LOGIC GAMES: not just tools, but models of interaction

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Abstract This paper is based on tutorials on 'Logic and Games' at the 7th Asian Logic Conference in Hsi-Tou, Taiwan, 1999, and until 2002 in Siena, Stuttgart, Trento, Udine, and Utrecht. We present logic games as a topic *per se*, giving models for dynamic interaction between agents. First, we survey some basic logic games. Then we show how their common properties raise general issues of game structure and 'game logics'. Next, we review logic games in the light of general game logic. Finally, we discuss more 'realistic' influences from game theory into logic games, including players' preferences, and imperfect information.

1 From products to activities: logic in games

Logical Dynamics Logic is often taken to be about propositions, truth, and proofs: abstract objects in Heaven, and their Platonic properties. But the discipline arose in Antiquity by studying *activities* on Earth: dialogue and argumentation. And the very terminology of logic still has a double meaning. 'Statement' is both a dynamic activity and the static product of that activity, 'proof' is a procedure of establishing a claim and a formal record of that procedure, etc. These activities are usually kept in the background, as mainly didactical motivation. Placing the dynamics at centre stage in logical theory is the program of 'logical dynamics' (van Benthem 1996).

Logic games The best source in modern logic for structured activities are *logic games*. These have been around since the Middle Ages, with the 'Obligatio' debates of authors like Walter Burleigh. Mathematical logic games were defined in the 1950s by Lorenzen, Ehrenfeucht–Fraïssé, and Hintikka – not accidentally in conjunction with the new wave in game theory. Today, two-person games are in wide use for logical tasks of evaluation of propositions in a model, comparing models, building models for assertions, or constructing proofs for claims. And new varieties are still appearing. For a survey with many references, see van Benthem 1999 – 2002.

This paper is based on van Benthem 1999. It presents logic games in one setting, to show their role as a model of 'intelligent interaction' which fits well with game theory. Our presentation is elementary, and mostly in the nature of a survey. We use existing results, plus an occasional new observation connecting up relevant strands, to paint a total picture. But first, let us look at some basic examples.

2 The basic logic games

2.1 Gamification In principle, any logical task can be 'gamified', by pulling it apart into roles for two players whose dynamic interaction tests the notion involved. Interaction involves *dependence*, and gamification works once we have an interplay between a universal quantifier ***A*** ('Abelard', 'Adam', 'Alter') and an existential ***E*** ('Eloise', 'Eve', "Ego'). Leibniz already explained the meaning of quantifier forms $\forall\varepsilon\, \exists\delta$ expressing dependence in mathematical settings in terms of a game:

> A challenger $\forall$ chooses some number ε at his discretion,
>
> and the defender $\exists$ has to produce a suitable response δ.

In what follows, we give some sketches of major logic games, referring to the literature for more detailed exposition and references: cf. van Benthem 1999 – 2002, 2005. Our aim is to make the reader aware of the ubiquity of these games, and give some examples for the general discussions later on in this paper.

2.2 Argumentation Perhaps the oldest example of a logic game is *argumentation*: one makes a claim against an opponent, upholding it in the face of objections. We all experience its game-like character of having to say the right thing at the right time, and also, the bitter taste of defeat when we have talked ourselves into a corner, contradicting ourselves. The latter are the typical losing stages in argumentation – being at the same time wins for the other player. Precise dialogue games for argumentation, in the style of Lorenzen, may be found in Rahman & Rueckert 2001.

Here is the key game-theoretic feature of argumentation – which probably led to the Greeks discovering logical patterns of reasoning in the first place. Roughly speaking,

> *Logically valid propositions* ϕ will be precisely those
> for which their proponent Eve has a *winning strategy*:

that is, a way of choosing her conversational moves against opponent Adam which guarantees that she never loses – no matter how Adam attacks her claim. Thus, Eve's winning strategy is a dynamic counterpart of a logical *proof* for the proposition ϕ. Now there may be more than one proof for a claim, and this reflects the diversity of rational behaviour. Players may have more than one strategy to win a debate. This adversarial strategy-based style of analysis holds across a wide range of logic games.

But players are on a par in games, and there is no need to glorify one over the other. It is their *interaction* which really matters. In particular, since not all ϕ are valid, there are argumentation games where Adam has a winning strategy for involving Eve in self-contradictions. Section 2.4 below takes up the matter what these look like.

2.3 Obligatio Most conversation is not about argumentation. We tend to believe things people tell us – even implausible ones like "I love you" – as long as they are *consistent*. Only in special settings will we be challenged to prove our assertions: in a juridical procedure, or in teaching mathematics. But maintaining consistency is itself a major logical task! Medieval training disputations often had a form like this:

> Eve has to maintain consistency when Adam confronts her with successive assertions ϕ, each of which she has to accept or reject. In the former case, ϕ is added to her cumulative store of commitments, otherwise $\neg\phi$ is added. Eve loses if at any stage, her commitments become inconsistent.

One may ask whether this sort of logic exam is fair. And indeed, in principle,

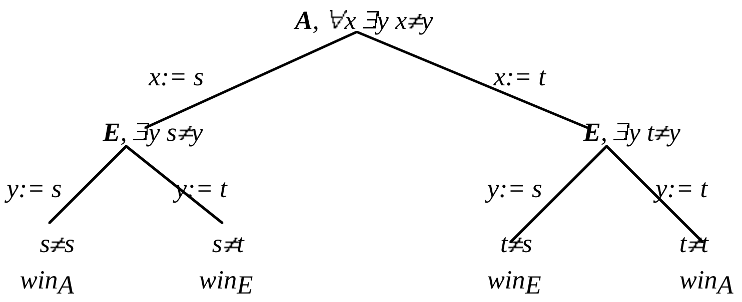
> Eve always has a *winning strategy*, being a string of *YES/NO* answers to any sequence of propositions, keeping her commitments consistent.

To see this, Eve's strategies for passing the exam are correlated with *logical models*: situations that make all her commitments true, any model will help her pass the test. Strategic insights into consistency are, alas, also logical skills that help us misinform and lie. For details of medieval disputations, involving Eve's initial knowledge, and an option of giving a third response of 'doubt' – cf. Dutilh-Novaes 2002.

2.4 Model checking Arguably the most basic logic game today occurs in a different setting. Let a first-order assertion ϕ be made about a model $\boldsymbol{M}$ and variable assignment s. A game $game(\phi, \boldsymbol{M}, s)$ of semantic evaluation or 'model checking' lets a 'Verifier' Eve claim the truth of ϕ, while a 'Falsifier' Adam defends its falsity:

> With atomic ϕ **,** one *checks* who is right about $\boldsymbol{M}, s$, and that player wins. Disjunctions are a *choice* for Eve, and play goes on with the disjunct chosen. Conjunctions are an initial choice for Adam. Negations trigger a *role switch*, with all turns interchanged between players. Existential quantifiers $\exists x\phi$ are a move for Eve who picks a witness object d in $\boldsymbol{M}$, and play continues with the formula $\phi(d)$. Universal quantifiers are Adam's choice of a challenge object.

The game for the formula $\forall x\, \exists y\, x{\neq}y$ in a model with two distinct objects s, t is this:



Note that ***V*** has a winning strategy here. Note how this perspective pulls one standard logical formula, say $\forall x\ \exists y\ \forall z\ \exists u\ \phi(x, y, z, u)$, apart into an interactive alternation $\forall_1 x\ \exists_2 y\ \forall_1 z\ \exists_2 u\ \phi(x, y, z, u)$. A good account of evaluation games is Hintikka & Sandu 1997, while Stirling 1999 has variants for fixed-point languages in computer science. Again, the central notion involves winning strategies (Hintikka 1973):

Fact 1 For all ***M***, s, ϕ, the following assertions are equivalent:

(a) ***M***, $s \models \phi$

(b) Eve has a winning strategy in *game*(ϕ, ***M***, s).

Adam has a winning strategy if ϕ is false in ***M***. Different winning strategies encode different *reasons* for the truth or falsity of an assertion ϕ. 'Reasons' are unusual logical objects – but with quantifier combinations such as a true $\forall x\ \exists y\ \forall z\ \exists u\ \phi(x, y, z, u)$, one can think of them as bunches of *Skolem functions* $f(x)$, $g(x, z)$ of the right arities providing Eve with her winning response in ϕ to Adam's successive choices.

2.5 Model construction The evaluation task of checking if ***M***, $s \models \phi$ starts from a given model and formula. But in the earlier task of checking consistency, only assertions are given, with the satisfiability question if one can find a model for these. This suggests a *model construction game* between a 'Builder' Eve who tries to create a model making some initial assertions true and others false, and a 'Critic' Adam who raises objections, making sure that every building task gets scheduled. In particular, Critic can force Builder to choose when a disjunct is to be made true, and he can keep calling new instances of initial universal quantifiers as Builder puts new objects into the model under construction. Builder loses at any stage if her current schedule tells her to make the same formula both true and false. A precise format for the game arises by dynamifying *semantic tableaus* (van Benthem 2006), with decomposition rules for logical operators as game moves. The result of all this is

Fact 2 The following assertions are equivalent in tableau construction games:

(a) A given set of first-order formulas Σ has a model,

(b) Builder has a winning strategy in the construction game for Σ.

Unlike first-order evaluation games, whose depth is bounded by the operator depth of the initial formula ϕ, construction games can have infinite runs. The reason is that some first-order formulas have only infinite models, making Builder go on forever. Again, there is a match between Builder's winning strategies and different *models* (if any) for Σ. More sophisticated construction games are used in Hodges 1985.

If Builder's winning strategies are like models, what about Critic's? The latter are guaranteed ways of blocking any construction attempt. It can be shown that these are essentially *proofs* of the negation of the initial assertion. Thus, the construction game is like the earlier argumentation game, when we reformulate the roles. Critic tries to prove some initial assertion, while Builder is looking for a *counter-example*. This answers our question in Section 2.2 about Adam. His winning strategies correlate with counter-examples to the claim put forward by Proponent. Thus, argumentation and model construction games are really two takes on the same logical process.

2.6 Model comparison In addition to model checking and satisfiability testing, other basic tasks for a modern logical system have to do with its expressive power. We measure the latter by seeing which models can be *distinguished* by our language. The most widely used logic game performs just this task. *Ehrenfeucht-Fraïssé games* cast Eve as a 'Duplicator' who claims that two models ***M***, ***N*** are similar, while the 'Spoiler' Adam claims they are different. Each round of the game starts with ***M, a*** – ***N, b***, where ***a, b*** are tuples of objects chosen according to the following procedure:

> In each round, Spoiler chooses a model ***M*** or ***N***, and an object *x* in it,
> Duplicator then chooses a corresponding object in the other model;
> and the link *x*–*y* is then added to the current match ***a*** – ***b***

Duplicator loses whenever the function from ***M*** into ***N*** defined by the current match of objects is no longer a partial isomorphism between the two models. She wins those runs of the game where this failure never occurs. We can play such games over a fixed finite number of rounds, or forever. An excellent standard textbook on Ehrenfeucht-Fraïssé games is Doets 1996. Here is the key result about the method:

Fact 3 For all models ***M***, ***N***, the following are equivalent:

(a) Duplicator has a winning strategy in a *k*-round comparison game

(b) ***M***, ***N*** satisfy the same first-order sentences up to quantifier depth *k*.

In games with no finite bound, Duplicator's winning strategies match up with *potential isomorphisms* between ***M***, ***N***: a notion of *structural similarity* for models. But as always in logic games, the viewpoint of the other player is of independent interest. In the *k*-round model comparison game, a winning strategy for Spoiler is essentially a first-order formula ϕ of depth $\leq k$ which is true in ***M*** and false in ***N***. We will make this more precise in Section 3. This reflects the original goal of testing the expressive power of our language. In this dual view of the game, then, Spoiler is the 'positive' player, claiming the language is rich, while Duplicator claims it is poor. A state-of-the-art exposition of model comparison games is Väänänen 2007.

2.7 Other logic games New logic games are still emerging today. E.g., Hirsch & Hodkinson 2002 study when a given abstract relational algebra ***A*** is representable as an algebra ***S*** of binary set relations over some set of individual objects *U*. Thus, we want to build a model *(U,* ***S****)* standing in some isomorphism-like relationship *E* to the model ***A***. Mixing ideas from model construction and model comparison games, the authors let players create 'networks', as stages of a representation in progress, with Builder responding to challenges made by Critic. An abstract relational algebra ***A*** is representable iff Builder has a winning strategy in this representation game. The result is a perspicuous new axiomatization of the representable relational algebras.

This concludes our survey of logic games. We now turn to their general features.

3 The unifying role of strategies

3.1 Strategies as a unifying notion Our games all had an 'Adequacy Theorem' stating that some standard notion obtains (truth, satisfiability, potential isomorphism) iff some designated player has a winning strategy. Thus, the typically game-theoretic notion of a strategy becomes a unifying idea across logic. This leads to surprising connections. E.g., in one and the same type of game, viz. that for model construction, we may encounter *proofs* as winning strategies for Critic, and *models* as winning strategies for Builder. Thus, logically very different notions turn out cousins after all. Sometimes also, strategies are new citizens asking for recognition in logic, such as 'semantic reasons' for truth or falsity in first-order evaluation games. These analogies suggest that underlying logic, there is a *calculus of strategies*, for combining them and proving their basic properties. We will return to this issue in Sections 6 and 11.

3.2 ∃-sickness**, **and its cure Despite their crucial role, strategies are often hidden in Adequacy Theorems. E.g., truth amounts to the existence of a winning strategy for

Verifier in an evaluation game, but we are not told how. Indeed, there is a wide-spread disease of *∃-sickness*: the wilful hiding of available specific information under existential quantifiers. Sure symptoms are indefinite articles “a”, or modal affixes “-ility”: cf. “having a strategy" or "winnab–ility". *∃*-sickness also afflicts completeness theorems relating "provabilit*y"* to validity, instead of a more informative match from *proofs* to semantic structures (but see the strong completeness theorems in Abramsky and Jagadeesan 1994). An early case is in Barwise & Perry 1983, who pointed out the self-inflicted problems of tense logic taking the past tense in “Lida fell down the stairs” as “at *some time* in the past”, losing the particular episode we have in mind.

Fortunately, the disease is often cured with a little exercise. Our first illustration shows how to make an existential quantifier explicit by analyzing a standard proof – in this case, adequacy for model comparison games – making the strategies explicit:

Theorem 4 There is an *explicit correspondence* between

(a) Winning strategies for ***S*** in the *k*-round comparison game for ***M**, **N***

(b) First-order sentences ϕ of quantifier depth *k* with $\boldsymbol{M} |= \phi$, *not* $\boldsymbol{N} |= \phi$

Proof From (b) to (a). Every such 'difference formula' ϕ of quantifier depth *k* defines a winning strategy for Spoiler in a *k*-round game between arbitrary models. Each round *k–m* starts with a match between linked objects chosen so far which differ on some subformula ψ of ϕ with quantifier depth *k–m*. By straightforward Boolean analysis, ***S*** then finds some existential subformula $\exists x \bullet \alpha$ of ψ with a matrix formula α of quantifier depth *k–m–1* on which the two models disagree. ***S***'s next choice is a witness in that model of the two where $\exists x \bullet \alpha$ holds.

From (a) to (b). Each winning strategy σ for Spoiler induces a distinguishing formula of proper depth. Let ***S*** make his first choice *d* in model ***M*** according to σ –

and write down an existential quantifier for *d*. Our formula under construction will be true in ***M***, and false in ***N***. We know that each choice of Duplicator for a corresponding object *e* in ***N*** gives a winning position for ***S*** in all remaining *k–1*-round games starting from an initial match *d–e*. By the inductive hypothesis, these induce distinguishing formulas of depth *k–1*. Now, the *Finiteness Lemma* for first-order logic over a fixed finite relational signature says that, for any fixed set of free variables and fixed quantifier depth, only finitely many non-equivalent formulas exist. In particular, only finitely many of the above distinguishing formulas can occur modulo logical equivalence. Some of these will start with 'their' first quantifier in ***M*** (say $A_1, .., A_r$) – others in ***N*** (say $B_1, .., B_s$). The total distinguishing formula for strategy $\boldsymbol{\sigma}$ is then the ***M***-true assertion $\exists x \bullet (A_1 \,\&\, .. \,\&\, A_r \,\&\, \neg B_1 \,\&\, ... \,\&\, \neg B_s)$. ♣

Thus, Spoiler’s winning strategies in a comparison game correspond to formulas, logical objects of prime interest. For Duplicator, the objects corresponding to her winning strategies might be called ‘analogies’, of a finite quality measured by the game length *k*. They are cut-off versions of *potential isomorphisms*.

Of course, even Theorem 4 is still *∃*-sick! But the remaining outer existential quantifier in its formulation may be harmless, in that its instantiation is the *proof*.

Our second illustration shows another way of high-lighting strategies, by analyzing the available *number of them*. Consider verification games for propositional logic. Here is an elementary, but at least, unusual observation on counting strategies.

Fact 5 One can count the number of verifying (falsifying) strategies, say $\#(V, \phi)$ $(\#(F, \phi))$ for any propositional formula ϕ as follows:

$\#(\mathbf{V}, p)$	=	1	$\#(\mathbf{F}, p)$	=	1
$\#(\mathbf{V}, \neg\phi)$	=	$\#(\mathbf{F}, \phi)$	$\#(\mathbf{F}, \neg\phi)$	=	$\#(\mathbf{V}, \phi)$

$$\#(\boldsymbol{V}, \phi \vee \psi) = 2\#(\boldsymbol{V}, \phi) \bullet \#(\boldsymbol{V}, \psi) \qquad \#(\boldsymbol{F}, \phi \vee \psi) = \#(\boldsymbol{F}, \phi) \bullet \#(\boldsymbol{F}, \psi)$$
$$\#(\boldsymbol{V}, \phi \wedge \psi) = \#(\boldsymbol{V}, \phi) \bullet \#(\boldsymbol{V}, \psi) \qquad \#(\boldsymbol{F}, \phi \wedge \psi) = 2\#(\boldsymbol{F}, \phi) \bullet \#(\boldsymbol{F}, \psi)$$

Proof The rationale for these clauses is immediate from the standard definition of strategies. in game trees as functions assigning unique moves to players' turns. ♣

Such counting becomes much more complex with full first-order evaluation games.

3.3 Strategies: actions or powers? Strategies are stepwise instructions for players to act. This detailed level of game structure was suppressed by existential quantifiers of 'having a strategy'. But upon reflection, one person's $\exists$-sickness may be another's sanity! In games, we are sometimes not interested in details of moves and actions, but just in the *control* that players have over possible *outcomes*. E.g., that Eve has a winning strategy really says that it is within her power – whatever Adam does – to make sure that the game ends in some specific set of runs or outcomes, designated as 'winning'. And players may have further powers in games: say, via losing strategies, or strategies that guarantee long runs. Such powers have been proposed as a natural level for describing influence in social settings (Pauly 2001).

This coarser level of control raises a general question. At what level do we want to describe games – in terms of global outcomes, or more detailed local actions? This choice is one of many general game-theoretic issues lying behind logic games:

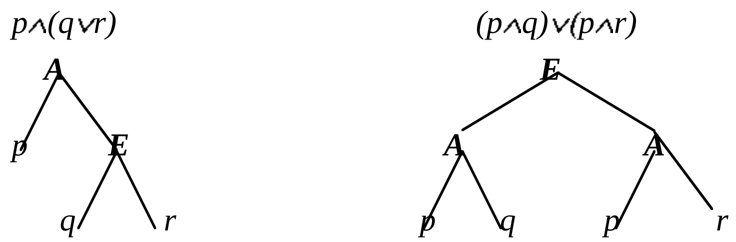
3.4 From logic games to game theory Logic games, though a very specialized class of activities, high-light issues which concern all games: playing at cards, competing in markets, or engaging in warfare (not *that* far removed from Academia). Some of these issues have arisen independently in game theory, others seem new. (A compact lucid source on game theory is Osborne & Rubinstein 1994.) In our next sections, we look at some pervasive ones – with heuristics inspired by the evaluation games of Section 2.4. An obvious bridge from logic to game theory are Adequacy Theorems like the one relating the logical notion of truth with the game-theoretic notion of a strategy. Unimaginative people interpret such results as a 'kiss of death' for the game-theoretic stance, as it 'just restates' what we know from standard logic already. But the opposite is true for the unprejudiced reader!

4 Game equivalences and game languages

4.1 When are two games the same? To raise our first question in a simple manner, consider the propositional distribution law for conjunction over disjunction:

$$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$$

The two finite trees in the following figure correspond to evaluation games for the two propositional formulas involved, letting ***A*** stand for Falsifier and ***E*** for Verifier.



This picture raises the following intuitive, and yet fundamental, question

When are two games the same?

In particular, does the logical validity of Distribution mean that the pictured games are the same in some natural sense? Clearly, the answer depends on our viewpoint:

I *If we focus on turns and moves, then the two games are not equivalent:* they differ in ‘etiquette’ (who gets to play first) and in choice structure.

This is the level of what game theorists call 'extensive games', with the familiar tree pictures involving details of choices and actions. But there are also 'strategic forms' of games, where we are just interested in listing outcomes that players can control.

E.g., the fact that the order of players' turns is reversed by applying Distribution is immaterial then. At the latter level, the answer to our question becomes 'Yes':

II *Both players have the same powers of achieving outcomes in both games*

A can force the outcome to fall in the sets *{p}, {q, r}*

E can force the outcome to fall in the sets *{p, q}, {p, r}*.

Here, a player's *powers* are those sets *U* of outcomes for which she has a *strategy* making sure that the game will end inside *U*, no matter what the other player does. On the left, ***A*** has strategies "left" and "right", yielding powers *{p}* and *{q, r}*. Player ***E*** also has two strategies, yielding powers *{p, q}, {p, r}*. On the right, ***E*** has two strategies "left" and "right", which give the same powers for ***E*** as on the left. By contrast, player ***A*** now has four strategies, which may be written ad-hoc as

"left: L, right: L", "left: L, right: R", "left: R, right: L", "left: R, right: R"

The first and fourth give the same powers for ***A*** as on the left, while the second and third strategy produce merely weaker powers subsumed by the former.

4.2 Game equivalences and game languages The general issue here are natural equivalences between games, setting coarser or finer levels of detail. Van Benthem 2002 uses an analogy with theories of computation, and *process equivalences* such as modal bisimulation. In particular, outcome–control views are like black-box input-output views of processes, while extensive games lie closer to views of computation endowing processes with richer internal states, including choices. Thus, structure theory of games is much like general multi-agent process theory.

As usual in mathematics, however, invariance relations between structures are one side of a coin. The other side are the matching properties of games one wants to define. E.g., for strategic games, it does not matter in which schedule ***E***, ***A*** took their turns: for extensive games, this is a relevant property. Such properties are expressed in *game languages* appropriate to the chosen 'equivalence level'. E.g., van Benthem 2002 shows that a good language for describing extensive game forms of perfect information is modal logic, plus fixed-point extensions like the modal μ–calculus. But the appropriate language for describing players' outcome powers only are modal logics over neighbourhood semantics, in the spirit of Parikh 1985.

4.3 Logic and games: the plot thickens We have reached a delicate point in our story. We started with specific games for logical notions. But now, logic now enters in a different guise, as different description levels for games correspond to different languages and associated logics. Thus, in one of those happy Hegelian inversions, in addition to *logic games*, there are also *game logics* describing games in general.

5 Players' powers and modal forcing languages

In this section, we focus on the input–output level of outcomes and players' powers – broadening the connections between logic and game theory. We start with the global level for games, since it seems most congenial to logic games (Section 9.2).

5.1 Determinacy In evaluation games, Verifier has a winning strategy if the relevant formula ϕ is true, and Falsifier has a winning strategy if ϕ is false. This means that these games have the following important game-theoretical property:

Fact 6 Evaluation games are *determined*: one of the two players
in *game* (ϕ, ***M***, *s*) must always have a winning strategy.

A general proof uses *Zermelo's Theorem* which says all zero-sum two-player games of *finite depth* are determined. Indeed all infinite games with topologically open winning conditions for one player are determined by the Gale-Stewart Theorem. This explains why our games of model construction or model comparison are determined, even though runs may be infinite. Critic (Spoiler) have open winning conditions, as

Builder's (Duplicator's) failures always arise at some finite stage. Finally, Martin's Theorem says that all infinite games are determined with winning conditions in the Borel Hierarchy of sets. Non-open Borel winning conditions occur with some games in computer science. Examples include *fairness* of runs for interactive game systems.

With non-Borel conventions, infinite games can become non-determined. We display one example, to make a general point about players' powers later on (Section 8.2). Take any free ultrafilter ***U**** on the natural numbers. Two players pick successive neighbouring closed intervals, of any finite sizes, producing a succession like this:

A: $[0, n_1]$, with $n_1>0$, ***E***: $[n_1+1, n_2]$, with $n_2> n_1+1$, etc.

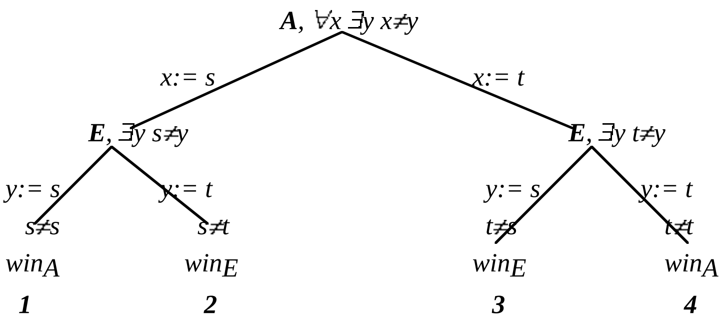
E wins if the union of all her intervals is in ***U****, otherwise, ***A*** wins. It is easy to see that winning sets in this game are not open for either player.

Fact 7 The interval selection game is not determined.

Proof If player ***A*** had a winning strategy, ***E*** could use that with a one-step delay to *copy* ***A***'s responses to her own moves disguised as her moves. Both resulting sets of intervals (disjoint up to some finite initial segment!) would then have their unions in ***U****: which cannot be, as ***U**** was free. Likewise, ***E*** has no winning strategy. ♣

There is a flourishing literature on determined games in descriptive set theory (cf. Löwe 2002), but we will concentrate on more general game issues here.

5.2 Powers and representation There is more to players' powers, even in logic games, then just abilities to win. Consider the game in Section 2.4 for $\forall x\ \exists y\ x{\neq}y$ in a first-order model with two distinct objects *s, t* – where we now number outcomes:



That ∀*x* ∃*y Rxy* is true is reflected in player ***E's*** having the obvious winning strategy "choose the object different from that chosen by ***A***". But players have more strategies in this game, and calculating as in Section 4.1 we get their true powers:

A can force the sets *{1, 2}, {3, 4}*
E can force the sets *{1, 3}, {1, 4}, {2, 3}, {2, 4}*

Power families like these satisfy the following general properties, which are reminiscent of the definition of an ultrafilter, but now 'spread out' over two players:

Monotonicity If *Y* is a power of ***i*** and *Y*⊆*Z*, then *Z* is also a power of ***i***

Consistency If *Y* is a power of ***A*** and *Z* a power of ***E***, then *Y*, *Z* overlap

Determinacy If *Y* is not a power of ***A (E)***, then its set complement –*Y* in the total space of outcomes is a power of ***E*** (***A***).

These are also *all* relevant properties:

Fact 8 An assignment of non-empty subsets of some set to two players represents their powers in some finite game iff these powers satisfy Monotonicity, Consistency, and Determinacy.

In non-determined games, the third condition drops out – after which we can do a similar representation result for monotone consistent families, in which their powers are realized in finite games of imperfect information (van Benthem 2001A; see also Section 10). A typical non-determined example would be the following specification:

powers of ***A*** *{1, 2}* powers of ***E*** *{1, 2)*

Alternatively, one can represent such families of non-determined powers using infinite games of perfect information – witness Section 8.

Digression Fact 8 allows for identifying different outcome states of a game. This can happen in first-order evaluation games for states with the same variable assignment, or in chess, for the same board configurations with different histories. If we insist on *unique* outcome states, additional conditions hold, reflecting a closer tie between strategies and powers. In particular, (a) the intersection of any two inclusion minimal power sets of two players is a singleton, and (b) each singleton outcome set can be obtained as such an intersection. A representation result for this case seems open (cf. van Benthem 2005). This issue will return for logic games in Section 9.

5.3 Modal forcing languages Games at the level of players' powers have a natural *modal game logic*. We just illustrate this – but cf. Parikh 1985 for motivation, and Blackburn, de Rijke & Venema 2001, Blackburn, van Benthem & Wolter 2006 for modal logic in general. The language has proposition letters, Boolean operators, and *forcing modalities* $\{G, \boldsymbol{i}\}\ \phi$ saying that player $\boldsymbol{i}$ has a strategy for playing the game G which guarantees a set of outcomes all of which satisfy ϕ. Here, ϕ may express winning, or any property of states. Next, games are associated with modal models $\boldsymbol{M}$, where states may interpret game-external proposition letters. We set

$\boldsymbol{M}, s \models \{G, \boldsymbol{i}\}\ \phi$ iff there exists a power X for player $\boldsymbol{i}$ in G played from state s such that for all $x \in X : \boldsymbol{M}, x \models \phi$

To bring this in line with modal semantics in its better-known 'neighbourhood' version, one might use binary state-to-set *forcing relations* $\rho_G{}^{\boldsymbol{i}}\ s, Y$, and set

there exists a set X with $\rho_G{}^{\boldsymbol{i}}\ s, Y$ and $\forall x \in X\ \boldsymbol{M}, x \models \phi$

The main effect of this at the level of validities is the following.

Fact 9 Modal logic with the forcing interpretation satisfies all principles of the minimal modal logic *K* except for distribution of *{}* over disjunctions.

In particular, $\{G, \boldsymbol{i}\}\,\phi$ is *upward monotone*:

if $\models \phi \rightarrow \psi$, then $\models \{G, \boldsymbol{i}\}\,\phi \rightarrow \{G, \boldsymbol{i}\}\,\psi$

But distribution over disjunctions is *not valid*:

$\{G, \boldsymbol{i}\}\,\phi \vee \psi \rightarrow \{G, \boldsymbol{i}\}\phi \vee \{G, \boldsymbol{i}\}\psi$

This is precisely the point of forcing. Other players may have powers that keep us from determining results precisely. I may have a winning strategy, but it may still be up to *you* exactly *where* my victory is going to take place. For instance, in the game of Section 5.2, ***E*** can force *{2, 3}*, but neither *{2}* nor *{3}*. Two further axioms relate powers of different players: matching the earlier Consistency and Determinacy.

Finally, this modal language has a matching notion of *bisimulation* between game models ***M***, which leaves truth of all modal forcing formulas invariant.

Definition A *power bisimulation* between two game models ***M***, ***N*** is a binary relation *E* between game states satisfying the following two conditions, for all ***i***:

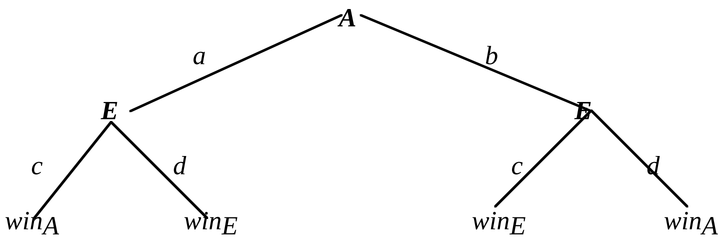
(1) if *xEy*, then *x*, *y* satisfy the same proposition letters

(2) if *xEy* and $\rho_{\boldsymbol{M}}^{\boldsymbol{i}}\,x,U$, then there exists a set *V* with $\rho_{\boldsymbol{N}}^{\boldsymbol{i}}\,y,V$ and $\forall v\in V\ \exists u\in U{:}\ uEv$; and vice versa.

We will us this notion later. The theory of game logic is much like that of standard modal logic. Cf. van Benthem 2003, Pauly 2001 for further details and results.

6 Extensive games and modal action logics

6.1 Extensive games as modal process models Moving beyond players' global powers, extensive games have a finer level of states, turns, and individual moves. Such games are like ordinary models for a standard modal language with moves as atomic actions, and some special predicates, like ***end*** for being an endpoint, or $\boldsymbol{turn}_i$ for turns of player ***i***. Consider again a simple game as in Section 5.2:



The assertion that ***E*** has a winning strategy can be expressed in detail here by means of the following modal formula true in the root, which records intermediate moves:

$$[a]\langle d\rangle win_E \wedge [b]\langle c\rangle win_E$$

Alternatively, these models are the labelled transition systems of computer science, with a state space for computations by several interactive processors. A systematic study of extensive games as process models for modal languages is made in van Benthem 2002. Here we just remark that the usual process of 'solving' a game by means of a Zermelo-style colouring algorithm (cf. the discussion of determinacy in Section 5.1) really amounts to stage-wise computing a smallest (or greatest) fixed-point definition for ***E***'s winning positions. E.g., to define the above forcing modality $\{G, \boldsymbol{E}\}\phi$ in more detail, one can use the following modal recursion:

$$\{\boldsymbol{E}\}\phi \leftrightarrow ((\boldsymbol{end} \,\&\, \phi) \vee (\boldsymbol{turn}_E \,\&\, \bigvee_a \langle a\rangle\{\boldsymbol{E}\}\phi) \vee (\boldsymbol{turn}_A \,\&\, \&_a\, [a]\{\boldsymbol{E}\}\phi))$$

In terms of the *modal μ–calculus*, this says (using greatest fixed-points) that

$$\{\boldsymbol{E}\}\phi \leftrightarrow \nu p\bullet ((\boldsymbol{end} \,\&\, \phi) \vee (\boldsymbol{turn}_E \,\&\, \bigvee_a \langle a\rangle p) \vee (\boldsymbol{turn}_A \,\&\, \&_a\, [a]p))$$

This means that all games can be solved by model checking some assertion in an appropriate game logic over them. Moreover, basic game-theoretic arguments, like the proof of Zermelo's Theorem or the Gale-Stewart Theorem can be formalized in modal fixed-point logics. Thus, one particular logic game from our list in Section 2, viz. model checking, may help us understand games in general!

6.2 Dynamic logic as strategy calculus Modal languages also have another use germane to games: they can define players' *strategies* explicitly. For, in extensive game trees, strategies for player ***i*** are nothing but *binary relations* which are functions on the turns for ***i***, while including all possible moves at other players' turns. Thus, they are of the same type as the above actions *a*, *b*. More generally, think of a *dynamic logic* with action expressions that can be formed from atomic ones using

> choice $\cup$, sequential composition ***;***, relational converse$^{\cup}$, finite iteration *, and the usual test program $(\phi)?$ for propositions ϕ.

For instance, then, the winning strategy for ***E*** in the above game may be defined as follows (with *T* for an assertion which is always true):

$$(\langle a^{\cup}\rangle T)?\ ;\ d)\cup(\langle b^{\cup}\rangle T)?\ ;\ c)$$

This means that propositional dynamic logic (Harel, Kozen & Tiuryn 2000) can be used as a general *calculus of strategies*, of the sort mentioned in Section 3.1. Thus, one particular modal game logic can help us understand logic games in general.

7 Game constructions

The preceding two sections were about game-internal properties, and ways of expressing these in modal languages. But we can also take an external view of games, more like in Process Algebra or Category Theory. For a start, games do not occur by themselves, they live in families. A natural general theme are natural game-forming constructions. Logic games provide many instances of this.

7.1 Logical game operations For a start, the evaluation games of Section 2.4 provide the following game-theoretical take on the basic logical operations:

(a) *conjunction and disjunction are choices* for players:

$G \cap H$ is ***A***'s choice, $G \cup H$ is ***E***'s choice of G or H

(b) *negation is a role switch*, leading to the dual game G^d

with all the turns and win markings reversed in *G*.

But clearly, choice and dual are completely general operations forming new games out of old. Here is another such operation which operates inside evaluation games. Consider the rule for an existentially quantified formula $\exists x \psi(x)$:

E must pick an object *d* in ***M***, and play then continues with $\psi(d)$

Properly understood, the existential quantifier $\exists x$ does not serve as a game operation here: it clearly denotes an independent atomic game of 'object picking' by Verifier. (The same point occurs in Abramsky 2006**.**) The general game operation in this clause hides behind the phrase "continues", which signals

(c) *sequential composition of games G ; H*

These are just a few of the natural operations that form new games out of old. To arrive at further ones, consider the intuitive idea of 'conjunction' of games. So far we have two candidates. *Boolean conjunction* $\cap$ forces a choice right at the start, with the game not chosen never accessed. *Sequential composition ;* may lead to play of both games, if the first is ever completed. But now consider the plight of academics playing games "family" and "career". Most of us compromise via a new operation:

(d) *parallel composition of games* $G \,||\, H$

This means playing a stretch in one game, then switching to the other, and so on – running around and trying to do the best in both. Logic games also provide examples of such interleaving (see Section 9.6). But even this is just one plausible parallel game construction: we might also proceed simultaneously in both games, and so on. Indeed, no complete operational repertoire is known either in logic or game theory.

7.2 Game algebra of sequential operations Game operations suggest game algebra: a calculus of equivalent game expressions. E.g., intuitively, the above choices for the two players are related by a De Morgan duality under role switch:

$$G \cap H = (G^d \cup H^d)^d$$

Or, typically, like composition of binary relations, game composition is associative. Another intuitive validity is left-distribution for composition over choice

$$(G \cup H) ; K = (G; K) \cup (H; K)$$

By contrast, the right-distribution law is not valid:

$$G ; (H \cup K) = (G; H) \cup (G; K)$$

E's choice on the left-, but not on the right-hand side may depend on the outcome of first playing game *G*. Another intuitively valid principle concerns role switch:

$$(G;H)^d = G^d ; H^d$$

These intuitions may be made precise using the notions of Section 5 (van Benthem 2003). The game equivalence that fits best with first-order validity looks at players' powers for determining outcomes. One can compute these inductively for complex games using choice, dual, and composition. We define complex forcing relations

$\rho^{\boldsymbol{i}}{}_G\, x, Y$ player ***i*** has a strategy in game *G* which makes sure that *G* ends in a state in *Y* when started from state *x*

Fact 10 The following equivalences hold for forcing relations:

$\rho^{\boldsymbol{E}}{}_{G \cup G'}\, x, Y$ iff $\rho^{\boldsymbol{E}}{}_G\, x, Y$ or $\rho^{\boldsymbol{E}}{}_{G'}\, x, Y$

$\rho^{\boldsymbol{A}}{}_{G \cup G'}\, x, Y$ iff $\exists Z, Z': \rho^{\boldsymbol{A}}{}_G\, x, Z$ and $\rho^{\boldsymbol{A}}{}_{G'}\, x, Z'$ and $Y = Z \cup Z'$

$\rho^{\boldsymbol{E}}{}_{G^{\boldsymbol{d}}}\, x, Y$ iff $\rho^{\boldsymbol{A}}{}_G\, x, Y$

$\rho^{\boldsymbol{A}}{}_{G^{\boldsymbol{d}}}\, x, Y$ iff $\rho^{\boldsymbol{E}}{}_G\, x, Y$

$\rho^{\boldsymbol{E}}{}_{G;G'}\, x, Y$ iff $\exists Z\colon \rho^{\boldsymbol{E}}{}_{G}\, x, Z$ and $\forall z{\in}Z\ \rho^{\boldsymbol{E}}{}_{G'}\, z, Y$

$\rho^{\boldsymbol{A}}{}_{G;G'}\, x, Y$ iff $\exists Z\colon \rho^{\boldsymbol{A}}{}_{G}\, x, Z$ and $\forall z{\in}Z\ \rho^{\boldsymbol{A}}{}_{G'}\, z, Y$

Using superset closure of powers, the second clause simplifies to

$\rho^{\boldsymbol{A}}{}_{G\cup G'}\, x, Y$ iff $\rho^{\boldsymbol{A}}{}_{G}\, x,Y$ and $\rho^{\boldsymbol{A}}{}_{G'}\, x, Y$

Remark If we also assume *determinacy*, in the earlier sense that for each set Y, either ***E*** can force Y, or ***A*** can force $W–Y$, then we just need to define forcing relations for player ***E***, because $\rho^{\boldsymbol{E}}{}_{G^{\boldsymbol{d}}}\, x, Y$ iff not $\rho^{\boldsymbol{E}}{}_{G}\, x, W–Y$. All powers for player ***A*** then follow by observing that $\rho^{\boldsymbol{A}}{}_{G^{\boldsymbol{d}}}\, x, Y$ iff not $\rho^{\boldsymbol{E}}{}_{G}\, x, W–Y$.

Now, take an algebraic language of game expressions starting with variables, and operations $\cup$, ^{d} , **;**. In addition take ι for the *idle game*, staying at the same state.

Definition Two game expressions G, H are *equivalent* (written $G = H$) if they have the same power relations for their players in all game models. We also write $G \leq H$ in case of a similar valid *inclusion* between the respective powers.

7.3 Excursion: a complete system Algebraic game validity (van Benthem 1999) validates the preceding observations. There is a simple complete equational system for this algebra of the game-theoretic analogues of the usual first-order operations (Goranko 2000). We display it here, to show a surprising fact behind our approach. Just underneath standard first-order logic, there lies a systematic game logic!

Theorem 11 *Basic Game Algebra* consists of the following principles:

(1)	the laws of *De Morgan algebra* for choice and dual			
(2)	$G\ ;\ (H\ ;K)$	=	$(G\ ;\ H)\ ;\ K$	*associativity*
	$(G \cup H)\ ;\ K$	=	$(G\ ;\ K) \cup (H\ ;\ K)$	*left-distribution*
	$(G\ ;\ H)^{d}$	=	$G^{d}\ ;\ H^{d}$	*dualization*
(3)	$G \leq H\ \rightarrow\ K\ ;\ G \leq K\ ;\ H$			*right-monotonicity*
(4)	$G\ ;\ \iota\ =\ G\ =\ \iota\ ;\ G$			

De Morgan algebra is essentially Boolean Algebra minus the special laws for *0, 1*. For our discussion in Section 9, we note that basic game algebra is *decidable*.

7.4 Dynamic game logic Basic game algebra can be embedded into the decidable *dynamic game logic* which extends the earlier modal forcing language with modalities $\{G, \boldsymbol{i}\}\phi$ with complex game terms G. Typical for these systems is the interplay of two ingredients in one language: a dynamic component with expressions G for games, and a static component with propositions ϕ about states of play. This is like dynamic logics in computer science which manipulate program expressions and propositions about computational states together. For more on dynamic game logic, cf. Parikh 1985, Pauly 2001, Pauly & van der Hoek 2006, van Benthem 1999–2002.

7.5 Logics of parallel game operations Parallel game constructions have been studied extensively in game semantics for *linear logic* whose tensor product involves interleaved games where Adam can switch between games. Under this interpretation, linear logic is a complete axiomatization of several central sequential and parallel game constructions. Cf. Blass 1992, Abramsky 1996, Girard 1997 for details.

8 Finite versus infinite games

8.1 The importance of infinite runs The emphasis so far has been on finite games and their outcome states. But several logic games in Section 2 support *infinite runs*, witness model construction, model comparison, and even model checking for first-order languages with fixed-point operators. The same move was behind the shift from Zermelo's Theorem to the Gale-Stewart Theorem, where players produce infinite runs, marked as winning or losing. And game theory also has more than just finite matrices or tree pictures. Infinite games model situations with ongoing behaviour, such as iterated Prisoner's Dilemma in studying the possible emergence of social cooperation. Such ongoing behaviour is just as important as finite termination.

A good analogy comes from computer science, using our process analogy for games. *Terminating programs* are meant to find some value, or finish some task. But there are also crucial non-terminating programs like *operating systems* which ensure the proper functioning of some device: the longer the better. (Such infinite computations are at centre stage in modern *co-algebra*, cf. Venema 2006.) Likewise, in linguistics, language games for conversational tasks should terminate. But there is also the Great Game of Language – with discourse as the 'operating system of cognition'. This should keep functioning forever. Both kinds of game, finite and infinite, make sense.

8.2 Extending the game logic perspective Infinite games can still be studied by the logical techniques of Sections 3–7. Outcomes are now the runs themselves. Over these, both outcome and action levels still make sense. Here are two illustrations.

First, consider *computation of powers*. Recall the interval selection game of Section 5.1, showing that non-determined games exist. Further interesting information can be extracted by looking into players' powers. We can then prove the general fact that

Fact 12 Identifying infinite runs that are equal up to finite initial segments, both players have *the same powers* in the interval selection game.

In particular, this perfect information game is about the simplest infinite realization of the following non-determined power specification from Section 5.2, viz.

powers of ***A****:* *{1, 2}*, powers of ***E****:* *{1, 2}*

For a finite game realization with imperfect information, see Section 10.

Our second example is *temporal logic of games*. Infinite games need more expressive game logics, in order to formalize well-known game-theoretic arguments. E.g., the proof of the Gale-Stewart Theorem involves the following result true for all games:

Weak Determinacy

Either player ***E*** has a winning strategy, or player ***A*** has a strategy which forces infinite branches on which player ***E*** never has a winning strategy.

The usual proof for the Gale-Stewart Theorem then runs as follows. Given that ***E***'s winning set is open, this Weak Determinacy implies that the branch forced by player ***A*** is a loss for ***E*** – so ***A*** has a winning strategy.

Fact 13 Weak Determinacy can be defined in a *branching temporal language*, evaluating formulas at pairs <*h*, *t*> of a current branch *h* and point *t* on it:

$\boldsymbol{M}, h, t \models \{G, \boldsymbol{E}\}\, \phi \vee \{G, \boldsymbol{A}\}\, \mathcal{A}\, \neg\{G, \boldsymbol{E}\}\, \phi$

Here, $\{G, \boldsymbol{i}\}\phi$ is a *modal-temporal forcing modality* extending the a-temporal one from of Section 5.3. It says at point *t* that player ***i*** has a strategy ensuring that only runs result with the current history *h* up to *t* as an initial segment, which satisfy the temporal logic formula ϕ. Moreover, temporal logic comes in explicitly once more through the use of the standard operator $\mathcal{A}$ ('always') in the right-hand disjunct. This says that a statement is *always true on the current branch*.

This temporal logic seems very powerful. Most reasonable winning conventions can be given in this format – such as the earlier *safety* and *liveness* properties, as well as winning conventions of infinite logic games. Also, like dynamic game logic (Section 7.4), temporal game logic involves a merge of internal and external game languages.

Infinite games are complex structures. In particular, they have huge unwieldy spaces of strategies – as high-lighted in the 'folk theorems' of game theory on a plethora of equilibria. Another general logical issue then is *finitization* (cf. Section 9). To which extent can we know infinite strategies in a full infinite game through their finite approximations? This may be a lost cause in general, but things look brighter if games have finite branching width, and we can use principles like König's Lemma.

9 From game logics to logic games

Having discussed game logics for a while, let us now return to logic games proper, and see what additional light is cast on these by Sections 6, 7, 8. We will see how all of the earlier themes make sense, and also suggest new results and questions.

9.1 Questioning game equivalence For a start, take the issue of 'equivalence' between logic games. The literature is full of statements to the effect that one logic game is 'equivalent' to another. E.g., Hodges 1999 says all games are equivalent to full infinite game trees, as we can disregard all undesired runs by calling them losses for ***E***. But this presupposes just one notion of game equivalence, and one biased in favour of just one player. Other authors 'reformulate' logic games without stating in what sense the new versions are equivalent. E.g., Barwise & van Benthem 1999 define infinite model comparison games inverting the schedule of Section 2.6:

> The game starts with one finite partial isomorphism between two models.
> Each round lets Duplicator ***D*** choose some family F of partial isomorphisms,
> followed by a selection by Spoiler ***S*** of one f in F. In the next round, ***D*** must
> select a set F^+ again, ***S*** then chooses a partial isomorphism in F^+ again,
> and so on. The back-and-forth property to be maintained by ***D*** is:
> *For every object a in one model, there exists an object b in the other*
> *model such that* $f \cup \{(a, b)\} \in F^+$ *– and likewise in the other direction.*

Fact 14 The inverted model comparison game is equivalent to standard Ehrenfeucht–Fraïssé games at the level of players' *powers*.

The trick is like the distributive law of Section 4.1, inverting scopes of logical operators. In each round, ***D*** offers ***S*** a panorama of all choices he could make, plus her own responses to them. ***S*** then selects his own move plus ***D***'s pre-packaged response – thereby setting the new stage. In human terms, ***D*** behaves like a colleague of mine, who tries to speed up department meetings by saying: "Now you're going to say *A*, and I will say *B* – or, you're going to say *C*, and then I will say *D* – etc."

More generally, in terms of Section 5 versus Section 6, logic games seem biased toward outcomes and powers, rather than the fine-structure of actions and turns. Most intuitive equivalences between such games can be justified in this manner.

9.2 The outcome perspective in logic games Our insights about general game logic have interesting implications in specific cases. Consider our running example of evaluation games for first-order logic. We give three concrete illustrations.

Richer denotations Any game ***game****(****M****, s, ϕ)* assigns a much more structured denotation to a formula ϕ than just a truth value, viz. the complete power structure of the two players. We can think of these as their forcing relations $\rho^{i}{}_{G}\, x, Y$ computed over the model ***M*** in the sense of Section 7.2. This suggest that there are several natural levels to assigning meaning, even for standard logical languages.

Power bisimulation At the level of powers, the right notion of equivalence between two models ***M****,* ***N*** is the power bisimulation of Section 5.3. Translating this back into standard first-order terms yields a variant of standard *potential isomorphism* (Section 2.6). This time, states to be related are not tuples of objects from the models, but variable assignments over them. The bisimulation is then a family of links satisfying obvious back-and-forth conditions for shifting values for variables. Well-understood, this also seems the more natural notion for first-order logic in general.

Representation of general games Perhaps most strikingly, evaluation games seem adequate for games in general!

Fact 15 Any extensive game is outcome-equivalent to one where players evaluate an associated game-logical assertion.

Proof We just give as an illustration. The game in section 6.1 is obviously outcome-equivalent to an evaluation game for its associated modal formula $[a]\langle d\rangle win_E \wedge [b]\langle c\rangle win_E$. Modulo outcome equivalence, we can re-arrange any finite extensive game to one with a uniform alternating schedule, while making all runs of equal length. This allows us to write up an associated modal formula in iterated $[]\langle\rangle$ form, whose evaluation game proceeds like the original game. ♣

Here is a more technical representation result showing how logic games are complete for game logics in one more sense (van Benthem 2003):

Theorem 16 The basic algebra of sequential operations on arbitrary games is precisely the game algebra of first-order or modal evaluation games.

Thus, any non-valid principle of basic game algebra (cf. Section 7.3) already has a counter-example in first-order evaluation games. More delicate issues arise when we demand *uniqueness* of outcomes (cf. Section 5.2): this causes no loss of generality in first-order evaluation games, but it does in modal ones (van Benthem 2003).

9.3 ***Fine-structure: the action level in logic games*** Despite the noted power bias, it also makes sense to look at the less well-studied fine-structure of logic games at their action level. Evidently, in this setting, far fewer games will be identified.

Finer levels of denotation Consider again first-order evaluation games. We now get much finer notions of denotation, leading to a subset of game equivalences, viz. those which leave the move and turn structure intact as far as ordinary modal logic cares about them. Some axioms of basic game algebra survive this: De Morgan laws, or left-distribution of composition over choice. Other axioms fail, as they merely preserved powers – e.g., the distribution law of Section 4.1 reversing players' turns.

Open question Determine the complete basic game algebra for modal equivalence.

This analysis suggests that there are many natural levels of equivalence for first-order formulas. And this again ties up neatly with a persistent philosophical tradition of looking for various levels of identifying 'propositions' (cf. Lewis 1972).

Strategy calculus in dynamic logic Individual actions also emerge in players' strategies in logic games (cf. Section 3). These strategies unified such diverse notions as formulas, 'reasons', proofs, models, or semantic analogies. Underlying all of these is the dynamic logic of Section 6.2. This gives a fresh look at known notions. E.g., in first-order evaluation games, the item closest to strategies are Skolem functions. Dynamic logic suggests a calculus of *definable Skolem functions*, taken as relations (a natural generalization). Its major operations are choice, composition, and iteration of binary relations, allowing, amongst others, the standard sequential program constructions *IF P THEN* π *ELSE* π' and *WHILE P DO* π (cf. van Benthem 2002).

Such an explicit format for analyzing strategies also makes sense in dynamic game logic (Section 7.4). That calculus was $\exists$–sick in the sense of Section 4.2, since it just says that a strategy exists without naming it. But we can remedy this with a modality

$$\{G, \mathbf{i}, \sigma\}\phi$$

stating that, *in game G, strategy* σ *for player* **i** *achieves a set of outcomes satisfying proposition* ϕ. Dynamic game logic in this guise still needs to be developed.

9.4 Digression: strategy calculus in type-theoretic format Formats other than dynamic logic may be attractive, too. Type theories (cf. Barendregt 1992) manipulate statements of the form $\sigma : G$ interpreted as 'σ is a proof of assertion G', or 'σ is an object having property G'. Strategy calculi might then manipulate statements

$\sigma : G$ $\quad\quad$ σ is a winning strategy for player ***E*** in game G

Such interpretations are given for linear logic games (cf. Section 7.5) in Abramsky & Jagadeesan 1994. Here is a simple example. Consider this sequent derivation for a propositional validity, whose steps are well-known valid inference rules:

$$A \Rightarrow A \qquad\qquad B \Rightarrow B$$
$$A, B \Rightarrow A \qquad\qquad A, B \Rightarrow B \qquad\qquad C \Rightarrow C$$
$$A, B \Rightarrow A \wedge B \qquad\qquad A, C \Rightarrow C$$
$$A, B \Rightarrow (A \wedge B) \vee C \qquad\qquad A, C \Rightarrow (A \wedge B) \vee C$$
$$A, B \vee C \Rightarrow (A \wedge B) \vee C$$
$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee C$$

We want to make the strategy calculus behind this derivation explicit. In particular, this requires the identification of strategy combinations supporting the proof steps. Here is a concrete format of analysis, written with some ad-hoc notation:

$$x{:}A \Rightarrow x{:}A \qquad\qquad y{:}B \Rightarrow y{:}B$$
$$x{:}A,\ y{:}B \Rightarrow x{:}A \qquad x{:}A,\ y{:}B \Rightarrow y{:}B \qquad\qquad z{:}X \Rightarrow z{:}C$$
$$x{:}A,\ y{:}B \Rightarrow (x, y){:}A \cap B \qquad\qquad x{:}A, z{:}C \Rightarrow z{:}C$$
$$x{:}A,\ y{:}B \Rightarrow \langle L, (x, y)\rangle{:}(A \cap B) \cup C \qquad\qquad x{:}A, z{:}C \Rightarrow \langle R, z\rangle{:}(A \cap B) \cup C$$
$$x{:}A,\ u{:}(B \cup C) \Rightarrow \textbf{IF } head(u)=L \textbf{ THEN } \langle x, tail(u)\rangle \textbf{ ELSE } tail(u){:}\ (A \cap B) \cup C$$
$$v{:}\ A \cap (B \cup C) \Rightarrow \textbf{IF } head((v)_2)=L \textbf{ THEN } \langle (v)_1, tail((v)_2)\rangle \textbf{ ELSE } tail((v)_2){:}\ (A \cap B) \cup C$$

This derivation composes strategies in complex games from those in sub-games. The operations for this task are completely general, not depending on proof theory:

storing strategies for a player who is not to move $\quad\quad$ $\langle\ ,\ \rangle$

using a strategy from a list $\quad\quad$ $(\ )_i$

computing the first recommendation of a strategy	*head()*
as well as the remaining strategy	*tail()*
making a choice dependent on some information	*IF THEN ELSE*

As strategies encode very different logical objects: proofs, models, analogies, etc., the above derivation can stand for quite different things. It is a recipe for constructing proofs, and the operations then encode what goes on as logical operations get added. But it also describes how any winning strategy for Verifier in an evaluation game for $A\wedge(B\vee C)$ can be turned into a winning strategy in $(A\wedge B)\vee C$. Not all logic games support operations of choice, though – and hence the above recipe may not make much sense (yet) for games of model construction, or model comparison.

9.5 Operations on logic games Logic games are not isolated activities, they can be combined (cf. Section 2.7). And then, they support not just sequential game operations, but also parallel ones beyond those found in the computational literature. Nice examples are the Wadge Game in descriptive set theory (Löwe 2002) and the interleaved fixed-point games found in the proof of the Stage Comparison Theorem of Moschovakis 1974. The operational structure behind logic games should be a good testing ground for general game algebra. Here, we only offer one illustration showing the interest of such matters (van Benthem 1999).

Relating evaluation and comparison games In this excursion, we relate two major logic games, viz. Ehrenfeucht games of model comparison with Hintikka games of evaluation. These have the same 'back and forth' idea in their object-picking moves. We make this precise in terms of game operations, proving the informal equation

$$\boldsymbol{E = H^2}$$

The 'squaring' operation here is interleaving games, and we can even correlate strategies in the two games directly. Recall the Adequacy Theorem for finite-depth

Ehrenfeucht-Fraïssé games of Section 2.6, $\exists$–cured in Section 4.2. This suggests an explicit link between strategies across comparison and evaluation games for models ***M***, ***N***. First-order formulas ϕ of quantifier depth *k* between ***M***, ***N*** drove winning strategies for Spoiler in the *k*-round comparison game between ***M***, ***N***. But we can do away with this intermediary! Let $\boldsymbol{M} \models \phi$, $\boldsymbol{N} \models \neg\phi$. This induces a winning strategy for Verifier in an evaluation game ***game(φ, M)*** *plus* one for Falsifier in ***game(φ, N)***:

Theorem 17 There exists an effective correspondence between

(a) Winning strategies for Spoiler in the *k*-round comparison game

(b) Pairs of winning strategies for Verifier and Falsifier in some *k*-round evaluation game, played in opposite models.

Proof Without loss of generality, formulas can be assumed to be constructed from atoms with negations, disjunctions, and existential quantifiers only. *From (b) to (a).* Let an *H-pair of depth k* consist of a formula ϕ of quantifier depth *k* plus a winning strategy σ for ***V*** in the ϕ-game in one of the models, and a winning strategy τ for ***F*** in the ϕ-game in the other model. We sketch how to *merge* σ, τ. Spoiler looks at the two evaluation games. Suppose ***V*** wins ϕ in ***M***, and ***F*** wins ϕ in ***N***. If ϕ is a negation $\neg\psi$, Spoiler switches to the obvious strategies for ***F*** and ***V*** w.r.t. ψ. (Note that this is internal computation: the opponent in the comparison game does not see any action yet.) If ϕ is a disjunction $\psi \vee \xi$, Spoiler uses his ***V***-strategy in the one model to choose a disjunct. His ***F***-strategy in the other model will also win against that disjunct. Proceeding in this way, the formula is broken down until an existential subformula $\exists x\psi$ is reached. Spoiler then uses his ***V***-strategy σ in the model where it lives, say ***M***, to pick a witness *d*. This model ***M*** and object *d* are his opening move in the first round of the Ehrenfeucht game. Next, what remains for Spoiler is still a winning strategy σ' for ψ in ***M*** after this first move. Now, let Duplicator respond with any object *e* in the other model ***N***. This choice can also be seen as a move by Verifier in the evaluation game for $\exists x\psi$ in ***N***. Now we know that Falsifier still has a winning strategy τ for ψ in ***N*** after this first move. So, by induction, we still have an *H*-pair of depth *k-1*, which can be merged into a follow-up winning strategy for Spoiler in the *(k-1)*-round comparison game between ***M*** and ***N***. The total effect is a *k*-round ***S***-strategy. This argument yields an algorithm for Spoiler's computation.

From (a) to (b). This direction seems harder, as we have to 'decompose' one object: Spoiler's winning strategy, into two separate ones that must form a suitable *H*-pair. One proof of this follows our earlier construction of a difference formula of depth *k* from an ***S***-strategy (Section 4.2). This formula induces two evaluation strategies effectively. Let us describe 'splitting' of a comparison strategy directly. Consider any winning strategy for ***S*** in the *k*-round comparison game between two models ***M***, ***N***. In the first move, ***S*** chooses, say, model ***M*** and object *d*. Our desired formula will then start with an existential quantifier, and ***V*** has the winning strategy in ***M***. Let Duplicator now make any response *e* in ***N***. We know that Spoiler still has a *(k-1)*-round winning strategy in the two expanded models *(**M**, d)* , *(**N**, e*). Inductively, we can find *H*-pairs of depth *k-1* for each choice *e* that Duplicator makes. Moreover, by the earlier Finiteness Lemma, only finitely many logically non-equivalent formulas can be involved in these pairs. Then, one over-all existential quantification over a suitable conjunction of formulas of depth *k-1* defines our desired *H*-pair of depth *k*. In particular, if it is ***V*** who has the winning strategy of a relevant *H*-pair ϕ in the model ***M***, put itself in the conjunction; otherwise, put its negation. ♣

Operations like interleaving are just the beginning of a logic of *game architecture*.

9.6 Finite versus infinite Logic games can have finite or infinite depth, leading to issues of *finitization* (Section 8). Say, model construction games can go on forever. Critic's winning strategies made Builder lose on each run, by a finite stage.

Fact 18 Critic's winning strategies in the Construction game are finite objects.

Proof How can this be? The reason is that the game tree for the model construction game is *finitely branching*. Hence by *König's Lemma*, there is some finite level at which Critic has already blocked every construction attempt. This closed game tree is a finite object. These strategies are associated with finite objects, viz. *proofs*. ♣

Such a reduction may fail in infinite comparison games, where Spoiler may be able to win against Duplicator, blocking each run at some finite stage, without there being a finite object encoding this. There must be a formula of *infinitary* first-order logic witnessing the relevant difference, but there need not be a standard first-order one.

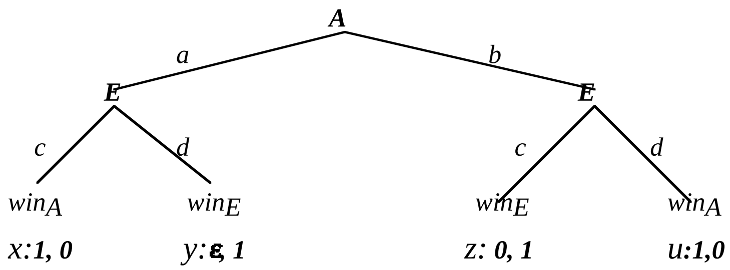
Finitization can be very useful. Hirsch and Hodkinson 2002 show the following for their game of Section 2.7, using finite branching for Builder's (though not Critic's!) moves: *Builder can win the infinite game if she can win all finite cut-offs*, and her winning strategy is easy to piece together from these. Then representability becomes equivalent to a set of first-order assertions expressing Builder's being able to win all finite cut-offs, which leads to a perspicuous axiomatization. Incidentally, in this game, by the same reasoning, Critic's winning strategy, if available, must be a *finite object* again. It is a sort of proof that the given algebra is not representable.

10 From game theory to logic games

Logic games are special, in that players' behaviour is much more constrained than in ordinary game theory. In real games, players have *preferences* between outcomes beyond winning or losing, they operate under *uncertainties* about what really happens during a move (think of card games), and there can be more than two players, leading to *coalitions*! Importing these concerns into logic games makes them more realistic, even though there is hardly any theory of the resulting activities.

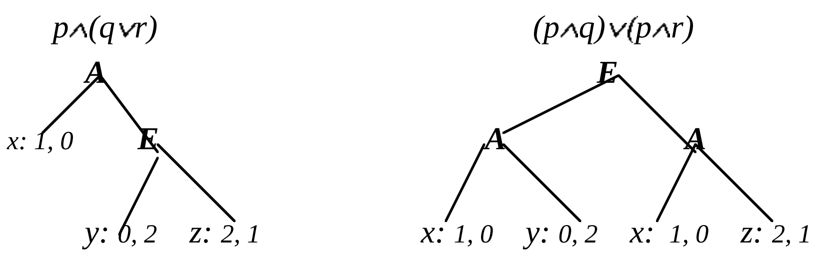
We will just skim a few issues here – but there are natural motivations for enriching logic games, even from the standpoint of reasoning and other logical core business.

10.1 Preferences Preferences in logic games allows for finer behaviour. Consider the game of Section 6.1, where ***A*** now has a slight preference for one site of defeat over the other (we write values for ***A***, ***E*** in that order underneath the outcome nodes):

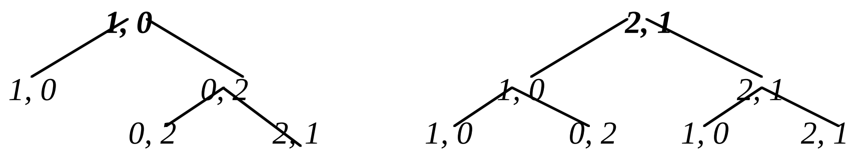


What will happen now? ***A*** can assume that, whatever he does, ***E*** will go for her most preferred outcome *y* or *z*. So, as he himself prefers *y* to *z*, he will choose “left”, forcing ***E*** to end up in *y*. Thus the game ends in the run ‘left-right’ with outcome *y*. Indeed, this pair of strategies σ, τ is in *Nash Equilibrium*: neither player can gain by deviating from his strategy, assuming the other strategy in the pair not change.

This story assumes a notion of 'rationality', as in the solution method of *Backward Induction* computing game values via a Maximin rule. Here is how this works on the evaluation games of Section 4.1, with a preference structure as indicated below:



We display all pairs (***A***–*value*, ***E***–*value)* computed bottom-up:



These trees correspond to different outcomes for the joint behaviour of the players. We predict outcome *x* on the left, but *z* on the right.

There are many new issues of general game logic for games with preferences. In general, these require combinations of modal logics for moves with *preference logics* (cf. van Benthem, van Otterloo & Roy 2006, van Benthem, Girard & Roy 2007, van Benthem & Liu 2006). Here we just look at one issue which also affects logic games.

Game algebra with preferences Call two game expressions equivalent if their Nash equilibrium solutions are the same for every concrete realization including players' utilities. The preceding example showed that the basic game algebra of Section 7.3 no longer qualifies: propositional distribution fails. Vice versa, invalid equivalences may hold for special preference values. Assuming rationality, games *A* and ***A∪B*** are preference-equivalent whenever ***E*** prefers *A* to *B*. Of special interest are antagonistic zero-sum games, where ***A*** evaluates all outcomes opposite to ***E***, as in logic games of winning and losing. Then some standard logic remains. Assume that different outcomes correspond to different preferences – otherwise, we might just as well identify them for game-theoretic purposes. Then it is easy to check the following

Fact 19 With zero-sum preferences Boolean Absorption $A = A \cap (A \cup B)$ is valid, whereas with general preferences, it is not.

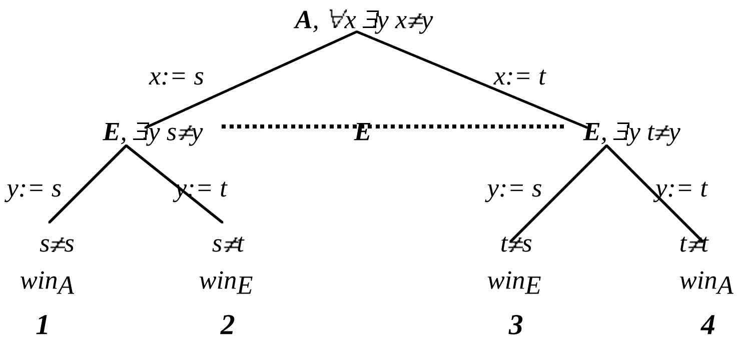
Open Problem Find the complete basic game algebra under Nash equilibrium equivalence in case of (a) arbitrary preferences, (2) zero-sum preferences.

Logic games with preferences Values and preferences also make of sense in logic itself, witness many-valued logics, or preference models for logics of belief and conditionals. Another concrete source refers to game dynamics. Let us introduce resource structure into logic games, and say that, other things being equal, *players prefer outcome nodes lying on a shorter path from the root*. This assumes that players want to get to a winning node as soon as possible, or if no such node exists, to a losing node with the least effort. Then we can make more definite predictions about the course of games, and adapt definitions of validity and truth accordingly. A final example are the more realistic logical dialogue games of van Benthem 2004 where arguments may lose 'force' from the moment they were first put on the table.

10.2 Solving games Making these connections does raise conceptual problems. In logic games, central notions are encoded in winning strategies, and in that case, the precise behaviour of the other player is unimportant: it will always be in Nash equilibrium with the winning strategy. But in game theory, a strategy models a type of behaviour vis-á-vis other players, and hence it is the *strategy profile for all players* that we are after. Nevertheless, multi-agent interactive behaviour seems crucial to logic, too. A proof is a long-term way of responding to objections, an isomorphism is a never-ending way of simulating one model in another, and sticking to the truth means emerging victorious no matter where your opponents in life try to push you. From a different perspective of resource-sensitivity, this also seems a major point of Girard 1997. For similar ideas in computer science, see Abramsky 2006. But then, we do seem to need a major change of perspective in logic. Based on preferences, game theory shows that Nash equilibria always exist for finite games if we are willing to admit *mixed strategies*, where pure actions are played with certain probabilities. What would be the logical point of such *probabilistic solutions*?

10.3 Imperfect information In standard logic game, at each stage, players know exactly where they are. But in real games, players may have imperfect information as to where they are in the game tree. E.g., in card games we do not know the complete distribution of the cards. Still players must move despite this partial ignorance.

Peculiarities of imperfect information games Games like this diverge from logic games in important respects (Hintikka & Sandu 1997). Consider the evaluation game for the first-order formula $\forall x\ \exists y\ x \neq y$ in Section 4.2. Now assume that Verifier is ignorant of the object chosen by Falsifier in his opening move. In game-theoretic notation, the new tree looks as follows, with a *dotted line* indicating ***E***'s uncertainty:



This game is quite different from its version with perfect information. In particular, if we allow only *uniform strategies* that can be played without resolving the uncertainty – as seems reasonable – ***E*** has only 2 of her original 4 strategies left in this game: 'left' and 'right'. Then *determinacy is lost*: neither player has a winning strategy!

Games with imperfect information like this still support game logics as in Sections 6, 7 above, at both power and action levels (cf. van Benthem 2001). For instance,

Fact 20 Player ***E***'s situation in the central nodes of the preceding game can be defined by the following formulas of an *epistemic-dynamic* action logic:

(a) $K_E(\langle y:=t\rangle win_E \vee \langle y:=s\rangle\, win_E)$

E knows that some move will make her win, picking either *s* or *t*

(b) $\neg K_E \langle y{:=}t\rangle win_E) \wedge \neg K_E \langle y{:=}s\rangle win_E)$

there is no particular move of which ***E*** knows that it will make her win.

This is the well–known *de re – de dicto* distinction from philosophical logic. For instance, I may know that the ideal partner for me is walking out right there in the street, without ever finding out which one of these people was that person.

On the other hand, we can also describe games like this at the global level of powers. E.g., with uniform strategies, players' powers in the above game are as follows:

Powers of ***A*** *{1, 2}, {3, 4}*, Powers of ***E*** *{1, 3}, {2, 4}*

The analysis of Section 5.2 can be extended to this situation. Families of powers satisfy Monotonicity and Consistency, though not Determinacy. And conversely, the former two conditions suffice for representability of given powers for two players as those realized in some game of imperfect information (van Benthem 2001).

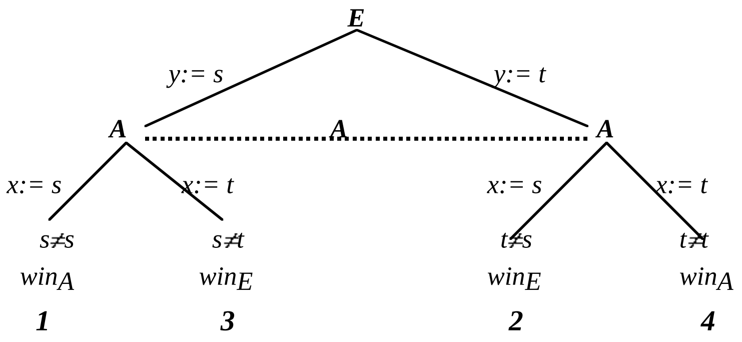
Logic games with imperfect information There is one exception to the above. Imperfect information has been added to logic games in the work of Hintikka & Sandu on *IF logic* (cf. Hintikka 2002 for a most recent version plus intended applications). In the slash notation of *IF* logic, the preceding game is written as

$\forall x\, \exists y/\mathbf{x}\; x{\neq}y$

expressing that the choice of the existential witness for *y* must not depend on the universal challenge for *x*. As in Section 4, a good test question for a notion of games like this is: when do two given *IF* formulas define the same games? For instance,

Is the game $\forall x\, \exists y/\mathbf{x}\, Rxy$ 'equivalent to' the game $\exists y\, \forall x\, Rxy$?

A popular answer is *YES*, as ***E*** has the same winning powers in both. But in games of imperfect information, the powers of one player do not automatically tell us all about the other. Hence ***A*** may not agree that these two games are the same. And indeed, the *Thompson transformations* of game theory (Osborne & Rubinstein 1994) say that the correct equivalent is rather this game, with switched scheduling:



In terms of *IF* logic, the real situation involves a much nicer quantifier exchange:

Fact 21 $\forall x\ \exists y/\mathbf{x}\ x{\neq}y$ is uniformly outcome-equivalent to the slash formula representing the above game, which has the form $\exists y\ \forall x/\mathbf{y}\ x{\neq}y$.

IF games have generated a lot of controversy (cf. van Benthem 2006 for an epistemic analysis). Even so, they show that introducing imperfect information into logic games is exciting and perhaps even useful. In particular, this may provide a normal form for validities in all imperfect information games, the way first-order evaluation games did for the general algebra of perfect information games (Section 9.2).

One might try similar moves with the other logic games mentioned in Section 2, provided motivations are found. One might speculate about proofs where participants have forgotten some things that has been said. Or, one can play model comparison games with a fixed number of *pebbles*, representing some finite memory (Immerman & Kozen 1987), where players will not be able to distinguish the same assignments of objects to these pebbles, even when they occur at different stages of the game.

Uncertainty about the future Finally, even in games of perfect information, where we know our position in the game tree at any time, we have 'forward ignorance' of the future course of play, since we need not know the strategy of the other player. Nash equilibrium makes some predictions, but in general, we are in deliberation about future actions and choices, based on beliefs about ourselves and others. There is a flourishing literature on this, invoking belief revision and counterfactual reasoning (cf. Stalnaker 1999, van Benthem 2007). The resulting game logics put ideas from philosophical logic on top of the mathematical logic of game structure. Such issues, too, make sense for studying logic games, but we forego them here.

10.4 More agents and coalitions Finally, many games involve more than two players, and hence the possibility of genuine *coalitions*: cf. Pauly 2001 for a study with tools from modal logic. Many agents are the reality in conversation and debate, and they also make sense in logic games when thinking about teams of players – viewing Adam and Eve rather as some sort of Bourbaki. This suggestion is in the air today among logicians. The 'team logic' of Väänänen 2006 is a serious contender.

11 Conclusions

This paper has shown how logic games are an interesting subclass of the totality of all games, raising new issues of game logic beyond standard game theory precisely since they are somewhat better delineated. On the other hand, the study of logic games might also benefit from importing ideas from general game theory. Either way, we think all this supports the idea of viewing logic as a study of the dynamics, rather than just statics, of statement, reasoning, and communication.

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