# A Practice-Based Critique of Reverse Mathematics

**MSc Thesis** *(Afstudeerscriptie)*

written by

**Jan W. Gronwald**
(born April 21st, 1998 in Szczecin, Poland)

under the supervision of **Prof Dr Michiel van Lambalgen** and **Dr Benno van den Berg**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

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**Members of the Thesis Committee:**
Prof Dr Michiel van Lambalgen (Supervisor)
Dr Benno van den Berg (Supervisor)
Dr Benedict Eastaugh
Dr Marianna Girlando
Dr Balder ten Cate (Chair)

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

**Abstract**

The present thesis studies the programmatic and formal choices made in the development of Reverse Mathematics (RM), a framework for the analysis and extraction of foundational assumptions underlying ordinary-mathematical theorems. It offers a critique of RM, based on its unfaithful representation of the latter. Among other issues, the main two problems preventing the classical RM to fulfil its foundational ambitions are the arbitrary techniques of encoding the informal mathematics, and the lack of distinction between a theorem and its proof that in practice leads to the possibility of calibrating a single theorem with different set-existence principles. Then, the Constructive RM is tried against the same questions. The thesis concludes that the constructive frameworks for RM offer a more fine-grained analysis together with a more faithful representation of *some* significant portions of informal mathematics, but they cannot analyze theorems whose constructive versions are viewed within classical mathematics as inequivalent to their classical counterparts.

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# A Practice-Based Critique of Reverse Mathematics

*Taking the Principle of the Excluded Middle from the mathematician is the same as prohibiting the boxer the use of his fists*

***Hilbert***

*Work in mathematics resembles the meticulous technique of a tailor rather than the crude disposition of a bruiser. One needs a precise scheme and nimble fingers, not fists in boxing gloves*

***A witty by-passer in my hometown of Szczecin***

# 0 | An Overview of the Big Five

## The Outline of the Thesis

For the purposes of this thesis I have to move back and forth between the technical results and features of different formal frameworks on the one hand, and the philosophical discussion of their meaning and consequences on the other. It is therefore sometimes challenging to square this with conciseness and clear exposition of the progression of the reasoning in my investigation. For these aims, I introduce the material as follows. The present chapter 0 is a presentation of the most important facts about the so-called Big Five subsystems of the Second Order Arithmetic ($\mathbf{Z_2}$). Then, I discuss the features of this framework with more depth and give a brief example of the reverse-mathematical practice.

In chapter 1 I introduce more context to the rise of reverse mathematics, a bit of its history and a review of its goals as a foundational endeavor together with the methods it employs to achieve some of these aims. Then, in chapter 2 I discuss the technical and philosophical problems occurring within the framework presented in chapter 0 in unison with using it as a case study to develop a series of constraints a suitable reverse-mathematical framework should satisfy.

Chapter 3 begins with an introduction of the constructive approaches to reverse mathematics and evaluates how they score against the constraints put forth in the previous chapter.

## 0.1 The Friedman-Simpson Program

The research project that goes by the name of the Friedman-Simpson style reverse mathematics (from now on I will call it the Classical RM), was started in the 70s by a series of papers by Harvey Friedman. Historically, second order arithmetic appeared way before, with [HA38] being sometimes cited as the first textbook treatment of it[1], but it seems that it was Friedman who recognized its usefulness for analysis of ordinary mathematical theorems. Put more precisely, the goal of Classical RM is to answer the question: *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?* Here, *non-set-theoretic mathematics* corresponds to a rough distinction between "countable" and "uncountable" mathematics[2] or

> that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts. We have in mind such branches as geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic, and computability theory. [Sim99, p. 1]

The objects of thus understood ordinary mathematics are almost always countable or separable, therefore it seems fitting to choose a framework "where countable objects occupy center stage" [Sim99, p. 2]. Moreover, the study of objects of areas such as calculus usually requires quantification over sets (e.g. when one wants to state something about a set of functions). For these reasons Classical RM is done in second order arithmetic.

$\mathbf{Z_2}$ is a system in a two-sorted language, with number- and set-variables. The atomic formulae are $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$ where $t_1, t_2$ are numerical terms (with number variables ranging over $\omega$) and $X$ is a set variable. It is axiomatized by (universal closures of) three kinds of axioms:

- the usual axioms for $+, \times$ and $<$

  $n + 1 \neq 0$

  $m + 1 = n + 1 \rightarrow m = n$

  $m + 0 = m$

[1] In fact, comprehension schemas go even earlier to none other than Gödel's Incompleteness paper [Göd86] or even Ramsey's [Ram26], [DW16].

[2] This delineation is indeed rather vague; I discuss its inherent problems further below in 2.2.

$m + (n + 1) = (m + n) + 1$

$m \cdot 0 = 0$

$m \cdot (n + 1) = (m \cdot n) + m$

$\neg m < 0$

$m < n + 1 \leftrightarrow (m < n \vee m = n)$

- induction axiom:

$$(0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$$

- comprehension scheme:

$$\exists X \forall n(n \in X \leftrightarrow \phi(n))$$

  where $\phi(n)$ is any second order formula in which $X$ does not occur freely. Comprehension schemes in the context of Reverse Mathematics (RM) are labelled as $\mathsf{CA}$ (as in "comprehension axiom").

A subsystem of that theory is axiomatized by the same set of axioms only that the defining comprehension scheme (or a different set-existence principle) is restricted to formulae from a designated level of the arithmetical hierarchy as defined in Definition 0.1.

For instance, the subsystem $\mathsf{RCA}_0$ is the system consisting of the basic axioms, the induction axiom (not the full scheme) and a comprehension scheme which is restricted to $\phi(n)$ being a $\Delta^0_1$ formula.

**Def. 0.1** ($\Sigma^0_k$ and $\Pi^0_k$ formulas)**.**

An $\mathcal{L}_2$ formula $\phi$ which is equivalent to a formula with only bounded quantifiers is said to be $\Sigma^0_0$ and $\Pi^0_0$.

An $\mathcal{L}_2$ formula $\phi$ is $\Sigma^0_{k+1}$ if it is of the form $\exists n_0 \cdots \exists n_m \theta$ where $\theta$ is $\Pi^0_k$.

An $\mathcal{L}_2$ formula $\phi$ is $\Pi^0_{k+1}$ if it is of the form $\forall n_0 \cdots \forall n_m \theta$ where $\theta$ is $\Sigma^0_k$.

If $\phi$ is both $\Sigma^0_k$ and $\Pi^0_k$ then we classify it as $\Delta^0_k$.

**Remark 0.2.** The index $n$ in $\Sigma^n_k$ and $\Pi^n_k$ signifies the number of set quantifiers, as opposed to number quantifiers in $\phi$. In Classical RM (Classical RM) we only speak of subsystems (including $\mathbf{Z_2}$) which only have one set quantifier. Therefore $\mathbf{Z_2} = \bigcup_{k\in\omega} \Pi^1_k\text{-}\mathsf{CA}_0$, but it is traditionally denoted as $\Pi^1_\infty\text{-}\mathsf{CA}_0$. ◀

$\mathsf{RCA}_0$ is one of the five subsystems of $\mathbf{Z_2}$ that hold a unique place among others. This group is called the Big Five: $\mathsf{RCA}_0$, $\mathsf{WKL}_0$, $\mathsf{ACA}_0$, $\mathsf{ATR}_0$, $\Pi^1_1\text{-}\mathsf{CA}_0$, each of which is strictly weaker (in the sense specified above) than the ones on the right-hand-side to it, thus forming a hierarchy reminiscent of the Gödel Hierarchy. The research had quickly revealed that significantly many key theorems of "undergraduate" mathematics are either provable in $\mathsf{RCA}_0$ or equivalent to one of its extensions in the Big Five[3]. Below, following [Sim99], I briefly discuss the contents of these subsystems.

### 0.1.1 $\mathsf{RCA}_0$

To begin with the weakest one, $\mathsf{RCA}_0$ comprises of the basic axioms, the induction restricted to $\Sigma^0_1$ formulae, $\Sigma^0_1\text{-}\mathsf{IND}$, and *recursive comprehension scheme*, also known as $\Delta^0_1$ comprehension scheme.

**Def. 0.3** ($\Sigma^0_1$ induction)**.** $\Sigma^0_1\text{-}\mathsf{IND}$ is the restriction of the second order induction scheme (as defined above) to $\mathcal{L}_2$-formulae $\phi(n)$ which are $\Sigma^0_1$. Thus we have the universal closure of

$$(\phi(0) \land \forall n(\phi(n) \to \phi(n+1))) \to \forall n \phi(n)$$

where $\phi(n)$ is a $\Sigma^0_1$ formula.

All subsystems of $\mathbf{Z_2}$ used in RM only have this restricted form of induction.[4]

**Def. 0.4** ($\Delta^0_1$ comprehension)**.** The $\Delta^0_1$ comprehension scheme consists of universal closures of the formulae of the form

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \phi(n))$$

where $\phi(n)$ is a $\Sigma^0_1$ formula, $\psi(n)$ is a $\Pi^0_1$ formula, $n$ is a number variable and $X$ is a set variable which does not occur freely in $\phi(n)$.

The minumum $\omega$-model of $\mathsf{RCA}_0$ is exactly the collection of subsets $A$ of $\omega$ that are recursive. That

[3] The "reversals" in the literature most often refer to the implication from a theorem to an axiom, but I will mean the whole equivalence between an ordinary-mathematical theorem and a subsystem.

[4] In $Z_2$, $\Sigma^0_1\text{-}\mathsf{IND}$ is equivalent to $\Pi^0_1\text{-}\mathsf{IND}$, as proved in (§II.3) of [Sim99].

is, given a set $B \in \mathcal{P}(\omega)$, there is a unique smallest $\omega$-model of $\mathsf{RCA}_0$ containing $B$, consisting of all sets $A \in \mathcal{P}(\omega)$ which are recursive in $B$. That is why this system roughly corresponds to recursive analysis and is perfect for formalizing recursive mathematics. More to this point, any theorem of the first order theory $\Sigma^0_1$-Peano Arithmetic (**PA**) (**PA** with the induction axiom restricted to $\Sigma^0_1$ formulae) is a theorem of $\mathsf{RCA}_0$ and vice versa, *i.e.* $\Sigma^0_1$-**PA** is the first order part of $\mathsf{RCA}_0$ [Sim99, p. 25]. Most importantly, $\mathsf{RCA}_0$ has the same consistency strength as, and is conservative over, Primitive Recursive Arithmetic (**PRA**) (for $\Pi^0_2$ sentences)[5] [Sim99, pp. 57–8]. Apart from that, being relatively weak, $\mathsf{RCA}_0$ usually serves as a base system of Classical RM: the reversals are proved as theorems of this system.

### 0.1.2 $\mathsf{WKL}_0$

$\mathsf{WKL}_0$ comprises of $\mathsf{RCA}_0$ + *Weak König's Lemma i.e.* the assertion that any infinite subtree of a binary tree has an infinite path[6]. It is strictly stronger than $\mathsf{RCA}_0$, as the latter's minimum $\omega$-model does not satisfy weak König Lemma. Regarding the mathematical motivation for this subsystem, it turns out that it has a close connection to the notion of compactness: it is equivalent to the Heine/Borel covering lemma: Every covering of the closed interval $[0, 1]$ by a sequence of open intervals has a finite subcovering. This concept, in a sense expressing that one can find a "well-behaved" (finite, controllable) substructure of a given structure falls nothing short of one of the most important of the modern mathematical practice. With this subsystem we reach the level of certain core theorems of *infinitary* mathematics. Many properties of continuous functions are provable in (and often equivalent to) $\mathsf{WKL}_0$: uniform continuity, the maximum principle or Riemann integrability. That is why [Sim84, p. 786] calls this system a theory of continuity.

A paramount and somewhat surprising result is the proof of $\mathsf{WKL}_0$'s conservativity over **PRA** for $\Pi^0_2$-sentences (*i.e.* essentially arithmetical statements)[7] [Sim99, p. 381]. This means that $\mathsf{WKL}_0$ has the same consistency strength as **PRA**.

Formally, $\mathsf{WKL}_0$ is equivalent to $\Sigma^0_1$-separation schema, which is a universal closure of the formulae

[5] **PRA** and $\Sigma^0_1 - $ **PA** have the same consistency strength and their proof-theoretic ordinal is $\omega^\omega$.
[6] A full binary tree is defined as a set of finite sequences of 0's and 1's, $\{0, 1\}^{<\mathbb{N}}$.
[7] Harrington even showed that $\mathsf{WKL}_0$ is $\Pi^1_1$-conservative over $\mathsf{RCA}_0$ [Sim99, p. 372].

of the form

$$(\forall n \neg(\phi(n) \wedge \psi(n))) \rightarrow \exists X (\forall n(\phi(n) \rightarrow n \in X) \wedge \forall n(\psi(n) \rightarrow n \notin X))$$

where $\phi(n)$, $\psi(n)$ are $\Sigma^0_1$ formulae and $X$ does not occur freely in $\phi$.

### 0.1.3 $\mathsf{ACA}_0$

This subsystem is given by the comprehension schema restricted to arithmetical formulae *i.e.* the formulae with no set quantifiers. A couple of useful results show that arithmetical comprehension, *i.e.* $\Pi^1_0$-$\mathsf{CA}_0$, is equivalent to $\Sigma^0_1$-$\mathsf{CA}_0$, which is in fact equivalent to any $\Sigma^0_k$-$\mathsf{CA}_0$ as well as to $\Pi^0_k$-$\mathsf{CA}_0$ for $1 \leq k \in \omega$ [Sim99, p. 26]. Arithmetical comprehension together with the induction axiom give rise to the arithmetical induction scheme.

**Def. 0.5** (arithmetical induction scheme)**.**

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n(\varphi(n))$$

where $\varphi(n)$ is any $\mathcal{L}_2$ arithmetical formula.

Importantly, it can be showed that any theorem of first order arithmetic **PA** is a theorem of $\mathsf{ACA}_0$ [Sim99, p. 7]. In model theoretic terms, we have that for any model $(\mathcal{M}, \mathcal{P}(\mathcal{M}), +_{\mathcal{M}}, \cdot_{\mathcal{M}}, 0_{\mathcal{M}}, 1_{\mathcal{M}}, <_{\mathcal{M}})$ of $\mathsf{ACA}_0$, its first order part, $(\mathcal{M}, +_{\mathcal{M}}, \cdot_{\mathcal{M}}, 0_{\mathcal{M}}, 1_{\mathcal{M}}, <_{\mathcal{M}})$ is a model of **PA**. It follows that $\mathsf{ACA}_0$ is a conservative extension of **PA** and its proof-theoretic ordinal is $\varepsilon_0$.

The minimum $\omega$-model of $\mathsf{ACA}_0$ consists of all subsets of $\omega$ that are definable over $(\omega, +, \cdot, 0, 1, <)$ or equivalently the set of those subsets of $\omega$ that are Turing-reducible to some $n$th Turing jump of the empty set.

In a sense, $\mathsf{ACA}_0$ plays a central role in all of Classical RM; it is significantly weaker than the theories "above" it and by virtue of that it often can serve as a base theory for reversals about some pretty strong principles, but at the same time it is strong enough to prove many key ordinary-mathematical theorems that are unprovable in weaker systems. It often serves as a reference point for the plethora of theories in RM [AD18].

[Sim84, p. 786] says that $\mathsf{ACA}_0$ provides a good theory of sequential completeness and convergence. And indeed it is equivalent over $\mathsf{RCA}_0$ to Bolzano/Weierstrass theorem: *Every bounded sequence or reals has a convergent subsequence*, the Ascoli lemma: *For a sequence of continuous mappings $\langle f_n : n \in \mathbb{N}\rangle$ of a compact metric space $A$ into a compact metric space $B$, $f_n : A \longrightarrow B$, there exists a uniformly convergent subsequence*, or to some combinatorial principles (e.g. *König's lemma*), as well as some important results in algebra, such as existence of maximal ideals for countable commutative rings. The exceptions to what is provable in $\mathsf{ACA}_0$ tend to involve ordinal numbers; most of these exceptions are provable in $\Pi^1_1$-$\mathsf{CA}_0$.

Philosophically, $\mathsf{ACA}_0$ is sometimes identified with predicative mathematics [Sim99, p. 42], as developed by Feferman in [Fef64] and [Fef68]. Following Weyl's conception, Feferman took $\mathbb{N}$ as given and in his system only admitted objects defined with parameters of previously defined sets, thus avoiding the arbitrary reference to *any* set in the set-theoretic universe $\mathsf{V}$, which is common in classical mathematics. This system closely corresponds to $\mathsf{ACA}_0$.

### 0.1.4 $\mathsf{ATR}_0$

$\mathsf{ATR}_0$ is an "intermediate" system that was historically discovered as a response to the needs of RM. Being strictly stronger than $\mathsf{ACA}_0$ and strictly weaker that $\Pi^1_1$-$\mathsf{CA}_0$, it provides an important enhancement to the calibration of the framework to the study of ordinary mathematics.

The system consists of $\mathsf{ACA}_0$ plus a set existence principle called *arithmetical transfinite recursion.*

**Def. 0.6** (arithmetical transfinite recursion)**.** Let $\varphi(n, X)$ be an arithmetical formula, possibly with parameters. It can be viewed as an "arithmetical operator" $\Phi : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ defined by

$$\Phi(X) = \{n \in \mathbb{N} : \varphi(n, X)\}$$

Now, let $(A, <_A)$ be some countable well-ordering. If we transfinitely iterate the operator $\Phi$ along $(A, <_A)$, we get some set $Y \subseteq \mathbb{N}$, satisfying the condition that for each $a \in A$, $Y^a$ is the result of iterating $\Phi$ along $(A, <_A)$ up to (but not including) $a$. *Arithmetical transfinite recursion* asserts the existence of a set $\mathbf{Y}$ which is the result of applying $\Phi$ one more time (*i.e.* $Y^a < \mathbf{Y}$ for any $a \in \mathbb{N}$).

More concisely, the axiom asserts that the Turing jump operator can be iterated along any countable well-ordering, starting at any set. Therefore the formation of sets in $\mathsf{ATR}_0$ is bounded by $\omega_1^{\mathrm{CK}}$, the first countable ordinal that is not the order type of a recursive well-ordering of $\omega$. Needless to say, then, that [Sim99, p. 48] calls it the theory of countable ordinals.

An $\omega$-model **HYP**, which is the collection of all hyperarithmetical subsets of $\omega$ is the intersection of all $\beta$-models of $\mathsf{ATR}_0$; **HYP**, however, is still too small to model $\mathsf{ATR}_0$ and the theory does not possess a minimum $\omega$-model.

$\mathsf{ATR}_0$ is conservative over Feferman's first-order theory of predicative analysis $\mathsf{IR}$ for $\Pi^1_1$-sentences and it has been shown that its proof-theoretic ordinal is $\Gamma_0$, the Feferman-Schütte ordinal. Therefore [Sim99, p. 41] identifies it philosophically with "predicative reductionism" (the program of reducing mathematics to predicative methods, analogously to Hilbert's finitism).

$\mathsf{ATR}_0$ is significantly stronger than $\mathsf{ACA}_0$ and there are important results provable in the former for which the latter does not suffice. For instance, $\mathsf{ACA}_0$ cannot even prove comparability of countable ordinals. $\mathsf{ATR}_0$ is the first theory on our hierarchy in which one can work out certain results about uncountable sets in topology or descriptive set theory; some important theorems include Souslin's theorem or the perfect set theorem. In fact, the latter is equivalent to the subsystem.

Notably, $\mathsf{ATR}_0$ is equivalent to the $\Sigma^1_1$-separation scheme, which yields an elegant order of strength:

$$\underset{\Delta^0_1\text{-}\mathsf{CA}_0}{\overset{\mathsf{RCA}_0}{\equiv}} < \underset{\Sigma^0_1\text{separation}}{\overset{\mathsf{WKL}_0}{\equiv}} < \underset{\Pi^1_0\text{-}\mathsf{CA}_0}{\overset{\mathsf{ACA}_0}{\equiv}} < \underset{\Sigma^1_1\text{separation}}{\overset{\mathsf{ATR}_0}{\equiv}} < \Pi^1_1\text{-}\mathsf{CA}_0$$ [8]

### 0.1.5 $\Pi^1_1$-$\mathsf{CA}_0$

$\Pi^1_1$-$\mathsf{CA}_0$ is the strongest of the Big Five that is the result of making possible to quantify over elements of sets:

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

(where $\varphi(n)$ is a $\Pi^1_1$ formula in which $X$ does not occur freely) is the defining comprehension scheme of the subsystem, which is equivalent to the existence of the hyperjump of every set and strong enough to establish existence of sets of functions.

[8] Sadly, $\mathsf{ACA}_0$ is not equivalent to $\Delta^1_1$-$\mathsf{CA}_0$, which in fact is somewhat stronger than it.

Similarly to the previous subsystem, $\Pi^1_1$-$\mathsf{CA}_0$ is too strong to have a minimum $\omega$-model. Instead, a subset $\mathcal{S}$ of $\omega$ is a $\beta$-model of $\Pi^1_1$-$\mathsf{CA}_0$ if and only if $A \in \mathcal{S}$ *implies* $\mathrm{HJ}(A) \in \mathcal{S}$, where $\mathrm{HJ}(A)$ denotes the hyperjump of $A$. And we have a minimum $\beta$-model for $\Pi^1_1$-$\mathsf{CA}_0$, namely

$$\{A \in \mathcal{P}(\omega) : \exists n \in \omega(A \leq_H \mathrm{HJ}(n, \varnothing))\}$$

where $A \leq_H B$ means that $A$ is hyperarithmetical in $B$. We also have minimum $\beta$-models for each $\Pi^1_k$-$\mathsf{CA}$, $1 \leq k \leq \infty$.

$\Pi^1_1$-$\mathsf{CA}_0$ is a strong system in which one can develop significant portions of topology and descriptive set theory, inaccessible to the previous systems. For instance, some (encodings of) classical results about Borel and analytic sets are doable in $\Pi^1_1$-$\mathsf{CA}_0$; notable examples are the Cantor/Bendixon Theorem (Every closed set in $\mathbb{N}^{\mathbb{N}}$ is a union of a perfect closed set and a countable set) or Kondo's Theorem (coanalytic sets in Cantor spaces have the uniformization property). In fact, both are equivalent to the subsystem's characteristic axiom[9]. Moreover, $\Pi^1_1$-$\mathsf{CA}_0$ gives some illuminating results about relations between the Ramsey Theorem and the Axiom of Determinacy (AD). This is generally because the system provides a good theory of countable well founded trees. [Sim99, p. 22] identifies it with impredicative mathematics.

In his [Tak67], Takeuti established the consistency strength of $\Pi^1_1$-$\mathsf{CA}$ , a strong subsystem with full induction, at the ordinal $\Psi_{\Omega_1}(\Omega_\omega)$ (where $\Psi$ is an appropriate collapsing function in the Buchholz-Schütte notation). Later the proof-theoretic strength of $\Pi^1_1$-$\mathsf{CA}_0$ was located at $\Psi_0(\Omega_\omega) = sup\{\Psi_0(\Omega_n) : n < \omega\}$[10] which is much bigger than $\Gamma_0$, but turns out to be a recursive ordinal still smaller than that of Kripke-Platek set theory ($\mathsf{KP}$).

To give more insight into the practice of RM, we continue with a remark on practice of proving reversals and a sketch of such a result following [Sim99, pp. 32–33].

[9] Admittedly though, the former holds over $\mathsf{ACA}_0$ and the latter over $\mathsf{ATR}_0$ and *not* over $\mathsf{RCA}_0$. We will discuss the meaning of this later in section 2.2.

[10] See [Tak13, ch. 5] and for its position in relation to other ordinals see [Mad17].

## 0.2 Reversals

Typically, a proof in RM begins with choosing a suitably weak base theory[11] $\mathbf{T}$ and formalizing a theorem $Thm$ under consideration in the language of the second order arithmetic $\mathcal{L}_2$, thus obtaining a formal version $\vartheta$ of $Thm$. This is done via encodings of mathematical objects in the conventional fashion, with $q \in \mathbb{Q}$ being an ordered pair of positive integers and a real being a sequence of rational numbers $\langle q_k : k \in \mathbb{N}\rangle$ such that $\forall k \forall i(|q_k - q_{k+i}| \leq 2^{-k})$ and functions being coextensional with "many-one" relations which are defined as sets, *etc.* (see *e.g.* [Sim99, pp. 73–76]).[12] If $\vartheta$ is not provable in $\mathbf{T}$ (which after Friedman began to be shown with model-theoretic tools), one takes a stronger extension $\mathbf{T}'$ of $\mathbf{T}$, and proves $\vartheta$ within it (often by a straightforward formalisation of the ordinary-mathematical proof of $Thm$ [Eas15, pp. 1–2]. Then, for the other direction of the equivalence, one assumes $\mathbf{T} + \vartheta$ and sets out to prove the stronger axioms of $\mathbf{T}'$. This establishes a reversal between $\vartheta$ and $\mathbf{T}'$ modulo the base theory $\mathbf{T}$. Let us now turn to an example of employing this strategy. I will later compare it with the constructive one in section 3.3.4.

### 0.2.1 An Example of a Reversal

**Def. 0.7** (Bolzano-Weierstraß)**.** Every bounded sequence of real numbers has a convergent subsequence.

First, we define a real number as above.

**Def. 0.8** (within $\mathsf{RCA}_0$)**.** A *real number* is defined to be a sequence of rational numbers $\langle q_k : k \in \mathbb{N}\rangle$ such that $\forall k \forall i(|q_k - q_{k+i}| \leq 2^{-k})$. Two real numbers $\langle q_k : k \in \mathbb{N}\rangle$, $\langle q'_k : k \in \mathbb{N}\rangle$ are *equal* if $\forall k(|q_k - q'_k| \leq 2^{-k+1})$.

Observe that this definition varies from the ordinary-mathematical one, where a real is identified with an equivalence class of Cauchy sequences. This choice of simplified representation is forced by the fact that $\mathcal{L}_2$ is ill-fitted to represent such an "infinitary" concept.

[11] In the present thesis whenever a reversal is presented and the base theory is not specified, we assume that it is $\mathsf{RCA}_0$, unless explicitly stated otherwise.

[12] Already at this point I want to stress that although *classical* methods of encoding are deeply cemented in the reverse-mathematical practice of formalization, they are by no means the only ones available. Alternative methods have a lasting presence and go at least as far back as to [GMR58], where Grzegorczyk *et al.* were able to avoid some of the coding machinery that must otherwise be used when only quantification over sets is permitted within a second order functional calculus.

Now, we define sequences of real numbers in $\mathcal{L}_2$.

**Def. 0.9** (within $\mathsf{RCA}_0$)**.** A *sequence of real numbers* is a function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$ such that for each $n \in \mathbb{N}$ the function $f_n : \mathbb{N} \longrightarrow \mathbb{Q}$ defined by $f_n(k) = f(k, n)$, is a real number.
We say that a sequence is *monotone* if it is increasing or decreasing, *i.e.* if either $f_n(k) < f_{n+1}(k)$ when $n < n + 1$ or $f_{n+1}(k) < f_n(k)$ when $n + 1 < n$, for all $n, n + 1 \in \mathbb{N}$. We say that such a sequence *converges to* $x$, (where $x = \lim_n x_n$) if $\forall_{0<\epsilon}\exists_n\forall_i\ (|x - x_{n+i}| < \epsilon)$. We say that a sequence is *convergent* if the limit $x$ exists.

We are now ready to state the reversal.

**Thm. 0.10** (Friedman)**.** *Over $\mathsf{RCA}_0$, the Bolzano-Weierstraß theorem is equivalent to the arithmetical comprehension scheme.*

*Proof.* First we show that the usual proof of the Bolzano-Weierstraß goes through in $\mathsf{ACA}_0$. For this, we show that if a sequence of reals is bounded, then by the monotone convergence theorem, it also has a convergent subsequence.

Let $f$ be an arbitrary bounded sequence of reals. Since $f$ is bounded, we can define $g : \mathbb{N} \longrightarrow \mathbb{N}$ by $g(k) =$ the largest $i < 2^k$ such that $i \cdot 2^{-k} \leq x_n \leq (i + 1) \cdot 2^{-k}$ for infinitely many $n \in \mathbb{N}$. $g$ exists by the $\Pi^1_0$ comprehension (similarly for the case of an infimum of decreasing sequences).

Now, set $x = \langle q_k : k \in \mathbb{N}\rangle$ where $q_k = g(k) \cdot 2^{-k}$. We can readily compare it with the definition 0.9 to verify that $\forall_{0<\epsilon}\exists_m\forall_n\ (m < n \rightarrow x_n < x + \epsilon)$ and that $\forall_{0<\epsilon}\forall m\exists n\ (m < n \wedge |x - x_{n|} < \epsilon)$, *i.e.* that $x = \lim\ sup_n x_n$.

Now define the subsequence $\langle x_n : n \in \mathbb{N}\rangle$ of $f$ to be the sequence satisfying $|x - x_n| \leq 2^{-k}$ of "peaks" of $f$, that is the sequence of $sup_n$. Clearly $x = \lim_k x_{n_k}$, so we have that $\mathsf{ACA}_0$ proves that a bounded sequence of reals has a subsequence converging to $x$, thus yielding the forward direction of the equivalence.[13]

For the second direction we reason in $\mathsf{RCA}_0$. Assume the Bolzano-Weierstraß theorem. Since

[13] Note that we used the sequential completeness of reals, which is equivalent to $\mathsf{ACA}_0$; in the key step we also used the monotone convergence theorem, which is equivalent to $\mathsf{ACA}_0$ (for proof through the sequential least upper bound principle, see [Sti18, pp. 113–118]).

$\mathsf{ACA}_0$ is equivalent to $\Sigma^0_1$-$\mathsf{CA}$, it suffices to prove for some $\Sigma^0_1$ formula $\varphi(n)$ stating the existence of $\{n : \varphi(n)\}$. So we choose

$$\varphi(n) \equiv \exists k \; \theta(k, n) \tag{1}$$

where $\theta$ is of course $\Sigma^0_0$, such that for each $k \in \mathbb{N}$ we define

$$c_k = \sum \{2^{-n} : n < k \wedge (\exists m < k \; \theta(m, n)\}$$

Then $\langle c_k : k \in \mathbb{N} \rangle$ is an increasing sequence of rational numbers which is bounded by $k$. This sequence exists by $\Delta^0_1$-comprehension (that we have in $\mathsf{RCA}_0$). By the Bolzano/Weierstraß theorem, $c = \lim_k c_k$ exists. So we have the formula

$$\forall n (\varphi(n) \leftrightarrow \forall k \; (|c - c_k| < 2^{-n} \rightarrow \exists m < k \; \theta(m, n)) \tag{2}$$

But this means that 1 and 2 are equivalent and since they are $\Sigma^0_1$ and $\Pi^0_1$ respectively, by $\Delta^0_1$-comprehension we have

$$\exists X \; \forall n \; (n \in X \leftrightarrow \varphi(n))$$

This establishes the $\Sigma^0_1$-comprehension and therefore also the arithmetical comprehension. ■

With this we conclude our bird's eye view over the Big Five. The following section offers a reading of the deeper goals underlying the reverse mathematical endeavor.

# 1 Preliminaries: the Motivations Behind Reverse Mathematics

## 1.1 The Roots of Reverse Mathematics

RM is a foundational program whose goal is most concisely described as finding the sufficient and necessary axioms to prove given mathematical theorems. It strives to give insight about what one might call “the epistemic weight” of studied theorems. The characteristic attitude of RM is not finding ways to prove a given statement by any means necessary, but trying to pin down the minimal collection of assumptions that imply it. In this way, the objective is to specify the “right” axioms in the sense that these axioms can be proved from the theorem [Fri75, p. 235] (hence *reverse* mathematics).

As much as philosophical considerations motivated the inception of RM as a branch of mathematical logic in the ’70s, the idea of finding the “right” axioms is seemingly as old as systematic mathematics itself. Conditioned by a century-long study of countless formal logics, today we tend to have a pretty relativistic inclinations with regard to axiom systems, but for generations of mathematicians “axiom” was identified with an intuitive and obvious assumption. That is why Euclid’s fifth postulate, believed to be far from trivial, troubled geometers since antiquity. Not being able to show that it is derivable from the first four axioms, they tried to prove that its negation contradicts these assumptions. This has lead to discoveries of equivalences between the postulate and several other statements (notably the existence of a rectangle, the sum of a triangle’s angles being equal to two right angles, or that three noncollinear points lie on a circle)[1], and ultimately to a demonstration

[1] For a broader historical commentary of the V Postulate, see [Hea56].

of its independence from the previous four axioms by Beltrami and Poincaré. These equivalences shed more light on how *strong* these theorems are, *i.e.* that given a *base theory* of the first four axioms, one cannot derive them without assuming something more. At the same time these results show how significant the fifth postulate is for the development of Euclidean geometry: the hallmark Pythagorean theorem is also equivalent to the parallel postulate, and therefore independent of the "obvious" axioms. These results were a by-product of a quest to find the most suitable assumptions for the Euclidean geometry or in other words, to find the *true* Euclidean axioms.

The ardent demand for foundational clarification was rekindled when the consequences of the use of the so-called infinitary methods began to be uncovered in the early 20th century. Until that era mathematics tended to be stubbornly algorithmic (and so constructive), but the new mathematics[2] developed in the fields such as real analysis, algebraic number theory, point-set topology and set theory by Heine, Borel, Russell, Cantor, Dedekind, Lebesgue, Steinitz, Hausdorff or J. Tannery was, mostly implicitly, using methods such as the Axiom of Choice (AC)[3] [Moo82, pp. 8–16, 21], thus resolutely departing from the constructive nature of the science as hitherto known. This direction naturally resulted in Cantor's claim of the well-orderability of any set (later proved by Zermelo to be equivalent with the full AC), which was treated scathingly by many thereafter. The titanic work [Sie18] by Sierpiński and the Lvov-Warsaw School unveiled how significant the axiom was by showing its indispensability in important results established even by its critics (e.g. Lebesgue). For that reason Sierpiński thought that it should be precisely determined which proofs depended upon the axiom, notwithstanding the foundational ideology one might endorse [Moo82, p. 200]. While Sierpiński did not use any formal framework (in today's sense of the term) to study AC, his investigation motive was extremely close to the basic inspiration of RM and arguably helped pave the way of the future research in the field due to his clear statement of the *pragmatic* dimension of the inquiry; this was different from the philosophical or methodological stimulus surrounding the analysis of the parallel postulate. The emphasis was put on finding the role the axiom plays in mathematics instead of finding the "true" extent of the necessity of AC[4]. But the case of AC is just one of many problems mathematics faced in the wake of the foundational crisis.

[2] Simpson might call it the "set-theoretic" sort.

[3] Though rarely in its uncountable version and most often in the form of employing denumerably many arbitrary choices or dependent choices. Countable and Dependent Choice is today widely accepted as constructive, but the infinitary tendency that followed soon after is also essential in this form of the axiom.

[4] Of course the conclusions about latter stem from those about the former, modulo one's foundational stance.

In 1902 Hilbert included the challenge for finding a finitistic proof of consistency of arithmetic on his list of the most pressing issues in mathematics of the day [Hil02] to settle all arguments about admissibility of the newly developed infinitistic methods. The idea was to show that even if mathematicians did apply the so-called ideal methods, the latter could be safely reduced to the "concrete" finitistic ones by showing in the abstract that arithmetic (and all other fields that were reducible to it) could be proved to be a consistent theory solely through finitistic means. That would effectively establish that the infinitary mathematics is merely a useful tool, a shorthand for the practically more arduous but "safe" finitistic portion and that it does not bring in any additional precarious content. The popular belief has it that Gödel's Incompleteness Results demolished such ambitions as the II Incompleteness Theorem (II GIT) states that if a given formal theory $T$ with a humble portion of arithmetic is consistent, the sentence expressing its consistency $Con(T)$ is unprovable in $T$[5]. But the story is not that simple: together with limitations, Gödel's results came with several paths forward[6]. One of them was called the "Gödel Hierarchy" [Sim10]. It is a hierarchy of theories $T_0, T_1, ...$ where any $T_i$ satisfies the conditions of the I Incompleteness Theorem (I GIT) and we write

$$T_i < T_j$$

for any two theories such that "$T_i$ is consistent" is a theorem of $T_j$. Then we say that *consistency strength* of $T_i$ is less than that of $T_j$ or that $T_j$ has an oracle for $T_i$, or that $T_i$ is *interpretable* in $T_j$ but not vice versa[7]. Notably, many important foundational theories are linearly ordered by $<$. An alternative ordering of theories stems from I GIT, where we write

$$T_i \subset T_j$$

meaning that the set of theorems of $T_i$ comprises a proper subset of theorems of $T_j$. In this way, we can think of $T_i \subset T_j$ as stating that $T_j$ is *stronger* than $T_i$. $\subset$- and $<$-orderings in many cases coincide, but that's not always the case. One can always "artificially" construct a pair of incomparable theories, but what makes the Hierarchy significant is that "mathematically natural" theories abide by a smooth order [Sim10, p. 3]. I am going to discuss what "mathematically natural"

[5] *Constructively* unprovable, as Gentzen gave an infinitary proof through cut-elimination.

[6] An important one that I will not deal with is described in Artemov's [Art19], where consistency of **PA** *is* proved in **PA**, but not by encapsulating the consistency property in a single formula.

[7] This hierarchy was anticipated in Tarski's work on truth predicate and formally studied in Turing's [Tur04] PhD thesis.

might actually mean in section 2.1 of the next chapter and for now will just focus on the meaning of the Hierarchy for RM.

From previous examples we know that if assumptions one uses to prove results are stated clearly enough, it is in principle possible to analyze them and compare them with theorems, gaining more insight into the nature of the studied axioms. For instance, consider the three following statements:

($a$) For any set $X$ of nonempty sets, there exists a choice function $f$ that is defined on $X$ and maps each set of $X$ to an element of that set.

($b$) Every filter on a set can be extended to an ultrafilter.[8]

($c$) For every set $X$ there is an ordinal $\alpha$ such that $X$ is equipollent to some subset of the power set of $\alpha$.[9]

What is the relation between them? The first two are well-known principles and ($c$) seems to be closely related to them. A logician will probably know that ($a$) is the strongest of them, but the relation between ($b$) and ($c$) is not at all obvious. It turns out that in **ZF**, ($a$) indeed implies the other two, but interestingly ($b$) and ($c$) are independent of each other despite the fact that both imply the Ordering Principle[10] [Moo82, p. 329]. From these results one can not only learn that AC is significantly stronger than both of these highly non-constructive principles, but also that the axiom has consequences in seemingly distant fields like abstract algebra and set theory, and that they have an important intersection in the form of the Ordering Principle. All of these methods could have been discovered without the foundational study, but these relations would have remained uncovered.

Now, what Gödel Hierarchy brings to the picture is the promise of a single "measure" that would give even more clarity to our understanding of the connections between these principles by relating them to a linear and relatively uniform hierarchy of consistency statements. This is an important enhancement because without it any result connecting theorems and axioms was being established in isolation, whereas now we have a single book-keeping device.

[8]The Ultrafilter Theorem is equivalent to the Boolean Prime Ideal Theorem.

[9]This statement is equivalent to the Kinna-Wagner Principle: For every set $X$ there is a function $f$ such that, for each subset $A$ of $X$ with two or more elements, $f(A)$ is a non-empty proper subset of $A$.

[10]"Every set can be ordered."

Somewhat anachronistically, one can compare that idea with different projects employing such ladders for foundational ends, such as infinitary logic, Gentzen's ordinal analysis or set-theoretic hierarchy of cardinal axioms, but it turns out that the "easy" partial order given by the Gödel Hierarchy and some of the related ladders is not enough to classify many results, and that some might fall far from any notion of comparability with it [NS17]. For these ends, one needs to refine the framework and place ordinary mathematics in focus. And that seems to be the distinguishing aspect of RM.

## 1.2 The Ascending and Descending Tendencies in Classical RM

In hindsight, it seems that for Friedman, Classical RM was but an intermediate stage of a grander program of "delivering incompleteness at the doorstep of all mathematicians", a plan aiming at showing that concrete and simple statements occurring in ordinary mathematical practice are sometimes independent of the accepted formal frameworks (such as **ZFC**); a way of arguing for integrating stronger assumptions such as some large cardinal axioms into popular mathematical practice. I call this objective "the ascending tendency" in RM, due to its orientation of incorporating ever-stronger principles into mathematics. Despite (or perhaps due to) the truly Gödelian nature of the idea, the research went in the opposite direction[11], namely showing how much of fundamental theorems in mathematics are provable in or equivalent to the relatively weak systems and in this way trying to recover Hilbert's Program[12], according to the idea laid down in [Kre68]. [Fef88] gives the survey of it as follows.

Let $T_1$ and $T_2$ be theories in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, both containing **PRA**. Let $\Phi \subseteq Form_{\mathcal{L}_1} \cap Form_{\mathcal{L}_2}$ be a primitive recursive set of formulae containing all closed terms $t_1 = t_2$. Then a proof-theoretic reduction of $T_2$ to $T_1$ which conserves $\Phi$ is a partial recursive function $f$ that given any

[11] Notably though, there were significant developments concerning various versions the finite Ramsey Theorem and related principles being independent of the casual systems [Yok23]. Some fruits of Friedman's pursuit are a massive unpublished manuscript [Fri11] and a more recent attempt [Fri17].

[12] To inspect the discrepancy in purposes compare Simpson's "*Unfortunately*, Gödel's [incompleteness] theorem shows that any such realization of [full ambitions of Hilbert's Program] is impossible" [Sim88, 352, italics added] with Friedman's "[W]e have continued the search for additional Concrete Mathematical Incompleteness that opens up new connections with normal mathematics. [...] The extent to which these new developments *invade* mathematics remains to be seen" [Fri11, 8–9, italics added].

$T_2$-proof of a sentence $\phi \in \Phi$ produces a $T_1$-proof of it. The core goal is to prove the existence of $f$ in $T_1$. Now, if $T_2$ proves "$0 = 1$" then $f$ will yield a $T_1$-proof of "$0 = 1$", from which it follows that $T_1$ proves (the formalisation of) "*If $T_1$ is consistent then $T_2$ is consistent*".

Such a relative consistency proof establishes a finitary reduction of $T_2$ to $T_1$ assuming that $T_1$ is a finitary system. These reductions are of great foundational significance as they give a base of finitary justification to portions of mathematical practice [Sim85]. This strategy admittedly follows Tait's thesis that "finitary" is to be identified with **PRA** [Tai81]. Simpson [Sim88, p. 352] asserts that

> There seems to be a certain naturalness about PRA which supports Tait's conclusion. PRA is certainly finitistic and "logic-free", yet sufficiently powerful to accommodate all elementary reasoning about natural numbers and manipulations of finite strings of symbols. PRA seems to embody just that part of mathematics which remains if we excise all infinitistic concepts and modes of reasoning. For my purposes here I am going to accept Tait's identification of finitism with PRA.

These heuristic remarks account for what Simpson says about the justification of Tait's thesis and having thus established clearly the system to which infinitary reasoning should be reduced to, he continues with a discussing the systems conservative over **PRA**. From the previous section we know that both $\mathsf{RCA}_0$ and $\mathsf{WKL}_0$ satisfy Simpson's aims, but it turns out that we can go even further. Let $\alpha$ denote a sequence of dense subcollections of $2^{\mathbb{N}}$ which is arithmetically definable from a given set. Simpson and Brown showed that $\mathsf{WKL}_0$ + "*there exists an infinite sequence of 0's and 1's which meets each of the given $\alpha_i$'s*"[13] is $\Pi^1_1$-conservative over $\mathsf{RCA}_0$. In conjunction with Parsons' result of conservation of $\mathsf{RCA}_0$ over **PRA**, this implies $\mathsf{WKL}_0^+$'s conservation over **PRA** for $\Pi^0_2$ sentences.

This is a tremendous result from the perspective of partial realization of Hilbert's Program, as [Sim88, p. 361] estimates that at least 85% of existing mathematics can be formalized within $\mathsf{WKL}_0$ or $\mathsf{WKL}_0^+$. I call Simpson's goal of founding as much mathematics as possible on the firm finitistic grounds "the descending tendency" of RM. Its divergence with Friedman's ideal is again acutely felt

[13] this subsystem is labelled as $\mathsf{WKL}_0^+$, and its characteristic principle is equivalent to a stronger version of Baire category theorem for Cantor space $2^{\mathbb{N}}$.

in context of the above "85% remark" when Friedman proclaims that "normal mathematical activity up to now represents only an infinitesimal portion of eventual mathematical activity" [Fri11, p. 8].

But this is not to suggest that these tendencies are essentially incompatible. It seems perfectly reasonable to try and delineate what portion of ordinary mathematics is reducible to finitistic – or possibly computable – methods, while simultaneously discovering ever-stronger principles that are independent of portions identified as decidedly infinitistic which are however useful for the practice.

The upshot of this distinction for the present thesis is that these tendencies are governed by different ambitions hence a philosophical evaluation of Classical RM must be informed by the intrinsic goal for which specific parts of the program are executed. For instance, the notion of mathematical practice must be viewed through the lens of this distinction, because the ascending tendency's notion of practice roughly refers to the methods mathematicians (now or in the future) would employ to prove more theorems, without any restriction as to what techniques should be used[14], whereas the central focus of the notion of practice at play in the descending tendency is the computational, recursive or numerical content of the theorems that are proved. The latter aims at basing what mathematicians do on the finitistic mathematical practice, the former at exploding what they do to reach new horizons. Therefore it seems important not to conflate these notions and apply the expectations of one to the results of another.

Nevertheless, what binds both tendencies into a philosophically unified enterprise is the apparent accommodating nature of Classical RM with regards to hitherto developed foundational programs. According to [Sim99, p. 42], one of the virtues of the Big Five is that they correspond to "various well known, philosophically motivated programs in foundations of mathematics", as suggested in table 1.1.

| **Subsystem** | **Program** | **Identified With** |
|---|---|---|
| $\mathsf{RCA}_0$ | constructivism | Bishop |
| $\mathsf{WKL}_0$ | finitistic reductionism | Hilbert |
| $\mathsf{ACA}_0$ | predicativism | Weyl, Feferman |
| $\mathsf{ATR}_0$ | predicative reductionism | Friedman & Simpson |
| $\Pi^1_1$-$\mathsf{CA}_0$ | impredicativism | Feferman *et. al.* |

Table 1.1: Foundational programs and the Big Five [Sim99, p. 42]

[14] The slogan of this attitude could be Wittgenstein's "Don't for heaven's sake, be afraid of talking nonsense! But you must pay attention to your nonsense" from *Culture and Value*.

As discussed in the outline of the Big Five, we already know what Simpson means by "finitistic reductionism" in regards to $\mathsf{WKL}_0$ and "predicativism" for $\mathsf{ACA}_0$. "Predicative reducibility" of $\mathsf{ATR}_0$ refers to Friedman's conservation result over Feferman's $\mathsf{IR}$, which implies that any $\Pi^1_1$ consequence of any theorem provable in $\mathsf{ATR}_0$ is predicatively true (see the discussion in [Sim85, pp. 154–156]). The impredicativity of $\Pi^1_1$-$\mathsf{CA}_0$ is pretty self-explanatory. It is true that *some* of these subsystems can be legitimately identified with some of the famous stances in the foundations. But the identification is far from complete. There are important projects such as Sam Buss's Bounded Arithmetic ([Bus86],[BS90]) connecting the study of feasible computability and complexity with questions about provability, that cannot be included as subsystems of $\mathbf{Z_2}$ due to it being too strong to handle feasibility even within $\mathsf{RCA}^*_0$.

However Simpson never claimed that the Big Five (or generally the subsystems of $\mathbf{Z_2}$) are an exhaustive representation of various foundational approaches problematic case is the identification of $\mathsf{RCA}_0$ with constructivism. Bishop-style analysis (that is claimed to be roughly represented by $\mathsf{RCA}_0$) and that in $\mathsf{RCA}_0$ diverge as soon as on the the definition of a real number and equality between two reals[15]. Friedman, Simpson and Smith say in [FSS83, p. 146] that "The axioms of $\mathsf{RCA}_0$ are 'constructive' in the sense that they are formally consistent with the statement that every total function from $\mathbb{N}$ into $\mathbb{N}$ is recursive", a rather deficient notion of constructivity indeed. Still, they do not fail to acknowledge three crucial formal differences[16]:

(*i*) The constructivists assume unrestricted induction on the natural numbers, while in $\mathsf{RCA}_0$ we only assume $\Sigma^0_1$ induction.

(*ii*) We always assume the law of the excluded middle, while [constructivists] deny it.

(*iii*) The meaning which the constructivists assign to the logical connectives and quantifiers is incompatible with our classical interpretation.

There are therefore some fundamental points of discord between the two programs, rendering the inclusion of constructive mathematics into Classical RM unattainable. This situation is undesirable for at least three reasons. First, from the perspective of philosophy of mathematics, it seems

[15]Bishop [BB85, pp. 18–19] defines a real number to be a *regular* sequence of rationals, a condition superfluous in case of $\mathsf{RCA}_0$, and gives a method of construction of an integer-valued index witnessing the transitivity of the equivalence relation between two real numbers, which [Sim99, p. 74] does not have to bother himself with.

[16][Sim99, p. 31] cites exactly the same differences.

that a foundational program claiming to subsume traditional schools within a single architecture which disregards intuitionism or constructivism (that are the approaches that arguably experienced the most extensive developments throughout the past century) is detrimentally incomprehensive. Second, the reason for which it should be preferable to employ constructive methods and theories in the analysis of ordinary mathematics is exactly the same as the reason for using a weak theory as a basis for proving reversals. Since constructive theories are typically weaker than the classical ones, intuitively they should provide a finer metric for RM's purposes; at the same time, there is little reasons to worry that they would be *too weak*, as we have many conservation results for systems based on intuitionistic logic over classical systems. Third, and this has to do with the very nature of constructive methods, the cause of (*iii*), the root of this incompatibility is constructivists' concern about the meaning of the terms they use. In Takeuti's phrasing,

> The fact that no contradiction arises does not explain what it means to say that a theorem is provable from the comprehension axiom. Nonconstructive proofs provide no insight into this important question. On the other hand, a constructive proof strengthens our intuition and adds meaning to the theorem. [Tak13, p. 298]

In other words, using constructive methods for reverse mathematics holds a promise of more insight into the substance of reversals and shedding more light on the structure of the equivalences between theorems and axiom schemes.

But without a solid presentation of the consequences of both the philosophical underpinnings and choice of framework of Classical RM that would provide a motivation to look into the constructive solutions, it would seem rather arbitrary to move on to them, save for sheer curiosity. After all, maybe Classical RM does what we expect of it just fine? This prompts the first main theme of the present thesis.

**Question 1.** *What are the formal or philosophical disadvantages of Classical RM that hinder the foundational analysis?* ◀

Now, if the results of this investigation will pose some problems for accepting Classical RM or specific parts of its framework, switching to constructivism is but one of many possible reactions. A thorough examination of all of these is beyond both the scope of the present work as well as my expertise. Studying possible constructive reactions should therefore be seen as a conscious choice of research direction rather than the only resort. I believe that taking this path is especially plausible given that there are several constructive approaches to finding the "proper" axioms for ordinary-mathematical theorems, such as reverse mathematics based on BISH as developed by Hajime Ishihara and others (see the overview in [Ish06]); Feferman's Explicit Mathematics (EM) based on Bishop's mathematics, but of a slightly different flavor (developed in [Fef75], see broader discussion in [Fef79]); work in second order Heyting Arithmetic (**HA**), proposed by Troelstra and Kreisel in [KT70] and developed by Ishihara, Nemoto and others. Therefore my work is going to amount to a practical and philosophical evaluation and comparison of different frameworks, rather than formulating an original system and reinventing the wheel. That being said, the answers to Question 1 will naturally guide towards a specific build-up of a plausible system for RM.

Finally, prompted by Benedict Eastaugh's [Eas15], I will come full circle and compare the facets of the classical and constructive approaches, trying to compare the advantages and disadvantages of these analyses. This poses the second main question of the present thesis:

**Question 2.** *Does Constructive RM face similar or additional issues and if so, which approach is preferable?* ◀

The rationale behind this question is that depending on the answers to Question 1, I will try to see how does the Constructive Reverse Mathematics (Constructive RM) score against the challenges of foundational analysis of ordinary mathematics that Classical RM faces. If it goes down well, I will try to see whether some portions of Classical RM can be recovered from the point of view of Constructive RM or conversely, whether there are any practical (*i.e.* stemming from the formal determinants of the systems considered) visions for accommodating *some* form of constructivist language within the Friedman-Simpson program – which hopefully will help dam up the "detrimental incomprehensiveness" of Classical RM. In the next chapter I examine the first Question, chapter 3 is devoted to a discussion of Constructive RM and attempts at answering the second Question.

# 2 | The Difficulties of Classical RM

## 2.1 The Shortcomings of the Build-up of the Framework

**Remark 2.1** (Heuristics and Preview)**.** The very phrasing of the title of this chapter begs the question of what should one qualify as a difficulty of a given formal framework. In what follows I endorse a somewhat pragmatic approach: The previous sections equipped us with a general understanding of both the formal aspects and the philosophical buildup of Classical RM and I believe that the most appropriate way to examine it is by trying it against its own goals and motivations; I believe that if a scientific theory's practice succeeds in tackling the problems it poses for itself, the theory is justified. I will therefore avoid pushing any full-blooded philosophical charges of a Brouwerian (or any other) ideological flavor.
Given the discussion in the preceding sections, we can identify three main goals of Classical RM:

(1) CALIBRATION Faithful representation of ordinary mathematical theorems, followed by the classification of their proof-theoretic strength.

(2) SIMPSON Realizing the Partial Hilbert's Program, *i.e.* showing that essentially infinitary theories can be founded upon the finitary ones through conservation results.

(3) FRIEDMAN Provision of practical reasons for accepting strong set-existence axioms by demonstrating that they are necessary for the ordinary-mathematical practice.

Throughout this chapter I present the philosophical and practical roots of Reverse Mathematics as a foundational program developed in the tradition of Hilbert's Program following the incompleteness theorems. The takeaway from Gödel's Hierarchy can be understood in two ways: either that a substantial portion of mathematics can be developed finitistically (the descending tendency) and therefore Hilbert's idea of ensuring the safety of mathematics is partially recovered, or that the infinitistic methods are needed to develop some (not necessarily numerous) important theorems or areas of mathematics (the ascending tendency) and therefore cannot be given up. Then, I introduce the technical and philosophical problems that the program as developed in the Friedman-Simpson tradition faces and I try to demonstrate their importance with respect to Question 1. I will claim that if the RM-framework in practice or by construction results in frustrating the faithful representation of informal mathematics, it should be reconsidered – either by changing its elements responsible for the problems or, in face of no alternatives – by giving up some of its ambitions.

In what follows I suggest that the popular base theory $\mathsf{RCA}_0$ is not so weak and therefore too coarse-grained for a faithful analysis of weak statements of ordinary mathematics, falling short of the CALIBRATION goal. Another miscarriage I note is that the underlying classical logic often blocks the direct delineation of the computational or combinatorial contents of theorems, most emphatically witnessed by its limitation to non-uniform versions of theorems and failure to distinguish the contrapositive versions of statements. Hence the presence of classical logic sometimes prevents a clear foundational analysis by imposing a coarse-grained metric on the studied theorems. In the same spirit, the Classical RM only allows for the extensional notion of a set which overlooks the intensional aspects of mathematical objects. Thus I arrive at a constraint stating that, for the sake of the possibly most accurate foundational analysis, the preferable framework for RM would be one that is as fine-grained as possible while retaining the expressive power and one that admits the study of intensional objects. Then, in a discussion of Simpson's idea of "ordinary mathematics" I observe that the informally done mathematics is too intertwined with the methods taken from logic and set theory for one to clearly distinguish between the "ordinary" and "unordinary" mathematics and therefore the distinction should be dropped. I note the significance of that constraint for SIMPSON'S goal of showing that most of "ordinary mathematics" can be done finitistically. Next, I give several examples of problems connected to formal representation of mathematics within $\mathbf{Z_2}$. Due to the limitations of the framework to countable/separable objects, RM has to resort to indirect encodings when it comes to the representation of more complex objects. There are cases when the coding machinery overwhelms the set-existence assumptions of the pertinent reversal, which is unfaithful with respect to informal mathematics. This is a self-inflicted wound, as the countable/separable mathematics typically do not need to be represented in these indirect ways within the subsystems up to $\mathsf{ACA}_0$. In response, I propose for reversals to be supplied with the strength of coding methods they use in addition to the base theory they are proved over. Finally, I will consider some examples indicating that the true subject matter of RM are proofs, not theorems, also noting the historical nature of the endeavor. ◀

### 2.1.1 Underlying Logic and Base Theory

Very much in the spirit of Simpson's remark about **PRA** cited above, in order to satisfy goal (1), one wants their framework to be as "logic-free" and neutral as possible. The assumptions introduced by the underlying logic should play a minimal role, otherwise the framework will be too coarse grained to discern differences between some statements. Coarse/fine grained-ness of a framework is rather important to my analysis, so let me give a few examples for its meaning.

**Example 2.2.** Forster and Truss showed in [FT07] that in **ZFC**, the König Lemma is equivalent to the Ramsey theorem for pairs ($RT^2_2$), while simultaneously it is well-known that in the framework of $\mathbf{Z_2}$, which is much weaker than the former, König Lemma is strictly stronger than $RT^2_2$. This is a consequence of a result in [SS95] showing that there is *no* noncomputable information that can be coded into a computable coloring of pairs that is recoverable from any homogenous set.
**ZFC**'s strong axioms, in particular the power set and the foundation, deprive this theory of a "high-resolution" insight otherwise accessible through methods in computability theory. ◀

Of course, fine-grained-ness of subsystems of $\mathbf{Z_2}$ also varies from one to another. The stronger the subsystem, the more coarse grained the metric, which allows one to prove more equivalences; but this might be as much because of them really holding as due to the "low resolution" of a strong theory.

**Example 2.3.** In their paper [MS05] initiating the RM of general topology, Mummert and Simpson use $\Pi^1_1$-$\mathsf{CA}_0$ as a base theory for the reversal between $\Pi^1_2$-$\mathsf{CA}$ and the statement *every countably based MF space* [a topological space whose points are maximal filters on some countable poset $P$] *which is regular, is homeomorphic to a complete separable metric space*. That is because the proof uses Kondo's uniformization theorem which is equivalent to $\Pi^1_1$-$\mathsf{CA}_0$ [Sim99, pp. 225–6][1] and no

[1]Interestingly, the reversal of Kondo's theorem to $\Pi^1_1$-$\mathsf{CA}_0$ uses the rather strong $\mathsf{ATR}_0$ as a base theory.

subsystem laying lower on the hierarchy would be able to attain that. Moreover, the notions such as $MF$ *spaces* are encodable only in $\mathsf{ACA}_0$ and not in, say, $\mathsf{WKL}_0$. Therefore it is only at the level of the more coarse-grained $\Pi^1_1$-$\mathsf{CA}_0$ that one can prove the reversal between the above statement and $\Pi^1_2$-$\mathsf{CA}$ , as the more fine-grained metric of, say, $\mathsf{ATR}_0$ would differentiate these two principles, encodings notwithstanding. ◀

In general, we can think of the class of models of a weaker theory $T$ as a class of models of a stronger theory $T'$ satisfying some restriction, *e.g.* the models of $\mathsf{ATR}_0$ are exactly those models of $\Pi^1_1$-$\mathsf{CA}_0$ that do not allow for an uncountable set to be well-ordered; the models of $\mathsf{RCA}_0$ are exactly the recursive sets within the class of $\mathsf{ACA}_0$'s models, *etc.* In short, because of finer-grained-ness of a weaker theory, more statements are false in it.[2]

The easy takeaway is that since a weaker theory is typically more fine-grained than a stronger one (although this is not always the case, for it often depends on the definition of "weaker"), one should choose the weakest possible base theory for their RM-setting. The issue however is not that simple, as this comes with a trade-off on the part of expressive force. Therefore one needs a theory weak enough not to "stick" to many theorems and subsystems together, but strong enough to encode the basic concepts that concern the reversals.

When it comes to Classical RM, despite the popular dubbing of $\mathsf{RCA}_0$ as a "weak base theory", it gives enough means to prove many important (formalizations of) theorems and define objects in various areas ordinary mathematics, some of which being far from elementary. For instance, it is strong enough to encode an analytic subset $A$ of a Cantor space, and even a point $X$ in $A$; it also establishes uncountability of reals and the uniqueness of comparison maps between sets[3]; in countable algebra, $\mathsf{RCA}_0$ proves the existence of a divisible closure of every Abelian group[4]; it is also strong enough to encode complete separable metric spaces and open sets in them, together with a measure for each such set. Interestingly, $\mathsf{RCA}_0$ is "almost strong enough" to prove some of the admittedly heavy nonrecursive results. For example, though Peano's existence theorem[5] is only

[2] Note that it is not exactly about $T$ being insufficient to prove a given statement that holds in $T'$, but about $T'$ recognizing equivalence between two statements and $T$ failing to achieve that (such cases of course form a subset of the former situations) – which is not directly caused by $T$'s confined tools, but rather their increased precision.

[3] We say that $f$ is a *comparison map from* $X$ *to* $Y$ if $f : |X| \leq |Y|$ or from $Y$ to $X$ if $f : |Y| \leq |X|$.

[4] For an Abelian group $D$ we say that it is *divisible* if for all $d \in D$ and all $n \geq 1$ there is $c \in D$ such that $n \cdot c = d$. A *divisible closure* is a morphism $h : D' \longrightarrow D$ s. t. for all nonzero $d \in D$ there is an $n \in \mathbb{N}$, $n \cdot d = h(d')$ for $d' \in D'$.

[5] "The initial value problem of an ordinary differential equation has a continuously differentiable solution on a rectangle".

provable in $\mathsf{WKL}_0$, $\mathsf{RCA}_0$ proves Picard's uniqueness and existence theorem [Sim99, pp. 158–159]:

**Thm. 2.4** (Picard)**.** *Let $f(x, y)$ have a modulus of uniform continuity $h : \mathbb{N} \longrightarrow \mathbb{N}$ and satisfying the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leq L \cdot |y_1 - y_2|$$

*and $|f(x, y)| \leq M$, where $M, L$ are positive real numbers and $M$ is the maximum of $f$. Then the initial value prbolem has a unique solution $y = \varphi(x)$ on the interval determined by the rectangle and $\varphi(x)$ has a modulus of uniform continuity. Moreover, this fact extends to the case involving any finite number of functions.*

The difference consisting in the assumption of the Lipschitz condition, which does not seem so heavy.

Another example is the Bolzano-Weierstraß Theorem, discussed in section 0.2.1. Although $\mathsf{RCA}_0$ is too weak to prove the sequential completeness of $\mathbb{R}$ (which would be equivalent to the least upper bound principle for $\mathbb{R}$ and to Bolzano-Weierstraß), it proves the *nested interval completeness* property for reals[6] which suffices to prove the Baire Category Theorem:

**Thm. 2.5** (Baire category theorem for $\mathbb{R}^k$)**.** *Let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence of dense open sets in $\mathbb{R}^k$. Then there exists $x \in \mathbb{R}^k$ such that $x \in U_n$ for all $n \in \mathbb{N}$.*

What lacks here for a statement of full Bolzano-Weierstraß is the claim that $\mathbb{R}$ is isomorphic to a nested interval in $\mathbb{R}$ (which gives us some idea of how strong this claim is). Although it is much weaker than Bolzano-Weierstraß, from the perspective of mathematical practice, Baire category theorem suffices to prove many basic results that have similar applications [Sim99, p. 76].

It seems that the major source of $\mathsf{RCA}_0$'s strength is the $\Sigma^0_1$ induction. Without it, some reversals do not hold, including that of $\mathsf{WKL}_0$ and Peano's existence theorem as well as one of the key results in RM, the equivalence between $\mathsf{ACA}_0$ and König's Lemma [Sim99, p. 411]. Indeed, even Simpson and Smith [SS86, p. 290] note that $\Sigma^0_1$-$\mathsf{IND}$ is a strong induction principle as it allows to define functions from $\mathbb{N}$ to $\mathbb{N}$ by primitive recursion, without which some basic results in countable algebra would be out of reach for $\mathsf{RCA}_0$. To give a few examples, let $K$ be any countable field and $f(x)$ be a

[6] *i.e.* the statement that any sequence of real numbers on an interval has a limit.

polynomial with integer coefficients in one variable, then the following facts are unprovable without $\Sigma^0_1$-IND [SS86, p. 290]:

(a) $f(x)$ has at least one factor over $K$ which is irreducible over $K$.

(b) $f(x)$ has a factorization into polynomials over $K$ each of which is irreducible over K.

(c) The set of roots of $f(x)$ in $K$ is finite.

In view of that, one might ask therefore, how *weak* is our "weak base theory" and whether all of that machinery is really needed. The reverse mathematics of $\mathsf{RCA}_0$ has been initiated by Friedman and Simpson in [FS00] where they introduced a theory $\mathsf{RCA}^*_0$ which has the induction axiom restricted to $\Sigma^0_0$ formulae and asserts the existence of the exponentiation function, whose first-order fragment is EFA, the elementary function arithmetic. This weaker system has poorer coding machinery which prevents the representation of some of the portions of analysis, but as [SS86] shows, several reversals over $\mathsf{RCA}_0$ can be reproduced in $\mathsf{RCA}^*_0$. Moreover, Takako Nemoto showed that most of the analysis of determinacy statements can be successfully carried out in this theory; on top of that, $\mathsf{RCA}^*_0$ has proved to be more fine-grained than $\mathsf{RCA}_0$: in [Nem09] Nemoto was able to separate two statements that are equivalent to $\mathsf{WKL}_0$ over $\mathsf{RCA}_0$. Therefore, the inability to establish equivalences between, say, König's Lemma and $\mathsf{ACA}_0$ over $\mathsf{RCA}^*_0$ might be viewed as a virtue of this subsystem, rather than a flaw. Yet, it has been suggested that $\mathsf{RCA}^*_0$'s power and convenience especially in countable algebra, the area for which it was designed, is extremely limited. Bounded primitive recursion is the best that we can get from this subsystem, which in context of algebra is a pretty weak principle; for the same reasons $\mathsf{RCA}^*_0$ has significant limitations for imitating the constructive results that can be carried out within it [Baz+24, p. 3].

One of the most convincing factors of doing reverse mathematics in $\mathsf{RCA}_0$ is its correspondence with the recursive methods, which directly provides us with visions of progress within Simpson's goal of partially realizing Hilbert's Program.

As much as this is true for mathematics within $\mathsf{RCA}_0$, when it comes to reversals over it, the issue is less obvious. In a sense, many reversals based on $\mathsf{RCA}_0$ are not done via computable methods, since in cases concerning stronger subsystems the models under consideration are the $\beta$-models, not the $\omega$-models that admit Turing reducibility. That is, these reversals over $\mathsf{RCA}_0$ are not proved through strictly computable methods. This is due to the simple fact that systems above $\mathsf{ACA}_0$ do not have a minimal $\omega$-model.[7] Having several examples of $\omega$-models of a statement is useful as it helps to produce separations when they are needed, and yields a more complete understanding of the theorem [DM22, p. 100]. In cases where the $\omega$-models of a theorem become less significant (because the theorem is too strong), we have less means of model-theoretic investigation and so poorer understanding of a theorem even in presence of a reversal.

### 2.1.2 Nonuniformity

This situation becomes even more pressing when we want to extract the core combinatorial content[8] from a reversal, aiming to understand the theorem even better. This aim is uniquely insightful as it enriches with the information not just about one problem, but a whole class of related problems. Such insights are attainable when we consider *uniform* versions of theorems.

**Def. 2.6.** A problem $\mathbf{P}$ *uniformly admits computable solutions* if there is a Turing functional $\Phi$ (a function taking functions as arguments) so that $\Phi(X)$ is a solution to every instance $X$ of $\mathbf{P}$.

In general, for every sentence $S$ of the form $\forall X(A(X) \to \exists Y(B(X,Y))$ its uniform version $\exists F \forall X(A(X) \to B(X, F(X))$ states the existence of a uniform procedure $F$ to construct a solution to each instance of the problem $X$. Informally, a uniform version of a theorem gives a proof which produces a choice function whose range provides witnesses to each instance of a theorem, where the function satisfies some desired definability property such as being recursive or (in weaker instances) belonging to some level of the arithmetical hierarchy. Uniformity, in a sense, "constructivises" the proof therefore giving more information about the objects involved.

Simply because Classical RM uses classical logic, this level of clarity cannot be attained by it.

> Classifications in reverse mathematics only capture the non-uniform content of problems,

[7] This important issue is brought to the forth in Shore's program of computable RM, where only the latter models are considered (see *e.g.* [Sho13]).

[8] This notion will be discussed in more depth below. For now, we roughly understand the *combinatorial core of a theorem* to be the basic structure of its proof. As Hirschfeld puts it: When we say that principles $P$ and $Q$ are equivalent over a theory $T$, we are saying that $P$ and $Q$ have the same "fundamental combinatorics" up to the combinatorial procedures that can be performed in $T$ , so we would like this class of procedures to be one we can understand and think of as natural in some sense [Hir15, p. 13]. A good example of this idea is the observation that Weak König's Lemma represents the combinatorial core of Lindenbaum's Lemma; see the discussion in [Hir15, pp. 7–14].

*i.e.* , the way output parameters depend on input parameters in the worst case. Again this is due to the usage of classical logic (opposed to intuitionistic logic). [BGP21, p. 41]

Moreover, Classical RM cannot distinguish between theorems and their contrapositions due to the lack of uniformity. An example of this is the Heine-Borel covering theorem, which is computable, but its contrapositive version is not; Classical RM automatically captures the stronger contrapositive statement [BGP21, p. 33]. Another well-known example of this phenomenon is Brouwer's fixed point theorem [Jos23]. Another issue with tracking computability by $\mathsf{RCA}_0$ is that it cannot keep a record of the number of parallel applications of rules in proofs thereof and fails to distinguish between a single, a finite number of consecutive applications or a finite number of parallel applications of a theorem, hence sometimes yielding results that are only computable in a very strong sense, mostly due to the usage of unbounded search (which is admissible in $\mathsf{RCA}_0$). This is again due to the underlying classical logic (as opposed to linear logic).

To sum up, classical logic together with the strength of $\mathsf{RCA}_0$ as a base theory appear to me as preventing Classical RM from revealing the combinatorial structure of the theorems under consideration. Given the tools developed in computable analysis in the last decades it seems only fitting to make use of them for the advantage of RM. As long as partial realizations of Hilbert's Program are concerned though, $\mathsf{RCA}_0$ augmented with conservation results over **PRA** seems to stand on solid ground. It is however debatable to me whether pursuing the idea of finitistic reductionism for its own sake is a worthwhile goal. RM was conceived from the ambition to find the *right* axioms for theorems in order to give a better understanding of their strength and structure they share with other statements. As long as this goal is definitely inspired by Hilbert's Program, it goes beyond it in the sense that one does not ask any more about a specific set of finitistic methods the theorems are to be reproduced with, but tries to classify the theorems with respect to *any* organised group of concepts that promises elucidation of combinatorial content thereof – be it set existence principles, function existence principles, degrees of (un-)computability or the "amount" of the law of excluded middle one has to apply. If the insights offered by alternative frameworks prove to be richer with respect to the discovery of structure and computational contents of ordinary mathematics, then sticking to the letter of finitism as a philosophy seems to fall short of the spirit of finitism as a practice rooted in Skolem's, Herbrand's, Gödel's and Kleene's inventions. Since it is the practice of ordinary mathematics that lays in the center of focus of reverse mathematics, it is also the practice of best methods accessible for the analysis of the theorems considered that should be put to use. The preferable framework for RM is one which possesses the finest resolution with respect to the combinatorial content of ordinary-mathematical theorems while simultaneously being strong enough to represent the objects of ordinary mathematics.

In the next section I will elaborate on the other facet entrenched in the framework of Classical RM that makes the analysis of computational contents of ordinary mathematics problematic.

### 2.1.3 Extensionality

A theory whose language has only quantifiers ranging over the number variables has traditionally been called an "arithmetic", while a theory with quantifiers for set variables, such as $\mathbf{Z_2}$, is informally referred to as "analysis". That is because the second order language in which every countable sequence of real numbers can be coded into a single real, and codes for real numbers can be quantified over in second order arithmetic make it much more natural to express statements about functions, which are the core of analysis. It is therefore natural to ask why are we using a *set*-based language to talk about *functions*, which are represented in a way specific to a formal theory like $\mathbf{Z_2}$?[9] The cause of this is the classical assumption of co-extensionality of functions and sets *i.e.* the identification of a function with an ordered pair which is of course a set. This treatment is somewhat troublesome when it comes to representing ordinary mathematics when we want to analyze the proof theoretic strength.

From a computational standpoint, if the sets under consideration are to be manipulated, we sometimes want to differentiate between two "names" or "symbolic representations" of what is usually considered "the same set" [Bee85, p. 166]. This intuition is captured in constructive mathematics where sets are given by their definitions (as opposed to identifying them with their elements): there can be two sets with the same elements but different definitions functioning as different methods for constructing the sets.

Although ordinary mathematics does not usually make such distinctions, the formal study of it should. One important example witnessing this is the conservation result between **IZF** and **ZF**

[9]Consider the important difference: the typical statement of the least upper bound principle is that *every set* of reals with an upper bound has a least upper bound; the version of the principle that is studied in RM states that every bounded *sequence* of real numbers has a least upper bound.

bearing on the double negation translation and the usage of intensional definitions. Since the goal of Classical RM is establishing the relations between the set-existence principles and representations of theorems in terms of proof-theoretic strength, it should make use of such tools exactly because they give a finer-grained metric for distinguishing between different statements.

One way for accounting for that is introducing a function-based framework, a tactic employed typically in constructive or recursive foundational theories. Quantifying over functions instead of sets does not necessarily carry the import of extensionality, which can be of course retrieved if need be. But if one wants to focus on the computational content of the theorems analysed – and extract more information that RM cares about – this strategy seems to be the most natural one [Baz+24, p. 4]. Indeed, Friedman originally introduced $\mathsf{RCA}_0$ in a functional language [Fri76], but shortly after opted for a set-based approach. [Baz+24, p. 4] speculate that the reason for this might be the import of primitive recursion into the system, which is otherwise proved (not assumed) in the set-based approach. This, however, does not seem to be a significant cost for a more natural framework, given that a system developed in a functional language can handle intensional definitions. In this way, a RM-framework would be fine-grained also with respect to rare yet important intensional objects in mathematics.

**Constraint 1.** A preferable framework for RM is one which admits a treatment of intensional objects for the aim of a more precise analysis of ordinary mathematics. ◀

## 2.2 Ordinary Mathematics and its Representation

I now turn to the discussion of the meaning of "*ordinary* mathematics". The Friedman-Simpson style RM presents itself as a descriptive foundational analysis of mathematics in the sense that it focuses on the areas possibly distant from the logical or foundational improts with metamathematical flavors. Simpson identifies

> as *ordinary* or *non-set-theoretic* that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts. We have in mind such branches as geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathe-

matical logic, and computability theory. [Sim99, p. 1]

The historical spirit of this identification is quite surprising and pincered by the path of development of mathematics both before and after the introduction of set-theoretic concepts.

As noted in the section 1.1 above, the "set-theoretic concepts" such as AC have been in use in various forms long before the introduction of the theory and the axiomatization of the latter only made this fact clear. If, however, Simpson here means the informal development of set theory, one must bear in mind that Cantor's work was not done in isolation from the mathematics of the era (and by that token it had not "introduced" any alien concepts hitherto unknown to the mathematical practice), but rather naturally arose from his research in analysis that had ties to and roots in the work of others (see *e.g.* [Dau91, pp. 30–36]). On the other hand, set theory has given mathematics some extremely powerful tools that got integrated into the practice of various other fields long before the beginnings of the Friedman-Simpson program. Today, the "the inter-penetration of intuitively different areas of mathematics [is] a fact of life" [Rya23, p. 2] emphatically witnessed by the use of methods from modular form theory in a proof of a purely arithmetical statement of Fermat's Last Theorem and many other examples; set theory is no exception from such interconnections. However there are theorems that are uncontroversially independent of set theory, other results belonging to the same area carry the influence of the abstract tools developed in it[10], rendering a clear distinction between set-theoretic and non-set-theoretic mathematics rather vague. The motivation of this differentiation was the rehabilitation of Hilbert's Program by showing that even if not all mathematics can be done only by finitist means, the traditional, non-foundational fragments of it, *i.e.* those fragments about whose "safety" we actually care (if we care at all), can be reduced to finitistic methods. And the tendency is indeed correct if one considers the fact that when it comes to reversals of theorems from areas deeply intertwined with set theory (such as classical topology or ordinal arithmetic) always occupy the higher strata of the Big Five, with the exception of a few basic results. Nevertheless, it cannot be said that systems such as $\Pi^1_1\text{-}\mathsf{CA}_0$ and above are inhabited only by the *unordinary* theorems with set-theoretic imports. If we understand "ordinariness" of a theorem as a property such that it concerns objects of traditional areas of mathematics and

[10]This is a common occurrence in analysis, number theory or model theory. Interestingly, the main goal of Friedman's conception of RM is to demonstrate exactly that: the indispensability of strong "set-theoretic" assumptions in practice of *prima facie* finitistic mathematics.

additionally relates to different such areas[11], then the Graph Minor Theorem is an example of one of the most ordinary statements recent mathematics has produced. R. Diestel comments in his book on graph theory [Die06, p. 315] that it is

> a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: *in every infinite set of graphs there are two such that one is a minor of the other.* This *graph minor theorem* (or *minor theorem* for short), inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within.

**Def. 2.7.** Let $e = xy$ be an edge of a graph $G = (V, E)$. By $G/e$ we denote a graph that is obtained from $G$ by *contracting* the the edge $e$ into a new vertex $v_e$, which becomes adjacent to all the former neighbors of $x$ and $y$. If $G'$ is obtained from $G$ by deleting some vertices and edges and then contracting some further edges, $G'$ is said to be the *minor* of $G$.

**Thm. 2.8** (Robertson and Seymour)**.** *For a countably infinite sequence $G_0, G_1, ...$ of finite graphs there exists $i < j$ such that $G_i$ is isomorphic to a minor of $G_j$.*

More than a decade before the theorem was proved, [FRS85] showed that the statement is independent of $\Pi^1_1$-$\mathsf{CA}_0$. Even if the GMT were the only "ordinary" theorem occupying the strata above the Big Five, its importance and robustness provides enough justification to the claim that one cannot discard these strong subsystems as unordinary or too set-theoretic. Therefore, there are little reasons to uphold the "ordinary–set-theoretic" distinction with respect to the subject matter of RM, as it proves to be arbitrary and detached from both the practice of ordinary mathematics and reverse mathematics itself[12].

**Constraint 2.** Reverse mathematics investigates the strength of theorems regardless of the area of mathematics they come from. ◀

**Remark 2.9.** This suggests that the "ascending tendency" might turn out to be the correct philosophical consequence of reverse mathematics. ◀

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[11] Following [Mon11, p. 432] one could identify the second condition as *robustness* of a theorem.

[12] Reverse mathematics of ostensibly "set-theoretic" areas such as general topology [MS05] or theory of ordinal numbers [Mad17] by now belong to the folklore.

This would doubtlessly be the case if the subject of RM were the ordinary-mathematical theorems *as they stand*. But as we have seen in 0.2.1, the contents of practice of RM are the *encoded* objects expressed in the formal language of $\mathbf{Z_2}$. This fact is a source of trouble, as there are several reasons for doubts about whether the codes faithfully represent the intended objects.

## 2.3 Problems with the Representation of Ordinary Mathematics

The problem of faithful representation of ordinary mathematics is a pivotal theme in RM. Since it is done in a formal language, it can only deal with *representations* of mathematical objects, not the objects themselves. The way this is typically done is by encoding such objects with naturals or sets of naturals in the traditional fashion of Gödel numbering. In turn, Classical RM can only work with "miniaturisations" of theorems, *i.e.* the versions stated in terms of countable or separable objects. Hence mainstream RM deals with theorems about countable groups, complete separable metric spaces, *etc.* This significantly limits the scope of analysis of ordinary mathematics, since some central fields of it deal with more complex objects. If we want to represent such objects in $\mathbf{Z_2}$, we have to resort to highly non-trivial coding techniques, which, it has been argued, will sometimes result in proofs whose claims verge on arbitrariness [DM22, p. 6]. Simpson himself notes in [Sim88, pp. 360–1] that

> The development of mathematics within $\mathbf{Z_2}$ or subsystems of $\mathbf{Z_2}$ involves a fairly heavy coding machinery. Doesn't this violate the claim of such subsystems to reflect mathematical practice?

To which he answers

> It is true that the language of $\mathbf{Z_2}$ requires mathematical objects such as real numbers, continuous functions, complete separable metric spaces, etc. to be encoded as subsets of $\mathbb{N}$ in a somewhat arbitrary way. [...] However, this coding in subsystems of $\mathbf{Z_2}$ is not more arbitrary or burdensome than the coding which takes place when we develop mathematics within, say, **ZFC**. Besides, the coding machinery could be eliminated by passing to appropriate conservative extensions with special variables ranging over

> real numbers, etc. If this were done, the codes would appear only in the proofs of the conservation results. I do not believe that the coding issue has any important effect on the program [...].

However, the claim that coding in $\mathbf{Z_2}$ is not more arbitrary than that in **ZFC** is to no avail: the former, as we have seen in example 2.2, is massively more coarse-grained than $\mathbf{Z_2}$. And the ready availability of methods of constructive and computable mathematics beneficial for the CALIBRATION of theorems that the project sets as its goal demand more precision than that of **ZFC**. When it comes to the tentative solution to the coding issues proposed above, the conservative extensions that Simpson talked of over 30 years ago have not yet arrived. Instead, it has become common to include the discussion of coding conundrums in introductions to all important works on Classical RM (see *e.g.* [DM22], [Hir15]). Moreover, in response to the coding issue the research in RM has seen an out-pour of alternative methods to ensure minimal occurrence of arbitrariness, peaking with Friedman proposing a program of *strict* reverse mathematics that is to try and provide a specific coding method for each area of mathematics (as if they were disconnected from one another). This should show how much effect the coding issue has had on the program.

Indeed, the usage of non-trivial coding methods leads to some truly pathological cases, which I bring up as examples below. It seems to me that all in all, such issues beg the question of faithfulness of the representations of the (miniaturisations of) ordinary-mathematical theorems.

**Example 2.10** (General Topology)**.** As mentioned in Example 2.3, [MS05] studies the reverse mathematics of $MF$ spaces with a countable basis that generates the topology. [Hun08, Proposition 2.15] investigated the strength of representational assumptions involved in the use of countable bases and showed that *the existence of a type-3 set of type-2 objects with cardinality* $\leq \beth_1$ *is equivalent to* $(\mathcal{E}_2)$. The axiom $(\mathcal{E}_2)$ is extremely strong: it implies the full second order comprehension $\Pi^1_\infty$-$\mathsf{CA}$, so, in other words, the existence of a countable representation of a topological space of cardinality $2^{\aleph}_0$ implies the entire second order arithmetic. As the original reversal is equivalent to "mere" $\Pi^1_2$-$\mathsf{CA}$, Hunter's result revealed how unfaithful to ordinary mathematics the methods used by Mummert and Simpson were and posed a serious doubt about the meaning of the reversal. ◀

It can be argued that since general topology is an area with heavy set-theoretic imports and hence does not belong to "ordinary mathematics" in Simpson's sense, the above problem is irrelevant to the issues of representing mathematical objects in $\mathbf{Z_2}$. It is however but the most evident example of heavy representational assumptions impairing a reversal. There is also another sense in which one would desire the representation system to be faithful: the minimal use of nontrivial coding techniques. A handbook example of trouble with that is the way continuous functions are being represented in $\mathbf{Z_2}$. One has to use highly nontrivial encodings of open (or closed) balls on complete separable metric spaces and ensure the satisfaction a number of relation conditions imposed on codes so that they meet an analogue of the natural $\epsilon - \delta$ requirements [DM22, Def. 10.3.3]. Now, it is true $\mathsf{RCA}_0$ suffices to do all of that[13], but this does not mean that the coding techniques used are obvious or naturally following the way we construct a continuous function in ordinary mathematics. Kohlenbach [Koh17] used the higher-order arithmetic to show that the coding of continuous functions fails to represent any given continuous functional $\Phi : \mathbb{N}^2 \longrightarrow \mathbb{N}$. This would not be an important issue had continuous functions not be of central interest to analysis in ordinary mathematics. Therefore, it comes as no surprise that the representation of continuity in $\mathbf{Z_2}$ implicitly smuggles higher-order objects into the pertinent statements [San15, Cor. 4.3].

Finally, there is a family of problems related to comparison of *(i)* different formal representations of a given theorem and *(ii)* ordinarily equivalent expressions of a given theorem. The discussion of the latter will naturally point to one of the most pressing issues of Classical RM, namely the apparent *(iii)* intensional nature of theorems, which, as I will argue, is not handled well in this framework.

### 2.3.1 Challenges of comparing theorems in a formal setting

*(i)* touches upon the issues that are more directly related to coding. A working logician has multiple coding methods at their disposal. For instance, reals can be represented as converging Cauchy sequences, Dedekind cuts, decimal expansions or in some constructivist settings as an arbitrarily close approximation to some number $x$; converting some of these representations into another cannot be carried out within $\mathsf{RCA}_0$ (tools from $\mathsf{WKL}_0$ or even $\mathsf{ACA}_0$ are sometimes needed [Hir07, §3]). On top of that the conversions are not uniform.

This poses a problem for comparison of reversals proved via different coding techniques. If the study of proof-theoretic strength of ordinary-mathematical theorems afforded by RM is to deliver

[13] If it were otherwise this would be the case of problems with representation bearing on the weight of assumptions in the proof-theoretic sense, discussed in the example 2.10 above.

a precise classification thereof, the research should ideally proceed within a unified coding practice, apart from the consensus about the set-existence principles the theorems are classified into. But in its development RM became a diverse area where there is no general agreement about the systems for classification; aside from the obvious fundamental differences in classifications with the Classical RM introduced by the computable and constructive frameworks, even the higher-order RM that in its build-up is much closer to the Friedman-Simpson style RM produces a notably different classification [San14, p. 2]. Different classifications stem from varying focus of the foundational aspects under investigation: Classical RM focuses on the proof-theoretic strength in the traditional sense, higher order reverse mathematics aims at distilling the effective content, computable reverse mathematics focuses on the computable content in the broader sense, while Constructive RM focuses on degrees of nonconstructivity in terms of the "amount" of the Law of Excludded Middle (LEM) indispensable in a proof of a given theorem. As long as the insights into different aspects of a theorem greatly enrich our foundational knowledge, they are obtained through different coding methods and in this sense speak of slightly different objects. Currently there is no sight of delineating the purely mathematical criteria to decide which coding system is best in order to unify the representations. Instead, some have proposed informal heuristics for trying different coding methods [DM22, pp. 6–7]:

(I) *Utility*: The utility of a coding system can be seen in the results it allows us to formalize and analyze. The more results can be expressed in a coding system, the more "useful" it is. For instance, the coding of real numbers with quickly converging Cauchy sequences leads to the operations being uniformly computable.

(II) *Local faithfulness*: It ensures a coding system's close relationship with the ordinary objects that are being represented. To give an example:

> There is significant [local faithfulness] in representing a countably infinite group by numbering the elements and representing the group operation as a function from $\omega \times \omega$ to $\omega$. On the other hand, our coding for continuous functions is arguably not as locally faithful as simply coding the function's value for each element in a dense subset. [DM22, p. 7]

The key aspect of a coding system being locally faithful or not is the amount of information it adds or removes from the original ordinary object. For example, representing a real not as an equivalence class of Cauchy sequences but instead as a single convergent Cauchy sequence, does remove some information from the original object.[14]

(III) *Global faithfulness*: It is the property that *every* object of the desired type has a code of the desired type in the standard model. This property ensures that a coding system does not exclude any actual objects of interest. For example, every real number is the limit of some quickly converging Cauchy sequence of rationals, and every open set of reals is the union of a sequence of open rational intervals. Thus, when interpreted in the standard model, a theorem referring to coded objects like these retains its original scope.

(IV) *Professional judgement*: The intuitive idea based on experience and expertise about what frameworks to use for what ends.

The last point is pretty revealing about the stage of youth the search for optimal coding framework finds itself in. But before that arrives, the research area would be much clearer if, in addition to the base theory a reversal is proved over, one would specify the proof-theoretic strength of the coding methods used for formalisation of the studied objects. This would simultaneously help in developing mathematical criteria for choosing a preferable coding system: when it turns out that in practice one system is used faithfully with respect to many important theorems, accepting it as a "base coding system" will only be natural.

**Constraint 3.** The general form of a reversal result is: *Over a base theory* $\mathbf{B}$*, the theorem* $\mathbf{T}$ *is equivalent to a set-existence principle* $X$ *modulo the encoding of strength* $Y$*.*

*(ii)* relates to the examination of strength of conversions between (ordinarily) equivalent statements of a theorem. It elucidates the assumptions at play in these conversions. Sometimes, as in the case of example 2.11 below, it turns out that the assumption of one form $T_a$ of a theorem is too little to straightforwardly prove (*i.e.* say, only using computable methods) the ordinarily equivalent form $T_b$ and one needs some more involved machinery to achieve that.

**Example 2.11** (Ergodic Theory)**.** In their paper [AS06], Simic and Avigad studied the reverse

[14] As noted above, the predominant issue here is that RM can only deal with countable/completely separable objects.

mathematics of some basic notions of analysis. They show that the mean ergodic theorem is naturally formalizable and provable in $\mathsf{ACA}_0$. Ordinary mathematics recognizes a number of statements equivalent to this theorem, but it takes the strength of $\mathsf{ACA}_0$ to state these equivalences [AS06, Cor. 15.2]. That is, the assertion of equivalence of two forms of a theorem requires as much proof-theoretic strength as a proof of the theorem itself. This means that either the equivalences are not that natural (as they need relatively strong assumptions in order to be proved) or that the combinatorial core of the mean ergodic theorem carries enough information so as to prove all ordinarily equivalent expressions of the theorem. The latter would be of course desirable, but given the coarse-grained machinery enforced by classical logic coupled with the fact that the core usually conveys the bare minimum for proving a given statement, it is the naturalness of equivalences that is brought into question. ◄

The relations between $T_i$'s will possibly be different when $T_i$'s are represented through different coding methods, but it is important to realise that the sole fact that they are not "easily" equivalent is a valuable insight afforded by reverse mathematics and does not necessarily mean that there is something wrong with the representation. If anything, the above example indicates that what is equivalent in ordinary mathematics is not obviously so under logical scrutiny.

### 2.3.2 Intensionality of theorems

This points the discussion towards a related problem *(iii)*: does RM deal with theorems or proofs thereof? Many of the reversals (the subsystem-to-theorem direction) are just formalizations of the informal proofs and require little special tricks to be implemented in the framework of $\mathbf{Z_2}$. Hence the reverse-mathematical folk parlance talks about reversing the theorems, be it as a shorthand or a misconception – for the practice decidedly suggests otherwise. Intuitively, reverse mathematics, just like any other area in the foundations, does not have any insight into the "objective" form of a theorem and can only work with the imagery brought about by the constructions provided by its proof. Let me clarify that with the some examples.

**Example 2.12** (Cauchy/Peano theorem)**.** The Cauchy/Peano theorem establishes the existence of solutions of the initial value problems for continuous functions on a rectangle. Traditionally it has been proved through the Ascoli lemma, which is equivalent to $\mathsf{ACA}_0$, by establishing the convergence of approximate solutions. But [Sim84] shows that Cauchy/Peano is equivalent to $\mathsf{WKL}_0$. The proof of this reversal goes through some coding tricks, the application of the contrapositive of weak König's lemma and a certain construction due to Aberth [Abe71][15] to avoid the usage of the Ascoli lemma. Simpson also shows that unlike $\mathsf{WKL}_0$, $\mathsf{ACA}_0$ is able to establish the existence of a *maximal* solution to an initial value problem. In consequence, Simpson offered an alternative proof of the Cauchy/Peano theorem that has provably distinct assumptions from the classical one. ◀

One might ask whether Simpson's result simply establishes the RM-analysis of Cauchy/Peano and thanks to the application of the tools of RM finds the true combinatorial core of the theorem. But this is only possible due to Abarth's prior analysis of the theorem from the perspective of tools available in $\mathsf{WKL}_0$. In other words, it is only the presence of a reversal of a weaker statement that enabled the claim of conclusive RM-analysis of the Cauchy/Peano theorem. The next example shows the situation where this is not attainable.

**Example 2.13** (Ergodic Szemeredi's theorem)**.** Szemeredi's theorem about arithmetic progressions in colorings of positive integers is a paramount result in Ramsey theory. Two years after it was established by Szemeredi, Furstenberg came up with an alternative proof of the statement, using methods from ergodic theory. Unlike the original combinatorial version, Furstemberg's proof uses rather strong infinitistic machinery. Thanks to the complexity-theoretic analysis by Ferenc Beleznay and Matthew Foreman [BF96], it has been proved that the Furstenberg stucture theorem, indispensable for the ergodic version of the proof, is very strong indeed (see [Avi09]):

**Thm. 2.14.** *Over $\mathsf{ACA}_0$, the Furstenberg structure theorem is equivalent to $\Pi^1_1$-$\mathsf{CA}_0$.*

On the other hand, Gowers gave an exponential upper bound on the growth rate of the function characterizing the original Szemeredi theorem; and indeed, despite the reversal of Szemeredi still standing as an open problem of RM, Gowers, Tao, Friedman and a number of other celebrated mathematicians believe that the original proof should go through with exclusive usage of primitive recursive methods. This would imply that the original proof of Szemeredi's theorem is provable in $\mathsf{RCA}^*_0$.[16] ◀

[15] It might be informative to note that the traditional proof of Cauchy/Peano fails in computable mathematics due to this result.

[16] A weak subsystem identified by the $\Sigma^0_0$-$\mathsf{IND}$ (instead of the $\Sigma^0_1$ induction of $\mathsf{RCA}_0$). It was first studied in [SS86].

The example distinguishes between two proofs of a single theorem that vary in proof theoretic strength, somewhat revealing the contextual or historical nature of RM: the analysis of the proof-theoretic strength of a theorem depends on the knowledge of its proofs at a time. This suggests that it is not the bare statements of theorems, but proofs, that is the subject matter of reversals. Understanding a theorem as a name for the collection of its proofs, one arrives at the conclusion that theorems are intensional objects, given the drastically different strength of the methods employed therein.

Let me turn to a possible objection to this diagnosis. For instance, Bolzano proved the Bolzano-Weierstraß theorem (equivalent to $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$) as a lemma in the proof of the Intermediate Value Theorem (equivalent to $\mathsf{RCA}_0$ over $\mathsf{RCA}_0$). That is, at that time the only known proof of the IVT was through a much stronger Bolzano-Weierstraß theorem. One might hope that had he have the tools of RM at his disposal, he would have realised that the "lemma" trivially implies the theorem (in the sense that any recursive set is also an arithmetical set). But this holds only if we have access to the reversal of the weaker theorem; in the realm of Platonic ideas where all theorems are supplied with the set-existence principles they are equivalent to (modulo base theory) RM is an analysis of theorems. But in the absence of reverse mathematics of, say, Szemeredi's theorem, one can only take the known RM-analysis of the proofs of it. The professional judgement that anticipates the theorem to be provable in $\mathsf{RCA}_0^*$ is of great importance for further progress, but it remains in the realm of speculation until the reversal arrives (after all, even Hilbert could be wrong about the capacity of finitistic methods).

Now, it can be argued that the RM-analysis of a statement reduces to the identification of the strongest lemma used in its proof followed by reversing this lemma and in this way establishing the reversal for the whole statement, thereby reducing the RM of proofs to RM of theorems. But the key step in this strategy is the process of identification of the strongest lemma, the procedure that assumes investigating the structure of the proof and finding its combinatorial core. So, in the process of reducing a proof (intuitively understood as a directed graph whose nodes are assumptions and lemmas and vertices are applications of certain rules) to its strongest lemma one still performs the work necessary for the RM of a proof.

This challenges the common perspective according to which

> it is the "combinatorial core" that the analysis of a theorem in reverse mathematics actually reveals. For this reason, equivalent theorems (in the sense of reverse mathematics) are sometimes said to have the same "underlying combinatorics" [DM22, p. 10].

If my argument is correct, we should speak of underlying combinatorics of *a proof* of a theorem. Hence, if two theorems are said to be equivalent in the above sense, it is in fact two proofs sharing the underlying combinatorics – and not the theorems themselves, since the subject matter of a single theorem can be constructed with methods implying radically different set-existence principles. Instead of identifying a theorem $T$ with its content, it seems more plausible to identify it with a body of ways the claimed content is constructed $Prf_i^T$'s, corresponding to treating proofs as types in the intensional frameworks. If that is correct, the claim of the constraint stated at the end of the section 2.1.3 is refined into a necessary condition for a faithful foundational RM-analysis of mathematics. For if the analysis of proofs of a single theorem yields different classifications, RM seems to be a futile endeavor. I view this as a strong argument for the change of way of thinking about the products given by the foundational analysis of RM, giving philosophical fuel for restating the goals of the program in more constructivist terms. It also has consequences on the way we understand Simpson's "Main Question": *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?* For if there is no general method of ensuring that an informal proof, which is the conceptual and structural base for a reversal, employs the weakest axioms and has the lowest possible complexity, this means that reversals are not unique solutions to the main question about a theorem $T$.

# 3 Does Constructive RM score better?

The conclusion of the last chapter has hopefully demonstrated that the intensional nature of theorems as studied in RM (the practice of which heavily bases upon formalizations of the proofs) prevents the classical framework from obtaining clear classifications. Together with the problems concerning coding, these two seem to be the most fundamental obstacles for Classical RM to be a successful foundational program. At the same time, the conclusions of the previous chapter give ready-made guidelines for examination of Constructive RM: *(i)* does it face similar problems related the way it handles intensional objects? *(ii)* does it face problems related to formal representation of mathematical objects? *(iii)* does it face any other issues that Classical RM is free of? The conclusions of the present chapter are that Constructive RM is closer to respecting the intensionality of theorems of informal mathematics, does enrich the foundational analysis with more exact methods of construction and insight into the numerical meaning of the theorems, but it is also limited in the sense that some of the classical theorems cannot be faithfully analyzed within this framework.

## 3.1 Systems for Constructive RM

Upon inspection, one very quickly realizes that the practice of Constructive RM is not firmly embedded in a single formal framework, as is the case with $\mathbf{Z_2}$ in Classical RM. There is a multitude of formal systems that could serve as base theories, such as W. Veldman's Basic Intuitionistic Mathematics (BIM) [Vel14], I. Loeb's "Weak Kleene-Vesley" (WKV) [Loe05], Kreisel's and Troelstra's **EL** [KT70], or Heyting Arithmetic in finite types ($\mathbf{HA}^{\omega}$) used in [Koh08]. Though they differ in formal buildup and sometimes in applications and proof-theoretic strength, each is based on intuitionistic logic and is developed in a functional language. Apart from the needless inflation of the size of the present thesis, there are at least two reasons why the careful discussion of each of them is not necessary.

First, it is possible to interpret the systems (in the sense of interpreting one system within another) in such a way that – modulo the strength of the interpretation function – some of these systems have the same proof-theoretic strength and can be treated as co-extensional. Moreover, they are conservative over some of the subsystems of the Classical RM, which makes them more familiar to us. Finding the interpretations between the systems is a powerful tool of relating them in terms of their provability conditions and as a method extends to establishing the conservativity (for a specified class of statements) result between the systems.[1]

To give a general understanding of the axiomatic build-up of some of the intuitionistic formal systems[2], we put:

- **EL** = successor, equations for all primitive recursive functions, $\lambda$-conversion, recursor axiom, $\Delta^0_0$-AC for number variables, full induction axiom.
- $\mathsf{EL}$ = axioms of basic arithmetic for $+$ and $\cdot$, full induction axiom, $\Delta^0_0$-AC for number variables.
- $\mathsf{EL}_0$ = axioms of basic arithmetic for $+$ and $\cdot$, $\Sigma^0_1$-IND axiom, $\Delta^0_0$-AC for number variables.
- $\mathsf{EL}^*_0$ = axioms of basic arithmetic and the defining equation for *exp*, $\Delta^0_0$-IND axiom, $\Delta^0_0$-AC for number variables, and the following *axiom for the restriction operator*

  $$\alpha \upharpoonright (0) = \langle\rangle \qquad\qquad \alpha \upharpoonright (x+1) = \alpha \upharpoonright (x) * \langle \alpha \upharpoonright (x) \rangle$$

  where $\langle\rangle$ is the (code of the) empty sequence, $*$ is the concatenation operator, and $\langle t \rangle$ is the one-length sequence ${}^1t$.
- $\mathsf{EL}^-_0$ is obtained from $\mathsf{EL}^*_0$ by replacing the $\Delta^0_0$-AC for number variables with the $\Delta^0_0$ bounded search axiom.

[1] One notable example of that is the Gödel-Gentzen translation of **PA** into **HA**.
[2] For exact definitions cf. [Nem23, pp. 670–1, 691].

The diagram below shows their relations (including some systems not introduced above) as summed up in [Nem23].

$$\mathsf{ACA}_0 \sim_{\Pi^0_2} \mathbf{EL} \sim \mathsf{EL} \sim \mathsf{BIM} \sim \mathsf{WKV}$$

$$\vee$$

$$\mathsf{RCA}_0 \sim_{\Pi^0_2} \mathbf{EL_0} \sim \mathsf{EL}_0$$

$$\vee$$

$$\mathsf{RCA}^*_0 \sim_{\Pi^0_2} \mathbf{EL_{ELEM}} \sim \mathsf{EL}^-_0 \sim \mathsf{EL}^*_0$$

where $\vee$ means that the group of the systems above it is proof-theoretically stronger than the group of the systems below it, $T \sim_{\Pi^0_2} T'$ means that $T'$ is conservative over $T$ for $\Pi^0_2$ sentences and $T \sim T'$ designates mutual interpretability of $T$ and $T'$ over a certain polynomial-time computable function.

Thanks to the relations between the intuitionistic and classical systems and the information about the latter from the previous chapters, one gets a rough understanding of the positioning of these systems in the context of my discussion. Moreover, given that many of these systems are interpretable in one another, there is no need to thoroughly introduce each one of them in order to carry out my philosophical investigation, as establishing interpretation is a stronger result than establishing conservativity for some class of sentences.

Second, Constructive RM can be developed in an informal fashion such that the results can always be formalized in a sufficiently strong system such as one of the above (though the options are much more abundant).

This has actually been the case in the first few decades after the inception of the program in the 1980s. Bishop's [BB85] gave a great momentum to research in constructive mathematics in general and constructive foundations is particular, and it was the framework developed in this book, now often referred to as BISH, that served as an informal "base theory" for the investigation. It was only in the early 2010s that the formal theories listed above began to be treated seriously in relation to Constructive RM. Even so, most of the results were proved in BISH and this direction is still very much in practice (see *e.g.* [Han18]). In the context of foundational analysis, *prima facie* it by default resolves all problems of formal representation that Classical RM struggles with. Therefore, if we treat Constructive RM as BISH, then the answer to the *(ii)* question stated at the beginning of this chapter is that Constructive RM is not immersed in problems of representation of ordinary mathematical objects like Classical RM is. However, guided by the rising popularity of the formal approach in Constructive RM, I believe that a closer look at the issue is still required.

## 3.2 Representation in Formal Systems

Section 2.3 of the previous chapter discussed the fact that Classical RM can only faithfully represent countable or separable objects of ordinary mathematics. This constraint is taken even more seriously in Constructive RM in the sense that while working in the "weaker" systems such as ones listed above, the research focuses on "arithmetized" versions of theorems and gives up the ambition of representing the higher-order objects through coding trickery. This is much easier to achieve in the constructive setting given that the objects defined with a method of their construction are commonly even more well-behaving and accessible than the countable and separable ones in the classical setting.

**Example 3.1** (**EL** and continuous bar induction)**.** $\mathsf{ACA}_0$ can be characterized as a subsystem of sequential convergence; being equivalent to the Bolzano-Weierstraß theorem that guarantees the existence of a limit of a bounded sequence (but does not necessarily provide a method to find it), it implies non-constructive existence claims. By contrast, **EL**, a system that is $\Pi^0_2$-conservative over $\mathsf{ACA}_0$ was used in [FK20] to characterize the continuous bar induction that guarantees that every point-wise continuous function from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ is induced by an inductively generated neighborhood function.

There is an obvious difference in the nature of the objects involved in these two statements. First, in case of $\mathsf{ACA}_0$, there is no restriction to the domain of the functions, *i.e.* they can *classically* range over the *classical* $\mathbb{R}$, while the functions involved in the claim of **EL** range over the very tame $\mathbb{N}^\mathbb{N}$ and pose no issues when it comes to formal representation. Secondly, the mathematical description of a limit of any bounded sequence of reals might go well beyond the "coarse grained" vocabulary of $\mathsf{ACA}_0$ (or any other subsystem of $\mathbf{Z_2}$ for that matter), while one has direct access to the values of the neighborhood function asserted by **EL** that is inductively generated. ◀

Similarly, if $\mathsf{WKL}_0$ is a system characterizing continuity ([Sim84, p. 786]), it does so only through (violently) encoding continuous functions into it, while the constructive principle FAN, stating that every bar is uniform, ensures the tameness of the involved objects from the get-go. This situation also transfers to the representation of more complex objects in constructive mathematics.

**Example 3.2** (Polish spaces in Constructive RM)**.** Complete separable metric spaces are represented as completions $(\hat{X}, \hat{d})$ of countable metric spaces. The latter are unproblematically represented via primitive recursive enumeration of elements in $X$ together with a functional representing a pseudo-metric $d_X$. Now, the completion $(\hat{X}, \hat{d})$ of $(X, d)$ is represented as the completion of $(\mathbb{N}^{\mathbb{N}}, d_X)$ whose element is given by a constructively definable function $h$ with a given modulus of uniform continuity $\omega_h$. The enriching of data by such a modulus $\omega_h$ is more convenient in practice than the classical presentation as a Cauchy sequence of polynomials having rational coefficients, since such a sequence is in general quite complicated to construct, whereas a modulus $\omega_h$ can often easily be written down [Koh08, p. 84]. Apart from the practical convenience, this method is in agreement with the constructive constraints on functions. ◀

The key lesson from this comparison is that it is not just the formal design of the systems, but the foundational assumptions underlying the mathematics developed in these systems that play the main role in avoiding representational conundrums. In fact, any result in constructive analysis is an example of this difference, which, in the context of formal representation, is of great value. The self-inflicted wound of Classical RM stemming from representing the infinitary objects in a finitary framework is here blocked by the restriction of the scope of investigation of mathematical objects. In short, since constructive mathematics eschews infinitary and uncountable objects, encapsulating them as "completed" objects via encodings is significantly rarer than in Classical RM. So, in comparison to the latter, there is no need to supplement a constructive reversal with the information about the coding machinery it uses, as suggested in Constraint 2.3.1.

**Remark 3.3** (Explicit Mathematics as RM)**.** These considerations concern the systems of relatively low proof-theoretic strength. Indeed, there are much stronger theories, such as ones developed in the tradition of Explicit Mathematics that could be taken into consideration as potential tools for foundational analysis in the sense of RM. However, the strongest theory $\mathbf{T_0}$ does not admit easy hierarchization of the pertinent axioms – the Elementary Comprehension and Inductive Generation – through, say, arithmetical hierarchy, which results in the lack of subsystems that could serve as reference points of reversals. On the other hand, the weakest base theory, $\mathbf{EM}_0$, has the same proof theoretic strength as **PA** or $\mathsf{ACA}_0$, which is far too strong for a proper foundational analysis. Hence the question of formal representation within EM should not even concern the investigation. ◀

After this brief detour through the constructive formal systems, let me resume the discussion of the informal Constructive RM regarding questions *(i)* and *(iii)*. Its conclusions will mostly transfer to the formal case.

## 3.3 Intensionality in BISH and Related Systems

### 3.3.1 BISH as RM

BISH is an informal framework that is based on intuitionistic logic and assumes the axiom of countable choice, dependent choice, unique choice. In it, every set $X$ comes with its own equality relation $=_X$ and every function is identified not with the collection of its values $Y$, but with the method of assembling these values in $Y$. Therefore, extensionally identical sets can be distinguished from each other by different methods of constructing them, which is reflected in their specific equality relations. This results in, for example, the empty set not being unique, as in conventional set theory, but there being many such uninhabited sets $\varnothing_{\mathrm{Z}}$, $\varnothing_{\mathrm{Q}}$ or $\varnothing_{\mathrm{R}}$, etc.

It can serve as a base theory in the way that it can be extended to intuitionistic mathematics (**INT**) by adding the principle of continuous choice and the **Fan Theorem**, so-called Russian recursive mathematics (**RUSS**) (by adding Markov's Principle (**MP**) and the Church's thesis), and to classical mathematics (**CLASS**) by adding LEM and the full axiom of choice [Ish06]. The flexibility afforded by BISH results in not a single, but three directions for classifications, each according to one of the modes of mathematics: intuitionistic, classical or recursive. The first important observation about reversals in BISH is that the principles mathematical statements are reversed to are either logical principles (such as LEM, Weak Law of Excluded Middle (**WLEM**), Limited Principle of Omiscience (**LPO**), Lesser Limited Principle of Omiscience (**LLPO**), **MP**), set-existence principles (Fan Theorem (**FAN**) and its variations), principles about the nature of continuity (*e.g.* "Every sequentially continuous map $f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}$ is point-wise continuous" ($\mathbf{BD} - \mathbf{N}$)) or principles describing recursive sets (*e.g.* anti-**FAN**). The first group of principles are the weakenings of LEM and this is what relates BISH to the study of classical mathematics. Since I am interested in comparing the constructive and classical reverse mathematics of ordinary mathematics (which by assumption is the common classical type), I will restrict the discussion only to this group of principles[3]. The principles are listed descendingly in terms of strength:

- LEM: For any wff $\varphi$, $\varphi \vee \neg\varphi$
- **WLEM**: For any wff $\varphi$, $\neg\varphi \vee \neg\neg\varphi$
- **LPO**: For any binary sequence $(\alpha_n)_{n\in\mathbb{N}}$ we can decide whether

$$\forall_{n\in\mathbb{N}}\ \alpha_n = 0 \vee \exists_{n\in\mathbb{N}}\ \alpha_n = 1$$

- **LLPO**: For any binary sequence $(\alpha_n)$ with at most one 1,

$$\forall_{n\in\mathbb{N}}\ \alpha_{2n} = 0 \vee \forall_{n\in\mathbb{N}}\ \alpha_{2n+1} = 0$$

And give an idea of the hierarchy of classifications to these principles [Man88], [Han18]:

**Proposition 3.4.** *LEM is equivalent to the statement that the supremum of every bounded, inhabited subset of reals exists.*

**Proposition 3.5.** ***WLEM** is equivalent to the first De Morgan's Law, $\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$.*

**Proposition 3.6.** ***LPO** is equivalent to the following statements:*

1. *The trichotomy for reals: $\forall_{x\in\mathbb{R}}\ x < 0 \vee x = 0 \vee x > 0$*
2. *Every real number is either rational or irrational.*

[3]Admittedly, though, some of the principles from the other groups *are* classically valid, but my ambitions of course do not encompass studying each case a classical statement is treated in BISH; instead I want to draw conclusions from representative examples.

3. *Every bounded monotone sequence of real numbers converges.*

4. *Bolzano-Weierstraß Theorem*

5. *Ascoli Lemma*

**Proposition 3.7.** ***LLPO*** *is equivalent to the following statements:*

1. *Every real number has a binary expansion.*

2. *the Intermediate Value Theorem*

3. *Weak König's Lemma*

In order for BISH to be a full-fledged alternative to Classical RM, one needs to show that the reversals are acceptable from the classical perspective. Ishihara [Ish06, p. 46] explains: constructive mathematicians have been making every effort, for a given classical theorem $Thm$, to find its constructive substitute $Thm^c$ such that[4]

$$\mathsf{BISH} \vdash Thm^c \text{ and } \mathbf{CLASS} \vdash Thm \leftrightarrow Thm^c.$$

Oftentimes one can find more than one such $Thm^c$. But he also notes that in some cases, we have to be content with $Thm^c$ such that $\mathsf{BISH} \vdash Thm^c$ and $\mathbf{CLASS} \vdash Thm \rightarrow Thm^c$. This can suffice for applications, but the absence of the classical implication $Thm^{c\prime} \rightarrow Thm$ signalises that the constructive theorem $Thm$ is too weak to demonstrate its classical counterpart. This is due to the fact that the domain of the classical objects is bigger than that of the constructive ones, making the classical theorems ranging beyond the concrete numerical meaning. Although in theory it is possible that a different $Thm^c$ that is equivalent to its classical version will be discovered, but given this domain discrepancy it is to be expected that there are theorems that cannot be convincingly analyzed by Constructive RM from the perspective of a believer of **CLASS**. In practice however, there is some compensation for that. For the goal of comparing the treatment of theorems in Constructive RM with that of Classical RM, I will now present the reversal between **LPO** and the

[4] Note that the symbol "$\vdash$" does not denote the formal provability, as both BISH and **CLASS** are informal frameworks. This, heartbreakingly to a logician, blocks showing conservativity of these frameworks for, say, the class of arithmetical statements.

Bolzano-Weierstraß theorem (due to Mandelkern [Man88]), analogous to that from 0.2.1. It will later serve as a case study for my philosophical discussion.

**Thm. 3.8.** *Over BISH, the following principles are equivalent:*

1. ***LPO***
2. *Any sequence of positive integers is either bounded or unbounded.*
3. *Any bounded sequence of positive integers has a constant subsequence.*
4. *Bolzano-Weierstraß theorem*
5. *Any bounded monotone sequence of real numbers converges.*

*Proof.* (1)⇒(2). Assuming **LPO**, given a sequence $(\alpha_n)$, define

$$\alpha_n^m =: \bigvee_{k=1}^{n} (0 \vee (\alpha_k - m \wedge 1))$$

then for each $n$, $\alpha_{n-1}^m$ is a sequence decided by **LPO**: if for all $n$, $\alpha_n^m = 0$, define $c_m = 1$; and if for some $n$, $\alpha_n^m = 1$, define $c_m = 0$. Then $(c_m)$ is a decision sequence. For if all $c_m = 0$, then the initial sequence $\alpha_n$ is unbounded, while if for some $m$, $c_m = 1$, then $\alpha_n$ is bounded by $m$.

(2)⇒(3) Define a sequence $(\beta_n^m)$ such that given $(\alpha_n)$ that is bounded by $M$, then for all $n$ s.t. $1 \leq m \leq M$, $\beta_m^n \equiv 1$ if $\alpha_n \neq m$, and $\beta_m^n \equiv n$ if $\alpha_n = m$. Observe that for each $m$, $(\beta_m^n)$ is either bounded or unbounded. If each of these is bounded, choose $i > \beta_m^n$ for all $m, n$. This must mean that $\alpha_n \neq m$ for all $m$, contradicting $(\alpha_n)$'s boundedness. Therefore there must exist an $m$ such that $(\beta_m^n)$ is unbounded and has a subsequence with all terms greater than 1. By construction of $(\beta_m^n)$ this means that $(\alpha_n)$ has a constant value $m$.

(3)⇒(4) We will go with the constructivisation of the usual "interval-halving" method.
Let $(\alpha_n)$ be a sequence of reals bounded by $l$. Constructively, we cannot divide the interval $[a_0, b_0]$ into halves and claim that they compose the original interval. However, the following lemma [BB85, Cor.2.17] helps:

**Lemma 3.9** (Constructive dichotomy)**.** *If $x, y$ and $z$ are real numbers with $y < z$, then either $x < z$ or $x > y$.*

*Proof.* Since $z - x + x - y = z - y > 0$, either $z - x > 0$ or $x - y > 0$, by [BB85, Prop. 2.16]. □

In this way we can use slightly overlapping intervals. So let $[a_0, b_0]$ be divided into two slightly overlapping subintervals $[a_0, z]$ and $[y, b_0]$. Now the key nonconstructive method in BW comes in: the decision that at least one of the subintervals contains infinitely many $a_n$'s from $(\alpha_n)$. Given a sequence of $a_n$'s within an interval $[a_0, b_0]$, decide, for each $n$, whether $a_i$ with $0 \leq i \leq n$ lies in the left $[a_0, z]$ or right $[y, b_0]$ subinterval, and define $\beta_n = 0$ or $\beta_n = 1$ accordingly. By continuing to divide the subintervals into ever-smaller slightly overlapping subintervals we construct the sequence of $\beta_n$'s, that again is bounded. By applying (3), we get an infinite sequence of 0's or of 1's based on $(\alpha_n)$. Hence it is either strictly increasing (1's) or strictly decreasing (0's).[5] Define $\alpha_{\beta_n}$ as a subsequence of $(\alpha_n)$ of $a_n$'s that comprises of the strictly increasing (*resp.* decreasing) elements within the ever-smaller subintervals. Since the subintervals are ever-smaller with each $a_n$, this subsequence is bounded and infinite. But since it's strictly increasing (decreasing), the distance $|a_n - an + 1|$ is ever smaller. So there is an $x$ s. t. $|a_n - x| \leq k^{-1}$ for each $k \in \mathbb{N}$. So $x$ is the limit that the subsequence $\alpha_{\beta_n}$ converges to.

(4)⇒(5) Let $(\alpha_n)$ be a bounded sequence of reals. We will show that its limit $L$ is also a limit of a monotone sequence. Without loss of generality, fix some $\epsilon > 0$ s. t. there is an $a_i \in (\alpha_n)$ that lies in the interval $[L - \epsilon, L]$. It is the case by the construction of $L$. Observe that the subsequence of $(\alpha_n)$ within this interval is monotone, as it is strictly increasing from $L - \epsilon$ towards $L$. Now, for the sake of contradiction, let the monotone sequence converge to a limit $M$ with $L < M$. Then there is an $a_j$ with $i < j$ beyond $L$; but then the subsequence of $(\alpha_n)$ also has to converge to $M$, so $L = M$, a contradiction. Therefore it must be the case that the monotone sequence converges to the limit $L$.

(5)⇒(1) Let $(\alpha_n) : \mathbb{N} \longrightarrow \{0, 1\}$. By the monotone convergence, the sequence $(\alpha_n)$ converges to

[5] This is exactly the amount of classical mathematics we need to overcome the constructive counterexample from [BB85, p. 29].

some limit $L = \sup\{a_n : n \in \mathbb{N}\}$. Define

$$(\beta_n) = \begin{cases} 0, & \text{if } \forall_{i \leq n}\ a_i = 0 \\ 1, & \text{if } \exists_{i \leq n}\ a_i = 1 \end{cases}$$

Now, observe that if $a_n = 0$ for all $n$, then $L = 0$, and the sequence is not monotone, but constant; that immediately yields **LPO**. And if there exists $n$ such that $a_n = 1$, then $x_n$ will be a strictly positive sum, and thus $L > 0$.

So let $(\beta_n)$ be monotone and bounded by 1. It is a decision sequence: for $a_0$ we have either $L < \frac{2}{3}$, in which case $a_0$=0, or $L > \frac{1}{3}$, in which case $a_0 = 1$ and there exists an $n$ for which $a_n = 1$. Continuing this method, we arrive at $\forall_{n \in \mathbb{N}}\ \beta_n = 0 \vee \exists_{n \in \mathbb{N}}\ \beta_n = 1$. ■

This concludes the presentation of Constructive RM in general and BISH in particular, equipping me to be able to discuss the question *(i)* about their handling of intensionality.

### 3.3.2 A remark on intensionality in Constructive RM

Intensionality, however consequential for the mode of mathematics developed in BISH, by itself has scarce effects on the constructiveness (or lack thereof) of the constructive formal theories. To see this, we can consider Heyting arithmetic in finite types supplemented with the axiom of extensionality, $\mathbf{E\text{-}HA}^{\omega}$ together with the full axiom of choice for finite types and a comprehension axiom for negative formulas in finite types. Kohlenbach shows in [Koh08, 119, Cor.7.10] that $\mathbf{E\text{-}HA}^{\omega}$ + AC + $\mathbf{CA}^{\omega}_{\neg}$ does not even prove $\Sigma^0_1$-LEM[6]. This result, in view of the well-known analogous result that constructive set theory with AC proves (by making a key use of the axiom of extensionality) full LEM, shows how little the inclusion of extensionality changes in the context of an ostensibly weaker theory such as $\mathbf{HA}^{\omega}$.[7] It can be therefore expected that the difference in the mathematics developed in intensional and extensional theories can only be brought about at a sufficient expressive power and strength of the axioms involved.

[6] In fact, it does not even prove the considerably weaker $\Pi^0_1$-LEM.

[7] Analogous phenomena are not foreign also to the Classical RM. For instance, [DM10, p. 5] show that some versions of Zorn's Lemma in $\mathbf{Z_2}$ are equivalent to $\mathsf{ACA}_0$ or even provable in $\mathsf{RCA}_0$.

### 3.3.3 Intensionality of theorems in Constructive RM

That being said, the "intensionality of theorems" that I care about is in a sense a meta-property and is not simply expressed by an axiom. A framework can be fully extensional but identify theorems with their proofs thus satisfying this property. To see if, or to what extent does BISH or related formal systems appreciate the distinction between a statement and its proof I have to inspect the *practice* of Constructive RM.

The way ordinary mathematics is handled in the constructive setting is by "constructivizing" the theorems, following Bishop's remark:

> Every theorem proved with idealistic [*i.e.* classical] methods presents a challenge: to find a constructive version, and to give it a constructive proof. [BB85, p. 3]

The typical way to do the first step is to add "extra data". For example, supplying a modulus of uniform continuity to each continuous function (as it is done in BISH) is exactly that. After giving a theorem a "numerical meaning", one proceeds with finding a constructive proof of it. At this point, what ensues in constructive mathematics can vary significantly from the practice of Constructive RM: in the former, one often has to find creative methods that are rewarded with more informative proofs[8]; in the latter, one has to find the constructively invalid principle in the proof and demonstrate its indispensability (this sometimes amounts to performing both tasks, when no "constructivized" version of the theorem is known), often however most steps in a proof are constructively valid and the only task is to show the equivalence of the non-constructive steps to one of the weakenings of LEM. In these cases, the reversals by default follow the steps in the original proofs, thus sticking, in the sense of practice, the theorems with their proofs. This is the case with many reversals of important theorems (*e.g.* Bolzano-Weierstraß and Ascoli Lemma to **LPO** or IVT and weak König's Lemma to **LLPO** [Han18, pp. 11, 22], [Ish90]).

When there is more work to be done than just identifying the non-constructive steps in proofs, the notorious feature of reversing theorems in BISH is, by design, following what might be called the constructive pattern of proof, *i.e.* basing the reasoning on the objects already constructed through giving them an exact numerical meaning. In this way, there is much less variation on the methods

[8] A beautiful example of that is Bishop's proof of irrationality of $\sqrt{2}$, which gives an upper bound on the distance of this number from the closest rational one.

used in proofs of a given statement, that in essence results in a closer identification of a statement with its proof.

Moreover, as the key non-constructive steps are logical principles, the argument amassed in a proof usually follows a specific line of reasoning that accounts for the usage of these laws. In contrast to the set-existence principles in Classical RM that do not emphasize the method of construction of such a set, the use of a logical principle like **LPO** stems from a specific line of reasoning about the sequences of numbers, that forces the preceding construction of objects enabling the use of the principle. In effect, the proofs follow patterns of argument based on the constructions and the reasoning necessary to admit the use of these principles, resulting in close relationship between a theorem and its proof.

Finally, having said that, one sociological qualification is in order: since constructive mathematics as a field of research is much less popular than conventional mathematics, it might be the case that it is yet-unexplored in the sense that improved or alternative ways of proof might be found, resulting in multiple proofs of a single theorem. That would fuel the inadequacy of speaking of, say, an equivalence between a principle and a theorem, just like in case of Classical RM as discussed in section 2.3.2. Even if that is to be the case however, it seems to me much less frequent than in the classical case, due to the method of constructive mathematics.

### 3.3.4 Comparison of Reversals

After these general remarks let me turn to the specific example by comparing the above reversal between **LPO** and the Bolzano-Weierstraß theorem with its classical counterpart with respect to two questions: *(a)* In what way do the "constructivized" objects change the (reverse) mathematics developed? *(b)* Is the constructive reversal more informative w.r.t. the numerical content of the objects involved?

Since the direction **LPO**$\Rightarrow$BW admits the use of the non-constructive principle, both proofs are relatively similar in this part. Hence I focus on the differences between the proofs. An obvious one is the use of the constructive dichotomy for reals instead of the division on an interval into two equal parts. This introduces a difference in the feasibility of finding the limit, since in general one has to make more decisions (divisions) if the subintervals overlap (*i.e.* are bigger than halves). This is a practical difference that does not influence the reversal since both frameworks admit countably-many iterations of a principle. Apart from that, both proofs use (in the forward direction) a decision method that yields the limit of the sequences.

But in the direction from the theorem to the principle ($\mathsf{ACA}_0/\mathbf{LPO}$) the difference of constructive and classical approach is fully fleshed out. Observe that in the classical case, the content of the formula 1 defining the subsequence is completely unrelated to the demonstration of the reversal; the only thing that matters is its logical complexity that lets one "plug it" into the $\Delta^0_1$-$\mathsf{CA}$, to prove the $\Sigma^0_1$ scheme. One could use any other $\Pi^0_1$ or $\Sigma^0_1$ formula to arrive at this conclusion. This suggests that either the $\Sigma^0_1$ sentences bear some essential determinants of formulas about the suprema of sequences of rationals, or that the reversal is not really informative about what is going on in BW. There is some definite merit to the former, since a supremum of a sequence of numbers can only be described by at least a $\Sigma^0_1$ or $\Pi^0_1$ formula; no $\Sigma^0_0$ formula can express it. Herein lies both the strength and weakness of the Classical RM: the sole focus on what sets exist under assumptions of different theorems enable making many theorems equivalent, but the *use* of the objects these theorems talk of plays a secondary role and sometimes is not significant at all.[9] Compare this with the proof that BW implies **LPO** though the monotone convergence theorem. The decision sequence $(\beta_n)$ that is exactly the **LPO** is constructed in close relation to the original monotone sequence $(\alpha_n)$, utilizing the division into subintervals.

In case of BW, the constructive handling does not introduce important changes to the mathematical objects, but this is by no means a rule, as discussed before. On the other hand, we get a clear method of constructing the decision sequence.

## 3.4 Evaluation and Conclusion

### 3.4.1 Resource-sensitivity

As to the question *(iii)* about possible other problems entrenched in Constructive RM, one has to look into resource-sensitivity. To reiterate, a resource-sensitive framework is one that distinguishes

[9]In the abstract, it is then possible that there is no obvious way of proving in informal classical mathematics that $Thm \to Thm'$ for $\vartheta \Leftrightarrow (\cdot) - \mathsf{CA}_0 \Leftrightarrow \vartheta'$ holding in Classical RM, where $\vartheta$ is a formalization of $Thm$. But I do not know of such examples.

between a single, a finite number or countable amount of consecutive or parallel applications of a principle.

Generally, the (non-)resource-sensitivity comes from the logic underlying a framework. Constructive RM, be it in BISH or formal systems, is even less resource-sensitive than its classical counterpart. This is mainly due to unrestricted use of the *axiom of countable choice*, which results in complete freedom of parallel usage of a principle (or a theorem) countably many times [BGP21, p. 42]. This is important because some of the reversals cannot be proved only with finite uses of a theorem. For example, **LLPO**⇔ weak König's Lemma requires iterating the principle countably many times [Han18, p. 24], making the result look weaker than it actually is from the perspective of what is *practically computable*[10]. Therefore, if a framework were to distinguish reversals based on resource-sensitivity, it would be even more fine-grained. An obvious path towards that would be using intuitionistic linear logic (that specifically focuses on the amount of uses of available tools) instead of intuitionistic one, but this approach has not yet been implemented. An alternative approach to this issue is presenting the reversals in the form $Thm \Leftrightarrow Principle + (some)\ axiom\ of\ choice$, emphasizing the use of non-resource sensitive axioms.

### 3.4.2 Different Mathematics?

It is sometimes claimed that since intuitionistic mathematics (unlike intuitionistic logic) involves constructions contradictory to ordinary mathematics, it cannot serve as a tool for foundational analysis of the latter. The stronger intuitionistic theories are *only* $\Pi^0_2$-conservative over the classical ones, which, in presence of their great expressive power and proof-theoretic strength amounts to little conservativity indeed. For if a theory such as $\mathbf{T_0}$ or **CZF**+$REA$'s[11] is $\Pi^0_2$ conservative over $\Delta^1_2$-CA+**BI**[12] [Rat17, p. 402], can such a result be viewed as any improvement of **HA**'s conservativity over **PA**? I do not think so. If a theory is fit to develop more complex areas of mathematics such as analysis and topology, its agreement on arithmetic with the classical counterparts seems to be a fact of obvious folklore at best and a waste of assets at worst. If the conservativity, which expresses the possibility of serving as a theory for both classical and constructive mathematics, only concerns

[10]The implication from Weak Limited Principle of Omniscience, that lays between **LPO** and **LLPO**, to WKL only requires a single use of the unique choice, making the result more feasible from the resource-sensitivity perspective at the expense of using a stronger classical principle.

[11]$REA$ is an axiom stating that every set is included in a regular set, effectively strengthening the theory.

[12]Bar Induction.

arithmetical statements, then there is no need for theories beyond the strength of **HA**.

On the other hand, these theories are admittedly too strong. By contrast, when it comes to intensionality and weak intuitionistic theories, they seem to be still too weak to produce problems when AC meets the axiom of extensionality in their context, that is, they do not imply LEM. These issues are expected to occur with vindication of the expressive power. Gödel's remark that in the context of arithmetic, intuitionism (and its implicit rejection of extensionality) makes no real difference remains in force. And it seems that this holds also for large portions of intuitionistic analysis, given its careful treatment of objects such as point-wise continuous functions. But one can expect that LEM will eventually begin to pop up if AC and extensionality will be used in stronger theories, appropriate to handle more involved objects of intuitionistic analysis and topology. In the context of my reasoning, this means that if one assumes the constructive framework to be the correct one for treating intensional objects of mathematics such as theorems, as I've argued above in section 3.3.3, the proofs can be assumed to be co-extensional with theorems within constructive frameworks up to high, *but not too high* strata of the hierarchy of proof-theoretic strength. It seems apparent from the constructive approach to mathematical objects that given a statement, in constructive mathematics there is less room for using proof methods of radically different strength in the process of demonstrating it, and this phenomenon might account for the uniqueness of proofs in the lower strata of intuitionistic theories.

Now, BISH can serve as a basis of foundational analysis of the ordinary classical mathematics due to its neutrality towards both the latter and intuitionistic mathematics. In this sense, it does not carry the burden of uncanny objects that Brouwer's mathematics had to be cleansed of. I believe that a different classification (from the classical one) of theorems that BISH arrives at stems from this framework's advantage of using the finer-grained base logic (that can, for instance distinguish a statement and its contrapositive) and the in-built uniformity and countability of objects. This claim goes against Simpson, who says that

> our approach in [Sim99] is to analyze the provability of mathematical theorems *as they stand*, passing to stronger subsystems of $\mathbf{Z_2}$ if necessary. [Sim99, 32, emphasis mine]

Since working in BISH on ordinary statements requires their constructivization, the changes introduced might seem suspicious from the perspective of faithfulness of representation. To this I respond that (1) these "changes" are not real definitional differences and from the classical perspective can be viewed as restrictions on the objects considered, not actual changes of the theorems. For instance, the constructive version of the Heine/Borel covering lemma will speak of a subset of continuous functions, namely the uniformly continuous functions, for which the result of course also holds in classical mathematics. Additionally, there is a reward for this restriction in the form of more precise numerical information on the behaviour of the functions and the construction of coverings. In this sense, the constructivization of statements also results in treating the theorems "as they stand", only for a smaller class of objects. And the restriction seems far from arbitrary. Apart from that, as [Eas15, p. 52] remarks

> an apparent enrichment is not always a genuine enrichment: a statement that employs an enriched notion may turn out to be equivalent to an alternative formalisation that does not.

An easy example of this is that the "enrichment" resulting in distinguishing the Fan Theorem from weak König's Lemma is silent from the classical perspective: in classical mathematics these statements are equivalent to each other.

Moreover, from the perspective of practice, the constructivizations in important cases do not change the sets of objects that mathematicians are actually interested in. Simpson [Sim99, pp. 136–7] notes that

> "any continuous function [from $\mathbb{R}^k$ into $\mathbb{R}$] which arises in practice" can be proved in $\mathsf{RCA}_0$ to have a modulus of uniform continuity on any closed bounded subset of its domain.

But then he insists

> This situation has prompted some authors, for example [BB85, p. 38], to build a modulus of uniform continuity into their definitions of continuous function. Such a procedure may be appropriate for Bishop since his goal is to replace ordinary mathematical theorems by their "constructive" counterparts. However ... our goal is quite different. Namely, we seek to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems *as they stand*. ... Thus Bishop's procedure would not

be appropriate for us. [Sim99, 137, original emphasis]

In other words, Simpson was eager to give up the precisification of the statements together with mathematical practice merely to preserve the expression of classical statements "as they stand". The misconception responsible for this choice is, I believe, expressed when Simpson says that "[Bishop's] goal is to replace ordinary mathematical theorems by their 'constructive' counterparts". This would be true if Simpson's discussion referred to Brouwer's constructivism, that indeed bluntly rejected huge portions of classical mathematics. However, one of the most influential of Bishop's contributions to constructivism was the observation that classical mathematics has a substantial underpinning of constructive truth [BB85, p. 12]. Bishop's constructivism is fully consistent with classical mathematics, which simply is an extension of it. Therefore, the "constructivized" versions of theorems within BISH are a part of classical mathematics.(2) The insistence on safeguarding classical mathematics as it stands seems all the more dazzling when one considers the formal acrobatics that have to be involved to represent classical mathematics in $\mathbf{Z_2}$. As discussed in chapter 2, encoding ordinary objects most often requires "arithmetization" of objects on top on other definitional changes which immediately should pose questions about the faithfulness of the representation. In any case, I believe it is clear that the statements Classical RM works on are far from being "as they stand" in the ordinary mathematical practice. Plainly put, the choice between Classical RM and Constructive RM in the context of faithfulness of representation is a choice between infinitary statements being forced into a finitary form and finitary statements. In this sense, Constructive RM is much more faithful to the mathematical practice as it stands.

# A Definitions

**Def. A.1** ($\omega$-model)**.** An $\omega$-model is a $\mathcal{L}_2$-structure

$$(\omega, S, +, \cdot, 0, 1, <)$$

where $\emptyset \neq S \subseteq \mathcal{P}(\omega)$. Thus an $\omega$-model differs from the intended model only by having a possibly smaller collection $S$ of sets to serve as the range of the set variables.

**Def. A.2** (Turing-reducibility)**.** Let $g$ be a finitary function on $\omega$. The class of functions *partial computable from* $g$ (the class of partial $g$computable functions) is the smallest class which includes every function that is primitive recursive in $g$ and is closed under composition, primitive recursion, and applying the minimum operator.

A function $f$ is computable from $g$ (or Turing reducible to $g$ written $f \leq_T g$, if it is partial computable from $g$ and total.

Turing reducibility can be characterized by $\Delta^0_1$ definability.

**Def. A.3** (Turing functional)**.** A *(Turing) functional* is a function $\Phi$ defined on $\omega^\omega$ such that for some $e \in \omega$,

$$\Phi(g) = \lambda \overrightarrow{n}\ \Xi(e, \overrightarrow{n}, g)$$

for all $g \in \omega^\omega$, where $\Xi$ is a universal computable function. We call $e$ an index for $\Phi$ and we say that $\Phi$ is $k$-ary, where $k$ is the length of $\overrightarrow{n}$.

**Def. A.4** (1-consistency of $T$)**.** A finite set of sentences $\mathcal{S}$ is said to be *1-consistent* if $\mathcal{S} + T_{\Pi^0_1}$ is consistent where $T^0_{\Pi 1}$ is the set of all true $\Pi^0_1$ sentences.

**Def. A.5** ($\beta$-model)**.** A $\beta$-model is an $\omega$-model $\mathcal{S} \in \mathcal{P}(\omega)$ with the following property. If $\sigma$ is any $\Pi^1_1$ or $\Sigma^1_1$ sentence possibly with parameters from $\mathcal{S}$, then $(\omega, \mathcal{P}(\mathcal{S}), +, \cdot, 0, 1, <)$ satisfies $\sigma$ iff the intended model, *i.e.* $(\omega, \mathcal{P}(\omega), +, \cdot, 0, 1, <)$, satisfies $\sigma$.

**Def. A.6** (hyperarithmetical set)**.** We define the hyperjump operation inductively:

$\mathrm{HJ}(0, X) = X$

$\mathrm{HJ}(n+1, X) = \mathrm{HJ}(\mathrm{HJ}(n, X))$

A set $X$ is *hyperarithmetical* in a set $Y$ if $X \leq_T Y^{\mathcal{O}}$ for some well ordering $\mathcal{O}$ computable from $Y$ If $X$ is hyperarithmetical in $Y$ we write $X \leq_{HYP} Y$.

Hyperarithmetical sets are exactly the $\Delta^1_1$-definable subsets of $\omega$.

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# Acronyms


**$\mathsf{RCA}_0$** Recursive Comprehension. 1, 3, 4, 5, 7, 9, 20, 26, 27, 28, 30, 32, 41, 42

**$\mathsf{WKL}_0$** Weak König Lemma. 1, 4, 5, 20, 26, 27, 28, 41

**$\mathsf{ACA}_0$** Arithmetic Comprehension. 1, 4, 6, 7, 9, 20, 26, 27, 28, 29, 40, 42

**$\mathsf{ATR}_0$** Arithmetic Transfinite Recursion. 1, 4, 7, 8, 9, 20, 25, 26

**$\Pi^1_1$-$\mathsf{CA}_0$** $\Pi^1_1$-Comprehension. 1, 4, 7, 8, 9, 20, 25, 26, 33, 34

**BISH** Bishop's Constructive Analysis. 1, 22, 46, 49, 50, 51, 52, 53, 54, 55, 56, 59, 61

**$\mathbf{Z_2}$** Second Order Arithmetic. 1, 2, 4, 25, 31, 35, 37, 40

**Classical RM** Classical RM. 2, 4, 5, 6, 17, 19, 20, 21, 22, 23, 26, 29, 30, 31, 32, 35, 38

**RM** Reverse Mathematics. 3, 6, 7, 9, 10, 13, 21, 22, 23, 25, 26, 27, 29, 30, 31, 32, 34, 35, 37, 38, 39

**PA** Peano Arithmetic. 5, 6

**PRA** Primitive Recursive Arithmetic. 5, 17, 25, 30

**AD** Axiom of Determinacy. 9

**KP** Kripke-Platek set theory. 9

**AC** Axiom of Choice. 14, 16, 33, 54, 59

**II GIT** II Incompleteness Theorem. 15

**I GIT** I Incompleteness Theorem. 15

**EM** Explicit Mathematics. 22, 48

**HA** Heyting Arithmetic. 22

**Constructive RM** Constructive Reverse Mathematics. 22, 38

$RT_2^2$ Ramsey theorem for pairs. 25

**LEM** Law of Excludded Middle. 38, 49, 50, 54, 59

$\mathbf{HA}^{\omega}$ Heyting Arithmetic in finite types. 44

**MP** Markov's Principle. 49

**LPO** Limited Principle of Omiscience. 49, 55, 56

**LLPO** Lesser Limited Principle of Omiscience. 49, 50, 55