is dually equivalent to the category of bimodal algebras. Similarly to Chapter 3, this duality allows us to obtain a geometric characterization of stable canonical rules for bimodal logics, which makes our construction in the proof of Lemma 4.1.19 possible.

Definition 4.1.7. A *bimodal space* is a triple (X, R_I, R_M) where (X, R_I) and (X, R_M) are modal spaces.

Let $\mathfrak{X}, \mathfrak{Y}$ be bimodal spaces, a map $h : X \to Y$ is *stable* if for $R \in \{R_I, R_M\}$ and any $x, y \in X$, xRy implies $f(x)Rf(y)$. Furthermore, h is a *bounded morphism* if h is stable, and for any $x, y \in X$, $f(x)Ry$ implies that there is a $z \in X$ such that xRz and $f(z) = y$ where $R \in \{R_I, R_M\}$.

A *valuation* on a bimodal space $\mathfrak{X}$ is a map $V : Prop \to Clop(X)$ which can be extended recursively to a map from $Form_{bi}$ to $Clop(X)$ in the usual way. For any bimodal logic L, a bimodal space $\mathfrak{X}$ is *an L-modal space* if it validates L. In particular, an $S4 \otimes K$-modal space $\mathfrak{X} = (X, R_I, R_M)$ is a bimodal space where (X, R_I) is an $S4$-space and (X, R_M) is a modal space.

Let **BMS** be the category of bimodal spaces with continuous bounded morphisms, and **BMA** be the category of bimodal algebras with their homomorphisms, the functor $(-)_* : \mathbf{BMA} \to \mathbf{BMS}$ and $(-)^* : \mathbf{BMS} \to \mathbf{BMA}$ that establish the duality are constructed as follows. For a bimodal algebra $\mathfrak{A} = (A, \Box_I, \Box_M)$, let $\mathfrak{A}_* = (A_*, R_I, R_M)$ where A_* is the Stone space of A and xR_Iy iff $\forall \Box_I a \in A(\Box_I a \in x \Rightarrow a \in y)$, and similarly we define R_M. For a bimodal space $\mathfrak{X} = (X, R_I, R_M)$, let $\mathfrak{X}^* = (X^*, \Box_{R_I}, \Box_{R_M})$ where X^* is the Boolean algebra of clopen sets of X and $\Box_{R_I} U = \{x \in X \mid R_I[x] \subseteq U\}$. Similarly we define $\Box_{R_M}$. The duals of maps are the same as those in Esakia duality (namely taking inverse images).

Theorem 4.1.8. **BMA** *is dually equivalent to* **BMS**, *which is witnessed by $(-)^*$ and $(-)_*$. In particular, for any bimodal algebra $\mathfrak{A}$, $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ witnessed by β where $\beta(a) = \{x \in A_* \mid a \in x\}$, and for any bimodal space $\mathfrak{X}$, $\mathfrak{X} \cong (\mathfrak{X}^*)_*$ witnessed by ϵ where $\epsilon(x) = \{U \in X^* \mid x \in U\}$.*

Similarly to Theorem 3.3.4, when restricted to those validating $S4 \otimes K$, the above theorem gives us the dual equivalence between the category of $S4 \otimes K$-modal spaces with continuous bounded morphisms and the category of $S4 \otimes K$-algebras with their homomorphisms.

Using Theorem 4.1.8, we can now easily exploit the geometric intuitions about the stable canonical rules for bimodal logics. The proof of the one for unimodal logics can be found in [6, Thm. 5.4].

Proposition 4.1.9. Let $\mathfrak{A} = (A, \Box_I, \Box_M)$ be a finite bimodal algebra, $D^I \subseteq A$ and $D^M \subseteq A$, and let $\mathfrak{B} = (B, \Box_I, \Box_M)$ be a bimodal algebra, then $\mathfrak{B} \not\models \mu(\mathfrak{A}, D^I, D^M)$ iff there is a continuous stable surjection $f : X_B \to X_A$ satisfying $\mathrm{CDC}_\Box$ for any $\beta(a)$ where $a \in D^I$ and for any $\beta(b)$ where $b \in D^M$[5].

Because of the above proposition, we are justified to write $\mu(\mathfrak{A}, D^I, D^M)$ as $\mu(\mathfrak{A}_*, \mathfrak{D}_I, \mathfrak{D}_M)$ where $\mathfrak{D}_I = \{\beta(a) \mid a \in D^I\}$ and $\mathfrak{D}_M = \{\beta(b) \mid b \in D^M\}$. Besides, since $\mathfrak{A}_*$ is finite, every subset of $\mathfrak{A}_*$ is clopen and thus of the form $\beta(a)$ for some $a \in \mathfrak{A}$ by Stone duality. Therefore, we are also justified to write $\mu(\mathfrak{Y}, \mathfrak{D}_I, \mathfrak{D}_M)$ where $\mathfrak{Y}$ is a finite bimodal space, and $\mathfrak{D}_I$, $\mathfrak{D}_M$ are sets of subsets of $\mathfrak{Y}$.

[5] $\mathrm{CDC}_\Box$ is given in Definition 3.3.9. When there is no ambiguity (in particular, for bimodal logics), we may just write CDC instead of $\mathrm{CDC}_\Box$.

4.1.2 The Gödel translation for intutionistic modal logics and related constructions

Now we introduce the operations from modal Esakia spaces to $S4 \otimes K$-modal spaces and vice versa. They are in essence the topological versions of those defined in [38] for general frames, and have an intimate relation with the Gödel translation.

For any $S4 \otimes K$-modal space $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$, we write $x \backsim y$ iff xR_Iy and yR_Iy, and then define $\rho : Y \to \mathcal{P}(Y)$ by $\rho(x) = \{y \in Y \mid x \backsim y\}$. And for convenience, we may also write $[x]$ for $\rho(x)$.

Definition 4.1.10.

- For any modal Esakia space $\mathcal{X} = (X, \leq, R, \mathcal{O})$ where $\mathcal{O}$ is the topology, we set $\sigma(\mathcal{X}) = (X, R_I, R_M, \mathcal{O})$ where $R_I =\leq$ and $R_M = R$.
- For any $S4 \otimes K$-modal space $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$, we set $\rho(\mathfrak{Y}) = (\rho[Y], \leq, [R_I \circ R_M \circ R_I], \rho[\mathcal{O}])$ where $\rho(x) \leq \rho(y)$ iff xR_Iy, $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ iff $xR_I \circ R_M \circ R_Iy$[6] and $\rho(\mathcal{O})$ is the quotient topology.

First, modal Esakia spaces have the following intrinsic property:

Proposition 4.1.11. For any modal Esakia space $\mathcal{X} = (X, \leq, R, \mathcal{O})$, $\leq \circ R = R \circ \leq = R$, i.e., $R = \leq \circ R \circ \leq$.

Proof. By the duality given in Theorem 3.3.3, it suffices to prove that for any modal Heyting algebra $\mathfrak{A} = (A, \Box)$, the relation holds on its dual space $\mathfrak{A}_*$. Namely, we need to prove that $\subseteq \circ R \circ \subseteq = R$. Let $x, y \in A_*$ where $x \subseteq \circ R \circ \subseteq y$, then there exists $z, z' \in A_*$ such that $x \subseteq z$, zRz' and $z' \subseteq y$. For any $\Box a \in A$, if $\Box a \in x$, then $\Box a \in z$. As zRz', it follows that $a \in z'$ and thus $a \in y$. Therefore, xRy. We get $\subseteq \circ R \circ \subseteq = R$ (the other direction is obvious). ■

Let $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$ be an $S4 \otimes K$-modal space, we say $U \subseteq Y$ is an *upset* if it is an upset w.r.t the quasi-order R_I, i.e., $U = \{y \in Y \mid xR_Iy \text{ for some } x \in U\}$. We can now check that the operations in Definition 4.1.10 transform modal Esakia spaces to bimodal spaces and vice versa.

Proposition 4.1.12. The following hold:

1) For any modal Esakia space $\mathcal{X}$, we have that $\sigma(\mathcal{X})$ is an $S4 \otimes K$-modal space.

[6]It is easy to check that they are well defined.

2) For any $S4 \otimes K$-modal space $\mathfrak{Y}$, we have that $\rho(\mathfrak{Y})$ is a modal Esakia space.

Proof. For 1), as $\mathcal{X} = (X, \leq, R, \mathcal{O})$ is a modal Esakia space, $(X, \leq, \mathcal{O})$ is an Esakia space. If we omit R, $\sigma(\mathcal{X})$ is defined exactly the same as that in [16, Def. 2.43], which is well-known to be an S4-modal space. Thus we only need to check that $(X, R_M, \mathcal{O})$ is a modal space. For this, it suffices to prove that if $U \subseteq X$ is clopen, then $\Box_{R_M} U = \Box_R U$ is also clopen. Let U be an arbitrary clopen subset of X, as $(X, \leq, \mathcal{O})$ is an Esakia space, $U = \bigcup_{1 \leq i \leq n}(U_i \setminus V_i)$ where U_i, V_i's are clopen upsets. $\Box_R U = \Box_R(\bigcup_{1 \leq i \leq n}(U_i \setminus V_i)) = \Box_R((U_1 \cap \bar{V_1}) \cup ... \cup (U_n \cap \bar{V_n}))$. By distributivity, $\Box_R U = \Box_R \bigcap_{1 \leq i \leq k}(U'_i \cup \bar{V}'_i)$ for some k where U'_i, V'_i's are clopen upsets. Then $\Box_R U = \bigcap_{1 \leq i \leq k} \Box_R(U'_i \cup \bar{V}'_i)$. Now, for any $1 \leq i \leq k$, as $V'_i \to U'_i \subseteq U'_i \cup \bar{V}'_i$, we have that $\Box_R(V'_i \to U'_i) \subseteq \Box_R(U'_i \cup \bar{V}'_i)$. Suppose $x \in \Box_R(U'_i \cup \bar{V}'_i)$, then $R[x] \subseteq U'_i \cup \bar{V}'_i$. For any xRy and $y \leq z$, as $R\circ \leq = R$ by Proposition 4.1.11, xRz, and thus $z \in U'_i \cup \bar{V}'_i$. Therefore, $y \in V'_i \to U'_i$, and thus $R[x] \subseteq V'_i \to U'_i$, namely $x \in \Box_R(V'_i \to U'_i)$. As $x \in \Box_R(U'_i \cup \bar{V}'_i)$ is arbitrary, we have that $\Box_R(U'_i \cup \bar{V}'_i) \subseteq \Box_R(V'_i \to U'_i)$, and thus $\Box_R(U'_i \cup \bar{V}'_i) = \Box_R(V'_i \to U'_i)$. As V'_i and U'_i are clopen upsets and $(X, \leq, \mathcal{O})$ is an Esakia space, $V'_i \to U'_i$ is a clopen upset. As $\mathcal{X}$ is a modal Esakia space, by item 1 in Definition 3.3.1, $\Box_R(U'_i \cup \bar{V}'_i) = \Box_R(V'_i \to U'_i)$ is clopen. $\Box_R U = \bigcap_{1 \leq i \leq k} \Box_R(U'_i \cup \bar{V}'_i)$ is also clopen.

For 2), let $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$ be an $S4 \otimes K$-modal space. If we omit R_M, $\rho(\mathfrak{Y})$ is defined exactly the same as that in [16, Def. 2.43], which is well-known to be an Esakia space. Thus it suffices to check that for any clopen upset U of $\rho(\mathfrak{Y})$, $\Box_{[R_I \circ R_M \circ R_I]} U$ is a clopen upset, and for any $x \in Y$, $[R_I \circ R_M \circ R_I][\rho(x)]$ is a closed upset.

Let U be an arbitrary clopen upset of $\rho(\mathfrak{Y})$, as $\rho(\mathcal{O})$ is the quotient topology, $\rho^{-1}(U)$ is clopen. By the definition of $\leq$, $\rho^{-1}(U)$ is an upset as U is an upset. Clearly, $\rho^{-1}(\Box_{[R_I \circ R_M \circ R_I]} U) = \Box_{R_I} \Box_{R_M} \Box_{R_I} \rho^{-1}(U)$. As $\rho^{-1}(U)$ is a clopen upset and $\mathfrak{Y}$ is an $S4 \oplus K$-modal space, $\rho^{-1}(\Box_{[R_I \circ R_M \circ R_I]} U) = \Box_{R_I} \Box_{R_M} \Box_{R_I} \rho^{-1}(U)$ is clopen. As R_I is transitive, obviously $\Box_{R_I} \Box_{R_M} \Box_{R_I} \rho^{-1}(U)$ is an upset. Therefore, $\rho^{-1}(\Box_{[R_I \circ R_M \circ R_I]} U)$ is a clopen upset, so is $\Box_{[R_I \circ R_M \circ R_I]} U$.

Let $x \in Y$ be arbitrary, clearly, $[R_I \circ R_M \circ R_I][\rho[x]]$ is an upset. By the definition of $\leq$, $\bigcup [R_I \circ R_M \circ R_I][\rho[x]] = (R_I \circ R_M \circ R_I)[x] = R_I[R_M[R_I[x]]]$. By the definition of modal spaces, $R_I[x]$ is closed. As $\mathfrak{Y}$ is an $S4 \otimes K$-modal space, $R_M[R_I[x]]$ is also closed[7], so is $R_I[R_M[R_I[x]]]$. Therefore, $[R_I \circ R_M \circ R_I][\rho(x)]$ is a closed upset.

[7]It is known that for any modal space $\mathfrak{X}$, if U is closed, then $R[U]$ is also closed:

■

Now, by duality, we can also give the dual of σ and ρ as follows. Let $\mathfrak{A}$ be an arbitrary modal Heyting algebra, $\mathfrak{A}_*$ is a modal Esakia space by the duality given in Theorem 3.3.3. $\sigma(\mathfrak{A}_*)$ is an $S4 \otimes K$-modal space by Proposition 4.1.12. By Theorem 4.1.8, $\sigma(\mathfrak{A}_*)^*$ is an $S4 \otimes K$-algebra. We set $\sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_*)^*$ for any modal Heyting algebra $\mathfrak{A}$. σ is then a map from the class of modal Heyting algebras to the class of $S4 \otimes K$-algebras. Similarly, we define $\rho(\mathfrak{B}) = \rho(\mathfrak{B}_*)^*$ for any $S4 \otimes K$-algebra. ρ is then a map from the class of $S4 \otimes K$-modal algebras to the class of modal Heyting algebras.

Then we can get the following proposition which says that the algebraic σ, ρ and the geometric σ, ρ are dual to each other respectively.

Proposition 4.1.13. The following hold:

1) For any modal Heyting algebra $\mathfrak{A}$, $(\sigma(\mathfrak{A}))_* \cong \sigma(\mathfrak{A}_*)$. Consequently, $\sigma((\mathcal{X})^*) \cong (\sigma(\mathcal{X}))^*$ for any modal Esakia space $\mathcal{X}$.

2) For any $S4 \otimes K$-algebra $\mathfrak{B}$, $(\rho(\mathfrak{B}))_* \cong \rho(\mathfrak{B}_*)$. Consequently, $(\rho(\mathfrak{X}))^* \cong \rho(\mathfrak{X}^*)$ for any $S4 \otimes K$-modal space $\mathfrak{X}$.

Proof. Simply by the definitions of σ and ρ and the duality.

■

Besides, since $R = \leq \circ R \circ \leq$ for any modal Esakia space $\mathcal{X} = (X, \leq, R, \mathcal{O})$, the following proposition is straightforward.

Proposition 4.1.14. For any modal Esakia space $\mathcal{X} = (X, \leq, R, \mathcal{O})$, $\rho(\sigma(\mathcal{X})) \cong \mathcal{X}$. Consequently, for any modal Heyting algebra $\mathfrak{A}$, $\rho(\sigma(\mathfrak{A})) \cong \mathfrak{A}$.

The above proposition is the analogue to the first half of [16, Prop. 2.45]. For the second half of [16, Prop. 2.45], the analogue does not hold for $S4 \otimes K$-algebras in general. We need an extra assumption.

Proposition 4.1.15. If $\mathfrak{B} = (B, \Box_I, \Box_M)$ is an $S4 \otimes K \oplus Mix$-algebra, then there is a homomorphic embedding of $\sigma(\rho(\mathfrak{B}))$ into $\mathfrak{B}$ (usually denoted as $\sigma(\rho(\mathfrak{B})) \rightarrowtail \mathfrak{B}$).

suppose $x \notin R[U]$, then $\Diamond_R\{x\} \cap U = \emptyset$. $\Diamond_R\{x\} = \Diamond_R \bigcap\{Y \mid x \in Y \in Clop(\mathfrak{X})\}$. By Esakia Lemma, $\Diamond_R \bigcap\{Y \mid x \in Y \in Clop(\mathfrak{X})\} = \bigcap\{\Diamond_R Y \mid x \in Y \in Clop(\mathfrak{X})\}$. Thus $\bigcap\{\Diamond_R Y \mid x \in Y \in Clop(\mathfrak{X})\} \cap U = \emptyset$. By compactness, $U \cap \Diamond_R Y_1 \cap ... \cap \Diamond_R Y_n = \emptyset$ for some $x \in Y_1, ..., Y_n \in Clop(\mathfrak{X})$. As $\Diamond_R(Y_1 \cap ... \cap Y_n) \subseteq \Diamond_R Y_1 \cap ... \cap \Diamond_R Y_n$, $U \cap \Diamond_R(Y_1 \cap ... \cap Y_n) = \emptyset$. As $x \in Y_1 \cap ... \cap Y_n$ is clopen and $Y_1 \cap ... \cap Y_n \cap R[U] = \emptyset$, $R[U]$ is closed.

Proof. Let $\mathfrak{B}$ be an $S4 \otimes K$-algebra which validates $\Box_I \Box_M \Box_I p \leftrightarrow \Box_M p$. By duality given in Theorem 4.1.8, $\mathfrak{B}_*$ is an $S4 \otimes K$-modal space on which $R_I \circ R_M \circ R_I = R_M$, and it suffices to show that there is a surjective continuous bounded morphism from $\mathfrak{B}_*$ onto $(\sigma(\rho(\mathfrak{B})))_*$. By Proposition 4.1.14, $(\sigma(\rho(\mathfrak{B})))_* \cong \sigma(\rho(\mathfrak{B}_*))$. We check that $x \mapsto [x]$ is a surjective continuous bounded morphism from $\mathfrak{X} = (X, R_I, R_M, \mathcal{O})$ onto $\sigma(\rho(\mathfrak{X}))$ for any $S4 \otimes K$-modal space $\mathfrak{X}$ on which $R_I \circ R_M \circ R_I = R_M$.

Clearly, the map is surjective and continuous (the topology is the quotient topology). If xR_Iy, then by definition $[x] \leq [y]$; If $[x] \leq [y]$, then by definition xR_Iy. Suppose xR_My, then as R_I is reflexive, $xR_I \circ R_M \circ R_Iy$ and thus $[x][R_I \circ R_M \circ R_I][y]$. Suppose $[x][R_I \circ R_M \circ R_I][y]$, then $xR_I \circ R_M \circ R_I y$ by definition. As $R_I \circ R_M \circ R_I = R_M$, xR_My. This proves that the map is a bounded morphism.

■

Now, we introduce the *Gödel translation* for intuitionistic modal logics, which is given in [38].

Definition 4.1.16. The *Gödel translation* for intuitionistic modal logics $t : Form_{i\Box} \to Form_{bi}$ is recursively defined as follows:

- $t(p) = \Box_I p$ where p is a propositional variable
- $t(\bot) = \Box_I \bot$
- $t(\top) = \Box_I \top$
- $t(\varphi \to \psi) = \Box_I(t(\varphi) \to t(\psi))$
- $t(\varphi \wedge \psi) = \Box_I(t(\varphi) \wedge t(\psi))$
- $t(\varphi \vee \psi) = \Box_I(t(\varphi) \vee t(\psi))$
- $t(\Box\varphi) = \Box_I \Box_M t(\varphi)$

It turns out that for the following results to hold, it makes no difference if we define $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$ and $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$ instead. When restricted to the set of intuitionistic propositional formulas, t is just the well-known Gödel translation. For any intuitionistic modal multi-conclusion rule Γ/Δ, we write $t(\Gamma/\Delta)$ for $\{t(\gamma) \mid \gamma \in \Gamma\}/\{t(\delta) \mid \delta \in \Delta\}$.

The following proposition is the counterpart to [27, Lem. 3.13].

Proposition 4.1.17. If $\mathfrak{A}$ is an $S4 \otimes K$-algebra, then $\mathfrak{A} \vDash t(\Gamma/\Delta)$ iff $\rho(\mathfrak{A}) \vDash \Gamma/\Delta$.

Proof. By Theorem 4.1.8, $\mathfrak{A} \vDash \Gamma/\Delta$ iff $\mathfrak{A}_* \vDash \Gamma/\Delta$ and $\rho(\mathfrak{A}) \vDash \Gamma/\Delta$ iff $\rho(\mathfrak{A}_*) = \rho(\mathfrak{A})_* \vDash \Gamma/\Delta$. Thus, it suffices to prove that for any $S4 \otimes K$-modal space $\mathfrak{X} = (X, R_I, R_M, \mathcal{O})$, $(X, R_I, R_M, \mathcal{O}) \vDash t(\Gamma/\Delta)$ iff $\rho(X, R_I, R_M, \mathcal{O}) \vDash \Gamma/\Delta$.

For the left-to-right direction, suppose $\rho(X, R_I, R_M, \mathcal{O}) \nvDash \Gamma/\Delta$, there is a valuation V on $\rho(X, R_I, R_M, \mathcal{O})$ such that $V(\varphi) = \rho[X]$ for any $\varphi \in \Gamma$ and $V(\psi) \neq \rho[X]$ for any $\psi \in \Delta$. Since $V(p)$ is a clopen upset in $\rho(X, R_I, R_M, \mathcal{O})$ for any propositional variable p, $\rho^{-1}(V(p))$ is a clopen upset in $(X, R_I, R_M, \mathcal{O})$. We define a valuation V' on $(X, R_I, R_M, \mathcal{O})$ by setting $V'(p) = \rho^{-1}(V(p))$. Then we prove by induction that for any formula φ, $V'(t(\varphi)) = \rho^{-1}(V(\varphi))$ (note that $V(\varphi)$ is a clopen upset for any formula φ):

If $\varphi = p$ for some propositional variable p, $V'(t(p)) = V'(\Box_I p) = \Box_I V'(p) = \Box_I \rho^{-1}(V(p))$. As $\rho^{-1}(V(p))$ is an upset and R_I is reflexive, $\Box_I \rho^{-1}(V(p)) = \rho^{-1}(V(p))$. Thus $V'(t(p)) = \rho^{-1}(V(p))$.

If $\varphi = \bot$, $V'(t(\bot)) = V'(\Box_I \bot) = \Box_I V'(\bot) = \emptyset = \rho^{-1}(V(\bot))$; if $\varphi = \top$, $V'(t(\top)) = V'(\Box_I \top) = \Box_I V'(\top) = X = \rho^{-1}(\rho[X]) = \rho^{-1}(V(\top))$.

If $\varphi = \psi \wedge \theta$,

$$\begin{aligned}
V'(t(\psi \wedge \theta)) &= V'(\Box_I(t(\psi) \wedge t(\theta))) \\
&= \Box_I V'(t(\psi) \wedge t(\theta)) \\
&= \Box_I(V'(t(\psi)) \cap V'(t(\theta))) \\
&= \Box_I(\rho^{-1}(V(\psi)) \cap \rho^{-1}(V(\theta))) \quad \text{(IH)} \\
&= \Box_I \rho^{-1}(V(\psi) \cap V(\theta)) \\
&= \Box_I \rho^{-1}(V(\psi \wedge \theta)) \\
&= \rho^{-1}(V(\psi \wedge \theta)).
\end{aligned}$$

Note that the last equality holds as $\rho^{-1}(V(\psi \wedge \theta))$ is an upset and R_I is reflexive. The case when $\varphi = \psi \vee \theta$ is similar.

If $\varphi = \psi \to \theta$, $V'(t(\psi \to \theta)) = V'(\Box_I(t(\psi) \to t(\theta))) = \Box_I(V'(t(\psi)) \to V'(t(\theta))) = \Box_I(X \setminus V'(t(\psi)) \cup V'(t(\theta)))$. By IH, $V'(t(\psi)) = \rho^{-1}(V(\psi))$ and $V'(t(\theta)) = \rho^{-1}(V(\theta))$. Thus $V'(t(\psi \to \theta)) = \Box_I(X \setminus \rho^{-1}(V(\psi)) \cup \rho^{-1}(V(\theta))) = \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$ (note $V(\psi)$ and $V(\theta)$ are upsets).

Suppose $x \in \rho^{-1}(\rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta)))$, then $\rho(x) \in \rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta))$. For any $xR_I y$, by definition $\rho(x) \leq \rho(y)$, and thus $\rho(y) \notin V(\psi) \setminus V(\theta)$. As $xR_I y$ is arbitrary, we have that $x \in \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$, and thus $\rho^{-1}(\rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta))) \subseteq \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$. Suppose $x \in \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$ while $x \notin \rho^{-1}(\rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta)))$. Then $\rho(x) \notin \rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta)))$, there exists $\rho(x) \leq \rho(y)$ such that $\rho(y) \in V(\psi) \setminus V(\theta)$. Thus $y \notin \rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta))$, and by definition $xR_I y$. This contradicts the assumption that $x \in \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$. Therefore, if $x \in \Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta)))$, then $x \in \rho^{-1}(\rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta)))$. Thus $\Box_I(\rho^{-1}(\rho[X] \setminus V(\psi) \cup V(\theta))) \subseteq \rho^{-1}(\rho[X] \setminus {\downarrow}(V(\psi) \setminus V(\theta)))$, and $\Box_I(\rho^{-1}(\rho[X] \setminus$

$V(\psi)\cup V(\theta))) = \rho^{-1}(\rho[X]\setminus{\downarrow}(V(\psi)\setminus V(\theta)))$. As $\rho^{-1}(\rho[X]\setminus{\downarrow}(V(\psi)\setminus V(\theta))) = \rho^{-1}(V(\psi\to\theta))$, this proves that $V'(t(\psi\to\theta)) = \rho^{-1}(V(\psi\to\theta))$.

If $\varphi = \Box\psi$, $V'(t(\Box\psi)) = V'(\Box_I\Box_M t(\psi)) = \Box_I\Box_M V'(t(\psi)) =$ (By IH) $\Box_I\Box_M\rho^{-1}(V(\psi))$. And $\rho^{-1}(V(\Box\psi)) = \rho^{-1}(\Box_{[R_I\circ R_M\circ R_I]}V(\psi)) = \Box_I\circ\Box_M\circ\Box_I\rho^{-1}(V(\psi))$. As $V(\psi)$ is an upset, so is $\rho^{-1}(V(\psi))$. And since R_I is reflexive, $\Box_I\rho^{-1}(V(\psi)) = \rho^{-1}(V(\psi))$, and thus $\Box_I\circ\Box_M\circ\Box_I\rho^{-1}(V(\psi)) = \Box_I\circ\Box_M\rho^{-1}(V(\psi))$. We have that $V'(t(\Box\psi)) = \rho^{-1}(V(\Box\psi))$. This finishes the induction.

Since $V'(t(\varphi)) = \rho^{-1}(V(\varphi))$, $V'(t(\varphi)) = X$ for any $\varphi\in\Gamma$ as $V(\varphi) = \rho[X]$; $V'(t(\psi)) \neq X$ for any $\psi\in\Delta$ as $V(\psi)\neq\rho[X]$. Thus V' is a valuation on $(X, R_I, R_M, \mathcal{O})$ which refutes $t(\Gamma/\Delta)$, $(X, R_I, R_M, \mathcal{O}) \not\models t(\Gamma/\Delta)$.

For the right-to-left direction, suppose $(X, R_I, R_M, \mathcal{O}) \not\models t(\Gamma/\Delta)$, there is a valuation V on $(X, R_I, R_M, \mathcal{O})$ such that $V(t(\varphi)) = X$ for any $\varphi\in\Gamma$ and $V(t(\psi))\neq X$ for any $\psi\in\Delta$. Define $V'(p) = \rho[V(\Box_I p)] = \rho[\Box_I V(p)]$ for any propositional variable p. As $\Box_I V(p)$ is a clopen upset in $(X, R_I, R_M, \mathcal{O})$, so $\rho[\Box_I V(p)]$ is a clopen upset in $\rho(X, R_I, R_M, \mathcal{O})$. Thus V' is indeed a valuation on $\rho(X, R_I, R_M, \mathcal{O})$. We prove by induction that $V'(\varphi) = \rho[V(t(\varphi))]$ for any formula φ.

The cases when φ is a propositional letters, $\bot$ or $\top$ are easy.

If $\varphi = \psi\vee\theta$, $V'(\psi\vee\theta) = V'(\psi)\cup V'(\theta) =$ (By IH) $\rho[V(t(\psi))]\cup\rho[V(t(\theta)))] = \rho[V(t(\psi))\cup V(t(\theta))] = \rho[V(t(\psi)\vee t(\theta))]$. As $V(t(\varphi))$ is an upset for any formula φ, so $V(t(\psi)\vee t(\theta)) = V(t(\psi))\cup V(t(\theta))$ is an upset. Thus $\rho[V(t(\psi)\vee t(\theta))] = \rho[\Box_I V(t(\psi)\vee t(\theta))] = \rho[V(\Box_I(t(\psi)\vee t(\theta)))] = \rho[V(t(\psi\vee\theta))]$. The case when $\varphi = \psi\wedge\theta$ is similar.

If $\varphi = \psi\to\theta$, $V'(\psi\to\theta) = \rho[X]\setminus{\downarrow}(V'(\psi)\setminus V'(\theta)) =$ (By IH) $\rho[X]\setminus{\downarrow}(\rho[V(t(\psi))]\setminus\rho[V(t(\theta))])$. And $\rho(V(t(\psi\to\theta))) = \rho(V(\Box_I(t(\psi)\to t(\theta)))) = \rho[\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))]$. Now suppose $\rho(x)\in\rho[X]\setminus{\downarrow}(\rho[V(t(\psi))]\setminus\rho[V(t(\theta))])$, namely $\rho(x)\notin{\downarrow}(\rho[V(t(\psi))]\setminus\rho[V(t(\theta))])$. As $V(t(\psi))$ and $V(t(\theta))$ are upsets, $\rho[V(t(\psi))]\setminus\rho[V(t(\theta))] = \rho[V(t(\psi))\setminus V(t(\theta))]$[8]. Thus for any $xR_I y$, by definition $\rho(x)\leq\rho(y)$, thus $\rho(y)\notin\rho[V(t(\psi))\setminus V(t(\theta))]$, and $y\notin V(t(\psi))\setminus V(t(\theta))$. Therefore, $x\in\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))$, and $\rho(x)\in\rho[\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))]$. This proves that $\rho[X]\setminus{\downarrow}(\rho[V(t(\psi))]\setminus\rho[V(t(\theta))])\subseteq\rho[\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))]$.

For the other inclusion, suppose $\rho(x)\in\rho[\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))]$ while $\rho(x)\in{\downarrow}(\rho[V(t(\psi))]\setminus\rho[V(t(\theta))])$. Then $x\backsim y$ for some $y\in\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))$, and thus $x\in\Box_I(X\setminus V(t(\psi))\cup V(t(\theta)))$. As $\rho(x)\in{\downarrow}(\rho[V(t(\psi))]\setminus$

[8] $\rho[X\setminus Y] = \rho[X]\setminus\rho[Y]$ for any upsets X, Y: clearly, $\rho[X]\setminus\rho[Y]\subseteq\rho[X\setminus Y]$. Suppose $a\in\rho[X\setminus Y]$, there is $x\in X\setminus Y$ such that $a = \rho(x)$. As X, Y are upsets, for any $x\backsim x'$, $x'\notin Y$. Thus $\rho(x)\notin\rho[Y]$, $a = \rho(x)\in\rho[X]\setminus\rho[Y]$. Thus $\rho[X\setminus Y]\subseteq\rho[X]\setminus\rho[Y]$, $\rho[X\setminus Y] = \rho[X]\setminus\rho[Y]$.

$\rho[V(t(\theta))]$), there exists $\rho(x) \leq \rho(z)$ such that $\rho(z) \in \rho[V(t(\psi))] \setminus \rho[V(t(\theta))]$. There exists $z' \in V(t(\psi)) \setminus V(t(\theta))$ such that $\rho(z) = \rho(z')$. As $\rho(x) \leq \rho(z')$, xR_Iz', which contradicts the fact that $x \in \Box_I(X \setminus V(t(\psi)) \cup V(t(\theta)))$. Thus if $\rho(x) \in \rho[\Box_I(X \setminus V(t(\psi)) \cup V(t(\theta)))]$, then $\rho(x) \notin {\downarrow}(\rho[V(t(\psi))] \setminus \rho[V(t(\theta))])$, namely $\rho[\Box_I(X \setminus V(t(\psi)) \cup V(t(\theta)))] \subseteq \rho[X] \setminus {\downarrow}(\rho[V(t(\psi))] \setminus \rho[V(t(\theta))])$. We thus have that $\rho[\Box_I(X \setminus V(t(\psi)) \cup V(t(\theta)))] = \rho[X] \setminus {\downarrow}(\rho[V(t(\psi))] \setminus \rho[V(t(\theta))])$. This proves that $V'(\psi \to \theta) = \rho(V(t(\psi \to \theta)))$.

If $\varphi = \Box\psi$, $V'(\Box\psi) = \Box_{[R_I \circ R_M \circ R_I]} V'(\psi) = (\text{IH})\Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi)]$. And $\rho[V(t(\Box\psi))] = \rho[V(\Box_I \Box_M t(\psi))] = \rho[\Box_I \Box_M V(t(\psi))]$. Suppose $a \in \rho[\Box_I \Box_M V(t(\psi))]$, then $a = \rho(x)$ for some $x \in \Box_I \Box_M V(t(\psi))$. As $V(t(\psi))$ is an upset, $x \in \Box_I \Box_M V(t(\psi)) = \Box_I \Box_M \Box_I V(t(\psi))$. Thus $a = \rho(x) \in \Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))]$. As $a \in \rho[\Box_I \Box_M V(t(\psi))]$ is arbitrary, we have that $\rho[\Box_I \Box_M V(t(\psi))] \subseteq \Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))]$.

For the other inclusion, suppose $\rho(x) \in \Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))]$ while $x \notin \Box_I \Box_M \Box_I V(t(\psi)) = \Box_I \Box_M V(t(\psi))$, then there exist xR_Ix_1, $x_1R_Mx_2$ and x_2R_Iy such that $y \notin V(t(\psi))$. Thus, $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ by definition. As $V(t(\psi))$ is an upset, $\rho(y) \notin \rho[V(t(\psi))]$ which contradicts the assumption that $\rho(x) \in \Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))]$. Thus if $\rho(x) \in \Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))]$, then $\rho(x) \in \rho[\Box_I \Box_M V(t(\psi))]$. Therefore, $\Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))] \subseteq \rho[\Box_I \Box_M V(t(\psi))]$, and thus $\Box_{[R_I \circ R_M \circ R_I]}\rho[V(t(\psi))] = \rho[\Box_I \Box_M V(t(\psi))]$. This proves that $V'(\Box\psi) = \rho[V(t(\Box\psi))]$ and finishes the induction.

Now as $V(t(\varphi)) = X$ for any $\varphi \in \Gamma$, $V'(\varphi) = \rho[V(t(\varphi))] = \rho[X]$ for any $\varphi \in \Gamma$. As $V(t(\psi)) \neq X$ for any $\psi \in \Delta$ and $V(t(\psi))$ is an upset, $V'(\psi) = \rho[V(t(\psi))] \neq \rho[X]$ for any $\psi \in \Delta$. Thus V' is a valuation on $\rho(X, R_I, R_M, \mathcal{O})$ which refutes Γ/Δ. Therefore, $\rho(X, R_I, R_M, \mathcal{O}) \not\models \Gamma/\Delta$.

■

When restricted to formulas, the above proposition is exactly [38, Lem. 5] or [37, Lem. 19] in algebraic terms. Thus it is a generalization of that result from formulas to multi-conclusion rules. Now we define the *modal companions* for intuitionistic modal logics. This concept connects intuitionistic modal logics with bimodal logics via the Gödel translation.

Definition 4.1.18. Let $L \in \mathbf{NExt}(\mathbf{IntK}_\Box^R)$ and $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R)$. Then M is a *modal companion* of L if $\Gamma/\Delta \in L \iff t(\Gamma/\Delta) \in M$. Moreover, let $L \in \mathbf{NExt}(\mathbf{IntK}_\Box)$ and $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K})$, then M is a *modal companion* of L if $\varphi \in L \iff t(\varphi) \in M$.

We then define the following maps between the lattices $\mathbf{NExt}(\mathbf{IntK}_\Box^R)$ and $\mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R)$. The analogues of them for the lattices of superintuitionistic logics and classical modal logics over **S4** are quite well-known and can be found in many textbooks on modal logic. See [15] for more details.

$$\tau : \mathbf{NExt}(\mathbf{IntK}^R_\Box) \to \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R)$$
$$\tau(L) = \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L\}$$

$$\sigma : \mathbf{NExt}(\mathbf{IntK}^R_\Box) \to \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R)$$
$$\sigma(L) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L\}$$

$$\rho : \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R) \to \mathbf{NExt}(\mathbf{IntK}^R_\Box)$$
$$\rho(M) = \{\Gamma/\Delta \mid t(\Gamma/\Delta) \in M\}$$ [9]

Similarly, these maps can be defined for logics:

$$\tau : \mathbf{NExt}(\mathbf{IntK}_\Box) \to \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix})$$
$$\tau(L) = \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix} \oplus \{t(\varphi) \mid \varphi \in L\}$$

$$\sigma : \mathbf{NExt}(\mathbf{IntK}_\Box) \to \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix})$$
$$\sigma(L) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix} \oplus \{t(\varphi) \mid \varphi \in L\}$$

$$\rho : \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}) \to \mathbf{NExt}(\mathbf{IntK}_\Box)$$
$$\rho(M) = \{\varphi \mid t(\varphi) \in M\}$$

Semantically, we can extend the algebraic mappings σ and ρ (given below Proposition 4.1.12) to universal classes:

$$\sigma : \mathbf{Uni}(\mathbf{MHA}) \to \mathbf{Uni}(\mathbf{S4} \otimes \mathbf{K})$$
$$\sigma(\mathcal{A}) = Uni(\{\sigma(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\})$$

$$\rho : \mathbf{Uni}(\mathbf{S4} \otimes \mathbf{K}) \to \mathbf{Uni}(\mathbf{MHA})$$

[9] To see why $\rho(M) \in \mathbf{NExt}(\mathbf{IntK}^R_\Box)$, note that $t(\Gamma/\Delta) \in M$ iff (by Theorem 2.4.7) $Alg(M) \models t(\Gamma/\Delta)$ iff (by Proposition 4.1.17) $\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in Alg(M)\} \models \Gamma/\Delta$ iff $Uni(\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in Alg(M)\}) \models \Gamma/\Delta$ iff (by Theorem 2.5.6) $\Gamma/\Delta \in Ru(Uni(\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in Alg(M)\}))$. Thus $\rho(M) = Ru(Uni(\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in Alg(M)\})) \in \mathbf{NExt}(\mathbf{IntK}^R_\Box)$. Similarly we can show that $\rho(M) \in \mathbf{NExt}(\mathbf{IntK}_\Box)$ where $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K})$.

$$\rho(\mathcal{W}) = \{\rho(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{W}\}$$

We can also define the semantic analogue to τ as follows:

$$\tau : \mathbf{Uni}(\mathbf{MHA}) \to \mathbf{Uni}(\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix})$$

$$\tau(\mathcal{A}) = \{\mathfrak{A} \text{ is a } S4 \otimes K \oplus Mix\text{-algebra} \mid \rho(\mathfrak{A}) \in \mathcal{A}\}^{10}.$$

4.1.3 Proof of the Blok-Esakia theorem

Now, we can start with the following lemma which is the counterpart to [16, Lem. 2.50] and plays a key role in the proof of the Blok-Esakia theorem.

Lemma 4.1.19. *Let $\mathfrak{A}$ be a $Grz \otimes K \oplus Mix$-algebra, then for any bimodal multi-conclusion rule Γ/Δ, we have that*

$$\mathfrak{A} \vDash \Gamma/\Delta \textit{ iff } \sigma(\rho(\mathfrak{A})) \vDash \Gamma/\Delta.$$

Proof. For the left-to-right direction, by Proposition 4.1.15, $\sigma(\rho(\mathfrak{A})) \rightarrowtail \mathfrak{A}$. Thus, the result follows immediately.

For the right-to-left direction, we prove that if $\mathfrak{A}_* \nvDash \Gamma/\Delta$, then $\sigma(\rho(\mathfrak{A}))_* \nvDash \Gamma/\Delta$. By Proposition 4.1.13, $\sigma(\rho(\mathfrak{A}))_* = \sigma(\rho(\mathfrak{A})_*) = \sigma(\rho(\mathfrak{A}_*))$. And by Proposition 4.1.6, we can assume that $\Gamma/\Delta = \mu(\mathfrak{B}, D^I, D^M)$ where $\mathfrak{B}$ is a finite $S4 \otimes K$-algebra which validates **Mix** and $D^I, D^M \subseteq B$. Suppose $\mathfrak{A}_* \nvDash \mu(\mathfrak{B}, D^I, D^M)$ where $R_I \circ R_M \circ R_I = R_M$ on $\mathfrak{A}_*$ as $\mathfrak{A}$ validates **Mix**, then by Proposition 4.1.9, there is a continuous stable surjection $f : \mathfrak{A}_* \to \mathfrak{B}_*$ satisfying CDC for $\{\beta(a) \mid a \in D^I\}$ and $\{\beta(b) \mid b \in D^M\}$. We construct a continuous stable surjection $g : \sigma(\rho(\mathfrak{A}_*)) \to \mathfrak{B}_*$ satisfying CDC for $\{\beta(a) \mid a \in D^I\}$ and $\{\beta(b) \mid b \in D^M\}$, and this would show that $\sigma(\rho(\mathfrak{A}_*)) \nvDash \mu(\mathfrak{B}, D^I, D^M)$ by Proposition 4.1.9 and thus finish the proof.

We use the construction in the proof of [16, Lem. 2.50]. Let $C \subseteq \mathfrak{B}_*$ be an R_I-cluster, consider $Z_C = f^{-1}(C)$. As f is continuous, Z_C is clopen in $\mathfrak{A}_*$. Since f is stable, Z_C does not cut any R_I-cluster. As $\sigma(\rho(\mathfrak{A}_*))$ has the quotient topology, $\rho[Z_C]$ is clopen. Assume $C = \{x_1, ...x_n\}$ (note $\mathfrak{B}_*$ is finite), $f^{-1}(x_i) \subseteq Z_C$ is clopen. As $\mathfrak{A}_*$ is a $Grz \otimes K$-space, by Proposition 2.6.13, we have that $M_i = max_{R_I}(f^{-1}(x_i))$ is closed and M_i does not cut

[10] It is easy to check that $\rho(\mathcal{W})$ is always a universal class by Proposition 4.1.17. To see that $\tau(\mathcal{A})$ a universal class, note that $\mathfrak{A} \in Alg(\tau(Ru(\mathcal{A})))$ (here τ is the syntactic one as you can tell) iff $\mathfrak{A}$ is a $S4 \otimes K \oplus Mix$-algebra such that $\mathfrak{A} \vDash t(\Gamma/\Delta)$ for any $\Gamma/\Delta \in Ru(\mathcal{A})$ iff (by Proposition 4.1.17) $\mathfrak{A}$ is a $S4 \otimes K \oplus Mix$-algebra such that $\rho(\mathfrak{A}) \vDash \Gamma/\Delta$ for any $\Gamma/\Delta \in Ru(\mathcal{A})$ iff (by Theorem 2.5.6) $\mathfrak{A}$ is a $S4 \otimes K \oplus Mix$-algebra such that $\rho(\mathfrak{A}) \in \mathcal{A}$ iff $\mathfrak{A} \in \tau(\mathcal{A})$. Thus $\tau(\mathcal{A}) = Alg(\tau(Ru(\mathcal{A})))$ is a universal class.

any R_I-cluster. As $\sigma(\rho(\mathfrak{A}_*))$ has the quotient topology, $\rho[M_i]$ is closed. And for $i \neq j$, $\rho[M_i] \cap \rho[M_j] = \emptyset$.

Then we find disjoint clopen sets $U_1, ..., U_n$ of $\sigma(\rho(\mathfrak{A}_*))$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$. Let $k \leq n$ and assume that U_i has been defined for all $i < k$. If $k = n$, let $U_n = \rho[Z_C] \setminus (\bigcup_{i<k} U_i)$. Otherwise, let $V_k = \rho[Z_C] \setminus (\bigcup_{i<k} U_i)$ and note that V_k contains $\rho[M_i]$ for $k \leq i \leq n$. As $\sigma(\rho(\mathfrak{A}_*))$ is a Stone space, by its separation properties, for each $k < i \leq n$, there is a clopen set U_{k_i} of $\sigma(\rho(\mathfrak{A}_*))$ such that $\rho[M_k] \subseteq U_{k_i}$ and $\rho[M_i] \cap U_{k_i} = \emptyset$. Then let $U_k = \bigcap_{k<i\leq n} U_{k_i} \cap V_k$. Now define $g_C : \rho[Z_C] \to C$ as follows: $g_C(z) = x_i$ iff $z \in U_i$. Finally, define $g : \sigma(\rho(\mathfrak{A}_*)) \to \mathfrak{B}_*$ as follows:

$$g(\rho(z)) = \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper } R_I\text{-cluster} \\ g_C(\rho(z)) & \text{where } C \text{ is the proper } R_I\text{-cluster containing } f(z) \end{cases}$$

As shown in the proof of [16, Lem. 2.50], g is surjective, continuous, and relation-preserving w.r.t R_I, and satisfies CDC for $\{\beta(a) \mid a \in D^I\}$. We only need to check that g is relation-preserving w.r.t R_M and satisfies CDC for $\{\beta(b) \mid b \in D^M\}$.

As $\mathfrak{B}$ validates **Mix**, $R_I \circ R_M \circ R_I = R_M$ on $\mathfrak{B}_*$. Now suppose $\rho(a)R_M\rho(b)$, as f is stable, $f(a)R_Mf(b)$. Since for any z, $f(z)$ and $g(\rho(z))$ are mapped to the same R_I-cluster, in any case, $g(\rho(a))R_If(a)$ and $f(b)R_Ig(\rho(b))$. As $R_I \circ R_M \circ R_I = R_M$ on $\mathfrak{B}_*$, $g(\rho(a))R_Mg(\rho(b))$. g is thus relation-preserving w.r.t R_M.

Suppose $g(\rho(x))R_My$ where $y \in \delta$ for some $\delta \in \{\delta_b \mid b \in D^M\}$[11]. As $g(\rho(x))$ and $f(x)$ belong to the same R_I-cluster and $R_I \circ R_M \circ R_I = R_M$ on $\mathfrak{B}_*$, it follows that $f(x)R_My$. Since f satisfies CDC for $\{\beta(b) \mid b \in D^M\}$, there exists $z \in \mathfrak{A}_*$ such that xR_Mz and $f(z) \in \delta$. Since $f^{-1}(f(z))$ is clopen in $\mathfrak{A}_*$, there exists $z' \in max_{R_I} f^{-1}(f(z))$ such that zR_Iz'. As $R_I \circ R_M \circ R_I = R_M$ on $\mathfrak{A}_*$, we have that xR_Mz' and $f(z') \in \delta$. And from $z' \in max_{R_I} f^{-1}(f(z))$, it follows that $f(z') = g(\rho(z'))$ by construction, and thus $g(\rho(z')) \in \delta$. As xR_Mz', $\rho(x)[R_I \circ R_M \circ R_I]\rho(z')$, g satisfies CDC for $\{\beta(b) \mid b \in D^M\}$.

■

Compared to that of [16, Lem. 2.50], in the above proof, we have to exploit the assumption that $\mathfrak{B}$ validates **Mix** heavily, which is only given by Proposition 4.1.6.

Theorem 4.1.20. *For every* $\mathcal{A} \in$ ***Uni***(**Grz** $\otimes$ **K** $\oplus$ ***Mix***), $\sigma(\rho(\mathcal{A})) = \mathcal{A}$.

[11] Note that δ_b is just the (set-theoretic) complement of $\beta(b)$ as defined in the preliminaries.

Proof. By Proposition 4.1.15, for every $\mathfrak{A} \in \mathcal{A}$, $\sigma(\rho(\mathfrak{A})) \rightarrowtail \mathfrak{A}$. Thus $\sigma(\rho(\mathcal{A})) \subseteq \mathcal{A}$. Now, suppose $\mathcal{A} \not\models \Gamma/\Delta$, there is $\mathfrak{A} \in \mathcal{A}$ such that $\mathfrak{A} \not\models \Gamma/\Delta$. As $\mathfrak{A}$ validates **Mix**, by Lemma 4.1.19, we have $\sigma(\rho(\mathfrak{A})) \not\models \Gamma/\Delta$. Thus $Ru(\sigma(\rho(\mathcal{A}))) \subseteq Ru(\mathcal{A})$, by Theorem 2.4.7, $\mathcal{A} \subseteq \sigma(\rho(\mathcal{A}))$. Thus $\sigma(\rho(\mathcal{A})) = \mathcal{A}$.

■

The above theorem is the counterpart to [35, Lem. 4.4], which was also proved later by the method of stable canonical rules in [16, Thm. 2.51]. Having this theorem, the following results can be obtained by well-known routine arguments shown in [16] and [27] for example.

Lemma 4.1.21. *For each $L \in$ **NExt**(**IntK**$^R_\Box$) and $M \in$ **NExt**(**S4** $\otimes$ **K**^{R}), the following hold:*

- $Alg(\tau(L)) = \tau(Alg(L))$
- $Alg(\sigma(L)) = \sigma(Alg(L))$
- $Alg(\rho(M)) = \rho(Alg(M))$

Proof. For any $S4 \otimes K \oplus Mix$-algebra $\mathfrak{A}$, $\mathfrak{A} \in Alg(\tau(L))$ iff $\mathfrak{A} \models t(\Gamma/\Delta)$ for any $\Gamma/\Delta \in L$ iff (by Proposition 4.1.17)$\rho(\mathfrak{A}) \models \Gamma/\Delta$ for any $\Gamma/\Delta \in L$ iff $\rho(\mathfrak{A}) \in Alg(L)$ iff $\mathfrak{A} \in \tau(Alg(L))$. Thus $Alg(\tau(L)) = \tau(Alg(L))$.

For the second one, for any modal Heyting algebra $\mathfrak{A}$, $\mathfrak{A}_*$ is a modal Esakia space. By Proposition 4.1.11, we know that $\sigma(\mathfrak{A}_*)$ validates **Mix**. For any modal Esakia space $\mathcal{X} = (X, \leq, R, \mathcal{O})$, by definition $(X, \leq, R)$ is just an Esakia space. If we dismiss R, then σ operates on $(X, \leq, R)$ exactly the same way as that in [16, Def. 3.32], which is known to give us a Grz-space. Thus $\sigma(\mathfrak{A}_*)$ is a $Grz \otimes K \oplus Mix$-modal space. By Proposition 4.1.13, $(\sigma(\mathfrak{A}))_* \cong \sigma(\mathfrak{A}_*)$, and thus $\sigma(\mathfrak{A})$ is a $Grz \otimes K \oplus Mix$-algebra. Therefore, by Theorem 4.1.20, it suffices to prove that for any $\mathfrak{A} = \sigma(\rho(\mathfrak{A}))$ which is a $Grz \otimes K \oplus Mix$-algebra, $\mathfrak{A} \in Alg(\sigma(L))$ iff $\mathfrak{A} \in \sigma(Alg(L))$.

Suppose $\mathfrak{A} = \sigma(\rho(\mathfrak{A})) \in \sigma(Alg(L))$ where $\mathfrak{A}$ is a $Grz \otimes K \oplus Mix$-algebra. For any $\mathfrak{B} \in Alg(L)$, we have that $\mathfrak{B} \models \Gamma/\Delta$ for any $\Gamma/\Delta \in L$. By Proposition 4.1.14, $\rho(\sigma(\mathfrak{B})) \models \Gamma/\Delta$. Then by Proposition 4.1.17, $\sigma(\mathfrak{B}) \models t(\Gamma/\Delta)$ for any $\mathfrak{B} \in Alg(L)$ and $\Gamma/\Delta \in L$. As $\sigma(Alg(L))$ is generated by $\{\sigma(\mathfrak{B}) \mid \mathfrak{B} \in Alg(L)\}$, we have that $\sigma(Alg(L)) \models t(\Gamma/\Delta)$ for any $\Gamma/\Delta \in L$. Thus $\mathfrak{A} \models t(\Gamma/\Delta)$ for any $\Gamma/\Delta \in L$, namely $\mathfrak{A} \in Alg(\sigma(L))$. For the other direction, suppose $\mathfrak{A} \in Alg(\sigma(L))$, $\mathfrak{A} \models t(\Gamma/\Delta)$ for any $\Gamma/\Delta \in L$. By Proposition 4.1.17, $\rho(\mathfrak{A}) \models \Gamma/\Delta$ for any $\Gamma/\Delta \in L$. Thus $\rho(\mathfrak{A}) \in Alg(L)$, $\mathfrak{A} = \sigma(\rho(\mathfrak{A})) \in \sigma(Alg(L))$.

For the third one, let $\mathfrak{A}$ be a modal Heyting algebra, if $\mathfrak{A} \in \rho(Alg(M))$, then $\mathfrak{A} = \rho(\mathfrak{B})$ for some $\mathfrak{B} \in Alg(M)$. For any $t(\Gamma/\Delta) \in M$, we have that $\mathfrak{B} \vDash t(\Gamma/\Delta)$. By Proposition 4.1.17, $\mathfrak{A} = \rho(\mathfrak{B}) \vDash \Gamma/\Delta$ for any $t(\Gamma/\Delta) \in M$. Thus $\mathfrak{A} \in Alg(\rho(M))$. This proves that $\rho(Alg(M)) \subseteq Alg(\rho(M))$. For the other direction, if $\rho(Alg(M)) \vDash \Gamma/\Delta$, then by Proposition 4.1.17, $Alg(M) \vDash t(\Gamma/\Delta)$. Thus $t(\Gamma/\Delta) \in M$, and $\Gamma/\Delta \in \rho(M)$. We have that $Ru(\rho(Alg(M))) \subseteq \rho(M)$, and thus by Theorem 2.5.6, $Alg(\rho(M)) \subseteq \rho(Alg(M))$. Therefore, $Alg(\rho(M)) = \rho(Alg(M))$

■

The above lemma is the analogue to [27, Thm. 5.9], whose proof, as we can see, heavily depends on Proposition 4.1.17. We then can give a characterization of modal companions of an intuitionistic modal logic as that in [16, Lem. 2.53].

Proposition 4.1.22. For any $L \in \mathbf{NExt}(\mathbf{IntK}_\Box^R)$, we have that $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R)$ is a modal companion of L iff $Alg(L) = \rho(Alg(M))$

Proof. For the left-to-right direction, if M is a modal companion of L, then $L = \rho(M)$. Thus $Alg(L) = Alg(\rho(M)) = \rho(Alg(M))$ by the above lemma. For the other direction, assume $Alg(L) = \rho(Alg(M))$, then $\Gamma/\Delta \in L$ iff $Alg(L) \vDash \Gamma/\Delta$ iff $\rho(Alg(M)) \vDash \Gamma/\Delta$ iff (by Proposition 4.1.17) $Alg(M) \vDash t(\Gamma/\Delta)$ iff (by Theorem 2.4.7) $t(\Gamma/\Delta) \in M$. M is thus a modal companion of L.

■

The following proposition is a generalization of [37, Thm. 27] from logics to multi-conclusion consequence relations.

Proposition 4.1.23. For every $L \in \mathbf{NExt}(\mathbf{IntK}_\Box^R)$, the modal companion of L which contains **Mix** form an interval $\{M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R) \mid \tau(L) \subseteq M \subseteq \sigma(L)\}$.

Proof. By Lemma 4.1.21, it suffices to prove that for any $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R)$, we have that M is a modal companion of L iff $\sigma(Alg(L)) \subseteq Alg(M) \subseteq \tau(Alg(L))$.

For the left-to-right direction, assume M is a modal companion of L which contains **Mix**. By Proposition 4.1.22, $Alg(L) = \rho(Alg(M))$. For any $\mathfrak{A} \in Alg(M)$, we have that $\rho(\mathfrak{A}) \in \rho(Alg(M))$. Thus $\rho(\mathfrak{A}) \in Alg(L)$, and $\mathfrak{A} \in \tau(Alg(L))$. Therefore, $Alg(M) \subseteq \tau(Alg(L))$. To see that $\sigma(Alg(L)) \subseteq Alg(M)$, as $\sigma(A)$ is a $Grz \otimes K \oplus Mix$-algebra (shown in the proof of Lemma 4.1.21), by Theorem 4.1.20, it suffices to prove that for any $\mathfrak{A} = \sigma(\rho(\mathfrak{A})) \in$

$\sigma Alg(L)$, we have that $\mathfrak{A} \in Alg(M)$. Suppose $\mathfrak{A} \cong \sigma(\rho(\mathfrak{A})) \in \sigma Alg(L)$, by Lemma 4.1.21, $\mathfrak{A} \in Alg(\sigma(L))$. Thus for any $\Gamma/\Delta \in L$, we have that $\mathfrak{A} \models t(\Gamma/\Delta)$. By Proposition 4.1.17, $\rho(\mathfrak{A}) \models \Gamma/\Delta$ for any $\Gamma/\Delta \in L$. Thus $\rho(\mathfrak{A}) \in Alg(L) = \rho(Alg(M))$, namely $\rho(\mathfrak{A}) \cong \rho(\mathfrak{B})$ for some $\mathfrak{B} \in Alg(M)$. $\mathfrak{A} = \sigma(\rho(\mathfrak{A})) \cong \sigma(\rho(\mathfrak{B})) \rightarrowtail \mathfrak{B} \in Alg(M)$. As $Alg(M)$ is a universal class which is closed under subalgebras, $\mathfrak{A} \in M$. Thus $\sigma(Alg(L)) \subseteq Alg(M)$.

For the other direction, assume that $\sigma(Alg(L)) \subseteq Alg(M) \subseteq \tau(Alg(L))$. By Proposition 4.1.14, $\rho(\sigma(Alg(L))) = Alg(L)$. As $\sigma(Alg(L)) \subseteq Alg(M)$, we have that $Alg(L) \subseteq \rho(Alg(M))$. As $Alg(M) \subseteq \tau(Alg(L))$, it follows that $\rho(Alg(M)) \subseteq \rho(\tau(Alg(L)))$. Assume $\mathfrak{A} \in \tau(Alg(L))$, then $\rho(\mathfrak{A}) \in Alg(L)$, and $\sigma(\rho(\mathfrak{A})) \in \sigma(Alg(L)) \subseteq Alg(M)$. Thus $\rho(\sigma(\rho(\mathfrak{A}))) \in \rho(Alg(M))$. By Proposition 4.1.14, $\rho(\mathfrak{A}) \in \rho(Alg(M))$. Therefore, $\rho(\tau(Alg(L))) \subseteq \rho(Alg(M))$, and $\rho(\tau(Alg(L))) = \rho(Alg(M))$. By definition, $\rho(\tau(Alg(L))) \subseteq Alg(L)$, and thus $\rho(Alg(M)) \subseteq Alg(L)$. Therefore, $\rho(Alg(M)) = Alg(L)$. M is a modal companion of L by Proposition 4.1.22.

■

Now, we obtain the Blok-Esakia theorem for intuitionistic modal logics:

Theorem 4.1.24.

1. *The mappings* $\sigma : \mathbf{NExt}(\boldsymbol{IntK}_\Box^R) \to \mathbf{NExt}(\boldsymbol{Grz} \otimes \boldsymbol{K} \oplus \boldsymbol{Mix}^R)$ *and* $\rho : \mathbf{NExt}(\boldsymbol{Grz} \otimes \boldsymbol{K} \oplus \boldsymbol{Mix}^R) \to \mathbf{NExt}(\boldsymbol{IntK}_\Box^R)$ *are lattice isomorphisms and mutual inverses.*

2. *The mappings* $\sigma : \mathbf{NExt}(\boldsymbol{IntK}_\Box) \to \mathbf{NExt}(\boldsymbol{Grz} \otimes \boldsymbol{K} \oplus \boldsymbol{Mix})$ *and* $\rho : \mathbf{NExt}(\boldsymbol{Grz} \otimes \boldsymbol{K} \oplus \boldsymbol{Mix}) \to \mathbf{NExt}(\boldsymbol{IntK}_\Box)$ *are lattice isomorphisms and mutual inverses.*

Proof. For item 1, by Lemma 4.1.21, it suffices to prove that $\sigma : \mathbf{Uni}(\mathbf{MHA}) \to \mathbf{Uni}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix})$ and $\rho : \mathbf{Uni}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}) \to \mathbf{Uni}(\mathbf{MHA})$ are lattice isomorphisms and mutual inverses.

We first prove that σ and ρ are mutual inverses and thus bijective. For any $\mathcal{A} \in \mathbf{Uni}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix})$, by Theorem 4.1.20, $\sigma(\rho(\mathcal{A})) = \mathcal{A}$. Let $\mathcal{A} \in \mathbf{Uni}(\mathbf{MHA})$ be arbitrary, by Proposition 4.1.14, it is clear that $\mathcal{A} \subseteq \rho(\sigma(\mathcal{A}))$. Suppose $\mathcal{A} \models \Gamma/\Delta$, namely for any $\mathfrak{A} \in \mathcal{A}$, we have that $\mathfrak{A} \models \Gamma/\Delta$. By Proposition 4.1.14, $\rho(\sigma(\mathfrak{A})) \models \Gamma/\Delta$, for any $\mathfrak{A} \in \mathcal{A}$. By Proposition 4.1.17, $\sigma(\mathfrak{A}) \models t(\Gamma/\Delta)$ for any $\mathfrak{A} \in \mathcal{A}$. Thus $\sigma(\mathcal{A}) \models t(\Gamma/\Delta)$. By Proposition 4.1.17, $\rho(\sigma(\mathcal{A})) \models \Gamma/\Delta$. Thus $Ru(\mathcal{A}) \subseteq Ru(\rho(\sigma(\mathcal{A})))$, by Theorem 2.4.7 $\rho(\sigma(\mathcal{A})) \subseteq \mathcal{A}$. Therefore, $\rho(\sigma(\mathcal{A})) = \mathcal{A}$.

We then prove that σ and ρ preserve joins and meets, and thus are lattice morphisms. Clearly, for any $\mathcal{A}, \mathcal{B} \in \mathbf{Uni}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix})$, we have that

$\rho(\mathcal{A}\cap\mathcal{B})\subseteq\rho(\mathcal{A})\cap\rho(\mathcal{B})$. Assume $\mathfrak{C}\in\rho(\mathcal{A})\cap\rho(\mathcal{B})$, there exist $\mathfrak{A}\in\mathcal{A}$ and $\mathfrak{B}\in\mathcal{B}$ such that $\mathfrak{C}=\rho(\mathfrak{A})=\rho(\mathfrak{B})$, and $\sigma(\rho(\mathfrak{A}))=\sigma(\rho(\mathfrak{B}))\in\mathcal{A}\cap\mathcal{B}$. Thus $\rho(\sigma(\rho(\mathfrak{A})))\in\rho(\mathcal{A}\cap\mathcal{B})$. By Proposition 4.1.14, $\mathfrak{C}=\rho(\mathfrak{A})\in\rho(\mathcal{A}\cap\mathcal{B})$. Thus $\rho(\mathcal{A})\cap\rho(\mathcal{B})\subseteq\rho(\mathcal{A}\cap\mathcal{B})$, and $\rho(\mathcal{A})\cap\rho(\mathcal{B})=\rho(\mathcal{A}\cap\mathcal{B})$. Clearly, for any $\mathcal{A},\mathcal{B}\in\mathbf{Uni}(\mathbf{Grz}\otimes\mathbf{K}\oplus\mathbf{Mix})$, we have that $\rho(\mathcal{A})\vee\rho(\mathcal{B})\subseteq\rho(\mathcal{A}\vee\mathcal{B})$. Suppose $\rho(\mathcal{A})\vee\rho(\mathcal{B})\models\Gamma/\Delta$, then $\rho(\mathcal{A})\models\Gamma/\Delta$ and $\rho(\mathcal{B})\models\Gamma/\Delta$. By Proposition 4.1.17, $\mathcal{A}\models t(\Gamma/\Delta)$ and $\mathcal{B}\models t(\Gamma/\Delta)$. Thus $\mathcal{A}\vee\mathcal{B}\models t(\Gamma/\Delta)$. By Proposition 4.1.17, $\rho(\mathcal{A}\vee\mathcal{B})\models\Gamma/\Delta$. Thus $Ru(\rho(\mathcal{A})\vee\rho(\mathcal{B}))\subseteq Ru(\rho(\mathcal{A}\vee\mathcal{B}))$, and by Theorem 2.5.6, $\rho(\mathcal{A}\vee\mathcal{B})\subseteq\rho(\mathcal{A})\vee\rho(\mathcal{B})$. Therefore, $\rho(\mathcal{A}\vee\mathcal{B})=\rho(\mathcal{A})\vee\rho(\mathcal{B})$, ρ preserves joins and meets. For any $\mathcal{A},\mathcal{B}\in\mathbf{Uni}(\mathbf{MHA})$, by what we have proved above $\sigma(\mathcal{A})\vee\sigma(\mathcal{B})=\sigma(\rho(\sigma(\mathcal{A})\vee\sigma(\mathcal{B})))=\sigma(\rho(\sigma(\mathcal{A}))\vee\rho(\sigma(\mathcal{B})))=\sigma(\mathcal{A}\vee\mathcal{B})$. Clearly, $\sigma(\mathcal{A}\cap\mathcal{B})\subseteq\sigma(\mathcal{A})\cap\sigma(\mathcal{B})$. Suppose $\sigma(\mathcal{A}\cap\mathcal{B})\models\Gamma/\Delta$ while $\sigma(\mathcal{A})\cap\sigma(\mathcal{B})\not\models\Gamma/\Delta$, there exists $\mathfrak{A}\in\sigma(\mathcal{A})\cap\sigma(\mathcal{B})$ such that $\mathfrak{A}\not\models\Gamma/\Delta$. By Proposition 4.1.14, $\rho(\mathfrak{A})\in\mathcal{A}\cap\mathcal{B}$. Then $\sigma(\rho(\mathfrak{A}))\in\sigma(\mathcal{A}\cap\mathcal{B})$, and thus $\sigma(\rho(\mathfrak{A}))\models\Gamma/\Delta$ which contradicts Lemma 4.1.19. Therefore, $Ru(\sigma(\mathcal{A})\cap\sigma(\mathcal{B}))\subseteq Ru(\sigma(\mathcal{A})\cap\sigma(\mathcal{B}))$, by Theorem 2.4.7, $\sigma(\mathcal{A})\cap\sigma(\mathcal{B})\subseteq\sigma(\mathcal{A})\cap\sigma(\mathcal{B})$. Therefore, $\sigma(\mathcal{A})\cap\sigma(\mathcal{B})=\sigma(\mathcal{A})\cap\sigma(\mathcal{B})$, σ preserves joins and meets. This finishes our proof.

Item 2 follows immediately from item 1 and Propositions 2.4.4 and 2.5.3. ■

The transformations σ and ρ are useful as they may allow us to transfer questions about intuitionistic modal logic to bimodal logics and vice versa. In particular, at this stage one can easily prove that ρ preserves decidability, Kripke completenss, the finite model property and tabularity as shown in [38, Thm. 11].

The second item of the above theorem is the Blok-Esakia theorem for intuitionistic modal logics proved in [37], so what we have got here is a generalization of it from logics to multi-conclusion consequence relations. However, one may note that [37, Cor. 28] is slightly stronger than the second item of the above theorem, which says that the Blok-Esakia theorem holds not only for normal extensions but also for weaker extensions (**K** could be replaced by **C** on both sides of the map σ and ρ)[12] . The reason why so far we can not prove the result for nonnormal modal logic is that we do not have a nice dual description of the algebraic semantics for such weak logics[13]. As a result, we can not exploit geometric intuitions which are quite

[12]**C** stands for "congruential". It is the least modal logic which has the algebraic semantics. See [37] for more details.

[13]This also explains the reason why unlike Kripke frames, what Wolter and Zakharyaschev called "frames" are quite algebraic.

crucial in Lemma 4.1.19.

Having said that, compared to the proofs in [37], our proofs are arguably much more self-contained. Since the construction in the proof of Lemma 4.1.19 is the same as that in [16, Lem. 2.50], this also indicates the robustness of this construction.

4.2 The Dummett-Lemmon conjecture

The Dummett-Lemmon conjecture for superintuitionistic logics states that a superintuitionistic logic is Kripke complete iff its least modal companion is Kripke complete. This conjecture was proved correct by Zakharyaschev in [41], which is a very important application of his canonical formulas. Using stable canonical rules, Cleani [16, Thm. 2.70] proved that the Dummet-Lemmon conjecture holds for superintuitionistic rules systems as well[14].

In this section, following a somewhat similar strategy, we prove the Dummett-Lemmon conjecture for intuitionistic modal multi-conclusion consequence relations which says that for any $L \in \mathbf{NExt}(\mathbf{IntK}^R_\Box)$, L is Kripke complete iff $\tau(L)$ (i.e. the least modal companion containing $\mathbf{Mix}$) is Kripke complete. The proof uses the stable canonical rules for intuitionistic modal logics (also those for bimodal logics) and the Blok-Esakia theorem proved in the previous section, and thus can be seen as the peak of what we have done so far.

We start with the definition of Kripke frames in the setting of intuitionistic modal logics.

Definition 4.2.1. An *intuitionistic modal Kripke frame* is a triple $(X, \leq, R)$ where X is a non-empty set, $\leq$ is a partial order on X and $R \subseteq X^2$ such that $\leq \circ R = R \circ \leq = R$.

It is easy to check that for any intuitionistic modal Kripke frame $(X, \leq, R)$, the set of all upsets of X is closed under $\Box_R$. A *valuation* on a intuitionistic modal Kripke frame $(X, \leq, R)$ is a map $V : Prop \to Up(X)$ which can be recursively extended to all intuitionistic modal formulas in the standard way. Besides, for any intuitionistic modal Kripke frame $\mathfrak{X} = (X, \leq, R)$, we use $\mathfrak{X}^*$ to denote the intuitionistic modal algebra of upsets of $\mathfrak{X}$ where $\Box_R$ is defined in the same way as that in Theorem 3.3.3.

For any $S4 \otimes K$ Kripke frame $\mathfrak{F} = (X, R_I, R_M)$, it is clear that $\rho(\mathfrak{F}) = (\rho[Y], \leq, [R_I \circ R_M \circ R_I])$ is an intuitionistic modal Kripke frame where ρ is

[14] Unfortunately, as mentioned in the introduction, the proof in his master thesis contains a gap that could be corrected.

exactly the same as that in Definition 4.1.10 (without the topology).

The following proposition can be proved in the same way as Proposition 4.1.17.

Proposition 4.2.2. For any $S4 \otimes K$ Kripke frame (X, R_I, R_M) and any intuitionistic modal multi-conclusion rule Γ/Δ, $(X, R_I, R_M) \vDash t(\Gamma/\Delta)$ iff $(\rho[X], \leq, [R_I \circ R_M \circ R_I]) \vDash \Gamma/\Delta$.

Proof. For the left-to-right direction, suppose $(\rho[X], \leq, [R_I \circ R_M \circ R_I]) \nvDash \Gamma/\Delta$, there is a valuation V such that $V(\varphi) = \rho[X]$ for any $\varphi \in \Gamma$ and $V(\psi) \neq \rho[X]$ for any $\psi \in \Delta$. Define a valuation V' on (X, R_I, R_M) as follows: $V(p) = \rho^{-1}(V(p))$ for any propositional variable p, and then check that V' refutes $t(\Gamma/\Delta)$ on (X, R_I, R_M).

For the right-to-left direction, suppose $(X, R_I, R_M) \nvDash t(\Gamma/\Delta)$, then there is a valuation V on (X, R_I, R_M) such that $V(t(\varphi)) = X$ for any $\varphi \in \Gamma$ and $V(t(\psi)) \neq X$ for any $\psi \in \Delta$. Define a valuation V' on $(\rho[X], \leq, [R_I \circ R_M \circ R_I])$ as follows: $V'(p) = \rho[V(\Box_I p)] = \rho[\Box_I V(p)]$ (note that $\rho[\Box_I V(p)]$ is an upset), and then check V' refutes Γ/Δ on $(\rho[X], \leq, [R_I \circ R_M \circ R_I])$.

■

We first obtain the easy direction of the Dummett-Lemmon conjecture as follows:

Proposition 4.2.3. For any $L \in \mathbf{NExt}(\mathbf{IntK}^R_\Box)$, if $\tau(L)$ is Kripke complete, then L is Kripke complete.

Proof. Suppose $\tau(L)$ is Kripke complete. For any intuitionistic modal multi-conclusion rule Γ/Δ, suppose $\Gamma/\Delta \notin L$, then there exists a modal Heyting algebra $\mathfrak{A}$ such that $\mathfrak{A} \vDash L$ while $\mathfrak{A} \nvDash \Gamma/\Delta$. If $\sigma(\mathfrak{A}) \vDash t(\Gamma/\Delta)$, then by Proposition 4.1.17 $\rho(\sigma(\mathfrak{A})) \vDash \Gamma/\Delta$. As $\rho(\sigma(\mathfrak{A})) \cong \mathfrak{A}$ by Proposition 4.1.14, $\mathfrak{A} \vDash \Gamma/\Delta$, we get a contradiction. Thus $\sigma(\mathfrak{A}) \nvDash t(\Gamma/\Delta)$. As $\rho(\sigma(\mathfrak{A})) \cong \mathfrak{A} \vDash L$, by Proposition 4.1.17, $\sigma(\mathfrak{A}) \vDash \tau(L)$. Thus $t(\Gamma/\Delta) \notin \tau(L)$. As $\tau(L)$ is Kripke complete, there exists an $S4 \otimes K \oplus Mix$ Kripke frame (X, R_I, R_M) such that $(X, R_I, R_M) \vDash \tau(L)$ while $(X, R_I, R_M) \nvDash t(\Gamma/\Delta)$. Now, by Proposition 4.2.2, $(\rho[X], \leq, [R_I \circ R_M \circ R_I]) \vDash L$ while $(\rho[X], \leq, [R_I \circ R_M \circ R_I]) \nvDash \Gamma/\Delta$. As $(\rho[X], \leq, [R_I \circ R_M \circ R_I])$ is an intuitionistic modal Kripke frame, this proves that L is Kripke complete.

■

Then we prove the so-called rule collapse lemma[15]. Informally speaking,

[15]Note that the one called "rule-collapse lemma" in Cleani's master thesis [16, Lem. 2.62] is problematic because the collapsed stable canonical rule for superintuitionistic logics is not of the right form. Here, we actually prove something weaker.

it describes what happens if we collapse R_I-clusters in the stable canonical rules for bimodal logics.

Lemma 4.2.4 (Rule collapse lemma)**.** *For any $S4 \otimes K$-modal space $\mathfrak{X}$, and any $S4 \otimes K \oplus Mix$-modal space $\mathfrak{A}_*$, let $\mathfrak{D}_I, \mathfrak{D}_M \subseteq \mathcal{P}(\mathfrak{A}_*)$, if $\mathfrak{X} \not\models \mu(\mathfrak{A}_*, \overline{\mathfrak{D}_I}, \overline{\mathfrak{D}_M})$, then $\sigma(\rho(\mathfrak{X})) \not\models \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\mathfrak{D}_I}, \overline{\rho\mathfrak{D}_M})$ where $\overline{\mathfrak{D}_I} = \{\bar{\delta} \mid \delta \in \mathfrak{D}_I\}$, $\overline{\mathfrak{D}_M} = \{\bar{\delta} \mid \delta \in \mathfrak{D}_M\}$, $\overline{\rho\mathfrak{D}_I} = \{\overline{\rho[\delta]} \mid \delta \in \mathfrak{D}_I\}$ and $\overline{\rho\mathfrak{D}_M} = \{\overline{\rho[\delta]} \mid \delta \in \mathfrak{D}_M\}$[16].*

Proof. Suppose $\mathfrak{X} \not\models \mu(\mathfrak{A}_*, \overline{\mathfrak{D}_I}, \overline{\mathfrak{D}_M})$, by Proposition 4.1.9, there exists $f : \mathfrak{X} \to \mathfrak{A}_*$ such that f is surjective, continuous, stable and satisfies CDC for $\overline{\mathfrak{D}_I}$ and $\overline{\mathfrak{D}_M}$. Define $g : \sigma(\rho(\mathfrak{X})) \to \sigma(\rho(\mathfrak{A}_*))$ as follows: $g(\rho(x)) = \rho(f(x))$. Suppose $\rho(x) = \rho(y)$, then xR_Iy and yR_Ix. As f is stable, it follows that $f(x)R_If(y)$ and $f(y)R_If(x)$, and thus $f(x) \backsim f(y)$. Therefore, g is well defined. As f is surjective, so is g. Then we check that g is stable, continuous and satisfies CDC for $\overline{\rho\mathfrak{D}_I}, \overline{\rho\mathfrak{D}_M}$.

Suppose $\rho(x)R_I\rho(y)$, then $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$, and xR_Iy in $\mathfrak{X}$. As f is stable, $f(x)R_If(y)$. Thus $\rho(f(x)) \leq \rho(f(y))$ in $\rho(\mathfrak{A}_*)$, and $\rho(f(x))R_I\rho(f(y))$ in $\sigma(\rho(\mathfrak{A}_*))$ by definition, namely $g(\rho(x))R_Ig(\rho(y))$. Suppose $\rho(x)R_M\rho(y)$, then $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ in $\rho(\mathfrak{X})$. Thus $xR_I \circ R_M \circ R_Iy$ in $\mathfrak{X}$ by definition. There exist x_1, x_2 such that $xR_Ix_1R_Mx_2R_Iy$. As f is stable, $f(x)R_If(x_1)R_Mf(x_2)R_If(y)$. Thus $f(x)R_I \circ R_M \circ R_If(y)$, by definition $\rho(f(x))[R_I \circ R_M \circ R_M]\rho(f(y))$, namely $\rho(f(x))R_M\rho(f(y))$ in $\sigma(\rho(\mathfrak{A}_*))$. Thus, $g(\rho(x))R_Mg(\rho(y))$. This proves that g is stable.

For any $\rho[F] \subseteq \rho(\mathfrak{A}_*)$ where $F \subseteq \mathfrak{A}_*$, we have that $x \in \rho^{-1}(g^{-1}(\rho[F]))$ iff $\rho(x) \in g^{-1}(\rho[F])$ iff $g(\rho(x)) \in \rho[F]$ iff $\rho(f(x)) \in \rho[F]$ iff $f(x) \in \rho^{-1}(\rho[F])$ iff $x \in f^{-1}(\rho^{-1}(\rho[F]))$. As f is continuous, we know that $\rho^{-1}(g^{-1}(\rho[F])) = f^{-1}(\rho^{-1}(\rho[F]))$ is clopen in $\mathfrak{X}$. Clearly, $\rho^{-1}(g^{-1}(\rho[F]))$ does not cut any R_I-cluster in $\mathfrak{X}$. As $\sigma(\rho(\mathfrak{X}))$ has the quotient topology, $g^{-1}(\rho[F]) = \rho[\rho^{-1}(g^{-1}(\rho[F]))] = \rho[f^{-1}(\rho^{-1}(\rho[F]))]$ is clopen in $\sigma(\rho(\mathfrak{X}))$. Therefore, g is continuous.

Suppose $R_M[g(\rho(x))] \cap \rho[\delta] \neq \emptyset$ where $\delta \in \mathfrak{D}_M$, then there exists $a \in \delta$ such that $g(\rho(x))R_M\rho(a)$. Thus $\rho(f(x))R_M\rho(a)$ in $\sigma(\rho(\mathfrak{A}_*))$, and $\rho(f(x))[R_I \circ R_M \circ R_I]\rho(a)$. By definition, $f(x)R_I \circ R_M \circ R_Ia$ in $\mathfrak{A}_*$. Since $\mathfrak{A}_*$ validates **Mix**, $R_I \circ R_M \circ R_I = R_M$. Thus $f(x)R_Ma$. As f satisfies CDC for $\overline{\mathfrak{D}_M}$, there exists y such that xR_My and $f(y) \in \delta$. As R_I is reflexive, $xR_I \circ R_M \circ R_Iy$, and by definition $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ in $\rho(\mathfrak{A}_*)$. Namely, $\rho(x)R_M\rho(y)$ in $\sigma(\rho(\mathfrak{A}_*))$ by the definition of σ, and $g(\rho(y)) = \rho(f(y)) \in \rho[\delta]$. Thus $g[R_M[\rho(x)]] \cap \rho[\delta] \neq \emptyset$. This proves that g satisfies CDC for $\overline{\rho\mathfrak{D}_M}$.

[16]Recall that we use "$^-$" to denote the set-theoretic complement.

Suppose $R_I[g(\rho(x))] \cap \rho[\delta] \neq \emptyset$ where $\delta \in \mathfrak{D}_I$, then there exists $a \in \delta$ such that $g(\rho(x)) R_I \rho(a)$. Thus $\rho(f(x)) = g(\rho(x) \leq \rho(a)$ in $\rho(\mathfrak{A}_*)$. By definition, $f(x) R_I a$. As f satisfies CDC or $\overline{\mathfrak{D}_I}$, there exists y such that $x R_I y$ and $f(y) \in \delta$. Then $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$, and by definition of σ, we have that $\rho(x) R_I \rho(y)$ in $\sigma(\rho(\mathfrak{X}))$ and $g(\rho(y)) = \rho(f(y)) \in \rho[\delta]$. This proves that g satisfies CDC for $\overline{\rho\mathfrak{D}_I}$.

Therefore, $\sigma(\rho(\mathfrak{X})) \not\models \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\mathfrak{D}_I}, \overline{\rho\mathfrak{D}_M})$.

■

Apart from the rule collapse lemma, we also need to show that the refutation conditions stated in Proposition 3.3.12 work essentially the same way for intuitionistic modal Kripke frames.

Theorem 4.2.5. *For any intuitionstic modal Kripke frame $(X, \leq, R)$, we have that $(X, \leq, R) \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$ iff there is a surjective stable order-preserving map $f : X \to X_A$ satisfying $CDC_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and satisfies $CDC_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$.*

Proof. For any intuitionistic modal Kripke frame $\mathcal{X} = (X, \leq, R)$, $\mathcal{X} \models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$ iff $\mathcal{X}^* \models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$ iff $(\mathcal{X}^*)_* \models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$.

Suppose $\mathcal{X} = (X, \leq, R) \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$, then $(\mathcal{X}^*)_* \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$. By Proposition 3.3.12, there is a surjective stable Priestley morphism $g : (\mathcal{X}^*)_* \to X_A$ satisfying $\text{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and $\text{CDC}_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$. Define $\epsilon : \mathcal{X} \to (\mathcal{X}^*)_*$ as follows: $\epsilon(x) = \{\uparrow x \subseteq U \mid U \text{ is an upset of } X\}$ (It is easy to see that $\epsilon(x)$ is a prime filter of upsets of X). Then we check that $g \circ \epsilon$ satisfies all the expected conditions.

For any $x \leq y$ in $\mathcal{X}$, $\uparrow y \subseteq \uparrow x$, and thus $\epsilon(x) \subseteq \epsilon(y)$, $g \circ \epsilon(x) \leq g \circ \epsilon(y)$. $g \circ \epsilon$ is order-preserving. Suppose xRy in $\mathcal{X}$, then for any $\Box_R U \in \epsilon(x)$ where U is an upset, by definition, $\uparrow x \subseteq \Box_R U$, and thus $x \in \Box_R U$, namely $R[x] \subseteq U$. Therefore, $y \in U$. As U is an upset, $\uparrow y \subseteq U$, and $U \in \epsilon(y)$. This means that $\epsilon(x) R^* \epsilon(y)$ in $(\mathcal{X}^*)_*$. As g is stable, $g(\epsilon(x)) R_A g(\epsilon(y))$ in $\mathfrak{A}_*$, and thus $g \circ \epsilon$ is stable.

For any $p \in X_A$, as g is continuous, we know that $g^{-1}(p)$ is a clopen set of $(\mathcal{X}^*)_*$. As $(\mathcal{X}^*)_*$ is an Esakia space if we dismiss R^*, by Remark 2.6.6 we have that $g^{-1}(p) = \bigcup_{1 \leq i \leq n} (\beta(U_i) \setminus \beta(V_i))$ where $U_i' s, V_i' s$ are upsets of X. Since g is surjective, $g^{-1}(p)$ is not empty, there exists $1 \leq i \leq n$ such that $\beta(U_i) \setminus \beta(V_i) \neq \emptyset$. Thus $U_i \setminus V_i \neq \emptyset$, there exists $x \in U_i \setminus V_i$. Then $\uparrow x \subseteq U_i$ while $\uparrow x \nsubseteq V_i$. Thus $\epsilon(x) \in \beta(U_i)$ while $\epsilon(x) \notin \beta(V_i)$. Therefore, $\epsilon(x) \in g^{-1}(p)$, and $g(\epsilon(x)) = p$. As $p \in X_A$ is arbitrary, this proves that $g \circ \epsilon$ is surjective.

Let $x \in X$ be arbitrary, suppose $R_A[g(\epsilon(x))] \nsubseteq \beta(a)$ while $(g \circ \epsilon)[R[x]] \subseteq \beta(a)$ where $a \in D^{\Box}$. As g satisfies CDC$_{\Box}$ for $\beta(a)$, we have that $g[R^*[\epsilon(x)]] \nsubseteq \beta(a)$, there exists $\epsilon(x)R^*\mathfrak{q}$ such that $g(\mathfrak{q}) \notin \beta(a)$, namely $a \notin g(\mathfrak{q})$. As $(\mathcal{X}^*)_* \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$, we know that $\mathcal{X}^* \not\models \rho(\mathfrak{A}, D^{\to}, D^{\Box})$. By Proposition 3.1.6, there is a stable bounded lattice embedding $h : A \to Up(X)$ satisfying CDC for $D^{\to}$ and $D^{\Box}$. By duality, we can assume that $g = h^{-1}$, and thus there exists an upset U of X (in fact just $h(a)$) such that for any prime filter $\mathfrak{p}$ of upsets of X (i.e any element of $(\mathcal{X}^*)_*$) $a \in g(\mathfrak{p})$ iff $U \in \mathfrak{p}$. As $(g \circ \epsilon)[R[x]] \subseteq \beta(a)$, for any xRy in X, we have that $g(\epsilon(y)) \in \beta(a)$, namely $a \in g(\epsilon(y))$ Thus $U \in \epsilon(y)$, and $y \in {\uparrow}y \subseteq U$. Therefore, $R[x] \subseteq U$. As $\epsilon(x)R^*\mathfrak{q}$, for any $\Box_R V \in \epsilon(x)$, it follows that $V \in \mathfrak{q}$ where V is an upset of X. For any $x \leq z$ in X, we have that $R[z] \subseteq R[x] \subseteq U$ as $\leq \circ R = R$ in $\mathcal{X}$. Thus ${\uparrow}x \subseteq \Box_R U$, and by definition $\Box_R U \in \epsilon(x)$. Thus $U \in \mathfrak{q}$ and $a \in g(\mathfrak{q})$, contradicting the assumption that $a \notin g(\mathfrak{q})$. Therefore, for any $x \in X$, we have that $(g \circ \epsilon)[R[x]] \subseteq \beta(a)$ implies that $R_A[g(\epsilon(x))] \subseteq \beta(a)$ where $a \in D^{\Box}$. This proves that $g \circ \epsilon$ satisfies CDC$_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$.

Let $x \in X$ be arbitrary, suppose ${\uparrow}g(\epsilon(x)) \cap \beta(a) \setminus \beta(b) \neq \emptyset$ where $(a, b) \in D^{\to}$, as g satisfies CDC$_{\to}$ for $\beta(a) \setminus \beta(b)$, we have that $g[{\uparrow}\epsilon(x)] \cap \beta(a) \setminus \beta(b) \neq \emptyset$. Thus there exists a prime filter $\mathfrak{p}$ of upsets of X such that $\epsilon(x) \subseteq \mathfrak{p}$ and $g(\mathfrak{p}) \in \beta(a) \setminus \beta(b)$, namely $a \in g(\mathfrak{p})$ and $b \notin g(\mathfrak{p})$. As we have proved above, there exist upsets U_a, U_b of X such that for any prime filters $\mathfrak{q}$ of upsets of X (i.e any element of $(\mathcal{X}^*)_*$) $a \in g(\mathfrak{q})$ iff $U_a \in \mathfrak{q}$, and $b \in g(\mathfrak{q})$ iff $U_b \in \mathfrak{q}$. Thus $U_a \in \mathfrak{p}$ and $U_b \notin \mathfrak{p}$. Suppose $(g \circ \epsilon)[{\uparrow}x] \cap \beta(a) \setminus \beta(b) = \emptyset$, then for any $x \leq y$, we have that $U_a \in \epsilon(y)$ implies that $U_b \in \epsilon(y)$, namely $y \in U_a$ implies that $y \in U_b$. Thus ${\uparrow}x \subseteq U_a \to U_b$. As $\epsilon(x) \subseteq \mathfrak{p}$, we get that $U_a \to U_b \in \mathfrak{p}$, contradicting the fact that $U_a \in \mathfrak{p}$ and $U_b \notin \mathfrak{p}$. Therefore, $(g \circ \epsilon)[{\uparrow}x] \cap \beta(a) \setminus \beta(b) \neq \emptyset$. This proves that $g \circ \epsilon$ satisfies CDC$_{\to}$ for $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$. Therefore, $g \circ \epsilon : X \to X_A$ is a surjective stable order-preserving map satisfying CDC$_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and satisfies CDC$_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$.

For the other direction, suppose there is a surjective stable order-preserving map $f : X \to X_A$ satisfying CDC$_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$ and satisfies CDC$_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$.

As $\mathfrak{A}$ is finite, every prime filter of $\mathfrak{A}$ (i.e elements of X_A) is principal and is given by a join-irreducible element. For each join-irreducible element of a of $\mathfrak{A}$, we denote the prime filter corresponding to it by P_a. Define $g : (\mathcal{X}^*)_* \to X_A$ as follows: $g(\mathfrak{p}) = \{a \in A \mid {\uparrow}f^{-1}(P_{a_i}) \in \mathfrak{p}$ for some join-irreducible element $a_i \leq a\}$. Clearly, $0 \notin g(\mathfrak{p}) \neq \emptyset$ and $g(\mathfrak{p})$ is upward-closed. Suppose $a, b \in g(\mathfrak{p})$, then there exist join-irreducible elements $a_i, b_j \in A$ such that $a_i \leq a$, $b_j \leq b$, $\uparrow f^{-1}(P_{a_i}) \in \mathfrak{p}$ and $\uparrow f^{-1}(P_{b_j}) \in \mathfrak{p}$. As $\mathfrak{p}$ is a prime filter, $\uparrow f^{-1}(P_{a_i}) \cap \uparrow f^{-1}(P_{b_j}) \neq \emptyset$, say $x \in \uparrow f^{-1}(P_{a_i}) \cap \uparrow f^{-1}(P_{b_j})$. There exist $y \leq x, z \leq x$ in X such that $f(y) = P_{a_i}$ and $f(z) = P_{b_j}$. As f is order-preserving, we have that $P_{a_i}, P_{b_j} \subseteq f(x)$, and thus $a_i, b_j \in f(x)$. As $f(x)$ is a prime filter of $\mathfrak{A}$, it follows that $a_i \wedge b_j \neq 0$. Let $c_1, ..., c_n$ enumerate all the join-irreducible elements of $\mathfrak{A}$ less than or equal to $a_i \wedge b_j$ (note $1 \leq n$ as $a_i \wedge b_j \neq 0$). For any $x' \in \uparrow f^{-1}(P_{a_i}) \cap \uparrow f^{-1}(P_{b_j})$, as we have shown $P_{a_i} \subseteq f(x')$ and $P_{b_j} \subseteq f(x')$. Thus $f(x') = P_{c_k}$ for some $1 \leq k \leq n$. Since $x' \leq x'$, $x' \in \uparrow f^{-1}(P_{c_1}) \cup ... \cup \uparrow f^{-1}(P_{c_n})$. As x' is arbitrary, this proves that $\uparrow f^{-1}(P_{a_i}) \cap \uparrow f^{-1}(P_{b_j}) \subseteq \uparrow f^{-1}(P_{c_1}) \cup ... \cup \uparrow f^{-1}(P_{c_n})$. As $\uparrow f^{-1}(P_{a_i}) \in \mathfrak{p}$, $\uparrow f^{-1}(P_{b_j}) \in \mathfrak{p}$ and $\mathfrak{p}$ is a prime filter, $\uparrow f^{-1}(P_{c_m}) \in \mathfrak{p}$ for some $1 \leq m \leq n$. As $c_m \leq a_i \wedge b_j \leq a \wedge b$ is join-irreducible, $a \wedge b \in g(\mathfrak{p})$. This proves that $g(\mathfrak{p})$ is a proper filter. By the definition of join-irreducibility, it is easy to see that $g(\mathfrak{p})$ is also prime. Therefore, g is well defined.

Clearly, if $\mathfrak{p} \subseteq \mathfrak{q}$, then $g(\mathfrak{p}) \subseteq g(\mathfrak{q})$, and thus g is order-preserving. Suppose $f(x) = P_a$ where $x \in X$ and $a \in \mathfrak{A}$ is join-irreducible. Then by definition $\uparrow f^{-1}(P_a) \in \epsilon(x)$, and thus $P_a \subseteq g(\epsilon(x))$. Now suppose $\uparrow f^{-1}(P_b) \in \epsilon(x)$ where b is join-irreducible, then $\uparrow x \subseteq \uparrow f^{-1}(P_b)$, there exists $y \leq x$ such that $f(y) = P_b$. As f is order-preserving, $f(y) = P_b \subseteq f(x) = P_a$. Thus $g(\epsilon(x)) \subseteq P_a$, and $g(\epsilon(x)) = f(x)$. As f is surjective, so is g. For any $P_a \in X_A$, let $\Gamma = \{b \in A \mid b \text{ is join-irreducible and } a \not\leq b\}$ (note that Γ is finite as $\mathfrak{A}$ is finite), then by definition $g^{-1}(P_a) = \beta(\uparrow f^{-1}(P_a)) \setminus \bigcup_{b\in\Gamma} \beta(\uparrow f^{-1}(P_b))$ which is clopen in $(\mathcal{X}^*)_*$. Thus g is continuous. Therefore, g is a surjective Priestley morphism.

Suppose $\mathfrak{p} R^* \mathfrak{q}$ in $(\mathcal{X}^*)_*$, then for any $\Box a \in g(\mathfrak{p})$, there exists a join-irreducible element $a_i \leq \Box a$ such that $\uparrow f^{-1}(P_{a_i}) \in \mathfrak{p}$. For any $x \in \uparrow f^{-1}(P_{a_i})$, there exists $y \leq x$ such that $f(y) = P_{a_i}$. As f is order-preserving, $f(y) \subseteq f(x)$, $a_i \in f(x)$, and thus $\Box a \in f(x)$. For any xRz in $\mathcal{X}$, as f is stable, $f(x) R_A f(z)$. Thus $a \in f(z)$, and $f(z) = P_{b_i}$ for some join-irreducible element $b_i \leq a$. As $x \in \uparrow f^{-1}(P_{a_i})$ is arbitrary, this proves that $\uparrow f^{-1}(P_{a_i}) \subseteq \Box_R(\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_n}))$ where $b_1, ..., b_n$ enumerate all the join-irreducible elements less than or equal to a. As $\uparrow f^{-1}(P_{a_i}) \in \mathfrak{p}$, it follows that $\Box_R(\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_n})) \in \mathfrak{p}$. As $\mathfrak{p} R^* \mathfrak{q}$, we know that $\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_n}) \in \mathfrak{q}$, and $\uparrow f^{-1}(P_{b_i}) \in \mathfrak{q}$ for some $1 \leq i \leq n$. Thus $a \in g(\mathfrak{q})$. As $\Box a \in g(\mathfrak{p})$ is arbitrary, this proves that $g(\mathfrak{p}) R_A g(\mathfrak{q})$ in X_A, g is thus stable.

Suppose $g[R^*[\mathfrak{p}]] \subseteq \beta(a)$ where $a \in D^{\Box}$, then for any $\mathfrak{p} R^* \mathfrak{q}$, we have that $g(\mathfrak{q}) \in \beta(a)$, namely $a \in g(\mathfrak{q})$. By definition, there exists a join-irreducible element $a_i \leq a$ such that $\uparrow f^{-1}(P_{a_i}) \in \mathfrak{q}$. As $\mathfrak{q}$ is a prime filter of upsets of X, we also get that $\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n}) \in \mathfrak{q}$ where $a_1, ..., a_n$ enumerate all the join-irreducible elements less than or equal to a. As $\mathfrak{q}$ is arbitrary, $\Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \in \mathfrak{p}$ (note that $\Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n}))$ is not empty as $\mathfrak{p}$ is a prime filter). For any $x \in \Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n}))$, by definition, $R[x] \subseteq \uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})$, and thus $f[R[x]] \subseteq \beta(a)$. As f satisfies CDC$_\Box$ for $\beta(a)$, we have that $R_A[f[x]] \subseteq \beta(a)$, namely $\Box a \in f(x)$, and thus $f(x) = P_{b_i}$ for some join-irreducible element $b_i \leq \Box a$. As $x \leq x$, $x \in \uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m})$ where $b_1, ..., b_m$ enumerate all join-irreducible elements less than or equal to $\Box a$. As $x \in \Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n}))$ is arbitrary, this proves that $\Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \subseteq \uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m})$. Since $\Box_R(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \in \mathfrak{p}$ and $\mathfrak{p}$ is a prime filter, $\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m}) \in \mathfrak{p}$, and thus $\uparrow f^{-1}(P_{b_j}) \in \mathfrak{p}$ for some $1 \leq j \leq m$. Therefore, $\Box a \in g(\mathfrak{p})$, and thus $R_A[g(\mathfrak{p})] \subseteq \beta(a)$. This proves that g satisfies CDC$_\Box$ for $\beta(a)$ where $a \in D^\Box$.

Suppose $g(\uparrow \mathfrak{p}) \cap \beta(a) \setminus \beta(b) = \emptyset$ where $(a, b) \in D^\to$. For any $\mathfrak{p} \subseteq \mathfrak{q}$ in $(\mathcal{X}^*)_*$, we have that $g(\mathfrak{q}) \notin \beta(a) \setminus \beta(b)$. Namely, if $a \in g(\mathfrak{q})$, then $b \in g(\mathfrak{q})$. Thus $\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n}) \in \mathfrak{q}$ implies that $\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m}) \in \mathfrak{q}$ where $a_1, ..., a_n$ enumerate all join-irreducible elements less than or equal to a while $b_1, ..., b_m$ enumerate all join-irreducible elements less than or equal to b. As $\mathfrak{p} \subseteq \mathfrak{q}$ is arbitrary, this means that $(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \to (\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m})) \in \mathfrak{p}$ (note that $(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \to (\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m}))$ is not empty). Suppose $x \in (\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \to (\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m}))$, for any $x \leq y$, if $y \in \uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})$, then $y \in \uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m})$. Thus $f[\uparrow x] \cap \beta(a) \setminus \beta(b) = \emptyset$. As f satisfies CDC$_\to$ for $\beta(a) \setminus \beta(b)$, we have that $\uparrow f(x) \cap \beta(a) \setminus \beta(b) = \emptyset$. Thus $a \to b \in f(x)$, and $f(x) = P_{c_j}$ for some join-irreducible element $c_j \leq a \to b$. As $x \leq x$, $x \in \uparrow f^{-1}(P_{c_j}) \subseteq \uparrow f^{-1}(P_{c_1}) \cup ... \cup \uparrow f^{-1}(P_{c_k})$ where $c_1, ..., c_k$ enumerate all join-irreducible elements less than or equal to $a \to b$. As $x \in (\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \to (\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m}))$ is arbitrary, this proves that $(\uparrow f^{-1}(P_{a_1}) \cup ... \cup \uparrow f^{-1}(P_{a_n})) \to (\uparrow f^{-1}(P_{b_1}) \cup ... \cup \uparrow f^{-1}(P_{b_m})) \subseteq \uparrow f^{-1}(P_{c_1}) \cup ... \cup \uparrow f^{-1}(P_{c_k})$. Then $\uparrow f^{-1}(P_{c_1}) \cup ... \cup \uparrow f^{-1}(P_{c_k}) \in \mathfrak{p}$, and thus $a \to b \in g(\mathfrak{p})$. Therefore, $\uparrow g(\mathfrak{p}) \cap \beta(a) \setminus \beta(b) = \emptyset$. Thus, if $\uparrow g(\mathfrak{p}) \cap \beta(a) \setminus \beta(b) \neq \emptyset$, then $g[\uparrow \mathfrak{p}] \cap \beta(a) \setminus \beta(b) \neq \emptyset$ where $(a, b) \in D^\to$. This proves that g satisfies CDC$_\to$ for $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^\to$.

Therefore, g is a surjective stable Priestley morphism satisfying CDC$_\Box$ for any $\beta(a)$ where $a \in D^\Box$ and satisfies CDC$_\to$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^\to$. By Proposition 3.3.12, $(\mathcal{X}^*)_* \not\models \rho(\mathfrak{A}, D^\to, D^\Box)$, and thus $\mathcal{X} = (X, \leq, R) \not\models \rho(\mathfrak{A}, D^\to, D^\Box)$.

■

Similarly (in fact, more easily), we get the following refutation conditions for bimodal Kripke frames.

Theorem 4.2.6. *For any bimodal Kripke frame* (X, R_I, R_M), $(X, R_I, R_M) \not\models \mu(\mathfrak{A}, D^I, D^M)$ *iff there is a surjective relation-preserving (or stable) map* $f : X \to X_A$ *satisfying CDC*$_\Box$ *for any* $\beta(a)$ *and* $\beta(b)$ *where* $a \in D^I$ *and* $b \in D^M$.

Then we prove the rule translation lemma, which is the counterpart to [16, Lem. 2.56] and shows that the Gödel translation connects the stable canonical rules for intuitionistic modal logics and the stable canonical rules for bimodal logics.

Lemma 4.2.7 (Rule translation lemma)**.** *For every stable canonical rule* $\rho(\mathfrak{B}, D^{\to}, D^{\Box})$ *and every* $S4 \otimes K \oplus Mix$*-modal space* $\mathfrak{X}$*, we have that* $\mathfrak{X} \models t(\rho(\mathfrak{B}_*, \mathfrak{D}_{\to}, \mathfrak{D}_M))$ *iff* $\mathfrak{X} \models \mu(\sigma(\mathfrak{B}_*), \mathfrak{D}_I, \mathfrak{D}_M)$ *where* $\mathfrak{D}_{\to} = \{\beta(a) \setminus \beta(b) \mid (a, b) \in D^{\to}\}$, $\mathfrak{D}_M = \{\beta(a) \mid a \in D^{\Box}\}$ *and* $\mathfrak{D}_I = \{\overline{\beta(a)} \cup \beta(b) \mid (a, b) \in D^{\to}\}$.

Proof. By Propositions 4.1.13 and 4.1.17, it suffices to prove that $\rho(\mathfrak{X}) \models \rho(\mathfrak{B}_*, \mathfrak{D}_{\to}, \mathfrak{D}_M)$ iff $\mathfrak{X} \models \mu(\sigma(\mathfrak{B}_*), \mathfrak{D}_I, \mathfrak{D}_M)$.

Suppose $\mathfrak{X} \not\models \mu(\sigma(\mathfrak{B}_*), \mathfrak{D}_I, \mathfrak{D}_M)$, then there is a continuous stable surjection $f : \mathfrak{X} \to \sigma(\mathfrak{B}_*)$ satisfying CDC$_\Box$ for any $\beta(a) \in \mathfrak{D}_M$ and for any $\beta(b) \in \mathfrak{D}_I$. Define $g : \rho(\mathfrak{X}) \to \mathfrak{B}_*$ by $g(\rho(x)) = f(x)$. Suppose $x \backsim y$ in $\mathfrak{X}$, then xR_Iy and yR_Ix. As f is stable, $f(x)R_If(y)$ and $f(y)R_If(x)$ in $\sigma(\mathfrak{B}_*)$, namely $f(x) \subseteq f(y)$ and $f(y) \subseteq f(x)$ in $\mathfrak{B}_*$. Thus $f(x) = f(y)$, g is well defined.

As f is surjective, so is g. Suppose $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$, then xR_Iy in $\mathfrak{X}$, as f is stable, $f(x)R_If(y)$ in $\sigma(\mathfrak{B}_*)$, namely $f(x) \subseteq f(y)$ in $\mathfrak{B}_*$. Thus $g(\rho(x)) \subseteq g(\rho(y))$, g is order-preserving. Suppose $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$, then there exist x_1, x_2 such that $xR_Ix_1R_Mx_2R_Iy$ in $\mathfrak{X}$. As f is stable, $f(x)R_If(x_1)R_Mf(x_2)R_If(y)$ in $\sigma(\mathfrak{B}_*)$, namely $f(x) \subseteq f(x_1)Rf(x_2) \subseteq f(y)$ in $\mathfrak{B}_*$. As $\mathfrak{B}_*$ is a modal Esakia space, by Proposition 4.1.11, $\subseteq \circ R \circ \subseteq = R$. Thus $f(x)Rf(y)$ in $\mathfrak{B}_*$, namely $g(\rho(x))Rg(\rho(y))$. This proves that g is stable.

For any $p \in \mathfrak{B}_*$, we have that $x \in \rho^{-1}(g^{-1}(p))$ iff $g(\rho(x)) = p$ iff $f(x) = p$ iff $x \in f^{-1}(p)$. As f is continuous, $\rho^{-1}(g^{-1}(p)) = f^{-1}(p)$ is clopen in $\mathfrak{X}$. Clearly, $\rho^{-1}(g^{-1}(p))$ does not cut any R_I-cluster, so $g^{-1}(p)$ is clopen in $\rho(\mathfrak{X})$ as the topology is the quotient topology. Thus g is continuous.

Suppose $\uparrow g(\rho(x)) \cap \beta(a) \setminus \beta(b) \neq \emptyset$ where $(a, b) \in D^{\to}$, then $\uparrow f(x) \cap \beta(a) \setminus \beta(b) \neq \emptyset$. Thus $R_I[f(x)] \not\subseteq \overline{\beta(a)} \cup \beta(b)$. As f satisfies CDC$_\Box$ for $\overline{\beta(a)} \cup \beta(b)$, we have that $f[R_I[x]] \not\subseteq \overline{\beta(a)} \cup \beta(b)$. There exists xR_Iy such that $f(y) \in \beta(a)$ while $f(y) \notin \beta(b)$. Then $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$, $g(\rho(y)) = f(y) \in \beta(a) \setminus \beta(b)$, and thus $g[\uparrow\rho(x)] \cap \beta(a) \setminus \beta(b) \neq \emptyset$. Therefore, g satisfies $\text{CDC}_{\rightarrow}$ any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\rightarrow}$.

Suppose $g[[R_I \circ R_M \circ R_I][\rho(x)]] \subseteq \beta(a)$ where $a \in D^{\Box}$, then for any xR_My in $\mathfrak{X}$, as R_I is reflexive, $xR_I \circ R_M \circ R_Iy$. By definition, $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ in $\rho(\mathfrak{X})$. As $g[[R_I \circ R_M \circ R_I][\rho(x)]] \subseteq \beta(a)$, it follows that $g(\rho(y)) = f(y) \in \beta(a)$. Thus $f[R_M[x]] \subseteq \beta(a)$, and as f satisfies $\text{CDC}_{\Box}$ for $\beta(a)$, we have that $R_M[f(x)] \subseteq \beta(a)$, namely $R[g(\rho(x))] \subseteq \beta(a)$ in $\mathfrak{B}_*$. Therefore, g satisfies $\text{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$. By Proposition 3.3.12, $\rho(\mathfrak{X}) \not\models \rho(\mathfrak{B}_*, \mathfrak{D}_{\rightarrow}, \mathfrak{D}_M)$.

For the other direction, suppose $\rho(\mathfrak{X}) \not\models \rho(\mathfrak{B}_*, \mathfrak{D}_{\rightarrow}, \mathfrak{D}_M)$, by Proposition 3.3.12, there is a surjective stable Priestley morphism $f : \rho(\mathfrak{X}) \to \mathfrak{B}_*$ satisfying CDC$\rightarrow$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\rightarrow}$ and $\text{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$. Define $g : \mathfrak{X} \to \sigma(\mathfrak{B}_*)$ as follows: $g(x) = f(\rho(x))$.

As f is surjective, so is g. Suppose xR_Iy in $\mathfrak{X}$, then $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$. As f is order-preserving, $f(\rho(x)) \subseteq f(\rho(y))$, namely $f(\rho(x))R_If(\rho(y))$ in $\sigma(\mathfrak{B}_*)$, and $g(x)R_Ig(y)$. Thus g is relation-preserving w.r.t R_I. Suppose xR_My in $\mathfrak{X}$, as R_I is reflexive, $xR_I \circ R_M \circ R_Iy$. By definition, $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ in $\rho(\mathfrak{X})$. As f is stable, $f(\rho(x))Rf(\rho(y))$ in $\mathfrak{B}_*$, namely $g(x)R_Mg(y)$ in $\sigma(\mathfrak{B}_*)$. g is thus relation-preserving w.r.t R_M.

For any $p \in \sigma(\mathfrak{B}_*)$, $g^{-1}(p) = \rho^{-1}(f^{-1}(p))$. As f is continuous, $f^{-1}(p)$ is clopen in $\rho(\mathfrak{X})$. As $\rho(\mathfrak{X})$ has the quotient topology, $\rho^{-1}(f^{-1}(p))$ is clopen in $\mathfrak{X}$. Thus g is continuous.

Suppose $g[R_I[x]] \subseteq \overline{\beta(a)} \cup \beta(b)$ where $(a, b) \in D^{\rightarrow}$, for any $\rho(x) \leq \rho(y)$ in $\rho(\mathfrak{X})$, xR_Iy in $\mathfrak{X}$, and $f(\rho(y)) = g(y) \in \overline{\beta(a)} \cup \beta(b)$. Thus $f[\uparrow\rho(x)] \cap \beta(a) \setminus \beta(b) = \emptyset$. As f satisfies $\text{CDC}_{\rightarrow}$ for $\beta(a) \setminus \beta(b)$, we have that $\uparrow f(\rho(x)) \cap \beta(a) \setminus \beta(b) = \emptyset$. Thus $R_I[g(x)] \cap \beta(a) \setminus \beta(b) = \emptyset$, namely $R_I[g(x)]] \subseteq \overline{\beta(a)} \cup \beta(b)$. This proves that g satisfies $\text{CDC}_{\Box}$ for any element in $\mathfrak{D}_I$.

Suppose $g[R_M[x]] \subseteq \beta(a)$ where $a \in D^{\Box}$. For any $\rho(x)[R_I \circ R_M \circ R_I]\rho(y)$ in $\rho(\mathfrak{X})$, by definition $xR_I \circ R_M \circ R_Iy$ in $\mathfrak{X}$. As $\mathfrak{X}$ validates **Mix**, $R_I \circ R_M \circ R_I = R_M$, and thus xR_My. As $g[R_M[x]] \subseteq \beta(a)$, it follows that $f(\rho(y)) = g(y) \in \beta(a)$. Thus $f[[R_I \circ R_M \circ R_I][\rho(x)]] \subseteq \beta(a)$. As f satisfies $\text{CDC}_{\Box}$ for $\beta(a)$, we have that $R[f(\rho(x))] \subseteq \beta(a)$. Thus $R_M[g(x)] = R[f(\rho(x))] \subseteq \beta(a)$. This proves that g satisfies $\text{CDC}_{\Box}$ for any $\beta(a)$ where $a \in D^{\Box}$. Therefore, $\mathfrak{X} \not\models \mu(\sigma(\mathfrak{B}_*), \mathfrak{D}_I, \mathfrak{D}_M)$ by Proposition 4.1.9.

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We can also state the above lemma in algebraic terms just like [16, Lem. 2.56]: for every stable canonical rule $\rho(\mathfrak{B}, D^{\rightarrow}, D^{\Box})$ and every $S4 \otimes K \oplus Mix$-algebra $\mathfrak{A}$, we have that $\mathfrak{A} \models t(\rho(\mathfrak{B}, D^{\rightarrow}, D^{\Box}))$ iff $\mathfrak{A} \models \mu(\sigma(\mathfrak{B}), D^I, D^M)$

where $D^I = \{\neg a \vee b \mid (a,b) \in D^{\to}\}$ and $D^M = D^{\Box}$.

Now, we can finish the proof of the Dummett-Lemmon conjecture which is the main result of this section.

Theorem 4.2.8. *For any $L \in \mathbf{NExt}(\boldsymbol{IntK}^R_{\Box})$, L is Kripke complete iff $\tau(L)$ is Kripke complete.*

Proof. The right-to-left direction is given by Proposition 4.2.3. For the other direction, let $L \in \mathbf{NExt}(\mathbf{IntK}^R_{\Box})$ be arbitrary, suppose L is Kripke complete and $\Gamma/\Delta \notin \tau(L)$ where Γ/Δ is a bimodal multi-conclusion rule. By Proposition 4.1.6, we can assume that $\Gamma/\Delta = \mu(\mathfrak{A}_*, \mathfrak{D}_I, \mathfrak{D}_M)$ where $\mathfrak{A}$ is a finite $S4 \otimes K \oplus Mix$-algebra and $\mathfrak{D}_I, \mathfrak{D}_M \subseteq \mathcal{P}(\mathfrak{A}_*)$. By Corollary 2.4.8 and Theorem 4.1.8, there exists an $S4 \otimes K \oplus Mix$-modal space $\mathfrak{X}$ such that $\mathfrak{X} \vDash \tau(L)$ and $\mathfrak{X} \nvDash \mu(\mathfrak{A}_*, \mathfrak{D}_I, \mathfrak{D}_M)$. By Lemma 4.2.4, $\sigma(\rho(\mathfrak{X})) \nvDash \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}})$ where $\overline{\rho\overline{\mathfrak{D}_I}} = \{\overline{\rho[\bar{\delta}]} \mid \delta \in \mathfrak{D}_I\}$ and $\overline{\rho\overline{\mathfrak{D}_M}} = \{\overline{\rho[\bar{\delta}]} \mid \delta \in \mathfrak{D}_M\}$. Note that $\sigma(\rho(\mathfrak{X}))$ is a $Grz \otimes K \oplus Mix$-modal space.

Now by Theorem 4.1.24, there exists an $L' \in \mathbf{NExt}(\mathbf{IntK}^R_{\Box})$ such that $\sigma(L') = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}})$, namely $\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}}) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L'\}$. Therefore, $\sigma(\rho(\mathfrak{X})) \nvDash t(\Gamma/\Delta)$ for some $\Gamma/\Delta \in L'$. By Theorem 3.1.9, we can assume that Γ/Δ is a stable canonical rule of the form $\rho(\mathfrak{B}, D^{\to}, D^{\Box})$. Thus $\sigma(\rho(\mathfrak{X})) \nvDash t(\rho(\mathfrak{B}, D^{\to}, D^{\Box}))$. By Proposition 4.1.17 and Proposition 4.1.14, $\rho(\mathfrak{X}) \nvDash \rho(\mathfrak{B}, D^{\to}, D^{\Box})$. As $\mathfrak{X} \vDash \tau(L)$, by Proposition 4.1.17, $\rho(\mathfrak{X}) \vDash L$. Therefore, $\rho(\mathfrak{B}, D^{\to}, D^{\Box}) \notin L$. As L is Kripke complete, there is an intuitionistic modal Kripke frame $\mathfrak{F} = (X, \leq, R)$ such that $\mathfrak{F} \vDash L$ while $\mathfrak{F} \nvDash \rho(\mathfrak{B}, D^{\to}, D^{\Box})$. Viewing $\mathfrak{F}$ as an $S4 \otimes K \oplus Mix$-bimodal Kripke frame, as $\rho(\mathfrak{F}) = \mathfrak{F} \vDash L$, by Proposition 4.2.2, $\mathfrak{F} \nvDash t(\rho(\mathfrak{B}, D^{\to}, D^{\Box}))$ and $\mathfrak{F} \vDash \tau(L)$.

Then we prove that $\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L'\} \subseteq \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}})$: let $\mathfrak{X}$ be an arbitrary $S4 \otimes K \oplus Mix$-modal space, suppose $\mathfrak{X} \nvDash t(\Gamma/\Delta)$ for some $\Gamma/\Delta \in L'$. By Theorem 3.1.9, we can assume Γ/Δ is a stable canonical rule of the form $\rho(\mathfrak{C}_*, \mathfrak{D}'_{\to}, \mathfrak{D}'_M)$. As $\mathfrak{X} \nvDash t(\rho(\mathfrak{C}_*, \mathfrak{D}'_{\to}, \mathfrak{D}'_M))$, by Lemma 4.2.7, $\mathfrak{X} \nvDash \mu(\sigma(\mathfrak{C}_*), \mathfrak{D}'_I, \mathfrak{D}'_M)$. By Lemma 4.2.4 and the fact that $\sigma(\rho(\sigma(\mathfrak{C}_*))) = \sigma(\mathfrak{C}_*)$, it follows that $\sigma(\rho(\mathfrak{X})) \nvDash \mu(\sigma(\mathfrak{C}_*), \mathfrak{D}'_I, \mathfrak{D}'_M)$. By Lemma 4.2.7, $\sigma(\rho(\mathfrak{X})) \nvDash t(\rho(\mathfrak{C}_*, \mathfrak{D}'_{\to}, \mathfrak{D}'_M))$. As $\sigma(\rho(\mathfrak{X}))$ is a $Grz \otimes K \oplus Mix$-modal space and $\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}}) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L'\}$, we get that $\sigma(\rho(\mathfrak{X})) \nvDash \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}})$. By Proposition 4.1.15 and the duality, $\mathfrak{X} \nvDash \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\rho\overline{\mathfrak{D}_I}}, \overline{\rho\overline{\mathfrak{D}_M}})$. This proves that for any $S4 \otimes K \oplus Mix$-modal space, if $\mathfrak{X} \nvDash \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L'\}$, then

$\mathfrak{X} \not\models \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\overline{\rho\mathfrak{D}_I}}, \overline{\overline{\rho\mathfrak{D}_M}})$. By Theorems 2.4.7 and 4.1.8, $\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L'\} \subseteq \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}^R \oplus \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\overline{\rho\mathfrak{D}_I}}, \overline{\overline{\rho\mathfrak{D}_M}})$.

Now, since $\mathfrak{F} \not\models t(\rho(\mathfrak{B}, D^{\to}, D^{\Box}))$ where $\rho(\mathfrak{B}, D^{\to}, D^{\Box}) \in L'$, we have that $\mathfrak{F} \not\models \mu(\sigma(\rho(\mathfrak{A}_*)), \overline{\overline{\rho\mathfrak{D}_I}}, \overline{\overline{\rho\mathfrak{D}_M}})$. Therefore, by Theorem 4.2.6, there is a surjective stable map $f : X \to \sigma(\rho(\mathfrak{A}_*))$ satisfying CDC for any element in $\overline{\overline{\rho\mathfrak{D}_I}}$ and $\overline{\overline{\rho\mathfrak{D}_M}}$. We construct a new bimodal Kripke frame $\mathfrak{F}' = (X', R_I, R_M)$ as follows: for any $x \in X$, say $f(x) = \{a_1, ..., a_n\}$ where $a_1, ..., a_n \in \mathfrak{A}_*$ (an R_I-cluster), replace x by n many copies of it, say $x_1, ..., x_n$, then let X' be the set of all such elements. For any $x_i, y_j \in X'$ (x_i is a copy of x and y_j is a copy of y), define $x_i R_I y_j$ iff $x \le y$ (in $\mathfrak{F}$), and $x_i R_M y_j$ iff $x R_M y$. It is easy to check that $\mathfrak{F}'$ is an $S4 \otimes K \oplus Mix$-Kripke frame. As $\rho(\mathfrak{F}') = \mathfrak{F} = \rho(\mathfrak{F}) \models L$, by Proposition 4.2.2, $\mathfrak{F}' \models \tau(L)$.

We then define $g : \mathfrak{F}' \to \mathfrak{A}_*$ as follows: $g(x_i) = a_i$ where x_i is copy of x and $f(x) = \{a_1, ..., a_n\}$. As f is surjective , g is surjective by the construction of X'. For any $x_i, y_j \in X'$(say $g(x_i) = a_i$, $g(y_j) = b_j$), suppose $x_i R_I y_j$ in $\mathfrak{F}'$, then $x \le y$ in $\mathfrak{F}$ (or $x R_I y$ in $\mathfrak{F}$ when viewed $\mathfrak{F}$ as a bimodal Krikpe frame). As f is stable, $f(x) R_I f(y)$ in $\sigma(\rho(\mathfrak{A}_*))$, namely $f(x) \le f(y)$ in $\rho(\mathfrak{A}_*)$. As a_i is an element of $f(x)$ and b_j is an element of $f(y)$ by the construction, $a_i R_I b_j$ in $\mathfrak{A}_*$, namely $g(x_i) R_I g(y_j)$. Suppose $x_i R_M y_j$, then $x R_M y$ in $\mathfrak{F}$. As f is stable, $f(x)[R_I \circ R_M \circ R_M] f(y)$. As $\mathfrak{A}_*$ validates **Mix**, $R_I \circ R_M \circ R_I = R_M$, and thus $f(x)[R_M] f(y)$. We have that $a_i R_M b_j$, namely $g(x_i) R_M g(b_j)$. Therefore, g is stable.

Suppose $R_M[g(x_i)] \cap \bar{\delta} \neq \emptyset$ where $\delta \in \mathfrak{D}_M$, there exists $p \in \bar{\delta}$ such that $g(x_i) R_M p$. Thus $\rho(p) \in \rho[\bar{\delta}]$, and $f(x) = \rho(g(x_i))[R_M]\rho(p)$. Since f satisfies CDC for $\overline{\rho[\bar{\delta}]}$, there exists $z \in \mathfrak{F}$ such that $x R_M z$ and $f(z) \in \rho[\bar{\delta}]$. By the construction, there exists z_j such that $g(z_j) \in \bar{\delta}$. Then as $x_i R_M z_j$, $g[R_M[x_i]] \cap \bar{\delta} \neq \emptyset$. Thus g satisfies CDC for any element in $\mathfrak{D}_M$. Similarly, we can prove that g satisfies CDC for any element in $\mathfrak{D}_I$. Therefore, by Theorem 4.2.6, $\mathfrak{F}' \not\models \mu(\mathfrak{A}_*, \mathfrak{D}_I, \mathfrak{D}_M)$. As $\mathfrak{F}' \models \tau(L)$, this proves that $\tau(L)$ is Kripke complete.

■

The above theorem is not only interesting for its own sake, but also strengthens the connection between the lattice of intuitionistic modal logics and the lattice of normal extensions of the bimodal logic $\mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{Mix}$. It improves our toolkit when we try to study intuitionistic modal logics or bimodal logics via the Gödel translation. In particular, it allows us to reduce the problem about Kripke completeness of an intuitionistic modal multi-conclusion consequence relation to the same problem about a bimodal multi-conclusion consequence relation and vice versa.

In conclusion, this chapter illustrates how stable canonical rules can be used in reasoning about logics. In particular, these rules allowed us to prove the Blok-Esakia theorem and the Dummett-Lemmon conjecture for intuitionistic modal logics.

Chapter 5

Stable canonical rules for Heyting-Lewis logics

After developing the machinery of stable canonical rules for intuitionistic modal logics, in this chapter, we apply further our techniques to Heyting-Lewis logics, which are superintuitionistic logics with a strict implication. We will first introduce Heyting-Lewis multi-conclusion consequence relations and their algebraic semantics. Then we continue to develop the corresponding stable canonical rules for them. In the end, we will try to mimic the proof of the Blok-Esakia theorem given in the fourth chapter and point out an underlying problem. This observation helps us find an error in the proof of the Blok-Esakia theorem for Heyting-Lewis logics given in [23].

5.1 Preliminaries

Heyting-Lewis logics are superintutionistic logics extended with a strict implication which is weaker than Heyting implication (i.e the implication in Heyting algebras). These logics were introduced by Litak and Visser in [29] and have appeared in various settings, ranging from preservativity logic of Heyting arithmetic to the generalization of intuitionistic epistemic logic.

The signature of Heyting-Lewis logics $hl = \{\wedge, \vee, \to, \top, \bot, \mathrel{-\!\!3}\}$, and the set of formulas $Form_{hl}$ is defined recursively as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \mathrel{-\!\!3} \varphi$$

Definition 5.1.1. A logic over $Form_{hl}$ is a *Heyting-Lewis logic* if the following hold:

- $\mathbf{IPC} \subseteq L$
- $(\varphi \dashv\!\!\!\!3\, \psi) \wedge (\varphi \dashv\!\!\!\!3\, \chi) \rightarrow (\varphi \dashv\!\!\!\!3\, (\psi \wedge \chi)) \in L$
- $(\varphi \dashv\!\!\!\!3\, \chi) \wedge (\psi \dashv\!\!\!\!3\, \chi) \rightarrow ((\varphi \vee \psi) \dashv\!\!\!\!3\, \chi) \in L$
- $(\varphi \dashv\!\!\!\!3\, \psi) \wedge (\psi \dashv\!\!\!\!3\, \chi) \rightarrow (\varphi \rightarrow \chi)$
- $\varphi \rightarrow \psi, \varphi \in L$ implies $\psi \in L$ (MP)
- $\varphi \rightarrow \psi \in L$ implies $\varphi \dashv\!\!\!\!3\, \psi \in L$ (arrow-Nec)

We denote the least Heyting-Lewis logic by $\mathbf{iA}$. It is not difficult to check that if we define $\Box\varphi$ as $\top \dashv\!\!\!\!3\, \varphi$, then we get a *normal* modal logic[1]. In fact, if we add $(\varphi \dashv\!\!\!\!3\, \psi) \rightarrow \Box(\varphi \rightarrow \psi)$ as an axiom, then $\Box$ and $\dashv\!\!\!\!3$ are interdefinable and we are thus back to the setting of intuitionistic modal logics [29, Lem. 4.4].

Next we introduce multi-conclusion consequence relations for Heyting-Lewis logics:

Definition 5.1.2. A *Heyting-Lewis multi-conclusion consequence relation* is a multi-conclusion consequence relation H over Rul_{hl}[2] satisfying the following conditions:

- $/\varphi \in H$ whenever $\varphi \in \mathbf{iA}$
- $\varphi \rightarrow \psi / \varphi \dashv\!\!\!\!3\, \psi \in H$
- $\varphi \rightarrow \psi, \varphi / \psi \in H$

Let L be a Heyting-Lewis logic, we use $\mathbf{Ext}(L)$ to denote the lattice of all Heyting-Lewis logics extending L with $\oplus$ as join and intersection as meet. Similarly, we define $\mathbf{Ext}(H)$ where H is a Heyting-Lewis multi-conclusion consequence relation. Other notations such as L^R are defined in the same way as for intuitionistic modal logics and bimodal logics.

Now we introduce the algebraic semantics for Heyting-Lewis logics:

Definition 5.1.3. A *Heyting-Lewis algebra* (*HL-algebra* for short) is a tuple $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, 0, 1, \dashv\!\!\!\!3)$ where $(A, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and $\dashv\!\!\!\!3 \colon A \times A \rightarrow A$ is a binary operation satisfying the following axioms:

- $(a \dashv\!\!\!\!3\, b) \wedge (a \dashv\!\!\!\!3\, c) = a \dashv\!\!\!\!3\, (b \wedge c)$

[1] See [10, Def. 1.42].

[2] Elements in Rul_{hl} are called *Heyting-Lewis multi-conclusion rules*.

- $(a \strictif c) \wedge (b \strictif c) = (a \vee b) \strictif c$
- $(a \strictif b) \wedge (b \strictif c) \leq (a \strictif c)$[3]
- $a \strictif a = 1$

For simplicity, we will write $\mathfrak{A} = (A, \strictif)$ where A is assumed to be a Heyting algebra. Let **HLA** be the class of all Heyting-Lewis algebras, by Theorem 2.2.18, **HLA** is a variety. Let **Var**(**HLA**) and **Uni**(**HLA**) denote the lattice of subvarieties and the lattice of universal subclasses of **HLA** respectively. We have the following standard correspondence results.

Theorem 5.1.4. *The following maps are pairs of mutually inverse isomorphisms:*

- *Alg:* $\mathbf{Ext}(\boldsymbol{iA}) \to \boldsymbol{Var}(\mathbf{HLA})$ *and Th:* $\boldsymbol{Var}(\mathbf{HLA}) \to \mathbf{Ext}(\boldsymbol{iA})$
- *Alg:* $\mathbf{Ext}(\boldsymbol{iA}^R) \to \boldsymbol{Uni}(\mathbf{HLA})$ *and Ru:* $\boldsymbol{Uni}(\mathbf{HLA}) \to \mathbf{Ext}(\boldsymbol{iA}^R)$

Corollary 5.1.5. The following hold:

- Every Heyting-Lewis logic is complete with respect to some variety of Heyting-Lewis algebras.
- Every Heyting-Lewis multi-conclusion consequence relation is complete with respect to some universal class of Heyting-Lewis algebras.

5.2 Stable canonical rules for Heyting-Lewis multi-conclusion consequence relations

In this section, we develop stable canonical rules for Heyting-Lewis multi-conclusion consequence relations. We follow the road map of Section 3.1 closely and make adjustments whenever necessary.

Definition 5.2.1. Let $\mathfrak{A} = (A, \strictif)$ and $\mathfrak{B} = (B, \strictif)$ be HL-algebras, and let $h : \mathfrak{A} \to \mathfrak{B}$ be a bounded lattice homomorphism, we say that h is *stable* if the following holds:

$$h(a \strictif b) \leq h(a) \strictif h(b) \text{ for any } a, b \in A.$$

[3]Equivalently, $((a \strictif b) \wedge (b \strictif c)) \to (a \strictif c) = 1$

Definition 5.2.2. Let $\mathfrak{A} = (A, \strictif)$, $\mathfrak{B} = (B, \strictif)$ be HL-algebras, $D^{\to} \subseteq A^2$ and $D^{\strictif} \subseteq A^2$. A bounded lattice embedding $h : A \to B$ satisfies

- the *closed domain condition* (CDC for short) for $D^{\to}$ if $h(a \to b) = h(a) \to h(b)$ for any $(a, b) \in D^{\to}$
- the *closed domain condition* (CDC for short) for $D^{\strictif}$ if $h(a \strictif b) = h(a) \strictif h(b)$ for any $(a, b) \in D^{\strictif}$.

It turns out that we can associate each Heyting-Lewis multi-conclusion rule with finitely many refutation patterns similarly to Proposition 3.1.3.

Proposition 5.2.3. For each Heyting-Lewis multi-conclusion rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{\strictif})$, ..., $(\mathfrak{A}_n, D_n^{\to}, D_n^{\strictif})$ such that each $\mathfrak{A}_i$ is a finite HL-algebra, $D_i^{\to}, D_i^{\strictif} \subseteq A_i^2$, and for each HL-algebra $\mathfrak{B} = (B, \strictif)$, we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\strictif}$.

Proof. The proof is almost the same as that of Proposition 3.1.3. We just need to consider the strict implication case for induction, and check that the filtrated algebra $\mathfrak{B}'$ is a HL-algebra.

Let Γ/Δ be an arbitrary Heyting-Lewis multi-conclusion rule. If $\Gamma/\Delta \in \mathbf{iA}^R$, take $n = 0$. Suppose $\Gamma/\Delta \notin \mathbf{iA}^R$, let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$. Assume $|\Theta| = m$, there are only finitely many pairs $(\mathfrak{A}, D^{\to}, D^{\strictif})$ satisfying the following two conditions up to isomorphism:

i) $\mathfrak{A} = (A, \strictif)$ is a finite HL-algebra such that $\mathfrak{A}|_{\{\wedge,\vee,1,0\}}$ is at most m-generated as a bounded distributive lattice and $\mathfrak{A} \not\models \Gamma/\Delta$.

ii) $D^{\to} = \{(V(\varphi), V(\psi)) \mid \varphi \to \psi \in \Theta\}$ and $D^{\strictif} = \{(V(\varphi), V(\psi)) \mid \varphi \strictif \psi \in \Theta\}\}$ where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \Gamma/\Delta$.

Let $(\mathfrak{A}_1, D_1^{\to}, D_1^{\strictif}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\strictif})$ be the enumeration of such pairs. For any HL-algebra $\mathfrak{B} = (B, \strictif)$, we prove that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\strictif}$.

For the right-to-left direction, suppose there is $i \leq n$ and a stable bounded lattice embedding $h : A_i \to B$ satisfying CDC for $D_i^{\to}$ and $D_i^{\strictif}$. Define a valuation V_B on $\mathfrak{B}$ by $V_B(p) = h(V_i(p))$ for any propositional letter p where V_i is the valuation on $\mathfrak{A}$. We then prove by induction that $V_B(\psi) = h(V_i(\psi))$ for any $\psi \in \Theta$. We only consider the case when ψ is of the form $\varphi \dashv \chi$ as other cases are either trivial or have been considered in the proof of Proposition 3.1.3:

If $\varphi = \psi \dashv \chi$, then as $\varphi \dashv \chi \in \Theta$, $(V_i(\varphi), V_i(\chi)) \in D_i^{\dashv}$.

$$\begin{array}{rcll} V_B(\varphi \dashv \chi) & = & V_B(\varphi) \dashv V_B(\chi) & \\ & = & h(V_i(\varphi)) \dashv h(V_i(\varphi)) & \text{(IH)} \\ & = & h(V_i(\varphi) \dashv V_i(\chi)) & \text{(CDC)} \\ & = & h(V_i(\varphi \dashv \chi)). & \end{array}$$

Then it is easy to see that V_B refutes Γ/Δ on $\mathfrak{B}$.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \Gamma/\Delta$. There exists a valuation V_B on B such that $V_B(\gamma) = 1_B$ for any $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for any $\delta \in \Delta$. Let B' be the bounded sublattice of B generated by $V_B(\Theta) = \{V_B(\varphi) \mid \varphi \in \Theta\}$. Let $D^{\dashv} = \{(V_B(\varphi), V_B(\psi)) \mid \varphi \dashv \psi \in \Theta\}$ and $D^{\to} = \{(V_B(\varphi), V_B(\psi)) \mid \varphi \to \psi \in \Theta\}$, we define $\to'$ and $\dashv'$ on B' as follows: $a \to' b = \bigvee\{d \in B' \mid d \wedge a \leq b\}$ for any $a, b \in B'$; $a \dashv' b = \bigvee\{d \in B' \mid d \leq a \dashv b\}$ for any $a, b \in B'$.

We first check that $\mathfrak{B}' = (B', \to', \dashv')$ is a HL-algebra. Since we know that $(B', \to')$ is a Heyting algebra, we only need to check that $\dashv'$ satisfies the corresponding axioms. For any $a \in B'$, as $a \dashv a = 1$ and $1 \in B'$, $a \dashv' a = 1$ by definition.

For any $a, b, c \in B'$, $(a \dashv' b) \wedge (a \dashv' c)$
$= \bigvee\{d \in B' \mid d \leq a \dashv b\} \wedge \bigvee\{e \in B' \mid e \leq a \dashv c\}$
$= \bigvee\{d \wedge e \mid d, e \in B', d \leq a \dashv b \text{ and } e \leq a \dashv c\}$(distributivity)
$= \bigvee\{d \in B' \mid d \leq (a \dashv b) \wedge (a \dashv c)\}$
$= \bigvee\{d \in B' \mid d \leq a \dashv (b \wedge c)\}$(as $(a \dashv b) \wedge (a \dashv c) = a \dashv (b \wedge c)$)
$= a \dashv' (b \wedge c)$.

For any $a, b, c \in B'$, $(a \dashv' c) \wedge (b \dashv' c)$
$= \bigvee\{d \in B' \mid d \leq a \dashv c\} \wedge \bigvee\{e \in B' \mid e \leq b \dashv c\}$
$= \bigvee\{d \wedge e \mid d, e \in B', d \leq a \dashv c \text{ and } e \leq b \dashv c\}$(distributivity)
$= \bigvee\{d \in B' \mid d \leq (a \dashv c) \wedge (b \dashv c)\}$
$= \bigvee\{d \in B' \mid d \leq (a \vee b) \dashv c\}$(as $(a \dashv c) \wedge (b \dashv c) = (a \vee b) \dashv c$)
$= (a \vee b) \dashv' c$

For any $a, b, c \in B'$, $(a \dashv' b) \wedge (b \dashv' c)$
$= \bigvee\{d \in B' \mid d \leq a \dashv b\} \wedge \bigvee\{e \in B' \mid e \leq b \dashv c\}$
$= \bigvee\{d \wedge e \mid d, e \in B', d \leq a \dashv b \text{ and } e \leq b \dashv c\}$(distributivity)
$= \bigvee\{d \in B' \mid d \leq (a \dashv b) \wedge (b \dashv c)\}$
$\leq \bigvee\{d \in B' \mid d \leq a \dashv c\}$ (as $(a \dashv b) \wedge (b \dashv c) \leq (a \dashv c)$) $= a \dashv' c$.

Therefore, $(B', \to', \dashv')$ is a HL-algebra. Let $h : (B', \to', \dashv') \to (B, \dashv)$ be the inclusion map, h is clearly a bounded lattice embedding as B' is a bounded sublattice of B. As we checked in the proof of Proposition 3.1.3,

h satisfies CDC for $D^{\to}$. Besides, for any $a, b \in B'$, $a \dashv\!\!\!\rightarrow' b \leq a \dashv\!\!\!\rightarrow b$ by definition. h is thus stable.

For any $(a, b) \in D^{\dashv\!\!\!\rightarrow}$, $a = V_B(\varphi)$ and $b = V_B(\psi)$ for some $\varphi \dashv\!\!\!\rightarrow \psi \in \Theta$. $V_B(\varphi \dashv\!\!\!\rightarrow \psi) = V_B(\varphi) \dashv\!\!\!\rightarrow V_B(\psi) = a \dashv\!\!\!\rightarrow b \in B'$. Thus $a \dashv\!\!\!\rightarrow b' = a \dashv\!\!\!\rightarrow b$ by the definition of $\dashv\!\!\!\rightarrow'$. Therefore, the stable bounded lattice embedding h satisfies CDC for $D^{\to}$ and $D^{\dashv\!\!\!\rightarrow}$.

Let V' be the valuation V_B restricted to B', we then prove that for any $\varphi \in \Theta$, $V'(\varphi) = V_B(\varphi)$ by induction on φ. We only consider the following case as others are either trivial or have been considered in the proof of Proposition 3.1.3:

If $\varphi = \psi \dashv\!\!\!\rightarrow \chi$, as $\psi \dashv\!\!\!\rightarrow \chi \in \Theta$, it follows that $\psi, \chi \in \Theta$ and $(V_B(\psi), V_B(\chi)) \in D^{\dashv\!\!\!\rightarrow}$. Thus $V_B(\psi) \dashv\!\!\!\rightarrow V_B(\chi) \in B'$.

$$\begin{aligned} V'(\psi \dashv\!\!\!\rightarrow \chi) &= V'(\psi) \dashv\!\!\!\rightarrow' V'(\chi) \\ &= V_B(\psi) \dashv\!\!\!\rightarrow' V_B(\chi) \quad \text{(IH)} \\ &= V_B(\psi) \dashv\!\!\!\rightarrow V_B(\chi) \quad \text{(By the definition of } \dashv\!\!\!\rightarrow'\text{)} \\ &= V_B(\psi \dashv\!\!\!\rightarrow \chi). \end{aligned}$$

Since V_B is a valuation which refutes Γ/Δ on $\mathfrak{B}$, V' is a valuation which refutes Γ/Δ on $(B', \to', \dashv\!\!\!\rightarrow')$ by the above result. Thus, $(B', \to', \dashv\!\!\!\rightarrow') \not\models \Gamma/\Delta$. It is then easy to see that $(B', \to', \dashv\!\!\!\rightarrow', D^{\to}, D^{\dashv\!\!\!\rightarrow})$ must be one of $(\mathfrak{A}_1, D_1^{\to}, D_1^{\dashv\!\!\!\rightarrow}), ..., (\mathfrak{A}_n, D_n^{\to}, D_n^{\dashv\!\!\!\rightarrow})$. As h is a stable bounded lattice embedding from B' to B satisfying CDC for $D^{\to}$ and $D^{\dashv\!\!\!\rightarrow}$, we get what we want.

■

As for intuitionistic modal logics and bimodal logics, in this case too we can encode the refutation pattern in a syntactic form:

Definition 5.2.4. Let $\mathfrak{A} = (A, \dashv\!\!\!\rightarrow)$ be a finite HL-algebra, $D^{\to} \subseteq A^2$ and $D^{\dashv\!\!\!\rightarrow} \subseteq A^2$. For each $a \in A$, we introduce a new propositional letter p_a and define the *stable canonical rule* $\delta(\mathfrak{A}, D^{\to}, D^{\dashv\!\!\!\rightarrow})$ based on $(\mathfrak{A}, D^{\to}, D^{\dashv\!\!\!\rightarrow})$ as follows:

$$\begin{aligned} \Gamma \quad = \quad & \{p_{a \vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \{p_0 \leftrightarrow \bot, p_1 \leftrightarrow \top\} \\ & \cup \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \{p_{a \dashv\!\!\!\rightarrow b} \to (p_a \dashv\!\!\!\rightarrow p_b) \mid a, b \in A\} \\ & \cup \{(p_a \dashv\!\!\!\rightarrow p_b) \to p_{a \dashv\!\!\!\rightarrow b} \mid (a, b) \in D^{\dashv\!\!\!\rightarrow}\} \\ & \cup \{p_{a \to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D^{\to}\} \end{aligned}$$

$$\Delta = \{p_a \leftrightarrow p_b \mid a \neq b \in A\}$$

$$\delta(\mathfrak{A}, D^{\to}, D^{\dashv\!\!\!\rightarrow}) = \Gamma/\Delta.$$

As the counterparts in the setting of Heyting-Lewis logics to Propositions 3.1.5 and 3.1.6, the following results can be proved in the same way.

Proposition 5.2.5. Let $\mathfrak{A} = (A, \twoheadrightarrow)$ be a finite HL-algebra, $D^{\to} \subseteq A^2$ and $D^{\twoheadrightarrow} \subseteq A^2$, then $\mathfrak{A} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\twoheadrightarrow})$.

Proposition 5.2.6. Let $\mathfrak{A} = (A, \twoheadrightarrow)$ be a finite HL-algebra, $D^{\to} \subseteq A^2$, $D^{\twoheadrightarrow} \subseteq A^2$, and $\mathfrak{B} = (B, \twoheadrightarrow)$ be a HL-algebra. Then $\mathfrak{B} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\twoheadrightarrow})$ iff there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^{\to}$ and $D^{\twoheadrightarrow}$.

Proof. For the right-to-left direction, suppose there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^{\to}$ and $D^{\twoheadrightarrow}$. Define V_B on B by $V_B(p_a) = h(a)$ for any $a \in A$.

As h is stable, for any $a, b \in A$, we have that $h(a \twoheadrightarrow b) \leq h(a) \twoheadrightarrow h(b)$. Thus $V_B(p_{a \twoheadrightarrow b}) \leq V_B(p_a) \twoheadrightarrow V_B(p_b)$, $V_B(p_{a \twoheadrightarrow b} \to (p_a \twoheadrightarrow p_b)) = 1$. As h satisfies CDC for $D^{\twoheadrightarrow}$, for any $(a, b) \in D^{\twoheadrightarrow}$, $h(a \twoheadrightarrow b) = h(a) \twoheadrightarrow h(b)$. Thus for any $(a, b) \in D^{\twoheadrightarrow}$,

$$\begin{aligned} V_B((p_a \twoheadrightarrow p_b) \to p_{a \twoheadrightarrow b}) &= V_B(p_a \twoheadrightarrow p_b) \to V_B(p_{a \twoheadrightarrow b}) \\ &= (V_B(p_a) \twoheadrightarrow V_B(p_b)) \to V_B(p_{a \twoheadrightarrow b}) \\ &= (h(a) \twoheadrightarrow h(b)) \to h(a \twoheadrightarrow b) \\ &= 1. \end{aligned}$$

Then by what we have checked in the proof of Proposition 3.1.6, we know that for any $\gamma \in \Gamma$, $V_B(\gamma) = 1$ while for any $\delta \in \Delta$, $V_B(\delta) \neq 1$. Thus V_B refutes Γ/Δ on $\mathfrak{B}$, and $\mathfrak{B} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\twoheadrightarrow})$.

For the left-to-right direction, suppose $\mathfrak{B} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\twoheadrightarrow})$. There exists a valuation V on B such that $V(\gamma) = 1$ for any $\gamma \in \Gamma$, and $V(\delta) \neq 1$ for any $\delta \in \Delta$. Define $h : A \to B$ by $h(a) = V(p_a)$ for any $a \in A$. Then by the proof of the Proposition 3.1.6, we know that h is a bounded lattice embedding which satisfies CDC for $D^{\to}$.

For any $a, b \in A$, as $V(p_{a \twoheadrightarrow b} \to (p_a \twoheadrightarrow p_b)) = 1$, it follows that $V(p_{a \twoheadrightarrow b}) \leq V(p_a) \twoheadrightarrow V(p_b)$. Thus $h(a \twoheadrightarrow b) = V(p_{a \twoheadrightarrow b}) \leq V(p_a) \twoheadrightarrow V(p_b) = h(a) \twoheadrightarrow h(b)$, h is stable.

For any $(a, b) \in D^{\twoheadrightarrow}$, as $V((p_a \twoheadrightarrow p_b) \to p_{a \twoheadrightarrow b}) = 1$, $V(p_a) \twoheadrightarrow V(p_b) \leq V(p_{a \twoheadrightarrow b})$. Thus $h(a) \twoheadrightarrow h(b) = V(p_a) \twoheadrightarrow V(p_b) \leq V(p_{a \twoheadrightarrow b}) = h(a \twoheadrightarrow b)$. $h(a) \twoheadrightarrow h(b) = h(a \twoheadrightarrow b)$ for any $(a, b) \in D^{\twoheadrightarrow}$. Therefore, h is a stable bounded lattice embedding which satisfies CDC for $D^{\to}$ and $D^{\twoheadrightarrow}$.

■

Combining Propositions 5.2.4 and 5.2.6 together, we obtain immediately that every Heyting-Lewis multi-conclusion consequence relations can be axiomatized by stable canonical rules.

Theorem 5.2.7. *For a Heyting-Lewis multi-conclusion rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1^{\to}, D_1^{⥽})$, ..., $(\mathfrak{A}_n, D_n^{\to}, D_n^{⥽})$ such that each $\mathfrak{A}_i$ is a finite HL-algebra, $D_i^{\to} \subseteq A_i^2$ and $D_i^{⥽} \subseteq A_i^2$, and for each HL-algebra $\mathfrak{B} = (B, ⥽)$, we have $\mathfrak{B} \models \Gamma/\Delta$ iff $\mathfrak{B} \models \delta(\mathfrak{A}_1, D_1^{\to}, D_1^{⥽})$, ..., $\delta(\mathfrak{A}_n, D_n^{\to}, D_n^{⥽})$.*

Theorem 5.2.8. *Every Heyting-Lewis multi-conclusion consequence relation is axiomatizable by stable canonical rules.*

Proof. By Theorem 5.2.7, the proof is similar to that for Theorem 3.1.8. ■

We proceed by obtaining a dual characterization of the stable canonical rules for Heyting-Lewis logics. We start with the duality between the category of HL-algebras and the category of strict implication spaces given in [13].

Definition 5.2.9. Let $(X, \leq, R)$ be a triple such that $(X, \leq)$ is an Esakia space and $R \subseteq X \times X$, then $(X, \leq, R)$ is a *strict implication space* if the following conditions hold:

- $R = \leq \circ R$
- For every $x \in X$, the set $R[x]$ is a closed upset.
- If U is a clopen subset of X, then $R^{-1}[U]$ is clopen as well[4].

The morphisms between strict implication spaces are defined as follows:

Definition 5.2.10. Let $(X_1, \leq, R_1)$ and $(X_2, \leq, R_2)$ be strict implication spaces, a map $f : X_1 \to X_2$ is called a $⥽$*-morphism* if the following conditions are satisfied for any $x, x', y \in X_1$ and $z \in X_2$:

1. f is continuous.
2. If $x \leq y$, then $f(x) \leq f(y)$.
3. If $f(x) \leq z$, then $f(x') = z$ for some $x \leq x'$.
4. If xR_1y, then $f(x)R_2f(y)$.
5. If $f(x)R_2z$, then $f(x') = z$ for some xR_1x'.

[4]Equivalently, $\Box_R[U] = \{x \mid R[x] \subseteq U\}$ is clopen for every clopen $U \subseteq X$.

Let **HLA** be the category of HL-algebras and HL-algebra homomorphisms, **SIS** be the category of strict implication spaces and $\prec$-morphisms, the functors $(-)_* : \mathbf{HLA} \to \mathbf{SIS}$ and $(-)^* : \mathbf{SIS} \to \mathbf{HLA}$ that establish the duality are constructed as follows. For a HL-algebra $\mathfrak{A} = (A, \prec)$, let $\mathfrak{A}_* = (A_*, R)$ where A_* is the dual Esakia space of A and xRy iff $\forall a, b \in A((a \prec b) \in x \text{ and } a \in y \implies b \in y)$. For a strict implication space $\mathcal{X} = (X, \leq, R)$, let $\mathcal{X}^* = (X^*, \prec_R)$ where X^* is the Heyting algebra of clopen upsets of X and $U \prec_R V = \{x \in X \mid R[x] \cap U \subseteq V\}$ for any $U, V \in X^*$. The duals of maps are exactly the same as that in Esakia duality. We spell out some useful details about the duality in the following theorem.

Theorem 5.2.11. *[13, Thm. 4.15]* **HLA** *is dually equivalent to* **SIS**, *which is witnessed by $(-)^*$ and $(-)_*$. In particular, for any HL-algebra $\mathfrak{A}$, $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ witnessed by β where $\beta(a) = \{x \in A_* \mid a \in x\}$, and for any strict implication space $\mathcal{X}$, $\mathcal{X} \cong (\mathcal{X}^*)_*$ witnessed by ϵ where $\epsilon(x) = \{U \in X^* \mid x \in U\}$.*

Next we recall the definition of stable maps. This time they are defined between strict implication spaces.

Definition 5.2.12. Let $(X, \leq, R)$ and $(Y, \leq, R)$ be strict implication spaces and $f : X \to Y$ be a Priestley morphism, f is *stable* if for any $x, y \in X$, xRy implies $f(x)Rf(y)$.

The following proposition is the counterpart to Proposition 3.3.6.

Proposition 5.2.13. Let $\mathfrak{A} = (A, \prec)$, $\mathfrak{B} = (B, \prec)$ be HL-algebras, $(X_A, \leq, R)$ and $(X_B, \leq, R)$ be the dual of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. For a bounded lattice homomorphism $h : A \to B$, h is stable iff $h_* : X_B \to X_A$ is stable.

Proof. By Priestley duality, h_* is a Priestley morphism. Thus, it suffices to prove that $h(a \prec b) \leq h(a) \prec h(b)$ for any $a, b \in A$ iff xRy implies that $h_*(x)Rh_*(y)$ for any $x, y \in X_B$.

For the left-to-right direction, suppose $h(a \prec b) \leq h(a) \prec h(b)$ for any $a, b \in A$. Let $x, y \in X_B$ be arbitrary, suppose xRy, $a \prec b \in h_*(x)$ and $a \in h_*(y)$ where $a, b \in A$. Then $h(a \prec b) \in x$ and $h(a) \in y$. As $h(a \prec b) \leq h(a) \prec h(b)$, $h(a) \prec h(b) \in x$, and thus $h(b) \in y$, namely $b \in h_*(y)$. This means that $h_*(x)Rh_*(y)$. Thus for any $x, y \in X_B$, we have that xRy implies that $h_*(x)Rh_*(y)$

For the other direction, suppose for any $x, y \in X_B$, xRy implies that $h_*(x)Rh_*(y)$. Suppose $x \in \beta(h(a \prec b))$, then $h(a \prec b) \in x$. For any

$y \in R[x] \cap \beta(h(a))$, we have that xRy and $h(a) \in y$. By the assumption, $h_*(x)Rh_*(y)$. As $h(a \multimap b) \in x$, it follows that $a \multimap b \in h_*(x)$. As $h(a) \in y$, we have that $a \in h_*(y)$. Thus $b \in h_*(y)$, and $h(b) \in y$. Therefore, $R[x] \cap \beta(h(a)) \subseteq \beta(h(b))$. By definition, $x \in \beta(h(a)) \multimap_R \beta(h(b))$. As $x \in \beta(h(a \multimap b))$ is arbitrary, this proves that $\beta(h(a \multimap b)) \subseteq \beta(h(a)) \multimap_R \beta(h(b))$. Since β is an isomorphism by Theorem 5.2.11, $h(a \multimap b) \leq h(a) \multimap h(b)$.

■

Then we show what the closed domain condition for $D^{\multimap}$ means dually.

Definition 5.2.14. Let $(X, \leq, R)$ and $(Y, \leq, R)$ be strict implication spaces, $f : X \to Y$ be a Priestley morphism, and D be a clopen subset of Y. We say that f satisfies the *strict implication closed domain condition* ($\mathrm{CDC}_{\multimap}$) for D if the following holds:

$$R[f(x)] \cap D \neq \emptyset \text{ implies } f[R[x]] \cap D \neq \emptyset.$$

Furthermore, let $\mathcal{D}$ be a collection of clopen subsets of Y, f satisfies the *strict implication closed domain condition* ($\mathrm{CDC}_{\multimap}$) for $\mathcal{D}$ if f satisfies ($\mathrm{CDC}_{\multimap}$) for each $D \in \mathcal{D}$.

Proposition 5.2.15. Let $\mathfrak{A} = (A, \multimap)$ and $\mathfrak{B} = (B, \multimap)$ be HL-algebras, $h : A \to B$ be a stable bounded lattice homomorphism, and $a, b \in A$, then the following are equivalent:

1. $h(a \multimap b) = h(a) \multimap h(b)$.
2. $h_* : X_B \to X_A$ satisfies $\mathrm{CDC}_{\multimap}$ for $\beta(a) \setminus \beta(b)$.

Proof. As h is stable and β is an isomorphism, $h(a \multimap b) = h(a) \multimap h(b)$ iff $h(a) \multimap h(b) \leq h(a \multimap b)$ iff $\beta(h(a)) \multimap_R \beta(h(b)) \subseteq \beta(h(a \multimap b))$.

Suppose $h(a \multimap b) = h(a) \multimap h(b)$, thus $\beta(h(a)) \multimap_R \beta(h(b)) \subseteq \beta(h(a \multimap b))$. Suppose $R[h_*(x)] \cap \beta(a) \setminus \beta(b) \neq \emptyset$, then there exists $y \in X_A$ such that $h_*(x)Ry$ and $y \in \beta(a) \setminus \beta(b)$. Namely $a \in y$ while $b \notin y$. By [13, Lem. 3.13], $a \multimap b \notin h_*(x)$, thus $h(a \multimap b) \notin x$, namely $x \notin \beta(h(a \multimap b))$. Since $\beta(h(a)) \multimap_R \beta(h(b)) \subseteq \beta(h(a \multimap b))$, it follows that $x \notin \beta(h(a)) \multimap_R \beta(h(b))$. By definition, $R[x] \cap \beta(h(a)) \nsubseteq \beta(h(b))$, namely there exists $z \in X_B$ such that xRz and $z \in \beta(h(a))$ while $z \notin \beta(h(b))$. Thus $h(a) \in z$ while $h(b) \notin z$, namely $a \in h_*(z)$ while $b \notin h_*(z)$. This means that $h_*[R[x]] \cap \beta(a) \setminus \beta(b) \neq \emptyset$. This proves that h_* satisfies $\mathrm{CDC}_{\multimap}$ for $\beta(a) \setminus \beta(b)$.

For the other direction, suppose h_* satisfies $\mathrm{CDC}_{\multimap}$ for $\beta(a) \setminus \beta(b)$. Suppose $x \in \beta(h(a)) \multimap_R \beta(h(b))$, then $R[x] \cap \beta(h(a)) \subseteq \beta(h(b))$. Namely, for any xRy, if $h(a) \in y$, then $h(b) \in y$. Thus $h_*[R[x]] \cap \beta(a) \setminus \beta(b) = \emptyset$. As h_* satisfies CDC$_{\dashv}$ for $\beta(a) \setminus \beta(b)$, we have that $R[h_*(x)] \cap \beta(a) \setminus \beta(b) = \emptyset$. Thus, $R[h_*(x)] \cap \beta(a) \subseteq \beta(b)$, and $h_*(x) \in \beta(a) \dashv_R \beta(b) = \beta(a \dashv b)$, namely $a \dashv b \in h_*(x)$. We get that $x \in \beta(h(a \dashv b))$. As $x \in \beta(h(a)) \dashv_R \beta(h(b))$ is arbitrary, this proves that $\beta(h(a)) \dashv_R \beta(h(b)) \subseteq \beta(h(a \dashv b))$. Therefore, $h(a \dashv b) = h(a) \dashv h(b)$. ■

Combining Propositions 5.2.13 and 5.2.15 together, we obtain the following dual description of the stable canonical rules for Heyting-Lewis logics:

Proposition 5.2.16. Let $\mathfrak{A} = (A, \dashv)$ be a finite HL-algebra, $D^{\to} \subseteq A^2$, $D^{\dashv} \subseteq A^2$, and $\mathfrak{B} = (B, \dashv)$ be a HL-algebra. Then $\mathfrak{B} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\dashv})$ iff there is a surjective stable Priestley morphism $f : X_B \to X_A$ satisfying CDC$_{\to}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\to}$ and CDC$_{\dashv}$ for any $\beta(a) \setminus \beta(b)$ where $(a, b) \in D^{\dashv}$.

Proof. By Proposition 5.2.6, $\mathfrak{B} \not\models \delta(\mathfrak{A}, D^{\to}, D^{\dashv})$ iff there is a stable bounded lattice embedding $h : A \to B$ satisfying CDC for $D^{\to}$ and $D^{\dashv}$. Then use Propositions 5.2.13 and 5.2.15.

■

5.3 Towards the Blok-Esakia theorem for Heyting-Lewis logics

Having developed the basic theory of the stable canonical rules for Heyting-Lewis logics, one may wonder whether we can repeat what we did in the Chapter 4. It is indeed true that we have an analogue of the Gödel translation in this setting. Thus it is natural to ask whether we can use stable canonical rules to prove the Blok-Esakia theorem and the Dummett-Lemmon conjecture for Heyting-Lewis logics.

In this section, we will follow the path of the last chapter as much as possible. However, this will in the end lead us to an intrinsic problem in proving the counterpart to Lemma 4.1.19. Furthermore, this observation helps us find an error in the proof of the Blok-Esakia theorem for Heyting-Lewis logics given in [23]. So far, this error has not been fixed and the observation gives us reasons to believe that the Blok-Esakia theorem may not hold in this case.

We start with the operations from strict implication spaces to bimodal spaces and vice versa.

For any $S4 \otimes K$-modal space $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$, we write $x \backsim y$ iff $xR_I y$ and $yR_I y$, and then define $\rho : Y \to \mathcal{P}(Y)$ by $\rho(x)$[5] $= \{y \in Y \mid x \backsim y\}$. We get the following definition analogous to Definition 4.1.10.

Definition 5.3.1.

- For any strict implication space $\mathcal{X} = (X, \leq, R, \mathcal{O})$ where $\mathcal{O}$ is the topology, we set $\sigma(\mathcal{X}) = (X, R_I, R_M, \mathcal{O})$ where $R_I =\leq$ and $R_M = R$.
- For any $S4 \otimes K$-modal space $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$, we set $\rho(\mathfrak{Y}) = (\rho[Y], \leq, [R_I \circ R_M], \rho[\mathcal{O}])$ where $\rho(x) \leq \rho(y)$ iff $xR_I y$, $\rho(x)[R_I \circ R_M]\rho(y)$ iff $xR_I \circ R_M y'$ for some $y' \backsim y$[6] and $\rho(\mathcal{O})$ is the quotient topology.

The only difference between the above definition and Definition 4.1.10 is that the relation R in $\rho(\mathfrak{Y})$ is defined as $[R_I \circ R_M]$ instead of $[R_I \circ R_M \circ R_M]$.

Proposition 5.3.2. The following hold:

1) For any strict implication space $\mathcal{X}$, we have that $\sigma(\mathcal{X})$ is an $S4 \otimes K$-modal space.

2) For any $S4 \otimes K$-modal space $\mathfrak{Y}$, we have that $\rho(\mathfrak{Y})$ is a strict implication space.

Proof. For 1), as $\mathcal{X} = (X, \leq, R, \mathcal{O})$ is an strict implication space, $(X, \leq, \mathcal{O})$ is an Esakia space. If we omit R, then $\sigma(\mathcal{X})$ is defined exactly the same as that in [16, Def. 2.43], which is well-known to be an S4-modal space. Then by the definition of strict implication spaces, $\sigma(\mathcal{X})$ is an $S4 \otimes K$-modal space.

For 2), let $\mathfrak{Y} = (Y, R_I, R_M, \mathcal{O})$ be an $S4 \otimes K$-modal space. If we omit R_M, then $\rho(\mathfrak{Y})$ is defined exactly the same as that in [16, Def. 2.43], which is well-known to be an Esakia space. Thus it suffices to check that for any clopen subset U of $\rho(\mathfrak{Y})$, $\Box_{[R_I \circ R_M]} U$ is clopen, and for any $x \in Y$, $[R_I \circ R_M][\rho(x)]$ is a closed.

Let U be an arbitrary subset of $\rho(\mathfrak{Y})$, for any $y \in Y$, $y \in \rho^{-1}(\Box_{[R_I \circ R_M]} U)$ iff $\rho(y) \in \Box_{[R_I \circ R_M] U}$ iff $[R_I \circ R_M][\rho(y)] \subseteq U$. By the definition of $[R_I \circ R_M]$, $[R_I \circ R_M][\rho(y)] \subseteq U$ iff $R_M[R_I[y]] \subseteq \rho^{-1}[U]$ iff $y \in \Box_{R_I} \Box_{R_M} \rho^{-1}[U]$. Thus $\rho^{-1}(\Box_{[R_I \circ R_M]} U) = \Box_{R_I} \Box_{R_M} \rho^{-1}[U]$. As U is clopen, so is $\rho^{-1}[U]$. As $\mathfrak{Y}$ is a modal space, $\Box_{R_I} \Box_{R_M} \rho^{-1}[U]$ is clopen as well. Thus $\Box_{[R_I \circ R_M]} U$ is clopen by the quotient topology.

[5] We may also write $[x]$ for $\rho(x)$.

[6] It is easy to check that they are well defined.

For any $x \in Y$, define $Z = R_M[R_I[x]]$, then $[R_I \circ R_M][\rho(x)] = \rho[Z]$ by definition. Let $\{W_i \mid i \in I\}$ be an open cover of $[R_I \circ R_M][\rho(x)]$, as we know that $\rho(\mathfrak{Y})$ is an Esakia space, by Remark 2.6.6 we can assume that for any $i \in I$, $W_i = U_i \backslash V_i$ where U_i, V_i are clopen upsets of $\rho(\mathfrak{Y})$. Thus $[R_I \circ R_M][\rho(x)] = \rho[Z] \subseteq \bigcup_{i \in I}(U_i \setminus V_i)$. And $Z \subseteq \rho^{-1}[\bigcup_{i \in I}(U_i \setminus V_i)] = \bigcup_{i \in I}(\rho^{-1}(U_i) \setminus \rho^{-1}(V_i))$ as U_i, V_i are upsets. Since $\mathfrak{Y}$ is an $S4 \otimes K$-modal space, $Z = R_M[R_I[x]]$ is closed and so is compact. Therefore, as $\rho^{-1}(U_i) \setminus \rho^{-1}(V_i)$'s are open sets, there exist $i_1, ..., i_n$ such that $Z \subseteq (\rho^{-1}(U_{i_1}) \setminus \rho^{-1}(V_{i_1})) \cup ... \cup (\rho^{-1}(U_{i_n}) \setminus \rho^{-1}(V_{i_n}))$. Thus $[R_I \circ R_M][\rho(x)] = \rho[Z] \subseteq (U_{i_1} \setminus V_{i_1}) \cup ... \cup (U_{i_n} \setminus V_{i_n})$. Therefore, $[R_I \circ R_M][\rho(x)]$ is compact. As $\rho(\mathfrak{Y})$ is an Esakia space and thus a Hausdorff space, $[R_I \circ R_M][\rho(x)]$ is closed.

■

Now, using Theorem 5.2.11, we can give the dual versions of σ and ρ as well. Let $\mathfrak{A}$ be an arbitrary HL-algebra, $\mathfrak{A}_*$ is a strict implication space by the duality given in Theorem 5.2.11. $\sigma(\mathfrak{A}_*)$ is an $S4 \otimes K$-modal space by Proposition 5.3.2. By Theorem 4.1.8, $\sigma(\mathfrak{A}_*)^*$ is an $S4 \otimes K$-algebra. We set $\sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_*)^*$ for any HL-algebra $\mathfrak{A}$. Then σ is a map from the class of HL-algebras to the class of $S4 \otimes K$-algebras. Similarly, we define $\rho(\mathfrak{B}) = \rho(\mathfrak{B}_*)^*$ for any $S4 \otimes K$-algebra $\mathfrak{B}$ where the ρ on the right-hand side is the one in Definition 5.3.1. The ρ on the left-hand side is then a map from the class of $S4 \otimes K$-algebras to the class of HL-algebras.

The following two propositions are then obvious:

Proposition 5.3.3. The following hold:

1. For any HL-algebra $\mathfrak{A}$, $(\sigma(\mathfrak{A}))_* \cong \sigma(\mathfrak{A}_*)$. Consequently, $\sigma((\mathcal{X})^*) \cong (\sigma(\mathcal{X}))^*$ for any strict implication space $\mathcal{X}$.
2. For any $S4 \otimes K$-algebra $\mathfrak{B}$, $(\rho(\mathfrak{B}))_* \cong \rho(\mathfrak{B}_*)$. Consequently, $(\rho(\mathfrak{X}))^* \cong \rho(\mathfrak{X}^*)$ for any $S4 \otimes K$-modal space $\mathfrak{X}$.

Proposition 5.3.4. For any strict implication space $\mathcal{X} = (X, \leq, R, \mathcal{O})$, $\rho(\sigma(\mathcal{X})) \cong \mathcal{X}$. Consequently, for any HL-algebra $\mathfrak{A}$, $\rho(\sigma(\mathfrak{A})) \cong \mathfrak{A}$.

Let **wMix**[7] denote the formula $\Box_I \Box_M p \leftrightarrow \Box_M p$, we then have the following counterpart to Proposition 4.1.15.

Proposition 5.3.5. If $\mathfrak{B} = (B, \Box_I, \Box_M)$ is an $S4 \otimes K \oplus wMix$-algebra, then there is a homomorphic embedding of $\sigma(\rho(\mathfrak{B}))$ into $\mathfrak{B}$.

[7] "w" stands for "weak" as it is weaker than **Mix**.

Proof. Let $\mathfrak{B}$ be an $S4 \otimes K$-algebra which validates $\Box_I \Box_M p \leftrightarrow \Box_M p$. By duality, $\mathfrak{B}_*$ is an $S4 \otimes K$-modal space on which $R_I \circ R_M = R_M$, and it suffices to show that there is a surjective continuous bounded morphism from $\mathfrak{B}_*$ onto $(\sigma(\rho(\mathfrak{B})))_*$. By Proposition 5.3.4, $(\sigma(\rho(\mathfrak{B})))_* \cong \sigma(\rho(\mathfrak{B}_*))$. We show that $x \mapsto [x]$ is a surjective continuous bounded morphism from $\mathfrak{X} = (X, R_I, R_M, \mathcal{O})$ onto $\sigma(\rho(\mathfrak{X}))$ for any $S4 \otimes K$-modal space $\mathfrak{X}$ on which $R_I \circ R_M = R_M$.

We only need to check the conditions about R_M as others are the same as those in the proof of Proposition 4.1.15: suppose $xR_M y$, then as R_I is reflexive, $xR_I \circ R_M y$ and thus $[x][R_I \circ R_M][y]$; suppose $[x][R_I \circ R_M][y]$, then $xR_I \circ R_M y'$ for some $y' \backsim y$ by definition. As $R_I \circ R_M = R_M$, we have that $xR_M y'$ and $[y'] = [y]$. ■

Now, we can introduce the *Gödel translation* for Heyting-Lewis logics.

Definition 5.3.6. The *Gödel translation* for Heyting-Lewis logics $t : Form_{hl} \to Form_{bi}$ is recursively defined as follows:

- $t(p) = \Box_I p$ where p is a propositional variable
- $t(\bot) = \Box_I \bot$
- $t(\top) = \Box_I \top$
- $t(\varphi \to \psi) = \Box_I(t(\varphi) \to t(\psi))$
- $t(\varphi \wedge \psi) = \Box_I(t(\varphi) \wedge t(\psi))$
- $t(\varphi \vee \psi) = \Box_I(t(\varphi) \vee t(\psi))$
- $t(\varphi \dashv\!\!3\, \psi) = \Box_I \Box_M(t(\varphi) \to t(\psi))$

As mentioned below Definition 5.1.1, $\dashv\!\!3$ has a modal flavor. Thus in the above translation, the clause for $\dashv\!\!3$ is quite natural. Besides, it turns out that the above translation is "good" in the following sense:

Theorem 5.3.7. *Let $\mathfrak{A}$ be an $S4 \otimes K$-algebra. Then $\mathfrak{A} \vDash t(\Gamma/\Delta)$ iff $\rho(\mathfrak{A}) \vDash \Gamma/\Delta$.*

Proof. By Theorem 5.2.11, $\mathfrak{A} \vDash \Gamma/\Delta$ iff $\mathfrak{A}_* \vDash \Gamma/\Delta$, and $\rho(\mathfrak{A}) \vDash \Gamma/\Delta$ iff $\rho(\mathfrak{A}_*) = \rho(\mathfrak{A})_* \vDash \Gamma/\Delta$. Thus it suffices to prove that for any $S4 \otimes K$-modal space $\mathfrak{X} = (X, R_I, R_M, \mathcal{O})$, we have that $(X, R_I, R_M, \mathcal{O}) \vDash t(\Gamma/\Delta)$ iff $\rho(X, R_I, R_M, \mathcal{O}) \vDash \Gamma/\Delta$.

For the left-to-right direction, suppose $\rho(X, R_I, R_M, \mathcal{O}) \not\models \Gamma/\Delta$, then there is a valuation V on $\rho(X, R_I, R_M, \mathcal{O})$ such that $V(\varphi) = \rho[X]$ for any $\varphi \in \Gamma$ and $V(\psi) \neq \rho[X]$ for any $\psi \in \Delta$. Since $V(p)$ is a clopen upset in $\rho(X, R_I, R_M, \mathcal{O})$ for any propositional variable p, it follows that $\rho^{-1}[V(p)]$ is a clopen upset in $(X, R_I, R_M, \mathcal{O})$. We define a valuation V' on $(X, R_I, R_M, \mathcal{O})$ by setting $V'(p) = \rho^{-1}[V(p)]$. Then we can prove by induction that for any formula φ, $V'(t(\varphi)) = \rho^{-1}[V(\varphi)]$ (note that $V(\varphi)$ is a clopen upset for any formula φ).

We only consider the case about $-\!\!\!\!3$ as others are the same as those in the proof of Theorem 4.1.17:

If $\varphi = \psi -\!\!\!\!3\, \theta$,

$$\begin{aligned} V'(t(\varphi)) &= V'(\Box_I\Box_M(t(\psi) \to t(\theta))) \\ &= \Box_I\Box_M V'(t(\psi) \to t(\theta)) \\ &= \Box_I\Box_M(X \setminus V'(t(\psi)) \cup V'(t(\theta))) \\ &= \Box_I\Box_M(X \setminus \rho^{-1}[V(\psi)] \cup \rho^{-1}[V(\theta)]) \quad \text{(IH)} \\ &= \Box_I\Box_M(\rho^{-1}[\rho[X] \setminus V(\psi) \cup V(\theta)]) \end{aligned}$$

Now, $\rho^{-1}[V(\varphi)] = \rho^{-1}[V(\psi -\!\!\!\!3\, \theta)] = \rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)]$. For any $a \in \rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)]$, we have that $\rho(a) \in V(\psi) -\!\!\!\!3\, V(\theta)$. Thus $[R_I \circ R_M][\rho(a)] \cap V(\psi) \subseteq V(\theta)$. For any $b \in R_M[R_I[a]]$, $aR_I \circ R_M b$, and thus $\rho(a)[R_I \circ R_M]\rho(b)$. As $[R_I \circ R_M][\rho(a)] \cap V(\psi) \subseteq V(\theta)$, we have that $\rho(b) \in \rho[X] \setminus V(\psi) \cup V(\theta)$. Therefore, $a \in \Box_I\Box_M(\rho^{-1}[\rho[X] \setminus V(\psi) \cup V(\theta)])$. As $a \in \rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)]$ is arbitrary, this proves that $\rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)] \subseteq \Box_I\Box_M(\rho^{-1}[\rho[X] \setminus V(\psi) \cup V(\theta)])$.

For any $a \notin \rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)]$, we have that $\rho(a) \notin V(\psi) -\!\!\!\!3\, V(\theta)$, and thus $[R_I \circ R_M][\rho(a)] \cap V(\psi) \not\subseteq V(\theta)$. Therefore, there exists $b \in X$ such that $\rho(a)[R_I \circ R_M]\rho(b)$, and $\rho(b) \in V(\psi)$ while $\rho(b) \notin V(\theta)$. By definition, there exists $b' \in X$ such that $b \backsim b'$ and $aR_I \circ R_M b'$. Then $\rho(b') = \rho(b) \notin \rho[X] \setminus V(\psi) \cup V(\theta)$. Thus $a \notin \Box_I\Box_M(\rho^{-1}[\rho[X] \setminus V(\psi) \cup V(\theta)])$. Therefore, $\Box_I\Box_M(\rho^{-1}[\rho[X] \setminus V(\psi) \cup V(\theta)]) \subseteq \rho^{-1}[V(\psi) -\!\!\!\!3\, V(\theta)]$. This proves that $V'(t(\psi -\!\!\!\!3\, \theta)) = \rho^{-1}[V(\psi -\!\!\!\!3\, \theta)]$.

Then we can check V' is a valuation on $(X, R_I, R_M, \mathcal{O})$ which refutes $t(\Gamma/\Delta)$ as we did in the proof of Theorem 4.1.17. Thus $(X, R_I, R_M, \mathcal{O}) \not\models t(\Gamma/\Delta)$.

For the other direction, suppose $(X, R_I, R_M, \mathcal{O}) \not\models t(\Gamma/\Delta)$, there is a valuation V on $(X, R_I, R_M, \mathcal{O})$ such that $V(t(\varphi)) = X$ for any $\varphi \in \Gamma$ and $V(t(\psi)) \neq X$ for any $\psi \in \Delta$. Define $V'(p) = \rho[V(\Box_I p)] = \rho[\Box_I V(p)]$ for any propositional variable p. As $\Box_I V(p)$ is a clopen upset in $(X, R_I, R_M, \mathcal{O})$, so $\rho[\Box_I V(p)]$ is a clopen upset in $\rho(X, R_I, R_M, \mathcal{O})$. Thus V' is indeed a valuation on $\rho(X, R_I, R_M, \mathcal{O})$. We prove by induction that $V'(\varphi) = \rho[V(t(\varphi))]$

for any formula φ. Again we only need to check the case about $\multimap$:

If $\varphi = \psi \multimap \theta$, then $V'(\varphi) = V'(\psi \multimap \theta) = V'(\psi) \multimap V'(\theta) =$ (IH)$\rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))]$. And we have that $\rho[V(t(\varphi))] = \rho[V(\Box_I \Box_M(t(\psi) \to t(\theta)))] = \rho[\Box_I \Box_M(V(t(\psi)) \to V(t(\theta)))] = \rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))]$.

Now for any $a \in \rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))]$, there exists $x \in \Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))$ such that $a = \rho(x)$. For any $\rho(x)[R_I \circ R_M]\rho(y)$, there exists $y' \backsim y$ such that $x R_I \circ R_M y'$. If $\rho(y) \in \rho[V(t(\psi))]$, then $y' \in V(t(\psi))$ as $V(t(\psi))$ is an upset. As $x \in \Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))$, we have that $y' \in V(t(\theta))$, and thus $\rho(y) = \rho(y') \in \rho[V(t(\theta))]$. This means that $a = \rho(x) \in \rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))]$. As $a \in \rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))]$ is arbitrary, we obtain that $\rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))] \subseteq \rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))]$.

For any $\rho(y) \in \rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))]$, we have that $[R_I \circ R_M][\rho(y)] \cap \rho[V(t(\psi))] \subseteq \rho[V(t(\theta))]$. For any $y R_I \circ R_M x$, by definition $\rho(y)[R_I \circ R_M]\rho(x)$. If $x \in V(t(\psi))$, then $\rho(x) \in \rho[V(t(\psi))]$, and thus $\rho(x) \in \rho[V(t(\theta))]$. As $V(t(\theta))$ is an upset, $x \in V(t(\theta))$. Therefore, $R_M[R_I[y]] \subseteq X \setminus V(t(\psi)) \cup V(t(\theta))$, and $y \in \Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))$. Thus $\rho(y) \in \rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))]$. As $\rho(y) \in \rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))]$ is arbitrary, we have that $\rho[V(t(\psi))] \multimap_{[R_I \circ R_M]} \rho[V(t(\theta))] \subseteq \rho[\Box_I \Box_M(X \setminus V(t(\psi)) \cup V(t(\theta)))]$. Thus $V'(\psi \multimap \theta) = \rho[V(t(\psi \multimap \theta))]$.

Then it is easy to check that V' is a valuation on $\rho(X, R_I, R_M, \mathcal{O})$ which refutes Γ/Δ, $\rho(X, R_I, R_M, \mathcal{O}) \not\models \Gamma/\Delta$.

■

The above proposition is the counterpart to Proposition 4.1.17 and a natural generalization of [23, Lem. 4.13] to multi-conclusion rules.

Now, we are ready to define *modal companions* for Heyting-Lewis logics in the same way as we did in Definition 4.1.18.

Definition 5.3.8. Let $L \in \mathbf{Ext}(\mathbf{iA}^R)$ and $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R)$. Then M is a *modal companion* of L if $\Gamma/\Delta \in L \iff t(\Gamma/\Delta) \in M$. Moreover, let $L \in \mathbf{Ext}(\mathbf{iA})$ and $M \in \mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K})$, then M is a *modal companion* of L if $\varphi \in L \iff t(\varphi) \in M$.

The following maps between the lattices $\mathbf{Ext}(\mathbf{iA}^R)$ and $\mathbf{NExt}(\mathbf{S4} \otimes \mathbf{K}^R)$ can also be defined similarly to intuitionistic modal logics:

$$\sigma : \mathbf{Ext}(\mathbf{iA}^R) \to \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix}^R)$$

$$\sigma(L) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix}^R \oplus \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in L\}$$

$$\rho : \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix}^R) \to \mathbf{Ext}(\mathbf{iA}^R)$$
$$\rho(M) = \{\Gamma/\Delta \mid t(\Gamma/\Delta) \in M\}.$$

For logics, these maps are as follows:

$$\sigma : \mathbf{Ext}(\mathbf{iA}) \to \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix})$$
$$\sigma(L) = \mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix} \oplus \{t(\varphi) \mid \varphi \in L\}$$

$$\rho : \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K}) \to \mathbf{Ext}(\mathbf{iA})$$
$$\rho(M) = \{\varphi \mid t(\varphi) \in M\}.$$

Now in order to repeat our proof strategy for the Blok-Esakia theorem in Chapter 4, the only missing piece is the counterpart of Lemma 4.1.19. Indeed, we can even obtain the result analogous to Proposition 4.1.6, which is needed in the proof of Lemma 4.1.19.

Proposition 5.3.9. For any bimodal multi-conclusion rule Γ/Δ, there exist tuples $(\mathfrak{A}_1, D_1^I, D_1^M), \dots, (\mathfrak{A}_n, D_n^I, D_n^M)$ such that each $\mathfrak{A}_i$ is a finite $S4 \otimes K \oplus wMix$-algebra, $D_i^I \subseteq A_i$ and $D_i^M \subseteq A_i$, and for each $S4 \otimes K \oplus wMix$-algebra $\mathfrak{B} = (B, \Box_I, \Box_M)$, $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is a stable embedding $h : A_i \to B$ satisfying CDC for D_i^I and D_i^M.

Proof. The proof is almost the same as that for Proposition 4.1.5.

Let Γ/Δ be an arbitrary bimodal multi-conclusion rule. If $\Gamma/\Delta \in \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{wMix}^R$, then take $n = 0$. Suppose $\Gamma/\Delta \notin \mathbf{S4} \otimes \mathbf{K} \oplus \mathbf{wMix}^R$, let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$, define $\Theta' = \Theta \cup \{\Box_I \varphi \mid \Box_M \varphi \in \Theta\}$. Clearly Θ' is finite and closed under subformulas, and for any formula $\Box_M \varphi$, we have that $\Box_M \varphi \in \Theta'$ implies that $\Box_I \varphi \in \Theta'$. There are only finitely many pairs $(\mathfrak{A}, D^I, D^M)$ satisfying the following two conditions up to isomorphism:

i) $\mathfrak{A} = (A, \Box_I, \Box_M)$ is a finite $S4 \otimes K \oplus wMix$-algebra such that A is an m-generated Boolean algebras and $\mathfrak{A} \not\models \Gamma/\Delta$.

ii) $D^I = \{V(\varphi) \mid \Box_I \varphi \in \Theta'\}$ and $D^M = \{V(\varphi) \mid \Box_M \varphi \in \Theta'\}$ where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \Gamma/\Delta$.

Let $(\mathfrak{A}_1, D_1^I, D_1^M), ..., (\mathfrak{A}_n, D_n^I, D_n^M)$ be an enumeration of such pairs. For any $S4 \otimes K \oplus wMix$-algebra $\mathfrak{B}$, we prove that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is a stable embedding $h : A_i \to B$ satisfying CDC for D_i^I and D_i^M.

For the left-to-right direction, we only need to note that in order to prove that $(B', \Box'_I, \Box'_M)$ validate **wMix**, the assumption that $\mathfrak{B}$ validates **wMix** is already sufficient. Others are exactly the same as those in the proof of Proposition 4.1.5.

■

Combining the above proposition and Proposition 4.1.4, we get the counterpart to Proposition 4.1.6. The only difference between them is that one is for $S4 \otimes K \oplus Mix$-algebras while the other one is for $S4 \otimes K \oplus wMix$-algebras.

Proposition 5.3.10. For any bimodal multi-conclusion rule Γ/Δ, there exist tuples $(\mathfrak{A}_1, D_1^I, D_1^M), \ldots, (\mathfrak{A}_n, D_n^I, D_n^M)$ such that each $\mathfrak{A}_i$ is a finite $S4 \otimes K \oplus wMix$-algebra, $D_i^I \subseteq A_i$ and $D_i^M \subseteq A_i$, and for each $S4 \otimes K \oplus wMix$-algebra $\mathfrak{B} = (B, \Box_I, \Box_M)$, $\mathfrak{B} \models \Gamma/\Delta$ iff $\mathfrak{B} \models \mu(\mathfrak{A}_1, D_1^I, D_1^M), ...,$ $\mu(\mathfrak{A}_n, D_n^I, D_n^M)$.

Having seen all of the theory developed for Heyting-Lewis logics so far, one is justified to believe that we can prove the counterpart to Lemma 4.1.19 and thus prove the Blok-Esakia theorem for Heyting-Lewis logics. However, it turns out that there is an intrinsic problem lying in the corresponding lemma:

Key Statement for the Blok-Esakia Theorem: *Let $\mathfrak{A}$ be a $Grz \otimes K \oplus wMix$-algebra, then for any bimodal multi-conclusion rule Γ/Δ, we have that*

$$\mathfrak{A} \models \Gamma/\Delta \textit{ iff } \sigma(\rho(\mathfrak{A})) \models \Gamma/\Delta.$$

Say we want to prove that if $\mathfrak{A}_* \not\models \Gamma/\Delta$, then $\sigma(\rho(\mathfrak{A}))_* \not\models \Gamma/\Delta$ where $\mathfrak{A}$ is a $Grz \otimes K \oplus wMix$-algebra (the other direction clearly holds by Proposition 5.3.5). By Proposition 5.3.3 $\sigma(\rho(\mathfrak{A}))_* = \sigma(\rho(\mathfrak{A})_*) = \sigma(\rho(\mathfrak{A}_*))$. And by Proposition 5.3.10, we can assume without loss of generality that $\Gamma/\Delta = \mu(\mathfrak{B}, D^I, D^M)$ where $\mathfrak{B}$ is a finite $S4 \otimes K$-algebra which validates **wMix** and $D^I, D^M \subseteq B$.

Thus suppose $\mathfrak{A}_* \not\models \mu(\mathfrak{B}, D^I, D^M)$ where $R_I \circ R_M = R_M$ on $\mathfrak{A}_*$ as $\mathfrak{A}$ validates **wMix**, then there is a continuous stable surjection $f : \mathfrak{A}_* \to \mathfrak{B}_*$ satisfying CDC for $\{\delta_a \mid a \in D^I\}$ and $\{\delta_a \mid a \in D^M\}$ by Proposition 4.1.9. We have to show that $\sigma(\rho(\mathfrak{A}_*)) \not\models \mu(\mathfrak{B}, D^I, D^M)$. By Proposition 4.1.9 again, this means that we need a continuous stable surjection $g : \sigma(\rho(\mathfrak{A}_*)) \to \mathfrak{B}_*$ satisfying CDC for $\{\delta_a \mid a \in D^I\}$ and $\{\delta_a \mid a \in D^M\}$.

Notice that in our construction in the proof of Lemma 4.1.19, for any $x \in \mathfrak{A}_*$, we have that $g(\rho(x))$ is mapped to an element of the R_I-cluster which contains $f(x)$. Namely, $g(\rho(x))R_I f(x)$ and $f(x)R_I g(\rho(x))$. Now, we have to check that g preserves R_M: suppose $\rho(x)R_M\rho(y)$ in $\sigma(\rho(\mathfrak{A}_*))$, namely $\rho(x)[R_I \circ R_M]\rho(y)$ in $\rho(\mathfrak{A}_*)$. By definition, there exists y' such that $xR_I{\circ}R_M y'$ and $y' \backsim y$. As f is stable, it follows that $f(x)R_I{\circ}R_M f(y')$ in $\mathfrak{B}_*$. And since $\mathfrak{B}$ validates **wMix**, $R_I{\circ}R_M = R_M$ in $\mathfrak{B}_*$. Thus $f(x)R_M f(y')$, and $g(\rho(x))R_M f(y')$. However, although $g(\rho(y)) \backsim f(y')$, we can not conclude that $g(\rho(x))R_M g(\rho(y))$[8]. An element a in $\mathfrak{B}_*$ may R_M-related to b but not R_M-related to every element of $[b]$ (the R_I-cluster of b)[9].

In fact, the above reasoning goes beyond falsifying our proof strategy there. The problem will occur as long as $g(\rho(x))$ is mapped to an element of the R_I-cluster which contains $f(x)$. And if $g(\rho(x))$ is not mapped to an element of the R_I-cluster which contains $f(x)$, then it is not even clear how such a construction could be relation-preserving w.r.t R_I.

What makes things more interesting is that the **Key Statement** is not only sufficient but also necessary for the Blok-Esakia theorem for Heyting-Lewis logics.

Proposition 5.3.11. The **Key Statement** is equivalent to the Blok-Esakia theorem for Heyting-Lewis logics.

Proof. As mentioned as above, if the **Key Statement** holds, by what we have proved above, it is then easy to check we can repeat our proof strategy of Chapter 4 and obtain the Blok-Esakia theorem for Heyting-Lewis logics.

Now, suppose the **Key Statement** is false while the Blok-Esakia theorem holds for Heyting-Lewis logics.[10] Then in particular, there exists a $Grz \otimes K \oplus wMix$-algebra $\mathfrak{A}$ such that $\mathfrak{A} \not\models \Gamma/\Delta$ while $\sigma(\rho(\mathfrak{A})) \models \Gamma/\Delta$ for some bimodal multi-conclusion rule Γ/Δ.

We first prove that $\sigma(\rho(Ru(\mathfrak{A})))$ and $Ru(\sigma(\rho(\mathfrak{A})))$ are modal companions of $\rho(Ru(\mathfrak{A}))$. By definition, if $\Gamma/\Delta \in \rho(Ru(\mathfrak{A}))$, then $t(\Gamma/\Delta) \in \sigma(\rho(Ru(\mathfrak{A})))$. Now, note that $\rho(Ru(\mathfrak{A})) = Ru(\rho(\mathfrak{A}))$: $\Gamma/\Delta \in \rho(Ru(\mathfrak{A}))$ iff $t(\Gamma/\Delta) \in Ru(\mathfrak{A})$, namely $\mathfrak{A} \models t(\Gamma/\Delta)$, which is equivalent to $\rho(\mathfrak{A}) \models \Gamma/\Delta$ by Theorem 5.3.7. Suppose $\Gamma/\Delta \notin \rho(Ru(\mathfrak{A})) = Ru(\rho(\mathfrak{A}))$, then $\rho(\mathfrak{A}) \not\models \Gamma/\Delta$. By Theorem 5.3.7, we thus have that $\mathfrak{A} \not\models t(\Gamma/\Delta)$. As

[8] The reason why we could make this conclusion in Lemma 4.1.19 is that there $\mathfrak{B}_*$ validates **Mix**. In particular, $R_M \circ R_I = R_M$ holds in $\mathfrak{B}_*$.

[9] This is also the reason why in Definition 5.3.1, we can not define that $\rho(x)[R_I{\circ}R_M]\rho(y)$ iff $xR_I \circ R_M y$.

[10] We would like to thank Rodrigo N. Almeida for bringing the following argument to our attention which holds first for superintuitionistic logics.

$\mathfrak{A} \vDash \{t(\Gamma/\Delta) \mid \Gamma/\Delta \in \rho(Ru(\mathfrak{A}))\}$ and $\mathfrak{A}$ is a $Grz \otimes K \oplus wMix$-algebra, it follows that $\mathfrak{A} \vDash \sigma(\rho(Ru(\mathfrak{A})))$. Thus $t(\Gamma/\Delta) \notin \sigma(\rho(Ru(\mathfrak{A})))$. This proves that $\Gamma/\Delta \in \rho(Ru(\mathfrak{A}))$ iff $t(\Gamma/\Delta) \in \sigma(\rho(Ru(\mathfrak{A})))$. Thus $\sigma(\rho(Ru(\mathfrak{A})))$ is a modal companion of $\rho(Ru(\mathfrak{A}))$.

For $Ru(\sigma(\rho(A)))$, suppose $\Gamma/\Delta \in \rho(Ru(\mathfrak{A})$, then $t(\Gamma/\Delta) \in Ru(\mathfrak{A})$ by definition. Namely, $\mathfrak{A} \vDash t(\Gamma/\Delta)$. By Proposition 5.3.5, $\sigma(\rho(\mathfrak{A})) \vDash t(\Gamma/\Delta)$. Suppose $\Gamma/\Delta \notin \rho(Ru(\mathfrak{A}))$, then by we have proved above, $\rho(\mathfrak{A}) \nvDash \Gamma/\Delta$. By Proposition 5.3.4 and Theorem 5.3.7, we have that $\sigma(\rho(\mathfrak{A})) \nvDash t(\Gamma/\Delta)$. Thus $t(\Gamma/\Delta) \notin Ru(\sigma(\rho(\mathfrak{A})))$. This proves that $\Gamma/\Delta \in \rho(Ru(\mathfrak{A}))$ iff $t(\Gamma/\Delta) \in Ru(\sigma(\rho(\mathfrak{A})))$. Thus $Ru(\sigma(\rho(A))$ is also a modal companion of $\rho(Ru(\mathfrak{A}))$.

Now, as $\mathfrak{A}$ is a $Grz \otimes K \oplus wMix$-algebra, by Proposition 5.3.5, we have that $Ru(\sigma(\rho(\mathfrak{A}))) \in \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix}^R)$. And clearly, $\sigma(\rho(Ru(\mathfrak{A}))) \in \mathbf{NExt}(\mathbf{Grz} \otimes \mathbf{K} \oplus \mathbf{wMix}^R)$. As $\sigma(\rho(Ru(\mathfrak{A})))$ and $Ru(\sigma(\rho(\mathfrak{A})))$ are modal companions of $\rho(Ru(\mathfrak{A}))$, it follows that $\Gamma/\Delta \in \rho(Ru(\sigma(\rho(\mathfrak{A}))))$ iff $t(\Gamma/\Delta) \in Ru(\sigma(\rho(\mathfrak{A})))$ iff $\Gamma/\Delta \in \rho(Ru(\mathfrak{A}))$. Therefore, $\rho(Ru(\sigma(\rho(\mathfrak{A})))) = \rho(Ru(\mathfrak{A}))$. We then have the following equations:

$$Ru(\mathfrak{A}) = \sigma(\rho(Ru(\mathfrak{A}))) = \sigma(\rho(Ru(\sigma(\rho(\mathfrak{A}))))) = Ru(\sigma(\rho(\mathfrak{A}))).$$

The first and the third ones hold because of the Blok-Esakia theorem. However, since $\mathfrak{A} \nvDash \Gamma/\Delta$ while $\sigma(\rho(\mathfrak{A})) \vDash \Gamma/\Delta$, we have that $Ru(\mathfrak{A}) \neq Ru(\sigma(\rho(\mathfrak{A})))$, a contradiction. Therefore, if the **Key Statement** is false, then the Blok-Esakia theorem for Heyting-Lewis logics does not hold.

■

Given the problem of our proof strategy and Proposition 5.3.11, we have reasons to believe that the Blok-Esakia theorem may not hold for Heyting-Lewis logics [11]. However, it is quite difficult to find a direct counterexample (if there is one). In particular, we do not know how to give a direct counterexample to the **Key Statement** or to the corresponding lemma in [23] (if there is one) as we can prove by a nontrivial argument for both of them that counterexamples cannot occur in finite cases[12]. And since our proof of the Dummett-Lemmon conjecture uses the Blok-Esakia theorem, right now,

[11] Although in [23], it was claimed that σ and ρ defined below Definition 5.3.8 are lattice isomorphisms, namely the Blok-Esakia theorem holds for Heyting-Lewis logics, we have found a gap in the proof of the key lemma [23, Lem. 4.18] which corresponds to our **Key Statement**: following Wolter and Zakharyaschev's proof strategy of [37], the problem occurs in the step for formulas of the form $\Box_M \psi$. This has been confirmed by the authors in personal communication.

[12] It is more easy to see that counterexamples (if there is one) to the **Key Statement** cannot occur in finite cases.

we do not know how to prove this conjecture for Heyting-Lewis logics (even if it holds).

Therefore, it seems that the Gödel translation for Heyting-Lewis logics is much more intriguing than that for intuitionistic modal logics. On the one hand, the Gödel translation does enjoy some nice properties such as the ones shown in Theorem 5.3.7. On the other hand, it seems that we cannot prove the Blok-Esakia theorem for Heyting-Lewis logics using the same proof strategy for intuitionistic modal logics (either by stable canonical rules or by Wolter and Zakharyaschev's proof strategy of [37]). Thus a much deeper analysis of the Gödel translation for Heyting-Lewis logics is needed in order to solve the open problems whether the Blok-Esakia theorem and the Dummett-Lemmon conjecture hold for Heyting-Lewis logics. We believe that our stable canonical rules (in particular, Theorem 5.2.8 and Proposition 5.3.10) may contribute to such an analysis.

Chapter 6

Conclusion

In this thesis we developed the method of stable canonical formulas and rules for intuitionistic modal logics and Heyting-Lewis logics. We proved that every intuitionistic modal multi-conclusion consequence relation is axiomatizable by stable canonical rules, and every intuitionistic modal logic over $\mathbf{IntS4}_{\Box}$ is axiomatizable by stable canonical formulas. Similarly, we proved that every Heyting-Lewis multi-conclusion consequence relation is axiomatizable by stable canonical rules. Besides, our rules have proved useful. In particular, with some adaptations to known techniques from [42] and [16], we provided an alternative proof of the Blok-Esakia theorem and proved an analogue of the Dummett-Lemmon conjecture for intuitionistic modal logics. Using the stable canonical rules for Heyting-Lewis logics, we also pointed out a gap in the proof of the Blok-Esakia theorem for Heyting-Lewis logics. Because of this, the existence of the Blok-Esakia isomorphism in this case remains open. Therefore, we hope that our work contributed to the theory of stable canonical rules and also provided a uniform method to study intuitionistic modal logics and Heyting-Lewis logics.

We end the thesis with some directions for further work.

- For reasons of space, we did not include stable canonical rules for intuitionistic modal logics with both $\Box$ and $\Diamond$ in this thesis. This can be done for $\mathbf{IntK}_{\Box,\Diamond}$ (i.e. $\mathbf{IntK}_{\Box}$ extended with $\Diamond$ which satisfies $\Diamond\bot = \bot$ and $\Diamond(p \vee q) = \Diamond p \vee \Diamond q$) with some adaptations to the techniques in Chapter 3. In particular, we can prove the corresponding completeness theorem with no difficulty and use the duality in [31] to give a dual description of the rules. However, once we require some interactions between $\Box$ and $\Diamond$, it is unclear how to do filtrations in the counterpart to Proposition 3.1.3. Solutions to some open problems about the finite model property of some intuitionistic modal logics with both $\Box$ and $\Diamond$ may help for this type of questions.

- In [9], Bezhanishvili *et al.* used stable canonical rules to give alternative proofs of decidability of admissibility for **IPC** and **S4**. In brief, the idea is as follows: for any multi-conclusion rule, we have an algorithm to transform it into finitely many stable canonical rules whose conjunction is equivalent to the original rule. Therefore, to determine whether the original rule is admissible[1] or not, we only need to determine whether those finitely many stable canonical rules are admissible. Thus an algorithm to determine whether a stable canonical rule is admissible or not is sufficient for the proof of decidability of admissibility.

  Since we have developed stable canonical rules for intuitionistic modal logics and Heyting-Lewis logics, one could try to use the same proof strategy to give a proof or an alternative proof of decidability of admissibility for some intuitionistic modal logics. The first step for this should be determining the admissible basis for a given intuitionistic modal logic in analogy with those given in [22].

- In [26], Illn gave a thorough analysis of *stable logics* in the settings of classical modal logics and superintuitionistic logics. Stable logics are just those logics axiomatized by stable canonical formulas without parameters (so we do not need to consider the CDC conditions anymore and the class of Kripke frames validating such a logic is closed under relation-preserving maps). One can develop the theory of stable logics for intuitionistic modal logics and Heyting-Lewis logics as well and show that these logics enjoy the finite model property. We leave this for future work.

[1] Informally speaking, a rule is admissible if adding it to the deductive system does not produce any new theorems.

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