EXTENDING ANTICHAINS IN THE POSET $\langle [\omega]^{<\omega}, \subseteq \rangle$

FRANCISCO SANTIAGO NIETO-DE LA ROSA, ULISES ARIET RAMOS-GARCÍA, ANA LUCÍA VARGAS-SANDOVAL, AND DICK DE JONGH

Abstract. We prove that for every antichain A in the poset $\langle [\omega]^{<\omega}, \subseteq \rangle$ the set of maximal antichains which extend A has finite size or the size of the continuum. As a consequence we prove a conjecture of de Jongh and Vargas-Sandoval about nepfi families of finite languages [2, 10].

1. Introduction

The notation and terminology in this note is mostly standard and follows [4, 5]. If X is a set and κ is a (possibly finite) cardinal, then the symbols $[X]^{\kappa}$, $[X]^{<\kappa}$ will denote the set of all subsets of X of cardinality κ and less than κ respectively. Recall that $A \subseteq [\omega]^{<\omega}$ is an $\subseteq$-*antichain* (or just *antichain*) if for all $a, b \in A$, neither $a \subseteq b$ nor $b \subseteq a$. Given $A, B \subseteq [\omega]^{<\omega}$ antichains, A *extends* B if $B \subseteq A$. We say that A is *compatible* with B if $A \cup B$ is an antichain.

In the context of Formal Learning Theory, a family of mathematical and computational frameworks for inductive inference, and with the goal to investigate *nepfi*[1] families with finite languages within the framework of Finite Identification, the authors in [2, 10] conjectured the following:

Conjecture 1.1 (de Jongh & Vargas-Sandoval [2, 10])**.** *Every nepfi family of finite languages has only finitely many or uncountably many maximal nepfi extensions.*

In the context of Finite Identification, antichains of finite languages (nepfi families of finite languages) have a maximal nepfi extension (see Theorem 5.4.4 in [10]), namely a maximal antichain which extends the original antichain. The aforementioned conjecture comes down to a purely combinatorial statement: The number of maximal antichains extending a fixed antichain with only finite languages is either finite or uncountable. In the same work, the conjecture is proved for some special cases, for instance, for *equinumerous families*, *i.e.,* families containing only n-tuples for a fixed $n \in \omega$.

In this note we show the following result:

Theorem 1.2. *Let $A \subseteq [\omega]^{<\omega}$ be an antichain. The set of maximal antichains which extend A has finite size or the size of the continuum.*

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[1] Here nepfi abbreviates non-effectively finitely identifiable from positive data.

As a consequence of this theorem we answer the Conjecture 1.1 positively. In fact, we prove a generalization of the conjecture to convex subclasses of $[\omega]^{<\omega}$, a concept that we will define later on.

In Section 2 we prove Theorem 1.2 stated above using a case by case strategy. In Section 3 we introduce the concept of convexity and show that the main theorem generalizes to classes of antichains within convex subclasses of $[\omega]^{<\omega}$, and we discuss how the theorem can be further extended. In Section 4 we present some concrete examples of antichains that illustrate the cases treated in the proof of the conjecture. In Section 5, we discuss the connection of the conjecture with Formal Learning Theory, in particular with the framework of finite identification, and its possible applications within.

2. Proof of the theorem

Fix A an antichain in the poset $\langle[\omega]^{<\omega},\subseteq\rangle$ and put $\overline{A}=\{x\in[\omega]^{<\omega}\colon \exists a\in A\,(a\subseteq x\vee x\subseteq a)\}$. Since $[\omega]^{<\omega}\setminus\overline{A}$ is the collection of all elements compatible with every element of A, to extend A to a maximal antichain is equivalent to find $B\subseteq[\omega]^{<\omega}\setminus\overline{A}$ a maximal antichain in the poset $\langle[\omega]^{<\omega}\setminus\overline{A},\subseteq\rangle$ and consider $A\cup B$.

If $[\omega]^{<\omega}\setminus\overline{A}$ is a finite set, then we can just get a finite number of antichains extending A. Thus, we assume $|[\omega]^{<\omega}\setminus\overline{A}|=\aleph_0$. In order to get antichains which extend A, we shall provide a digraph structure to the set $([\omega]^{<\omega}\setminus\overline{A})\cup\{\emptyset\}$, say $D=\langle V,E\rangle$, where $V=([\omega]^{<\omega}\setminus\overline{A})\cup\{\emptyset\}$ and $E=\{\langle x,y\rangle\in V^2\colon x\subsetneq y\}$.

Fact 2.1. If $x,y\in V$ and $x\subseteq z\subseteq y$, then $z\in V$. Furthermore, if $x,y\in V$ and $x\subsetneq y$, there exists $n\in\omega$ such that $x\cup\{n\}\subseteq y$ and $x\cup\{n\}\in V$.

Proof of the fact. Suppose that $z\notin V$. Then, for some $a\in A$, $a\subseteq z$ or $z\subseteq a$. In the first case $a\subseteq y$, in the second $x\subseteq a$, which both fail. The second part follows immediately. ∎

Case 1. D has an infinite directed path (*i.e.,* there is an infinite subset of V, call it $\{x_n\colon n<\omega\}$, such that $\langle x_n,x_{n+1}\rangle\in E$ for all $n<\omega$). (See Example 4.1 from Section 4 to illustrate this case.)

Let $P\subset E$ be an infinite directed path, with corresponding vertex set V_P. Since any two elements of V_P are of different size, we can enumerate V_P as $\{x_n\colon n<\omega\}$ so that $x_n\subsetneq x_{n+1}$ for $n<\omega$, and hence $P=\{\langle x_n,x_{n+1}\rangle\colon n<\omega\}$. Without los of generality we assume each $x_0\neq\emptyset$.

For any $F\in[\omega\setminus 2]^{<\omega}$ we put

$$x_F:=x_0\cup\bigcup_{n\in F}(x_n\setminus x_{n-1}).$$

It is clear by Fact 2.1 that $x_F\notin\overline{A}$ for all $F\in[\omega\setminus 2]^{<\omega}$, since $x_0\subseteq x_F\subseteq x_{max(F)}$ and $x_0\in V$ and $x_{max(F)}\in V$.

We now consider the set $A_f=\{x_{[f(i),f(i+1))}\colon i<\omega\}$ for each $f\in\mathcal{F}:=\{f\in\omega^\omega\colon f$ is an increasing function with $f(0)=2\}$. Since $\{[f(i),f(i+1))\colon i<\omega\}$ forms an interval partition of $\omega\setminus 2$ and $x_n\setminus x_{n-1}\neq\emptyset$ for $n\geqslant 1$, it follows that A_f is an antichain in the poset $\langle[\omega]^{<\omega}\setminus\overline{A},\subseteq\rangle$. However, if $f,g\in\mathcal{F}$ and $f\neq g$, then A_f and A_g are incompatible (*i.e.,* $A_f\cup A_g$ is not an antichain) because there exists $i\in\omega$ such that either $[f(i),f(i+1))\subseteq[g(i),g(i+1))$, or $[g(i),g(i+1))\subseteq[f(i),f(i+1))$.

For every $f \in \mathcal{F}$ choose a maximal antichain $B_f \subseteq [\omega]^{<\omega}$ such that $A \cup A_f \subseteq B_f$. It follows that if $f \neq g$, then $B_f \neq B_g$. Therefore, since $|\mathcal{F}| = \mathfrak{c}$,[2] we conclude that in this case A can be extended to a maximal antichain in the poset $\langle [\omega]^{<\omega}, \subseteq \rangle$ in continuum many ways.

Case 2. D has no infinite directed paths.

Define a rank function on V by letting

$$\mathsf{rank}(x) = \text{length of the longest directed path from } \emptyset \text{ to } x. \tag{2.1}$$

Clearly $\mathsf{rank}^{-1}(k)$ is a (possibly empty) antichain in the poset $\langle V, \subseteq \rangle$ for all $k \in \omega$. In particular, $\mathsf{rank}^{-1}(0) = \{\emptyset\}$.

In order to make our next arguments more comprehensible, given $x \in V$, we introduce the following notation:

$$\begin{aligned} \mathrm{succ}_D(x) &= \{y \in V : \langle x, y \rangle \in E \,\&\, \mathsf{rank}(y) = \mathsf{rank}(x) + 1\}, \\ \mathrm{pred}_D(x) &= \{y \in V : \langle y, x \rangle \in E \,\&\, \mathsf{rank}(x) = \mathsf{rank}(y) + 1\} \text{ and} \\ \mathrm{pred}_D(k) &= \bigcup_{x \in \mathsf{rank}^{-1}(k)} \mathrm{pred}_D(x), \text{ for } k \in \omega. \end{aligned}$$

It is good to realize that the above clause $\mathsf{rank}(y) = \mathsf{rank}(x) + 1$ for succ_D is not superfluous. The following fact lists some basic observations about $\mathrm{succ}_D(x)$ and $\mathrm{pred}_D(x)$.

Fact 2.2. Assume that $x \in \mathsf{rank}^{-1}(k+1)$ and $\{y, y'\} \subseteq \mathsf{rank}^{-1}(k)$ for some $k \in \omega$. Then:

(1) $|\mathrm{pred}_D(x)| < \aleph_0$;
(2) If $x \in \mathrm{succ}_D(y)$, then $x = y \cup \{n\}$ for some n;
(3) if $x \in \mathrm{succ}_D(y) \cap \mathrm{succ}_D(y')$, then $x = y \cup y'$;
(4) $|\mathrm{succ}_D(y) \cap \mathrm{succ}_D(y')| \leq 1$.

Proof of the fact. (1) is clear because x is finite and predecessors are subsets of x.
(2) Follows immediately from Fact 2.1 and the notion of rank.
To prove (3), note that $y, y' \in \mathrm{pred}_D(x)$. Thus $y \subseteq y \cup y' \subseteq x$. By Fact 2.1 then $y \cup y' \in V$. Also note that, since x and $y \neq y'$ differ one in rank, by (2) they also differ one in size. Thus, it is obvious that x has to be identical to $y \cup y'$.
Finally, (4) is an easy consequence of (3). ■

Claim 2.3. There exists $k \geqslant 1$ such that $\mathsf{rank}^{-1}(k)$ is infinite.

Proof of the claim. Assume, for the sake of contradiction, that $\mathsf{rank}^{-1}(k)$ is a finite set for all $k \in \omega$. For each node $v \in V$ define $U_v = \{u \in V : v \subseteq u\}$. We construct an infinite directed path $\{x_n : n < \omega\}$ in D contradicting the assumption of Case 2. We construct this path by recursion in such a way that for all n both U_{x_n} is infinite and $\mathsf{rank}(x_n) = n$. Let $x_0 = \emptyset$, satisfying the induction hypothesis. Suppose $x_n \in V$ is defined, we choose $x_{n+1} \in \mathrm{succ}_D(x_n)$ such that

[2]We write $\mathfrak{c} = 2^{\aleph_0}$ for the cardinality of the continuum.

$U_{x_{n+1}}$ is infinite. We always can make that choice because U_{x_n} is infinite but $\mathrm{succ}_D(x_n) \subseteq \mathsf{rank}^{-1}(n+1)$ is finite by assumption. ■

Subcase 2.1. There exists $k \in \omega \setminus 2$ such that both $\mathsf{rank}^{-1}(k)$ and $\mathrm{pred}_D(k)$ are infinite. (See Examples 4.2 and 4.3 from Section 4 to illustrate this subcase.)

In this subcase, we will build recursively two sequences $\langle y_n \rangle_{n\in\omega} \subseteq \mathrm{pred}_D(k)$ and $\langle x_n \rangle_{n\in\omega} \subseteq \mathsf{rank}^{-1}(k)$ such that for each $n \neq m$ we have that $\langle y_n, x_n \rangle \in E$ and $\langle y_n, x_m \rangle \notin E$.

Once $\langle y_n \rangle_{n\in\omega}$ and $\langle x_n \rangle_{n\in\omega}$ are defined, we can show that there are $\mathfrak{c}$ many maximal antichains which extend A. Indeed, since $\langle x_n \rangle_{n\in\omega} \subseteq \mathsf{rank}^{-1}(k)$ and $\mathsf{rank}^{-1}(k)$ forms an antichain, there are no $i, j < \omega$ with $i \neq j$ such that $\langle x_i, x_j \rangle \in E$. Similarly, $\langle y_i, y_j \rangle \notin E$, for $i, j < \omega$ with $i \neq j$. Thus, every $f \in 2^\omega$ provides an antichain A_f, defined by

$$A_f = \{y_i \colon f(i) = 0\} \cup \{x_i \colon f(i) = 1\}.$$

By construction, if $f \neq g$, then $A_f \cup A_g$ is not an antichain, so maximal extensions of A_f and A_g will be distinct.

In order to get the required sequences, we divided this subcase into two (sub) subcases.

Subcase 2.1.1. The set $Y := \{y \in \mathrm{pred}_D(k) \colon |\mathrm{succ}_D(y)| = \aleph_0\}$ is infinite. (See Example 4.2 from Section 4 to illustrate this case.)

By Fact 2.2(4), note that there is at most one element $y \in \mathrm{pred}_D(k)$ such that $\mathrm{succ}_D(y)$ is cofinite in $\mathsf{rank}^{-1}(k)$. We ignore such a y, i.e. we will not use this y in the construction of the sequence. Thus, without loss of generality, we will assume that the set $\mathsf{rank}^{-1}(k) \setminus \mathrm{succ}_D(y)$ is infinite for all $y \in \mathrm{pred}_D(k)$.

Keeping this in mind, take any $y \in Y$ as y_0 and take $x_0 \in \mathsf{rank}(k)^{-1}$ with $y_0 \in \mathrm{pred}_D(x_0)$. Assume that $\langle y_i \rangle_{i<n}$ and $\langle x_i \rangle_{i<n}$ have been constructed satisfying $\langle y_i, x_i \rangle \in E$ and $\langle y_i, x_j \rangle, \langle y_j, x_i \rangle \notin E$ for every $i < j < n$. Choose $y_n \in Y \setminus \bigcup_{i<n} \mathrm{pred}_D(x_i)$, and take x_n an element of $\mathrm{succ}_D(y_n) \setminus (\bigcup_{i<n} \mathrm{succ}_D(y_i))$, noting that the first set is infinite by Fact 2.2 (1). Then, $\langle y_n, x_i \rangle, \langle y_i, x_n \rangle \notin E$ for all $i < n$.

Subcase 2.1.2. The set Y is finite. (See Example 4.3 from Section 4 to illustrate this case.)

Take any y_0 from $\mathrm{pred}_D(k) - Y$, an infinite set. Then take any x_0 such that $y_0 \in \mathrm{pred}_D(x_0)$. Assume that $\langle y_i \rangle_{i<n}$ and $\langle x_i \rangle_{i<n}$ have been constructed satisfying $\langle y_i, x_i \rangle \in E$ and $\langle y_i, x_j \rangle, \langle y_j, x_i \rangle \notin E$ for every $i < j < n$, and $\mathrm{succ}_D(y_i)$ is finite for each $i < n$. Choose y_n from $\mathrm{pred}_D(k) - (Y \cup \bigcup\{\mathrm{pred}_D(x) \colon x \in \mathrm{succ}_D(y_i) \,\&\, i < n\})$. This is an infinite set by the assumptions made, and because $\mathrm{pred}_D(x)$ is always finite by Fact 2.2(1). Now take x_n to be an element of $\mathrm{succ}_D(y_n)$. By construction, $y_n \notin \mathrm{pred}_D(x_i)$ and $x_n \notin \mathrm{succ}_D(y_i)$ for all $i < n$, *i.e.* $\langle y_n, x_i \rangle, \langle y_i, x_n \rangle \notin E$ for all $i < n$.

Subcase 2.2. For every $k \in \omega$ it follows that if $\mathsf{rank}^{-1}(k)$ is infinite, then $\mathrm{pred}_D(k)$ is finite. (See Example 4.4 from Section 4 to illustrate this case.)

Under the hypothesis of this case, we will show that there are finitely many antichains which extend A. We start by proving a number of claims.

Claim 2.4. For every $k \in \omega$, the set $\mathrm{pred}(k)$ is finite.

Proof of the claim. By contradiction, suppose $\mathrm{pred}_D(k)$ is infinite for some k. Then the set of successors of $\mathrm{pred}_D(k)$ must be finite (otherwise it contradicts condition of the Subcase 2.2). But that means that some successors share some predecessors but not more than two of them, because of Fact 2.2(4). From that it follows that $\mathrm{pred}_D(k)$ is finite, contradicting our assumption. Note that the antichain of rank k could still be infinite, but all but finitely many of the elements have no successors. ∎

Then it follows almost immediately that rank is bounded and that pred is finite. We prove this in the following claim.

Claim 2.5. The function rank is bounded.

Proof of the claim. Suppose not. We construct an infinite directed chain starting with $\emptyset$. This contradicts the assumption of Case 2.

$\emptyset$ is a starting point of directed chains reaching to arbitrary n (let us call this infinity property). Assume we have a chain $\langle x_0,\ldots,x_k\rangle$ in V with $\mathsf{rank}(x_k)=k$ and x_k having the infinity property. First assume $\mathsf{rank}(k+1)^{-1}$ is finite. Then we can find y of rank $k+1$ such that $\langle x_k,y\rangle\in E$ with the infinity property. Take y as x_{k+1}. Next assume $\mathsf{rank}(k+1)^{-1}$ is infinite, and $\mathsf{rank}(k+2)^{-1}$ is also infinite. Then $\mathrm{pred}_D(k+2)$ is finite, which means that only finitely many members of $\mathsf{rank}(k+1)^{-1}$ can be starting points of chains, and we are in the same situation as in the first case. The final case which is now left is that $\mathsf{rank}(k+1)^{-1}$ is infinite, and $\mathsf{rank}(k+2)^{-1}$ is finite. In that case there have to be y,z of ranks $k+1,k+2$ such that $\langle x_k,y\rangle\in E$, $\langle y,z\rangle\in E$ and z has the infinity property. We can extend the chain $\langle x_0,\ldots,x_k\rangle$ with $\langle y,z\rangle$. An infinite directed chain will be produced. ∎

Claim 2.6. The set $\mathrm{pred}:=\{x\in V:\mathrm{succ}_D(x)\neq\emptyset\}$ is finite.

Proof of the claim. Suppose, by contradiction, that there are infinitely many points with direct successors. This should happen at some rank, say k. But that means by Claim 2.1 that there are infinitely many successors in rank $k+1$. By the condition of Subcase 2.2, we know that then the number of predecessors of elements in rank $k+1$ is finite, thus contradicting our initial assumption about the set of elements of rank k. ∎

Claim 2.7. Let B and C be two maximal antichains in the poset $\langle[\omega]^{<\omega}\setminus\overline{A},\subseteq\rangle$. Then $\mathrm{pred}\cap B\neq\mathrm{pred}\cap C$.

Proof of the claim.

Assume, to obtain a contradiction, without loss of generality that $\mathrm{pred}\cap B=\mathrm{pred}\cap C$ and $x\in B, x\notin C$. Then $x\notin\mathrm{pred}$. So, there is no y in the poset such that $x\subset y$. Assume there is y such that $y\subset x$ and $y\in C$. Then $y\in\mathrm{pred}$, so, by assumption, $y\in B$, impossible, since B is an antichain. So, there is no $y\in C$ such that $y\subset x$ or $x\subset y$. Since C is a maximal antichain, this, finally, implies $x\in C$. ∎

Therefore, there exists an one-to-one function from the set of maximal antichains in the poset $\langle[\omega]^{<\omega}\setminus\overline{A},\subseteq\rangle$ into the power set of pred which is finite. This finishes the proof.

3. Convexity

In this section we discuss generalizations of Theorem 1.2. As we can see in the previous section, Fact 2.1 is the key to several arguments in the proof. We can isolate this property as follows:

Definition 3.1. If $B \subseteq [\omega]^{<\omega}$. we say that B is *convex* if for every $x, y \in B$ and every z such that $x \subseteq z \subseteq y$, it holds that $z \in B$.

Obviously, $[\omega]^{<\omega}$ is a convex set. Naturally, For each $B \subseteq [\omega]^{<\omega}$, $\overline{B}$ is a convex set. Fact 2.1 shows that V is convex. Furthermore, we can generalize Theorem 1.2 to convex sets.

Theorem 3.2. *If B is a convex set and $A \subseteq B$ an antichain, the set of maximal antichains contained in B which extend A has finite size or the size of the continuum.*

To be more specific, we can define a closure operator. If $A \subseteq B$, let $\overline{A}^B = \{b \in B : \exists a \in A(b \subseteq a \vee a \subseteq b)\}$. We will write $\overline{B}$ to mean $\overline{B}^{[\omega]^{<\omega}}$.

The proof of Theorem 3.2 is analogous to the proof of Theorem 1.2 because we can replace Fact 2.1 with next obvious lemma.

Lemma 3.3. *If B is a convex set and $A \subseteq B$, then $B \setminus \overline{A}^B$ is convex.*

For this lemma we do not need A to be an antichain.

Even Theorem 3.2 implies Theorem 1.2, we preferred to present the proof of Theorem 1.2 of the simpler conjecture first. However, convexity is an "almost" essential hypothesis in Theorem 3.2 as the next example shows.

Example 3.4. Let $B = \{\{2k : k \leq m\} : m \in \omega\} \cup \{\{1\}\}$. It is easy to see that B is not convex, $\{\{1\}\}$ is an antichain and $\{\{1\} \cup \{2k : k \leq m\} : m \in \omega\}$ is the set of all maximal antichains which extend $\{1\}$ and are contained in B. This set is countable.

Nevertheless we can refine Theorem 3.2 as next proposition does.

Definition 3.5. If $B \subseteq [\omega]^{<\omega}$, we define the *convex closure* by $Conv(B) = \bigcap\{C : B \subseteq C \text{ convex}\}$

Of course $Conv(B)$ is always well defined and a convex set. Let us call B *almost convex* if $Conv(B) \setminus B$ is finite.

Proposition 3.6. *Let $B \subseteq [\omega]^{<\omega}$. If B is almost convex and $A \subseteq B$ is an antichain, then the set of maximal antichains which extend A within B has finite size or the size of the continuum.*

Proof. Let $F = Conv(B) \setminus B$. We set $T = \{C \subseteq B : A \subseteq C \ \&\ C \text{ is a maximal antichain in } B\}$ and $S = \{C \subseteq Conv(B) : A \subseteq C \ \&\ C \text{ is a maximal antichain in } Conv(B)\}$. To start we note that $|T| \leq |S|$ because $f : T \to S$ defined as follows is an injection:

$$f(C) = \begin{cases} C & \text{if} \quad \forall a \in F\, \exists c \in C (a \subseteq c \vee c \subseteq a) \\ C \cup H & \text{else,} \end{cases}$$

where $H \in \{G \subseteq F : G \text{ is a maximal antichain in } F \ \& \ \forall c \in C\, \forall g \in G\, (c \not\subseteq g \text{ or } g \not\subseteq c)\}$.

By Lemma 3.3 and Theorem 3.2 we know that $|S| \in \omega \cup \{\mathfrak{c}\}$. By the last paragrah we are already done if S is finite.

Assume not, i.e., $|S| = \mathfrak{c}$. Note that, for each $C \in S$, the set $C \cap B$ is a maximal antichain in B containing A. Thus, $C \cap B \in T$. Since F is finite, there is $G \subseteq F$ such that $S' = \{C \in S : C \cap F = G\}$ has $\mathfrak{c}$ many elements. Thus, $\{C \cap B : C \in S'\} \subseteq T$ and $|\{C \cap B : C \in S'\}| = \mathfrak{c}$. □

One might wonder whether B being almost convex is not only sufficient but also necessary for the theorem to go through. The following example shows that it is not. A nice necessary as well as sufficient condition seems hard to find.

Example 3.7. Let $B = \{\{3m\}, \{3m, 3m+1, 3m+2\}\} : m \in \omega\}$. Clearly B is not almost convex, there are infinitely many gaps. Antichains contained in B need to have exactly one of $\{3m\}$ and $\{3m, 3m+1, 3m+2\}$ of some segments of B, maximal antichains in B exactly one of $\{3m\}$ and $\{3m, 3m+1, 3m+2\}$ of all segments of B. It is easy to see that any antichain in B has either finitely many or uncountably many maximal extensions.

4. Examples

In what follows, we illustrate the cases and subcases treated in the proof with some concrete examples. First, some useful notation as in [10].

Let A^2 and A^3 be the family of all pairs and triples respectively. For any $Y \subseteq A^3$, we denote $NUM(Y)$ as the set of all the numbers that appear in any triple of Y, and the set $TRIPLES(Y)$ as the set of all triples that can be formed with the numbers in $NUM(Y)$.

We call $G \subseteq A^3$ a cluster in A^3 if $TRIPLES(G) = G$ and $|G| > 1$.

These conventions will be used in the following example.

Example 4.1 (Case 1)**.** Let $A := A_p \cup A_t$ s.t.

$$A_p := \{\{e_1, e_2\} : e_i \in EVEN\} \cup \{\{e, o\} : e \in EVEN \text{ and } o \in ODD\}$$

and

$$A_t := \{\{o_1, o_2, o_3\} : o_i \in ODD\}$$

First note that A is a maximal antichain and that A_t is an infinite cluster.

Let $A' := A \setminus A_t$. By Proposition 5.5.50 in [10](page 190), we have that A' has uncountably many maximal extensions.

Let's look at the set of possible extensions of A' as a directed graph D. Note that any n-tuple with $n > 1$ of odd numbers can be added to A' , so we can construct a directed infinite path of the form $\{o_1\} \subset \{o_1, o_2\} \subset \{o_1, o_2, o_3, o_4\} \ldots \subset \{o_1, o_2, ..., o_n\}, \ldots$ where any element in the path can properly extend A' and the infinite path is part of the directed graph of languages that can extend A'.

Example 4.2 (Case 2 - Subcase 2.1.1)**.** Let $A := A_p \cup A_t$ as in the prevvious example. Let $K_2 := \{\{2, x\} : x \in \omega \setminus \{2\}\}$. Clearly, $K_2 \subseteq A_p \subseteq A$.

Let $A' := A \setminus K_2$. By Proposition 5.5.52 (1) in [10](page 190), we know that A' has uncountably many maximal extensions (following the same construction as in the proof of 5.5.52 (1) in [10]).

Let's look at the set of possible extensions of A' as a directed graph D. First note that the only element of rank 1 is the set $\{2\}$, and every pair $\{2, n\}$ with $n \in \omega \setminus \{2\}$ has rank 2. Note also that every triple of the form $\{2, o_1, o_2\}$ with $o_i \in ODD$ can be considered an extension of A', has rank 3, and is a direct successor of $\{2, o_1\}$ and $\{2, o_2\}$. Note also that any other n-tuple larger than a triple cannot be added to

A' without impairing the antichain condition. Thus, the directed graph D ends at rank 3, since for any other $k > 3$, $\mathsf{rank}^{-1}(k) = \emptyset$. Thus, D has no infinite directed path. Observe that at rank 3, the conditions of subcase 2.1 are fulfilled, because $\mathsf{rank}^{-1}(3)$ and $\mathrm{pred}_D(3)$ are both infinite. Also, since we have infinitely many pairs of the form $\{2, o_i\}$ with $o_i \in ODD$ at rank 2 and since for each of those pairs it follows that the set $|\mathrm{succ}_D(\{2, o_i\})| = \aleph_0$, we have met the conditions of Subcase 2.1.1, namely that $Y := \{y \in \mathrm{pred}_D(k) : |\mathrm{succ}_D(y)| = \aleph_0\}$ is infinite.

Example 4.3 (Case 2 - Subcase 2.1.2). Let $A' := \{\{i, n\} : i, n \in \omega \setminus \{0, 1\}\}$. As noted in Example 5.5.4 in [10] (pages 165-166), this family has uncountably many maximal extensions.

Let's look at the set of possible extensions of A' as a directed graph D. First note that the only elements that have rank 1 are $\{0\}$ and $\{1\}$. So the family of all nodes of rank 1 is finite.

The node $\{0\}$ has infinitely many successors of rank 2, namely all pairs of the form $\{0, n\}$ with $n \neq 0 \in \omega$. From all those pairs, only $\{0, 1\}$ has infinitely many direct successors of rank 3, namely triples of the form $\{0, 1, n\}$ with $n \neq 0, 1$. Any other pair of the form $\{0, n\}$ with $n \neq 1$ has only one successor of rank 3, namely $\{0, n, 1\}$. This is because any other triple (and therefore any larger tuple) extending $\{0, n\}$ with $n \neq 1$ will impair the antichain condition of A'. Formally, for any $n \in \omega$ s.t. $n \neq 1$ it follows $|\mathrm{succ}_D(\{0, n\})| = 1$.

By a similar reasoning, we also have that $\{1\}$ has infinitely many successors of rank 2, namely all pairs of the form $\{1, n\}$ with $n \neq 1 \in \omega$. From all those pairs, only $\{0, 1\}$ has infinitely many direct successors of rank 3, namely triples of the form $\{0, 1, n\}$ with $n \neq 0, 1$. As observed before, for any $n \in \omega$ s.t. $n \neq 0$ it follows $|\mathrm{succ}_D(\{1, n\})| = 1$ since any other triple (and therefore any larger tuple) extending $\{1, n\}$ with $n \neq 0$ will impair the antichain condition of A'.

Altogether, we have that for $k = 3$, the sets $\mathsf{rank}^{-1}(3)$ and $\mathrm{pred}_D(3)$ are infinite (fulfilling the initial conditions of the Subcase 2.1). To see that this case fulfills the conditions of Subcase 2.1.2 note that since only one pair, $\{0, 1\} \in \mathrm{pred}_D(3)$, has infinitely many successors, the set $Y := \{y \in \mathrm{pred}_D(3) : |\mathrm{succ}_D(y)| = \aleph_0\}$ has only one element and therefore is finite.

Example 4.4 (Case 2 - Subcase 2.2)**.** The following corresponds to the second family mentioned in Example 5.5.4 in [10] (page 166). This example fulfills the condition of Subcase 2.2: "For every $k \in \omega$, if $\mathsf{rank}^{-1}(k)$ is infinite then $\mathrm{pred}_D(k)$ is finite."

Let $A' := \{\{i, n\} : i, n \in \omega - \{0\}\}$. As noted in Vargas-Sandoval [10], this family has only two many maximal extensions.

Let's look at the set of possible extensions of A' as a directed graph D. First note that the only element that has rank 1 is $\{0\}$. So the family of all nodes of rank 1 is finite.

The node $\{0\}$ has infinitely many successors of rank 2, namely all pairs of the form $\{0, n\}$ with $n \neq 0 \in \omega$. Note that any triple (and therefore any larger tuple) extending $\{0, n\}$ with $n \neq 0$ will impair the antichain condition of A'. Thus, the largest directed path in the directed graph D has rank 2, i.e., for any $k > 2$, $\mathsf{rank}^{-1}(k) = \emptyset$. Now, note that $\mathsf{rank}^{-1}(2)$ is an infinite antichain, so $|\mathsf{rank}^{-1}(2)| = \aleph_0$, and that $\mathrm{pred}_D(2)$ is finite, namely $|\mathrm{pred}_D(2)| = 1$. Altogether, it follows that the condition of Subcase 2.2 is fulfilled.

5. Discussion on Finite Identification

Formal Learning Theory is an umbrella term for a family of mathematical and computational frameworks that study *inductive inference* (or *inductive learning*) by means of a learning function. Such a term refers to the process of hypothesis change using incoming information that may result in stabilizing on an accurate hypothesis. Motivations of studying inductive learning range from modelling *children language acquisition* (inferring a grammar from inductively given examples of a language) and *scientific inquiry* (inferring a general hypothesis from an inductively given a stream of empirical data).[3]

With the emergence of artificial intelligence and machine learning in the 1950's, the study of inductive inference gained attention in the computer science community (for a general overview see e.g., [11, 1]). More recent work on this stems from the pioneering formal studies of [9], [12, 13], and [3]. With the aim of modelling children language acquisition, Gold's framework *identification in the limit* (or *learning in the limit*) marked the beginning of a mathematical and computational treatment for inductive learning.

The learning task in Gold's model consists of identifying a language (represented by a set of symbols) amidst a collection of languages on the basis of an infinite stream of examples from the language. The stream of examples consists either of positive data (an enumeration of all members of the language) or of complete data (positive and negative data, labelling all sentences as belonging to the language or not). Learning in the limit considers a learner to be *successful* if it stabilizes on a correct hypothesis after only finitely many mind changes. The fact that such a learner keeps conjecturing forever (even when she already stabilized on a correct hypothesis) suggests that she does not necessarily know when her conjecture is correct. On a slightly simpler approach, *finite identification* (or *learning with certainty*) considers a more restricted notion of a successful learner (first mentioned by [3], performed by [14], and formalized by [8] and, simultaneously, by [6, 7]). In this framework, a learner can produce just one conjecture that must be correct immediately.

[3]Formal learning theory often uses recursion-theoretic tools to reason about inductive inference with a computational learner, represented by a *recursive function*, an *algorithm* (or *effective procedure*) or an *inference machine*.

In [10] (and previously in [2]) the author focuses purely on finite identification. She develops a fine-grained theoretical analysis of the distinction between finite identification with positive information (pfi) and with complete information (cfi).

In Chapter 6, she focuses on the structural differences of families of languages that are pfi and families that are cfi without taking into account the computational aspects (non-effective families). She investigates whether any finitely identifiable family is contained in a maximal finitely identifiable one, first in the positive data case, and focuses on the conjecture, addressed in this note, that she partially resolves in [10]: any non-effective positively identifiable family of finite languages either has only finitely many maximal positively identifiable extensions or continuously many.

The answer to the aforementioned conjecture, now a fact, sheds light on a more refined characterization for nepfi (and pfi) families of finite languages, namely to distinguish the *finitely extendable* families from the ones that are *continuously extendable.*

In a way, maximal antichains give us a sense of being "complete" since we cannot add any other set to the family without losing its identifiability, it is the best we can do. On the one hand, families which have only finitely many maximal extensions give us a sense of being "almost complete" and the ways to "complete them" can be enumerated and tracked. On the other hand, families with continuously many maximal extensions seem to be very far from complete, moreover, their maximal extensions cannot be enumerated or tracked.

We illustrate this with a very simple example from [10].

Consider the set $X = \{2, 4, 6, 8, 10\}$ and the antichain of singletons $A = \{\{n\} : n \in \omega \setminus X\}$. Clearly, A is not maximal since $A \cup \{X\}$ is an antichain. In fact, $A \cup \{Y\}$ is an antichain for every $Y \subseteq X$. Let $set(A) = \{n \in \omega : \{n\} \in A\}$. First note that if $A \cup Y$ is an antichain then $Y \subseteq \omega \setminus set(A) \subseteq X$. Since X is finite, the number of antichains that we can construct with languages in the power set of X, $\mathcal{P}(X)$, is finite. Moreover, there are only finitely many distinct maximal antichains M with elements in $\mathcal{P}(X)$.

Consider the set $X = \{n \in \omega : n \text{ is even}\}$ and the antichain of singletons $A = \{\{n\} : n \in \omega \setminus X\}$. Clearly, A is not maximal since $A \cup \{2, 4\}$ is an antichain. In fact, $A \cup \{Y\}$ is an antichain for every finite set of even numbers $Y \subseteq X$. Let $set(A) = \{n \in \omega : \{n\} \in A\}$, note that $set(A)$ is the set of odd numbers. Note that if $A \cup \{Y\}$ is an antichain of finite sets then Y is finite and $Y \subseteq \omega \setminus set(A) \subseteq X$, i.e., Y must be a finite set of even numbers. Since $\omega \setminus set(A) = X$ is countable (i.e., $set(A)$ is not cofinite), by a combinatorial argument, we will construct continuously many maximal antichain extensions for A. For every $Z \subseteq \omega \setminus set(A) = X$, consider the family $A' = A \cup \{\{z\} : z \in Z\} \cup \{\{w_1, w_2\} : w_1, w_2 \in X \setminus Z\}$. Clearly A' is an antichain of finite languages that extends A. Adding any other language to A' will impair the antichain condition of the family and therefore A' is a maximal antichain.

References

1. Selmer Bringsjord and Naveen Sundar Govindarajulu, *Artificial intelligence*, The Stanford Encyclopedia of Philosophy (Edward N. Zalta, ed.), Metaphysics Research Lab, Stanford University, winter 2019 ed., 2019.
2. Dick de Jongh and Ana Lucia Vargas-Sandoval, *Finite identification with positive and with complete data*, Language, logic, and computation, Lecture Notes in Comput. Sci., vol. 11456, Springer, Berlin, 2019, pp. 42–63. MR 3981842
3. E Mark Gold, *Language identification in the limit*, Information and control **10** (1967), no. 5, 447–474.
4. Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597
5. Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980, An introduction to independence proofs. MR 597342
6. Steffen Lange and Thomas Zeugmann, *Monotonic versus non-monotonic language learning*, Nonmonotonic and Inductive Logic, Second International Workshop, Reinhardsbrunn Castle, Germany, December 1991 (G. Brewka, K.P. Jantke, and P.H. Schmitt, Eds.), Lecture Notes in Artificial Intelligence 659, Springer-Verlag, 1991, pp. 254–269.
7. ———, *Types of monotonic language learning and their characterization*, Proceedings of the 5th Workshop on Computational Learning Theory, ACM Press, 1992, pp. 377–390.
8. Yasuhito Mukouchi, *Characterization of finite identification*, Proceedings of the International Workshop on Analogical and Inductive Inference (AII 1992), Springer-Verlag, 1992, pp. 260–267.
9. Hilary Putnam, *Degree of confirmation and inductive logic*, The Philosophy of Rudolf Carnap (Paul Arthur Schilpp, ed.), Open Court: La Salle., 1963, pp. 761–783.
10. Ana Lucia Vargas Sandoval, *On the path to the truth: Logical & computational aspects of learning*, Ph.D. thesis, University of Amsterdam, 2020.
11. Oliver Schulte, *Formal learning theory*, The Stanford Encyclopedia of Philosophy (Edward N. Zalta, ed.), Metaphysics Research Lab, Stanford University, spring 2018 ed., 2018.
12. Ray J Solomonoff, *A formal theory of inductive inference. Part I*, Information and control **7** (1964), no. 1, 1–22.
13. ———, *A formal theory of inductive inference. Part II*, Information and control **7** (1964), no. 2, 224–254.
14. B.A. Trakhtenbrot and Ya. M. Barzdin, *Finite automata—behaviour and synthesis*, Fundamental Studies in Computer Science, vol. 1, Amsterdam: North-Holland, 1973.

Posgrado Conjunto en Ciencias Matemáticas UNAM-UMSNH, Morelia, Michoacán, México 58089

Email address: fsnieto@matmor.unam.mx, francisco.s.nieto@ciencias.unam.mx

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Morelia, Michoacán, México 58089

Email address: ariet@matmor.unam.mx

Email address: ana.varsa@gmail.com

Institute for Logic, Language and Computation, University of Amsterdam

Email address: d.h.j.dejongh@uva.nl