On the Provability Logic of Constructive Arithmetic

The Σ_1-provability logics of fragments of Heyting Arithmetic

MSc Thesis *(Afstudeerscriptie)*

written by

Borja Sierra Miranda
(born June 9th, 1999 in Sevilla, Spain)

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Members of the Thesis Committee:
Dr Bahareh Afshari (Supervisor)
Dr Maria Aloni (Chair)
Prof Dr Lev Beklemishev
Dr Benno van der Berg
Prof Dr Albert Visser (Supervisor)

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

We study provability logic in the context of intuitionistic arithmetics. In particular, we focus on the Σ_1-provability logics of subtheories of Heyting Arithmetic HA. In order to do so, we analyze the tools developed by Visser and Zoethout in [13] and a method for constructing so-called slow provability predicates introduced by Visser in [12]. We also study a theory distinct from HA, $\mathsf{i}\mathsf{I}\Sigma_1^+$, for which we can calculate its Σ_1-provability logic.

A Fernando,
espero que sigas estando orgulloso de tu hermano pequeño.

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Contents

Chapter 0

Introduction and Background

Provability logic arises from an interaction between proof theory and modal logic. The basic idea is to interpret the box in modal logic as provability in some theory. As an example, the principle K from modal logic, i.e. $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ can be understood as saying that the theory we are considering is closed under modus ponens: if $\phi \to \psi$ is provable then ϕ provable implies ψ provable. Note that box can be applied multiple times. This means that the theory we are interpreting box in must, in some sense, encode logic and its own axiomatization. Examples of such theories are theories as strong as ZFC (Zermelo-Fraenkel set theory), or as weak as EA (elementary arithmetic).

The first great step in provability logic was finding the axioms of GL, Gödel-Löb's logic, in 1955 by Löb. At this point this were stated as conditions of $\mathsf{prov}_{\mathsf{PA}}$, the provability predicate of Peano Arithmetic, not as a modal logic. Solovay proved in 1976 of the completeness of GL with respect the arithmetical semantics. This logic is defined as the minimal (normal) modal logic which contains the following principle: $\Box(\Box\phi \to \phi) \to \Box\phi$. Although this principle may appear complex, if we interpret $\Box$ as provability, its meaning is clear. $\Box\phi \to \phi$ is related to soundness, it can be read as "if ϕ is provable then ϕ holds (is true)". Then, $\Box(\Box\phi \to \phi)$ means that the theory proves soundness for the particular formula ϕ. Then, the whole implication, means that if the theory proves soundness of a formula, then the theory proves the formula. In other words, the theory only can prove the soundness of the theorems it already knows to hold. Thanks to Solovay's result it was proven that the provability logic of PA is exactly GL. In fact, GL is the provability logic of many arithmetical theories: any Σ_1-sound arithmetical theory T extending $\mathsf{I}\Delta_0 + \mathrm{EXP}$ (induction for bounded formulas and the exponentiation axiom).

Once the provability logic of PA was found, the question of calculating the provability logic of Heyting Arithmetic, HA, was a natural step. HA is nothing more than the same theory as PA, but changing the ambient logic from classical to constructive. This problem has been remarkably hard to solve, being open for four decades. Recently, in [6], Mojtaba Mojtahedi solved the problem, calculating the provability logic of HA for the first time.

Let us briefly explain the idea behinds Mojtahedi's proof. In order to do so, we need to introduce Σ_1-provability logic. Note that to fully interpret modal formulas in another first order theory like PA, it is not enough to give a meaning to $\Box$. One also need to deal with propositional variables. The idea is straightforward: give an interpretation that translates propositional variables into sentences of the theory, in our case arithmetical theories. Σ_1-provability logic is the modal logic that arises if we just keep the interpretations of this propositional variables to be Σ_1-formulas, in other words, a formula starting by an existential and then having all quantifiers (non-trivially) bounded.

The idea behinds Mojtahedi's proof is to divide the proof of completeness in two steps. First, one calculates the Σ_1-provability logic of the theory of interest, then one lifts the result from the Σ_1-provability logic to the full provability logic. In particular, he transforms a realization making a modal formula ϕ invalid into a Σ_1-realization making the modal formula also invalid. This clearly signals the importance of the Σ_1-provability logic to the calculation of the full provability logic, at least in the intuitionistic case.

Our goal is to study provability logic in the context of intuitionistic arithmetics. In particular, we focus on the Σ_1-provability logic of subtheories of HA. In order to do so, we will use tools develop by Albert Visser and Jetze Zoethout in [13], which where inspired by the ones developed by Mohammed Ardeshir and Mojtaba Mojtahedi in [1]. The long term objective of this study, which is impossible to completely solve in this exploration due to time constrains, is to check whether the same uniformity of provability logics that occurs in the classical arithmetical case also holds in the intuitionistic case.

Chapter 1

Preliminaries

In this chapter we are going to introduce the concepts that form the base of the text. Clearly. before introducing the concepts of provability logic, we need some background in arithemtical theories, modal logic and how to formalize logic in arithmetic. Since we are going to study intuitionistic arithmetic, we also need to briefly introduce intuitionistic logic. In particular, the structure of the chapter is as follows:

1. Section 1.1 introduces intuitionistic logic via a Hilbert system.

2. Section 1.2 introduces the concepts we need from Arithmetic and also the theories we will mainly work with.

3. Section 1.3 introduces the syntax and semantics of intuitionistic modal logic and the ony modal logic that we are going to need iGLC.

4. Section 1.4 explains how to codify different constructions of logic inside arithmetical theories.

5. Finally, Section 1.5 introduces the concepts of provability logic that we need.

Although the reader may be familiar with these concepts already, we encourage to read Section 1.5. In that section we introduce some notation that will be heavily used through all the text.

1.1 Intuitionistic Logic

Officially, our formulas are natural numbers, lists of formualas are also natural numbers and so on. The idea is that we use a codification of these concepts in natural numbers that can be developed in iEA, i.e. intuitionistic elementary arithmetic. In this way, we bring the metatheory we reason in and the arithmetical theory at hand a little closer. For details about these codifications, check the last chapter of [5]. In the book it is done in a classical setting, but all the codification can be performed in intuitinistic arithmetic.

We define a Hilbert system for intuitionistic first order logic, taken from [8].

Definition 1. We define the Hilbert system for intuitionistic predicative logic iFOL as:
Axioms:

1. $\bot \to \phi$;
2. $\phi \wedge \psi \to \phi, \phi \wedge \psi \to \psi$;
3. $\phi \to \psi \to \phi \wedge \psi$;
4. $\phi \to \phi \vee \psi, \psi \to \phi \vee \psi$;
5. $(\phi \to \chi) \to (\psi \to \chi) \to (\phi \vee \psi \to \chi)$;
6. $\phi \to \psi \to \phi$;
7. $(\phi \to \psi \to \chi) \to (\phi \to \psi) \to (\phi \to \chi)$;
8. $(\forall x \,.\, \phi) \to \phi[x/t]$, where t is free for x in ϕ;
9. $(\forall x \,.\, \psi \to \phi) \to \psi \to (\forall y \,.\, \phi[x/y])$, where $x \notin \mathsf{fv}(\psi)$ and $y = x$ or $y \notin \mathsf{fv}(\phi)$;
10. $\phi[x/t] \to \exists x \,.\, \phi$, where t is free for x in ϕ;
11. $(\forall x \,.\, \phi \to \psi) \to (\exists y \,.\, \phi[x/y]) \to \psi$, where $x \notin \mathsf{fv}(\psi)$ and $y = x$ or $y \notin \mathsf{fv}(\phi)$;
12. $\forall x \,.\, x \approx x$;
13. For any n-ary function symbol f and any $i \leq n$ we have the axiom:
$$\forall x_0, \dots, x_n \,.\, x_i \approx x_n \to f(\dots, x_i, \dots) \approx f(\dots, x_n, \dots);$$
14. For any n-ary relation symbol R and any $i \leq n$ we have the axiom:
$$\forall x_0, \dots, x_n \,.\, x_i \approx x_n \to (R(\dots, x_i, \dots) \leftrightarrow R(\dots, x_n, \dots)).$$

Rules:

$$\frac{\phi \to \psi \qquad \phi}{\psi}\ \mathrm{MP}$$

$$\frac{\phi}{\forall x . \phi}\ \mathrm{Gen}$$

We will write $\Gamma \vdash \phi$ to mean that there is a derivation of with assumptions in Γ. In this case, for Gen we need the additional condition that $x \notin \mathsf{fv}(\Gamma)$. ■

It is trivially true that $\Gamma \vdash \phi \to \psi$ implies $\Gamma, \phi \vdash \psi$. The reverse direction is the deduction theorem:

Lemma 2 (**Deduction theorem**). $\Gamma, \phi \vdash \psi$ implies that $\Gamma \vdash \phi \to \psi$.

A theory will be just a set of sentences of a given signature. Then, saying that a theory proves a formula will be saying that there is a derivation of the formula using as assumptions the axioms of the theory. Note that we impose that the axioms of a theory are sentences because we want to be able to use the rule Gen when reasoning inside a theory.

Definition 3. A theory T consists in a set Ax_T of sentences of a given signature. We will write $\Gamma \vdash_T \phi$ to mean that there is a finite subset $\Delta \subseteq \mathrm{Ax}_T$ such that $\Delta, \Gamma \vdash \phi$. ■

1.2 Arithmetical theories

In this text we will mainly work with two types arithmetical theories. The first kind are formulated in a language with addition, multiplication and exponentiation. The second kind are formulated in a language with all primitive recursive functions. When there is no danger of confusion, we will write simply $\mathcal{L}_1$ to refer to the first order language. We will assume that all the theories we work with are in one of these languages and are computably enumerable.

Both of this languages come equipped with a standard model whose domain is $\mathbb{N}$. If ϕ is an arithmetical formula, there is no ambiguity in the meaning of $\vDash_{\mathbb{N}} \phi$. Let us define some concepts that are going to be useful for our arithmetical theories. We start with the definition of the arithmetical hierarchy.

Definition 4. Assume we have a fixed signature with the binary relation symbol $\leq$. Then we define the class of Δ_0 formulas as the smallest class such that:

1. Atomic formulas belong to Δ_0.
2. Δ_0 is closed under conjunction, disjunction and implication.
3. If ϕ is a formula in Δ_0, x is a variable and τ is a term (of the given signature) such that $x \notin \mathrm{fv}(\tau)$, then $\forall x \leq \tau \,.\, \phi, \exists x \leq \tau \,.\, \phi$ are in Δ_0. These are called bounded quantifiers.

One we have the class of Δ_0-formulas, we can define $\Sigma_0 := \Pi_0 := \Delta_0$ and:

$$\Sigma_{n+1} := \{\exists x \,.\, \phi \mid \phi \in \Pi_n\},$$
$$\Pi_{n+1} := \{\forall x \,.\, \phi \mid \phi \in \Sigma_n\}.$$ ■

Definition 5. Let T be an arithmetical theory and let Γ be a set of formulas from the language of T. We say that:

1. T is Δ_0-sound (Σ_1-sound) if for any sentence $\phi \in \Delta_0$ ($\phi \in \Sigma_1$), $\vdash_T \phi$ implies $\vDash_{\mathbb{N}} \phi$.

2. T is Δ_0-complete (Σ_1-complete) if for any sentence $\phi \in \Delta_0$ ($\phi \in \Sigma_1$), $\vDash_{\mathbb{N}} \phi$ implies $\vdash_T \phi$.

3. T is Δ_0-decidable if for any formula $\phi \in \Delta_0$, we have that $\vdash_T \phi \vee \neg\phi$.

■

Both languages we are going to consider have a constant $\overline{0}$, representing the number 0 and function symbols $\mathsf{S}, +, \times$, which in the standard model are successor, addition and multiplication. Note that with $\overline{0}$ and S alone, we can get a closed term representing any natural number in the standard model: to represent the number n just apply S n times to $\overline{0}$. This is an unary representation of the natural numbers. For matters that we are going to discuss in Section 1.4, we need a more efficient coding of natural numbers as terms. For that purpose, we are going to use a binary representation of the natural numbers.

Definition 6. We define the function $\overline{\cdot} : \mathbb{N} \longrightarrow \mathrm{Term}$ as follows:

$$
\begin{aligned}
&0 \mapsto \overline{0};\\
&1 \mapsto \mathsf{S}(\overline{0});\\
&2n \mapsto (\overline{1}+\overline{1}) \times \overline{n}, \text{ where } n > 1;\\
&2n+1 \mapsto \mathsf{S}((\overline{1}+\overline{1}) \times \overline{n}), \text{ where } n > 1.
\end{aligned}
$$

If $n \in \mathbb{N}$, then $\overline{n}$ is called the *numeral of n*. ■

Definition 7 (Induction). Given a formula ϕ and a variable x, we define the sentence $I_{\phi,x}$ as the universal closure of:

$$\phi[x/0] \wedge (\forall x.\phi \to \phi[x/\mathsf{S}x]) \to \forall x.\phi,$$

where $x \in \mathsf{fv}(\phi)$. If Γ is a set of formulas, we define

$$I\Gamma = \{I_{\phi,x} \mid \phi \in \Gamma \text{ and } x \in \mathsf{fv}(\phi)\}.$$ ■

Definition 8 (Collection). Given a formula ϕ and variables x, y, we define the sentence $B_{\phi,x,y}$ as the universal closure of:

$$(\forall x \leq u\, \exists y\, .\, \phi) \to (\exists v \forall x \leq u\, \exists y \leq v\, .\, \phi),$$

where $u, v \notin \mathsf{fv}(\phi)$. If Γ is a set of formulas, we define

$$B\Gamma = \{B_{\phi,x,y} \mid \phi \in \Gamma, x, y \in \mathrm{Var}\}.$$ ■

Theories in the elementary language of Arithmetic

Definition 9. We define the *elementary language of Arithmetic*, $\mathcal{L}_{\mathsf{exp}}$, as the first order language with the following symbols:

1. A constant $\overline{0}$, called zero.

2. Two unary function symbols S, exp, called *succesor* and *exponentiation*, respectively.

3. Two binary function symbols $+, \times$, called addition and multiplication, respectively.

4. A binary relation symbol $\leq$.

■

We start with intuitionistic Robinson's Arithmetic with exponentiation. This theory only contains the axioms claiming that the successor function is injective but not surjective; the recursive definitions of addition, multiplication and exponentiation; and the definition of the order relation.

Definition 10. The theory $\mathsf{iQ}^{\mathsf{exp}}$ is the theory over iFOL in the language of $\mathcal{L}_{\mathsf{exp}}$ with axioms:

1. $\forall x \,.\, \mathsf{S}(x) \not\approx \overline{0}$.

2. $\forall x, y \,.\, \mathsf{S}(x) \approx \mathsf{S}(y) \to x \approx y$.

3. $\forall x \,.\, x \approx \overline{0} \vee \exists y \,.\, x \approx \mathsf{S}y$.

4. $\forall x \,.\, x + \overline{0} \approx x$.

5. $\forall x, y \,.\, x + \mathsf{S}(y) \approx \mathsf{S}(x + y)$.

6. $\forall x \,.\, x \times \overline{0} \approx \overline{0}$.

7. $\forall x, y \,.\, x + \mathsf{S}(y) \approx \mathsf{S}(x + y)$.

8. $\mathsf{exp}\left(\overline{0}\right) \approx \overline{1}$.

9. $\forall x \,.\, \mathsf{exp}(\mathsf{S}(x)) \approx \overline{2} \times \mathsf{exp}(x)$.

10. $\forall x, y \,.\, x \leq y \leftrightarrow \exists z \,.\, x + z \approx y$.

11. $\forall x, y \,.\, x \approx y \vee x \not\approx y$.

Also, let $\mathsf{iQ}^{\mathsf{exp}}_{-}$ be the theory with these axioms without 3 and 11. ■

We will mainly work with three extensions of this theory. The first one, intuitionistic Elementary Arithmetic, is just adding Δ_0-induction to this theory. The second one does not have a name, and is just the extension of intuitionistic Elementary Arithmetic via Σ_1-collection. Finally, the last extension we are going to work with is induction for Σ_1-formulas. Let us put the three together in the following definition.

Definition 11. We define the following theories over iFOL:

$$\begin{aligned} &\mathsf{iEA} := \mathsf{iQ}_{-}^{\mathsf{exp}} + I\Delta_0, \\ &\mathsf{iEA} + B\Sigma_1, \\ &\mathsf{iI}\Sigma_1 := \mathsf{iQ}_{-}^{\mathsf{exp}} + I\Sigma_1. \end{aligned}$$

■

In Chapter 5 we also work with an extension of $\mathsf{iI}\Sigma_1$, but its definition is postponed until that chapter.

Theories in the language of Primitive Recursive Functions

First, we define the function symbols of the language of Primitive Recursive Arithmetic.

Definition 12 (Primitive Recursive Function Symbol). We define the set of *primitive recursive function symbols*, $\mathscr{Pr}$, as the set with the following function symbols

1. Zero,S are a unary function symbols in $\mathscr{Pr}$.
2. Proj_i^n where $i < n$ is an n-ary function symbol in $\mathscr{Pr}$.
3. If f is an n-ary function symbol in $\mathscr{Pr}$ and $g_0, \dots, g_{n-1}$ are m-ary function symbols in $\mathscr{Pr}$, then $\mathsf{Comp}_{f,g_0,\dots,g_{n-1}}$ is a m-ary function symbol in $\mathscr{Pr}$.
4. If f is an n-ary function symbol in $\mathscr{Pr}$ and g is an $n+2$-ary function symbol in $\mathscr{Pr}$, then $\mathsf{Rec}_{f,g}$ is an $n+1$-ary function symbol in $\mathscr{Pr}$.

The first order language consisting in these function symbols, the constant $\overline{0}$ and the binary relation symbol $\leq$ is denoted as $\mathcal{L}(\mathscr{Pr})$. ■

Once we have the function symbols, we associate an axiom with each function symbol:

Definition 13. Let f be a function symbol in $\mathscr{Pr}$, we define the sentence ax_f recursively in f as:

$$\begin{aligned} &\mathsf{ax}_{\mathsf{Zero}} := \forall x \,.\, \mathsf{Zero}(x) \approx \overline{0}. \\ &\mathsf{ax}_{\mathsf{S}} := \mathsf{S}\left(\overline{0}\right) \not\approx \overline{0}. \\ &\mathsf{ax}_{\mathsf{Proj}_j^n} := \forall x_0, \dots, x_{n-1} \,.\, \mathsf{Proj}_j^n(x_0, \dots, x_{n-1}) \approx x_j. \\ &\mathsf{ax}_{\mathsf{Comp}_{f,g_0,\dots,g_{n-1}}} := \forall x_0, \dots, x_{m-1} \,. \\ &\quad \mathsf{Comp}_{f,g_0,\dots,g_{n-1}}(x_0, \dots, x_{m-1}) \approx f(g_0(x_0, \dots, x_{m-1}), \dots, g_{n-1}(x_0, \dots, x_{m-1})). \\ &\mathsf{ax}_{\mathsf{Rec}_{f,g}} := \left(\forall x_1, \dots, x_n \,.\, \mathsf{Rec}_{f,g}\left(\overline{0}, x_1, \dots, x_n\right) \approx f(x_1, \dots, x_n)\right) \wedge \\ &\quad (\forall x_0, \dots, x_n \,.\, \mathsf{Rec}_{f,g}(\mathsf{S}(x_0), x_1, \dots, x_n) \approx g(x_0, \mathsf{Rec}_{f,g}(x_0, x_1, \dots, x_n), x_1, \dots, x_n)). \end{aligned}$$

■

Definition 14 (Primitive Recursive Arithmetic). We define the first order theory iPRA as the theory over iFOL in the language $L(\mathscr{Pr})$ with the following axioms:

1. For any $f \in \mathscr{Pr}$, ax_f.
2. Induction for quantifier-free formulas, i.e. if ϕ is quantifier free the universal closure of:

$$\phi[x/0] \wedge (\forall x . \phi \to \phi[x/\mathsf{S}x]) \to \forall x . \phi.$$

■

We remind the Diagonalization lemma, which is a fundamental result for provability logic.

Lemma 15. Let $\phi(x_0, \dots, x_{n-1}, y)$ be a formula. Then there exists a formula $\psi(x_0, \dots, x_{n-1})$ such that

$$\mathsf{iEA} \vdash \psi(x_0, \dots, x_{n-1}) \leftrightarrow \phi(x_0, \dots, x_{n-1}, \overline{\psi}).$$

Note that if T extends iEA then the equivalence also holds in T.

1.3 Intuitionistic Modal Logic

We are going to introduce some concepts of Intuitionistic Modal Logic (IML) that we require. We will not write any proofs, but the interested reader should consult [13]. First, let us introduce the syntax of modal logic that we are going to use.

Definition 16. We define the *modal logic language*, $\mathcal{L}_{\mathrm{m}}$, as the given by the following BNF:

$$\mathcal{L}_{\mathrm{m}}: \qquad \phi ::= p \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \Box \phi,$$

where p is a propositional variable. ■

Now we start by defining the models of intuitionistic modal logic.

Definition 17. A *frame for intuitionistic modal logic* is a triple $\mathcal{F} = (W, \preccurlyeq, \sqsubset)$ where W is a non-empty set, $\preccurlyeq, \sqsubset$ are binary relations on W and we have the following properties:

1. $\preccurlyeq$ is a partial order, i.e. it is reflexive, transitive and antisymmetric.
2. (Model property) For any $w, v, u \in W$ we have that $w \preccurlyeq v \sqsubset u$ implies $w \sqsubset u$.

A *model for intuitionistic modal logic* is a quadruple $\mathcal{M} = (W, \preccurlyeq, \sqsubset, V)$ such that:

1. $(W, \preccurlyeq, \sqsubset)$ is a frame of modal intuitionistic logic.

2. V is a relation between W and propositional letters.

3. (Preservation of knowledge) If $w, v \in W$ and p is a propositional letter such that $w \preccurlyeq v$ and $w\, V p$, then $v\, V p$.

■

The definition of the semantics is straightforward. All the logical connectives have their usual definition, except for $\to$ and $\Box$. For these connectives we just need to note that there are two binary relations. For $\to$ we use the usual clause (in intuitionistic logic) using the intuitionistic relation, i.e.

$$\mathcal{M}, w \vDash \phi \to \psi \text{ iff for any } v \succcurlyeq w,\ \mathcal{M}, v \nvDash \phi \text{ or } \mathcal{M}, v \vDash \psi.$$

And for $\Box$ we use the usual clause using the modal relation, i.e.

$$\mathcal{M}, w \vDash \Box\phi \text{ iff for any } v \sqsupset w,\ \mathcal{M}, v \vDash \phi.$$

The preservation of knowledge gives the following result:

Lemma 18. Let $\mathcal{M} = (W, \preccurlyeq, \sqsubset, V)$ be a model for IML. If $w, v \in W$ and $\phi \in \mathcal{L}_{\mathrm{m}}$ such that $\mathcal{M}, w \vDash \phi$ and $w \preccurlyeq v$ then $\mathcal{M}, v \vDash \phi$.

Now we introduce the modal logic we are interested in: iGLC. iGL is the intuitionistic version of Gödel-Löb logic. As we already discussed in the introduction, the original GL is really important in (classical) provability logic. It is the provability logic of a wide range of arithmetical theories, such as $\mathsf{EA}, \mathsf{I}\Sigma_1$ and PA. The C in iGLC comes from the completeness principle, with are the formulas of shape $\phi \to \Box\phi$.

Definition 19. The set iGLC $\subseteq \mathcal{L}_{\mathrm{m}}$ is the smallest set that contains:

1. All $\mathcal{L}_{\mathrm{m}}$-substitution instances of theorems of iPC (intuitionistic propositional logic).

2. For any $\phi, \psi \in \mathcal{L}_{\mathrm{m}}$, $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$.

3. For any $\phi \in \mathcal{L}_{\mathrm{m}}$, $\Box(\Box\phi \to \phi) \to \Box\phi$.

4. For any $\phi \in \mathcal{L}_{\mathrm{m}}$, $\phi \to \Box\phi$.

And it is closed under the rules

1. (Modus ponens) If $\phi \to \psi, \phi \in$ iGLC, then $\psi \in$ iGLC.

2. (Necessitation) If $\phi \in$ iGLC, then $\Box\phi \in$ iGLC.

We will write $\vdash_{\mathsf{iGLC}} \phi$ to mean that $\phi \in$ iGLC and $\Gamma \vdash_{\mathsf{iGLC}} \phi$ to mean that there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\vdash_{\mathsf{iGLC}} \bigwedge \Gamma_0 \to \phi$. ■

We want to have a characterization of the models of iGLC. With that purpose in mind, we introduce the following properties of IML frames.

Definition 20. Let $\mathcal{F} = (W, \preccurlyeq, \sqsubset)$ be a frame for IML.

1. We say that $\mathcal{F}$ is *irreflexive* iff $\sqsubset$ is irreflexive.
2. We say that $\mathcal{F}$ is *transitive* iff $\sqsubset$ is transitive.
3. We say that $\mathcal{F}$ is *realistic* iff $\sqsubset \subseteq \preccurlyeq$.
4. We say that $\mathcal{F}$ is *conversely well-founded* iff $\sqsubset$ is conversely well-founded, i.e. every non-empty subset of W has a $\sqsubset$-maximal element.

■

Note that thanks to the model property, any realistic frame is automatically transitive. We have the following theorem, that characterizes the models of iGLC.

Theorem 21. Let $\phi \in \mathcal{L}_{\mathrm{m}}$. Then

1. If $\vdash_{\mathsf{iGLC}} \phi$ then ϕ is valid on all realistic and conversely well-founded frames.
2. $\vdash_{\mathsf{iGLC}} \phi$ iff ϕ is valid on all finite irreflexive realistic frames.

1.4 Arithmetizing Logic

When we have a function $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ it may be possible to represent it inside an arithmetical theory T. With this, we mean to have a formula $\phi_f(x_0, \ldots, x_{k-1}, y)$ such that:

1. $\vdash_T \forall x_0, \ldots, x_{k-1} \exists! y . \phi_f(x_0, \ldots, x_{k-1}, y)$.
2. $\vdash_T \phi_f(\overline{n_0}, \ldots, \overline{n_{k-1}}, \overline{m})$ iff $f(n_0, \ldots, n_{k-1}) = m$.

Note that this is the minimum we will ask, but depending on the kind of function that f is we may ask more. For example, if f is define recursively via equations, we also add the conditions that these equations are provable in T. Since our weakest theory is iEA, we know that all the theories we work with represents elementary functions. Whenever we have such a function f we may write f to indicate the term representing this function inside T. In case the name of f does not contain any letter, but just symbols, we will usually add $\bullet$ as a superscript, resulting in $f^{\bullet}$. We will follow a similar approach with predicates and if we have a predicate P we will write P for the version defined inside arithmetical theories.

In order to bring the metatheory and the arithmetical theories closer, we will assume that all the representation of constructions used in the arithmetical theories are the official definition of those concepts in the metatheory. This means that finite lists, formulas or proofs are numbers in the metatheory and they are defined as we defined them in arithmetical theories. This means that whenever we have an object defined by a number n, $\overline{n}$ is a term representing that object inside the arithmetical theories. As an example, we assume that we use a representation of formulas that works for iEA. This means that we will have formulas like $\mathsf{term}, \mathsf{form}, \mathsf{sent}$ claiming that a number is a term, a formula or a sentence respectively and terms like $\rightarrow^\bullet, \wedge^\bullet, \vee^\bullet, \forall^\bullet, \exists^\bullet, \bot^\bullet, \overline{0}^\bullet, +^\bullet, \times^\bullet$. We also have the usual $\mathsf{proof}_\alpha(p,A)$ and $\mathsf{prov}_\alpha(A)$ predicates, where α is a formula which represents some set of sentences. Then $\mathsf{proof}_\alpha(p,A)$ means that p is a proof from axioms in α of the formula A, and $\mathsf{prov}_\alpha(A) := \exists p \,.\, \mathsf{proof}_\alpha(p,A)$. We assume that $\mathsf{proof}_\alpha(p,A)$ implies that $(p)_{\mathsf{lenght}(p) \dot{-} \overline{1}} \approx A$, i.e. that the last element of p is A. This implies that the proofs are single-conclusion proofs. We will also write $\mathsf{der}_\alpha(X,A)$ to symbolize the derivability predicate, in other words, provability from a (finite) set of assumptions X.

Some elementary functions that are important in this context are also fv, bv, of free variables and bound variables of a term/formula. Their respective terms are fv and bv. Also, the representation of the function defining numeral, which will be denoted as num. In case it is applied to a variable x we will denote it as $\dot{x}$.

Another elementary function that we are going to use widely is the representation of substitution, $\mathrm{subst}(\phi, s_0, s_1)$. The idea is that ϕ is a term or a formula, s_0 is a sequence of variables and s_1 a sequence of terms, both sequences of the same length. Then $\mathrm{subst}(\phi, s_0, s_1)$ means the substitution in ϕ of the first variable of s_0 for the first term of s_1, the second variable of s_0 for the second term of s_1, and so on. In case we write things like $\mathrm{subst}(A, x_0, \tau_0, \dots, x_n, \tau_n)$ where x_i are variables and τ_i are terms, we mean $\mathrm{subst}(\phi, \langle v_0, \dots, v_n\rangle, \langle \tau_0, \dots, \tau_n\rangle)$. The function can be represented in all the theories we work with, with its recursive equations, and as we said earlier, we will refer to it as subst.

So, for example, the theories T we work with fulfill that:

$$\vdash_T \mathsf{subst}\left(\overline{\phi}, \overline{\langle x_0, \dots, x_n\rangle}, \overline{\langle \tau_0, \dots, \tau_n\rangle}\right) \approx \overline{\mathrm{subst}(\phi, \langle x_0, \dots, x_n\rangle, \langle \tau_0, \dots, \tau_n\rangle)},$$

and

$$\vdash_T \mathsf{num}(\overline{n}) \approx \overline{\overline{n}}.$$

More concrete examples of these equalities, are:

$$\vdash_T \mathsf{subst}\left(\overline{x \approx y}, \overline{x}, \overline{x}, \overline{y}, \overline{\overline{3}}\right) \approx \overline{x \approx \overline{3}},$$

$$\vdash_T \mathsf{num}\left(\overline{3}\right) \approx \overline{\overline{3}},$$

and

$$\begin{aligned}
\vdash_T \mathsf{S}^\bullet \left(\left(\mathsf{S}^\bullet \overline{\mathsf{0}}^\bullet +^\bullet \mathsf{S}^\bullet \overline{\mathsf{0}}^\bullet \right) \times^\bullet \mathsf{S}^\bullet \overline{\mathsf{0}}^\bullet \right) &\approx \mathsf{S}^\bullet \left(\left(\overline{\mathsf{S}\overline{\mathsf{0}}} +^\bullet \overline{\mathsf{S}\overline{\mathsf{0}}} \right) \times^\bullet \overline{\mathsf{S}\overline{\mathsf{0}}} \right) \\
&\approx \mathsf{S}^\bullet \left(\overline{\mathsf{S}\overline{\mathsf{0}} + \mathsf{S}\overline{\mathsf{0}}} \times^\bullet \overline{\mathsf{S}\overline{\mathsf{0}}} \right) \\
&\approx \mathsf{S}^\bullet \left(\overline{\left(\mathsf{S}\overline{\mathsf{0}} + \mathsf{S}\overline{\mathsf{0}} \right) \times \mathsf{S}\overline{\mathsf{0}}} \right) \\
&\approx \overline{\mathsf{S} \left(\left(\mathsf{S}\overline{\mathsf{0}} + \mathsf{S}\overline{\mathsf{0}} \right) \times \mathsf{S}\overline{\mathsf{0}} \right)} \\
&\approx \overline{\overline{\mathsf{3}}} \approx \mathsf{num}\left(\overline{\mathsf{3}} \right).
\end{aligned}$$

Let $\phi(x_0, \dots, x_{n-1})$ be a formula whose free variables are among $x_0, \dots, x_{n-1}$. That formula can be understood as a function $\phi : \mathrm{Term}^n \longrightarrow \mathrm{Form}$ such that:

$$(\tau_0, \dots, \tau_{n-1}) \mapsto \phi(\tau_0, \dots, \tau_{n-1}).$$

This function can be arithmetized in iEA, so we have a term $\phi^\bullet$ such that for any terms $\tau_0, \dots, \tau_{n-1}$

$$\vdash_{\mathsf{iEA}} \phi^\bullet \left(\overline{\tau_0}, \dots, \overline{\tau_{n-1}} \right) \approx \overline{\phi(\tau_0, \dots, \tau_{n-1})}.$$

Similarly, given a term $\tau(x_0, \dots, x_{n-1})$ we have a term $\tau^\bullet$ such that

$$\vdash_{\mathsf{iEA}} \tau^\bullet \left(\overline{\tau_0}, \dots, \overline{\tau_{n-1}} \right) \approx \overline{\tau(\tau_0, \dots, \tau_{n-1})}.$$

Finally, let us discuss how to define $\Box\phi$ in the context of arithmetical theories. Assume we have a formula $\phi(x, y_0, \dots, y_{n-1})$, then we can define a function $\mathrm{box}_\phi : \mathrm{Form} \times \mathrm{Term}^{n-1} \longrightarrow \mathrm{Form}$ as

$$\mathrm{box}_\phi(\psi(z_0, \dots, z_{m-1}), \tau_0, \dots, \tau_{n-1}) := \phi(\mathsf{subst}(\overline{\psi}, \overline{z_0}, \dot{z}_0, \dots, \overline{z_{m-1}}, \dot{z}_{m-1}), \tau_0, \dots, \tau_{n-1}).$$

Note that the free variables of $\mathrm{box}_\phi(\psi, \tau_0, \dots, \tau_{n-1})$ are the union of the free variables of $\psi, \tau_0, \dots, \tau_{n-1}$. In case ϕ is a formula, whose only free variable is x, we will call box_ϕ a box function, and we will usually denote it by $\Box$ or $\triangle$ with subscripts. For example, if we have a predicate α with an associated provability predicate prov_α, we will write $\Box_\alpha$ for $\mathrm{box}_{\mathsf{prov}_\alpha}$. We will say that box_ϕ is Σ_1 to mean that ϕ is Σ_1.

Given a formula ϕ and the function box_ϕ, it can be represented inside iEA. As usual, the term representing this function will be denoted by box_ϕ and in case $\Box$ is an alternative notation for box_ϕ, $\Box^\bullet$ will be an alternative notation for box_ϕ. Let us $\phi(x)$ be a formula with only one free variable and $\Box$ denote its box function. Then we have the following properties:

$$\vdash_{\mathsf{iEA}} \forall A \,.\, \mathsf{form}(A) \to \mathsf{form}(\Box^\bullet A).$$

$$\vdash_{\mathsf{iEA}} \forall A \,.\, \mathsf{fv}(A) \approx \mathsf{fv}(\Box^\bullet A).$$

$$\vdash_{\mathsf{iEA}} \forall A \,.\, \mathsf{sent}(A) \to \mathsf{prov}_{\mathsf{iEA}}(\Box^\bullet A \leftrightarrow^\bullet \phi^\bullet(\dot{A})).$$

So the usual properties of box_ϕ are iEA-provable.

1.5 Provability logic

Finally, after introducing the necessary concepts from arithmetical theories and modal logic, we can introduce the fundamental concepts of provability logic.

1.5.1 Definition of provability logic of a theory

We will be working with a box function$\Box$ that comes from an arithmetical formula $P(x)$. We remember that in the previous section we talked about how to formalized a box function $\Box$ inside a theory T obtaining a term $\Box^{\bullet}$.

Definition 22. A *realization* is a function σ from propositional variables to arithmetical sentences. If $\Box : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$, we can extend σ with $\Box$ to $\sigma_\Box : \mathcal{L}_\mathrm{m} \longrightarrow$ Sent as follows:

$$
\begin{aligned}
&\sigma_\Box(p) := \sigma(p),\\
&\sigma_\Box(\top) := \top,\\
&\sigma_\Box(\bot) := \bot,\\
&\sigma_\Box(\phi \to \psi) := \sigma_\Box(\phi) \to \sigma_\Box(\psi),\\
&\sigma_\Box(\phi \wedge \psi) := \sigma_\Box(\phi) \wedge \sigma_\Box(\psi),\\
&\sigma_\Box(\phi \vee \psi) := \sigma_\Box(\phi) \vee \sigma_\Box(\psi),\\
&\sigma_\Box(\Box\phi) := \Box(\sigma_\Box(\phi)).
\end{aligned}
$$

When it is clear from context we may drop the parenthesis in $\sigma_\Box(\phi)$ and write $\sigma_\Box\phi$ directly.

A Σ_1-*realization* is a realization σ of shape $\sigma : \mathcal{L}_1 \longrightarrow \Sigma_1$-Sent. ∎

Note that in the clause $\sigma_\Box(\Box\phi) := \Box(\sigma_\Box(\phi))$ one has to be careful. The $\Box$ at the left hand side of $:=$ is part of the syntax of modal logic. On the contrary, the $\Box$ at the right hand side of $:=$ is a function from arithmetical formulas into arithmetical formulas. With realizations and its extensions one can define the provability logic of a theory with a box function.

Definition 23. Let T be an arithemtical theory and $\Box : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$. We define the set $\mathbb{PL}(T,\Box)$ as follows:

$$\mathbb{PL}(T,\Box) := \{\phi \in \mathcal{L}_\mathrm{m} \mid \text{ for any realization } \sigma,\ \vdash_T \sigma_\Box\phi\}.$$

We also define the set Σ_1-$\mathbb{PL}(T,\Box)$ as follows:

$$\Sigma_1\text{-}\mathbb{PL}(T,\Box) := \{\phi \in \mathcal{L}_\mathrm{m} \mid \text{ for any } \Sigma_1\text{-realization } \sigma,\ \vdash_T \sigma_\Box\phi\}.$$ ∎

Usually, theories T will have a selected provability predicate prov_T (more on this in Subsection). When that is the case, we talk about the *provability logic* of T or the Σ_1-*provability logic* of T, written $\mathbb{PL}(T)$ and Σ_1-$\mathbb{PL}(T)$ respectively, to mean $\mathbb{PL}(T,\Box_T)$ and Σ_1-$\mathbb{PL}(T,\Box_T)$.

1.5.2 Gödel-Löb and Hilbert-Bernays conditions

We define some conditions that we impose to box functions to make them adequate for provability logic. This conditions are natural conditions that would be impose to a provability notion. The names of the conditions are the names of the ones who proposed them first.

Definition 24 (Gödel-Löb conditions). Let U, T be theories and $\Box$ be a box function. Then we define the *uniform Gödel-Löb conditions for* $(U, T, \Box)$ as the following three conditions:

1. (Necessitation) For any $\phi \in \mathcal{L}_1$, if $\vdash_T \phi$ then $\vdash_U \Box\phi$.
2. (K) For any $\phi, \psi \in \mathcal{L}_1$, $\vdash_U \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$.
3. (Trans) For any $\phi \in \mathcal{L}_1$, $\vdash_U \Box\phi \to \Box\Box\phi$.

We will write $\mathrm{GL}_{U,T,\Box}$ to mean that $(U, T, \Box)$ satisfy the uniform Gödel-Löb conditions. If we restrict the formulas in the conditions to be sentences we obtain the *Gödel-Löb conditions for* $(U, T, \Box)$. When $(U, T, \Box)$ satisfy these conditions we will write $\mathrm{Gl}_{U,T,\Box}$. When $U = T$ we will write $\mathrm{GL}_{T,\Box}$ and $\mathrm{Gl}_{T,\Box}$ instead of $\mathrm{GL}_{T,T,\Box}$ and $\mathrm{Gl}_{T,T,\Box}$. ■

Definition 25 (Hilbert-Bernays conditions). Let U, T be theories and $\Box$ be a box function. Then we define the *uniform Hilbert-Bernays conditions for* $(U, T, \Box)$ as the uniform Gödel-Löb conditions plus the additional condition

4. (Completeness) For any $\phi \in \Sigma_1$, $\vdash_U \phi \to \Box\phi$.

We will write $\mathrm{HB}_{U,T,\Box}$ to mean that $(U, T, \Box)$ satisfy the uniform Hilbert-Bernays conditions. If we restrict the formulas in the conditions to be sentences we obtain the *Hilbert-Bernays conditions for* $(U, T, \Box)$. When $(U, T, \Box)$ satisfy these conditions we will write $\mathrm{Hb}_{U,T,\Box}$. When $U = T$ we will write $\mathrm{HB}_{T,\Box}$ and $\mathrm{Hb}_{T,\Box}$ instead of $\mathrm{HB}_{T,T,\Box}$ and $\mathrm{Hb}_{T,T,\Box}$. ■

Lemma 26. Let U, T be theories and $\Box$ be a Σ_1 box function. Then$\mathrm{HB}_{U,T,\Box}$.Compl implies $\mathrm{HB}_{U,T,\Box}$.Trans. The same holds if instead of HB we have Hb.

Proof. Just notice that if $\Box\phi \in \Sigma_1$ then by $\mathrm{HB}_{U,T,\Box}$.Compl we get $\vdash_U \Box\phi \to \Box\Box\phi$. □

Lemma 27. Assume that $\mathrm{HB}_{U,T,\Box}$. Then, for any $\phi \in \mathcal{L}_1$ if $\vdash_{\mathsf{iFOL}} \phi$ then $\vdash_U \Box\phi$.

Proof. Trivial, since $\vdash_{\mathsf{iFOL}} \phi$ implies $\vdash_T \phi$ and then by $\mathrm{HB}_{U,T,\Box}$ we get $\vdash_U \Box\phi$. □

Lemma 28. Assume that $\mathrm{HB}_{U,T,\Box}$ and for any $\phi \in \mathcal{L}_1$, $\mathrm{fv}(\phi) = \mathrm{fv}(\Box\phi)$. Then

1. $\vdash_U \Box(\forall x \,.\, \phi) \to (\forall x \,.\, \Box\phi)$,
2. $\vdash_U (\exists x \,.\, \Box\phi) \to \Box(\exists x \,.\, \phi)$.

Proof. Note that $\vdash_{\mathsf{iFOL}} (\forall x \,.\, \phi) \to \phi$, so by $\mathrm{HB}_{U,T,\Box}$ and Lemma 27 we get

$$\vdash_U \Box(\forall x \,.\, \phi) \to \Box\phi.$$

By generalization rule we get

$$\vdash_U \forall x . \Box(\forall x \,.\, \phi) \to \Box\phi.$$

which implies

$$\vdash_U (\forall x \,.\, \Box(\forall x \,.\, \phi)) \to (\forall x \,.\, \Box\phi).$$

But since $x \notin \mathrm{fv}(\forall x \,.\, \phi)$ we have that $x \notin \mathrm{fv}(\Box(\forall x \,.\, \phi))$. This implies

$$\vdash_U \Box(\forall x \,.\, \phi) \to (\forall x \,.\, \Box\phi).$$

The proof of (2) is analogous. $\Box$

Lemma 29 (Monotonicity in the conditions)**.** Let C be HB or GL (Hb or Gl). We have that

1. If $U_0 \subseteq U_1$, then $C_{U_0,T,\Box}$ implies $C_{U_1,T,\Box}$.
2. If $T_0 \subseteq T_1$, then $C_{U,T_1,\Box}$ implies $C_{U,T_0,\Box}$.
3. Assume that for any formula (sentence) ϕ, $\vdash_U \triangle\phi \to \Box\phi$ and $\Box$ is a Σ_1 box function. Then, $\mathrm{HB}_{U,T,\triangle}$ ($\mathrm{Hb}_{U,T,\triangle}$) implies $\mathrm{HB}_{U,T,\Box}$ ($\mathrm{Hb}_{U,T,\Box}$).

Proof. (1) and (2) are trivial. Let us talk about (3). We only show transitivity since it is the trickiest. The idea is that since $\Box\phi$ is Σ_1 and $\mathrm{HB}_{U,T,\triangle}$ we have that $\vdash_U \Box\phi \to \triangle\Box\phi$. But by assumption $\vdash_U \triangle\Box\phi \to \Box\Box\phi$ so we get the desired $\vdash_U \Box\phi \to \Box\Box\phi$. $\Box$

Finally, let us talk about how to formalized this conditions in arithmetical theories. The idea is that instead of having a triple of two theories and a function, we have a triple of two unary predicates and a unary term. In this case we will denote the conditions with sans-serif font. For example, let us have $\phi(x), \psi(x)$ unary predicates and $\tau(x)$ an unary term. Then $\mathsf{HB}_{\phi,\psi,\tau}$ denotes the conjunction of the following four formulas:

1. $\forall A . \mathsf{form}(A) \wedge \psi(A) \to \phi(\tau(A))$.
2. $\forall A, B . \mathsf{form}(A) \wedge \mathsf{form}(B) \to \phi(\tau(A \to^{\bullet} B) \to^{\bullet} \tau(A) \to^{\bullet} \tau(B))$.
3. $\forall A . \mathsf{form}(A) \to \phi(\tau(A) \to^{\bullet} \tau(\tau(A)))$.

4. $\forall A.\Sigma_1\text{-form}(A) \rightarrow \phi(A \rightarrow^{\bullet} \tau(A))$.

We do the same with GL to get GL. Note that to get Hb and Gl it suffices to change form to sent. Then,

$$\vdash_T \mathsf{HB}_{\phi,\psi,\tau}$$

means that T shows that (ϕ, ψ, τ) fulfills the Hilbert-Bernays conditons.

1.5.3 Reflection and Absorption

In this subsection we define the reflection and absorption principles. These princples are a fundamental part of our tools.

Definition 30 (Reflection). Let $\Box$ be a box function, then we define the set $\mathrm{RFN}_{\Box}$ of *uniform reflection principles*, as

$$\mathrm{RFN}_{\Box} := \{\Box\phi \rightarrow \phi \mid \phi \in \mathcal{L}_1\}.$$

In case we want to restrict the formulas ϕ's to belong to a particular set Γ we will write Σ_1-$\mathrm{RFN}_{\Box}$. Of particular importance is the set of *sentential reflection principles*, i.e. Sent-$\mathrm{RFN}_{\Box}$, which we will denote simply as $\mathrm{Rfn}_{\Box}$.

We will write things like $\mathrm{RFN}_{T,\Box}$ and $\mathrm{Rfn}_{T,\Box}$, where T is an arithmetical theory, to mean that $\vdash_T \mathrm{RFN}_{T,\Box}$ and $\vdash_T \mathrm{Rfn}_{\Box}$, respectively. ■

Note that reflection is in some sense an internatization of soundness. One of our main use for reflection will be to prove another principle that will be fundamental for our tools: absorption.

Definition 31 (Absorption). Let $\Box, \triangle$ be box functions. We define the set of sentence $\mathrm{Abs}_{\Box,\triangle}$ as:

$$\mathrm{Abs}_{\Box,\triangle} := \{\Box\triangle\phi \rightarrow \Box\phi \mid \phi \in \mathrm{Sent}\}.$$

Also, if $\Gamma \subseteq \mathcal{L}_1$, we define the set of sentences Γ-$\mathrm{Abs}_{\Box,\triangle}$ as:

$$\mathrm{Abs}_{\Box,\triangle} := \{\Box\triangle\phi \rightarrow \Box\phi \mid \phi \in \mathrm{Sent} \cap \Gamma\}.$$

Finally, we will write $\mathrm{Abs}_{T,\Box\triangle}$ to mean that $\vdash_T \mathrm{Abs}_{\Box,\triangle}$ and similarly with Γ-$\mathrm{Abs}_{\Box,\triangle}$. ■

Lemma 32. Let $\Box$ be a box function and $\triangle$ be a Σ_1 box function. Then, $\mathrm{Hb}_{T,\Box}$ and $\mathrm{Abs}_{T,\Box,\triangle}$ implies that for any sentence ϕ, we have that $\vdash_T \triangle\phi \rightarrow \Box\phi$.

Proof. Let ϕ be a sentence, we know that $\triangle\phi \in \Sigma_1$. Then, by $\mathrm{Hb}_{T,\Box}$, we get that $\vdash_T \triangle\phi \rightarrow \Box\triangle\phi$. Since $\mathrm{Abs}_{T,\Box,\triangle}$, we also get that $\vdash_T \Box\triangle\phi \rightarrow \Box\phi$. Using these two facts, we get $\vdash_T \triangle\phi \rightarrow \Box\phi$, as desired. □

1.5.4 Fefermanian predicates

Definition 33. Let U, T be theories and α be a formula. We say that α enumerates T in U if

$$\phi \in \mathrm{Ax}_T \text{ iff } \vdash_U \alpha(\overline{\phi}).$$

We say that α truly enumerates T if

$$\phi \in \mathrm{Ax}_T \text{ iff } \vDash_{\mathbb{N}} \alpha(\overline{\phi}).$$

We will say that a theory T is *U-arithmetized* iff there is a formula α such that α enumerates T in U. In this case we will refer to the α as ax_T.

If Γ is a class of formulas, we will say that T is (Γ, U)-arithmetized if $\mathsf{ax}_T \in \Gamma$. In case $U = T$ we will say that it is self arithmetized and in case there is an α that truly enumerates T we will say that it is truly arithmetized and we will refer to this α as ax_T. ■

In the context of an arithmetized theory T we will write $\mathsf{proof}_T, \mathsf{prov}_T$ and $\Box_T$ to mean $\mathsf{proof}_{\mathsf{ax}_T}, \mathsf{prov}_{\mathsf{ax}_T}$ and $\Box_{\mathsf{ax}_T}$, respectively. We put here some lemmas that allow us to derive HB conditions from provability predicates defined by an axiomatization formula that enumerates a theory. We start with the modus ponens condition.

Lemma 34. Let T, U be theories such that

1. $\mathsf{iEA} \subseteq U$.

Then, for any formulas ϕ, ψ

$$\vdash_U \Box_\alpha(\phi \to \psi) \to \Box_\alpha \phi \to \Box_\alpha \psi.$$

The necessitation condition, which is proven by induction (performed in the metatheory) on the proof of $\vdash_T \phi$.

Lemma 35. Let T, U be theories such that

1. $\mathsf{iEA} \subseteq U$.

2. α enumerates T in U.

Then, for any formula ϕ

$$\vdash_T \phi \text{ implies } \vdash_U \Box_\alpha \phi.$$

The formalized completeness principle.

Lemma 36. Let U be theory in the language of $\mathcal{L}(\mathsf{exp})$ such that

1. $\mathsf{iEA} \subseteq U$.

2. $\vdash_U \forall A \,.\, \mathsf{prov}_{\mathsf{iQ}^{\mathsf{exp}}}(A) \to \mathsf{prov}_\alpha(A)$.

Then for any formula $\phi \in \Sigma_1$ we have that:

$$\vdash_U \phi \to \Box_\alpha \phi.$$

If we want to have all the HB conditions we need one additional detail. We need that U shows that $\Box_\alpha \phi$ is Σ_1. There are two alternatives: either α is Δ_0 or α is Σ_1 but U proves Σ_1-collection. Then, we can put this corollary:

Corollary 37. Let T, U be theories in $\mathcal{L}(\mathsf{exp})$ such that

1. Either α is Δ_0 and $\mathsf{iEA} \subseteq U$ or α is Σ_1 and $\mathsf{iEA} + B\Sigma_1 \subseteq U$.
2. α enumerates T in U.
3. $\vdash_U \forall A\,.\,\mathsf{prov}_{\mathsf{iQ}^{\mathsf{exp}}}(A) \to \mathsf{prov}_\alpha(A)$.

Then $\mathrm{HB}_{U,T,\Box_\alpha}$.

In case U, T are theories in $\mathcal{L}(\mathscr{Pr})$ we just need to change $\mathsf{iQ}^{\mathrm{exp}}$ to $\mathsf{iQ}^{\mathscr{Pr}}$ which is the theory based on iFOL whose axioms are those of PRA without induction and replacing $\overline{0} \neq \mathsf{S}\overline{0}$ for $\forall x\,.\,\overline{0} \neq \mathsf{S}x$. In particular, $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathsf{iPRA}}}$.

The proofs needed for these lemmas can be carried inside iEA, so

Corollary 38. Let T, U, V be theories in $\mathcal{L}(\mathsf{exp})$ such that

1. V is an extension of iEA.
2. We have one of the following:
   (a) α is Δ_0 and $\vdash_V \mathsf{prov}_{\mathsf{iEA}}(A) \to \mathsf{prov}_U(A)$
   (b) α is Σ_1 and $\vdash_V \mathsf{prov}_{\mathsf{iEA}+B\Sigma_1}(A) \to \mathsf{prov}_U(A)$.
3. $\vdash_V \forall A\,.\,\mathsf{ax}_T(A) \to \mathsf{prov}_U(\alpha^\bullet(\dot{A}))$.
4. $\vdash_V \Box_U(\forall A\,.\,\mathsf{prov}_{\mathsf{iQ}^{\mathsf{exp}}}(A) \to \mathsf{prov}_\alpha(A))$.

Then $\vdash_V \mathsf{HB}_{\mathsf{prov}_U,\mathsf{prov}_T,\Box^\bullet_\alpha}$

Finally, we put two lemmas which allow to change the base theory used for the enumeration.

Lemma 39. Let U_0, U_1 be Δ_0-complete and consistent and α be Δ_0. Then

$$\alpha \text{ enumerates } T \text{ in } U_0 \text{ iff } \alpha \text{ enumerates } T \text{ in } U_1.$$

Lemma 40. Let U_0, U_1 be Σ_1-sound and Σ_1-complete. Further, assume that $\vdash_{U_i} B\Sigma_1$. Then

$$\alpha \text{ enumerates } T \text{ in } U_0 \text{ iff } \alpha \text{ enumerates } T \text{ in } U_1.$$

Truly arithmetized

Now, let us connect theories arithmetized in other theories with truly arithmetized theories.

Lemma 41. Let T,U be theries and $\alpha \in \Gamma$. Assume that U is Γ-sound and Γ-complete. Then

$$\alpha \text{ enumerates } T \text{ in } U \text{ iff } \alpha \text{ truly enumerates } T.$$

Proof. It suffices to show that for any sentence ϕ,

$$\vdash_U \alpha\left(\overline{\phi}\right) \text{ iff } \vDash_{\mathbb{N}} \alpha\left(\overline{\phi}\right).$$

But this is trivial since $\alpha \in \Gamma$ and U is Γ-sound and Γ-complete. $\square$

Lemma 42. Assume that α truly enumerates T. Then for any formula ϕ,

$$\vdash_T \phi \text{ iff } \vDash_{\mathbb{N}} \mathsf{prov}_\alpha\left(\overline{\phi}\right).$$

Proof. Left to right is proven by a simple induction in the proof of ϕ. For right to left we know, by definition of prov_T, that there is a sequence of formulas $\langle \phi_0, \dots, \phi_n \rangle$ such that $\vDash_{\mathbb{N}} \mathsf{proof}\left(\overline{\langle \phi_0, \dots, \phi_n \rangle}, \overline{\phi}\right)$. Then we just need to proceed by strong induction (in the metatheory) over the length of the sequence. $\square$

Chapter 2

Translations

In this chapter we give the definition of various known translations of first-order formulas. With a translation we mean a function from $\mathcal{L}_1 \longrightarrow \mathcal{L}_1$. All these translations will have a theorem which claims that, under certain conditions on the theories, if a theory T proves ϕ then a theory U proves the translation of ϕ. What is more, all the results we are going to see here are formalizable inside $\mathsf{iI}\Sigma_1$[1]. This is of great importance, since we will use mainly the formalized version of the theorems in the rest of chapters. All these results are widely known, we write them here for completeness and because some of them need light modifications in the condititions, to be more general.

We have only included the proofs related to Visser's translation, since it is the most important translation for our purposes. The proofs related to the rest of the translations can be found in Appendix A. The reader can skip this chapter in a first reading and consult it only when needed.

2.1 Gödel's and Friedman's Translations

First, we define a concept that is going to be fundamental for most of the results in this chapter.

Definition 43. Let T be a theory and $f : \mathcal{L}_1(T) \longrightarrow \mathcal{L}_1(T)$. We say that T *is closed under* f iff

$$\text{For any } \phi \in \mathrm{Ax}_T, \quad \vdash_T f(\phi).$$

■

We define Gödel's double negation translation. This translation is really useful to relate classical arithmetical theories with intuitionistic arithmetical theories.

[1] In fact, they would also be provable in iEA, but seeing that they are formalizable in $\mathsf{iI}\Sigma_1$ is straightforward

Definition 44 (Gödel's translation, Double negation translation). Let ϕ be a formula. We define recursively the formula $(\phi)_{\mathrm{G}}$ as:

$$
\begin{aligned}
(\phi)_{\mathrm{G}} &:= \phi \text{ if } \phi \text{ atomic},\\
(\phi \wedge \psi)_{\mathrm{G}} &:= (\phi)_{\mathrm{G}} \wedge (\psi)_{\mathrm{G}},\\
(\phi \vee \psi)_{\mathrm{G}} &:= \neg\neg((\phi)_{\mathrm{G}} \vee (\psi)_{\mathrm{G}}),\\
(\phi \to \psi)_{\mathrm{G}} &:= (\phi)_{\mathrm{G}} \to (\psi)_{\mathrm{G}},\\
(\forall x \,.\, \phi)_{\mathrm{G}} &:= \forall x \,.\, (\phi)_{\mathrm{G}},\\
(\exists x \,.\, \phi)_{\mathrm{G}} &:= \neg\neg(\exists x \,.\, (\phi)_{\mathrm{G}}).
\end{aligned}
$$
■

Lemma 45. Let T be a Δ_0-decidable theory. Then for any $\phi \in \Delta_0$ we have that

$$\vdash_T (\phi)_{\mathrm{G}} \leftrightarrow \phi.$$

Theorem 46. Let T be a theory closed under $(\_)_{\mathrm{G}}$ and let $\phi \in \mathcal{L}_1$. Then,

$$\vdash_{\mathsf{FOL},T} \phi \text{ implies } \vdash_{i\mathsf{FOL},T} (\phi)_{\mathrm{G}}.$$

This is $\mathsf{iI}\Sigma_1$ verifiable.

Using deduction theorem it is easy to conclude the following corollary.

Corollary 47. Let T be a theory closed under $(\_)_{\mathrm{G}}$ and let $\Gamma \subseteq \mathcal{L}_1$ and $\phi \in \mathcal{L}_1$. Then,

$$\Gamma \vdash_{\mathsf{FOL},T} \phi \text{ implies } (\Gamma)_{\mathrm{G}} \vdash_{i\mathsf{FOL},T} (\phi)_{\mathrm{G}}.$$

Now we proceed with the Frieman's translation. The idea of this translation is to traverse the formula until it arrives the atomic subformulas and then introduce a disjunction.

Definition 48 (Friedman's translation). Let ψ be a formula, we define the formula $(\phi)^{\psi}_{\mathrm{F}}$ recursively in ϕ as:

$$
\begin{aligned}
&(\phi)^{\psi}_{\mathrm{F}} := \phi \vee \psi, \text{ if } \phi \text{ is atomic;}\\
&(\phi_0 \circ \phi_1)^{\psi}_{\mathrm{F}} := (\phi_0)^{\psi}_{\mathrm{F}} \circ (\phi_1)^{\psi}_{\mathrm{F}}, \text{ where } \circ \in \{\to, \vee, \wedge\};\\
&(Qx \,.\, \phi)^{\psi}_{\mathrm{F}} := Qx \,.\, (\phi)^{\psi}_{\mathrm{F}}, \text{ where } Q \in \{\forall, \exists\}.
\end{aligned}
$$
■

We prove a fundamental property of this translation. The idea behind this property, is the same as the axiom $\psi \to \phi \vee \psi$. Note however, that the equivalent of $\phi \to \phi \vee \psi$, i.e. $\phi \to (\phi)^{\psi}_{\mathrm{F}}$, does not hold in general.

Lemma 49. Let ϕ, ψ be formulas such that no free variable of ψ occurs bounded in ϕ. Then:

$$\vdash \psi \to (\phi)^{\psi}_{\mathrm{F}}.$$

Now, two lemmas establishing that for Δ_0 or Σ_1 formulas and Δ_0-decidable theoreis, the translation is equivalent to the disjunction.

Lemma 50. Let T be a Δ_0-decidable theory. Then for any Δ_0-formula ϕ and any ψ, such that the free variables of ψ are not bounded in ϕ, we have

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}} \leftrightarrow \phi \vee \psi.$$

Lemma 51. Let T be a Δ_0-decidable theory. Then, for any Σ_1-formula ϕ and any ψ, such that no free variable of ψ appears bounded in ϕ, we have that:

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}} \leftrightarrow \phi \vee \psi.$$

Now we need two technical lemmas, one about variables and the other one about substitutions.

Lemma 52. For any formulas ϕ, ψ, such that no free variable of ψ appears bounded in ϕ:

1. $\mathrm{fv}((\phi)^{\psi}_{\mathrm{F}}) = \mathrm{fv}(\phi) \cup \mathrm{fv}(\psi)$.
2. If x is free for τ in ϕ, ψ then x is free for τ in $(\phi)^{\psi}_{\mathrm{F}}$.

Lemma 53. Let ϕ, ψ be formulas, τ a term and x a variable. Assume that no free variable of ψ occurs bounded in ϕ and $x \notin \mathrm{fv}(\psi)$. Then

$$(\phi[x/\tau])^{\psi}_{\mathsf{F}} = (\phi)^{\psi}_{\mathsf{F}}[x/\tau].$$

Finally, the main theorem:

Theorem 54. Let ϕ, ψ be formulas. Assume that we have a proof π of $\vdash \phi$ such that no free variables of ψ appears bounded in the formulas of π and that T is closed under $(\_)^{\psi}_{\mathrm{F}}$. Then

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}}.$$

Note that in a proof we can always rearrange the name of the bound variables to make the theorem applicable. As a corollary of the theorem, using the deduction theorem.

Corollary 55. Assume that T is closed under $(\_)^{\psi}_{\mathrm{F}}$ and we have a proof π of $\Gamma \vdash_T \phi$ such that no free variable of ψ apears in π or in Γ bounded. Then

$$(\Gamma)^{\psi}_{\mathrm{F}} \vdash_T (\phi)^{\psi}_{\mathrm{F}}.$$

Corollary 56. Let T be closed under $(\_)^{\psi}_{\mathrm{F}}$. If we have a proof π of $\vdash_T \phi_0 \leftrightarrow \phi_1$, where no free variable of ψ appears bounded in π, then $\vdash_T (\phi_0)^{\psi}_{\mathrm{F}} \leftrightarrow (\phi_1)^{\psi}_{\mathrm{F}}$.

Finally, we state the formalized version inside any extension of $\mathsf{iI}\Sigma_1$.

Corollary 57. Let U be a theory such that

1. $\mathsf{iI}\Sigma_1 \subseteq U$.

2. $\vdash_U \forall A \,.\, \alpha(A) \rightarrow \mathsf{sent}(A)$.

Then, we have that

$$\begin{aligned}
&\forall C \,.\, \alpha(C) \rightarrow \mathsf{prov}_\alpha(C)^B_{\mathsf{F}},\\
&\forall v \,.\, \neg(v \in \mathsf{fv}(A) \wedge \exists C \in p \,.\, v \in \mathsf{bv}(C)),\\
&\mathsf{proof}_\alpha(p, A)\\
&\vdash_U \mathsf{prov}_\alpha(A)^B_{\mathsf{F}}.
\end{aligned}$$

We note that we have written $\mathsf{prov}_\alpha(C)^B_{\mathrm{F}}$ instead of $\mathsf{prov}_\alpha((C)^B_{\mathrm{F}})$. This kind of omission of brackets will be used frequently, even with other translations.

Π_2-conservativity over classical theory

Finally, this lemma claims that with Gödel's and Friedman's translations one can prove the Π_2-conservativity of classical theories over intuitionistic theories. We note that all of this is provable inside $\mathsf{i}\mathsf{I}\Sigma_1$.

Lemma 58. Let T be a theory such that it is closed under Gödel's translation and under Friedman's translation for Σ_1-formulas. Then, T with classical logic is Π_2-conservative over T with intuitionistic logic.

2.2 Visser's Translation

In this section we will define Visser's translation. This translation will not only be defined over formulas, but also over theories. Then, the method for calculating the Σ_1-provability logic of a intuitionistic theory that we use here can be understood as calculating the Σ_1-provability logic of the Visser translated theory and then lift the calculation to the original theory.

2.2.1 Definition and general properties

If we are given a box function $\Box$, we will define $\boxdot : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$ as

$$\boxdot\phi := \phi \wedge \Box\phi.$$

We define the main translation for our purposes: Visser's translation.

Definition 59 (Visser's Translation). Let $\Box : \mathrm{Form} \longrightarrow \mathrm{Form}$. We define the function $(\_)^{\Box}_{\mathrm{V}} : \mathrm{Form} \longrightarrow \mathrm{Form}$ by recursion as:

$$\begin{aligned}
(\phi)^{\Box}_{\mathrm{V}} &:= \phi, \text{ if } \phi \text{ atomic;}\\
(\phi \wedge \psi)^{\Box}_{\mathrm{V}} &:= (\phi)^{\Box}_{\mathrm{V}} \wedge (\psi)^{\Box}_{\mathrm{V}};\\
(\phi \vee \psi)^{\Box}_{\mathrm{V}} &:= (\phi)^{\Box}_{\mathrm{V}} \vee (\psi)^{\Box}_{\mathrm{V}};\\
(\phi \to \psi)^{\Box}_{\mathrm{V}} &:= \boxdot((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}});\\
(\exists x\,.\,\phi)^{\Box}_{\mathrm{V}} &:= \exists x\,.\,(\phi)^{\Box}_{\mathrm{V}};\\
(\forall x\,.\,\phi)^{\Box}_{\mathrm{V}} &:= \boxdot(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}}).
\end{aligned}$$

■

The following lemma is the fundamental property of Visser's translation.

Lemma 60. Let T be a theory such that $\mathrm{HB}_{T,\Box}$. Then, for any formula ϕ

$$\vdash_T (\phi)^{\Box}_{\mathrm{V}} \to \Box(\phi)^{\Box}_{\mathrm{V}}.$$

Proof. We proceed by induction on the complexity of ϕ. If ϕ is atomic we use $\mathrm{HB}_{T,\Box}.\mathrm{Compl}$.

Let $\phi = \phi_0 \wedge \phi_1$. Then by the induction hypothesis

$$\vdash_T (\phi_i)^{\Box}_{\mathrm{V}} \to \Box(\phi_i)^{\Box}_{\mathrm{V}}.$$

Then

$$\begin{aligned}
\vdash_T (\phi_0 \wedge \phi_1)^{\Box}_{\mathrm{V}} &= (\phi_0)^{\Box}_{\mathrm{V}} \wedge (\phi_1)^{\Box}_{\mathrm{V}}\\
&\to \Box(\phi_0)^{\Box}_{\mathrm{V}} \wedge \Box(\phi_1)^{\Box}_{\mathrm{V}} && \text{(by I.H.)}\\
&\to \Box((\phi_0)^{\Box}_{\mathrm{V}} \wedge (\phi_1)^{\Box}_{\mathrm{V}}) && \text{(by } \mathrm{HB}_{T,\Box}\text{)}\\
&= \Box(\phi_0 \wedge \phi_1)^{\Box}_{\mathrm{V}}
\end{aligned}$$

Let $\phi = \phi_0 \vee \phi_1$. Then by the induction hypothesis

$$\vdash_T (\phi_i)^{\Box}_{\mathrm{V}} \to \Box(\phi_i)^{\Box}_{\mathrm{V}}.$$

Then

$$\begin{aligned}
\vdash_T (\phi_0 \vee \phi_1)^{\Box}_{\mathrm{V}} &= (\phi_0)^{\Box}_{\mathrm{V}} \vee (\phi_1)^{\Box}_{\mathrm{V}}\\
&\to \Box(\phi_0)^{\Box}_{\mathrm{V}} \vee \Box(\phi_1)^{\Box}_{\mathrm{V}} && \text{(by I.H.)}\\
&\to \Box((\phi_0)^{\Box}_{\mathrm{V}} \vee (\phi_1)^{\Box}_{\mathrm{V}}) && \text{(by } \mathrm{HB}_{T,\Box}\text{)}\\
&= \Box(\phi_0 \vee \phi_1)^{\Box}_{\mathrm{V}}
\end{aligned}$$

Let $\phi = \phi_0 \to \phi_1$. Then

$$\begin{aligned}
\vdash_T (\phi_0 \to \phi_1)^{\Box}_{\mathrm{V}} &= \boxdot((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) \\
&\to \Box((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) \\
&\to \Box\boxdot((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) && (\text{by } \mathrm{HB}_{T,\Box}) \\
&= \Box(\phi_0 \to \phi_1)^{\Box}_{\mathrm{V}}
\end{aligned}$$

Let $\phi = \exists x \,.\, \phi_0$. Then by the induction hypothesis

$$\vdash_T (\phi_0)^{\Box}_{\mathrm{V}} \to \Box(\phi_0)^{\Box}_{\mathrm{V}}.$$

Then

$$\begin{aligned}
\vdash_T (\exists x \,.\, \phi_0)^{\Box}_{\mathrm{V}} &= \exists x \,.\, (\phi_0)^{\Box}_{\mathrm{V}} \\
&\to \exists x \,.\, \Box(\phi_0)^{\Box}_{\mathrm{V}} && (\text{by I.H.}) \\
&\to \Box(\exists x \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) && (\text{by } \mathrm{HB}_{T,\Box}) \\
&= \Box(\exists x \,.\, \phi_0)^{\Box}_{\mathrm{V}}
\end{aligned}$$

Let $\phi = \forall x \,.\, \phi_0$. Then

$$\begin{aligned}
\vdash_T (\forall x \,.\, \phi_0)^{\Box}_{\mathrm{V}} &= \boxdot(\forall x \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) \\
&\to \Box(\forall x \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) \\
&\to \Box\boxdot(\forall x \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) && (\text{by } \mathrm{HB}_{T,\Box}) \\
&= \Box(\forall x \,.\, \phi_0)^{\Box}_{\mathrm{V}}
\end{aligned}$$

$\Box$

We prove multiple lemmas that ease the calculation of the translation of a formula.

Lemma 61. Let T be a theory such that $\mathrm{HB}_{T,\Box}$. Then:

$$\vdash_T (\forall x_0, \dots, x_{n-1} \,.\, \phi)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot(\forall x_0, \dots, x_{n-1} \,.\, (\phi)^{\Box}_{\mathrm{V}}).$$

Proof. By induction in n. The case where $n = 0$ is thanks to Lemma 60.

Case $n+1$. By the induction hypothesis we have that

$$\vdash_T (\forall x_1, \dots, x_n \,.\, \phi)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot(\forall x_1, \dots, x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}).$$

Using iFOL reasoning

$$\vdash_T (\forall x_0 \,.\, (\forall x_1, \dots, x_n \,.\, \phi)^{\Box}_{\mathrm{V}}) \leftrightarrow (\forall x_0 \,.\, \boxdot(\forall x_1, \dots, x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})), \tag{i}$$

and by $\mathrm{HB}_{T,\Box}$ we also get

$$\vdash_T \Box(\forall x_0 \,.\, (\forall x_1,\dots,x_n \,.\, \phi)^{\Box}_{\mathrm{V}}) \leftrightarrow \Box(\forall x_0 \,.\, \boxdot(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})). \tag{ii}$$

Then

$$\begin{aligned}\vdash_T (\forall x_0,\dots,x_n \,.\, \phi)^{\Box}_{\mathrm{V}} &= \boxdot(\forall x_0 \,.\, (\forall x_1,\dots,x_n \,.\, \phi)^{\Box}_{\mathrm{V}})\\ &\leftrightarrow \boxdot(\forall x_0 \,.\, \boxdot(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})) && \text{(by (i) and (ii))}\\ &\overset{*}{\leftrightarrow} \boxdot(\forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})\end{aligned}$$

Let us show $\overset{*}{\leftrightarrow}$ in detail.

$\overset{*}{\rightarrow}$. Note that

$$\vdash_T \boxdot(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}) \rightarrow \forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}},$$

so

$$\vdash_T (\forall x_0 \,.\, \boxdot(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})) \rightarrow \forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}},$$

and finally by $\mathrm{HB}_{T,\Box}$ we can conclude:

$$\vdash_T \boxdot(\forall x_0 \,.\, \boxdot(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})) \rightarrow \boxdot(\forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}).$$

$\overset{*}{\leftarrow}$. It suffices that we show

$$\vdash_T \Box(\forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}) \rightarrow \boxdot(\forall x_0 \,.\, \Box(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})) \tag{iii}$$

Note that by Lemma 28 we have that

$$\vdash_T \Box(\forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}) \rightarrow \forall x_0 \,.\, \Box(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}).$$

Then by $\mathrm{HB}_{T,\Box}$ we have that:

$$\vdash_T \Box\Box(\forall x_0,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}}) \rightarrow \Box(\forall x_0 \,.\, \Box(\forall x_1,\dots,x_n \,.\, (\phi)^{\Box}_{\mathrm{V}})).$$

But $\vdash_T \Box(\forall x_0,\dots,x_n \,.\, (\phi)^{P}_{\mathrm{V}}) \rightarrow \Box\Box(\forall x_0,\dots,x_n \,.\, (\phi)^{P}_{\mathrm{V}})$, so we have (iii). $\Box$

Lemma 62. Let T be a theory such that $\mathrm{HB}_{T,\Box}$. Then:

$$\vdash_T \left(\bigwedge_{i\leq n} \phi_i \rightarrow \psi_i\right)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot\left(\bigwedge_{i\leq n} (\phi_i)^{\Box}_{\mathrm{V}} \rightarrow (\psi_i)^{\Box}_{\mathrm{V}}\right).$$

Proof. By induction in n. If $n=0$, then it is trivial by definition.

Case $n+1$. By the induction hypothesis we have that

$$\vdash_T \left(\bigwedge_{i\leq n} \phi_i \rightarrow \psi_i\right)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot\left(\bigwedge_{i\leq n} (\phi_i)^{\Box}_{\mathrm{V}} \rightarrow (\psi_i)^{\Box}_{\mathrm{V}}\right).$$

Then

$$
\begin{aligned}
\vdash_T \left(\bigwedge_{i\le n+1} \phi_i \to \psi_i\right)^{\Box}_{\mathrm{V}} &= \left(\bigwedge_{i\le n} \phi_i \to \psi_i\right)^{\Box}_{\mathrm{V}} \wedge (\phi_{n+1} \to \psi_{n+1})^{\Box}_{\mathrm{V}} \\
&= \left(\bigwedge_{i\le n} \phi_i \to \psi_i\right)^{\Box}_{\mathrm{V}} \wedge \boxdot((\phi_{n+1})^{\Box}_{\mathrm{V}} \to (\psi_{n+1})^{\Box}_{\mathrm{V}}) \\
&\leftrightarrow \boxdot\left(\bigwedge_{i\le n} (\phi_i)^{\Box}_{\mathrm{V}} \to (\psi_i)^{\Box}_{\mathrm{V}}\right) \wedge \boxdot((\phi_{n+1})^{\Box}_{\mathrm{V}} \to (\psi_{n+1})^{\Box}_{\mathrm{V}}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by I.H.)} \\
&\leftrightarrow \left(\bigwedge_{i\le n+1} (\phi_i)^{\Box}_{\mathrm{V}} \to (\psi_i)^{\Box}_{\mathrm{V}}\right) \wedge \Box\left(\bigwedge_{i\le n+1} (\phi_i)^{\Box}_{\mathrm{V}} \to (\psi_i)^{\Box}_{\mathrm{V}}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by HB}_{T,\Box}\text{)} \\
&= \boxdot\left(\bigwedge_{i\le n+1} (\phi_i)^{\Box}_{\mathrm{V}} \to (\psi_i)^{\Box}_{\mathrm{V}}\right)
\end{aligned}
$$

□

Lemma 63. Let T be such that $\mathrm{HB}_{T,\Box}$.

$$\vdash_T (\forall x \,.\, \phi \to \psi)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot(\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}).$$

Proof. We have that

$$
\begin{aligned}
\vdash_T (\forall x \,.\, \phi \to \psi)^{\Box}_{\mathrm{V}} &= \boxdot(\forall x \,.\, (\phi \to \psi)^{\Box}_{\mathrm{V}}) \\
&= \boxdot(\forall x \,.\, \boxdot((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})) \\
&\overset{*}{\leftrightarrow} \boxdot(\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})
\end{aligned}
$$

Let us show $\overset{*}{\leftrightarrow}$ in detail.

$\overset{*}{\to}$. We have that

$$\vdash_T \boxdot((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}},$$

but then by iFOL reasoning

$$\vdash_T (\forall x \,.\, \boxdot((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to (\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}), \tag{i}$$

and then by $\mathrm{HB}_{T,\Box}$ we get

$$\vdash_T \Box(\forall x \,.\, \boxdot((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \Box(\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}). \tag{ii}$$

(i) and (ii) give the desire implication.

$\overset{*}{\leftarrow}$. It suffices to show that

$$\vdash_T \Box(\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \boxdot(\forall x \,.\, \Box((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})). \tag{i}$$

First, note that thanks to Lemma 28 we have that

$$\vdash_T \Box(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \forall x\,.\,\Box((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}). \tag{ii}$$

Then by $\mathrm{HB}_{T,\Box}$ we get

$$\vdash_T \Box\Box(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \Box(\forall x\,.\,\Box((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})).$$

Since $\vdash_T \Box(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \Box\Box(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})$ we get

$$\vdash_T \Box(\forall x\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}) \to \Box(\forall x\,.\,\Box((\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}})). \tag{iii}$$

(ii) and (iii) gives (i). □

With the help of Lemmas 61 and 63 is not hard to give a prove of the following corollary.

Corollary 64. Let T be such that $\mathrm{HB}_{T,\Box}$.

$$\vdash_T (\forall x_0,\dots,x_n\,.\,\phi \to \psi)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot(\forall x_0,\dots,x_n\,.\,(\phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}}).$$

Lemma 65. Let T be such that $\mathrm{HB}_{T,\Box}$. Then for $n \geq 1$ we have that:

$$\vdash_T (\phi_n \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}).$$

Proof. By induction in n. If $n = 1$, te result is trivial by definition of $(\_)^{\Box}_{\mathrm{V}}$. Now, let us prove the inductive step. So assume, by the induction hypothesis that:

$$\vdash_T (\phi_n \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}} \leftrightarrow \boxdot((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}),$$

which implies

$$\vdash_T \Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to (\phi_n \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}}\Big) \leftrightarrow \Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \boxdot((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}})\Big). \tag{i}$$

Also, thanks to $\mathrm{HB}_{T,\Box}$ we get

$$\vdash_T \Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to (\phi_n \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}}\Big) \leftrightarrow \Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \boxdot((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}})\Big). \tag{ii}$$

Then we have:

$$\begin{aligned}\vdash_T (\phi_{n+1} \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}} &= \boxdot\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to (\phi_n \to \cdots \to \phi_0)^{\Box}_{\mathrm{V}}\Big)\\ &\leftrightarrow \boxdot\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \boxdot((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}})\Big)\\ &\qquad\text{(by (i) and (ii))}\\ &\overset{*}{\leftrightarrow} \boxdot\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}\Big).\end{aligned}$$

Let us show $\overset{*}{\leftrightarrow}$ in detail. $\overset{*}{\rightarrow}$ is easy, so let us show $\overset{*}{\leftarrow}$. It suffices to show

$$\vdash_T \Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}\Big) \to \boxdot\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \Box((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}})\Big).$$

Thanks to $\mathrm{HB}_{T,\Box}$ and $\vdash_T \Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}\Big) \to \Box\Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}\Big)$ we have that it suffices to show

$$\vdash_T \Box\Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}}\Big) \to \Big((\phi_{n+1})^{\Box}_{\mathrm{V}} \to \Box((\phi_n)^{\Box}_{\mathrm{V}} \to \cdots \to (\phi_0)^{\Box}_{\mathrm{V}})\Big). \qquad \text{(i)}$$

But (i) is easily proven using Lemma 60, since then $\vdash_T (\phi_{n+1})^{\Box}_{\mathrm{V}} \to \Box(\phi_{n+1})^{\Box}_{\mathrm{V}}$. $\Box$

We prove a pair of lemmas that says that under certain conditions Visser's translation over Δ_0 and Σ_1 formulas is equivalent to the identity.

Lemma 66. Let us have $\mathrm{HB}_{T,\Box}$. Then, for any $\phi \in \Delta_0$ we have that

$$\vdash_T (\phi)^{\Box}_{\mathrm{V}} \leftrightarrow \phi.$$

Proof. We proceed by induction of the definition of Δ_0-formulas. If ϕ is atomic then it is trivial since $(\phi)^{\Box}_{\mathrm{V}} = \phi$.

The cases where ϕ is a conjunction, disjunction or an existential quantification are trivial using the induction hypothesis

Let $\phi = \phi_0 \to \phi_1$. By the induction hypothesis

$$\vdash_T ((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) \leftrightarrow (\phi_0 \to \phi_1) \qquad \text{(i)}$$

By $\mathrm{HB}_{T,\Box}$ we also have

$$\vdash_T \Box((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) \leftrightarrow \Box(\phi_0 \to \phi_1) \qquad \text{(ii)}$$

So we can conclude using (i) and (ii)

$$\vdash_T \boxdot((\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}) \leftrightarrow \boxdot(\phi_0 \to \phi_1).$$

And the RHS is T-equivalent to $\phi_0 \to \phi_1$ thanks to $\mathrm{HB}_{T,\Box}$.Compl on the formula $\phi_0 \to \phi_1$, since it is Δ_0.

Let $\phi \equiv \forall x \leq \tau . \phi_0$. By the induction hypothesis we have that

$$\vdash_T (\phi_0)^{\Box}_{\mathrm{V}} \leftrightarrow \phi_0.$$

Then by iFOL reasoning

$$\vdash_T (\forall x \leq \tau \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) \leftrightarrow (\forall x \leq \tau \,.\, \phi_0). \qquad \text{(i)}$$

By $\mathrm{HB}_{T,\Box}$ we also get

$$\vdash_T \Box(\forall x \leq \tau \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) \leftrightarrow \Box(\forall x \leq \tau \,.\, \phi_0). \qquad \text{(ii)}$$

And then we can conclude

$$\begin{aligned}
\vdash_T (\forall x \leq \tau \,.\, \phi_0)^{\Box}_{\mathrm{V}} &\leftrightarrow \boxdot(\forall x \,.\, (x \leq \tau)^{\Box}_{\mathrm{V}} \to (\phi_0)^{\Box}_{\mathrm{V}}) && \text{(by Lemma 63)} \\
&= \boxdot(\forall x \leq \tau \,.\, (\phi_0)^{\Box}_{\mathrm{V}}) \\
&\leftrightarrow \boxdot(\forall x \leq \tau \,.\, \phi_0). && \text{(by (i) and (ii))}
\end{aligned}$$

□

Corollary 67. Let us have $\mathrm{HB}_{T,\Box}$. Then, for any $\phi \in \Sigma_1$ we have that:

$$\vdash_T (\phi)^{\Box}_{\mathrm{V}} \leftrightarrow \phi.$$

Proof. Since ϕ is a Σ_1 formula we have that $\phi \equiv \exists x.\phi_0$ for some Δ_0-formula ϕ_0. By lemma 66 we have that $\vdash_T \phi_0^{\Box} \leftrightarrow \phi_0$ from which we can derive by iFOL reasoning that $\vdash_T (\exists x.\phi_0^{\Box}) \leftrightarrow (\exists x.\phi_0)$, as desired. □

We need to prove some technical lemmas related to the behaviour of Visser's translation with variables and subtitutions. In order to do so, we first need to prove the following lemma about the behaviour of box functions and substitutions.

Lemma 68 (Substitution). Assume $\mathrm{HB}_{U,T,\Box}$. Then, for any formula ϕ, variable x and term τ free for x in ϕ:

$$\vdash_U (\Box\phi)[x/\tau] \leftrightarrow \Box(\phi[x/\tau]).$$

Proof. First, let us assume that $x \notin \mathrm{fv}(\tau)$. By $\mathrm{HB}_{U,T,\Box}$.Compl we have that

$$\vdash_U \tau \approx x \to \Box(\tau = x). \tag{i}$$

We also have that $\vdash_{\mathsf{iFOL}} \tau \approx x \to (\phi[x/\tau] \leftrightarrow \phi)$. Using $\mathrm{HB}_{U,T,\Box}$ and Lemma 27 we get that

$$\vdash_U \Box(\tau = x) \to (\Box(\phi[x/\tau]) \leftrightarrow \Box\phi). \tag{ii}$$

(i) and (ii) implies that $\vdash_U \tau \approx x \to (\Box(\phi[x/\tau]) \leftrightarrow \Box\phi)$. But then

$$\vdash_U \tau \approx \tau \to (\Box(\phi[x/\tau]) \leftrightarrow \Box\phi)[x/\tau],$$

where we used that $x \notin \mathrm{fv}(\tau)$. Using $x \notin \mathrm{fv}(\tau)$ again, we get the following equality of formulas

$$(\Box(\phi[x/\tau]) \leftrightarrow \Box\phi)[x/\tau] = \Box(\phi[x/\tau]) \leftrightarrow (\Box\phi)[x/\tau],$$

so we get the desired conclusion (since $\tau \approx \tau$ is provable).

If $x \in \mathrm{fv}(\tau)$, first we do the same proof for $\psi := \phi[x/y]$ where y is a new variable and then note that $\psi[y/\tau] = \phi[x/\tau]$ and $(\Box\psi)[y/\tau] = (\Box\phi)[x/\tau]$. □

Finally, the technical lemma about variables and substitutions.

Lemma 69. For any formula ϕ, variable x, and term τ, if $\mathrm{HB}_{T,\Box}$ then

1. ϕ and $(\phi)^{\Box}_{\mathrm{V}}$ have the same free variables.
2. τ free for x in ϕ iff τ free for x in $(\phi)^{\Box}_{\mathrm{V}}$.
3. If τ free for x in ϕ, then $\vdash_T (\phi)^{\Box}_{\mathrm{V}}[x/\tau] \leftrightarrow (\phi[x/\tau])^{\Box}_{\mathrm{V}}$.

Proof. Each of the statemens is proved by induction in ϕ. The first two are trivial and for the third it suffices to use Lemma 68. $\Box$

2.2.2 Translating theories

As we said in the introduction to this chapter, we want to show a theorem claiming that if U proves a formula ϕ, then a theory T proves the formula $(\phi)^{\Box}_{\mathrm{V}}$. For Gödel's translation U is the classical version of T and for Friedman's translation $U = T$. In case of Visser's translation we need to define a new theory.

Definition 70. Let T be theory and $\Box$ be a box function. Then we define the arithmetized theory $(T)^{\Box}_{\mathrm{V}}$ as:

$$(T)^{\Box}_{\mathrm{V}} := T + \{\phi \in \mathsf{Sent} \mid \vdash_T (\phi)^{\Box}_{\mathrm{V}}\}.$$

In addition, if T is an arithmetizable theory, we define the formula:

$$\mathsf{ax}_{(T)^{\Box}_{\mathrm{V}}}(A) := \mathsf{ax}_T(A) \vee \mathsf{prov}_T(A)^{\Box^{\bullet}}_{\mathrm{V}},$$ ■

The following result follows directly from the definition of $(T)^{\Box}_{\mathrm{V}}$.

Lemma 71. For any formula ϕ,

$$\vdash_T (\phi)^{\Box}_{\mathrm{V}} \text{ implies } \vdash_{(T)^{\Box}_{\mathrm{V}}} \phi.$$

Proof. By definition of $(T)^{\Box}_{\mathrm{V}}$ we will have that $\phi \in \mathrm{Ax}_{(T)^{\Box}_{\mathrm{V}}}$. $\Box$

And now the desired result. It is the reverse direction of Lemma 71.

Theorem 72. Let T be a theory such that $\mathrm{HB}_{T,\Box}$ and let T be closed under $(\_)^{\Box}_{\mathrm{V}}$. For any formula ϕ, we have that

$$\vdash_{(T)^{\Box}_{\mathrm{V}}} \phi \text{ implies } \vdash_T (\phi)^{\Box}_{\mathrm{V}}.$$

Proof. We proceed by strong induction in the length of the proof of $\vdash_{(T)^{\Box}_{\mathrm{V}}} \phi$ and cases in the justification for the last formula. Note that thanks to the hypothesis if ϕ is a non-logical axiom of $(T)^{\Box}_{\mathrm{V}}$ we are covered. We need to check all logical axioms.

Propositional logic axioms (1 to 7) are easy, using Lemma 65 when necessary to simplify the calculation of the translation.

Let us assume we have the axiom $(\forall x \,.\, \phi) \to \phi[x/\tau]$ where x is free for τ in ϕ. The translation gives

$$\boxdot((\forall x \,.\, \phi)^{\Box}_{\mathrm{V}} \to (\phi[x/\tau])^{\Box}_{\mathrm{V}}),$$

since $\mathrm{HB}_{T,\Box}$ it suffices to show

$$\vdash_T (\forall x \,.\, \phi)^{\Box}_{\mathrm{V}} \to (\phi[x/\tau])^{\Box}_{\mathrm{V}}.$$

Note that thanks to Lemma 69 we know that x is free for τ in $(\phi)^{\Box}_{\mathrm{V}}$. Then we have:

$$\begin{aligned}
\vdash_T (\forall x.\phi)^{\Box}_{\mathrm{V}} &= \boxdot(\forall x \,.\, (\phi)^{\Box}_{\mathrm{V}}) \\
&\to \forall x \,.\, (\phi)^{\Box}_{\mathrm{V}} \\
&\to (\phi)^{\Box}_{\mathrm{V}}[x/\tau] && \text{(since } x \text{ free for } \tau \text{ in } (\phi)^{\Box}_{\mathrm{V}}) \\
&\leftrightarrow (\phi[x/\tau])^{\Box}_{\mathrm{V}} && \text{(by Lemma 69)}
\end{aligned}$$

Let us assume we have the axiom $(\forall x \,.\, \psi \to \phi) \to \psi \to (\forall y \,.\, \phi[x/y])$ where $x \notin \mathrm{fv}(\psi)$ and ($x = y$ or $y \notin \mathrm{fv}(\phi)$). Using Lemma 65 and that $\mathrm{HB}_{T,\Box}$ we know that it suffices to show that:

$$(\forall x \,.\, \psi \to \phi)^{\Box}_{\mathrm{V}} \to (\psi)^{\Box}_{\mathrm{V}} \to (\forall y \,.\, \phi[x/y])^{\Box}_{\mathrm{V}}.$$

Using Lemma 63 we know that this is T-equivalent to

$$\boxdot(\forall x \,.\, (\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}}) \to (\psi)^{\Box}_{\mathrm{V}} \to \boxdot(\forall y \,.\, (\phi[x/y])^{\Box}_{\mathrm{V}}).$$

Thanks to Lemma 69 and $\mathrm{HB}_{T,\Box}$ we get that this is T-equivalent to

$$\boxdot(\forall x \,.\, (\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}}) \to (\psi)^{\Box}_{\mathrm{V}} \to \boxdot(\forall y \,.\, (\phi)^{\Box}_{\mathrm{V}}[x/y]). \tag{i}$$

Notice that thanks to Lemma 69 we have that $x \notin \mathrm{fv}((\psi)^{\Box}_{\mathrm{V}})$ and, using the same lemma, if $y \neq x$ we have that $x \notin \mathrm{fv}((\phi)^{\Box}_{\mathrm{V}})$. This means that

$$(\forall x \,.\, (\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}}) \to (\psi)^{\Box}_{\mathrm{V}} \to (\forall y \,.\, (\phi)^{\Box}_{\mathrm{V}}[x/y]), \tag{ii}$$

since it is another instance of the same logical axiom. In addition, thanks to $\mathrm{HB}_{T,\Box}$ and (ii), we have that

$$\Box(\forall x \,.\, (\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}}) \to \Box(\psi)^{\Box}_{\mathrm{V}} \to \Box(\forall y \,.\, (\phi)^{\Box}_{\mathrm{V}}[x/y]),$$

Thanks to Lemma 69 we have that $\vdash_T (\psi)^{\Box}_{\mathrm{V}} \to \Box(\psi)^{\Box}_{\mathrm{V}}$. From this we get

$$\Box(\forall x\,.\,(\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}}) \to (\psi)^{\Box}_{\mathrm{V}} \to \Box(\forall y\,.\,(\phi)^{\Box}_{\mathrm{V}}[x/y]), \tag{iii}$$

and (ii) and (iii) gives the desired (i).

The existential quantifier axioms are analogous to the universal quantifier axioms.

All equality axioms are a block of universal quantifiers and a Δ_0 formula ϕ. But if ϕ is Δ_0 then

$$\begin{aligned} \vdash_T (\forall x_0, \dots, x_{n-1}.\phi)^{\Box}_{\mathrm{V}} &\leftrightarrow \boxdot(\forall x_0, \dots, x_{n-1}.(\phi)^{\Box}_{\mathrm{V}}) && \text{(by Lemma 61)} \\ &\leftrightarrow \boxdot(\forall x_0, \dots, x_{n-1}.\phi) && \text{(by } \mathsf{HB}_{T,\Box} \text{ and Lemma 66)} \end{aligned}$$

But the last formula of the equivalence is clearly provable in T using $\mathrm{HB}_{T,\Box}$.

Modus ponens. So assume that we have $\vdash_T \phi$ and we have shorter proofs of $\vdash_T \psi$ and $\vdash_T \psi \to \phi$. Applying the induction hypothesis to this shorter proofs we obtain that $\vdash_T (\psi)^{\Box}_{\mathrm{V}}$ and $\vdash_T (\psi \to \phi)^{\Box}_{\mathrm{V}}$, in other words, $\vdash \boxdot((\psi)^{\Box}_{\mathrm{V}} \to (\phi)^{\Box}_{\mathrm{V}})$. From these is easy to derive $\vdash_T (\phi)^{\Box}_{\mathrm{V}}$ using iFOL reasoning.

Generalization. Assume that we have $\vdash_T \forall x.\phi$ and there is a shorter proof of $\vdash_T \phi$. By the induction hypothesis we obtain that $\vdash_T (\phi)^{\Box}_{\mathrm{V}}$, using generalization we derive that $\vdash_T \forall x.(\phi)^{\Box}_{\mathrm{V}}$. Since $\mathrm{HB}_{T,\Box}$, we also have that $\vdash_T \Box(\forall x.(\phi)^{\Box}_{\mathrm{V}})$ so we can conclude that $\vdash_T (\forall x.\phi)^{\Box}_{\mathrm{V}}$, as desired. □

A corollary that is easy to prove thanks to the deduction theorem.

Corollary 73. Let T be a theory such that $\mathrm{HB}_{T,\Box}$ and let T be closed under $(\_)^{\Box}_{\mathrm{V}}$. Then

$$\Gamma \vdash_{(T)^{\Box}_{\mathrm{V}}} \phi \text{ implies } (\Gamma)^{\Box}_{\mathrm{V}} \vdash_T (\phi)^{\Box}_{\mathrm{V}}$$

Theorem 72 also allows to establish a conservativity result from $(T)^{\Box}_{\mathrm{V}}$ to T.

Corollary 74. Let us have a theory T such that $\mathrm{HB}_{T,\Box}$ and T is closed under $(\_)^{\Box}_{\mathrm{V}}$. For any ϕ Σ_1-formula

$$\vdash_{(T)^{\Box}_{\mathrm{V}}} \phi \text{ implies } \vdash_T \phi.$$

In particular:

$$T \text{ is consistent if and only if } (T)^{\Box}_{\mathrm{V}} \text{ is consistent.}$$

Proof. Thanks to invariance of Σ_1 formulas under translation (Lemma 67) and Theorem 72. □

And finally, the formalized version of the theorem.

Corollary 75. Let T, U be theories and let $\Box$ be the box function of a Σ_1-formula with one free variable. Assume that

1. $\mathsf{iI}\Sigma_1 \subseteq U$.
2. $\vdash_U \mathsf{HB}_{\mathsf{prov}_T, \Box^*}$.
3. $\vdash_U \forall A\,.\,\mathsf{ax}_T(A) \to \mathsf{prov}_T(A)^{\Box^*}_{\mathsf{V}}$.

Then

$$\vdash_U \forall A.\mathsf{prov}_{(T)^{\triangle}_{\mathsf{V}}}(A) \leftrightarrow \mathsf{prov}_T(A)^{\triangle^*}_{\mathsf{V}}.$$

Proof. We can replicate the arguments of the section of Visser translation general properties and the theorem of translating theories but inside of U, since in the metatheory we only need Σ_1-induction to show these arguments. The conclusion, is just the formalization of the Theorem 72. $\Box$

2.2.3 Translating a provability predicate

From provability predicates of T we need to be able to construct provability predicates suitable for $(T)^{\triangle}_{\mathsf{V}}$. The following definition explains how.

Definition 76. Let us have two formulas $P(x), Q(x)$, where x is a designated variable of the formula and $\Box, \triangle$ its box translations respectively. We define the formula $P^Q(A)$ as $P(A)^{\triangle^*}_{\mathsf{V}}$. The box function of this formula will be denoted as $\Box^{\triangle}\phi$. ■

Note that if P is Σ_1 then P^Q is Σ_1, since Visser's translation is primitive recursive.

With this lemma we prove that the new constructed box function $\Box^{\triangle}$ is suitable for $(T)^{\triangle}_{\mathsf{V}}$.

Lemma 77. Let us have box functions $\Box$ and $\triangle$, such that

1. $\Box, \triangle$ are Σ_1.
2. $\mathrm{HB}_{T,\Box}$ and $\mathrm{HB}_{T,\triangle}$.
3. T is closed under $(\_)^{\triangle}_{\mathsf{V}}$.

Then we have that $\mathrm{Hb}_{T,\Box^{\triangle}}$ and $\mathrm{Hb}_{(T)^{\triangle}_{\mathsf{V}},\Box^{\triangle}}$.

Proof. Since $\Box$ is Σ_1 we know that $\Box^{\triangle}$ is also Σ_1, so to check that $\mathrm{Hb}_{T,\Box^{\triangle}}, \mathrm{Hb}_{(T)^{\triangle}_{\mathsf{V}},\Box^{\triangle}}$ it suffices to check necessity, formalized modus ponens and formalized completeness. Since $T \subseteq (T)^{\triangle}_{\mathsf{V}}$ it is easy to see that it suffices to check formalized modus ponens and formalized Σ_1-completeness for T and necessity in the form:

$$\vdash_{(T)^{\triangle}_{\mathsf{V}}} \phi \text{ implies } \vdash_T \Box^{\triangle}\phi.$$

Necessity. Let ϕ be a sentence such that $\vdash_{(T)^{\triangle}_{\mathsf{V}}} \phi$. By Theorem 72 we have that $\vdash_T (\phi)^{\triangle}_{\mathsf{V}}$ and since $\mathrm{HB}_{T,\Box}$ we get $\vdash_T \Box(\phi)^{\triangle}_{\mathsf{V}}$. But since ϕ is a sentence, we get $\vdash_T \Box^{\triangle}\phi$, as desired.

Formalized modus ponens. Let ϕ, ψ be sentences, we have to show that

$$\vdash_T \Box^{\triangle}(\phi \to \psi) \to \Box^{\triangle}\phi \to \Box^{\triangle}\psi. \tag{i}$$

We have that:

$$\vdash_T \left(\overline{\phi}\right)^{\triangle^\bullet}_{\mathsf{V}} \approx \overline{(\phi)^{\triangle}_{\mathsf{V}}},$$
$$\vdash_T \left(\overline{\psi}\right)^{\triangle^\bullet}_{\mathsf{V}} \approx \overline{(\psi)^{\triangle}_{\mathsf{V}}},$$
$$\vdash_T \left(\overline{\phi \to \psi}\right)^{\triangle^\bullet}_{\mathsf{V}} \approx \overline{(\phi \to \psi)^{\triangle}_{\mathsf{V}}},$$

so (i) is equivalent to show that

$$\vdash_T \Box(\phi \to \psi)^{\triangle}_{\mathsf{V}} \to \Box(\phi)^{\triangle}_{\mathsf{V}} \to \Box(\psi)^{\triangle}_{\mathsf{V}},$$

but this is easy to show by definition of $(\_)^{\triangle}_{\mathsf{V}}$, $\mathrm{HB}_{T,\Box}$ and a little of iFOL reasoning.

Σ_1-completeness. We have to show that for any Σ_1-sentence

$$\vdash_T \phi \to \Box^{\triangle}\phi.$$

Since ϕ is a sentence, it is equivalent to show that

$$\vdash_T \phi \to \Box(\phi)^{\triangle}_{\mathsf{V}}.$$

Since ϕ is Σ_1, by Lemma 67 with the assumption that $\mathrm{HB}_{T,\triangle}$ we obtain $\vdash_T \phi \leftrightarrow (\phi)^{\triangle}_{\mathsf{V}}$. By $\mathrm{HB}_{T,\Box}$ we have $\vdash_T \Box\phi \leftrightarrow \Box(\phi)^{\triangle}_{\mathsf{V}}$, so it suffices to show

$$\vdash_T \phi \to \Box\phi,$$

which holds by $\mathrm{HB}_{T,\Box}$.Compl. $\Box$

2.2.4 Formulas whose translation implies the original formula

For reasons that will become clear in Chapter 4, we need to study when the Visser's translation of a formula implies the original formula. In other words, we want to prove that for certain class of formulas, we have that $\vdash_T (\phi)^{\Box}_{\mathsf{V}} \to \phi$. With that purpose, we define the class of formulas $\mathcal{A}$.

Definition 78. Let $\mathcal{A}$ be the set of $\mathcal{L}_1$-formulas described by the following Backus-Naur form:

$$\mathcal{A} ::= \phi \mid \mathcal{A} \wedge \mathcal{A} \mid \mathcal{A} \vee \mathcal{A} \mid \forall x \,.\, \mathcal{A} \mid \exists x \,.\, \mathcal{A} \mid \psi \to \mathcal{A},$$

where ϕ is an atomic formula, x is a variable and ψ is a Σ_1-formula. ■

With this definition we can prove the desired lemma.

Lemma 79. Let $\phi \in \mathcal{A}$ and also assume that $\mathrm{HB}_{T,\Box}$. We have that:

$$\vdash_T (\phi)^{\Box}_{\mathrm{V}} \to \phi.$$

Proof. We proceed by induction in ϕ. If it is atomic the result is trivial. The cases of conjunction, disjunction and universal or existential quantification are easy using the induction hypothesis. Let us show the implication case. We have that $\phi = \phi_0 \to \phi_1$, where ϕ_0 is Σ_1 and ϕ_1 is $\mathcal{A}$. By the induction hypothesis we have that $\vdash_T (\phi_1)^{\Box}_{\mathrm{V}} \to \phi_1$ and since ϕ_0 is Σ_1, by Lemma 67 we have that $\vdash_T \phi_0 \leftrightarrow (\phi_0)^{\Box}_{\mathrm{V}}$. We also have that $\vdash_T (\phi_0 \to \phi_1)^{\Box}_{\mathrm{V}} \to (\phi_0)^{\Box}_{\mathrm{V}} \to (\phi_1)^{\Box}_{\mathrm{V}}$, so using these three deductions we conclude

$$\vdash_T (\phi_0 \to \phi_1)^{\Box}_{\mathrm{V}} \to \phi_0 \to \phi_1.$$ □

2.3 De Jongh's Translation

The last translation needed for our purposes is De Jongh's translation. First, let us define it. Note that it is similar to Visser's translation, but with two differences. The first one is that it requires an extra formula ψ. The second difference, and the one which makes Visser's translation and De Jongh's translation different in their nature, is that the translation is not recursively applied inside $\Box$.

Definition 80 (De Jongh's Translation). Let ϕ, ψ be formulas and let $\Box$ be a box function. We define $[\psi]_{\Box}\phi$ recursively in ϕ as:

$$\begin{aligned}
&[\psi]_{\Box}\phi = \phi, \text{ if } \phi \text{ is atomic},\\
&[\psi]_{\Box}(\phi_0 \wedge \phi_1) = [\psi]_{\Box}\phi_0 \wedge [\psi]_{\Box}\phi_1,\\
&[\psi]_{\Box}(\phi_0 \vee \phi_1) = [\psi]_{\Box}\phi_0 \vee [\psi]_{\Box}\phi_1,\\
&[\psi]_{\Box}(\phi_0 \to \phi_1) = ([\psi]_{\Box}\phi_0 \to [\psi]_{\Box}\phi_1) \wedge \Box(\psi \to \phi_0 \to \phi_1),\\
&[\psi]_{\Box}(\forall x . \phi) = (\forall x \,.\, [\psi]_{\Box}\phi) \wedge \Box(\psi \to \forall x . \phi),\\
&[\psi]_{\Box}(\exists x \,.\, \phi) = \exists x \,.\, [\psi]_{\Box}\phi.
\end{aligned}$$ ■

First, we establish a lemma that proves the fundamental property of De Jongh's translation.

Lemma 81. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi, \psi \in \mathcal{L}_1$ such that there is no free variable of ψ bounded in ϕ. Then:

$$\vdash_U [\psi]_\Box \phi \to \Box(\psi \to \phi).$$

We establish some lemmas that ease the calculation of De Jongh's translation.

Lemma 82. Assume that $\mathrm{HB}_{U,T,\Box}$. Then we have that:

$$\vdash_U [\chi]_\Box \left(\bigwedge_{i=0}^{m} \phi_i \to \psi_i \right) \leftrightarrow \left(\bigwedge_{i=0}^{m} [\chi]_\Box \phi_i \to [\chi]_\Box \psi_i \right) \wedge \Box \left(\chi \to \left(\bigwedge_{i=0}^{m} \phi_i \to \psi_i \right) \right).$$

Lemma 83. Let $\mathrm{HB}_{U,T,\Box}$ and $x_0, \dots, x_m$ be variables not free in χ. Then we have that:

$$\vdash_U [\chi]_\Box (\forall x_0, \dots, x_m . \phi) \leftrightarrow (\forall x_0, \dots, x_m . [\chi]_\Box \phi) \wedge \Box(\chi \to \forall x_0, \dots, x_m . \phi).$$

Lemma 84. Let $\mathrm{HB}_{U,T,\Box}$ and assume that $x \notin \mathrm{fv}(\chi)$. Then

$$\vdash_U [\chi]_\Box (\forall x . \phi \to \psi) \leftrightarrow (\forall x . [\chi]_\Box \phi \to [\chi]_\Box \psi) \wedge \Box(\chi \to \forall x . \phi \to \psi).$$

Lemma 85. Let $\mathrm{HB}_{U,T,\Box}$, then

$$\vdash_U [\chi]_\Box (\phi_m \to \cdots \to \phi_0) \leftrightarrow ([\chi]_\Box \phi_m \to \cdots \to [\chi]_\Box \phi_0) \wedge \Box(\chi \to \phi_m \to \cdots \to \phi_0)$$

A pair of lemmas proving that under certain conditions De Jongh's translation for Δ_0 and Σ_1 formulas is equivalent to the identity.

Lemma 86. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi \in \Delta_0$, such that no free variable of χ appears bounded in ϕ. Then

$$\vdash_U \phi \leftrightarrow [\chi]_\Box \phi.$$

Lemma 87. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi \in \Sigma_1$, such that no free variable of χ appears bounded in ϕ. Then

$$\vdash_U \phi \leftrightarrow [\chi]_\Box \phi.$$

Two technical lemmas proving properties of De Jongh translation related to variables and substitutions.

Lemma 88. We have that

1. $\mathrm{fv}([\chi]_\Box \phi) \subseteq \mathrm{fv}(\chi) \cup \mathrm{fv}(\phi)$.
2. x is free for τ in ϕ iff x is free for τ in $[\chi]_\Box \phi$.

Lemma 89. Assume that $\mathrm{HB}_{U,T,\Box}$. Let x be free for τ in ϕ, and $x \notin \mathrm{fv}(\chi)$. Then

$$\vdash_U ([\chi]_\Box \phi)[x/\tau] \leftrightarrow [\chi]_\Box \phi[x/\tau].$$

The main theorem of the De Jongh's translation.

Theorem 90. Let χ be a formula, let T be such that if $\phi \in \mathrm{Ax}_T$ then $\vdash_U [\chi]_\Box(\phi)$ and $\mathrm{HB}_{U,T,\Box}$. Then if π is a proof of $\vdash_T \phi$ where no free variable of χ appears bounded, we have that

$$\vdash_U [\chi]_\Box \phi.$$

An easy corollary using the deduction theorem.

Corollary 91. Let χ be a sentence, let T and U be such that $\phi \in \mathrm{Ax}_T$ implies $\vdash_U [\chi]_\Box \phi$ and $\mathrm{HB}_{U,T,\Box}$. Assume that we have a proof π of $\Gamma \vdash_T \phi$ such that no free variable of ψ appears bounded in π or in Γ. Then

$$[\chi]_\Box \Gamma \vdash_U [\chi]_\Box \phi.$$

And finally the formalized version of the main theorem for De Jongh's translation.

Corollary 92. Let V, U, T be theories and $\Box$ be a box function of a formula (so $\Box^\bullet$ exists). Assume that

1. $\mathsf{i}\mathsf{I}\Sigma_1 \subseteq V$.
2. $\vdash_V \mathsf{HB}_{\mathsf{prov}_U, \mathsf{prov}_T, \Box^\bullet}$.

Then

$$\begin{aligned}
&\forall C \,.\, \mathsf{ax}_T(C) \to \mathsf{prov}_U([B]_{\Box^\bullet} C),\\
&\forall v \,.\, \neg(v \in \mathsf{fv}(A) \wedge \exists C \in p \,.\, v \in \mathsf{bv}(C)),\\
&\mathsf{proof}_T(p, A)\\
&\vdash_V \mathsf{prov}_U([B]_{\Box^\bullet} A).
\end{aligned}$$

2.3.1 Auxiliary translations

Let us define some translations resembling De Jongh translation, but from propositional formulas to first-order formulas. These definitions will be useful in Chapter 4.

Definition 93. Let $\phi, \psi \in \mathcal{L}_\mathrm{p}$, σ be a realization and $\Box$ be a box function. Then, we define $[\psi, \sigma]_\Box \phi \in \mathcal{L}_1$ recursively in ϕ as:

$$\begin{aligned}
&[\psi,\sigma]_\Box p := \sigma(p),\\
&[\psi,\sigma]_\Box \top := \top,\\
&[\psi,\sigma]_\Box \bot := \bot,\\
&[\psi,\sigma]_\Box(\phi_0 \wedge \phi_1) := [\psi,\sigma]_\Box \phi_0 \wedge [\psi,\sigma]_\Box \phi_1,\\
&[\psi,\sigma]_\Box(\phi_0 \vee \phi_1) := [\psi,\sigma]_\Box \phi_0 \vee [\psi,\sigma]_\Box \phi_1,\\
&[\psi,\sigma]_\Box(\phi_0 \to \phi_1) := ([\psi,\sigma]_\Box \phi_0 \to [\psi,\sigma]_\Box \phi_1) \wedge \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1).
\end{aligned}$$

Similarly, we define $[\psi,\sigma]^{\circ}_{\Box}\phi \in \mathcal{L}_1$ recursively in ϕ as:

$$\begin{aligned}
&[\psi,\sigma]^{\circ}_{\Box} p := \sigma(p),\\
&[\psi,\sigma]^{\circ}_{\Box} \top := \top,\\
&[\psi,\sigma]^{\circ}_{\Box} \bot := \bot,\\
&[\psi,\sigma]^{\circ}_{\Box}(\phi_0 \wedge \phi_1) := [\psi,\sigma]^{\circ}_{\Box}\phi_0 \wedge [\psi,\sigma]^{\circ}_{\Box}\phi_1,\\
&[\psi,\sigma]^{\circ}_{\Box}(\phi_0 \vee \phi_1) := [\psi,\sigma]^{\circ}_{\Box}\phi_0 \vee [\psi,\sigma]^{\circ}_{\Box}\phi_1,\\
&[\psi,\sigma]^{\circ}_{\Box}(\phi_0 \to \phi_1) := \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1).
\end{aligned}$$

■

The following lemmas are needed in Chapter 4.

Lemma 94. Let $\mathrm{HB}_{U,T,\Box}$. Then for any $\phi,\psi \in \mathcal{L}_{\mathrm{p}}$ and any Σ_1-realization σ we have

$$\vdash_U [\sigma\psi]_{\Box}\sigma\phi \leftrightarrow [\psi,\sigma]_{\Box}\phi.$$

Lemma 95. Let $\phi,\psi \in \mathcal{L}_{\mathrm{p}}$, σ be a realization and $\Box\colon \mathcal{L}_1 \longrightarrow \mathcal{L}_1$. Then

$$\vdash_U [\psi,\sigma]_{\Box}\phi \to [\psi,\sigma]^{\circ}_{\Box}\phi.$$

Lemma 96. Assume that $\mathrm{Rfn}_{U,\Box}$. Then, for any $\phi,\psi \in \mathcal{L}_{\mathrm{p}}$ and realization σ we have

$$\vdash_U [\psi,\sigma]^{\circ}_{\Box}\phi \to \sigma([\psi]\phi).$$

Lemma 97. Let $\Box$ be Σ_1 and σ a Σ_1-realization. Then $[\psi,\sigma]^{\circ}_{\Box}\phi$ is U-provably equivalent to a Σ_1-formula.

Chapter 3

Solovay in the Intuitionistic Case

In this chapter we will introduce the method from [13]. It adapts the Solovay's construction from classical logic to intuitionistic logic. Since in intuitionistic modal logic there are two binary relations, an intuitionistic relation and a modal relation, we will need two provability predicates. In addition, we will ned that they fulfill the absorption law for Σ_1-sentence, i.e. $\Box\triangle\phi \to \Box\phi$ where ϕ is a Σ_1-sentence. For this reason we include a section with a method from construcing $\triangle$ from $\Box$ that satisfies the absorption law. Informally, we call $\triangle$ a slow version of $\Box$. This is since thanks to the definition we have that $\triangle\phi \to \Box\phi$, while $\triangle$ representing the same provability predicate as $\Box$ in the standard arithmetical model. The trick is that the arithmetical theory is incapable of showing that they are the same provability predicate.

The idea behind this construction of the slow predicate is taken from [12]. While originally this construction was used to construct non-uniform provability predicates (i.e. predicates that only fulfill Hb), we show that it also work for constructing uniform predicates. Finally, we will apply the intuitionistic Solovay construction and create a slow version of $\Box_{\mathsf{iPRA}}$, called $\Box_{\mathcal{S}\mathsf{iPRA}}$, to prove that

$$\mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathsf{iGLC}.$$

3.1 Equivalence of completeness and strong Löb

First, we are going to define two principles that we will need: the completeness principle (the C of iGLC) and the Strong Solovay principle. We will show that in fact, both are equivalent under some standard assumptions.

Definition 98. We define the set of formulas of the *sentential completeness principle for* $\Box$, $\mathrm{Cp}_{\Box}$ as the set:

$$\mathrm{Cp}_{\Box} := \{\phi \to \Box\phi \mid \phi \in \mathrm{Sent}\}.$$

Similarly, we define the set of formulas of the *strong Löb's principle for* $\Box$, $\mathrm{Slp}_{\Box}$, as the set:

$$\mathrm{Slp}_{\Box} := \{(\Box\phi \to \phi) \to \phi \mid \phi \in \mathrm{Sent}\}.$$

If T is an arithmetical theory, we will write $\mathrm{Cp}_{T,\Box}$ to mean $\vdash_T \mathrm{Cp}_{\Box}$ and $\mathrm{Slp}_{T,\Box}$ to mean $\vdash_T \mathrm{Slp}_{\Box}$. ■

Now we show Löb's rule using his reasoning.

Lemma 99. Assume $\mathrm{Gl}_{T,\Box}$ and let ϕ be a sentence. Then,

$$\vdash_T \Box\phi \to \phi \text{ implies } \vdash_T \phi.$$

Proof. Assume that $\Box$ is the box function of $P(x)$. We start using diagonalization in T for the formula $P(x) \to \phi$. With this we obtain a sentence ψ such that $\vdash_T \psi \leftrightarrow (\Box\psi \to \phi)$ (i). In particular $\vdash_T \psi \to \Box\psi \to \phi$, using that $\mathrm{Gl}_{T,\Box}$ it is easy to derive that $\vdash_T \Box\psi \to \Box\Box\psi \to \Box\phi$ (ii). But since $\vdash_T \Box\psi \to \Box\Box\psi$, by $\mathrm{Gl}_{T,\Box}$.Trans, we have that from (ii) we can derive $\vdash_T \Box\psi \to \Box\phi$. Since $\vdash_T \Box\phi \to \phi$ we also have that $\vdash_T \Box\psi \to \phi$ (iii), but by (i), this implies $\vdash_T \psi$. By necessitation, we get $\vdash_T \Box\psi$, so using (iii) we can conclude that $\vdash_T \phi$, as wanted. □

We remind the reader that Gl are the sentential Gödel-Löb conditions. These are weaker conditions than Hb, and are explained in Section 1.5.

By an analogous reasoning, but having two sentences ϕ_0 ,ϕ_1 and ψ being the fixpoint of $P(x) \to \Box\phi_0 \to \phi_1$, we obtain a variation of the rule:

Lemma 100. Assume $\mathrm{Gl}_{T,\Box}$ and ϕ be a sentence. Then

$$\vdash_T \Box\phi_0 \to \Box\phi_1 \to \phi_1 \text{ implies } \vdash_T \Box\phi_0 \to \phi_1.$$

Using this strengthened version, we can show Löb's axiom.

Lemma 101. Assume $\mathrm{Gl}_{T,\Box}$ and let ϕ be a sentence. Then

$$\vdash_T \Box(\Box\phi \to \phi) \to \Box\phi.$$

Proof. Just note that by $\mathrm{Gl}_{T,\Box}$.K we have that $\vdash_T \Box(\Box\phi \to \phi) \to \Box\Box\phi \to \Box\phi$ and using Lemma 100 directly give us: $\vdash_T \Box(\Box\phi \to \phi) \to \Box\phi$. □

The second part of the following proof follows Dick de Jongh's proof of axiom 4 from Löb's principle.

Lemma 102. Let $\mathrm{Gl}_{T,\Box}$, then we have that $\mathrm{Cp}_{T,\Box}$ and $\mathrm{Slp}_{T,\Box}$ are equivalent.

Proof. Let us assume that $\mathrm{Cp}_{T,\Box}$. Since $\mathrm{Gl}_{T,\Box}$, we have that by Lemma 101:

$$\vdash_T \Box(\Box\phi \to \phi) \to \Box\phi.$$

So, by $\mathrm{Cp}_{T,\Box}$, we also have:

$$\vdash_T (\Box\phi \to \phi) \to \Box\phi.$$

And using intuitionistic propositional reasoning we can conclude

$$\vdash_T (\Box\phi \to \phi) \to \phi.$$

Let us assume that $\mathrm{Slp}_{T,\Box}$. Since $\mathrm{Gl}_{T,\Box}$, we have that

$$\vdash_T \Box(\phi \wedge \Box\phi) \to \Box\phi,$$

so

$$\vdash_T \phi \to (\Box(\phi \wedge \Box\phi) \to \phi \wedge \Box\phi).$$

Using $\mathrm{Slp}_{T,\Box}$ we have that

$$\vdash_T \phi \to \phi \wedge \Box\phi,$$

so we can conclude by intuitionistic propositional reasoning that

$$\vdash_T \phi \to \Box\phi. \qquad \Box$$

3.2 Good pair theorem

In this section we are going to closely follow [13]. We just need to guarantee the following restrictions:

1. The base theory is iEA, instead of $\mathsf{i}\mathsf{I}\Sigma_1$.
2. Instead of assuming Fefermanian predicates we just require Hilbert-Bernays predicates.

Definition 103. Let T be a theory and $\Box, \triangle$ be Σ_1 box functions. We say that $(\Box, \triangle)$ is a *good pair* for T iff

1. $\mathrm{Hb}_{T,\Box}, \mathrm{Hb}_{T,\triangle}$.
2. For any sentence ϕ, if $\models_{\mathbb{N}} \Box\phi$ then $\vdash_T \phi$.
3. $\mathrm{Slp}_{T,\triangle}$.
4. Σ_1-$\mathrm{Abs}_{T,\Box,\triangle}$, i.e. the Σ_1 sentential absorption principle.

■

We note that in the previous definition all the principles we use, apply to sentences only. For the rest of this section we assume that $(\Box, \triangle)$ is a good pair for T, a Δ_0-decidable extension of iEA.

Note that since $\Box, \triangle$ are Σ_1 box functions we have Σ_1 formulas $P(x), Q(x)$ such that $\Box$ is the box function of $P(x)$ and $Q(x)$. Then $P(x) := \exists p \,.\, P_0(p,x)$ and $Q(x) := \exists p \,.\, Q_0(p,x)$, where P_0 and Q_0 are Δ_0. We will refer to them as $\mathsf{proof}_{\Box}(p,x)$ and $\mathsf{proof}_{\triangle}(p,x)$, respectively.

We know that iGLC is sound and complete with respect to finite irreflexive realistic models. That means the following:

Theorem 104. Let ϕ be a modal formula. Then

$$\vdash_{\mathsf{iGLC}} \phi \text{ iff } \{\mathcal{F} \mid \mathcal{F} \text{ is finite, irreflexive and realistic}\} \vDash \phi.$$

For our purposes let us assume that ϕ is a formula such that $\nvdash_{\mathsf{iGLC}} \phi$. This means that there is a finite, irreflexive and realistic model $\mathcal{M}_0 = (M_0, \sqsubset_0, \preccurlyeq_0, V_0)$ and a world r such that $\mathcal{M}_0, r \nvDash \phi$. Since the model is finite we are going to assume, without loss of generality, that its worlds are the set $\{1, \ldots r\}$. We will extend this model, calling the resulting model $\mathcal{M}$:

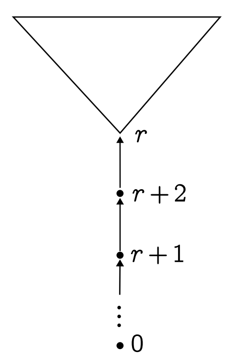
$$M := \mathbb{N},$$

$$\sqsubset := \sqsubset_0 \cup \{(0,j) \mid 0 < j\} \cup \{(i,j) \mid r < i \text{ and } 1 \leq j < i\},$$

$$\preccurlyeq := \preccurlyeq_0 \cup \{(0,j) \mid j \in \mathbb{N}\} \cup \{(i,j) \mid r < i \text{ and } 1 \leq j \leq i\},$$

$$V(p) := V_0(p).$$

Note that the idea behind this definition is to add a tail to the original finite model. We can picture this new model as:



The triangle represents the original modem $\mathcal{M}_0$, with root at r. The arrows represents both, the modal and intuitionistic relation and we have to imagine them as beein transitively closed. The dots (the new words) have a reflexive intuitionistic arrow but neither have a modal reflexive arrow. The 0 is the new root of the model, and thanks to transitivity it has an arrow to any other node.

T has Δ_0-definitions of the modal and intuitionistic relations of $\mathcal{M}$, defined as:

$$x \preccurlyeq y = \left(\bigvee_{(i,j)\in \preccurlyeq_0} x \approx \overline{i} \wedge y \approx \overline{j} \right) \vee x \approx \overline{0} \vee (\overline{r} < x \wedge \overline{1} \leq y \leq x),$$

$$x \sqsubset y = \left(\bigvee_{(i,j)\in \sqsubset_0} x \approx \overline{i} \wedge y \approx \overline{j} \right) \vee (x \approx \overline{0} \wedge y > \overline{0}) \vee (\overline{r} < x \wedge \overline{1} \leq y < x).$$

In particular T knows that:

1. $\preccurlyeq$ is a partial order, i.e. T proves the conjunction of the following formulas:

$$\begin{aligned}
&\forall x \,.\, x \preccurlyeq x,\\
&\forall x, y \,.\, x \preccurlyeq y \wedge y \preccurlyeq x \rightarrow x \approx y,\\
&\forall x, y, z \,.\, x \preccurlyeq y \wedge y \preccurlyeq z \rightarrow x \preccurlyeq z.
\end{aligned}$$

2. $\sqsubset$ is irreflexive:

$$\vdash_T \forall x \,.\, x \not\sqsubset x.$$

3. The model property holds:

$$\vdash_T \forall x, y, z \,.\, x \preccurlyeq y \wedge y \sqsubset z \rightarrow x \sqsubset z.$$

4. The frame is realistic:

$$\vdash_T \forall x, y \,.\, x \sqsubset y \rightarrow x \preccurlyeq y.$$

If the reader wants to remind what was the definitions of the model property or of a frame being realistic, we remember that these are defined in Section 1.3.

Let us define the formulas $\chi_0(s,x,A), \chi_1(s,x,A), \chi_2(s,x,A)$ as:

$$\begin{aligned}
\chi_0(s,x,A) :=\ & (s)_{x \dot{-} \overline{1}} \sqsubset (s)_x \wedge \\
& \mathsf{proof}_{\Box}\left(x \dot{-} \overline{1}, \exists^\bullet \overline{s}\, \exists^\bullet \overline{z} \leq^\bullet \mathsf{length}^\bullet(\overline{s}) \,.\, A \wedge^\bullet \neg^\bullet (\overline{s})^\bullet_{\overline{z}} \sqsubseteq^\bullet \mathsf{num}((s)_x)\right),\\
\chi_1(s,x,A) :=\ & (s)_{x \dot{-} \overline{1}} \prec (s)_x \wedge \\
& \mathsf{proof}_{\triangle}\left(x \dot{-} \overline{1}, \exists^\bullet \overline{s}\, \exists^\bullet \overline{z} \leq^\bullet \mathsf{length}^\bullet(\overline{s}) \,.\, A \wedge^\bullet \neg^\bullet (\overline{s})^\bullet_{\overline{z}} \sqsubseteq^\bullet \mathsf{num}((s)_x)\right),\\
\chi_2(s,x,A) :=\ & \neg \chi_0(s,x,A) \wedge \neg \chi_1(s,x,A) \wedge (s)_{x \dot{-} \overline{1}} \approx (s)_x.
\end{aligned}$$

Now we define the formula $\phi(s,A)$ as:

$$\begin{aligned}
\phi(s,A) :=\ & \mathsf{seq}(s) \wedge \mathsf{length}(s) > \overline{0} \wedge (s)_{\overline{0}} \approx \overline{0} \wedge \\
& \big(\forall x < \mathsf{length}(s) \,.\, x \not\approx \overline{0} \rightarrow \chi_0(s,x,A) \vee \chi_1(s,x,A) \vee \chi_2(s,x,A)\big).
\end{aligned}$$

Let $\theta(s)$ be the fixpoint of this formula given by the diagonal lemma. In other words, we have that:

$$\vdash_T \theta(s) \leftrightarrow \phi\left(s, \overline{\theta(s)}\right). \tag{H0}$$

Let us explain the meanin of θ. Basically, $\theta(s)$ holds iff s is a sequence calculating an initial part of Solovay's function. There is a little different, now Solovay's function can move in two different relations: the intuitionistic and the modal. It moves through the modal relation when the input codes a $\Box$-proof claiming that it will not stay in its position. Similarly, it moves through the intuitionistc relation when the input codes a $\triangle$-proof claiming that it will not stay in its position. In case the input codes neither, it stays in its position.

Thanks to H0, we know that $\theta(s)$ is equivalent in T to a Δ_0-formula. One trivial consequence of the definition of θ is:

$$\vdash_T \forall s_0, s_1 \,.\, s_0 \subseteq s_1 \wedge \theta(s_1) \rightarrow \theta(s_0). \tag{H1}$$

Now, we can show (using induction) that

$$\vdash_T \forall s_0, s_1 \,.\, \mathsf{length}(s_0) \approx \mathsf{length}(s_1) \wedge \theta(s_0) \wedge \theta(s_1) \rightarrow s_0 \approx s_1. \tag{H2}$$

$$\vdash_T \forall x \, \exists s \,.\, \mathsf{length}(s) \approx x \wedge \theta(s). \tag{H3}$$

To show the first formula we use induction in:

$$\forall k \forall s_0, s_1 < k \,.\, \mathsf{length}(s_0) \approx \mathsf{length}(s_1) \wedge \theta(s_0) \wedge \theta(s_1) \rightarrow s_0 \approx s_1.$$

For the second, we need to bound s. The idea is to show that $s \leq (0,1,2,\ldots,x)$ and show that the function $x \mapsto (0,\ldots,x)$ exists inside T, which does since $\mathsf{iEA} \subseteq T$ and this function is of order x^x.

Now, let us define a function h such that:

$$h(x) \approx y := \exists s \,.\, \mathsf{length}(s) > x \wedge \theta(s) \wedge y \approx (s)_x.$$

Thanks to H0, H1, H2 and H3 we have that

$$\vdash_T \forall x \, \exists! y \,.\, h(x) \approx y.$$

So $h(x) \approx y$ defines a Σ_1-function inside T. Informally, we can understand $h : \mathbb{N} \longrightarrow \mathbb{N}$ as defined by $h(0) = 0$ and:

$$h(k+1) = \begin{cases} m & \text{if } h(k) \sqsubset m \text{ and } \mathsf{proof}_{\Box}\left(\overline{k}, \overline{\exists x \,.\, \neg(h(x) \sqsubseteq \overline{m})}\right), \\ n & \text{if } h(k) \prec n \text{ and } \mathsf{proof}_{\triangle}\left(\overline{k}, \overline{\exists x \,.\, \neg(h(x) \preccurlyeq \overline{n})}\right), \\ h(k) & \text{otherwise.} \end{cases}$$

Inside T we can show that

$$\vdash_T \forall s \forall x_0, x_1 < \mathsf{length}(s) \,.\, \theta(s) \wedge x_0 < x_1 \rightarrow (s)_{x_0} \preccurlyeq (s)_{x_1}.$$

Which easily implies

$$\vdash_T \forall x_0, x_1 \,.\, x_0 \leq x_1 \rightarrow h(x_0) \preccurlyeq h(x_1). \tag{H4}$$

Since T knows that the frame is realistic it can easily show that for any $k \in \mathbb{N}$:

$$\vdash_T \left(\exists y \,.\, \neg h(y) \preccurlyeq \overline{k}\right) \to \left(\exists x \,.\, \neg h(x) \sqsubset \overline{k}\right)$$

But then using that $\mathrm{HB}_{T,\triangle}$ we have that:

$$\vdash_T \triangle \left(\exists y \,.\, \neg h(y) \preccurlyeq \overline{k}\right) \to \triangle \left(\exists x \,.\, \neg h(x) \sqsubset \overline{k}\right).$$

Since $\left(\exists x \,.\, \neg h(x) \sqsubset \overline{k}\right)$ is equivalent to a Σ_1 formula modulo T we have that we have absorption for it. Then by $\mathrm{Hb}_{T,\Box}$ we have that

$$\vdash_T \triangle \left(\exists x \,.\, \neg h(x) \sqsubset \overline{k}\right) \to \Box \triangle \left(\exists x \,.\, \neg h(x) \sqsubset \overline{k}\right),$$

which by absorption we conclude that:

$$\vdash_T \triangle \left(\exists y \,.\, \neg h(y) \preccurlyeq \overline{k}\right) \to \Box \left(\exists x \,.\, \neg h(x) \sqsubseteq \overline{k}\right). \tag{H5}$$

We also have the following two results for $i \in \mathbb{N}$:

$$\vdash_T \neg \left(x \preccurlyeq \overline{i}\right) \leftrightarrow \bigvee_{j \in M, j \not\preccurlyeq i} x \approx \overline{j}, \tag{H6}$$

$$\vdash_T \neg \left(x \sqsubset \overline{i}\right) \leftrightarrow \bigvee_{j \in M, j \not\sqsubset i} x \approx \overline{j}. \tag{H7}$$

Note that this is defined since given i there are only finitely many j such that $j \not\preccurlyeq i$ or $j \not\sqsubset i$.

By definition of $\mathcal{M}$ we have that:

Lemma 105. Given any formula ϕ either $[\![\phi]\!]^{\mathcal{M}}$ is finite or $[\![\phi]\!]^{\mathcal{M}} = \mathbb{N}$.

The previous lemma justifies this definition:

Definition 106. For any sentence $\phi \in \mathcal{L}_{\mathrm{m}}$, we define the $\mathcal{L}_1$-sentence $[\phi]$ as:

$$[\phi] = \begin{cases} \bigvee_{i \in [\![\phi]\!]^{\mathcal{M}}} \exists x \,.\, h(x) \approx \overline{i} & \text{if } [\![\phi]\!]^{\mathcal{M}} \text{ is finite,} \\ \top & \text{if } [\![\phi]\!]^{\mathcal{M}} = \mathbb{N}. \end{cases}$$ ■

From now own we will omit the $^{\mathcal{M}}$ in $[\![\phi]\!]^{\mathcal{M}}$ and simply write $[\![\phi]\!]$. After all this work we are ready for the main part of the proof. We want to show that [_] commutes with all the operators in propositional modal logic.

Lemma 107 (Disjunction)**.** For any $\phi, \psi \in \mathcal{L}_{\mathrm{m}}$, we have that

$$\vdash_T [\phi \vee \psi] \leftrightarrow [\phi] \vee [\psi].$$

Proof. If $[\![\phi]\!] = \mathbb{N}$ then $[\![\phi \vee \psi]\!] = \mathbb{N}$ and then the equivalence is trivial. If $[\![\psi]\!] = \mathbb{N}$ it is analogous, so the only case left is $[\![\phi]\!]$ and $[\![\psi]\!]$ finite. However, this case is trivial by definition of [_]. □

Lemma 108 (Conjunction). For any $\phi, \psi \in \mathcal{L}_{\mathrm{m}}$, we have that

$$\vdash_T [\phi \wedge \psi] \leftrightarrow [\phi] \wedge [\psi].$$

Proof. Suppose that $[\![\phi]\!]$ is $\mathbb{N}$. Then $[\phi] = \top$ and $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!] = [\![\psi]\!]$, so $[\phi \wedge \psi] = [\psi]$. Then

$$\begin{aligned} \vdash_T [\phi \wedge \psi] &= [\psi] \\ &\leftrightarrow \top \wedge [\psi] \\ &= [\phi] \wedge [\psi]. \end{aligned}$$

The case of $[\![\psi]\!] = \mathbb{N}$ is analogous, so assume that both $[\![\phi]\!]$ and $[\![\psi]\!]$ are finite.

$\rightarrow$ is trivial, so we only show $\leftarrow$. For this direction it suffices to show that if $i \in [\![\phi]\!]$ and $j \in [\![\psi]\!]$

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right) \rightarrow [\phi \wedge \psi]. \tag{i}$$

First, we show that

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right) \rightarrow \bar{i} \preccurlyeq \bar{j} \vee \bar{j} \preccurlyeq \bar{i}. \tag{ii}$$

We reason inside T. So assume we have x, y such that $h(x) \approx \bar{i}$ and $h(y) \approx \bar{j}$. Since $x \leq y \vee y \leq x$ using Lemma H4 we have that $h(x) \preccurlyeq h(y) \vee h(y) \preccurlyeq h(x)$ which implies $\bar{i} \preccurlyeq \bar{j} \vee \bar{j} \preccurlyeq \bar{i}$. We leave T.

If i and j are incomparable, by Σ_1-completeness, we have that $\vdash_T \neg(\bar{i} \preccurlyeq \bar{j} \vee \bar{j} \preccurlyeq \bar{i})$. By (ii) we get $\vdash_T \neg\left(\left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right)\right)$ in which case (i) is trivial. If i, j are comparable, without loss of generality assume $i \preccurlyeq j$. By knowledge persistance (outside T), $j \in [\![\phi \wedge \psi]\!]$, so

$$\begin{aligned} \vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right) &\rightarrow \left(\exists y \,.\, h(y) \approx \bar{j}\right) \\ &\rightarrow [B \wedge C]. && \text{(by } j \in [\![\phi \wedge \psi]\!]) \end{aligned}$$

$\square$

Lemma 109 (Implication). For any $\phi, \psi \in \mathcal{L}_{\mathrm{m}}$, we have that

$$\vdash_T [\phi \rightarrow \psi] \leftrightarrow ([\phi] \rightarrow [\psi]).$$

Proof. If $[\![\phi \rightarrow \psi]\!] = \mathbb{N}$, then $[\![\phi]\!] \subseteq [\![\psi]\!]$. By simple propositional reasoning, we can show that $\vdash_T ([\phi] \rightarrow [\psi]) \leftrightarrow \top$, so we will have the desired equivalence. So we can assume that $[\phi \rightarrow \psi]$ is finite.

First we prove left to right. We always have that $[\![(\phi \rightarrow \psi) \wedge \phi]\!] \subseteq [\![\psi]\!]$. So by simple propositional reasoning

$$\vdash_T [(\phi \rightarrow \psi) \wedge \phi] \rightarrow [\psi].$$

By Lemma 108 we have that

$$\vdash_T [\phi \to \psi] \wedge [\phi] \to [(\phi \to \psi) \wedge \phi],$$

and from this and the previous implication we get the left to right direction.

Now we prove right to left. Thanks to $\mathrm{Slp}_{T,\triangle}$, it suffices to show that

$$\vdash_T ([\phi] \to [\psi]) \to \triangle[\phi \to \psi] \to [\phi \to \psi]. \tag{i}$$

We show (i). Let $j_0, \dots, j_{s-1}$ be the maximal elements of M that do not belong to $[\![\phi \to \psi]\!]$ (by the shape of the model we know that these exist and that there are only finitely many). In particular, this implies that $j_t \in [\![\phi]\!]$ and $j_t \notin [\![\psi]\!]$. Using that M is conversely well-founded and preservation of knowledge, we can show that for any $i \in M$,

$$i \in [\![\phi \to \psi]\!] \text{ iff (for any } t < s, i \not\preccurlyeq j_t). \tag{ii}$$

Note that if $i \in [\![\phi \to \psi]\!]$ we get, for any $t < s$, $\vdash_T h(x) \approx \bar{i} \to \neg \left(h(x) \preccurlyeq \overline{j_t} \right)$ thanks to (ii). Using this and iFOL reasoning we can derive that for any $t < s$:

$$\vdash_T [\phi \to \psi] \to \left(\exists y \,.\, \neg \left(h(y) \preccurlyeq \overline{j_t} \right) \right).$$

Using this, $\mathrm{Hb}_{T,\triangle}$ we get that for any $t < s$:

$$\triangle[\phi \to \psi] \vdash_T \triangle \left(\exists y \,.\, \neg \left(h(y) \preccurlyeq \overline{j_t} \right) \right). \tag{iii}$$

Now we work inside T. Assume that $[\phi] \to [\psi]$ and $\triangle[\phi \to \psi]$, we have thanks to (iii), that there are k_t (for any $t < s$) such that $\mathrm{proof}_\triangle \left(k_t, \overline{\exists y \,.\, \neg \left(h(y) \preccurlyeq \overline{j_t} \right)} \right)$. Inside T we have that for any $t < s$, $h(k_t) \prec \overline{j_t} \vee h(k_t) = \overline{j_t} \vee \neg \left(h(k_t) \preccurlyeq \overline{j_t} \right)$, since $x \preccurlyeq y$ is a Δ_0-formula. This allows us to separate the rest of the proof in 3 cases.

1. For some $t < s$, $h(k_t) \prec \overline{j_t}$. By definition of h, we get $h(k_t + 1) \approx \overline{j_t}$. But $\overline{j_t} \in [\![\phi]\!]$ so $[\phi]$ holds. Since $[\phi] \to [\psi]$ we have that $[\psi]$ holds and since $[\![\psi]\!] \subseteq [\![\phi \to \psi]\!]$ we can conclude that $[\phi \to \psi]$ holds.

2. For some $t < s$, $h(k_t) \approx \overline{j_t}$. Then, $[\phi \to \psi]$ follows similarly to the previous case.

3. For all $t < s$, $\neg(h(k_t) \preccurlyeq \overline{j_t})$. Let $k \approx \max(k_0, \dots, k_{s-1})$. Then, we also have that
$$\text{for all } t < s, \neg \left(h(k) \preccurlyeq \overline{j_t} \right). \tag{iv}$$

   Let us see why. Let $t < s$ such that $h(k) \preccurlyeq \overline{j_t}$. Since $k_t \leq k$ we have by H4 that $h(k_t) \preccurlyeq h(k) \preccurlyeq \overline{j_t}$. Since T knows that $\preccurlyeq$ is transitive, we get that $h(k_t) \preccurlyeq \overline{j_t}$, contrary to this case assumption. But then, using H6 we get:
$$\bigvee_{j \in M, j \not\preccurlyeq j_t} h(k) \approx \bar{j}.$$

And putting these together for each $t < s$, modulo some iFOL reasoning, we have that

$$\bigvee_{j \in M,\ \text{for any t} < \text{s}: j \not\preccurlyeq j_t} h(k) \approx \bar{j}.$$

But note that, outside T, $\{j \in M \mid \text{ for any } t < s . j \not\preccurlyeq j_t\} = [\![\phi \rightarrow \psi]\!]$ by (ii). Then, inside T, we got $[\phi \rightarrow \psi]$.

$\square$

For the case of box we need two auxiliary lemmas:

Lemma 110. Let $i \in M - \{0\}$. Then,

$$\vdash_T \left(\exists x . h(x) \approx \bar{i}\right) \rightarrow \Box \left(\exists y . \bar{i} \prec h(y)\right).$$

Proof. To show the desired lemma we are going to show two auxiliary facts.

$$\vdash_T \left(\exists y . \neg\left(h(y) \preccurlyeq \bar{i}\right)\right) \wedge \left(\exists x . h(x) \approx \bar{i}\right) \rightarrow \left(\exists y . \bar{i} \prec h(y)\right). \qquad \text{(i)}$$

$$\vdash_T \left(\exists y . h(y) \sqsubseteq \bar{i}\right) \wedge \left(\exists x . h\left(x \dot{-} \bar{1}\right) \approx \bar{i} \wedge h(x) \sqsubset \bar{i}\right) \rightarrow \left(\exists y . \bar{i} \prec h(y)\right). \qquad \text{(ii)}$$

Proof of lemma from (i) and (ii). We reason inside T. We assume that $\exists x . h(x) \approx \bar{i}$. There exists an x which is the minimum number such that $h(x) \approx \bar{i}$. Note that taking this minimum is allowed in T, since $\exists x . h(x) \approx \bar{i}$ means that there is a sequence s such that $\theta(s)$ and $(s)_x \approx \bar{i}$. Then, it suffices by the properties of θ and the definition of h to find the least x such that $(s)_x \approx \bar{i}$, which can be done in iEA. Since $i > 0$ we have that $x > \bar{0}$ and then by the definition of h, $h\left(x \dot{-} \bar{1}\right) \prec \bar{i}$. We make a case distinction in $h\left(x \dot{-} \bar{1}\right) \sqsubset \bar{i}$, since $x \sqsubset y$ is a Δ_0-formula.

1. Assume $\neg\left(h(x \dot{-} 1) \sqsubset \bar{i}\right)$. Then since $h(x) \approx \bar{i}$, by definition of h this implies that $\triangle\left(\exists y . \neg\left(h(y) \preccurlyeq \bar{i}\right)\right)$. Since $\exists x . h(x) \approx \bar{i}$ is a Σ_1-sentence, we have that $\triangle\left(\exists x . h(x) \approx \bar{i}\right)$. Using (i) and $\mathrm{Hb}_{T,\triangle}$ we can conclude $\triangle\left(\exists y . \bar{i} \prec h(y)\right)$. Since $\exists y . \bar{i} \prec h(y)$ is a Σ_1-sentence, we also have that $\triangle\left(\exists y . \bar{i} \prec h(y)\right) \rightarrow \Box\left(\exists y . \bar{i} \prec h(y)\right)$ by H5, so we can conclude the desired $\Box(\exists y . \bar{i} \prec h(y))$.
2. Asssume $h\left(x \dot{-} \bar{1}\right) \sqsubset \bar{i}$. Since $h(x) \approx \bar{i}$ by definition of h we have that $\Box\left(\exists x . \neg h(x) \sqsubseteq \bar{i}\right)$ or $\triangle\left(\exists x . \neg h(x) \preccurlyeq \bar{i}\right)$. Using Lemma H5 we can get that $\Box\left(\exists x . \neg h(x) \sqsubseteq \bar{i}\right)$ in both cases. By assumption of the case, we have

$\exists x \,.\, h(x) \approx \overline{i} \wedge h\left(x \dot{-} \overline{1}\right) \sqsubset \overline{i}$ and since this is a Σ_1-sentence we conclude $\Box\left(\exists x \,.\, h(x) \approx \overline{i} \wedge h(x \dot{-} 1) \sqsubset \overline{i}\right)$. Finally, using (ii) and $\mathrm{Hb}_{T,\Box}$ we conclude the desired $\Box\left(\exists y \,.\, \overline{i} \prec h(y)\right)$.

Proof of (i). First, we show that

$$\vdash_T \neg\left(h(y) \preccurlyeq \overline{i}\right) \wedge h(x) \approx \overline{i} \to \overline{i} \prec h(y). \tag{iii}$$

We reason inside T. Assume that $\neg\left(h(y) \preccurlyeq \overline{i}\right) \wedge h(x) \approx \overline{i}$. If $y < x$ by Lemma H4 we have that $h(y) \preccurlyeq h(x) \approx \overline{i}$, which is impossible. So $x \leq y$, which again by lemma H4 we have that $\overline{i} \approx h(x) \preccurlyeq h(y)$. But since $\neg\left(h(y) \preccurlyeq \overline{i}\right)$ we must have that $\neg\left(h(y) \approx \overline{i}\right)$ so from $\overline{i} \preccurlyeq h(y)$ we conclude $\overline{i} \prec h(y)$ and we have proven (iii). From (iii) and some reasoning in iFOL we can conclude (i).

Proof of (ii). First, we show that

$$\vdash_T \neg\left(h(y) \sqsubseteq \overline{i}\right) \wedge h(x) \approx \overline{i} \wedge h\left(x \dot{-} \overline{1}\right) \sqsubset \overline{i} \to \overline{i} \prec h(y). \tag{iv}$$

We reason in T, assume $\neg\left(h(y) \sqsubseteq \overline{i}\right) \wedge h(x) \approx \overline{i} \wedge h\left(x \dot{-} \overline{1}\right) \sqsubset \overline{i}$. Now, suppose that $y < x$, then $y \leq x \dot{-} \overline{1}$, so by Lemma H4 we have that $h(y) \preccurlyeq h\left(x \dot{-} \overline{1}\right) \sqsubset \overline{i}$. Since T shows the model property, we have that $h(y) \sqsubset \overline{i}$, contradiction. Then $y \geq x$, so again by Lemma H4 we have that $\overline{i} \approx h(x) \preccurlyeq h(y)$ and $\neg\left(\overline{i} \approx h(y)\right)$, so $\overline{i} \prec h(y)$. This establishes (iv), and from it and some iFOL reasoning we can conclude (ii). $\Box$

Lemma 111. Let $i, j \in \mathbb{N}$ such that $i \prec j$ and $i \not\sqsubset j$. Then

$$\vdash_T \left(\exists x \,.\, h(x) \approx \overline{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \overline{j}\right) \to \triangle\left(\exists z \,.\, \overline{j} \prec h(z)\right).$$

Proof. If $i \prec j$ and $i \not\sqsubset j$ then since T extends iEA we have (by Σ_1-completeness) that $\vdash_T \overline{i} \prec \overline{j} \wedge \neg\left(\overline{i} \sqsubset \overline{j}\right)$. We reason inside T and assume that $\left(\exists x \,.\, h(x) \approx \overline{i}\right)$ and $\left(\exists y \,.\, h(y) \approx \overline{j}\right)$. As we did in the proof of Lemma 110 we can consider the least y such that $h(y) \approx \overline{j}$. Since $i \prec j$ (outside T) we have that $j > 0$ (outside T), but by Σ_1-completeness we have that $\overline{j} > \overline{0}$ inside T. Then, inside T, we have that $y > \overline{0}$ by the definition of h. By the definition of h we also have that $h\left(y \dot{-} \overline{1}\right) \prec \overline{j}$. By assumption, we have an x such that $h(x) \approx \overline{i}$. Assume that $y \leq x$, then by lemma H4 we have that $\overline{j} \approx h(y) \preccurlyeq h(x) \approx \overline{i} \prec \overline{j}$. But then since T knows that $\preccurlyeq$ is antisymmetric we would conclude $\overline{i} \approx \overline{j}$, which is a contradiction since $\overline{i} \prec \overline{j}$ inside T. Then $x < y$, so $x \leq y \dot{-} \overline{1}$ and then by H4 we have that $\overline{i} \approx h(x) \preccurlyeq h(y-1)$.

If $h(y-1) \sqsubset \overline{j}$, then $\overline{i} \preccurlyeq h(y-1) \sqsubset \overline{j}$, so $\overline{i} \sqsubset \overline{j}$. But we also have that $\neg(\overline{i} \sqsubset \overline{j})$, a contradiction, so it must be the case that $\neg\left(h(y-1) \sqsubset \overline{j}\right)$. Following the same reasoning as in Lemma 110 in 1. we can conclude $\triangle\left(\exists z \,.\, \overline{j} \prec h(z)\right)$, as desired. $\square$

Finally, we show the $\Box$ case. In the proof we can notice the motivation of making the model infinite to be able of establishing the equivalence.

Lemma 112 (Box). For any $\phi \in \mathcal{L}_{\mathrm{m}}$, we have that

$$\vdash_T [\Box\phi] \leftrightarrow \Box[\phi].$$

Proof. If $[\![\phi]\!] = \mathbb{N}$, then $[\![\Box\phi]\!] = \mathbb{N}$ and the equivalence is trivial by definition. Assume that $[\![\phi]\!]$ is finite. This implies that

$$0 \notin [\![\Box\phi]\!], \tag{i}$$

since 0 has infinitely many successors.

First we show right to left. Let $j_0, \ldots, j_{s-1}$ be the $\sqsubset$-maximal elements j of M such that $j \notin [\![\phi]\!]$. Note that for any $t < s$, $j_t \neq 0$, since if $j_t = 0$ is maximal not belonging to $[\![\phi]\!]$, then any $n > 0$ belongs to $[\![\phi]\!]$. This implies that $0 \in [\![\Box\phi]\!]$, contrary to (i). Also, note that there are only finitely many maximal j such that $j \notin [\![\phi]\!]$ by the shape of the model. Finally, note that for any $t < s$ we have that

$$j_t \in [\![\Box\phi]\!], \tag{ii}$$

by the definition of the j_t's.

Let us show that

$$\text{for any } t < s, \vdash_T [\phi] \to \exists x \,.\, \neg\left(h(x) \sqsubseteq \overline{j_t}\right). \tag{iii}$$

It suffices to show that $i \in [\![\phi]\!]$ then $i \not\sqsubseteq j_t$. So let $i \in [\![\phi]\!]$ but $i \sqsubseteq j_t$. Since the frame is realistic, $i \preccurlyeq j_t$ and by preservation of knowledge $j_t \in [\![\phi]\!]$. This is a contradiction with the definitions of j_t, as wanted.

We also have the following property:

$$(\text{for all } t < s, i \not\sqsubseteq j_t) \text{ implies that } i \in [\![\phi]\!]. \tag{iv}$$

This can be shown using that $\sqsubset$ is transitive and conversely well-founded.

Finally, we reason inside T. Suppose that $\Box[\phi]$, By (iii) and $\mathrm{Hb}_{T,\Box}$ we get that for all $t < s$, $\Box\left(\exists x \,.\, \neg\left(h(x) \sqsubseteq \overline{j_t}\right)\right)$. So we have that for each $t < s$ there exists k_t be such that $\mathrm{proof}_\Box\left(k_t, \overline{\exists x \,.\, \neg\left(h(x) \sqsubseteq \overline{j_t}\right)}\right)$. Since $x \sqsubseteq y$ is Δ_0 formula, we can distinguish 3 cases:

1. For some $t < s$, $h(k_t) \sqsubset \overline{j_t}$. By the definition of h, we get $h\left(k_t \dot{-} \overline{1}\right) \approx \overline{j_t}$, and $[\Box\phi]$ follows thanks to (ii).

2. For some $t < s$, $h(k_t) \approx \overline{j_t}$. Then $[\Box\phi]$ follows thanks to (ii).

3. For all $t < s$, $\neg(h(k_t) \sqsubseteq \overline{j_t})$. Let $k \approx \max(k_0, \dots, k_{s-1})$. If $h(k) \approx \overline{j_t}$ for some $t < s$ we get a contradiction to the hypothesis of the case (in particular we would be in case 2.). Similarly, if $h(k) \sqsubset \overline{j_t}$ for some $t < s$, thanks to H4 we get that $h(k_t) \preccurlyeq h(k) \sqsubset \overline{j_t}$. Since T knows that M fulfills the model property, we have that $h(k_t) \sqsubset \overline{j_t}$ which contradicts the hypothesis of the case (it returns to case the first case). We can conclude that for any $t < s$,
$$\neg\left(h(k) \sqsubseteq \overline{j_t}\right).$$
By H7 we get that for any $t < s$
$$\bigvee_{j \in M, j \not\sqsubseteq j_t} h(k) \approx \overline{j}.$$
Using iFOL reasoning we get that
$$\bigvee_{j \in M,\ \text{for all } t<s : j \not\sqsubseteq j_t} h(k) \approx \overline{j}.$$
Outside T we have that $\{j \in M \mid \text{ for all } t < s : j \not\sqsubseteq j_t\} = [\![\phi]\!] \subseteq [\![\Box\phi]\!]$, the inclusion thanks to $\mathcal{M}$ being realistic. So, inside T, we conclude $[\Box\phi]$.

Now we show the left to right direction, i.e $\vdash_T [\Box\phi] \to \Box[\phi]$. Let $i \in [\![\Box\phi]\!]$, by (i) we have that $i > 0$. By Lemma 110 we get that
$$\vdash_T \left(\exists x \,.\, h(x) \approx \overline{i}\right) \to \Box\left(\exists y \,.\, \overline{i} \prec h(y)\right). \tag{v}$$
By the shape of $\mathcal{M}$, we know that any $k > 0$ there is a greatest $n \in \mathbb{N}$ such that there exists a sequence $k = k_0 \prec k_1 \prec \cdots \prec k_n$. We call this n the $\prec$-rank of k. Let a be the $\prec$-rank of i. For any $b \in \mathbb{N}$ we define the set:
$$U_b = \{j \in \mathbb{N} \mid i \prec j, i \not\sqsubset j \text{ and } \prec\text{-rank}(j) < b\}.$$
Note that for any b, U_b is finite since $i \neq 0$ implies that there are only finitely many j such that $i \prec j$. By the definition of $\prec$ we know that
$$\vdash_T \overline{i} \prec h(y) \to \bigvee_{j : i \prec j} h(y) \approx \overline{j}. \tag{vi}$$
But $i \preccurlyeq j$ implies that $\prec\text{-rank}(j) < \prec\text{-rank}(i)$. This means that if $i \preccurlyeq j$ and $i \not\sqsubset j$ we get that $j \in U_a$ and if $i \sqsubset j$ then $j \in [\![\phi]\!]$. So (vi) can be written as
$$\vdash_T \overline{i} \prec h(y) \to \left(\bigvee_{j \in [\![\phi]\!]} h(y) \approx \overline{j}\right) \vee \left(\bigvee_{j \in U_a} h(y) \approx \overline{j}\right).$$

By iFOL reasoning we get

$$\vdash_T \left(\exists y \,.\, \bar{i} \prec h(y)\right) \to [\phi] \vee \left(\bigvee_{j \in U_a} \exists y \,.\, h(y) \approx \bar{j}\right).$$

This together with (v) and thanks to $\mathrm{Hb}_{T,\Box}$ we get

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \to \Box \left([\phi] \vee \bigvee_{j \in U_a} \exists y \,.\, h(y) \approx \bar{j}\right). \tag{vii}$$

Our objective now will be to get rid of the right disjunction inside $\Box$ in (vii). Let $j \in U_b$ for some $b \geq 1$. By Lemma 111 we have that

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right) \to \triangle \left(\exists z \,.\, \bar{j} \prec h(z)\right). \tag{viii}$$

Also, as we did previously for i, we have that

$$\vdash_T \left(\exists z \,.\, \bar{j} \prec h(z)\right) \to [\phi] \vee \left(\bigvee_{k \in U_{b-1}} \exists z \,.\, h(z) \approx \bar{k}\right)$$

Using (viii) and $\mathrm{Hb}_{T,\triangle}$ we get that

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\exists y \,.\, h(y) \approx \bar{j}\right) \to \triangle \left([\phi] \vee \bigvee_{k \in U_{b-1}} \exists z \,.\, h(z) \approx \bar{k}\right)$$

This holds for any $j \in U_b$, so (changing also the name of some bound variables)

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left(\bigvee_{j \in U_b} \exists y \,.\, h(y) \approx \bar{j}\right) \to \triangle \left([\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}\right)$$

Since $[B]$ is T-equivalent to a Σ_1-sentence, we have that $\vdash_T [\phi] \to \triangle[\phi]$. Also $\vdash_T \triangle[\phi] \to \triangle \left([\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}\right)$, so

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left([\phi] \vee \bigvee_{j \in U_b} \exists y \,.\, h(y) \approx \bar{j}\right) \to \triangle \left([\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}\right). \tag{ix}$$

We also have that $\exists x \,.\, h(x) \approx \bar{i}$ is T-equivalent to a Σ_1-sentence, so thanks to $\mathrm{Hb}_{T,\Box}$ we have that $\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \to \Box \left(\exists x \,.\, h(x) \approx \bar{i}\right)$. With this we can perform the following reasoning:

$$\begin{aligned}
&\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \Box\left([\phi] \vee \bigvee_{j \in U_b} \exists y \,.\, h(y) \approx \bar{j}\right) \\
&\quad \to \Box\left(\left(\exists x \,.\, h(x) \approx \bar{i}\right) \wedge \left([\phi] \vee \bigvee_{j \in U_b} \exists y \,.\, h(y) \approx \bar{j}\right)\right) \\
&\quad \to \Box \triangle \left([\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}\right) && \text{(by (ix))} \\
&\quad \to \Box\left([\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}\right) && (\Sigma_1\text{-absorption})
\end{aligned}$$

where for the last implication we used that $[\phi] \vee \bigvee_{j \in U_{b-1}} \exists y \,.\, h(y) \approx \bar{j}$ is T-equivalent to a Σ_1-sentence.

If we apply this reasoning to (vii) for $b = a, \ldots, 1$ we end getting

$$\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) \to \Box\left([\phi] \vee \bigvee_{j \in U_0} \exists y \,.\, h(y) \approx \bar{j}\right).$$

But $U_0 = \emptyset$, so in fact we get

$$\begin{aligned}
\vdash_T \left(\exists x \,.\, h(x) \approx \bar{i}\right) &\to \Box([\phi] \vee \bot) \\
&\leftrightarrow \Box[\phi].
\end{aligned}$$

This holds for any $i \in [\![\Box\phi]\!]$, so we can conclude that $\vdash_T [\Box\phi] \to \Box[\phi]$. $\square$

Theorem 113. Let σ be the Σ_1-realization given by $\sigma(p) = [p]$. For any $\phi \in \mathcal{L}_{\mathrm{m}}$ we have that:

$$\vdash_T \sigma_\Box(\phi) \leftrightarrow [\phi].$$

Proof. By induction in ϕ using lemmas 107, 108, 109 and 112. $\square$

We establish one fundamental property of Solovay's function: in the standard model it can be shown that it never advances from the starting node.

Lemma 114. Suppose that T is Σ_1-sound. Then $\vDash_{\mathbb{N}} h(x) \approx \bar{0}$.

Proof. Since M is conversely well-founded, we know that h must have a certain limit $i \in M$. Then $\left(\exists x \,.\, h(x) \approx \bar{i}\right)$ is a true Σ_1-sentence, so by Σ_1-completeness of T we get that $\vdash_T \exists x \,.\, h(x) \approx \bar{i}$. Now, assume that $i > 0$. By Lemma 110 we get that $\vdash_T \Box\left(\exists y \,.\, \bar{i} \prec h(y)\right)$. Since $\Box\left(\exists y \,.\, \bar{i} \prec h(y)\right)$ is a Σ_1-sentence and T is Σ_1-sound we have that $\vDash_{\mathbb{N}} \Box\left(\exists y \,.\, \bar{i} \prec h(y)\right)$ and then, by assumption of good pair, we get $\vdash_T \exists y \,.\, \bar{i} \prec h(y)$ and since this sentence is Σ_1 and T is Σ_1-sound we get $\vDash_{\mathbb{N}} \exists y \,.\, \bar{i} \prec h(y)$. This is impossible, since i is the limit of h. So $i \leq 0$, i.e. $i = 0$, and thanks to Lemma H4 it must be the case that h is the constant function 0. $\square$

Theorem 115. Let T be a Σ_1-sound theory. Let $(\square, \triangle)$ be a good pair for T and assume that $\mathrm{CP}_{T,\square}$. Then:

$$\Sigma_1\text{-}\mathbb{PL}(T,\square) = \mathbb{PL}(T,\square) = \mathsf{iGLC}.$$

Proof. First, let us focus in Σ_1-$\mathbb{PL}$. Tt is easy to show that $\supseteq$, i.e. soundness. Note that to prove soundness of the completness principle we need $\mathrm{Cp}_{T,\square}$, even if we are dealing only with Σ_1-realizations (thanks to be dealing with Σ_1-realization we do not need it for $p \to \square p$, but we need it for $\neg p \to \square(\neg p)$, for example). We focus in proving $\subseteq$, i.e. completeness.

Let $\phi \in \mathcal{L}_{\mathrm{m}}$ such that $\nvdash_{\mathsf{iGLC}} \phi$. By Theorem 104 we have that there exist a finite, irreflexive, realistic model $\mathcal{M}_0$ and an $r \in \mathcal{M}_0$ such that $\mathcal{M}_0, r \nvDash \phi$. It is clear that going from $\mathcal{M}_0$ to $\mathcal{M}$, we will still have an intuitionistic iGLC conversely well-founded, irreflexive and realistic model and ϕ is still not true at r, i.e. $\mathcal{M}, r \nvDash \phi$. We want to show that there exists a realization σ such that $\nvdash_T \sigma_\square \phi$.

We define the Solovay function h as described previously and the Σ_1-realization σ as $\sigma(p) = [p]$. Assume that $\vdash_T \sigma_\square \phi$. By Theorem 113, we have that $\vdash_T [\phi]$. Since $[\phi]$ is equivalent to a Σ_1-sentence in T and T is Σ_1-sound we have that $\mathbb{N} \vDash [\phi]$. By Lemma 114 we also have that $\mathbb{N} \vDash h(x) \approx 0$. This implies that $0 \in [\![\phi]\!]$, but $0 \preccurlyeq r$ and, by preservation of knowledge, we get that $r \in [\![\phi]\!]$. But by assumption $r \notin [\![\phi]\!]$, a contradiciton. So $\nvdash_T \sigma_\square \phi$, as wanted.

Let us see why it also holds for $\mathbb{PL}$ with the extra assumption. To show completeness we have shown that if $\nvdash_{\mathsf{iGLC}} \phi$ then there exists a Σ_1-realization σ such that $\nvdash_T \sigma_\square(\phi)$. This trivially implies that there exists a realization σ such that $\nvdash_T \sigma_\square(\phi)$, so we have completeness. We also need soundness. Clearly, the only worrying axiom is the completeness principle, but since we have $\mathrm{Cp}_{T,\square}$ it is sound. $\square$

3.3 Theories with provability logic iGLC

First, we are going to use the good pair theorem to show a general method to obtain predicates whose (Σ_1-)$\mathbb{PL}$ is iGLC.

Theorem 116. Assume that

1. T is a Σ_1-sound, Σ_1-complete, Δ_0-decidable. We also assume that T is Δ_0-self arithmetizable and extends iEA or that T is Σ_1-self arithmetizable and extends $\mathsf{iEA} + B\Sigma_1$.

2. Let $\triangle$ be a Σ_1 box function and assume that $\mathrm{HB}_{T,\triangle}$.

3. For any Σ_1-sentence ϕ, $\vdash_T \Box_T \triangle\phi \to \Box_T \phi$.

4. For any sentence ϕ, $\vdash_T \triangle\phi \to \Box_T \phi$.

5. T is closed under $(\_)^{\triangle}_{\mathrm{V}}$.

Then

$$\Sigma_1\text{-}\mathbb{PL}((T)^{\triangle}_{\mathrm{V}}, \Box^{\triangle}_{T}) = \mathbb{PL}((T)^{\triangle}_{\mathrm{V}}, \Box^{\triangle}_{T}) = \mathsf{iGLC}.$$

Proof. Note that $\mathsf{iEA} \subseteq T \subseteq (T)^{\triangle}_{\mathrm{V}}$, so $(T)^{\triangle}_{\mathrm{V}}$ is an extension of iEA. Our objective is to apply Theorem 115. We divide the proof in 3 parts.

$(\Box^{\triangle}_{T}, \triangle^{\triangle})$ is a $(T)^{\triangle}_{\mathrm{V}}$-good pair. First note that the assumption of self-arithmetization of T gives us that $\Box^{\triangle}_{T}$ is Σ_1 and also that $\mathrm{HB}_{T,\Box_T}$. The assumption that $\triangle$ is Σ_1 gives us that $\triangle^{\triangle}_{T}$ is Σ_1.

Thanks to Lemma 77 we have that $\mathrm{Hb}_{(T)^{\triangle}_{\mathrm{V}},\Box^{\triangle}_{T}}$ and $\mathrm{Hb}_{(T)^{\triangle}_{\mathrm{V}},\triangle^{\triangle}}$.

Let ϕ be a sentence such that $\vDash_{\mathbb{N}} \Box^{\triangle}_{T}\phi$. Since ϕ is a sentence this means that $\vDash_{\mathbb{N}} \Box(\phi)^{\triangle}_{\mathrm{V}}$. Since by the assumption of self-arithmetization of T and that T is Σ_1-sound and Σ_1-complete, we get by Lemma 41 that T is truly arithmetizable. This means by Lemma 42 that $\vDash_{\mathbb{N}} \Box_T(\phi)^{\triangle}_{\mathrm{V}}$ implies $\vdash_T (\phi)^{\triangle}_{\mathrm{V}}$. Then $\phi \in \mathrm{Ax}_{(T)^{\triangle}_{\mathrm{V}}}$, so $\vdash_{(T)^{\triangle}_{\mathrm{V}}} \phi$, as wanted.

Let ϕ be a sentence and let us show that $\vdash_{(T)^{\triangle}_{\mathrm{V}}} \phi \to \triangle^{\triangle}\phi$. Using Lemma 71 we know that it suffices to show that $\vdash_T (\phi \to \triangle^{\triangle}\phi)^{\triangle}_{\mathrm{V}}$. But, thanks to $\mathrm{HB}_{T,\triangle}$, it suffices to show $\vdash_T (\phi)^{\triangle}_{\mathrm{V}} \to (\triangle^{\triangle}\phi)^{\triangle}_{\mathrm{V}}$. Since $\triangle^{\triangle}\phi$ is Σ_1, thanks to Lemma 67 we get that we only need to show $\vdash_T (\phi)^{\triangle}_{\mathrm{V}} \to \triangle^{\triangle}\phi$. Since ϕ is a sentence, this is equivalent to showing $\vdash_T (\phi)^{\triangle}_{\mathrm{V}} \to \triangle(\phi)^{\triangle}_{\mathrm{V}}$. But this is just Lemma 60.

Finally, we need to show $\Sigma_1\text{-}\mathrm{Abs}_{(T)^{\triangle}_{\mathrm{V}},\Box^{\triangle}_{T},\triangle^{\triangle}}$. So let ϕ be a sentence, we have that:

$$\begin{aligned}
\vdash_T \Box^{\triangle}_{T}\triangle^{\triangle}\phi &\leftrightarrow \Box_T(\triangle^{\triangle}\phi)^{\triangle}_{\mathrm{V}} && \text{(since } \phi \text{ is a sentence)}\\
&\leftrightarrow \Box_T \triangle^{\triangle}\phi && \text{(by Lemma 67 since } \triangle^{\triangle}\phi \text{ is } \Sigma_1 \text{ and } \mathrm{Hb}_{T,\Box_T})\\
&\leftrightarrow \Box_T \triangle(\phi)^{\triangle}_{\mathrm{V}} && \text{(since } \phi \text{ is a sentence)}\\
&\leftrightarrow \Box_T \triangle\phi && \text{(by Lemma 67 since } \phi \text{ is } \Sigma_1\text{, } \mathrm{Hb}_{T,\Box_T} \text{ and } \mathrm{Hb}_{T,\triangle})\\
&\to \Box_T \phi && \text{(by } \Sigma_1\text{-}\mathrm{Abs}_{T,\Box_T,\triangle})\\
&\leftrightarrow \Box_T(\phi)^{\triangle}_{\mathrm{V}} && \text{(by Lemma 67 since } \phi \text{ is } \Sigma_1 \text{ and } \mathrm{Hb}_{T,\Box_T})\\
&\leftrightarrow \Box^{\triangle}_{T}\phi. && \text{(since } \phi \text{ is a sentence)}
\end{aligned}$$

This suffices, since $T \subseteq (T)^{\triangle}_{\mathrm{V}}$.

$(T)^{\triangle}_{\mathrm{V}}$ is Σ_1-sound. This is direct since T is Σ_1-sound and we can apply Corollary 74.

$\mathrm{Cp}_{(T)^{\triangle}_{\mathrm{V}},\Box^{\triangle}_{T}}$. Let ϕ be a sentence. Since it is a sentence it suffices to show that $\vdash_{(T)^{\triangle}_{\mathrm{V}}} \phi \to \Box_T(\phi)^{\triangle}_{\mathrm{V}}$. While proving that $(\Box^{\triangle}_{T}, \triangle^{\triangle})$ is a good pair for $(T)^{\triangle}_{\mathrm{V}}$ we showed that $\vdash_{(T)^{\triangle}_{\mathrm{V}}} \phi \to \triangle(\phi)^{\triangle}_{\mathrm{V}}$. Since $\vdash_T \triangle\psi \to \Box_T\psi$ for any sentence ψ, we have that $\vdash_T \triangle(\phi)^{\triangle}_{\mathrm{V}} \to \Box_T(\phi)^{\triangle}_{\mathrm{V}}$ and then it is easy to get the desired result. $\Box$

Let us show that this result can also give us *theories* whose Σ_1-provability logic is iGLC or whose full provability logic is iGLC. The main idea is we need to guarantee that $\Box^{\triangle}_{T}$ and $\Box_{(T)^{\triangle}_{\mathrm{V}}}$ are provably equal. We have the following lemma:

Lemma 117. Let T be a theory such that

1. T is Δ_0 self arithmetizable and extends iEA or T is Σ_1 self arithmetizalbe and extends $\mathsf{iEA}+B\Sigma_1$.
2. $\mathrm{HB}_{T,\triangle}$.
3. For any sentence ϕ,
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \Box_{(T)^{\triangle}_{\mathrm{V}}} \phi \leftrightarrow \Box^{\triangle}_{T}\phi.$$

Then
$$\Sigma_1\text{-}\mathbb{PL}((T)^{\triangle}_{\mathrm{V}}, \Box^{\triangle}_{T}) = \Sigma_1\text{-}\mathbb{PL}((T)^{\triangle}_{\mathrm{V}}).$$
and
$$\mathbb{PL}((T)^{\triangle}_{\mathrm{V}}, \Box^{\triangle}_{T}) = \mathbb{PL}((T)^{\triangle}_{\mathrm{V}}).$$

Proof. It suffices to show that for any realization σ and any $\phi \in \mathcal{L}_{\mathrm{m}}$, we have that
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \sigma_{\Box^{\triangle}_{T}}\phi \leftrightarrow \sigma_{\Box_{(T)^{\triangle}_{\mathrm{V}}}}\phi.$$
We proceed by induction in ϕ. The case of ϕ a propositional variable or $\bot$ is trivial. The cases where ϕ is a conjunction, disjunction of implication are easy thanks to the induction hypothesis. Finally, let us assume that $\phi = \Box\phi_0$. By induction hypothesis we have that
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \sigma_{\Box^{\triangle}_{T}}\phi_0 \leftrightarrow \sigma_{\Box_{(T)^{\triangle}_{\mathrm{V}}}}\phi_0.$$
By Lemma 77, we have that $\mathrm{Hb}_{(T)^{\triangle}_{\mathrm{V}},\Box_{(T)^{\triangle}_{\mathrm{V}}}}$. With this we get
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \Box_{(T)^{\triangle}_{\mathrm{V}}}(\sigma_{\Box^{\triangle}_{T}}\phi_0) \leftrightarrow \Box_{(T)^{\triangle}_{\mathrm{V}}}(\sigma_{\Box_{(T)^{\triangle}_{\mathrm{V}}}}\phi_0).$$
Thanks to the assumption applied to the left hand side
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \Box^{\triangle}_{T}(\sigma_{\Box^{\triangle}_{T}}\phi_0) \leftrightarrow \Box_{(T)^{\triangle}_{\mathrm{V}}}(\sigma_{\Box_{(T)^{\triangle}_{\mathrm{V}}}}\phi_0),$$
in other words
$$\vdash_{(T)^{\triangle}_{\mathrm{V}}} \sigma_{\Box^{\triangle}_{T}}(\Box\phi_0) \leftrightarrow \sigma_{\Box_{(T)^{\triangle}_{\mathrm{V}}}}(\Box\phi_0),$$
as desired. $\Box$

We put the previous theorem and the previous lemma together in a corollary:

Corollary 118. Assume that

1. T is a Σ_1-sound, Σ_1-complete, Δ_0-decidable theory extending iEA. We also assume that T is Σ_1-self arithmetizable and $\vdash_T B\Sigma_1$ or T is Δ_0-self arithmetizable.
2. We have a Σ_1 box function $\triangle$ and assume that $\mathrm{HB}_{T,\triangle}$.
3. For any Σ_1-sentence ϕ, $\vdash_T \Box_T \triangle\phi \to \Box_T \phi$.
4. For any sentence ϕ, $\vdash_T \triangle\phi \to \Box_T \phi$.
5. T is closed under $(\_)^{\triangle}_{\mathrm{V}}$.
6. For any sentence ϕ, $\vdash_{(T)^{\triangle}_{\mathrm{V}}} \Box_{(T)^{\triangle}_{\mathrm{V}}} \phi \leftrightarrow \Box^{\triangle}_{T} \phi$.

Then

$$\Sigma_1\text{-}\mathbb{PL}((T)^{\triangle}_{\mathrm{V}}) = \mathbb{PL}((T)^{\triangle}_{\mathrm{V}}) = \mathsf{iGLC}.$$

3.4 Constructing predicates with absorption

We start with a provability predicate $\Box$ and we want to construct a provability predicate $\triangle$ such that $\mathrm{Abs}_{\Box,\triangle}$ holds. In order to do this we use the construction proposed by Visser in [12]. Note that the construction proposed by Visser is more general that ours. This is because, due to the nature of the examples we are going to study, our particular case suffices.

Let true be the Σ_1-truth predicate, definable in iEA. It is of shape $\exists y\, .\, \mathsf{true}_0(y,x)$, where true_0 is $\Delta_0(\mathsf{exp})$. We will write $\mathsf{true}^z(x) := \exists y \leq z\, .\, \mathsf{true}_0(y,x)$.

Definition 119. Let $\alpha(x)$ be a Δ_0-formula such that $\vdash_{\mathsf{iEA}} \forall A\, .\, \alpha(A) \to \mathsf{sent}(A)$. We define:

1. $\mathsf{prov}_{\alpha,(x)}(y) := \exists p \leq x\, .\, \mathsf{proof}_{\alpha}(p,y)$. We will denote its box function as $\Box_{(\_)}(\_) : \mathrm{Term} \times \mathrm{Form} \longrightarrow \mathrm{Form}$.
2. $\Sigma_1\text{-}\mathsf{refl}_{\alpha}(x) := \forall S\, .\, \Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{\alpha,(x)}(S) \to \mathsf{true}(S)$.
3. $\mathcal{S}_{\alpha}(x) := \exists z \forall S \leq x\, .\, \Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{\alpha,(x)}(S) \to \mathsf{true}^z(S)$.

■

$\mathcal{S}_{\alpha}$ will be our fundamental tool to construct the new provability predicate. Note that the definition can be seen as a modification of Σ_1-refl, i.e. Σ_1-reflection. The modification is needed to work without $B\Sigma_1$, as the following lemma shows.

Lemma 120. We have that

$$\vdash_{\mathsf{iEA}} \forall x\,.\,\mathcal{S}_\alpha(x) \to \Sigma_1\text{-}\mathsf{refl}_\alpha(x),$$
$$\vdash_{\mathsf{iEA}+B\Sigma_1} \forall x\,.\,\mathcal{S}_\alpha(x) \leftrightarrow \Sigma_1\text{-}\mathsf{refl}_\alpha(x).$$

Proof. Let us omit the subscript α for the proof. The first statement is trivial. So it suffices that we prove:

$$\vdash_{\mathsf{iEA}+B\Sigma_1} \forall x\,.\,\Sigma_1\text{-}\mathsf{refl}(x) \to \mathcal{S}(x).$$

Let us show the equivalent $\Sigma_1\text{-}\mathsf{refl}(x) \vdash_{\mathsf{iEA}+B\Sigma_1} \mathcal{S}(x)$. First note that Σ_1-refl is the following formula:

$$\forall S\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S) \to \exists z\,.\,\mathsf{true}^z(S). \tag{i}$$

Since $\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S)$ is Δ_0-formula we have that $\mathsf{iEA}+B\Sigma_1$ proves that it is decidable. Then we have that (i) is iEA-equivalent to

$$\forall S\,\exists z\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S) \to \mathsf{true}^z(S). \tag{ii}$$

And note since part of the antecedent of the implication is $\mathsf{prov}_{(x)}(S)$, we have that $S \leq x$ and then (ii) is iEA-equivalent to

$$\forall S \leq x\,\exists z\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S) \to \mathsf{true}^z(S). \tag{iii}$$

Using $B\Sigma_1$ then (iii) implies:

$$\exists y \forall S \leq x\,\exists z \leq y\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S) \to \mathsf{true}^z(S), \tag{iv}$$

where y is a new variable. But since, $\vdash_{\mathsf{iEA}} z_0 \leq z_1 \to \mathsf{true}^{z_0}(x) \to \mathsf{true}^{z_1}(x)$ we can conclude from (iv) that:

$$\exists z \forall S \leq x\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{(x)}(S) \to \mathsf{true}^z(S). \qquad \square$$

The idea is that, if we are given a theory T, we are going to axiomatize a new theory $\mathcal{S}T$. In the metatheory both theories coincide, inside iEA we only have $\mathsf{prov}_{\mathcal{S}T}(A) \to \mathsf{prov}_T(A)$ but not the reverse implication. Even T cannot show the reverse implication. The idea for doing this is take the axiomatization of T and put as axioms of $\mathcal{S}T$ the axioms of T that are "small". Small is nothing more that the arithmetical predicate $\mathcal{S}_\alpha(x)$ carefully designed for which $\vdash_T \mathcal{S}_\alpha(\overline{n})$ holds for any $n \in \mathbb{N}$ but $\nvdash_T \forall x\,.\,\mathcal{S}_\alpha(x)$. This makes that in the metatheory T and $\mathcal{S}T$ have the same axiomatization, but T is not aware of this.

For technical reasons we will assume that the axioms of the theory T are split in two, the α-axioms and the β-axioms. The idea is that when constructing $\mathcal{S}T$ we only require the β-axioms to be small. This distinction is used in Chapter 5.

Definition 121. Let $\gamma(x) := (\alpha(x), \beta(x))$, where α, β are Δ_0. Assume that $\vdash_{\mathsf{iEA}} \forall A.\alpha(A) \vee \beta(A) \to \mathsf{sent}(A)$ and $\vdash_{\mathsf{iEA}} \forall A.\mathsf{prov}_{\mathsf{iEA}}(A) \to \mathsf{prov}_{\gamma}(A)$. Then we define the formulas:

$$\begin{aligned}
&\mathsf{ax}_{\gamma}(A) := \alpha(A) \vee \beta(A),\\
&\mathsf{ax}_{\mathcal{S}\gamma}(A) := \alpha(A) \vee (\beta(A) \wedge \mathcal{S}_{\gamma}(A)),\\
&\mathsf{ax}_{\gamma \leq x}(A) := \alpha(A) \vee (\beta(A) \wedge A \leq x).
\end{aligned}$$

■

Note that $\mathsf{ax}_{\gamma \leq x}$ is nothing more that the axiomatization provided by taking all the α-axioms and only the β-axioms of ax_{γ} which are smaller than x. The idea of this theory is similar the the theory consisting in taking all the α-axioms and only the first x β-axioms of ax_{γ}.

For the rest of the section we will assume that we have an arbitrary fixed γ that fulfills the conditons of the previous definition. We need the following result from [12].

Lemma 122. We have that

$$\vdash_{\mathsf{iEA}} \forall x.\Box_{\gamma}\mathcal{S}_{\gamma}(x).$$

Now, there are two ways of making a small theory from γ. We can say that all the β-axioms are small, as in $\mathsf{ax}_{\mathcal{S}\gamma}$; or we can say that all the β-axioms are bounded by a small number. The following lemma proves that the two approaches are, in fact, equivalent.

Lemma 123. We have that

$$\vdash_{\mathsf{iEA}} \forall A.\mathsf{prov}_{\mathcal{S}\gamma}(A) \leftrightarrow \exists x.\mathsf{prov}_{\gamma \leq x}(A) \wedge \mathcal{S}_{\gamma}(x).$$

Proof. We work inside iEA. Let us show left to right. Let p be a witness of $\mathsf{prov}_{\mathcal{S}\gamma}(A)$. Since $\alpha(x)$ and $\beta(x)$ are Δ_0 we can pick the biggest axiom in p that fulfills β but not α, let it be B. Note that by definition of $\mathcal{S}\gamma$ it must be the case that $\mathcal{S}_{\gamma}(B)$. For any other axiom C in p we have that either $\alpha(C)$ or $\beta(C) \wedge C \leq B$. This implies that p is also a witness of $\mathsf{prov}_{\gamma \leq B}$. Since $\mathcal{S}_{\gamma}(B)$ we can conclude that p is a witness of $\exists x.\mathsf{prov}_{\gamma \leq x}(C) \wedge \mathcal{S}_{\gamma}(x)$, as wanted.

Finally, let us show right to left. Let x be such that $\mathsf{prov}_{\gamma \leq x}(A) \wedge \mathcal{S}_{\gamma}(x)$ and let q be a witness of $\mathsf{prov}_{\gamma \leq x}(A)$. Since iEA knows that $\mathcal{S}_{\gamma}(x)$ is downward persistent, we can conclude that q is also a witness of $\mathsf{prov}_{\mathcal{S}\gamma}$, as wanted. □

The following theorem allows us to show that $\Box_{\mathcal{S}\gamma}$ fulfills the absorption principle with respect to $\Box_{\gamma}$ in the cases we use this construction.

Theorem 124 (Absorption). Assume that $\mathrm{HB}_{\mathsf{iEA},\Box_{\gamma}}$ and let ϕ be a sentence. We have that:

$$\forall x.\Box_{\gamma}\Box_{\gamma \leq x}\phi \to \Box_{\gamma}\phi \vdash_{\mathsf{iEA}} \Box_{\gamma}\Box_{\mathcal{S}\gamma}\phi \to \Box_{\gamma}\phi.$$

Proof. We use diagonalization to find a sentence χ such that

$$\vdash_{\mathsf{iEA}} \chi \leftrightarrow (\exists x\,.\,\Box_{\gamma \le x}\phi \wedge \forall y \le x\,.\,\neg\mathsf{proof}_{\gamma}(y,\overline{\chi})).$$

Note that χ is a Σ_1-sentence.

Since $\Box_{\gamma,(x)}\chi$ is a Δ_0-formula, we have that

$$\vdash_{\mathsf{iEA}} \Box_{\gamma,(x)}\chi \vee \neg\Box_{\gamma,(x)}\chi. \tag{i}$$

But we have that $\vdash_{\mathsf{iEA}} \Sigma_1\text{-}\mathsf{sent}(\overline{\chi})$. Then

$$\begin{aligned}
\vdash_{\mathsf{iEA}} \mathcal{S}_{\gamma}(x) &\to \Sigma_1\text{-}\mathsf{refl}_{\gamma}(x)\\
&= (\forall S\,.\,\Sigma_1\text{-}\mathsf{sent}(S) \wedge \mathsf{prov}_{\gamma,(x)}(S) \to \mathsf{true}(S))\\
&\to \Sigma_1\text{-}\mathsf{sent}(\overline{\chi}) \wedge \Box_{\gamma,(x)}\chi \to \mathsf{true}(\overline{\chi})\\
&\to \Box_{\gamma,(x)}\chi \to \chi \qquad\qquad (\chi \text{ is a } \Delta_0\text{-sentence})
\end{aligned}$$

in other words

$$\mathcal{S}_{\gamma}(x), \Box_{\gamma,(x)}\chi \vdash_{\mathsf{iEA}} \chi. \tag{ii}$$

We also have that

$$\neg\Box_{\gamma,(x)}\chi \vdash_{\mathsf{iEA}} \forall y \le x\,.\,\neg\mathsf{proof}_{\gamma}(y,\overline{\chi}),$$

so, thanks to the fixpoint equivalence of χ, we have that

$$\Box_{\gamma\le x}\phi, \neg\Box_{\gamma,(x)}\chi \vdash_{\mathsf{iEA}} \chi. \tag{iii}$$

(i), (ii) and (iii) together gives us that

$$\Box_{\mathcal{S}\gamma}\phi \vdash_{\mathsf{iEA}} \chi.$$

By assumption, we have that $\mathrm{HB}_{\mathsf{iEA},\Box_{\gamma}}$, so

$$\Box_{\gamma}\Box_{\mathcal{S}\gamma}\phi \vdash_{\mathsf{iEA}} \Box_{\gamma}\chi. \tag{iv}$$

We have that

$$\begin{aligned}
\vdash_{\mathsf{iEA}} \Box_{\gamma}\chi &\to \exists p\,.\,\mathsf{proof}_{\gamma}(p,\overline{\chi})\\
&\to \exists p\,.\,\Box_{\gamma,(p)}\chi\\
&\to \exists p\,.\,\Box_{\gamma}\Box_{\gamma,(p)}\chi. \qquad\qquad (\text{thanks to } \mathrm{HB}_{\mathsf{iEA},\Box_{\gamma}})
\end{aligned}$$

Let us call this implication (v). Also by definition of χ and using $\mathrm{HB}_{\mathsf{iEA},\Box_{\gamma}}$,

$$\Box_{\gamma}\chi \vdash_{\mathsf{iEA}} \Box_{\gamma}\left(\exists x\,.\,\Box_{\gamma\le x}\phi \wedge \forall y \le x\,.\,\neg\mathsf{proof}_{\gamma}(y,\overline{\chi})\right). \tag{vi}$$

By (v), (vi) and the upward persistence of $\Box_{\gamma\le x}$, provable inside iEA, we get that

$$\Box_{\gamma}\chi \vdash_{\mathsf{iEA}} \exists p\,.\,\Box_{\gamma}\Box_{\gamma\le p}\phi.$$

Then

$$\Box_{\gamma}\chi \vdash_{\mathsf{iEA}+(\forall x.\Box_{\gamma}\Box_{\gamma\le x}\phi\to\Box_{\gamma}\phi)} \Box_{\gamma}\phi.$$

This with (iv) gives us the desired

$$\Box_{\gamma}\Box_{\mathcal{S}\gamma}\phi \vdash_{\mathsf{iEA}+(\forall x.\Box_{\gamma}\Box_{\gamma\le x}\phi\to\Box_{\gamma}\phi)} \Box_{\gamma}\phi. \qquad \Box$$

The following lemma is needed in Chapter 5 for technical purposes.

Lemma 125 (Formalized Hilbert-Bernays from base). Let T be an extension of iEA. Let α enumerate T in T and assume that $\mathsf{HB}_{T,\Box_\alpha}$ and $\vdash_T \mathsf{HB}_{\alpha,\Box^\bullet_\alpha}$. Then

$$\vdash_T \mathsf{HB}_{\mathsf{prov}_\alpha,\Box^\bullet_{\mathcal{S}\gamma}}.$$

Proof. Since T is an extension of iEA and α enumerates T in T, it is easy to see that

$$\vdash_T \forall A, B\,.\,\mathsf{prov}_\alpha(\Box^\bullet_{\mathcal{S}\gamma}(A \to^\bullet B) \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} B)$$

just by definition of $\Box_{\mathcal{S}\gamma}$.

Since $\vdash_T \mathsf{HB}_{\alpha,\Box_\alpha}$ we have that

$$\vdash_T \forall A.\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_\alpha(A \to^\bullet \Box^\bullet_\alpha A). \tag{i}$$

Also, note that

$$\vdash_T \forall A.\mathsf{prov}_\alpha(A) \to \mathsf{prov}_{\mathcal{S}\gamma}(A),$$

and since $\mathrm{HB}_{T},\Box_\alpha$ we have that

$$\vdash_T \Box_\alpha(\forall A.\mathsf{prov}_\alpha(A) \to \mathsf{prov}_{\mathcal{S}\gamma}(A)).$$

This gives

$$\vdash_T \forall A.\mathsf{prov}_\alpha(\Box^\bullet_\alpha A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A). \tag{ii}$$

(i) and (ii) gives

$$\vdash_T \forall A.\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_\alpha(A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A).$$

Since $\vdash_T \mathsf{HB}_{\alpha,\Box^\bullet_\alpha}$ we have that

$$\vdash_T \forall A\,.\,\mathsf{prov}_\alpha(A) \to \mathsf{prov}_\alpha(\Box^\bullet_\alpha(A)). \tag{iii}$$

But (iii) with (ii) implies that

$$\vdash_T \forall A\,.\,\mathsf{prov}_\alpha(A) \to \mathsf{prov}_\alpha(\Box^\bullet_{\mathcal{S}\gamma}(A)).$$ □

Apart from the absoprtion principle we want that $\Box_{\mathcal{S}\gamma}$ fulfills the uniform Hilbert-Bernays conditons. In fact, we need that these conditions are provable in certain arithmetical theories, as the following theorem establishes.

Theorem 126 (Formalized uniform Hilbert Bernays). Assume that:

1. $\mathsf{iEA} \subseteq U$.
2. $\mathrm{HB}_{\mathsf{iEA},\Box_\gamma}$.
3. $\vdash_U \forall A\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \exists x\,.\,\mathsf{prov}_\gamma(A \to^\bullet \Box^\bullet_{\gamma \leq \dot{x}} A)$.

Then:

$$\vdash_U \mathsf{HB}_{\mathsf{prov}_\gamma, \Box^\bullet_{\mathcal{S}\gamma}}.$$

Proof. We need to check three properties. Let us start with necessitation, we have to show that

$$\vdash_U \forall A \,.\, \mathsf{prov}_\gamma(A) \to \mathsf{prov}_\gamma(\Box^\bullet_{\mathcal{S}\gamma}(A)). \tag{i}$$

Note that thanks to Lemma 123, $\mathrm{HB}_{\mathsf{iEA},\gamma}$ and $\mathsf{iEA} \subseteq U$ we get that

$$\vdash_U \Box_\gamma \left(\forall A.\mathsf{prov}_{\mathcal{S}\gamma}(A) \leftrightarrow \exists x \,.\, \mathsf{prov}_{\gamma \le x}(A) \wedge \mathcal{S}(x) \right).$$

This implies that to show (i) it suffices to show

$$\vdash_U \forall A \,.\, \mathsf{prov}_\gamma(A) \to \mathsf{prov}_\gamma(\exists^\bullet \overline{x} \,.\, \Box^\bullet_{\gamma \le \overline{x}} A \wedge^\bullet \mathcal{S}^\bullet(\overline{x})).$$

But for this it suffices to show:

$$\vdash_U \forall A \,.\, \mathsf{prov}_\gamma(A) \to \exists x \,.\, \mathsf{prov}_\gamma(\Box^\bullet_{\dot{x}} A \wedge^\bullet \mathcal{S}^\bullet(\dot{x})).$$

Thanks to Lemma 122, to show this we just need

$$\vdash_U \forall A \,.\, \mathsf{prov}_\gamma(A) \to \exists x \,.\, \mathsf{prov}_\gamma(\Box^\bullet_{\dot{x}} A).$$

But note that:

$$\begin{aligned}
\vdash_U \mathsf{prov}_\gamma(A) \to \mathsf{prov}_\gamma(\mathsf{uc}(A)) & \\
&= \exists p \,.\, \mathsf{proof}_\gamma(p, \mathsf{uc}(A)) \\
&\to \exists p \,.\, \mathsf{prov}_{\gamma \le p}(\mathsf{uc}(A)) \\
&\to \exists p \,.\, \mathsf{prov}_\gamma(\mathsf{prov}^\bullet_{\gamma \le \dot{p}}(\mathsf{num}(\mathsf{uc}(A)))) && (\text{by } \mathrm{HB}_{\mathsf{iEA},\Box_\gamma}) \\
&\to \exists p \,.\, \mathsf{prov}_\gamma(\Box^\bullet_{\gamma \le \dot{p}} A). && (\text{instantiation inside } \mathsf{prov}^\bullet_{\gamma \le \dot{p}})
\end{aligned}$$

Now, we need to check that

$$\vdash_U \forall A, B \,.\, \mathsf{prov}_\gamma \Box^\bullet_{\mathcal{S}\gamma}(A \to^\bullet B) \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} B.$$

But this is straightforward by definition of $\Box^\bullet_{\mathcal{S}\gamma}$.

Finally, to show that

$$\vdash_U \forall A \,.\, \Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_\gamma(A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A),$$

we just need to use the third assumption of the theorem and perform a similar reasoning to the one performed for the necessition rule. The essence of both reasonings is that, thanks to lemmas 122 and 123, with the hypothesis of $\mathrm{HB}_{\mathsf{iEA},\Box_\gamma}$; it is easy to show that

$$\vdash_{\mathsf{iEA}} \mathsf{prov}_\gamma(\Box^\bullet_{\gamma \le \dot{x}} A \to^\bullet \Box^\bullet_{\mathcal{S}\gamma} A) \qquad \Box$$

If the theory U that we use to establish this theorem is sound we get the usual uniform Hilbert-Bernays conditions.

3.5 Application: iPRA

Let us apply the intuitionistic Solovay construction to iPRA. iPRA was chosen since we need a theory with at least Σ_1-reflection with respect to its finite subtheories. This does not hold with finite axiomatizable theories of HA like iEA or $\mathsf{iI}\Sigma_1$.

Slow predicate

We will construct a slow predicate for iPRA using the construction of Section 3.4. In this case we will not need any α-axioms.

Definition 127. We define

$$\begin{aligned}\mathsf{ax}_{\mathcal{S}\mathsf{iPRA}}(A) &:= \mathsf{ax}_{(\perp,\mathsf{ax}_{\mathsf{iPRA}})}(A),\\ \mathsf{ax}_{\mathsf{iPRA}\leq x}(A) &:= \mathsf{ax}_{(\perp,\mathsf{ax}_{\mathsf{iPRA}})\leq x}(A).\end{aligned}$$ ■

We start by proving a lemma that will give the desired Σ_1-absorption.[1]

Lemma 128 (Formalized Σ_1-reflection of iPRA)**.** We have that:

$$\vdash_{\mathsf{iPRA}} \forall A, x\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_{\mathsf{iPRA}}(\Box^{\bullet}_{\mathsf{iPRA}\leq\dot{x}} A \to^{\bullet} A).$$

Proof. Note that the sentence we want to prove in iPRA is iPRA-equivalent to a Π_2 sentence. By Π_2-conservativity of PRA over iPRA it suffices to show that

$$\vdash_{\mathsf{PRA}} \forall A, x\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_{\mathsf{iPRA}}(\Box^{\bullet}_{\mathsf{iPRA}\leq\dot{x}} A \to^{\bullet} A).$$

Note however, that PRA is capable of showing the Π_2-conservativity of PRA over iPRA, since it only needs to use Gödel's and Friedman's translation. In other words, we have that:

$$\vdash_{\mathsf{PRA}} \forall A\,.\,\Pi_2\text{-}\mathsf{sent}(A) \wedge \mathsf{prov}_{\mathsf{PRA}}(A) \to \mathsf{prov}_{\mathsf{iPRA}}(A).$$

PRA can show that $\Box^{\bullet}_{\mathsf{iPRA}\leq\dot{x}} A \to^{\bullet} A$ is $\mathsf{prov}_{\mathsf{iPRA}}$-equivalent to a Π_2-formula, thanks to Σ_1-$\mathsf{form}(A)$. Then, it suffices that we show:

$$\vdash_{\mathsf{PRA}} \forall A, x\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_{\mathsf{PRA}}(\mathsf{uc}(\Box^{\bullet}_{\mathsf{iPRA}\leq\dot{x}} A \to^{\bullet} A)). \tag{i}$$

Let us reason inside PRA. Let A be a Σ_1-formula, then (note that since we are reasoning inside PRA, $\vdash_{\mathsf{PRA}}$ should be understood as $\mathsf{prov}_{\mathsf{PRA}}$):

$$\begin{aligned}\vdash_{\mathsf{PRA}} \Box^{\bullet}_{\mathsf{iPRA}\leq\dot{x}} A &\to^{\bullet} \Box^{\bullet}_{\mathsf{PRA}\leq\dot{x}} A\\ &\to^{\bullet} A\end{aligned}$$

Where the last implication holds thanks to Σ_1-reflection principle of PRA with respect to it finite subtheories. This principle can be formalized in PRA (in particular, it can be formalized in EA, as it is said in [3], page 103 Example 7). Clearly, from the provability of this formula we can derive the provability of the universal closure which will give us (i) □

[1] We will use some conservativity results of iPRA and $\mathsf{iI}\Sigma_1$ over its classical versions. These results can be shown with the tools provided by Gödel's and Friedman's translations. The intereseted reader can find the proofs in [14]

We have used some conservativity principles over classical theories to prove the result we were interested in. Developing a direct proof of this or checking whether the original proof can be carried to the intuitionistic setting would be interesting as future work.

Corollary 129. Let ϕ be a Σ_1-sentence. Then

$$\vdash_{\mathsf{iPRA}} \Box_{\mathsf{iPRA}} \Box_{\mathcal{S}\mathsf{iPRA}} \phi \to \Box_{\mathsf{iPRA}} \phi.$$

Proof. We want to apply Theorem 124. Note that iPRA is (Δ_0, iEA)-arithmetizable, so $\mathrm{HB}_{\mathsf{iEA},\mathsf{iPRA},\Box_{\mathsf{iPRA}}}$. But since $\mathsf{iEA} \subseteq \mathsf{iPRA}$ we get that $\mathrm{HB}_{\mathsf{iEA},\Box_{\mathsf{iPRA}}}$. So all we need to show is that for any Σ_1-sentence ϕ,

$$\vdash_{\mathsf{iPRA}} \forall x \,.\, \Box_{\mathsf{iPRA}} \Box_{\mathsf{iPRA}_{\le x}} \phi \to \Box_{\mathsf{iPRA}} \phi.$$

In particular, we have to show that:

$$\vdash_{\mathsf{iPRA}} \forall x \,.\, \mathsf{prov}_{\mathsf{iPRA}} \left(\mathsf{subst} \left(\overline{\mathsf{prov}_{\mathsf{iPRA}_{\le x}} \left(\overline{\phi} \right)}, \overline{x}, \dot{x} \right) \right) \to \mathsf{prov}_{\mathsf{iPRA}} \left(\overline{\phi} \right),$$

but thanks to properties of subst this is equivalent to showing

$$\vdash_{\mathsf{iPRA}} \forall x \,.\, \mathsf{prov}_{\mathsf{iPRA}} \left(\mathsf{subst} \left(\overline{\mathsf{prov}_{\mathsf{iPRA}_{\le x}} (A)}, \overline{x}, \dot{x}, \overline{A}, \overline{\overline{\phi}} \right) \right) \to \mathsf{prov}_{\mathsf{iPRA}} \left(\overline{\phi} \right).$$

But instantiating A as $\overline{\phi}$ in Lemma 128, and since ϕ is a sentence, we obtain

$$\vdash_{\mathsf{iPRA}} \mathsf{prov}_{\mathsf{iPRA}} \left(\mathsf{prov}^{\bullet}_{\mathsf{iPRA}_{\le \dot{x}}} \left(\overline{\overline{\phi}} \right) \to^{\bullet} \overline{\phi} \right)$$

i.e.

$$\vdash_{\mathsf{iPRA}} \mathsf{prov}_{\mathsf{iPRA}} \left(\mathsf{subst} \left(\overline{\mathsf{prov}_{\mathsf{iPRA}_{\le x}} (A)}, \overline{x}, \dot{x}, \overline{A}, \overline{\overline{\phi}} \right) \to^{\bullet} \overline{\phi} \right)$$

which clearly implies the desired formula. $\square$

Lemma 130. We have that:

$$\vdash_{\mathsf{iPRA}} \mathsf{HB}_{\mathsf{prov}_{\mathsf{iPRA}}, \Box^{\bullet}_{\mathcal{S}\mathsf{iPRA}}}$$

As a corollary, since iPRA is sound, we can conclude that $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$.

Proof. We want to apply Theorem 126. We have that $\mathsf{iEA} \subseteq \mathsf{iPRA}$, and we also have that iPRA is (Δ_0, iEA)-arithmetizable, which gives $\mathrm{HB}_{\mathsf{iEA},\mathsf{iPRA},\Box_{\mathsf{iPRA}}}$, so $\mathrm{HB}_{\mathsf{iEA},\Box_{\mathsf{iPRA}}}$. All left to show is that

$$\vdash_{\mathsf{iPRA}} \forall A \,.\, \Sigma_1\text{-form}(A) \to \exists x \,.\, \mathsf{prov}_{\mathsf{iPRA}} (A \to^{\bullet} \Box^{\bullet}_{\mathsf{iPRA}_{\le \dot{x}}} A).$$

Note that this sentence is Π_2, so it suffices to show it in $\mathsf{iI}\Sigma_1$.

Let us do an unformalized proof of this, which is straightforward to formalize in iPRA. So let ϕ be a Σ_1-formula and we have to show that there is an n such that

$$\vdash_{\mathsf{iPRA}} \phi \to \Box_{\mathsf{iPRA}_{\leq \overline{n}}} \phi.$$

Since this formula is Π_2, we note that the idea to prove this is the same as the classical idea to show formalized completeness in PA: use induction in ϕ and use that we have all the recursive axioms of the function symbols. The induction is no problem, since it would be a Σ_1-induction in $\mathsf{iI}\Sigma_1$. The problem arises when we want all the axioms of the function symbols in ϕ. We should be careful here, in the traditional PA case there are finitely many function symbols in the language, so for any formula ϕ we can find an uniform n that fulfills the requirements. This is not the case in iPRA.

In this case we must go through the formula ϕ, find the biggest function symbol and take a big enough n to have the axioms of this function symbol and all the axioms of smaller function symbols. This will be enough, because in case we have a function symbols whose axioms depend on other function symbol by how we defined our language the second function symbol will be smaller than the first one. It is not hard to see that the calculation of this n can be carried in iPRA. Just to measure how big the axioms of a function symbol can be depending on the size of the function symbol and take an upper bound for the biggest function symbol in ϕ. $\Box$

Closure under Visser translation

Lemma 131. iPRA is closed under $(\_)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}}$. This is verifiable in $\mathsf{iI}\Sigma_1$.

Proof. First, remember that thanks to Lemma 130 we have that $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$ and this is $\mathsf{iI}\Sigma_1$ verifiable.

All non-induction axioms of iPRA are of shape $\forall x_0, \dots, x_{n-1} \,.\, \phi_0$ where ϕ_0 is Δ_0. Then by Lemma 61, we have that

$$\vdash_{\mathsf{iPRA}} (\forall x_0, \dots, x_{n-1} \,.\, \phi_0)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}} \leftrightarrow \boxdot_{\mathcal{S}\mathsf{iPRA}}(\forall x_0, \dots, x_{n-1} \,.\, (\phi_0)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}}).$$

It suffices to show $\vdash_{\mathsf{iPRA}} \forall x_0, \dots, x_{n-1} \,.\, \phi_0^{\Box_{\mathcal{S}\mathsf{iPRA}}}$. Thanks to ϕ_0 being Δ_0 and Lemma 66, we have that $\vdash_{\mathsf{iPRA}} (\phi_0)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}} \leftrightarrow \phi_0$, so we can derive the desired result.

Finally, let us assume that ϕ is an induction axiom. Then, we have a quantifier-free formula ψ such that

$$\phi = \psi[x/\overline{0}] \wedge (\forall x \,.\, \psi \to \psi[x/\mathsf{S}(x)]) \to (\forall x \,.\, \psi).$$

To show that $\vdash_{\mathsf{iPRA}} (\phi)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}}$, by Lemma 63 and $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$, it suffices to show that:

$$\vdash_{\mathsf{iPRA}} (\psi)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}}[x/\overline{0}] \wedge \boxdot_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, (\psi)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}} \to (\psi)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}}[x/\mathsf{S}(x)]) \to \boxdot_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, (\psi)^{\Box_{\mathcal{S}\mathsf{iPRA}}}_{\mathrm{V}})$$

Since ψ is Δ_0 we have that $\vdash_{\mathsf{iPRA}} (\psi)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}} \leftrightarrow \psi$, it suffices to show:

$$\vdash_{\mathsf{iPRA}} \psi[x/\bar{0}] \wedge \boxdot_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, \psi \to \psi[x/\mathsf{S}(x)]) \to \boxdot_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, \psi). \tag{i}$$

Since it is an induction axiom, we already have that

$$\vdash_{\mathsf{iPRA}} \psi[x/\bar{0}] \wedge (\forall x \,.\, \psi \to \psi[x/\mathsf{S}(x)]) \to (\forall x \,.\, \psi). \tag{ii}$$

Using $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$ we also get:

$$\vdash_{\mathsf{iPRA}} \Box_{\mathcal{S}\mathsf{iPRA}}\left(\psi[x/\bar{0}]\right) \wedge \Box_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, \psi \to \psi[x/\mathsf{S}(x)]) \to \Box_{\mathcal{S}\mathsf{iPRA}}(\forall x \,.\, \psi). \tag{iii}$$

Since $\psi[x/\bar{0}]$ is Δ_0, by $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$.Compl we also get

$$\vdash_{\mathsf{iPRA}} \psi[x/\bar{0}] \to \Box_{\mathcal{S}\mathsf{iPRA}}\left(\psi[x/\bar{0}]\right). \tag{iv}$$

Clearly (ii), (iii) and (iv) gives (i).

For verifiability inside $\mathsf{i}\mathsf{I}\Sigma_1$, we note that all the lemmas used in the proof are $\mathsf{i}\mathsf{I}\Sigma_1$ verifiable. □

Lemma 132. We have that:

$$\vdash_{\mathsf{iPRA}} \forall A \,.\, \mathsf{prov}_{(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}}(A) \leftrightarrow \mathsf{prov}_{\mathsf{iPRA}}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{iPRA}}}.$$

Proof. Note that right to left is trivial by definition of $\mathsf{ax}_{(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}}$. It suffices that we show:

$$\vdash_{\mathsf{iPRA}} \forall A \,.\, \mathsf{prov}_{(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}}(A) \to \mathsf{prov}_{\mathsf{iPRA}}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{iPRA}}}.$$

But note that the sentence is iPRA-equivalent to a Π_2-sentence (as it is the universal quantification of an implication of Σ_1-formulas). By Π_2 conservativity of $\mathsf{i}\mathsf{I}\Sigma_1$ over iPRA we get that it suffices to show:

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \forall A \,.\, \mathsf{prov}_{(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}}(A) \to \mathsf{prov}_{\mathsf{iPRA}}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{iPRA}}}.$$

But we obtain this from applying Corollary 75, and lemmas 130 and 131 provide the necessary hypothesis. □

Σ_1-provability logic

Finally, we are able to apply Solovay's construction.

Theorem 133.

$$\Sigma_1\text{-}\mathbb{PL}((i\mathsf{PRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathbb{PL}((\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathsf{iGLC}.$$

Proof. First, let us apply Theorem 116 to obtain that

$$\Sigma_1\text{-}\mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}, \Box_{\mathsf{iPRA}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}, \Box_{\mathsf{iPRA}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathsf{iGLC}. \tag{i}$$

Note that iPRA is a sound theory, Δ_0-decidable and Σ_1-complete extending iEA. Also, it is self Δ_0-arithmetizable. By Lemma 130 we have that $\mathrm{HB}_{\mathsf{iPRA},\Box_{\mathcal{S}\mathsf{iPRA}}}$. By Corollary 129 we get that absorption law for Σ_1-sentences. By definition of $\mathrm{ax}_{\mathcal{S}\mathsf{iPRA}}$, we have that for any sentence ϕ, $\vdash_{\mathsf{iPRA}} \Box_{\mathcal{S}\mathsf{iPRA}}\, \phi \to \Box_{\mathsf{iPRA}}\, \phi$. Finally, by Lemma 131 we have that it is closed under $(\_)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}$.

Now, thanks to Lemma 132 we get that for any sentence ϕ,

$$\vdash_{\mathsf{iPRA}} \Box_{(\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}}\, \phi \leftrightarrow \Box_{\mathsf{iPRA}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}\, \phi. \tag{ii}$$

(ii) and Lemma 117 gives us that:

$$\Sigma_1\text{-}\mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}, \Box_{\mathsf{iPRA}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \Sigma_1\text{-}\mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}). \tag{iii}$$

$$\mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}, \Box_{\mathsf{iPRA}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}) = \mathbb{PL}((\mathsf{iPRA})_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}). \tag{iv}$$

Using (i),(iii) and (iv) gives the desired result. $\Box$

Chapter 4

NNIL Algorithm

In this chapter we are going to see how to calculate the Σ_1-provability logic of T from the Σ_1-provability logic of $((T)^{\triangle}_{\mathsf{V}}, \Box^{\triangle}_{T})$. For this we will need to use the NNIL algorithm. The definition of this algorithm and the proof of its main properties is from [11].

4.1 Σ_1-provability logic of T from $(T)^{\triangle}_{\mathsf{V}}$

In this section we will see how assuming two conditions, $\mathcal{A}_{T,\Box_T}$ and $\mathcal{B}_{T,\Box_T,\triangle}$, we get the Σ_1-provability logic of T from the Σ_1-provability logic of $((T)^{\Box}_{\mathsf{V}}, \Box^{\triangle}_{T})$.

We introduce the class of NNIL propositional formulas, No Nestings of Implications to the Left.

Definition 134. We define the classes $\mathsf{NNIL} \subseteq \mathcal{L}_{\mathrm{p}}$ and $\mathsf{NI} \subseteq \mathcal{L}_{\mathrm{p}}$ (No Implications) of propositional formulas with the following Back-Naus form:

$$\begin{array}{rl} \mathsf{NNIL}: & \phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \psi \to \phi, \\ \mathsf{NI}: & \psi ::= p \mid \bot \mid \top \mid \psi \wedge \psi \mid \psi \vee \psi. \end{array}$$

■

The NNIL algorithm is an elementary function $(\_)^* : \mathcal{L}_{\mathrm{p}} \longrightarrow \mathsf{NNIL}$, which has the following properties:

1. For any ϕ, $\vdash_{\mathsf{IPC}} \phi^* \to \phi$.
2. For any $\phi \in \mathcal{L}_{\mathrm{p}}$, $\psi \in \mathsf{NNIL}$ if $\vdash_{\mathsf{IPC}} \psi \to \phi$ then $\vdash_{\mathsf{IPC}} \psi \to \phi^*$.

Definition 135. Let us have box functions $\Box, \triangle$. We define the sets:

$$\mathscr{A}_{\Box} = \{\Box(\sigma\phi) \leftrightarrow \Box(\sigma\phi^*) \mid \phi \in \mathcal{L}_{\mathrm{p}}, \sigma \in \Sigma_1\text{-real}\},$$

$$\mathscr{B}_{\Box,\triangle} = \{\Box(\sigma\phi) \leftrightarrow \Box^{\triangle}(\sigma\phi) \mid \phi \in \mathsf{NNIL}, \sigma \in \Sigma_1\text{-real}\},$$

where Σ_1-real is the set of Σ_1 realizations.

If T is a theory, we will write $\mathscr{A}_{T,\Box}$ to mean $\vdash_T \mathscr{A}_{\Box}$ and $\mathscr{B}_{T,\Box,\triangle}$ to mean $\vdash_T \mathscr{B}_{\Box,\triangle}$. ■

Having $\mathscr{A}_{T,\Box_T}$ and $\mathscr{B}_{T,\Box_T,\triangle}$ are going to be fundamental to lift the calculation of Σ_1-$\mathbb{PL}$ from $((T)^{\triangle}_{\mathrm{V}},\Box^{\triangle}_{T})$ to (T).

We need to define an extension of the NNIL-algorithm to modal formulas. In order to define it we need to define what it means that a propositional logic formula ψ is a propositional skeleton of a modal logic formula ϕ.

Definition 136 (Propositional skeleton). Let $\phi(\vec{p})$ be a modal formula. We say that a propositional formula $\psi(\vec{p}, q_0, \ldots, q_{n-1})$ (with the list of q variables in that order) is a propositional skeleton of ϕ iff there are unique $\chi_0, \ldots, \chi_{n-1} \in \mathcal{L}_{\mathrm{m}}$ distinct with

$$\phi = \psi(\vec{p}, \Box\chi_0, \ldots, \Box\chi_{n-1}).$$

■

Note that propositional skeletons are unique up to renaming of $q_0, \ldots, q_{n-1}$ variables by other variables not appearing in the modal formula.

We extend the definition of NNIL formulas to modal logic. This class is going to be called TNNIL, Thoroughly No Nestings of Implications to the Left.

Definition 137 (TNNIL)**.** We define the classes $\mathsf{TNNIL} \subseteq \mathcal{L}_{\mathrm{m}}$ and $\mathsf{NI} \subseteq \mathcal{L}_{\mathrm{m}}$ (No Outside Implications) of modal formulas with the following Back-Naus form:

$$\begin{aligned} \mathsf{TNNIL}: &\qquad \phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \psi \to \phi \mid \Box\phi, \\ \mathsf{NOI}: &\qquad \psi ::= p \mid \bot \mid \top \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Box\chi, \end{aligned}$$

where χ is any $\mathcal{L}_{\mathrm{m}}$ formula. ■

With this we can extend the NNIL algorithm to the modal case.

Definition 138. We define the TNNIL-algorithm by recursion. Let $\phi(\vec{p}) \in \mathcal{L}_{\mathrm{m}}$, let $\psi(\vec{p}, q_0, \ldots, q_{n-1})$ be a propositional skeleton of ϕ and $\chi_0, \ldots, \chi_{n-1}$ be the only modal formulas satisfying the skeleton condition. Then we define

$$\phi^+ = \psi^*(\vec{p}, \Box\chi_0^+, \ldots, \Box\chi_{n-1}^+).$$ ■

According to our definition the TNNIL-algorithm is not an algorithm. These comes from 2 problems:

1. A formula has infinitely many propositional skeletons, which may cause distinct outputs depending on the choice of propositional skeleton.

2. We do not indicate how to calculate a propositional skeleton of a modal formula.

It is easy to see that the particular choice of propositional skeletons that we make is unimportant for the resulting formula, so in fact the first objection is not a problem. The second objection is not a problem neither, since it is easy to came up with algorithms to calculate a propositional skeleton of a modal formula.

We have the following lemmas related to the TNNIL-algorithm:

Lemma 139. Assume that

1. $\mathscr{A}_{T,\Box}$.
2. $\mathrm{Gl}_{T,\Box}$.

Then for any $\phi \in \mathcal{L}_{\mathrm{p}}$ and $\sigma \in \Sigma_1$-real, we have that:

$$\vdash_T \Box(\sigma_\Box \phi) \leftrightarrow \Box(\sigma_\Box \phi^+).$$

Proof. We proceed by strong induction in the box depth of ϕ. If ϕ has no boxes, this is simply the assumption $\mathscr{A}_{T,\Box}$. Let ψ be a propositional skeleton of ϕ and $\psi(\vec{p},\Box\chi_0,\dots,\Box\chi_{n-1}) = \phi(\vec{p})$. By induction hypothesis we have that for any $i < n$:

$$\vdash_T \Box(\sigma_\Box \chi_i) \leftrightarrow \Box(\sigma_\Box \chi_i^+). \tag{i}$$

We define the following Σ_1-realization:

$$\begin{aligned}\tau(p_i) &= \sigma(p_i),\\ \tau(q_i) &= \Box(\sigma_\Box(\chi_i)).\end{aligned}$$

We have the following equalites:

$$\begin{aligned}
\sigma_\Box\phi &= \sigma_\Box(\psi(\vec{p},\Box\chi_0,\dots,\Box\chi_{n-1}))\\
&= \psi(\sigma(\vec{p}),\Box(\sigma_\Box\chi_0),\dots,\Box(\sigma_\Box\chi_{n-1}))\\
&= \tau\psi\\
\sigma_\Box\phi^+ &= \sigma_\Box(\psi^*(\vec{p},\Box\chi_0^+,\dots,\Box\chi_{n-1}^+))\\
&= \psi^*(\sigma(\vec{p}),\Box(\sigma_\Box\chi_0^+),\dots,\Box(\sigma_\Box\chi_{n-1}^+))\\
\tau\psi^* &= \psi^*(\tau(\vec{p}),\tau(\vec{q}))\\
&= \psi^*(\sigma(\vec{p}),\Box(\sigma_\Box\chi_0),\dots,\Box(\sigma_\Box\chi_{n-1}))
\end{aligned}$$

With these equalities in mind, we have the following equivalences:

$$\begin{aligned}
\vdash_T \Box(\sigma_\Box\phi^+) &\leftrightarrow \Box(\tau\psi^*) && \text{(by (i) and } \mathrm{Gl}_{T,\Box})\\
&\leftrightarrow \Box(\tau\psi) && \text{(by } \mathscr{A}_{T,\Box})\\
&= \Box(\sigma_\Box\phi).
\end{aligned}$$

$\Box$

Lemma 140. Assume $\mathscr{B}_{T,\Box,\triangle}$ and $\mathrm{Gl}_{T,\Box^\triangle}$. Then, for any $\phi \in \mathsf{TNNIL}$ and $\sigma \in \Sigma_1$-real, we have that

$$\vdash_T \Box(\sigma_\Box\phi) \leftrightarrow \Box^\triangle(\sigma_{\Box^\triangle}\phi).$$

Proof. We proceed by strong induction in the box depth of ϕ. If ϕ has no boxes, this is simply the assumption $\mathscr{B}_{T,\Box,\triangle}$. Now, assume that $\psi(\vec{p},\vec{q})$ is a propositional skeleton of $\phi(\vec{p})$ and $\psi(\vec{p},\Box\chi_0,\dots,\Box\chi_{n-1}) = \phi$, which thanks to

$\phi \in \mathsf{TNNIL}$ implies that $\psi \in \mathsf{NNIL}$ and $\chi_0, \ldots, \chi_{n-1} \in \mathsf{TNNIL}$. By the induction hypothesis we have that for any $i < n$:

$$\vdash_T \Box(\sigma_\Box \chi_i) \leftrightarrow \Box^\triangle(\sigma_{\Box^\triangle} \chi_i). \tag{i}$$

We define τ, a Σ_1-real, as

$$\begin{aligned} \tau(p_i) &= \sigma(p_i), \\ \tau(q_i) &= \Box(\sigma_\Box \chi_i). \end{aligned}$$

We have the following equalities:

$$\begin{aligned} \sigma_\Box \phi &= \sigma_\Box(\psi(\vec{p}, \Box\chi_0, \ldots, \Box\chi_{n-1})) \\ &= \psi(\sigma(\vec{p}), \Box(\sigma_\Box\chi_0), \ldots, \Box(\sigma_\Box\chi_{n-1})) \\ &= \tau\psi. \\ \sigma_{\Box^\triangle} \phi &= \sigma_{\Box^\triangle}(\psi(\vec{p}, \Box\chi_0, \ldots, \Box\chi_{n-1})) \\ &= \psi(\sigma(\vec{p}), \Box^\triangle(\sigma_{\Box^\triangle}\chi_0), \ldots, \Box^\triangle(\sigma_{\Box^\triangle}\chi_{n-1})) \end{aligned}$$

With these equalities in mind, we have the following equivalences:

$$\begin{aligned} \vdash_T \Box^\triangle(\sigma_{\Box^\triangle}\phi) &\leftrightarrow \Box^\triangle(\tau\psi) && \text{(by (i) and } \mathrm{Gl}_{T,\Box^\triangle}\text{)} \\ &\leftrightarrow \Box(\tau\psi) && \text{(by } \mathscr{B}_{T,\Box,\triangle}\text{)} \\ &= \Box(\sigma_\Box\phi). \end{aligned}$$

□

Theorem 141. Assume that

1. The theory T is Σ_1-complete, Δ_0-decidable and Π_2-sound. Also, assume that either T is a self Δ_0-arithmetizable extension of iEA or T is a self Σ_1-arithmetizable extension of $\mathsf{iEA} + B\Sigma_1$.
2. $\mathrm{HB}_{T,\triangle}$.
3. $\mathscr{A}_{T,\Box_T}, \mathscr{B}_{T,\Box_T,\triangle}$.
4. T is closed under $(\_)^\triangle_{\mathrm{V}}$.

Then

$$\Sigma_1\text{-}\mathbb{PL}(T) = \{\phi \in \mathcal{L}_{\mathrm{m}} \mid \phi^+ \in \Sigma_1\text{-}\mathbb{PL}((T)^\triangle_{\mathrm{V}}, \Box^\triangle_T)\}.$$

Proof. We have to show that

$$\phi \in \Sigma_1\text{-}\mathbb{PL}(T) \text{ iff } \phi^+ \in \Sigma_1\text{-}\mathbb{PL}((T)^\triangle_{\mathrm{V}}, \Box^\triangle_T). \tag{i}$$

By Corollary 37, we have that $\mathrm{HB}_{T,\Box_T}$; and by Lemma 77, we get $\mathrm{Hb}_{T,\Box^\triangle_T}$. Also, by Lemma 42, we get that for any sentence ψ,

$$\vdash_T \psi \text{ iff } \vDash_{\mathbb{N}} \Box_T\psi. \tag{ii}$$

Let σ be a Σ_1-realization and $\phi \in \mathcal{L}_{\rm m}$, we have that:

$$\begin{aligned} \vdash_T \Box_T(\sigma_{\Box_T}\phi) &\leftrightarrow \Box_T(\sigma_{\Box_T}\phi^+) && \text{(by Lemma 139)} \\ &\leftrightarrow \Box_T^{\triangle}(\sigma_{\Box_T^{\triangle}}\phi^+). && \text{(by Lemma 140)} \end{aligned}$$

We will denote this equivalence as (iii).

With this we can conclude:

$$\begin{aligned} \vdash_T \sigma_{\Box_T}\phi \text{ iff } &\vDash_{\mathbb{N}} \Box_T(\sigma_{\Box_T}\phi) && \text{(by (ii))} \\ \text{iff } &\vDash_{\mathbb{N}} \Box_T^{\triangle}(\sigma_{\Box_T^{\triangle}}\phi^+) && \text{(by (iii) and } \Pi_2\text{-soundness)} \\ \text{iff } &\vDash_{\mathbb{N}} \Box_T\left(\sigma_{\Box_T^{\triangle}}\phi^+\right)^{\triangle}_{\rm V} && \text{(since } \sigma_{\Box_T^{\triangle}}\phi^+ \text{ is a sentence)} \\ \text{iff } &\vdash_T (\sigma_{\Box^{\triangle}}\phi^+)^{\triangle}_{\rm V} && \text{(by (ii))} \\ \text{iff } &\vdash_{(T)^{\triangle}_{\rm V}} \sigma_{\Box_T^{\triangle}}\phi^+. && \text{(by Theorem 72)} \end{aligned}$$

Finally, thanks to σ and ϕ begin arbitrary, we have proven (i) so we can conclude the desired result. □

4.2 How to get $\mathscr{A}$

In this section we are going to study some conditions under which we can ensure $\mathscr{A}_{T,\Box_T}$. In other words, how to prove that for any formula ϕ and Σ_1-realization σ we have that:

$$\vdash_T \Box_T(\sigma\phi) \leftrightarrow \Box_T(\sigma\phi^*).$$

First, we establish a little lemma that guarantees the right to left direction.

Lemma 142. Let T be a theory such that $\mathrm{Hb}_{T,\Box}$. Then for any propositional formula ϕ and Σ_1-realization σ we have that:

$$\vdash_T \Box(\sigma\phi^*) \to \Box(\sigma\phi).$$

Proof. By the properties of the NNIL-algorithm we know that $\vdash_{\mathsf{iPC}} \phi^* \to \phi$. Then it is clear that:

$$\vdash_T \sigma\phi^* \to \sigma\phi.$$

Finally, using $\mathrm{Hb}_{T,\Box}$ it is easy to conclude

$$\vdash_T \Box(\sigma\phi^*) \to \Box(\sigma\phi).$$

□

Now we focus in getting the left to right direction of $\mathscr{A}$. In order to prove it, we need to introduce some definitions and results from [11]. First, we define a translation between propositional formulas:

Definition 143. We define the function $[\_]\_ : \mathcal{L}_\mathrm{p} \times \mathcal{L}_\mathrm{p} \longrightarrow \mathcal{L}_\mathrm{p}$ recursively in the second argument as:

$$\begin{aligned}
[\psi]\phi &:= \phi \text{ if } \phi \text{ atomic},\\
[\psi](\phi_0 \wedge \phi_1) &:= [\psi]\phi_0 \wedge [\psi]\phi_1,\\
[\psi](\phi_0 \vee \phi_1) &:= [\psi]\phi_0 \vee [\psi]\phi_1,\\
[\psi](\phi_0 \rightarrow \phi_1) &:= \psi \rightarrow \phi_0 \rightarrow \phi_1.
\end{aligned}$$

■

And we also need the concept of σ-relation:

Definition 144. A σ-relation is a binary relation $\rhd \subseteq \mathcal{L}_\mathrm{p} \times \mathcal{L}_\mathrm{p}$ such that:

1. $\phi \vdash_{\mathsf{iPC}} \psi$ implies $\phi \rhd \psi$.
2. $\phi \rhd \psi, \psi \rhd \chi$ implies $\phi \rhd \chi$.
3. $\chi \rhd \phi, \chi \rhd \psi$ implies $\chi \rhd \phi \wedge \psi$.
4. $\phi \rhd \chi, \psi \rhd \chi$ implies $\phi \vee \psi \rhd \chi$.
5. Let us have propositional formulas $\phi_0, \dots, \phi_k, \psi_0, \dots, \psi_{k-1}$, then
$$\left(\bigwedge_{i<k} \phi_i \rightarrow \psi_i\right) \rightarrow \phi_k \rhd \bigvee_{i \le k} \left[\bigwedge_{i<k} \phi_i \rightarrow \psi_i\right] \phi_i.$$
6. $\phi \rhd \psi$ implies $(p \rightarrow \phi) \rhd (p \rightarrow \psi)$.

The smallest σ-relation will be denoted $\blacktriangleright_\sigma$. ■

We have the following result from [11]:

Theorem 145. For any $\phi \in \mathcal{L}_\mathrm{p}$ we have that

$$\phi \blacktriangleright_\sigma \phi^*.$$

As a corollary, we have that for any σ-relation $\rhd$,

$$\phi \rhd \phi^*.$$

We need one more ingredient:

Definition 146. Let $\phi, \psi \in \mathcal{L}_\mathrm{p}$. Then, we say that $\phi \rhd_T \psi$ iff

For any $\chi \in \Sigma_1$-sent and any $\sigma \in \Sigma_1$-real, $\chi \vdash_T \sigma\phi$ implies $\chi \vdash_T \sigma\psi$.

We also define the formalization of this relation, defining the formula:

$$A \rhd_T B := \forall C, f\,.\, \Sigma_1\text{-}\mathsf{sent}(C) \wedge \Sigma_1\text{-}\mathsf{real}(f, \mathsf{fv}(A) \cup^{\bullet} \mathsf{fv}(B)) \wedge \mathsf{der}_T(C, f(A)) \\ \to \mathsf{der}_T(C, f(B)).$$ ■

To define the formula $A \rhd_T B$ we need to use a the formula $\Sigma_1\text{-}\mathsf{real}(f, X)$. This formula claims that f is a Σ_1-realization whose domain is the finite set X. We need to specify its domain as a finite set, since only finite functions can be represented as a number.

We note that, following [11], $\rhd_T$ is just a preservativity relation. For classical theories, preservativity and conservativity are dual notions. The insterested reader should consult Remark 4.1 at page 11 in [11].

If we are capable of showing that $\rhd_T^T = \left\{ (\phi, \psi) \mid\ \vdash_T \overline{\phi} \rhd_T \overline{\psi} \right\}$ is a σ-relation we would finish, since then by Theorem 145:

$$\vdash_T \Box_T(\top \to \sigma\phi) \to \Box_T(\top \to \sigma\phi^*).$$

But if $\mathrm{Hb}_{T,\Box_T}$ we can conclude

$$\vdash_T \Box_T(\sigma\phi) \to \Box_T(\sigma\phi^*),$$

which is the desired result. It is clear, that if we prove the internalized versions of properties 1 to 6 of σ-relations inside T we would have that $\rhd_T^T$ fulfills properties 1 to 6, and then it is a σ-relation.

To avoid making things too dense we are going to do the following. First, we are going to show what conditions are necessary to show that $\vdash_T$ fulfills properties 1 to 6. Then, we will see which conditions must have T to have those arguments formalized inside T. With all this done, we will put our results together in a theorem.

Lemma 147 (Property 1).

$$\phi \vdash_{\mathsf{iPC}} \psi \text{ implies } \phi \rhd_T \psi.$$

Proof. This lemma is trivial, since our theories are formulated in iFOL so the assumptions implies that $\sigma(\phi) \vdash_T \sigma(\psi)$. From this and some propositional reasoning we can easily conclude that $\chi \to \sigma(\phi) \vdash_T \chi \to \sigma(\psi)$, which implies the desired result. □

Lemma 148 (Property 2).

$$\text{If } \phi_0 \rhd_T \phi_1 \text{ and } \phi_1 \rhd_T \phi_2, \text{ then } \phi_0 \rhd_T \phi_2.$$

Proof. Assume that $\phi_0 \rhd_T \phi_1$ (i), $\phi_1 \rhd_T \phi_2$ (ii) and $\chi \vdash_T \sigma\phi_0$ (iii). Using (i) and (iii) we get that $\chi \vdash_T \sigma\phi_1$ and using this and (ii) we get the desired $\chi \vdash_T \sigma\phi_2$. □

Lemma 149 (Property 3).

$$\text{If } \phi \rhd_T \psi_0 \text{ and } \phi \rhd_T \psi_1, \text{ then } \phi \rhd_T (\psi_0 \wedge \psi_1).$$

Proof. Assume that $\phi \rhd_T \psi_0$ (i), $\phi \rhd_T \psi_1$ (ii) and $\chi \vdash_T \sigma\phi$ (iii). From (i) and (iii) we obtain that $\chi \vdash_T \sigma\psi_0$ and from (ii) and (iii) that $\chi \vdash_T \sigma\psi_1$. But then, concatenating both proofs and doing a little of propositional reasoning we obtain $\chi \vdash_T \sigma\psi_0 \wedge \sigma\psi_1$, i.e. $\chi \vdash_T \sigma(\psi_0 \wedge \psi_1)$ as wanted. □

Properties 4 and 5 are going to be harder to prove. The idea is that we will need a Σ_1-predicate Q with box function $\triangle$ such that $\mathrm{Rfn}_{T,\triangle}$ and use the relation of $\rhd_T$ with Σ_1-formulas. Clearly, we cannot have that $\triangle = \Box_T$, since then $\vdash_T \Box_T \bot \to \bot$, i.e. $\vdash_T \neg\Box_T \bot$, contraty to Gödel's incompleteness theorems. For this reason we will need to work with theories $T_n \subset T$, but at the same time we will need that $\bigcup_{n\in\mathbb{N}} T_n = T$. This last equality will be needed since the idea will be to assume that $\vdash_T \phi$ and then get that for some $n \in \mathbb{N}$, $\vdash_{T_n} \phi$. With this we will be allowed to use De Jongh translation, which will be fundamental to put $\Box_{T_n}$ in crucial places and then eliminate it via $\mathrm{Rfn}_{T,\Box_{T_n}}$. We start with property 4[1]:

Lemma 150 (Property 4). Let us have theories $T, (T_n)_{n\in\mathbb{N}}$, such that:

1. $\bigcup_{n\in\mathbb{N}} T_n = T$.
2. For any n, $\mathrm{HB}_{T,T_n,\Box_{T_n}}$.
3. For any n, $\phi \in \mathrm{Ax}_{T_n}$ implies $\vdash_T [\top]_{\Box_{T_n}}(\phi)$ (translation defined in 2.3).
4. For any n, $\mathrm{Rfn}_{T,\Box_{T_n}}$.

Then

$$\phi_0 \rhd_T \psi \text{ and } \phi_1 \rhd_T \psi \text{ implies } \phi_0 \vee \phi_1 \rhd_T \psi.$$

Proof. Let us assume that $\phi_0 \rhd_T \psi$ and $\phi_1 \rhd_T \psi$, and $\chi \vdash_T \sigma\phi_0 \vee \sigma\phi_1$ where χ is a Σ_1-formula. Then, we know thanks to $\bigcup_{n\in\mathbb{N}} T_n = T$ that for some $n \in \mathbb{N}$, $\chi \vdash_{T_n} \sigma\phi_0 \vee \sigma\phi_1$. Then by Theorem 90 we have that

$$[\top]_{\Box_{T_n}}\chi \vdash_T [\top]_{\Box_{T_n}}\sigma\phi_0 \vee [\top]_{\Box_{T_n}}\sigma\phi_1. \tag{i}$$

Thanks to χ being a Σ_1-formula and Lemma 87 we have that $\vdash_T [\top]_{\Box_{T_n}}\chi \leftrightarrow \chi$ and using Lemma 81 with $\mathrm{HB}_{T,T_n,\Box_{T_n}}$ we have that $\vdash_T [\top]_{\Box_{T_n}}\phi_i \to \Box_{T_n}\phi_i$ so from (i) we obtain:

$$\chi \vdash_T \Box_{T_n}(\sigma\phi_0) \vee \Box_{T_n}(\sigma\phi_1). \tag{ii}$$

Finally, note that by $\mathrm{Rfn}_{T,\Box_{T_n}}$ we have that $\Box_{T_n}(\sigma\phi_i) \vdash_T \sigma\phi_i$ and using that σ is a Σ_1-realization, $\Box_{T_n}(\sigma\phi_i)$ is a Σ_1-formula and $\phi_i \rhd_T \psi$ we get that $\Box_{T_n}(\sigma\phi_i) \vdash_T \sigma\psi$. This means that from (ii) we can conclude

$$\chi \vdash_T \sigma\psi,$$

as wanted. □

[1]Property 4 has also an alternative proof using q-realizability, see [11]

Lemma 151 (Property 5). Let us have theories $T,(T_n)_{n\in\mathbb{N}}$ and propositional formulas $\phi_0,\ldots,\phi_k,\psi_0,\ldots,\psi_{k-1}$. Let us define $I:=\bigwedge_{i<k}\phi_i\to\psi_i$ and $H:=\bigvee_{i\leq k}[I,\sigma]^{\circ}_{\Box_{T_n}}\phi_i$. Assume that we have

1. $\bigcup_{n\in\mathbb{N}}T_n=T$.
2. For any n, $\mathrm{HB}_{T,T_n,\Box_{T_n}}$.
3. For any n, $\phi\in\mathrm{Ax}_{T_n}$ implies $\vdash_T[I]_{\Box_{T_n}}(\phi)$ (translation defined in 2.3).
4. For any n, $\mathrm{Rfn}_{T,\Box_{T_n}}$.
5. T is closed under $(\_)^H_{\mathrm{F}}$ (translation defined in 2.1).

Then

$$\left(\bigwedge_{i<k}\phi_i\to\psi_i\right)\to\phi_k\rhd_T\bigvee_{i\leq k}\left[\bigwedge_{i<k}\phi_i\to\psi_i\right]\phi_i.$$

Proof. Assume that

$$\chi\vdash_T\left(\bigwedge_{i<k}\sigma\phi_i\to\sigma\psi_i\right)\to\sigma\phi_k,$$

where χ is a Σ_1-formula and σ a Σ_1-realization. Since $\bigcup_{n\in\mathbb{N}}T_n=T$ we have that there is a $n\in\mathbb{N}$ such that

$$\chi\vdash_{T_n}\left(\bigwedge_{i<k}\sigma\phi_i\to\sigma\psi_i\right)\to\sigma\phi_k,$$

and then, using Theorem 90, we have

$$[\sigma I]_{\Box_{T_n}}\chi\vdash_{T_n}[\sigma I]_{\Box_{T_n}}\left(\bigwedge_{i<k}\sigma\phi_i\to\sigma\psi_i\right)\to[\sigma I]_{\Box_{T_n}}\sigma\phi_k. \tag{i}$$

Since χ is a Σ_1-formula by Lemma 87 we have that $\vdash_T[\sigma I]_{\Box_{T_n}}\chi\leftrightarrow\chi$. Also, by Lemma 82 we have that

$$[\sigma I]_{\Box_{T_n}}\left(\bigwedge_{i<k}\sigma\phi_i\to\sigma\psi_i\right)\leftrightarrow\left(\bigwedge_{i<k}[\sigma I]_{\Box_{T_n}}\sigma\phi_i\to[\sigma I]_{\Box_{T_n}}\sigma\psi_i\right)\wedge\Box_{T_n}(I\to I).$$

But since $\mathrm{HB}_{T,T_n,\Box_{T_n}}$ we have that $\vdash_T\Box_{T_n}(I\to I)$. So from (i) we get

$$\chi\vdash_{T_n}\left(\bigwedge_{i<k}[\sigma I]_{\Box_{T_n}}\sigma\phi_i\to[\sigma I]_{\Box_{T_n}}\sigma\psi_i\right)\to[\sigma I]_{\Box_{T_n}}\sigma\phi_k. \tag{ii}$$

But, by Lemma 94, we have that $\vdash_T [\sigma I]_{\Box_{T_n}} \sigma\phi_i \leftrightarrow [I,\sigma]_{\Box_{T_n}} \phi_i$ and by Lemma 95 we have that $\vdash_T [I,\sigma]_{\Box_{T_n}} \phi_i \to [I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i$. So from (ii) we get

$$\chi \vdash_{T_n} \left(\bigwedge_{i<k} [I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i \to [\sigma I]_{\Box_{T_n}} \sigma\psi_i \right) \to [I,\sigma]^{\circ}_{\Box_{T_n}} \phi_k. \tag{iii}$$

Now, using Theorem 54, we get:

$$(\chi)^H_{\mathrm{F}} \vdash_{T_n} \left(\bigwedge_{i<k} \left([I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i\right)^H_{\mathrm{F}} \to \left([\sigma I]_{\Box_{T_n}} \sigma\psi_i\right)^H_{\mathrm{F}} \right) \to \left([I,\sigma]^{\circ}_{\Box_{T_n}} \phi_k\right)^H_{\mathrm{F}}. \tag{iv}$$

For any $i \geq k$ we note that $[I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i$ is T-equivalent to a Σ_1-formula so

$$\begin{aligned} \vdash_T \left([I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i\right)^H_{\mathrm{F}} &\leftrightarrow [I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i \vee H && \text{(by Lemma 51)} \\ &\leftrightarrow H. && \text{(by definition of } H\text{)} \end{aligned}$$

Also $\vdash_T (\chi)^H_{\mathrm{F}} \leftrightarrow \chi \vee H$. So in fact (iv) gives

$$\chi \vee H \vdash_T \left(\bigwedge_{i<k} H \to \left([\sigma I]_{\Box_{T_n}} \sigma\psi_i\right)^H_{\mathrm{F}} \right) \to H. \tag{v}$$

Since $\vdash_T \chi \to \chi \vee H$ and by Lemma 49 $\vdash_T H \to \left([\sigma I]_{\Box_{T_n}} \sigma\psi_i\right)^H_{\mathrm{F}}$. Then (v) gives

$$\chi \vdash_T H,$$

in other words

$$\chi \vdash_T \bigvee_{i \leq k} [I,\sigma]^{\circ}_{\Box_{T_n}} \phi_i.$$

Then by Lemma 96 $\chi \vdash_T \bigvee_{i \leq k} [I,\sigma]\phi_i$, i.e $\chi \vdash_T \sigma([I]\phi_i)$, as wanted. □

Lemma 152 (Property 6). For any propositional formulas ϕ_0, ϕ_1 and any propositional variable p we have that

$$\phi_0 \rhd_T \phi_1 \text{ implies } (p \to \phi_0) \rhd_T (p \to \phi_1).$$

Proof. Assume that $\phi \rhd_T \psi$ and $\chi \vdash_T \sigma(p) \to \sigma\phi_0$ (i), where σ is a Σ_1-realization and χ is a Σ_1-sentence. Let us define $\psi := \sigma(p)$, note that it is a Σ_1-sentence. By (i) we have that

$$\chi \wedge \psi \vdash_T \sigma\phi_0. \tag{ii}$$

But since iEA $\subseteq T$, using codified pairs we know that $\chi \wedge \psi$ is T-equivalent to a Σ_1-sentence. Then, using that $\phi_0 \rhd_T \phi_1$ and (ii) we get

$$\chi \wedge \psi \vdash_T \sigma\phi_1,$$

which implies the desired

$$\chi \vdash_T \psi \to \sigma\phi_1,$$

in other words

$$\chi \vdash_T \sigma(p) \to \sigma\phi_1.$$

$\square$

If $\mathsf{iEA} \subseteq U$, lemmas 147, 148, 149 are easily provable inside U. For property 6 we need that U is capable of proving that:

$$\vdash_U \forall A, B. \Sigma_1\text{-}\mathsf{sent}(A) \wedge \Sigma_1\text{-}\mathsf{sent}(B) \to \exists C. \Sigma_1\text{-}\mathsf{sent}(C) \wedge \mathsf{prov}_T(A \wedge^\bullet B \leftrightarrow^\bullet C).$$

But again, if U extends iEA, it is capable of showing this. Finally, let us analyze properties 4 and 5. Notice that we need the lemmas and main theorem of the De Jongh translation. We have already seen that in order to have those lemmas having Σ_1-induction suffices[2]. So if the hypothesis of the theorems are provable in U and U can perform Σ_1-induction we will have the formalizations of properties 4 and 5 inside U. So, we have the following lemma:

Lemma 153. Let U be a theory extending $\mathsf{i I}\Sigma_1$ such that:

1. ax_T enumerates T over U.
2. $\vdash_U \forall A \,.\, \mathsf{ax}_T(A) \leftrightarrow \exists x \,.\, \mathsf{ax}_{T_x}(A)$.
3. $\vdash_U \forall x \,.\, \mathsf{HB}_{\mathsf{prov}_T, \mathsf{prov}_{T_x}, \square^\bullet_{T_x}}$.
4. $\vdash_U \forall x, A, B \,.\, \mathsf{sent}(B) \wedge \mathsf{ax}_{T_x}(A) \to \mathsf{prov}_T([B]_{\square^\bullet_{T_{\dot{x}}}} A)$.
5. $\vdash_U \forall A, B \,.\, \mathsf{sent}(B) \wedge \mathsf{ax}_T(A) \to \mathsf{prov}_T(A)^B_{\mathrm{F}}$.
6. $\vdash_U \forall A, x \,.\, \mathsf{sent}(A) \to \mathsf{prov}_T(\square^\bullet_{T_{\dot{x}}} A \to^\bullet A)$.

Then $\rhd^U_T$ is a σ-relation.

4.3 How to get $\mathscr{B}$

We want to see how we can prove $\mathscr{B}_{T, \square_T, \triangle}$, i.e. that for any $\phi \in \mathsf{NNIL}$ and Σ_1-realization σ:

$$\vdash_T \square(\sigma\phi) \leftrightarrow \square^\triangle(\sigma\phi).$$

First, we focus in the right to left direction. We will check that under weak conditions, in particular conditions that were already necessary to apply the first part of the construction i.e. Theorem 116, are enough.

[2] In fact, it can be shown that iEA suffices, since the existential variable of the formula where we perform Σ_1-induction is exponentially bounded. We do not assert this because we do not want to enter in the burden of proving it, and it is not necessary for the theories we work with.

Lemma 154. Let $\phi \in \mathsf{NI}$ and σ be a Σ_1-realization. Then $\sigma\phi$ is iEA-equivalent to a Σ_1-formula.

Proof. By induction in the formula ϕ. If it is an atomic propositional formula the result is trivial. The cases of conjunction and disjunction are easy, using that the class of Σ_1-formulas is closed under conjunction and disjunction up to iEA-equivalence. $\square$

Lemma 155. Let $\phi \in \mathsf{NNIL}$ and σ be a Σ_1-realization. Then $\sigma\phi$ is iEA-equivalent to a $\mathcal{A}$ formula.

Proof. If ϕ is an atomic propositional formula the result is trivial, since Σ_1 formulas are iEA-equivalent to $\mathcal{A}$-formulas. The case where ϕ is a conjunction or a disjunction is trivial. We can just use the induction hypothesis and that $\mathcal{A}$ is closed under conjunction and disjunction. Finally, for the implication case we have that $\phi = \phi_0 \to \phi_1$ where $\phi_0 \in \mathsf{NI}$ and $\phi_1 \in \mathsf{NNIL}$. By the induction hypothesis we get that there is a formula $\psi_1 \in \mathcal{A}$ such that

$$\vdash_{\mathsf{iEA}} \sigma\phi_1 \leftrightarrow \psi_1.$$

Also, by Lemma 154 there is a $\psi_0 \in \Sigma_1$ such that

$$\vdash_{\mathsf{iEA}} \sigma\phi_0 \leftrightarrow \psi_0.$$

But then,

$$\vdash_{\mathsf{iEA}} (\phi_0 \to \phi_1) \leftrightarrow (\psi_0 \to \psi_1),$$

and $\psi_0 \to \psi_1 \in \mathcal{A}$. $\square$

Lemma 156. Let $\phi \in \mathsf{NNIL}$ and σ be a Σ_1-realization. Let T be such that:

1. T is a theory Δ_0-decidable, Σ_1-complete and extends iEA. In addition, T is self Δ_0-arithmetizable or $\vdash_T B\Sigma_1$ and T is self Σ_1-arithmetizable.
2. $\mathrm{HB}_{T,\triangle}$.
3. T is closed under $(\_)^{\triangle}_{\mathrm{V}}$.

Then

$$\vdash_T \Box^{\triangle}(\sigma\phi) \to \Box(\sigma\phi).$$

Proof. By Corollary 37, we get $\mathrm{HB}_{T,\Box_T}$. By Lemma, 155 there is $\psi \in \mathcal{A}$ such that $\vdash_T \sigma\phi \leftrightarrow \psi$. Since $\psi \in \mathcal{A}$, by Lemma 79 we know that $\vdash_T (\psi)^{\triangle}_{\mathrm{V}} \to \psi$. Note that $\vdash_T \sigma\phi \leftrightarrow \psi$ implies, by Theorem 72 , that $\vdash_T (\sigma\phi \leftrightarrow \psi)^{\triangle}_{\mathrm{V}}$, which implies $\vdash_T (\sigma\phi)^{\triangle}_{\mathrm{V}} \leftrightarrow (\psi)^{\triangle}_{\mathrm{V}}$. Then, we have that

$$\vdash_T (\sigma\phi)^{\triangle}_{\mathrm{V}} \to \sigma\phi.$$

Finally, using that $\mathrm{HB}_{T,\Box_T}$ and that $\sigma\phi$ is a sentence, we can conclude:

$$\vdash_T \Box^{\triangle}_T(\sigma\phi) \to \Box_T(\sigma\phi).$$ $\square$

To get the left to right direction of $\mathscr{B}_{T,\Box,\triangle}$, we can use Corollary 75 together with $\vdash_{\mathsf{iEA}} \mathsf{prov}_T(A) \to \mathsf{prov}_{(T)^{\triangle}_{\mathrm{V}}}(A)$, which holds by definition of $(T)^{\triangle}_{\mathrm{V}}$.

Chapter 5

$\mathsf{i}\mathsf{I}\Sigma_1$ adding sentential reflection

In Chapters 3 and 4 we have provided a method for calculating the Σ_1-provability logic of intuitionistic theories. We have noticed that in both chapter we needed sentential reflection of the theory for its finite subtheories. This principle is fulfilled by HA, but it is not fulfilled by finite axiomatizable theories such as iEA or $\mathsf{i}\mathsf{I}\Sigma_1$. For this reason, it is hard to find another theory for which the method can be applied.

In this chapter, we define such a theory. The theory will be $\mathsf{i}\mathsf{I}\Sigma_1$ plus sentential reflection for $\mathsf{i}\mathsf{I}\Sigma_1$. This results in a theory, $\mathsf{i}\mathsf{I}\Sigma_1^+$, that proves its sentential reflection princple for its finite subtheories. We apply the method to obtain its Σ_1-provability logic.

5.1 Definition and basic properties

In this section we define $\mathsf{i}\mathsf{I}\Sigma_1^+$ and we prove that some box functions fulfill the Hilbert-Bernays conditions with respect this theory.

Definition 157. We define the theory

$$\mathsf{i}\mathsf{I}\Sigma_1^+ := \mathsf{i}\mathsf{I}\Sigma_1 + \mathrm{Rfn}_{\Box_{\mathsf{i}\mathsf{I}\Sigma_1}}.$$

Let us define a (α,β)-arithmetization of this theory as:

$$\begin{aligned}\alpha(A) &:= \mathsf{ax}_{\mathsf{i}\mathsf{I}\Sigma_1}(A),\\ \beta(A) &:= \exists B \leq A \,.\, \mathsf{sent}(B) \wedge A \approx \Box^{\bullet}_{\mathsf{i}\mathsf{I}\Sigma_1} B \rightarrow^{\bullet} B.\end{aligned}$$

We assume that $\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1}(A)$ is a self Δ_0-arithmetization of $\mathrm{i}\mathsf{I}\Sigma_1$. We will write

$$\begin{aligned}
&\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A) := \alpha(A) \vee \beta(A),\\
&\mathsf{ax}_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}(A) := \alpha(A) \vee (\beta(A) \wedge \mathcal{S}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A)),\\
&\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \leq x}(A) := \alpha(A) \vee (\beta(A) \wedge A \leq x).
\end{aligned}$$
■

Lemma 158. $\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A)$ enumerates $\mathrm{i}\mathsf{I}\Sigma_1^+$ in $\mathrm{i}\mathsf{I}\Sigma_1$.

Proof. This is straightforward by definition of $\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A)$. □

Corollary 159. Let T be a Δ_0-complete and consistent extension of iEA. Then $\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A)$ enumerates $\mathrm{i}\mathsf{I}\Sigma_1^+$ in T. This implies that $\mathsf{HB}_{T,\mathrm{i}\mathsf{I}\Sigma_1^+,\Box_{\mathrm{i}\mathsf{I}\Sigma_1^+}}$.

Proof. Just use lemmas 158 and 39. The Hilbert-Bernays property can be shown using Corollary 37. □

Lemma 160. We have that

1. $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1^+},\Box^\bullet_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$.
2. $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1},\Box^\bullet_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$.

Proof. Proof of (1). We want to apply Theorem 126. We have that $\mathsf{iEA} \subseteq \mathrm{i}\mathsf{I}\Sigma_1$, thanks to Corollary 158 we get $\mathsf{HB}_{\mathsf{iEA},\Box_{\mathrm{i}\mathsf{I}\Sigma_1^+}}$. All left to show is

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \exists x\,.\,\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1^+}(A \to^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \leq \dot{x}} A). \tag{i}$$

But since $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1},\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1}}$ we have that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A\,.\,\Sigma_1\text{-}\mathsf{form}(A) \to \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(A \to^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1} A). \tag{ii}$$

But $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A\,.\,\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(A) \to \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \leq \bar{0}}(A)$, so by $\mathsf{HB}_{\mathrm{i}\mathsf{I}\Sigma_1,\Box_{\mathrm{i}\mathsf{I}\Sigma_1}}$ we get

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A\,.\,\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1} A \to^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \leq \dot{\bar{0}}} A) \tag{iii}$$

But (ii) and (iii) gives (i).

Proof of (2). We want to apply Lemma 125. Note that $\mathsf{iEA} \subseteq \mathrm{i}\mathsf{I}\Sigma_1$, $\mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1}$ enumerates $\mathrm{i}\mathsf{I}\Sigma_1$ in $\mathrm{i}\mathsf{I}\Sigma_1$, $\mathsf{HB}_{\mathrm{i}\mathsf{I}\Sigma_1,\Box_{\mathrm{i}\mathsf{I}\Sigma_1}}$ and $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1},\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1}}$. So all the properties are fulfilled. □

Lemma 161. We have that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall x \,.\, \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}, \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le x}, \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le x}}.$$

Proof. Necessitation. We have to show that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A \,.\, \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le x}(A) \rightarrow \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}} A).$$

This is provable by induction in the formula

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall p \forall A \le p \,.\, \mathsf{proof}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le x}(p, A) \rightarrow \exists q \,.\, \mathsf{proof}_{\mathrm{i}\mathsf{I}\Sigma_1}(q, \mathsf{prov}^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}}(\dot{A})),$$

using that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall x, A \,.\, \mathsf{ax}_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le x}(A) \rightarrow \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(\mathsf{ax}^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}}(\dot{A})),$$

by $\mathrm{HB}_{\mathrm{i}\mathsf{I}\Sigma_1, \Box_{\mathrm{i}\mathsf{I}\Sigma_1}}$.Compl.

Modus ponens. We have to show that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A, B \,.\, \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}}(A \rightarrow^\bullet B) \rightarrow^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}} A \rightarrow^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}} B).$$

This is easily provable just by definition of $\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}}$.

Completeness. We have to show that

$$\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A \,.\, \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(A \rightarrow^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}} A). \tag{i}$$

Note that $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \forall A \,.\, \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(A \rightarrow^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1} A)$ since $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}, \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1}}$. Also, by definition, is easy to show that $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}(\Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1} A \rightarrow^\bullet \Box^\bullet_{\mathrm{i}\mathsf{I}\Sigma_1^+ \le \dot{x}} A)$. These gives the desired (i). $\Box$

5.2 Closure under translations

We prove the closure of $\mathrm{i}\mathsf{I}\Sigma_1^+$ under the translations defined in Chapter 2.

5.2.1 Closure under Visser translation

Lemma 162. $\mathrm{i}\mathsf{I}\Sigma_1$ is closed under $(\_)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$. This is verifiable in $\mathrm{i}\mathsf{I}\Sigma_1$.

Proof. Thanks to Lemma 160 we have that $\vdash_{\mathrm{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathrm{i}\mathsf{I}\Sigma_1}, \Box^\bullet_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$. Since $\mathrm{i}\mathsf{I}\Sigma_1$ is sound we also get $\mathrm{HB}_{\mathrm{i}\mathsf{I}\Sigma_1, \Box_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$. Note that all the axioms distinct from induction are of shape $\forall x_0, \dots, x_n \,.\, \phi$ where ϕ is Δ_0. Proving that the translation of this axioms is provable is easy using $\mathrm{HB}_{\mathrm{i}\mathsf{I}\Sigma_1, \Box_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$ with lemmas 61 and 66.

The translation of the induction axioms is also easy using $\mathrm{HB}_{\mathrm{i}\mathsf{I}\Sigma_1, \Box_{\mathcal{S}\mathrm{i}\mathsf{I}\Sigma_1^+}}$ and lemmas 63 and 67. Note that for this we use that the induction axiom is only of Σ_1-formulas. $\Box$

Corollary 163. For any sentence ϕ, $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \to \Box_{\mathsf{i}\mathsf{I}\Sigma_1} (\phi)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. This is verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$.

Proof. By Lemma 162 we have that

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \forall A\,.\,\mathsf{ax}_{\mathsf{i}\mathsf{I}\Sigma_1}(A) \to \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}. \tag{i}$$

We also have by Lemma 160

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \mathsf{HB}_{\mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1},\Box^{\bullet}_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}. \tag{ii}$$

Then by Corollary 75 with (i) and (ii) we get

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \forall A\,.\,\mathsf{prov}_{(\mathsf{i}\mathsf{I}\Sigma_1)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}}(A) \leftrightarrow \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}.$$

Since $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A) \to \mathsf{prov}_{(\mathsf{i}\mathsf{I}\Sigma_1)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}}(A)$ by definition, we get

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \forall A\,.\,\mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A) \to \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A)_{\mathsf{V}}^{\Box^{\bullet}_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}, \tag{i}$$

so if ϕ is a sentence, instantiating A with $\overline{\phi}$ we get

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \to \Box_{\mathsf{i}\mathsf{I}\Sigma_1} (\phi)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}.$$

From (i), using $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1,\Box_{\mathsf{i}\mathsf{I}\Sigma_1}}$ and Lemma 28, we get the verifiability in $\mathsf{i}\mathsf{I}\Sigma_1$. $\Box$

Theorem 164. $\mathsf{i}\mathsf{I}\Sigma_1^+$ is closed under $(\_)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. This is verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$.

Proof. Thanks to $\mathsf{i}\mathsf{I}\Sigma_1 \subseteq \mathsf{i}\mathsf{I}\Sigma_1^+$ and Lemma 162 we only need to worry about the axioms of $\mathsf{i}\mathsf{I}\Sigma_1^+$ that are not axioms of $\mathsf{i}\mathsf{I}\Sigma_1$. In particular, let ϕ be a sentence we have to show that the translation of $\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \to \phi$ is $\mathsf{i}\mathsf{I}\Sigma_1^+$-provable.

Now, by Lemma 160 and since $\mathsf{i}\mathsf{I}\Sigma_1$ is sound we have that $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. Using this we get that it suffices to show:

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+} \left(\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi\right)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}} \to (\phi)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}.$$

But using Lemma 67 since $\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi$ is a Σ_1 formula, we note that it suffices to show

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \to (\phi)_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}.$$

But we have

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \rightarrow \Box_{\mathsf{i}\mathsf{I}\Sigma_1} (\phi)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}} \qquad \text{(by Corollary 163)}$$

$$\rightarrow (\phi)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}. \qquad \text{(by axiom of } \mathsf{i}\mathsf{I}\Sigma_1^+\text{)}$$

The reasoning is clearly verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$ and all the results used are verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$. Some of them are originally verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$ and used in the proof using sound, in the formalized proof we will use them without applying soundness. □

5.2.2 Closure under Friedman translation

Lemma 165. Let ψ be a sentence. Then $\mathsf{i}\mathsf{I}\Sigma_1$ is closed under $(\_)_{\mathrm{F}}^{\psi}$. This is verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$.

Proof. This is simple using Lemma 51 and that $\mathsf{i}\mathsf{I}\Sigma_1$ is Δ_0-decidable, both things are $\mathsf{i}\mathsf{I}\Sigma_1$-verifiable. □

Corollary 166. Let ϕ, ψ be sentence. Then

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \rightarrow \Box_{\mathsf{i}\mathsf{I}\Sigma_1} (\phi)_{\mathrm{F}}^{\psi}.$$

This is verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$.

Proof. Thanks to Lemma 165 we can apply Corollary 57 to obtain that

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \forall A, B \,.\, \mathsf{sent}(B) \wedge \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A) \rightarrow \mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1}(A)_{\mathsf{F}}^{B}. \qquad \text{(i)}$$

Then, the desired results comes from instantiating A with $\overline{\phi}$ and B with $\overline{\psi}$.

Verifiabilty in $\mathsf{i}\mathsf{I}\Sigma_1$ comes from (i) with $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1, \Box_{\mathsf{i}\mathsf{I}\Sigma_1}}$ and Lemma 28. □

Theorem 167. Let ψ be a sentence. $\mathsf{i}\mathsf{I}\Sigma_1^+$ is closed under $(\_)_{\mathrm{F}}^{\psi}$. This is verifiable in $\mathsf{i}\mathsf{I}\Sigma_1$.

Proof. Since $\mathsf{i}\mathsf{I}\Sigma_1$ is closed under $(\_)_{\mathrm{F}}^{\psi}$ we only need to worry about the axioms of $\mathsf{i}\mathsf{I}\Sigma_1^+$ that are not axioms of $\mathsf{i}\mathsf{I}\Sigma_1$. Let ϕ be a sentence, note that the translation of $\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \rightarrow \phi$ is $\left(\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi\right)_{\mathrm{F}}^{\psi} \rightarrow (\phi)_{\mathrm{F}}^{\psi}$. But $\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi$ is Σ_1, so by Lemma 51 this formula is $\mathsf{i}\mathsf{I}\Sigma_1$-equivalent to

$$\Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \vee \psi \rightarrow (\phi)_{\mathrm{F}}^{\psi}. \qquad \text{(i)}$$

By Lemma 49 we have that $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1} \psi \rightarrow (\phi)_{\mathrm{F}}^{\psi}$. Then, to show (i) it suffices that we show

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+} \Box_{\mathsf{i}\mathsf{I}\Sigma_1} \phi \rightarrow (\phi)_{\mathrm{F}}^{\psi}.$$

And we have the following reasoning proving this

$$\vdash_{\mathsf{iI\Sigma_1^+}} \Box_{\mathsf{iI\Sigma_1}} \phi \to \Box_{\mathsf{iI\Sigma_1}} (\phi)^{\psi}_{\mathrm{F}} \qquad \text{(by Corollary 166)}$$
$$\to (\phi)^{\psi}_{\mathrm{F}}. \qquad \text{(axiom of } \mathsf{iI\Sigma_1^+}\text{)}$$

□

5.2.3 Closure under De Jongh translation

Lemma 168. Let ψ be a sentence and $n \in \mathbb{N}$. $\mathsf{iI\Sigma_1}$ is closed under $[\psi]_{\Box_{\mathsf{iI\Sigma_1^+} \leq \overline{n}}}(\_)$. This is verifiable in $\mathsf{iI\Sigma_1}$.

Proof. By Lemma 161 and soundness of $\mathsf{iI\Sigma_1}$, we have that $\mathrm{HB}_{\mathsf{iI\Sigma_1}, \mathsf{iI\Sigma_1} \leq n, \Box_{\mathsf{iI\Sigma_1} \leq \overline{n}}}$. Then, for non-induction axioms, since these are the universal closure of Δ_0-formulas, we just need to use Lemmas 83 and 86. For the induction axioms we just need to use Lemmas 84 and 87.

For the verifiability in $\mathsf{iI\Sigma_1}$, just use Lemma 161 without applying soundness and the sames lemmas about the De Jong's translation, since they are verifiable in $\mathsf{iI\Sigma_1}$. □

Corollary 169. Let ϕ, ψ be sentences and $n \in \mathbb{N}$. Then

$$\vdash_{\mathsf{iI\Sigma_1}} \Box_{\mathsf{iI\Sigma_1}} \phi \to \Box_{\mathsf{iI\Sigma_1}} \left([\psi]_{\Box_{\mathsf{iI\Sigma_1^+} \leq \overline{n}}} \phi \right).$$

This is verifiable in $\mathsf{iI\Sigma_1}$.

Proof. Using Lemmas 161 and 168 to get the hypothesis of Corollary 92, we get that

$$\vdash_{\mathsf{iI\Sigma_1}} \forall A, B, x \,.\, \mathsf{sent}(B) \wedge \mathsf{prov}_{\mathsf{iI\Sigma_1^+} \leq x}(A) \to \mathsf{prov}_{\mathsf{iI\Sigma_1}}([B]_{\Box^{\bullet}_{\mathsf{iI\Sigma_1^+} \leq \dot{x}}} A). \qquad \text{(i)}$$

If ϕ, ψ are sentences and $n \in \mathbb{N}$, instantiation A with $\overline{\phi}$, B with $\overline{\psi}$ and x with $\overline{n}$ we get

$$\vdash_{\mathsf{iI\Sigma_1}} \Box_{\mathsf{iI\Sigma_1^+} \leq \overline{n}} \phi \to \Box_{\mathsf{iI\Sigma_1}} ([\psi]_{\Box_{\mathsf{iI\Sigma_1^+} \leq \overline{n}}} \phi),$$

as desired.

From (i), using $\mathrm{HB}_{\mathsf{iI\Sigma_1}, \Box_{\mathsf{iI\Sigma_1}}}$ and Lemma 28, we get the verifiability in $\mathsf{iI\Sigma_1}$. □

Theorem 170. Let $n \in \mathbb{N}$ and ϕ, ψ be sentences such that $\phi \in \mathrm{Ax}_{\mathsf{iI\Sigma_1^+} \leq \overline{n}}$. Then $\vdash_{\mathsf{iI\Sigma_1^+}} [\psi]_{\Box_{\mathsf{iI\Sigma_1^+} \leq \overline{n}}} \phi$. This is verifiable in $\mathsf{iI\Sigma_1}$.

Proof. Thanks to Lemma 168, we know that we only need to worry about the axioms of shape $\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi$, where χ is a sentence. We need to show two things:

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}[\psi]_{\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}}(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi)\to[\psi]_{\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}}\chi, \tag{i}$$

and

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}(\psi\to\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi). \tag{ii}$$

(ii) is trivial thanks to $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1,\mathsf{i}\mathsf{I}\Sigma_1^+\leq n,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}}$ and to $\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi$ being an axiom of $\mathsf{i}\mathsf{I}\Sigma_1^+\leq n$. To show (i) we just need to use Corollary 169 and Lemma 87, in a similar way we did in the proof of this theorem for Visser's translation (Theorem 164).

This proof can be carried in $\mathsf{i}\mathsf{I}\Sigma_1$ straightforwardly, since one the results we are $\mathsf{i}\mathsf{I}\Sigma_1$ verifiable. $\Box$

5.3 Σ_1-provability logic

Finally, using the results of the previous sections, we are able to calculate the Σ_1-provability logic of $\mathsf{i}\mathsf{I}\Sigma_1^+$.

The following lemma just expresses that $\mathsf{i}\mathsf{I}\Sigma_1^{++}$ proves the sentential reflection principles for $\mathsf{i}\mathsf{I}\Sigma_1^+$ for the subtheories $\mathsf{i}\mathsf{I}\Sigma_1^+\leq n$. Note that this theories have all the axioms of $\mathsf{i}\mathsf{I}\Sigma_1$ but just a finite amount of the axioms of shape $\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\psi\to\psi$.

Lemma 171 (Reflection).

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\forall A,x\,.\,\mathsf{sent}(A)\to\mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1^+}(\Box^{\bullet}_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\dot{x}}A\to^{\bullet}A).$$

Proof. We show the unformalized version of this. From the proof it will become clear that it can be shown inside $\mathsf{i}\mathsf{I}\Sigma_1$, so also in $\mathsf{i}\mathsf{I}\Sigma_1^+$. So let us have a sentence ϕ and a number n. We have that

$$\begin{aligned}\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}\phi\to\Box_{\mathsf{i}\mathsf{I}\Sigma_1}&\left(\bigwedge_{\psi\in\{\chi\in\mathrm{Sent}\,\mid\,(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi)\leq n\}}\left(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\psi\to\psi\right)\to\phi\right)\\ &\to\bigwedge_{\psi\in\{\chi\in\mathrm{Sent}\,\mid\,(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi)\leq n\}}\left(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\psi\to\psi\right)\to\phi\quad&&(\text{axiom of }\mathsf{i}\mathsf{I}\Sigma_1^+)\\ &\to\phi&&(\text{axiom of }\mathsf{i}\mathsf{I}\Sigma_1^+)\end{aligned}$$

Let us explain why the first implication holds. We work inside $\mathsf{i}\mathsf{I}\Sigma_1^+$, assume p is a witness of $\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq\overline{n}}\phi$. Then, the same p is a witness of

$$\mathsf{der}_{\mathsf{i}\mathsf{I}\Sigma_1}\left(\left\{\overline{\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\psi\to\psi}\mid\psi\in\{\chi\in\mathrm{Sent}\mid(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi)\leq n\}\right\}^{\bullet},\overline{\phi}\right).$$

Using some propositional reasoning we obtain a witness q of

$$\mathsf{der}_{\mathsf{i}\mathsf{I}\Sigma_1}\left(\overline{\bigwedge_{\psi\in\{\chi\in\mathrm{Sent}\mid(\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\chi\to\chi)\leq n\}}\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\psi\to\psi}, \overline{\phi}\right).$$

Then, we can obtain a proof of the desired formula using the formalized deduction theorem. $\square$

Lemma 172 (Absorption). Let ϕ be a sentence, then

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}\phi\to\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}\phi.$$

Proof. Let ϕ be a sentence. We want to apply Theorem 124. We have that $\mathrm{HB}_{\mathsf{iEA},\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}}$ thanks to Corollary 159. So all we need to show is

$$\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\forall x\,.\,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq x}\phi\to\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}\phi.$$

But this is a consequence of Lemma 171. $\square$

Lemma 173. $\mathsf{i}\mathsf{I}\Sigma_1^+$ is sound.

Proof. $\mathsf{i}\mathsf{I}\Sigma_1$ is sound, so we only need to show that for any sentence ϕ, $\vDash_{\mathbb{N}}\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\phi\to\phi$. Assume that $\vDash_{\mathbb{N}}\Box_{\mathsf{i}\mathsf{I}\Sigma_1}\phi$, thanks to Lemma 42 we get that $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1}\phi$. Since $\mathsf{i}\mathsf{I}\Sigma_1$ is sound we conclude $\vDash_{\mathbb{N}}\phi$, as desired. $\square$

Theorem 174. We have that:

$$\Sigma_1\text{-}\mathbb{PL}\left(\left(\mathsf{i}\mathsf{I}\Sigma_1^+\right)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}},\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}\right)=\mathbb{PL}\left(\left(\mathsf{i}\mathsf{I}\Sigma_1^+\right)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}},\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}\right)=\mathsf{iGLC}.$$

Proof. We want to apply Theorem 116. Note that $\mathsf{i}\mathsf{I}\Sigma_1^+$ is Σ_1-complete, Δ_0-decidable and extends iEA. Also, it is sound by Lemma 173 and it is self Δ_0-arithmetizable by Corollary 159.

By Lemma 160 and thanks to soundness of $\mathsf{i}\mathsf{I}\Sigma_1$ we get that $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. We also have absortion, i.e. $\mathrm{Abs}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+},\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$, by Lemma 172. By Lemma 32 we also get that for any sentence ϕ, $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1^+}\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}\phi\to\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}\phi$. Finally, by Lemma 164, $\mathsf{i}\mathsf{I}\Sigma_1^+$ is closed under $(\_)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. $\square$

Lemma 175. We have that $\mathscr{A}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}}$.

Proof. Right to left direction is trivial by $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}}$ and Lemma 142.

For left to right direction our objective is to apply Lemma 153. The arithmetization condition is clear by Lemma 158. It is also clear that $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1}\forall A\,.\,\mathsf{ax}_{\mathsf{i}\mathsf{I}\Sigma_1^+}(A)\leftrightarrow\exists x\,.\,\mathsf{ax}_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq x}(A)$. By Lemma 161 we get $\vdash_{\mathsf{i}\mathsf{I}\Sigma_1}\forall x\,.\,\mathsf{HB}_{\mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1^+},\mathsf{prov}_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq x},\Box^{\bullet}_{\mathsf{i}\mathsf{I}\Sigma_1^+\leq x}}$.

Conditions related to closure under translations are fulfilled by lemmas 167 and 170. Finally, formalized reflection is thanks to Lemma 171. $\square$

Lemma 176. We have that $\mathscr{B}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+},\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$.

Proof. We show the two directions of $\mathscr{B}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+},\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. Left to right is proven just using that $\mathsf{i}\mathsf{I}\Sigma_1$ proves that $\mathsf{i}\mathsf{I}\Sigma_1^+$ is closed under Visser translation and Lemma 160 with Corollary 75

For the right to left direction we just apply Lemma 156. $\square$

Theorem 177.

$$\Sigma_1\text{-}\mathbb{PL}(\mathsf{i}\mathsf{I}\Sigma_1^+) = \{\phi \in \mathcal{L}_m \mid \phi^+ \in \mathsf{iGLC}\}.$$

Proof. Our objective is to apply Theorem 141. Note that $\mathsf{i}\mathsf{I}\Sigma_1^+$ is Σ_1-complete, Δ_0-decidable and, by Lemma 173, it is also sound. In addition, by Lemma 159 it is self Δ_0-arithmetizable.

By Lemma 160 and soundness of $\mathsf{i}\mathsf{I}\Sigma_1$ we get $\mathrm{HB}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. By Lemma 175 we get $\mathscr{A}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}}$ and by Lemma 176 we get $\mathscr{B}_{\mathsf{i}\mathsf{I}\Sigma_1^+,\Box_{\mathsf{i}\mathsf{I}\Sigma_1^+},\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$. By Theorem 164, we also have that $\mathsf{i}\mathsf{I}\Sigma_1^+$ is closed under $(\_)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}$.

With this we get that

$$\Sigma_1\text{-}\mathbb{PL}(\mathsf{i}\mathsf{I}\Sigma_1^+) = \left\{\phi \in \mathcal{L}_m \mid \phi^+ \in \Sigma_1\text{-}\mathbb{PL}\left(\left(\mathsf{i}\mathsf{I}\Sigma_1^+\right)_{\mathrm{V}}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}, \Box_{\mathsf{i}\mathsf{I}\Sigma_1^+}^{\Box_{\mathcal{S}\mathsf{i}\mathsf{I}\Sigma_1^+}}\right)\right\}.$$

But by Theorem 174 we get the desired result. $\square$

Chapter 6

Conclusions and future work

Conclusions

This work initiates the study of provability logic of subtheories of Heyting Arithmetic, after the advances made by Mojtahedi in [6], where he calculated the provability logic of Heyting Arithmetic. Due to restriction of time, we have mainly studied Σ_1-provability logic instead of full provability logic. We hope this work helps to solve in the future if the provability logic of subtheories of Heyting Arithmetic is as uniform as in the classical case.

In Chapter 2, we have defined some well known translations in Intuitionistic Arithmetic with some of its fundamental properties. This translations provides the basic conditions for the rest of the method.

In Chapter 3, we have lifted some conditions of the intuitionistic Solovay's construction from [13] to a more general setting. In particular, we have weakened the minimum theory from $\mathsf{iI}\Sigma_1$ to iEA (as it was already known that this change was possible) and we have rewritten the assumptions of some lemmas and theorems to use the Hilbert-Bernays conditions, since these offer a little of more generality. We have also used the construction of [12] to construct predicates with absorption (as it was suggested by Pakhomov in page 733, Remark 42 of [13]). Finally, we have used all of this to construct a logic, $(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}\mathsf{iPRA}}}$, whose ($\Sigma_1$-)provability logic is iGLC.

In Chapter 4, we have analyzed which properties of the NNIL algorithm allows us to calculate the Σ_1-provability logic of a theory T from the Σ_1-provability logic of $(T)_{\mathsf{V}}^{\Box_{\mathcal{S}T}}$.

Finally, in Chapter 5, we have constructed a theory for which the whole method is applicable, both the intuitionistic Solovay's construction and the NNIL algorithm part. For this, we have added a principle that during the writing of Chapter 3 and Chapter 4 we found in essential parts of the proofs:

sentential reflection. With this, we have provided a theory whose Σ_1-provability logic is the same as the Σ_1-provability logic of HA.

Future work

Our initial aim was to calculate the Σ_1-provability logic of more mainstream arithmetical theories, such as iEA or $\mathsf{iI}\Sigma_1$. However, during our research we have found that the actual method is really hard to apply to these theories. The main problem is that both theories are finitely axiomatizable, and as we found sentential reflection for the finite subtheories seems to be essential in both parts of the methods. However, finite axiomatizable theories cannot show this principle, due to Gödel's second incompleteness theorem. Then, the biggest open question is to study a way of calculating the Σ_1-provability logic of finitely axiomatized subtheories of HA. After this study, it is unknown to us if they will have the same Σ_1-provability logic as HA or if the finite axiomatizability will change the Σ_1-provability logic in the intuitionistic setting.

A more amenable open question which is the calculation of the Σ_1-provability logic if iPRA. We have just applied the first half of the method, obtaining the Σ_1-provability of $(\mathsf{iPRA})_{\mathsf{V}}^{\Box_{\mathcal{S}}\mathsf{iPRA}}$. This was possible since for this part of the method only Σ_1-reflection is needed. For now, the NNIL part needs full sentential reflection. We believe that the NNIL part of the method can be improved and that then it will be applicable to iPRA, obtaining that it has the same Σ_1-provability logic as HA.

Finally, an easy endevour that was not done here due to lack of time is to generalize the construction of Chapter 5. In that chapter we add sentential reflection as axioms to $\mathsf{iI}\Sigma_1$. This was in order to create a theory distinct from HA for which the method of intuitionistic Solovay's construction + NNIL algorithm can be applied. It seems that the choice of $\mathsf{iI}\Sigma_1$ is largely non-fundamental and it was done just for matters of time and concreteness. Rewritting the section to generalize $\mathsf{iI}\Sigma_1$ to an arbitrary theory T and see what conditons are needed to apply the result of Chapter 5 can put some light over the method. Also, harder tasks, are to study if it is possible to obtain some information about the Σ_1-provability logic of T from the Σ_1-provability logic of $T + \mathrm{Rfn}_{\Box_T}$ and to study if the method of intuitionistic Solovay + NNIL is applicable to $T + \bigcup_{n\in\omega} \mathrm{Rfn}_{\Box_{T\leq\overline{n}}}$, which is a weaker theory than $T + \mathrm{Rfn}_{\Box_T}$

Appendix A

Proofs about translations

A.1 Gödel's and Friedman's Translations

Lemma 45. Let T be a Δ_0-decidable theory. Then for any $\phi \in \Delta_0$ we have that

$$\vdash_T (\phi)_{\mathrm{G}} \leftrightarrow \phi.$$

Proof. This is a simple induction in the shape of ϕ. □

Theorem 46. Let T be a theory closed under $(\_)_{\mathrm{G}}$ and let $\phi \in \mathcal{L}_1$. Then,

$$\vdash_{\mathsf{FOL},T} \phi \text{ implies } \vdash_{i\mathsf{FOL},T} (\phi)_{\mathrm{G}}.$$

This is $\mathsf{iI}\Sigma_1$ verifiable.

Proof. This is proved by a simple induction in the proof witnessing $\vdash_{\mathsf{FOL},T} \phi$. □

Lemma 49. Let ϕ, ψ be formulas such that no free variable of ψ occurs bounded in ϕ. Then:

$$\vdash \psi \to (\phi)^{\psi}_{\mathsf{F}}.$$

Proof. By induction in ϕ. If ϕ is atomic, a conjunction, or a disjunction the result is trivial.

Case $\phi = \phi_0 \to \phi_1$. By the induction hypothesis, we have that $\vdash \psi \to (\phi_1)^{\psi}_{\mathsf{F}}$. From this, $\vdash \psi \to (\phi_0)^{\psi}_{\mathsf{F}} \to (\phi_1)^{\psi}_{\mathsf{F}}$, i.e. $\vdash \psi \to (\phi_0 \to \phi_1)^{\psi}_{\mathsf{F}}$ as wanted.

Case $\phi = \forall x \,.\, \phi_0$. By assumption, we have that $x \notin \mathrm{fv}(\psi)$. By the induction hypothesis, we have $\vdash \psi \to (\phi_0)^{\psi}_{\mathsf{F}}$. Since $x \notin \mathrm{fv}(\psi)$ from this we can conclude that $\vdash \psi \to \forall x \,.\, (\phi_0)^{\psi}_{\mathsf{F}}$, i.e. $\vdash \psi \to (\forall x \,.\, \phi_0)^{\psi}_{\mathsf{F}}$ as wanted.

Case $\phi = \exists x \,.\, \phi_0$. By the induction hypothesis, we have that $\vdash \psi \to (\phi_0)^{\psi}_{\mathsf{F}}$. By iFOL reasoning, $\vdash (\phi_0)^{\psi}_{\mathsf{F}} \to \exists x \,.\, (\phi_0)^{\psi}_{\mathsf{F}}$. Putting these two facts together we get that $\vdash \psi \to \exists x \,.\, (\phi_0)^{\psi}_{\mathsf{F}}$, in other words, $\vdash \psi \to (\exists x \,.\, \phi_0)^{\psi}_{\mathsf{F}}$, as wanted. □

Lemma 50. Let T be a Δ_0-decidable theory. Then for any Δ_0-formula ϕ and any ψ, such that the free variables of ψ are not bounded in ϕ, we have

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}} \leftrightarrow \phi \vee \psi.$$

Proof. We proceed by induction on the shape of Δ_0 formulas. The case where ϕ is atomic is trivial.

Case $\phi = \phi_0 \vee \phi_1$. By the induction hypothesis, we have that

$$\vdash_T (\phi_i)^{\psi}_{\mathrm{F}} \leftrightarrow \phi_i \vee \psi.$$

So we have

$$\begin{aligned} \vdash_T (\phi_0 \vee \phi_1)^{\psi}_{\mathrm{F}} &= (\phi_0)^{\psi}_{\mathrm{F}} \vee (\phi_1)^{\psi}_{\mathrm{F}} \\ &\leftrightarrow (\phi_0 \vee \psi) \vee (\phi_1 \vee \psi) && \text{(by I.H.)} \\ &\leftrightarrow (\phi_0 \vee \phi_1) \vee \psi. \end{aligned}$$

Case $\phi = \phi_0 \wedge \phi_1$. By induction hypothesis, we have that

$$\vdash_T (\phi_i)^{\psi}_{\mathrm{F}} \leftrightarrow \phi_i \vee \psi.$$

So we have

$$\begin{aligned} \vdash_T (\phi_0 \wedge \phi_1)^{\psi}_{\mathrm{F}} &= (\phi_0)^{\psi}_{\mathrm{F}} \wedge (\phi_1)^{\psi}_{\mathrm{F}} \\ &\leftrightarrow (\phi_0 \vee \psi) \wedge (\phi_1 \vee \psi) && \text{(by I.H.)} \\ &\leftrightarrow (\phi_0 \wedge \phi_1) \vee \psi. \end{aligned}$$

Case $\phi = \phi_0 \to \phi_1$. Before proving this case we need a little lemma. If ϕ, χ are Δ_0-formulas and ψ is any formula, then

$$\vdash_T (\phi \vee \psi \to \chi \vee \psi) \leftrightarrow ((\phi \to \chi) \vee \psi). \tag{i}$$

The implication from right to left is easy, so we will only prove the other. Note that:

$$\begin{aligned} &\chi \vdash_T \phi \to \chi, \\ &\neg\phi \vdash_T \phi \to \chi, \\ &\phi \vee \psi \to \chi \vee \psi, \phi, \neg\chi \vdash_T \psi. \end{aligned}$$

From these we can conclude that:

$$\phi \vee \psi \to \chi \vee \psi, \phi \vee \neg\phi, \chi \vee \neg\chi \vdash_T (\phi \to \chi) \vee \psi.$$

But using that ϕ, χ are Δ_0, so $\vdash_T \phi \vee \neg\phi$ and $\vdash_T \chi \vee \neg\chi$, we can conclude that:

$$\phi \vee \psi \to \chi \vee \psi \vdash_T (\phi \to \chi) \vee \psi.$$

Now, we can show the implication case as follows. By the induction hypothesis, we have that

$$\vdash_T (\phi_i)^{\psi}_{\mathrm{F}} \leftrightarrow \phi_i \vee \psi.$$

So

$$\begin{aligned}\vdash_T (\phi_0 \to \phi_1)^{\psi}_{\mathrm{F}} &= (\phi_0)^{\psi}_{\mathrm{F}} \to (\phi_1)^{\psi}_{\mathrm{F}} \\ &\leftrightarrow \phi_0 \vee \psi \to \phi_1 \vee \psi && \text{(by I.H.)} \\ &\leftrightarrow (\phi_0 \to \phi_1) \vee \psi. && \text{(by (i))}\end{aligned}$$

Case $\phi = \forall x \leq \tau \,.\, \phi_0$. We assume that $x \notin \mathrm{fv}(\psi)$ and by induction hypothesis we have that

$$\vdash_T (\phi_0)^{\psi}_{\mathrm{F}} \leftrightarrow \phi_i \vee \psi.$$

We will use the lemma (i) proven in the previous case. Also we need the following lemma: let ϕ be a Δ_0-formula, ψ be a formula and τ a term such that $x \notin \mathrm{fv}(\psi, \tau)$, we have that

$$\vdash_T (\forall x \,.\, (x \leq \tau \to \phi) \vee \psi) \leftrightarrow (\forall x \leq \tau \,.\, \phi) \vee \psi. \tag{ii}$$

Right to left is trivial. For left to right note that $\forall x \leq \tau \,.\, \phi$ is a Δ_0-formula so we know that $\vdash_T (\forall x \leq \tau \,.\, \phi) \vee \neg(\forall x \leq \tau \,.\, \phi)$. Also, $\vdash_T \neg(\forall x \leq \tau \,.\, \phi) \to \exists x \leq \tau \,.\, \neg\phi$ which holds because we can do cases in $\exists x \leq \tau \,.\, \neg\phi$ (since it is a Δ_0-formula) and since in intuitionistic logic we have that $\vdash \neg(\exists x \,.\, \chi) \to (\forall x \,.\, \neg\chi)$ (and also since $\vdash_T \phi \vee \neg\phi$ we have that $\vdash_T \neg\neg\phi \to \phi$). Now, if assume $\forall x \leq \tau \,.\, \phi$ the left to right is trivial, otherwise we can assume the negation which implies $\exists x \leq \tau \,.\, \neg\phi$, but note that:

$$x \leq \tau \wedge \neg\phi, \forall x \,.\, (x \leq \tau \to \phi) \vee \psi \vdash_T \psi.$$

But since $x \notin \mathrm{fv}(\psi)$ we have that

$$\exists x \leq \tau \,.\, \neg\phi, \forall x \,.\, (x \leq \tau \to \phi) \vee \psi \vdash_T \psi.$$

With the lemmas (i) and (ii) we are finally able to prove this case:

$$\begin{aligned}\vdash_T (\forall x \leq \tau \,.\, \phi_0)^{\psi}_{\mathrm{F}} &= \big(\forall x \,.\, (x \leq \tau)^{\psi}_{\mathrm{F}} \to (\phi)^{\psi}_{\mathrm{F}}\big) \\ &= \big(\forall x \,.\, x \leq \tau \vee \psi \to (\phi)^{\psi}_{\mathrm{F}}\big) \\ &\leftrightarrow \big(\forall x \,.\, x \leq \tau \vee \psi \to \phi \vee \psi\big) && \text{(by I.H.)} \\ &\leftrightarrow \big(\forall x \,.\, (x \leq \tau \to \phi) \vee \psi\big) && \text{(by (i))} \\ &\leftrightarrow \big((\forall x \leq \tau \,.\, \phi) \vee \psi\big). && \text{(by (ii))}\end{aligned}$$

Case $\phi = \exists x \leq \tau \,.\, \phi_0$. By the induction hypothesis, we have that:

$$\vdash_T (\phi_0)^{\psi}_{\mathrm{F}} \leftrightarrow \phi_0 \vee \psi.$$

Then:

$$\begin{aligned}
\vdash_T (\exists x \leq \tau \,.\, \phi_0)^{\psi}_{\mathrm{F}} &= \big(\exists x \,.\, (x \leq \tau \vee \psi) \wedge (\phi_0)^{\psi}_{\mathrm{F}}\big) \\
&\leftrightarrow \big(\exists x \,.\, (x \leq \tau \vee \psi) \wedge (\phi_0 \vee \psi)\big) && \text{(by I.H.)} \\
&\leftrightarrow \big(\exists x \,.\, (x \leq \tau \wedge \phi_0) \vee \psi\big) \\
&\leftrightarrow \big((\exists x \leq \tau \,.\, \phi_0) \vee \psi\big).
\end{aligned}$$

Where in the last equivalence we used that x is not free in ψ. □

Lemma 51. Let T be a Δ_0-decidable theory. Then, for any Σ_1-formula ϕ and any ψ, such that no free variable of ψ appears bounded in ϕ, we have that:

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}} \leftrightarrow \phi \vee \psi.$$

Proof. Let ϕ be a Σ_1-formula. Then it is of shape $\exists x \,.\, \phi_0$ where ϕ_0 is a Δ_0-formula. We have that:

$$\begin{aligned}
\vdash_T (\exists x \,.\, \phi_0)^{\psi}_{\mathsf{F}} &= (\exists x \,.\, (\phi_0)^{\psi}_{\mathsf{F}}) \\
&\leftrightarrow \exists x \,.\, \phi_0 \vee \psi && \text{(by lemma 50)} \\
&\leftrightarrow (\exists x \,.\, \phi_0) \vee \psi. && \text{(since } x \notin \mathrm{fv}(\psi))
\end{aligned}$$

□

Lemma 52. For any formulas ϕ, ψ, such that no free variable of ψ appears bounded in ϕ:

1. $\mathrm{fv}((\phi)^{\psi}_{\mathrm{F}}) = \mathrm{fv}(\phi) \cup \mathrm{fv}(\psi)$.

2. If x is free for τ in ϕ, ψ then x is free for τ in $(\phi)^{\psi}_{\mathrm{F}}$.

Proof. By induction in ϕ both statements are easy to show. □

Lemma 53. Let ϕ, ψ be formulas, τ a term and x a variable. Assume that no free variable of ψ occurs bounded in ϕ and $x \notin \mathrm{fv}(\psi)$. Then

$$(\phi[x/\tau])^{\psi}_{\mathsf{F}} = (\phi)^{\psi}_{\mathsf{F}}[x/\tau].$$

Proof. By induction in ϕ. □

Theorem 54. Let ϕ, ψ be formulas. Assume that we have a proof π of $\vdash \phi$ such that no free variables of ψ appears bounded in the formulas of π and that T is closed under $(\_)^{\psi}_{\mathrm{F}}$. Then

$$\vdash_T (\phi)^{\psi}_{\mathrm{F}}.$$

Proof. We do induction in the proof of $\vdash \phi$. We need to check that for logical axioms ϕ we have that $\vdash \phi^{\psi}$. We consider each logical axiom subsequently.

Assume we have the logical axiom $\perp \to \phi$, after the translation it is $\perp \vee \psi \to (\phi)^{\psi}_{\mathrm{F}}$. Thanks to lemma 49 we have that $\vdash \psi \to (\phi)^{\psi}_{\mathrm{F}}$, so we find the desired formula.

The other propositional axioms are easy since they are axiom schemas, and after the translation we obtain another formula of the axiom schema.

Assume we have the logical axiom $(\forall x.\phi) \to \phi[x/\tau]$ where τ is free for x in ϕ. After the translation, we have the formula $(\forall x.(\phi)^{\psi}_{\mathrm{F}}) \to (\phi[x/\tau])^{\psi}_{\mathrm{F}}$. By assumption we have that $x \notin \mathrm{fv}(\psi)$ so we can apply lemma 53 and then it is enough to show that:

$$\vdash (\forall x.(\phi)^{\psi}_{\mathrm{F}}) \to (\phi)^{\psi}_{\mathrm{F}}[x/\tau],$$

but this is just an instance of the same logical axiom.

Assume we have the logical axiom $(\forall x.\phi_0 \to \phi_1) \to \phi_0 \to (\forall y.\phi_1[x/y])$ where $x \notin \mathrm{fv}(\phi_0)$ and ($y = x$ or $y \notin \mathrm{fv}(\phi_1)$). After the translation we have that formula:

$$(\forall x.(\phi_0)^{\psi}_{\mathrm{F}} \to (\phi_1)^{\psi}_{\mathrm{F}}) \to (\phi_0)^{\psi}_{\mathrm{F}} \to (\forall y.(\phi_1)^{\psi}_{\mathrm{F}}[x/y]),$$

where we used that $x \notin \mathrm{fv}(\psi)$ with lemma 53. Since we assume that in the proof no bounded variable appears in ψ, we have that $x, y \notin \mathrm{fv}(\psi)$. Then, $x \notin \mathrm{fv}(\phi_0) \cup \mathrm{fv}(\psi) = \mathrm{fv}((\phi_0)^{\psi}_{\mathrm{F}})$ by Lemma 52. Now, we have that either $y = x$ or $y \notin \mathrm{fv}(\phi_1)$, so again by Lemma 52 $y \notin \mathrm{fv}((\phi_1)^{\psi}_{\mathrm{F}})$. In any case, after the translation the formula is just another instance of the same logical axiom.

The existential axioms have a similar proof to the universal axioms.

Finally, let us treat about the equality axioms. However, note that $(\forall x.x \approx x)^{\psi}_{\mathrm{F}} = (\forall x.x \approx x \vee \psi)$ is easily provable. Let us show an example of the functional equality axiom and other of the relational equality axiom. Let $f(x, y)$ be a binary function symbol, we have the axiom

$$\forall x, y, z.y \approx z \to f(x, y) \approx f(x, z).$$

After the translation we get:

$$\forall x, y, z.y \approx z \vee \psi \to f(x, y) \approx f(x, z) \vee \psi,$$

but this is clearly provable by doing cases in the disjunction of the antecedent of the implication. Let $R(x, y)$ be a binary relation symbol and assume we have the axiom

$$\forall x, y, z.y \approx z \to (R(x, y) \leftrightarrow R(x, z)).$$

After the translation it is

$$\forall x, y, z. y \approx z \vee \psi \to (R(x,y) \vee \psi \leftrightarrow R(x,z) \vee \psi)$$

It suffices that we show that the following formula is provable

$$\forall x, y, z. y \approx z \vee \psi \to (R(x,y) \vee \psi \to R(x,z) \vee \psi)$$

But note that:

$$y \approx z, R(x,y) \vdash R(x,z)$$

thanks to the equality axiom. By doing cases in the disjunctions of the antecedent of the implication the result is easy to prove.

Now, assume that $\vdash \phi$ where there is a χ such that $\vdash \chi$ and $\vdash \chi \to \phi$ with shorter proofs. By induction hypotheis $\vdash (\chi)^{\psi}_{\mathrm{F}}$ and $\vdash (\chi)^{\psi}_{\mathrm{F}} \to (\phi)^{\psi}_{\mathrm{F}}$. So by applying MP we have that $\vdash (\phi)^{\psi}_{\mathrm{F}}$, as wanted.

Finally, assume that $\vdash \forall x.\phi$ where there is a shorter proof of $\vdash \phi$. Then by induction hypothesis $\vdash (\phi)^{\psi}_{\mathrm{F}}$. Finally, we can conclude using generalization that $\vdash \forall x.(\phi)^{\psi}_{\mathrm{F}}$, in other words, $\vdash (\forall x.\phi)^{\psi}_{\mathrm{F}}$. $\square$

Corollary 56. Let T be closed under $(\_)^{\psi}_{\mathrm{F}}$. If we have a proof π of $\vdash_T \phi_0 \leftrightarrow \phi_1$, where no free variable of ψ appears bounded in π, then $\vdash_T (\phi_0)^{\psi}_{\mathrm{F}} \leftrightarrow (\phi_1)^{\psi}_{\mathrm{F}}$.

Proof. Just applying Corollary 55. $\square$

Π_2-conservativity over classical theory

Lemma 58. Let T be a theory such that it is closed under Gödel's translation and under Friedman's translation for Σ_1-formulas. Then, T with classical logic is Π_2-conservative over T with intuitionistic logic.

Proof. First, let us show that for any $\phi \in \Delta_0$ we have that

$$\vdash_{\mathsf{iFOL},T} \neg\neg\exists y \,.\, \phi \text{ implies } \vdash_{\mathsf{iFOL},T} \exists y \,.\, \phi.$$

Note that we can assume that no free variable of $\exists y \,.\, \phi$ occurs bounded in the proof of $\vdash_{\mathsf{iFOL},T} \neg\neg\exists y \,.\, \phi$ since we can suitably rename bounded variables. Then, by Theorem 54, we have that

$$\vdash_{\mathsf{iFOL},T} (\neg\neg\exists y \,.\, \phi)^{\exists y.\phi}_{\mathrm{F}},$$

equivalently

$$\vdash_{\mathsf{iFOL},T} ((\exists y \,.\, \phi)^{\exists y.\phi}_{\mathrm{F}} \to \exists y \,.\, \phi) \to \exists y \,.\, \phi.$$

Thanks to Lemma 51 we have that

$$\vdash_{\mathsf{iFOL},T} (\exists y \,.\, \phi)^{\exists y.\phi}_{\mathrm{F}} \leftrightarrow (\exists y \,.\, \phi) \vee (\exists y \,.\, \phi),$$

so we get

$$\vdash_{\mathsf{iFOL},T} (quant\exists y\phi \to \exists y\,.\,\phi) \to \exists y\,.\,\phi.$$

But the antecedent of the implication is clearly provable, so

$$\vdash_{\mathsf{iFOL},T} \exists y\,.\,\phi.$$

Now, assume that $\vdash_{\mathsf{FOL}} \forall x\,\exists y\,.\,\phi$ where $\phi \in \Delta_0$. Then $\vdash_{\mathsf{FOL}} \exists y\,.\,\phi$. By Theorem 46 and Lemma 45,

$$\vdash_{\mathsf{iFOL},T} \neg\neg\exists y\,.\,\phi,$$

but thanks to previous result

$$\vdash_{\mathsf{iFOL},T} \exists y\,.\,\phi.$$

By generalization we get the desired

$$\vdash_{\mathsf{iFOL},T} \forall x\,\exists y\,.\,\phi. \qquad \square$$

A.2 De Jongh Translation

Lemma 81. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi, \psi \in \mathcal{L}_1$ such that there is no free variable of ψ bounded in ϕ. Then:

$$\vdash_U [\psi]_\Box \phi \to \Box(\psi \to \phi).$$

Proof. By induction in ϕ. If ϕ is atomic, then we have that, by $\mathrm{HB}_{U,T,\Box}$.Compl, $\vdash_U \phi \to \Box\phi$. Also, by Lemma 27 and $\mathrm{HB}_{U,T,\Box}$, $\vdash_U \Box\phi \to \Box(\psi \to \phi)$, we can conclude $\vdash_U \phi \to \Box(\psi \to \phi)$.

Let $\phi = \phi_0 \wedge \phi_1$, by the induction hypothesis for $i = 0, 1$:

$$\vdash_U [\psi]_\Box \phi_i \to \Box(\psi \to \phi_i).$$

Then

$$\begin{aligned}
\vdash_T [\psi]_\Box(\phi_0 \wedge \phi_1) &= [\psi]_\Box \phi_0 \wedge [\psi]_\Box \phi_1 \\
&\to \Box(\psi \to \phi_0) \wedge \Box(\psi \to \phi_1) && \text{(by I.H.)} \\
&\to \Box((\psi \to \phi_0) \wedge (\psi \to \phi_1)) && \text{(by Lemma 27 and } \mathrm{HB}_{U,T,\Box}) \\
&\to \Box(\psi \to \phi_0 \wedge \phi_1). && \text{(by Lemma 27 and } \mathrm{HB}_{U,T,\Box})
\end{aligned}$$

Let $\phi = \phi_0 \vee \phi_1$, by the induction hypothesis for $i = 0, 1$:

$$\vdash_U [\psi]_\Box \phi_i \to \Box(\psi \to \phi_i).$$

Note that using Lemma 27 and $\mathrm{HB}_{U,T,\Box}$

$$\vdash_U \Box(\psi \to \phi_i) \to \Box(\psi \to \phi_0 \vee \phi_1),$$

Then

$$\begin{aligned}\vdash_U [\psi]_\Box(\phi_0 \vee \phi_1) &= [\psi]_\Box\phi_0 \vee [\psi]_\Box\phi_1 \\ &\to \Box(\psi \to \phi_0 \vee \phi_1).\end{aligned}$$

Let $\phi = \exists x.\phi_0$, by the induction hypothesis

$$\vdash_U [\psi]_\Box\phi_0 \to \Box(\psi \to \phi_0).$$

Then

$$\begin{aligned}\vdash_U [\psi]_\Box(\exists x\,.\,\phi_0) &= \exists x\,.\,[\psi]_\Box\phi_0 \\ &\to \exists x\,.\,\Box(\psi \to \phi_0) && \text{(by I.H.)} \\ &\to \Box(\exists x\,.\,\psi \to \phi_0) && \text{(by Lemma 27 and } \mathrm{HB}_{U,T,\Box}) \\ &\to \Box(\psi \to \exists x\,.\,\phi_0). && \text{(by Lemma 27 and } \mathrm{HB}_{U,T,\Box})\end{aligned}$$

In the last implication we used that $x \notin \mathrm{fv}(\psi)$.

The cases of implication and universal quantification are trivial. □

Lemma 82. Assume that $\mathrm{HB}_{U,T,\Box}$. Then we have that:

$$\vdash_U [\chi]_\Box\left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right) \leftrightarrow \left(\bigwedge_{i=0}^{m}[\chi]_\Box\phi_i \to [\chi]_\Box\psi_i\right) \wedge \Box\left(\chi \to \left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right)\right).$$

Proof. By induction in m. If $m = 0$, then it is trivial by definition. Assume it is true for m, we need to show the case $m+1$. We have:

$$\begin{aligned}\vdash_U [\chi]_\Box\left(\bigwedge_{i=0}^{m+1}\phi_i \to \psi_i\right) &= [\chi]_\Box\left(\left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right) \wedge (\phi_{m+1} \to \psi_{m+1})\right) \\ &= [\chi]_\Box\left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right) \wedge [\chi]_\Box(\phi_{m+1} \to \psi_{m+1}) \\ &\leftrightarrow \left(\bigwedge_{i=0}^{m}[\chi]_\Box\phi_i \to [\chi]_\Box\psi_i\right) \wedge \Box\left(\chi \to \left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right)\right) \\ &\qquad \wedge ([\chi]_\Box\phi_{m+1} \to [\chi]_\Box\psi_{m+1}) \wedge \Box(\chi \to \phi_{m+1} \to \psi_{m+1}) \\ &\leftrightarrow \left(\bigwedge_{i=0}^{m+1}[\chi]_\Box\phi_i \to [\chi]_\Box\psi_i\right) \\ &\qquad \wedge \Box\left(\left(\chi \to \left(\bigwedge_{i=0}^{m}\phi_i \to \psi_i\right)\right) \wedge (\chi \to \phi_{m+1} \to \psi_{m+1})\right) \\ &\leftrightarrow \left(\bigwedge_{i=0}^{m+1}[\chi]_\Box\phi_i \to [\chi]_\Box\psi_i\right) \wedge \Box\left(\chi \to \left(\bigwedge_{i=0}^{m+1}\phi_i \to \psi_i\right)\right)\end{aligned}$$

□

Lemma 83. Let $\mathrm{HB}_{U,T,\Box}$ and $x_0,\dots,x_m$ be variables not free in χ. Then we have that:

$$\vdash_U [\chi]_\Box(\forall x_0,\dots,x_m.\phi) \leftrightarrow (\forall x_0,\dots,x_m.[\chi]_\Box\phi) \wedge \Box(\chi \to \forall x_0,\dots,x_m.\phi).$$

Proof. By induction in m. If $m=0$ it is trivial by definition. Now, assume that it is true for m let us show the case for $m+1$.

$$\begin{aligned}
\vdash_U [\chi]_\Box(\forall x_0,\dots,x_{m+1}.\phi) &= (\forall x_0.[\chi]_\Box(\forall x_1,\dots,x_{m+1}.\phi)) \wedge \Box(\chi \to \forall x_0,\dots,x_{m+1}.\phi)\\
&\leftrightarrow (\forall x_0,\dots,x_{m+1}.[\chi]_\Box\phi) \wedge (\forall x_0.\Box(\chi \to \forall x_1,\dots,x_{m+1}.\phi))\\
&\quad \wedge \Box(\chi \to \forall x_0,\dots,x_{m+1}.\phi)\\
&\leftrightarrow (\forall x_0,\dots,x_{m+1}.[\chi]_\Box\phi) \wedge \Box(\chi \to \forall x_0,\dots,x_{m+1}.\phi)
\end{aligned}$$

For the last equivalence, to show right to left we used $\mathrm{HB}_{U,T,\Box}$ and $x_0 \notin \mathrm{fv}(\chi)$. $\Box$

Lemma 84. Let $\mathrm{HB}_{U,T,\Box}$ and assume that $x \notin \mathrm{fv}(\chi)$. Then

$$\vdash_U [\chi]_\Box(\forall x.\phi \to \psi) \leftrightarrow (\forall x.[\chi]_\Box\phi \to [\chi]_\Box\psi) \wedge \Box(\chi \to \forall x.\phi \to \psi).$$

Proof.

$$\begin{aligned}
\vdash_U [\chi]_\Box(\forall x.\phi \to \psi) &= (\forall x.[\chi]_\Box(\phi \to \psi)) \wedge \Box(\chi \to \forall x.\phi \to \psi)\\
&= (\forall x.[\chi]_\Box\phi \to [\chi]_\Box\psi) \wedge (\forall x.\Box(\chi \to \phi \to \psi)) \wedge \Box(\chi \to \forall x.\phi \to \psi)\\
&\leftrightarrow (\forall x.[\chi]_\Box\phi \to [\chi]_\Box\psi) \wedge \Box(\chi \to \forall x.\phi \to \psi)
\end{aligned}$$

For the last equivalence we use that $\mathrm{HB}_{U,T,\Box}$ and that $x \notin \mathrm{fv}(\chi)$. $\Box$

Lemma 85. Let $\mathrm{HB}_{U,T,\Box}$, then

$$\vdash_U [\chi]_\Box(\phi_m \to \cdots \to \phi_0) \leftrightarrow ([\chi]_\Box\phi_m \to \cdots \to [\chi]_\Box\phi_0) \wedge \Box(\chi \to \phi_m \to \cdots \to \phi_0)$$

Proof. If $m=0$ it is trivial by definition, so let us show the inductive case. By the induction hypothesis we have that

$$\vdash_U [\chi]_\Box(\phi_m \to \cdots \to \phi_0) \leftrightarrow ([\chi]_\Box\phi_m \to \cdots \to [\chi]_\Box\phi_0) \wedge \Box(\chi \to \phi_m \to \cdots \to \phi_0).$$

Then

$$\begin{aligned}
\vdash_U [\chi]_\Box(\phi_{m+1} \to \cdots \to \phi_0) &= ([\chi]_\Box\phi_{m+1} \to [\chi]_\Box(\phi_m \to \cdots \to \phi_0)) \wedge \Box(\chi \to \phi_{m+1} \to \cdots \phi_0)\\
&\leftrightarrow \Big([\chi]_\Box\phi_{m+1} \to ([\chi]_\Box\phi_m \to \cdots \to [\chi]_\Box\phi_0) \wedge \Box(\chi \to \phi_m \to \cdots \to \phi_0)\Big)\\
&\quad \wedge \Box(\chi \to \phi_{m+1} \to \cdots \to \phi_0) && \text{(by I.H.)}\\
&\leftrightarrow ([\chi]_\Box\phi_{m+1} \to [\chi]_\Box\phi_m \to \cdots \to [\chi]_\Box\phi_0)\\
&\quad \wedge ([\chi]_\Box\phi_{m+1} \to \Box(\chi \to \phi_m \to \cdots \to \phi_0))\\
&\quad \wedge \Box(\chi \to \phi_{m+1} \to \cdots \to \phi_0) && \text{(by iFOL reasoning)}\\
&\leftrightarrow^* ([\chi]_\Box\phi_{m+1} \to \cdots \to [\chi]_\Box\phi_0) \wedge \Box(\chi \to \phi_{m+1} \to \cdots \to \phi_0)
\end{aligned}$$

Let us show $\leftrightarrow^*$ in detail. Left to right is trivial, to show right to left let us prove that

$$\Box(\chi \to \phi_{m+1} \to \cdots \to \phi_0), [\chi]_\Box \phi_{m+1} \vdash_U \Box(\chi \to \phi_m \to \cdots \to \phi_0). \tag{i}$$

Note that by Lemma 81 we have that

$$[\chi]_\Box \phi_{m+1} \vdash_U \Box(\chi \to \phi_{m+1}).$$

Then, using Lemma 27 and $\mathrm{HB}_{U,T,\Box}$

$$\Box(\chi \to \phi_{m+1} \to \cdots \to \phi_0), [\chi]_\Box \phi_{m+1} \vdash_U \Box(\chi \to \phi_{m+1} \wedge (\phi_{m+1} \to \cdots \to \phi_0)) \tag{ii}$$

By another application of Lemma 27 and $\mathrm{HB}_{U,T,\Box}$, we get

$$\Box(\chi \to \phi_{m+1} \wedge (\phi_{m+1} \to \cdots \to \phi_0)) \vdash_U \Box(\chi \to \phi_m \to \cdots \to \phi_0). \tag{iii}$$

Clearly (ii) and (iii) gives (i). $\Box$

Lemma 86. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi \in \Delta_0$, such that no free variable of χ appears bounded in ϕ. Then

$$\vdash_U \phi \leftrightarrow [\chi]_\Box \phi.$$

Proof. By induction in ϕ. If ϕ is an atom it is trivial. Conjunction, disjunction and bounded existential are easy applying the induction hypothesis.

Case $\phi = \phi_0 \to \phi_1$. By the induction hypothesis

$$\vdash_U \phi_i \leftrightarrow [\chi]_\Box \phi_i.$$

Then

$$\begin{aligned} \vdash_U [\chi]_\Box(\phi_0 \to \phi_1) &= ([\chi]_\Box \phi_0 \to [\chi]_\Box \phi_1) \wedge \Box(\chi \to \phi_0 \to \phi_1) \\ &\leftrightarrow (\phi_0 \to \phi_1) \wedge \Box(\chi \to \phi_0 \to \phi_1) && \text{(by I.H.)} \\ &\leftrightarrow (\phi_0 \to \phi_1), \end{aligned}$$

where we used:

$$\begin{aligned} \vdash_U (\phi_0 \to \phi_1) &\to \Box(\phi_0 \to \phi_1) && (\Sigma_1\text{-completeness}) \\ &\to \Box(\chi \to \phi_0 \to \phi_1). && (\text{by Lemma 27 and } \mathrm{HB}_{U,T,\Box}) \end{aligned}$$

Case $\phi = \forall x < \tau.\phi_0$. By Lemma 84, since $x \notin \mathrm{fv}(\chi)$, it suffices to show that

$$\vdash_U \forall x < \tau.[\chi]_\Box \phi_0 \wedge \Box(\chi \to \forall x < \tau.\phi_0).$$

By the induction hypothesis,

$$\vdash_U \phi_0 \leftrightarrow [\chi]_\Box \phi_0.$$

Then

$$\vdash_U (\forall x < \tau.\phi_0) \leftrightarrow (\forall x < \tau.[\chi]_\Box \phi_0).$$

All left to show is that

$$\vdash_U (\forall x < \tau.\phi_0) \to \Box(\chi \to \forall x < \tau.\phi_0).$$

But this is trivial using $\mathrm{HB}_{U,T,\Box}$ and Lemma 27. Note that in calculating the translation we are using Lemma 84. $\Box$

Lemma 87. Assume that $\mathrm{HB}_{U,T,\Box}$. Let $\phi \in \Sigma_1$, such that no free variable of χ appears bounded in ϕ. Then

$$\vdash_U \phi \leftrightarrow [\chi]_\Box \phi.$$

Proof. Applying previous lemma. $\Box$

Lemma 88. We have that

1. $\mathrm{fv}([\chi]_\Box \phi) \subseteq \mathrm{fv}(\chi) \cup \mathrm{fv}(\phi)$.
2. x is free for τ in ϕ iff x is free for τ in $[\chi]_\Box \phi$.

Proof. The first point is a simple induction in ϕ. Let us show the second point, also by induction in ϕ. If it is an atom it is trivial, conjunction and disjunction are trivial by the induction hypothesis. For the implication case it suffices to apply the induction hypothesis and note that for any formula ϕ, x is free for any term in $\Box\phi$ (here we assume that we rename the bound variables of the formula defining $\Box$ if necessary). Now, let $\phi = \exists y.\phi_0$, and assume that x is free for τ in ϕ and $x \notin \mathrm{fv}(\chi)$. If $x = y$, the result trivially holds since $[\psi]_\Box \exists y.\phi_0 = \exists y.[\psi]_\Box \phi_0$. If $x \neq y$ we have that x is free for τ in ϕ_0 and $y \notin \mathrm{fv}(\tau)$. By induction hypothesis we have that x is free for τ in $[\chi]_\Box \phi_0$, so it is also free for τ in $\exists y.[\chi]_\Box \phi_0 = [\chi]_\Box \exists y.\phi_0$.

The case where $\phi = \forall x.\phi_0$ is as $\exists x.\phi$, but again we need to use that x is free for any term in $\Box(\chi \to \forall y.\phi_0)$ (module renaming bounded variables of the formula defining $\Box$). $\Box$

Lemma 89. Assume that $\mathrm{HB}_{U,T,\Box}$. Let x be free for τ in ϕ, and $x \notin \mathrm{fv}(\chi)$. Then

$$\vdash_U ([\chi]_\Box \phi)[x/\tau] \leftrightarrow [\chi]_\Box \phi[x/\tau].$$

Proof. We proceed by induction in ϕ. If ϕ is atomic the result is trivial. Conjunction and disjunction are easy using the induction hypothesis. Let $\phi = \phi_0 \to \phi_1$, by the induction hypothesis we have that:

$$\vdash_U ([\chi]_\Box \phi_i)[x/\tau] \leftrightarrow [\chi]_\Box \phi_i[x/\tau].$$

Also note that:

$$\begin{aligned} \vdash_U \Box(\chi \to \phi_0 \to \phi_1)[x/\tau] &\leftrightarrow \Box((\chi \to \phi_0 \to \phi_1)[x/\tau]) && \text{(by Lemma 68)} \\ &= \Box(\chi \to \phi_0[x/\tau] \to \phi_1[x/\tau]) && (\text{since } x \notin \mathrm{fv}(\chi)) \end{aligned}$$

Let us call this equivalence (i). Then:

$$\begin{aligned} \vdash_U [\chi]_\Box(\phi_0 \to \phi_1)[x/\tau] &= ([\chi]_\Box \phi_0[x/\tau] \to [\chi]_\Box \phi_1[x/\tau]) \wedge \Box(\chi \to \phi_0 \to \phi_1)[x/\tau] \\ &\leftrightarrow ([\chi]_\Box \phi_0[x/\tau] \to [\chi]_\Box \phi_1[x/\tau]) \wedge \Box(\chi \to \phi_0[x/\tau] \to \phi_1[x/\tau]) \\ &\qquad \text{(by I.H. and (i))} \\ &= [\chi]_\Box(\phi_0[x/\tau] \to \phi_1[x/\tau]) \\ &= [\chi]_\Box((\phi_0 \to \phi_1)[x/\tau]) \end{aligned}$$

If $\phi = \exists y.\phi_0$ we have two cases. If $x = y$ the result is trivial, if $x \neq y$ it suffices to apply the inductive hypothesis. Note that to do this is fundamental that if x is free for τ in $\exists y.\phi_0$ and $x \neq y$ then x is free for τ in ϕ_0.

Finally, if $\phi = \forall y.\phi_0$ is a mixture between the existential and the implication case. First, if $x = y$ the result is trivial and if $x \neq y$ we proceed similarly to the implication case. Again, we need to use the induction hypothesis which we can apply since if $x \neq y$ and x is free for τ in $\forall y.\phi_0$ we can conclude that x is free for τ in ϕ_0. $\square$

Theorem 90. Let χ be a formula, let T be such that if $\phi \in \mathrm{Ax}_T$ then $\vdash_U [\chi]_\Box(\phi)$ and $\mathrm{HB}_{U,T,\Box}$. Then if π is a proof of $\vdash_T \phi$ where no free variable of χ appears bounded, we have that

$$\vdash_U [\chi]_\Box \phi.$$

Proof. By induction in the proof of $\vdash_T \phi$. If ϕ is a non-logical axiom we have the desired result by hypothesis. Let us prove the case where ϕ is a logical axiom. For that we check all the logical axioms individually.

1. Case $\bot \to \phi$. After the translation we have the formula

   $$(\bot \to [\chi]_n \phi) \wedge \Box_n(\chi \to \bot \to \phi).$$

   The left conjunct is another instance of the axiom, so it is provable. The right conjunct is provable thanks to $\mathrm{HB}_{U,T,\Box}$ and Lemma 27.

2. Case $\phi \wedge \psi \to \phi$. Thanks to Lemma 85 it suffices to show

   $$\vdash_U ([\chi]_\Box \phi \wedge [\chi]_\Box \psi \to [\chi]_\Box \phi) \wedge \Box(\chi \to \phi \wedge \psi \to \phi).$$

   The left conjunct is another instance of the axiom, so it is clearly T-provable. For the right conjunct we just need to use $\mathrm{HB}_{U,T,\Box}$.

3. Case $\phi \to \psi \to \phi \wedge \psi$. Similar proof to the two first cases.

4. Case $\phi \to \phi \vee \psi$. Similar proof to the two first cases.

5. Case $(\phi_0 \to \psi) \to (\phi_1 \to \psi) \to (\phi_0 \vee \phi_1 \to \psi)$. Thanks to Lemma 85 it suffices to show that

   $$\vdash_U [\chi]_\Box(\phi_0 \to \psi) \to [\chi]_\Box(\phi_1 \to \psi) \to [\chi]_\Box(\phi_0 \vee \phi_1 \to \psi) \tag{i}$$

   and

   $$\vdash_U \Box(\chi \to (\phi_0 \to \psi) \to (\phi_1 \to \psi) \to (\phi_0 \vee \phi_1 \to \psi)) \tag{ii}$$

   (ii) is clearly provable using $\mathrm{HB}_{U,T,\Box}$. Let us show (i). But, to show (i) it suffices to show:

   $$\vdash_U ([\chi]_\Box \phi_0 \to [\chi]_\Box \psi) \to ([\chi]_\Box \phi_1 \to [\chi]_\Box \psi) \to ([\chi]_\Box \phi_0 \vee [\chi]_\Box \phi_1 \to [\chi]_\Box \psi), \tag{iii}$$

and

$$\vdash_U \Box_{\Box}(\chi \to \phi_0 \to \psi) \to \Box_{\Box}(\chi \to \phi_1 \to \psi) \to \Box_{\Box}(\chi \to (\phi_0 \vee \phi_1) \to \psi) \qquad \text{(iv)}$$

The formula in (iii) is just another instance of the axiom, so it is provable. To show (iv) it suffices to use $\mathrm{HB}_{U,T,\Box}$.

6. Case $\phi \to \psi \to \phi$. Similar proof to the two first cases.

7. Case $(\phi \to \psi \to \chi) \to (\phi \to \psi) \to (\phi \to \chi)$. Similar proof to case 5.

8. Case $(\forall x . \phi) \to \phi[x/\tau]$, where τ is free for x in ϕ. After the translation we get the formula

$$([\chi]_{\Box} \forall x . \phi \to [\chi]_{\Box} \phi[x/\tau]) \wedge \Box()$$

The right conjunct is proved by $\mathrm{HB}_{U,T,\Box}$ and Lemma 27. Let us show the left conjunct in U-provable. Since x is free for τ in ϕ and by assumption $x \notin \mathrm{fv}(\chi)$, by Lemma 89 we have that it suffices to show

$$\vdash_U (\forall x . [\chi]_{\Box} \phi) \wedge \Box(\chi \to \forall x . \phi) \to ([\chi]_{\Box} \phi)[x/\tau].$$

However, $(\forall x . [\chi]_{\Box} \phi) \to ([\chi]_{\Box} \phi)[x/\tau]$ is another instance of this logical axiom. In order to have this, we use that x is free for τ in $[\chi]_{\Box} \phi$ and Lemma 88.

9. Case $(\forall x . \psi \to \phi) \to \psi \to (\forall y . \phi[x/y])$, where $x \notin \mathsf{fv}(\psi)$ and $y = x$ or $y \notin \mathsf{fv}(\phi)$. By Lemma 85 it suffices to show that

$$\vdash_U [\chi]_{\Box}(\forall x . \psi \to \phi) \to [\chi]_{\Box} \psi \to [\chi]_{\Box}(\forall y . \phi[x/y]) \qquad \text{(i)}$$

and

$$\vdash_U \Box \Big(\chi \to (\forall x . \psi \to \phi) \to \psi \to (\forall y . \phi[x/y]) \Big). \qquad \text{(ii)}$$

(ii) can be easily proven using $\mathrm{HB}_{U,T,\Box}$ and Lemma 27. Thanks to Lemma 84, to show (i) it suffices to show

$$\vdash_U (\forall x . [\chi]_{\Box} \psi \to [\chi]_{\Box} \phi) \to [\chi]_{\Box} \psi \to \forall y . [\chi]_{\Box}(\phi[x/y]) \qquad \text{(iii)}$$

and

$$\vdash_U \Box(\chi \to \forall x . \psi \to \phi) \to [\chi]_{\Box} \psi \to \Box(\chi \to \forall y . \phi[x/y]). \qquad \text{(iv)}$$

(iv) can be proven using $\mathrm{HB}_{U,T,\Box}$ and lemmas 27 and 81. All left to show is (iii). Notice that, since $x \notin \mathrm{fv}(\psi)$ and $x \notin \mathrm{fv}(\chi)$ due to x appearing bounded in the proof, we can conclude via Lemma 88 that $x \notin \mathrm{fv}([\chi]_{\Box} \psi)$. Then if $x = y$ (iii) is an instance of the same logical axiom, so it is provable. Otherwie $y \notin \mathrm{fv}(\phi)$, which again using Lemma 88 implies that $y \notin \mathrm{fv}([\chi]_{\Box} \phi$. This makes the formula an instance of the same logical axiom.

10. Case $\phi[x/t] \to \exists x . \phi$, where t is free for x in ϕ. Similar proof to case 8.

11. Case $(\forall x.\phi\to\psi)\to(\exists y.\phi[x/y])\to\psi$, where $x\notin \mathsf{fv}(\psi)$ and $y=x$ or $y\notin \mathsf{fv}(\phi)$. Similar proof to case 9.

12. Finally, let us cover the equality axioms together. Notice that all the equality axioms are of shape $\forall x_0,\ldots,x_n.\phi_0$, where ϕ_0 is Δ_0. This makes easy to prove them using lemmas 83 and 86.

Assume that ϕ is obtained by modus ponens of $\psi\to\phi$ and ψ. By I.H. we have that:

$$\vdash_U [\chi]_\Box(\psi\to\phi), \tag{i}$$

$$\vdash_U [\chi]_\Box\psi. \tag{ii}$$

Then by (i):

$$\vdash_U [\chi]_\Box\psi\to[\chi]_\Box\phi,$$

so we can conclude the desired conclusion by (ii) and MP.

Assume that the last step is generalization of x in ϕ, so $\vdash_T \phi$ with a shorter proof. By the induction hypothesis we have that:

$$\vdash_U [\chi]_\Box\phi.$$

We can apply generalization to obtain:

$$\vdash_U \forall x\,.\,[\chi]_\Box\phi. \tag{i}$$

Note also that by assumption of this case $\vdash_T \forall x\,.\,\phi$, so $\vdash_U \Box(\forall x\,.\,\phi)$ and then

$$\vdash_U \Box(\chi\to\forall x\,.\,\phi) \tag{ii}$$

(i) and (ii) gives the desired result. $\square$

Corollary 91. Let χ be a sentence, let T and U be such that $\phi\in\mathrm{Ax}_T$ implies $\vdash_U [\chi]_\Box\phi$ and $\mathrm{HB}_{U,T,\Box}$. Assume that we have a proof π of $\Gamma\vdash_T\phi$ such that no free variable of ψ appears bounded in π or in Γ. Then

$$[\chi]_\Box\Gamma\vdash_U [\chi]_\Box\phi.$$

Proof. Just use the deduction theorem and Theorem 90. $\square$

A.2.1 Auxiliary translations

Lemma 94. Let $\mathrm{HB}_{U,T,\Box}$. Then for any $\phi,\psi\in\mathcal{L}_\mathrm{p}$ and any Σ_1-realization σ we have

$$\vdash_U [\sigma\psi]_\Box\sigma\phi\leftrightarrow[\psi,\sigma]_\Box\phi.$$

Proof. By induction in ϕ. If it is $\top$ or $\bot$ the result is trivial. If $\phi = p$, by Lemma 87, we get $\vdash_U [\sigma\psi]_\Box \sigma(p) \leftrightarrow \sigma(p)$. Then

$$\begin{aligned}\vdash_U [\sigma\psi]_\Box \sigma(p) &\leftrightarrow \sigma(p)\\ &= [\psi,\sigma]_\Box p.\end{aligned}$$

Conjunction and disjunction are trivial by the induction hypothesis. Finally, let $\phi = \phi_0 \to \phi_1$. By the induction hypothesis we get

$$\begin{aligned}\vdash_U [\sigma\psi]_\Box(\sigma\phi_0 \to \sigma\phi_1) &= ([\sigma\psi]_\Box\sigma\phi_0 \to [\sigma\psi]_\Box\sigma\phi_1) \wedge \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1)\\ &\leftrightarrow ([\psi,\sigma]_\Box\phi_0 \to [\psi,\sigma]_\Box\phi_1) \wedge \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1) \qquad \text{(by I.H.)}\\ &= [\psi,\sigma]_\Box(\phi_0 \to \phi_1).\end{aligned}$$

□

Lemma 95. Let $\phi, \psi \in \mathcal{L}_\mathrm{p}$, σ be a realization and $\Box : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$. Then

$$\vdash_U [\psi,\sigma]_\Box\phi \to [\psi,\sigma]_\Box^\circ\phi.$$

Proof. By definition in ϕ. If ϕ is an atomic propositional formula the result is trivial. If it is a conjunction or disjunction it is trivial using the induction hypothesis.

Finally, let $\phi = \phi_0 \to \phi_1$. Then

$$\begin{aligned}\vdash_U [\psi,\sigma]_\Box(\phi_0 \to \phi_1) &= ([\psi,\sigma]_\Box\phi_0 \to [\psi,\sigma]_\Box\phi_1) \wedge \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1)\\ &\to \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1)\\ &= [\psi,\sigma]_\Box^\circ(\phi_0 \to \phi_1).\end{aligned}$$

□

Lemma 96. Assume that $\mathrm{Rfn}_{U,\Box}$. Then, for any $\phi, \psi \in \mathcal{L}_\mathrm{p}$ and realization σ we have

$$\vdash_U [\psi,\sigma]_\Box^\circ\phi \to \sigma([\psi]\phi).$$

Proof. By induction in ϕ. In case ϕ is an atomic propositional formula the result is clear. The cases of conjunction and disjunction are easy applying the induction hypothesis.

Let $\phi = \phi_0 \to \phi_1$. Then

$$\begin{aligned}\vdash_U [\psi,\sigma]_\Box^\circ(\phi_0 \to \phi_1) &= \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1)\\ &\to \sigma\psi \to \sigma\phi_0 \to \sigma\phi_1 \qquad \text{(by } \mathrm{Rfn}_{U,\Box}\text{)}\\ &= \sigma([\psi](\phi_0 \to \phi_1)).\end{aligned}$$

□

Lemma 97. Let $\Box : \mathcal{L}_1 \longrightarrow \Sigma_1$ and σ be a Σ_1-realization. Then $[\psi,\sigma]_\Box^\circ\phi$ is iEA-equivalent to a Σ_1-formula.

Proof. By induction in ϕ, if it is a propositional atomic formula then it is trivial. The conjunction and disjunction are easy the by induction hypothesis. For the conjunction we need to use the codification of pairs. Finally, if $\phi = \phi_0 \to \phi_1$ we have that

$$[\psi,\sigma]^{\circ}_{\Box}(\phi_0 \to \phi_1) = \Box(\sigma\psi \to \sigma\phi_0 \to \sigma\phi_1),$$

which is Σ_1 by hypothesis. $\Box$

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