UPDATING UNCERTAIN EVIDENCE
Conditionalization and Complexity in Multi-Layer Belief Models

MSc Thesis (*Afstudeerscriptie*)

written by

Giuseppe Manes

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Date of the public defense:	**Members of the Thesis Committee:**
June 29, 2026	Dr. Balder Ten Cate (chair)
	Dr. Ronald de Haan (supervisor)
	Dr. Mina Young Pedersen (supervisor)
	Dr. Aybüke Özgün
	Dr. Daira Pinto Prieto

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Effectus cognitio a cognitione causæ dependet et eandem involvit.

Ethica, Ordine Geometrico Demostarta
Baruch de Spinoza

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Abstract

In (Pinto Prieto, 2024) it is shown how Dempster-Shafer theory can be combined with Topological models of evidence into a unified Multi-Layer Belief Model that efficiently handles the combination of uncertain, partial and mutually contradictory pieces of evidence. More precisely, the topological structure captures specific kinds of doxastic attitudes towards evidence, while a belief function reconstructed on top of this structure represents the uncertainty of the evidence. This model has been shown to achieve results analogous to both Dempster-Shafer theory and Topological models of evidence, while being strictly more powerful, as it subsumes the essential properties of both frameworks. However, as originally defined, the model does not include any updating procedure for *conditionalization*, i.e. a way for revising the model after learning new certain evidence. This is the problem we address in this work.

Both in Dempster-Shafer theory (Shafer, 1976) and in Topological models of evidence (Özgün, 2017) there exist different ways to update beliefs. Drawing on the intuitions behind those approaches, we define an updating procedure for the Multi-Layer Belief Model that captures conditionalization of evidence while preserving the model's expressivity and its ability to recover the foundational frameworks, now also at the dynamic level.

We approach the problem from two main dimensions. First epistemically, we investigate how the update respects the agent's original evidential standards and how it relates to established notions of conditional belief. Then computationally, we analyze the complexity of the update algorithms and provide a concrete implementation. Thus, the Multi-Layer Belief Model, extended with conditionalization, offers a unified framework for reasoning with uncertain, partial, and possibly inconsistent evidence under dynamic information.

More concretely, in Chapter 1 we first establish the basic concepts underlying the frameworks we consider and analyze how they model an agent's doxastic demands. Then, in Chapter 2, we define how to update each layer of the model upon learning a proposition $\Phi \subseteq X$. For the qualitative layer, we restricted its elements via a public announcement-like operation, and the frame of justification is recomputed on the latter (Section 2.2). At the level of the quantitative layer the update is performed using a novel merging function δ^t that combines the weights of evidence pieces sharing the same intersection with Φ (Section 2.3). Finally, the bridging layer recomputes the evidence allocation function and the final belief on the updated structure (Section 2.5). Furthermore, we show that the procedure satisfies four natural desiderata: restriction of the state space, renormalization of the mass function, preservation of the agent's doxastic standards, and iterability (Section 2.6). All the updating definitions have been encoded in explicit polynomial-time algorithms, for which we proved correctness, and we provide an implementation available in a public repository (Manes, 2026).

Finally, in Chapter 3 we relate our conditionalization to the conditioning methods of Dempster-Shafer theory and Topological models of evidence. Under suitable choices of the frame of justification and the evidence allocation function, our procedure recovers Dempster-Shafer conditioning via Dempster's rule of combination, both at the level of the evidence set and at the level of the final basic probability assignment (Section 3.1), and replicates the belief dynamics of public announcement and conditional belief in topological models (Section 3.2).

Contents

Introduction

In various symbolic reasoning tasks, we usually model an agent's beliefs using pieces of evidence, which they can process to form beliefs within the landscape of their own *epistemic horizon*. Moreover, we are not only interested in modeling static beliefs, but also in how these beliefs can change upon learning new evidence over time.

In this regard, consider this claim by Baruch de Spinoza from *Ethica, Ordine Geometrico Demonstrata*, Part 1, Axiom 3:

> Ex data causa determinata necessario sequitur effectus et contra si nulla detur determinata causa, impossibile est ut effectus sequatur.[1]

This sentence asserts that when a specific *causa determinata* is given, an *effectus* necessarily follows. This idea can be readily applied to the context of formal learning, where we receive new evidence, i.e. our *causa*, from a source which causes an *effectus* in the context of our epistemic horizon. What does it mean for a cause to be *determinata*? It means that the source is, in some sense, reliable and the evidence is certain.

If we consider such a case in the context of learning from evidence, it points to a specific type of inductive inference: receiving certain evidence that will, in a determinate way, shape what is admissible within our epistemic horizon. Consequently, it will *necessarily* determine what the *effects* of learning this new information are.

In this work we want to consider this specific type of learning from evidence. If we analyze the literature, this type of learning can take different names, depending on the tradition to which the symbolic reasoning method in question ascribes. If we look at probabilistic reasoning, this type of inference takes the name of *conditionalization* or simply *conditional belief*, as we can read in (Halpern, 2003; Titelbaum, 2022). In the field of epistemic logic, this dynamic is encoded in the concept of *public announcement*, as shown in (Baltag and Renne, 2016).

However, reasoning with evidence in a formal setting brings many problems when we want to allow our evidence to be as permissible as possible. For example, when we want to model reasoning in the context of real-world data, often our body of evidence is *inconsistent* and *partial*. To better treat this kind of evidence, different systems have been proposed: some quantitative like Dempster-Shafer theory or imprecise probabilities (Halpern, 2003; Shafer, 1976), and others qualitative in the context of dynamic epistemic logic, or more recently in topological semantics applied to epistemic logics (Baltag and Renne, 2016; Özgün, 2017).

A precise model called *Multi-Layer Belief Model*, which aims to combine these two aspects of evidential reasoning, has been proposed in (Pinto Prieto et al., 2023; Pinto Prieto, 2024). The latter integrates qualitative topological structures with quantitative Dempster-Shafer weights, allowing the representation and combination of evidence that may be inconsistent, partial, and originating from distinct sources. While it supports multiple agent definitions

1 Personal translation: "From a given determined cause an effect necessarily follows, and conversely if no determined cause is given, it is impossible that an effect follows."

and a flexible notion of justification, it provides no mechanism to condition the agent's epistemic horizon to a specific *causa determinata*. This means that it lacks an operation that, given certain evidence, necessarily determines the resulting belief state.

This thesis fills that gap by extending the Multi-Layer Belief Model itself with a full conditionalization procedure. The work is structured as follows. In Chapter 1, we define the theoretical foundations on which the Multi-Layer Belief Model itself is built, namely Topological Models of Evidence, as defined in (Özgün, 2017), and Dempster-Shafer Theory, as presented in (Shafer, 1976), and we present the Multi-Layer Belief Model itself. In Chapter 2, we introduce our conditionalization procedure, showing how to update each layer of the model upon learning a proposition and proving that the operation satisfies natural *desiderata*. In Chapter 3, we relate our proposal to other ways of determining beliefs on new evidence from other formal reasoning theories, demonstrating that under suitable conditions our procedure replicates Dempster-Shafer conditioning, the belief operator in public announcement models and the conditional belief operator from topological evidence models.

In line with the original motivation of the Multi-Layer Belief Model, as explained in (Pinto Prieto, 2024), which was designed to handle real-world data and therefore to be applicable in practice, we do not limit our investigation to the logical and probabilistic properties of the conditionalization procedure. We also examine it from a computational perspective. A belief model that aims to be useful for reasoning about actual evidence must account not only for the epistemic demands of the agent but also for the reliability and efficiency of the underlying computations. Accordingly, we define explicit algorithms for updating each layer of the model, prove their correctness with respect to the intended semantics, and analyze their computational complexity, showing that they run in polynomial time in the size of the input. Furthermore, we also implemented the algorithms and most important results presented in this thesis in Python in a public repository, available at (Manes, 2026). This computational analysis and practical implementation complement the theoretical development and ensures that the proposed conditionalization procedure is not only conceptually sound but also practically feasible.

1 Theoretical background

Consider the following sentence extracted from *Ethica, Ordine Geometrico Demonstrata*, Part 2, Proposition 40, Scholium 2 by Spinoza, where the author outlines the different types of *perceptions* and *knowledge*, among which:

> [...] ex singularibus nobis per sensus mutilate, confuse et sine ordine ad intellectum repræsentatis [...] et ideo tales perceptiones cognitionem ab experientia vaga vocare consuevi.[1]

What Spinoza describes as *cognitio ab experientia vaga*, i.e. *knowledge from vague experience*, represents those perceptions that are mutilated, confused, and lacking order, and captures precisely the kind of uncertain, fragmentary information that comes in our experience, which formal models of belief and evidence aim to handle systematically. In (Pinto Prieto et al., 2023; Pinto Prieto, 2024), we can find the definition of one such model, the so-called *Multi-Layer Belief Model* (MLBM), constructed from the combination of two very different frameworks: the *Topological Models of Evidence* (TME) defined in (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2017; Baltag, Bezhanishvili, Özgün, et al., 2018) and *Dempster-Shafer Theory* (DST), exposed in (Shafer, 1976; Halpern, 2003).

We begin with defining the basic notions that are shared across the different theoretical basis, and that allow us to work and define the combined model MLBM.

To understand the basic notions behind MLBM, we first need to understand what is the aim of this belief model. As said in (Pinto Prieto, 2024, pg.7), the goal of the work (and so also of the belief model designed in it) is to understand:

> the problem of combining non-ideal evidence, understanding non-ideal as *uncertain*, *partial* and *possibly mutually contradictory*. The ultimate objective of combining evidence is to draw robust conclusion from a body of evidence.

To approach this aim, as we will see, TME and DST offer independent but complementary tools, offering the possibility to study the problem from different perspectives. Following (Pinto Prieto, 2024), we will outline these similarities in the following points:

1. they both consider as a primitive object a set of possible states, that represent the space of all possible observations made by our agent,
2. they both represent the basic pieces of evidence as subsets of the set of possible states,
3. both theories provide basic tools for combining pieces of evidence, but in a different way:
    - DST aggregates evidence among three dimensions: two quantitative, i.e. number of pieces of evidence and the level of uncertainty associated with each piece of evidence, and one qualitative, i.e. consistency among different pieces, which is represented by non-empty intersections among them,

1 Personal translation: "[...] from singularities represented to us through the senses in a mutilated, confused, and disorderly manner relative to the intellect [...] and therefore these perceptions of cognition I call *knowledge from vague experience*."

- TME takes a pure qualitative approach, elaborating it in further details, resulting in a very strong notion of belief based on evidence.

Thus, in what follows we will define the basic notions of *set of possible states* and *piece of evidence*. These concepts represent the common ground on which the two frameworks build on their different conception of belief.

However, we adopt a deliberately broad understanding of the notions of *agent* and *source*, imposing as few restrictions as possible. An agent may be any ideal being capable of observing and gathering pieces of evidence. A source may be anything that provides the agent with information about some possible evidence within the landscape of possibilities represented by our set of possible states. These assumptions are needed in order to be able to make sense of what some objects in our theories represent epistemically. Moreover, for technical purposes, we always assume to have a unique ideal agent and to work with independent sources.

Therefore, consider the following definition:

Definition 1 (Set of possible states)**.** We call X the set of all possible states observed by our agent.[2]

Thus, the basic elements of our construction are always states, or equivalently points or worlds. Throughout this work, we will also use the term *space* as an alternative name to the set of possible states.

An important feature of these theories is that both are suited to deal with basic pieces of evidence and their dynamics (both for combination and update purposes). The notion of evidence, however, is treated with as few requirements as possible to ensure that we can model the widest range of situations. In order to define this notion we need firstly to define what is a proposition in our setting. However, this brings some problems regarding what a proposition means in our theoretical basis. For now and for most of this work, we will assume the naïve definition used in DST, where we do not have any kind of valuation function, but we will see that also in TME, we have a same concept of proposition but more refined. Therefore, we will define a proposition as follows:

Definition 2 (Proposition)**.** Let X be a set of possible states. Then we call any $P \in \wp(X)$ a proposition.

Consequently, a piece of evidence will be a particular proposition in our framework.

This conception of propositions, on the one hand, has epistemic significance, as it allows us to define a set of evidence over our set of states. On the other hand, it is convenient for formal reasons, since the logical connectives will assume, in both theories and in our combined model, the standard set-theoretical interpretation.

So far, we have discussed pieces of evidence and sets of evidence without formally defining them. This issue is widely debated in the philosophical literature, concerning what should be conceived as evidence and what it means for something to count as evidence. These debates reflect different treatments of the concept in its formalization. For a general overview of the concept for the philosophically inclined reader we refer to (Kelly, 2016). From our perspective, we will not engage in discussions that exceed the scope of this work. We will place as few restrictions as possible on what can be considered evidence, as our goal is to keep the framework as expressive as possible. Hence, we define a collection of evidence as follows:

Definition 3 (Set of evidence)**.** We call any $\mathcal{E} \subseteq \wp(X) \setminus \{\varnothing\}$ a set of evidence and any $E \in \mathcal{E}$ a piece of evidence.

2 Throughout this thesis, we will assume this set to be finite. We assume this to work on the same type of structure on which the MLBM framework, as defined in (Pinto Prieto et al., 2023; Pinto Prieto, 2024), is built. As said in (Pinto Prieto, 2024, pg. 22), it was assumed in the construction that "the set of possible states is *finite*, reflecting our ultimate motivation of applying these constructs to real-life scenarios." Even if in this thesis we are not focusing on the applicability of such models, we are going to be faithful in our work to the original formalism and also be able to build directly on top of the results achieved in (Pinto Prieto, 2024), thus being able to assume all the results without worrying about possible complications from trying to include the infinite case.

In (Benthem and Pacuit, 2011) an equivalent notion is defined with the further restriction of always considering the whole set a piece of evidence. This restriction is avoided both in TME and DST for different reasons:

- in TME, we make consideration on the whole topological space and having the whole space in our evidence is formally a redundant requirement,
- in DST, as we will see, the belief computed via our belief function on the whole space will always have belief 1.

Now we will briefly introduce both TME and DST highlighting the most important features to describe MLBM.

1.1 *Topological Models of Evidence*

Let's start from TME. Our main source for this framework is (Özgün, 2017). This type of models used topological properties to characterize the semantics of the basic modal logic. The first attempt for this kind of semantic approach traces back to (McKinsey, 1941; McKinsey and Tarski, 1944). However, in order to introduce how this semantics is defined, we will firstly recall some basic definitions in topology, we will use as our reference (Engelking, 1989; Dugundji, 1966).

First, we define what is a topology:

Definition 4 (Topological space)**.** A topological space is a pair (X, τ), where X is a set and $\tau \subseteq \wp(X)$ is the topology on X, such that

1. $\emptyset \in \tau$ and $X \in \tau$
2. for any $U, V \in \tau$ it holds that $U \cap V \in \tau$
3. for any $\mathcal{A} \subseteq \tau$, it holds that $\bigcup_{U \in \mathcal{A}} U \in \tau$

Now, we go on introducing other basic notions in topology:

Definition 5 (Open and closed set and neighborhoods)**.** Given a topological space (X, τ):

- we call any $U \in \tau$ an open set and its complement $\overline{U}$ is called a closed set.
- for any $x \in X$ and any $U \in \tau$ if $x \in U$, then we call U an open neighborhood of x.

Furthermore, we denote with $\overline{\tau}$ the family of closed sets in (X, τ), i.e. $\overline{\tau} = \{X \setminus U \mid U \in \tau\}$.

Definition 6 (Interior and Closure operator)**.** Let (X, τ) be a topological space and any $U \subseteq X$. Then we can define the interior and closure operators, respectively as:

$$\mathsf{int}\,(U) = \bigcup\{V \in \tau \mid V \subseteq U\} \quad \text{and} \quad \mathsf{cl}\,(U) = \bigcap\{V \in \overline{\tau} \mid U \subseteq V\}$$

We can also define particular sub-collections of the topology called base and subbase:

Definition 7 (Base of a topological space)**.** Given a topological space (X, τ), we call a family of set $\mathcal{B} \subseteq \tau$ a base of (X, τ) if every $U \in \tau$ such that $U \neq \emptyset$ is the union of a subfamily $\mathcal{A} \subseteq \mathcal{B}$.

Consider now the following proposition, as presented in (Dugundji, 1966, p. 65, Theorem 3.1):

Proposition 1. Given any family $\mathcal{E} \subseteq \wp(X)$, there exists a unique smallest topology $\tau_{\mathcal{E}} \supseteq \mathcal{E}$, that consists of $\varnothing, X$ and all finite intersections of any $U \in \mathcal{E}$ and arbitrary unions of these intersections. We call $\mathcal{E}$ a subbase for $\tau_{\mathcal{E}}$, and $\tau_{\mathcal{E}}$ the evidential topology generated by $\mathcal{E}$.

We notice from Proposition 1 that we can define a base arbitrarily for any topological space, since a subbase that is closed under finite intersections forms a base for the topology it generates.

Consequently, we can now apply Proposition 1 to define the basic structure of our models. Let X be a space and let $\mathcal{E}$ be a family of subsets of X representing a set of evidence. Using the procedure described above, we can generate a topology $\tau_{\mathcal{E}}$ from $\mathcal{E}$. When $\mathcal{E}$ satisfies the conditions of Definition 3, we call $\tau_{\mathcal{E}}$ the *evidential topology*.

The basis of the so generated topological space captures the *combination of evidence*. Intuitively, starting from a collection of evidence, the evidential topology generated by it consists of all the arguments that can be derived from that evidence. The corresponding basis then represents the result of combining all the initial evidence.

A remark is needed regarding the conception of a proposition in this setting. In Definition 2, we defined a proposition as a subset of the space. However, this notion is not entirely suitable in the context of TME, where we work with a language consisting of formulae that include atomic ones, normally called propositions, which are syntactic signs denoting a set of states in our space, i.e. a subset P of X. Thus, more formally, in the context of TME, we typically refer to a proposition as a part of the language.

To avoid confusion, we reserve the term proposition for subsets of our space. What is usually called a proposition in the context of TME will be referred to as an atomic formula, denoted by lowercase Latin letters (e.g., *p*). As we will see, such an atomic formula denotes a specific set of states, which we will denote by the corresponding uppercase Latin letter (e.g., *P*). For formulae of higher complexity, we adopt the same convention using lowercase and uppercase Greek letters, respectively.

Now, we go on defining a what are the admissible formulae in our language. This language will be the standard language for doxastic logics, where we have boolean values and a box modality for belief. We do not consider the knowledge case since it is not in our interest.

Definition 8 (Syntax)**.** We call $\mathcal{L}$ the set of well-formed formulae in our language, and we say that $\varphi \in \mathcal{L}$ if and only if

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{B}\varphi.$$

The other Boolean connectives are defined as usual, i.e. $\varphi \vee \psi = \neg(\neg\varphi \wedge \psi)$ and $\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi)$. Furthermore, we call the set of atomic formulae in $\mathcal{L}$, Atom.

When we normally speak of doxastic logic we usually refer to logic axiomatized by the system KD45. Under the standard relational semantics, this system is well known to be sound and complete with respect to the class of serial and Euclidean frames. However, we do not wish to work within the relational framework.

However, as introduced above, we will refer to another type of semantics, defined over topological models of evidence.

Definition 9 (Topological models of evidence)**.** We call $\mathfrak{M} = (X, \mathcal{E}, \tau_{\mathcal{E}}, V)$ a topological model of evidence or a topo-e-model if

- X is a set of states
- $\mathcal{E}$ is a set of evidence, according to Definition 3

- $\tau_{\mathcal{E}}$ is the evidential topology generated by $\mathcal{E}$ as a subbase (as claimed in Proposition 1)
- $V : \mathsf{Atom} \longrightarrow \wp(X)$ is a valuation function

Furthermore, we call $\mathfrak{F} = (X, \mathcal{E}, \tau_{\mathcal{E}})$ a topological evidence frame or a topo-e-frame.

Finally, we can define the semantics for our language. There are multiple ways to define it, using topological models, however in this context we use the semantics defined in (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2022) for justified belief. Other approaches are the ones that can be found in different sources, among the ones we mention (Baltag, Bezhanishvili, Özgün, et al., 2017; Baltag, Bezhanishvili, Özgün, et al., 2018; Benthem and Pacuit, 2011; Özgün, 2013), or for a more general overview on the topics the reader can always use as reference (Özgün, 2017).

This type of semantics as explained in (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2022) is modelled on the notion of justification. This choice of definition is motivated by the limitations of the topological approach developed in (Özgün, 2013; Baltag, Bezhanishvili, Özgün, et al., 2013), where the restriction to extremally disconnected spaces prevented a smooth extension of the framework to include dynamic updates such as public announcements. To overcome this obstacle, we turn to a semantic grounded in the topological concept of denseness, which allows us to preserve the desired logical properties while accommodating informational dynamics.

Definition 10 (Dense set)**.** Let (X, τ) be a topological space. We say that a $U \subseteq X$ is dense in X if and only if for all $T \in \tau \setminus \{\varnothing\}$, $U \cap T \neq \varnothing$.

An equivalent characterization is given by the closure operator: a set $U \subseteq X$ is dense in X if and only if $\mathsf{cl}\,(U) = X$. We will use alternatively the two characterization of denseness, depending on the context.

Definition 11 (Semantics)**.** Given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}, \tau, V)$, we firstly define an interpretation map $\|*\| : \mathcal{L} \longrightarrow \wp(X)$, defined recursively as follows:

$$
\begin{aligned}
\|\top\| &= X \\
\|p\| &= V(p) \\
\|\neg\varphi\| &= X \setminus \|\varphi\| \\
\|\varphi \wedge \psi\| &= \|\varphi\| \cap \|\psi\| \\
\|\mathsf{B}\varphi\| &= \{x \in X \mid \forall U \in \tau_{\mathcal{E}} \setminus \{\varnothing\}(\exists U' \in \tau_{\mathcal{E}} \setminus \{\varnothing\}(U' \subseteq U \cap \|\varphi\|))\} \\
&= \{x \in X \mid \exists U \in \tau_{\mathcal{E}}(U \subseteq \|\varphi\| \text{ and } \mathsf{cl}\,(U) = X)\}
\end{aligned}
$$ [3]

(Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2022) establish that the semantics so defined is sound and complete with respect to KD45. Moreover, the same works show as an agent can believe a formula φ if and only if $\mathsf{int}\,(\varphi)$ is dense in X, i.e.

$$\mathfrak{M}, x \models \mathsf{B}\varphi \iff \mathsf{cl}\,(\mathsf{int}\,(\|\varphi\|)) = X.$$

This is important because, as we can see from the definition of the semantics, the requirement for belief at a state $x \in X$ is that there exists an open set in the topology contained in the set of states where the proposition is true.

3 The equivalence between the possible semantic definitions of $\mathsf{B}\varphi$ has been proven in (Özgün, 2017, Proposition 5.3.4). We here mention the ones that are relevant for our proofs later on and will use them interchangeably.

That is, there must be an argument coherent with our formula that is dense, which is equivalent to saying it is coherent with all the arguments in the topological space. We notice that this is a global property of a model, and is independent of the state that is being considered. If a proposition is believed in a state then it is believed in X.

1.2 *Dempster-Shafer Theory*

Dempster-Shafer theory, and the correlated belief functions, are a possible generalization of probability theory. There are multiple ways to introduce this kind of framework. One of these reads belief function as *lower probability measures*[4], i.e. the greatest lower bound across all probability measures that are compatible with the available evidence. However, we will present this framework following the original construction of (Shafer, 1976). Nevertheless it is important to understand why this setting generalize probabilities, even if in a naïve way. Therefore, in what follows we will introduce the definition of a probability measure and of belief functions

Thus consider the following definitions:

Definition 12 (Probability measures and probability space)**.** Given a set of states X, we call $\mathcal{A} \subseteq \wp(X)$ an *algebra over* X if $X \in \mathcal{A}$ and

- if $U, V \in \mathcal{A}$ then $U \cup V \in \mathcal{A}$,
- if $U \in \mathcal{A}$ then $\overline{U} \in \mathcal{A}$.

We call an algebra $\mathcal{A}$ a σ*-algebra* if it is also closed under countable union, i.e. for any countable family $\{U_i\}_{i\in I} \subseteq \mathcal{A}$ we have $\bigcup_{i\in I} U_i \in \mathcal{A}$.

Given a set X and an algebra $\mathcal{A}$ over it, we call every $U \in \mathcal{A}$ a *measurable set*. A *probability measure* over X is a function $\mu : \mathcal{A} \longrightarrow [0, 1]$ satisfying:

1. $\mu(\varnothing) = 0$,
2. $\mu(X) = 1$,
3. for all $U, V \in \mathcal{A}$ with $U \cap V = \varnothing$, $\mu(U \cup V) = \mu(U) + \mu(V)$.

We call $(X, \mathcal{A}, \mu)$ a *probability space*. We denote with $\mathbb{P}$ the set of all probability measures $\mu : \mathcal{A} \longrightarrow [0, 1]$ over X.

Now consider the definition of a belief function.

Definition 13 (Belief function)**.** Given a set of states X, $\mathsf{Bel} : \wp(X) \longrightarrow [0, 1]$ is a belief function if it satisfies the following requirements:

$$\begin{aligned} \mathsf{Bel}(\varnothing) &= 0 \\ \mathsf{Bel}(X) &= 1 \\ \mathsf{Bel}\left(\bigcup_{i=1}^{n} U_i\right) &\geqslant \sum_{i=1}^{n} \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=i}} (-1)^{i+1}\mathsf{Bel}\left(\bigcap_{j\in I} U_j\right) \qquad \text{where } U_i, U_j \subseteq X \end{aligned}$$

4 For more information about lower probability measures and imprecise probabilities in general, we suggest the interested reader to consult (Augustin et al., 2014; Walley, 1991), and to know more about the relationship between belief functions and lower probability measures we refer to (Halpern, 2003; Halpern and Fagin, 1991; Halpern and Fagin, 1992; Voorbraak, 1992).

Notice how, according to Definition 12, given a set X and an algebra $\mathcal{A}$ over it, each probability measure μ defined on X is in fact a belief function according to the definition above. This holds trivially, since the last condition in Definition 13 is precisely the generalized version of the last condition in Definition 12, where the equality sign is replaced by a less-than-or-equal-to sign.

In general we can read belief functions as a lower bound on the likelihood of U. Alongside belief functions, we can define the dual upped bound on the likelihood of U, the *plausibility function* defined as

$$\mathsf{Pl}(U) = 1 - \mathsf{Bel}(\overline{U}).$$

Notice that this function can be axiomatized in a similar way to axiomatization given for belief functions in Definition 13. However, the third axiom is replaced by a dual inclusion–exclusion inequality that provides a lower bound on the plausibility of an intersection in terms of the plausibilities of unions, following the same alternating sum pattern as the belief axiom but with unions and intersections swapped.

Now, we move to describe how we can construct a belief function over a space from a set of evidence, defined according to Definition 3. More formally, in the following way of conceptualizing belief functions, we are more focused on the evidence, which support certain events with some degree of belief, for each subset of X. This is captured by a *basic probability assignment*:

Definition 14 (Basic probability assignment)**.** Given a set of states X, we call $m : \wp(X) \to [0, 1]$ a basic probability assignment if it satisfies the following properties:

$$\begin{aligned} m(\varnothing) &= 0 \\ \sum_{U \subseteq X} m(U) &= 1 \end{aligned}$$

Given a basic probability assignment we call the sets that have a positive value focal elements. [5]

Intuitively, this functions describe to which extent the evidence supports a proposition U. This is formally expressed as follows:

Definition 15 (Belief function based on a basic probability assignment)**.** Given a basic probability assignment m then we will have that

$$\mathsf{Bel}_m(U) = \sum_{\{U' | U' \subseteq U\}} m(U') \quad \text{and} \quad \mathsf{Pl}_m(U) = \sum_{\{U' | U' \cap U \neq \varnothing\}} m(U').$$

We can think of $\mathsf{Pl}_m(U)$ as the sum of the probabilities of the evidence that is compatible with the actual world being in U. It can be checked that both Bel_m and Pl_m are indeed Belief and Plausibility functions. Moreover, given a belief function Bel on some set X, there is a unique mass function m on X such that $\mathsf{Bel} = \mathsf{Bel}_m$.

We now turn to a fundamental operation in DST: Dempster's rule of combination. This rule takes two belief functions, representing distinct bodies of evidence, and produces a combined belief function that aggregates the information in a manner akin to a normalized conjunction of the underlying evidence.

5 In the DST literature this types of functions is also called mass function. However, since we will use the term mass function in a different context, we will refer to these functions as basic probability assignment.

Definition 16 (Dempster Rule of Combination). Let X be a finite set of states, and let m_1 and m_2 be mass functions (basic belief assignments) over X. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$, where $I, J \subseteq \mathbb{N}$, be the focal elements of m_1 and m_2, respectively, such that

$$m_1(A_i) > 0 \quad \text{for all } i \in I, \qquad m_2(B_j) > 0 \quad \text{for all } j \in J.$$

We call the total conflict the following sum of the products of the masses of all pairs of focal elements that are disjoint:

$$\sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j),$$

and we assume it to be less than 1.
Then, the combined mass function $m_1 \oplus m_2$ under Dempster's Rule of Combination is given by

$$m_1 \oplus m_2(\emptyset) = 0,$$
$$m_1 \oplus m_2(C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i)\, m_2(B_j)}{1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i)\, m_2(B_j)}.$$

For all $D \subseteq X$ that $A_i \cap B_j \neq D$ for any A_i, B_j, we define $m_1 \oplus m_2(D) = 0$.

1.3 *Multi-Layer Belief Model*

Now, we introduce the Multi-Layer Belief Model (MLBM) introducing the basic definitions and explaining how this model is able con combine and extend the features of both TME and DST

The structure of this model is very complex, thus we will first give an intuitive explanation of the different part of the models, that has the name suggests can be divided in 3 layers: the qualitative one, the quantitative one and the bridging one.

The first layer is the *qualitative layer*. This layer works with the propositional content of the basic pieces of evidence, which are elements of $\mathcal{E}$, and identifies a set of justifications representing the agent's evidential demand. The second layer is the *quantitative layer*. Here, we combine the degrees of uncertainty about evidence, drawing inspiration from the combination rules used in DST. The combined mass function is defined over the sets of pieces of evidence. The goal of this layer is to compute degrees of belief based on justifications that will be defined as sets of possible states. However, the combined uncertainty values are assigned to family of sets of possible states instead. Therefore, we need to transfer these values from the family of sets of possible states to the space. To accomplish this, we define the third layer: the *bridging layer*. This layer connects the values of the mass function obtained in the second layer to the justifications of the first one. As a result, we obtain a belief function Bel which is able to compute degrees of belief according to the evidential demands of the agent and based on a possibly mutually contradictory, partial, and uncertain body of evidence.

We now present each layer in detail, expanding on the overview given above. To introduce the new components of the model, we consider an arbitrary set of evidence $\mathcal{E}$, defined as in Definition 3. We begin by defining the basic frames that underlie the model.

Definition 17 (Qualitative and quantitative evidence frame). Let X be a set of possible states, $\mathcal{E} \subseteq \wp(X) \setminus \{\emptyset\}$ be a non-empty set of evidence and $\mathcal{E}_{\mathrm{Q}} \subseteq \mathcal{E} \times (0,1)$ be a nonempty subset containing exactly one element for each element of $\mathcal{E}$.

- We will say that $(X, \mathcal{E})$ is a *qualitative evidence frame*.
- We call the tuple $\left(X, \mathcal{E}^{\mathrm{Q}}\right)$ a *quantitative evidence frame*, and $\mathcal{E}^{\mathrm{Q}}$ *quantitative set of evidence*.

Now we start from the qualitative layer. In this layer we work using qualitative evidence frames of the form $(X, \mathcal{E})$. More precisely we first construct the topology on X given by the set of evidence $\mathcal{E}$, following the procedure given in Proposition 1, and then we define a notion of justification. The notion of justification is more general than the one used in (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2022), where the set of justifications is defined as the set of arguments, i.e. the set of non-empty open sets in the evidential topology. In our case, we will allow for different types of justifications, which will be defined as subsets of the evidential topology. This generalization is motivated by the need to accommodate different types of evidential demands that an agent might have, which may not be adequately captured by the notion of arguments alone.

Definition 18 (Frame of justification)**.** Given a qualitative evidence frame $(X, \mathcal{E})$, we call $\mathcal{J} \subseteq \tau_{\mathcal{E}} \setminus \{\varnothing\}$ a *frame of justification*.

This approach grants greater flexibility to the agent, enabling the modelling of diverse types of evidential demands. Consequently, the agent can compute beliefs even when confronted with bodies of evidence that are partial, uncertain, or possibly mutually contradictory. This flexibility is particularly important, as accommodating different types of justifications will prove crucial for reconstructing the patterns of reasoning found DST and TME. For this reason we characterize the following types of frame of justification:

Definition 19 (Types of frames of justification)**.** Given a qualitative evidence frame $(X, \mathcal{E})$, we define:

1. the *Dempster-Shafer frame of justification*, denoted by $\mathcal{J}_{\mathrm{DS}}$, is the set of arguments, i.e.

$$\mathcal{J}_{\mathrm{DS}} = \tau_{\mathcal{E}} \setminus \{\varnothing\},$$

2. the *strong denseness frame of justification* , denoted by $\mathcal{J}^{\mathrm{SD}}$, is the set of all arguments consistent with $\tau_{\mathcal{E}}$, that is, the set of all dense open sets in X, i.e.

$$\mathcal{J}_{\mathrm{SD}} = \{A \in \tau_{\mathcal{E}} \setminus \{\varnothing\} \mid \text{for all } U \in \tau_{\mathcal{E}} \setminus \{\varnothing\},\ A \cap U \neq \varnothing\}.$$

Now, we move on to the quantitative layer. In this layer we work with quantitative evidence frames of the form $\left(X, \mathcal{E}^{\mathrm{Q}}\right)$. The main goal of this layer is to combine the degrees of uncertainty about pieces of evidence, drawing inspiration from the combination rules used in DST. To do so, we define a mass function over the sets of evidence, which will allow us to compute degrees of belief based on justifications that will be defined as sets of possible states.

In order to combine the degrees of belief about the evidence, we need to define a mass function over the sets of evidence. This mass function will allow us to compute degrees of belief based on justifications that will be defined as sets of possible states. The definition of mass function over the sets of evidence is a generalization of the standard definition of basic probability assignment in DST, which is defined over the subsets of the set of states. This generalization is necessary to accommodate the fact that we are working with a more complex structure, where the evidence is not just a set of states but also includes degrees of uncertainty.

Definition 20 (Mass Function over a set of evidence)**.** Let $\mathcal{E}$ be a set of evidence. Then we call $m_{\mathcal{E}} : \wp(\mathcal{E}) \longrightarrow [0, 1]$ a *mass function over a set of evidence* if it holds that

$$\sum_{A \subseteq \mathcal{E}} m_{\mathcal{E}}(A) = 1$$

Now, we define a mass function over the sets of evidence, which will allow us to compute degrees of belief based on justifications that will be defined as sets of possible states.

Definition 21 (δ function)**.** Let $\left(X, \mathcal{E}^{\mathrm{Q}}\right)$ be a quantitative evidence frame with n pieces of evidence, i.e. $\mathcal{E}^{\mathrm{Q}} = \left\{(E_i, p_i)_{1 \leqslant i \leqslant n}\right\}$, and let be $\mathcal{E}$ its underlying set of evidence. We define a mass function $\delta : \wp(\mathcal{E}) \longrightarrow [0, 1]$ to merge the certainty values p_i with $1 \leqslant i \leqslant n$:

$$\delta(E) = \prod_{E_i \in E} p_i \prod_{E_i \notin E} (1 - p_i)$$

In (Pinto Prieto, 2024) it is shown, respectively in Proposition 3.1 and Proposition 3.2, that the function δ defined above is indeed a mass function over the sets of evidence, i.e. it satisfies the requirements in Definition 20, and that given a quantitative evidence frame $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ and the mass function δ defined over it, then it holds that for every piece of evidence $E \in \mathcal{E}$

$$\sum_{E_i \in E} \delta(E_i) = p_i.$$

This function is ready as merely a system to distribute the certainty of a single piece of evidence to all the ways it can be observing combination with the other pieces of evidence.

Now we define the last layer which is the one responsible to connect the qualitative and quantitative layers to calculate degrees of belief based on a quantitative evidence frame $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ and a given frame of justification $\mathcal{J}$.

In order to do this we will follow these steps:

1. Define a family of functions to map $\wp(\mathcal{E})$, i.e. the domain of the δ function, to $\tau_{\mathcal{E}}$
2. Define two mass functions, one with domain $\tau_{\mathcal{E}}$ and another with domain $\mathcal{J}$.

We are equipped at this point with a frame of justification $\mathcal{J}$ and a merged measure of certainty distributed over the elements of $\wp(\mathcal{E})$ over which we defined a function δ. Now we have to link the mass values defined over the elements of $\wp(\mathcal{E})$ by δ to the elements of $\tau_{\mathcal{E}}$, and so to $\mathcal{J}$.

However, mapping these two sets is not trivial, since we can map different values to the element of the topology based on the different interpretations the agent gives to the values. The choice among which interpretation to choose depends on the context of the agent, and reliability of the sources that will make the agent establish some assumptions or others.

We capture different ways of interpreting the mass values provided by δ via a particular type of functions that we call *evidence allocation functions*.

Definition 22 (Evidence allocation functions)**.** Let $(X, \mathcal{E})$ be a qualitative evidence frame. A set of *evidence allocation functions* $\mathfrak{F}$ on $(X, \mathcal{E})$ is a set of functions from $\wp(\mathcal{E})$ to $\tau_{\mathcal{E}}$ such that for all $f, g \in \mathfrak{F}$:

1. $f(\varnothing) = X$

2. for all $E \in \mathcal{E}$ such that $E \neq \varnothing$, $f(E) \in \tau_E$ (the topology generated by E) and it is dense in $\bigcup E$ with respect to τ_E or $f(E) = \varnothing$
3. for all $E \subseteq \mathcal{E}$ and every $f, g \in \mathfrak{F}$, $f(E) \subseteq g(E)$ or $g(E) \subseteq f(E)$

Note that since $E \subseteq \mathcal{E}$, we have $\tau_E \subseteq \tau_{\mathcal{E}}$

We now give some examples of functions, which are proven to be evidence allocation functions, according to Definition 22, in (Pinto Prieto, 2024).

Definition 23 (Types of evidence allocation function)**.** Let $(X, \mathcal{E})$ be a qualitative evidence frame. The set of evidence allocation functions $\mathfrak{F} = \{i, u, d\}$ is defined as follows for any $E \in \wp(\mathcal{E})$:

$$i(E) = \begin{cases} \bigcap E & \text{if } E \neq \varnothing, \\ X & \text{otherwise,} \end{cases}$$

$$u(E) = \begin{cases} \bigcup E & \text{if } E \neq \varnothing, \\ X & \text{otherwise,} \end{cases}$$

$$d(E) = \begin{cases} \min(\text{dense}(E), \subseteq) & \text{if } E \neq \varnothing, \\ X & \text{otherwise,} \end{cases}$$

where $\text{dense}(E)$ denotes the set of dense elements of τ_E within $\bigcup E$.

All three functions i, u, and d defined in Definition 23 are indeed evidence allocation functions, as established in (Pinto Prieto, 2024). Moreover, they satisfy additional properties that we summarize below:

- The functions i and u are, respectively, the minimal and maximal elements of the poset $(\mathfrak{F}, \subseteq)$, for any set $\mathfrak{F}$ of evidence allocation functions. That is, for every $f \in \mathfrak{F}$ and any $E \in \wp(\mathcal{E})$, we have

$$i(E) \subseteq f(E) \subseteq u(E).$$

- The function d is well-defined for all $E \in \wp(\mathcal{E})$, by Lemma 3.1 in (Pinto Prieto, 2024). This lemma guarantees that, in any qualitative evidence frame, the set of dense elements of the topology generated by a collection of evidence in $\bigcup E$ always contains a minimal element under the subset ordering.

Evidence allocation functions can be seen as contextual parameters that reflect an agent's epistemic attitude toward reliability and informativeness. Different evidence allocation functions capture distinct attitudes. For example, the function i represents agents who assume that each piece of evidence individually contains the actual world; while the function u corresponds to agents who believe that at least one piece of evidence does. The function d captures agents who assume that the actual world is consistent with the strongest argument that all pieces of evidence can jointly produce.

Thanks to evidence allocation functions, we can map elements of $\wp(\mathcal{E})$ into the topology generated by $\mathcal{E}$, i.e., $\tau_{\mathcal{E}}$. This leads us to the second part of our problem: to construct a function that associates subsets of $\mathcal{E}$ with elements of $\tau_{\mathcal{E}}$, enabling the computation of degrees of belief based on a body of evidence that may be incomplete, uncertain, or even mutually inconsistent, while respecting the agent's evidential constraints. Thus, consider the new function δ_τ defined as follows:

Definition 24. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and $\mathfrak{F}$ a set of evidence allocation functions. Then we define the function $\delta_\tau : \mathfrak{F} \times \wp(X) \to [0,1]$ such that

$$\delta_\tau(f, T) = \begin{cases} \displaystyle\sum_{E: f(E) = T} \delta(E) & \text{if } T \in \tau_{\mathcal{E}}, \\ 0 & \text{otherwise.} \end{cases}$$

If we fix $f \in \mathfrak{F}$ we get that $\delta_\tau(f, \cdot)$ is a mass function over the space X since δ is a mass function over $\mathcal{E}$. Furthermore, if we normalize the previous function with respect to a frame of justification $\mathcal{J}$, we can define the function $\delta_{\mathcal{J}}$ as follows

Definition 25. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame, $\mathcal{J}$ a frame of justification over $\tau_{\mathcal{E}}$ and $\mathfrak{F}$ a set of evidence allocation functions. Then we define the function $\delta_{\mathcal{J}} : \mathfrak{F} \times \wp(X) \to [0,1]$ such that

$$\delta_{\mathcal{J}}(f, A) = \begin{cases} \dfrac{\delta_\tau(f,A)}{\displaystyle\sum_{T \in \mathcal{J}} \delta_\tau(f, T)} & \text{if } A \in \mathcal{J} \\ 0 & \text{otherwise} \end{cases}$$

For every frame of justification $\mathcal{J}$ and every evidence allocation function f, we can use the function $\delta_{\mathcal{J}}(f, \cdot)$ to get degrees of belief additively. Formally, this is expressed by the following claim, proven in (Pinto Prieto et al., 2023):

Proposition 2. Given a qualitative evidence frame $(X, \mathcal{E}_Q)$, a frame of justification $\mathcal{J}$ and an evidence allocation function $f \in \mathfrak{F}$, the function $\delta_{\mathcal{J}}(f, \cdot)$ as defined in Definition 25 is a mass function over X such that $\delta_{\mathcal{J}}(f, \varnothing) = 0$.

This proposition states that $\delta_{\mathcal{J}}(f, \cdot)$ is in fact a basic probability assignment according to Definition 14 Now we can finally define the degree of belief for a proposition $\Phi \subseteq X$ for our Multi-Layer Belief Model. Thus consider the following definition:

Definition 26 (Multi-layer belief function)**.** Given a quantitative evidence frame $(X, \mathcal{E}_Q)$, a frame of justification $\mathcal{J}$ and a set of evidence allocation functions $\mathfrak{F}$. Then for any $P \subseteq X$, we call the function

$$\mathrm{Bel}_{\mathcal{J}}(f, P) = \sum_{A \subseteq P} \delta_{\mathcal{J}}(f, A)$$

a *multi-layer belief function*. When $\mathcal{J}$ and f are clear from the context we will simply write $\mathrm{Bel}(P)$

In order to conclude the construction we have to show that this function is in fact a belief function, according to Definition 13, and this falls trivially from the fact that $\delta_{\mathcal{J}}(f, \cdot)$, is a basic probability assignment, and by Definition 15.

Furthermore, we note that different combinations of a frame of justification and an evidence allocation function, as shown in (Pinto Prieto, 2024), can recreate, under certain conditions, the behavior of the systems on which MLBM is built. Specifically:

- The first combination ($\mathcal{J}^{\mathrm{DS}}$ and i) aligns with DST. This is shown in Proposition 3.8 to be equivalent to Dempster's rule of combination, thereby recovering the method of evidence aggregation in DST.

- The second combination ($\mathcal{J}^{\mathsf{SD}}$ and d) mirrors the behavior of B in TME. Proposition 3.9 shows that the proposition Φ has positive belief if and only if $\mathsf{B}\varphi$ is true, in the topological models which are constructed from the evidence set that underlies the quantitative set of evidence used for counting belief.

1.4 *Conclusions*

In this chapter, we analyzed the background frameworks that will be used throughout the rest of this work. We introduced the common notions shared by *Topological Models of Evidence* (TME) and *Dempster-Shafer Theory* (DST). We then presented the core elements of both theories, exploring basic concepts in topology and probability theory. Building on this theoretical foundation, we examined the construction of the *Multi-Layer Belief Model* (MLBM), understanding what it inherits from the aforementioned theories and how it can both extend and recover each of them.

In the next chapter, we will proceed by analyzing the motivations and the technical and philosophical details of the conditionalization procedure, which lies at the core of the following work.

It is worth noting that practical implementations of the MLBM have been developed as part of the work presented in (Pinto Prieto et al., 2023) and (Compagno et al., 2026). Those code-bases will serve as the foundation from which we will implement the conditionalization procedure explored in the next chapters.

2 Conditionalization on the Multi-Layer Belief Model

2.1 *Conditionalize on a piece of evidence*

In *Ethica, Ordine Geometrico Demonstrata*, Part 3, Proposition 26, Spinoza suggests that:

> Quicquid ex ratione conamur, nihil aliud est quam intelligere nec mens quatenus ratione utitur, aliud sibi utile esse judicat nisi id quod ad intelligendum conducit.[1]

Thus, as suggested by the *God-Intoxicated Man*, any rational agent pursues the acquisition of new information. But what does it mean for an agent to acquire new information? And, how should this new information be processed by a belief system?

In this context, we can speak of different doxastic dynamics: gaining new evidence to add to our beliefs, or removing old evidence that no longer holds. However, to capture how an agent's beliefs would change upon receiving specific evidence, we require a more precise notion.

Consider the case when we learn a particular piece of evidence Φ. Prior to learning Φ, the agent may already have dispositions about what to believe if Φ were to turn out true. This disposition is captured by the notion of a *conditional belief*. To say that an agent has a conditional belief in a proposition Ψ given evidence Φ is to say that, should the agent come to learn Φ (and nothing more informative or misleading), the agent would then believe Ψ. After actually observing Φ, the conditional belief becomes an unconditional belief in Ψ. The central question, then, is: how can we capture this notion of conditional belief in MLBM?

To do so we can think of the most famous and intuitive notion of such conditional belief that is captured by conditional probability. Suppose that an agent's uncertainty is represented by a probability measure μ on a finite set of possible states X. The agent then observes or learns that the actual states are in $\Phi \subseteq X$. How should μ be updated to a new probability measure $\mu|_\Phi$ that takes this new information into account?

Clearly, if the agent learns that Φ is true, then it seems reasonable to require that:

$$\mu|_\Phi(\overline{\Phi}) = 0,$$

i.e., all states outside Φ become impossible. What about states in Φ? What should their probability be? One reasonable intuition here is that if all that the agent has learned is Φ, then the relative likelihood of states in Φ should remain unchanged. (This presumes that the way the agent learns Φ does not itself give the agent additional information;

1 Personal translation: "Whatsoever we endover guided by our reason, is nothing else than to understand, and as long as the mind is using reason it does not judge anything to be useful for itself unless what leads to understanding."

otherwise, relative likelihoods may indeed change.) That is, for any $\Psi_1, \Psi_2 \subseteq \Phi$ with $\mu(\Psi_2) > 0$, we require:

$$\frac{\mu(\Psi_1)}{\mu(\Psi_2)} = \frac{\mu|_\Phi(\Psi_1)}{\mu|_\Phi(\Psi_2)}.$$

These conditions completely determine $\mu|_\Phi$ whenever $\mu(\Phi) > 0$, yielding the standard definition of conditional probability:

$$\mu|_\Phi(\Psi) = \frac{\mu(\Psi \cap \Phi)}{\mu(\Phi)} \quad \text{for all } \Psi \subseteq X.$$

With this machinery in place, we can define conditional belief in terms of conditional probability.

Definition 27 (Conditional Belief via Conditional Probability)**.** Let μ be a probability measure over a finite set of possible states X. For any propositions $\Psi \subseteq X$ and evidence $\Phi \subseteq X$ with $\mu(\Phi) > 0$, the conditional probability of Ψ given Φ is:

$$\mu|_\Phi(\Psi) = \mu(\Psi \mid \Phi) = \frac{\mu(\Psi \cap \Phi)}{\mu(\Phi)}.$$

Having introduced the probabilistic notion of conditional belief via conditional probability, we now want to investigate how we can introduce such a notion in our model. To do so, we will first try to capture this notion in our theoretical bases, namely TME and DST, and then examine how these treatments can be integrated within the MLBM framework.

In (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2016; Baltag, Bezhanishvili, Özgün, et al., 2022), we can find different notions that captures what the agent would believe upon learning Φ. The first one that we introduce is a static notion, i.e. we define a belief conditioned on a proposition Φ without actually performing the update.

Definition 28 (Conditional Belief in TME)**.** Let $\mathfrak{M} = (X, \mathcal{E}, \tau_\mathcal{E}, V)$ be a topo-e-model, according to Definition 9, and consider any proposition $\Phi, \Psi \subseteq X$. For any formula $\psi \in \mathcal{L}$ such that $\|\psi\| = \Psi$, the agent has a conditional belief in ψ given Φ if and only if, for every Φ-consistent argument we can strengthen it to a Φ-consistent argument for $\Phi \to \Psi$. Formally:

$$\begin{aligned} \mathfrak{M}, x \models \mathsf{B}^\Phi \psi \iff & \text{for all } V \in \tau_\mathcal{E} \text{ with } V \cap \Phi \neq \varnothing \\ & \text{there exists } V' \in \tau_\mathcal{E} \text{ with } V' \cap \Phi \neq \varnothing \text{ and } V' \subseteq V \cap (\Phi \to \Psi), \end{aligned}$$

where $\Phi \to \Psi = \overline{\Phi} \cup \Psi$.[2]

This notion captures the dispositional attitude of conditional belief: it is defined entirely within the agent's current epistemic state, without requiring the agent to have actually learned Φ. By contrast, we can define a dynamic to capture this behavior. This dynamic is the public announcement operation, defined below, which produces a genuinely new model representing the agent's epistemic state after receiving the information.

Definition 29 (Public Announcement Model)**.** Let $\mathfrak{M} = (X, \mathcal{E}, \tau_\mathcal{E}, V)$ be a topo-e-model, according to Definition 9, and consider a proposition $\Phi \subseteq X$. Then we call the model $\mathfrak{M}^{!\Phi} = \left(X^{!\Phi}, \mathcal{E}^{!\Phi}, \tau_\mathcal{E}^{!\Phi}, V^{!\Phi}\right)$ a *public announcement model*, where

2 There are various ways to characterize this notion. We chose this one only for technical purposes. For more details the interested reader can refer to (Özgün, 2017, Chapter 5, Section 3.3). Furthermore, in (Baltag, Bezhanishvili, Özgün, et al., 2016, Observation 4 (see extended version)) it is noted how this static notion is equivalent to an evidence dynamics called *evidence addition*, as defined in (Özgün, 2017, Definition 5.4.2) or equivalently in (Baltag, Bezhanishvili, Özgün, et al., 2016, Definition 5). Since this exceeds the scope of this work we do not investigate this notion further, but the interested reader may consult the aforementioned sources for more insights.

- $X^{!\Phi} = \Phi$
- $\mathcal{E}^{!\Phi} = \{E \cap \Phi \mid E \in \mathcal{E} \text{ and } E \cap \Phi \neq \varnothing\}$
- $\tau_{\mathcal{E}}^{!\Phi} = \{U \cap \Phi \mid U \in \tau_{\mathcal{E}}\}$
- $V^{!\Phi}(p) = V(p) \cap \Phi$, for all $p \in \mathsf{Atom}$

Furthermore, we call $\mathfrak{F}^{!\Phi} = \left(X^{!\Phi}, \mathcal{E}^{!\Phi}, \tau_{\mathcal{E}}^{!\Phi}\right)$ a public announcement frame.

Importantly, the public announcement operation behaves, in flavor, like conditional probability. Just as conditional probability $\mu|_U$ restricts the original probability space to U and renormalizes, the public announcement model $\mathfrak{M}^{!\Phi}$ restricts the original set of states X to Φ, the evidence set $\mathcal{E}$ to $\{E \cap \Phi \mid E \in \mathcal{E}\}$, and the topology $\tau_{\mathcal{E}}$ to $\{U \cap \Phi \mid U \in \tau_{\mathcal{E}}\}$. In both cases, the entire original structure is transformed into a new structure restricted to the learned proposition, and the resulting object serves as the new epistemic state for future reasoning.

This is not the case for conditional belief as defined in Definition 28. Conditional belief only ascribes a disposition about a specific proposition Ψ given Φ, without modifying the underlying model. It does not produce a new evidence set, a new topology, or a new valuation function. Consequently, conditional belief cannot support iterated conditioning: after asking what the agent would believe given Φ, one cannot then ask what the agent would believe given a further proposition Ψ without additional machinery. The public announcement operation, by contrast, yields a full model of the same type, enabling sequential updates in a way that mirrors the behavior of conditional probability.

We now turn to the treatment of conditional belief in Dempster-Shafer theory. Two distinct notions appear in (Shafer, 1976; Halpern, 2003; Halpern and Fagin, 1992; Halpern and Fagin, 1991).[3] For the scope of this work we will focus on the notion which defines conditional belief via Dempster's rule of combination, as in Definition 16. Thus consider the following definition:

Definition 30 (Conditional Dempster-Shafer Belief)**.** Let Bel be a Dempster-Shafer belief function based on a basic probability assignment m defined on a finite set of possible states X, and let $\Phi \subseteq X$ be a proposition such that $\mathsf{Pl}(\Phi) > 0$. Define the mass function m_Φ by:

$$m_\Phi(A) = \begin{cases} 1 & \text{if } A = \Phi \\ 0 & \text{otherwise} \end{cases}.$$

Then the conditional Dempster-Shafer belief of a proposition $\Psi \subseteq X$ given Φ is the function $\mathsf{Bel}|_\Phi(\Psi) = \mathsf{Bel}(\Psi \mid \Phi)$ based on the combined mass function $m \oplus m_\Phi$, i.e.,

$$\mathsf{Bel}|_\Phi(\Psi) = \mathsf{Bel}(\Psi \mid \Phi) = \sum_{A \subseteq \Psi} (m \oplus m_\Phi)(A),$$

where $\oplus$ denotes Dempster's rule of combination as defined in Definition 16.

We believe that this notion is more appropriate for our context since, as argued in (Halpern, 2003, pg. 96), it is more natural when we have a belief function that arises from combining explicit bodies of evidence. Notice that

3 The other notion is based on how we define conditional belief in the context of imprecise probabilities and, more precisely, how we condition on a new evidence a *lower probability measure*. As we did in Section 1.2, we refer the interested reader to the relevant sources for more information. We especially recommend (Halpern and Fagin, 1991; Halpern and Fagin, 1992), where we believe it is possible to find a compelling and exhaustive analysis of the subject matter.

the resulting belief function represents a new epistemic state of the same type as the original, allowing for iterated conditioning, and that in case our belief function is also a probability measure the conditional Dempster-Shafer belief is indeed a conditional probability measure, according to Definition 27.

However, we cannot simply adopt this notion directly for updating the qualitative layer of MLBM. As shown and argued in (Pinto Prieto et al., 2023; Pinto Prieto, 2024), MLBM is a more general model that integrates both qualitative and quantitative dimensions of evidence. The qualitative layer involves a topological structure generated by the evidence set $\mathcal{E}$, while the quantitative layer involves a mass function δ. These qualitative and quantitative elements are brought together at the bridging layer, where we obtain a basic probability assignment $\delta_{\mathcal{J}}(f, \cdot)$ derived from the quantitative evidence frame.

Conditioning on a new piece of evidence Φ must respect both layers simultaneously: the topological structure must be restricted (as in the public announcement operation in TME), and the mass function must be renormalized. Neither the TME conditional belief nor the DST conditionalization notions alone suffice, because each operates on only one layer. Therefore, MLBM requires its own way to update the weights and the qualitative structure in a unified manner.

Having examined how conditional belief is treated in TME and in DST, we now turn to defining a conditionalization procedure for MLBM. As argued above, neither existing notion can be directly adopted as the only element of our conditional procedure, because MLBM integrates both a qualitative topological layer and the notion of a mass function, as employed in the quantitative layer in Definition 20. This latter notion is different from the basic probability assignment in Definition 14, which appears only at the final step of the bridging layer.

Thus, analyzing the different ways of conditioning on a piece of evidence listed in this section, we claim that a proper conditionalization operation for MLBM must satisfy the following *desiderata*:

1. Upon learning a proposition $\Phi \subseteq X$, the new set of possible states should be restricted to Φ.
2. The basic probability assignment $\delta_{\mathcal{J}}(f, \cdot)$ should be conditioned on Φ by discarding all mass assigned to subsets not contained in Φ and renormalizing the remaining mass.
3. The frame of justification $\mathcal{J}$ and the evidence allocation function f should be renormalized on a topological space and a set of evidence that has as total space Φ, while preserving the epistemic demands they encode.
4. The conditionalization operation should produce a new MLBM of the same type as the original, allowing for iterated conditioning.

In the following sections, we propose a conditionalization procedure for MLBM that satisfies these *desiderata*. To do so we will follow the structure of the model updating each layer individually and then computing the total belief at the end updating the bridging layer.

2.2 *Conditionalization on the qualitative layer*

We start presenting here the conditional update at the level of the qualitative layer. Remember from Section 1.3 that in this layer we work with a qualitative evidence frame $(X, \mathcal{E})$, an evidential topology $\tau_{\mathcal{E}}$ and a frame of justification $\mathcal{J}$, that can be any frame of justification according to Definition 18.

We want to understand how to condition this layer on a proposition $\Phi \subseteq X$. In Section 2.1 we saw how the best way to condition a topo-e-model to a new proposition is represented by the public announcement update presented in Definition 29. Furthermore notice that the elements from which we start our qualitative layer, i.e.

the qualitative evidence frame and the evidential topology are exactly the frame of a topo-e-model according to Definition 9. Thus we propose to update these elements exactly following the definition of the frame of the public announcement model built with our qualitative evidence frame and our topology, defined in Definition 29.

To show that the updated qualitative layer respect the definition, we will then introduce a known result, as it plays a crucial role in our construction. Specifically, the result shows that after a public announcement, we can restrict the topology to a subspace defined by the announced subset, thereby simplifying the model while preserving the relevant evidential structure.

Proposition 3 (Subspace Topology)**.** Let (X, τ) be a topological space, and let $\Phi \subseteq X$ be any subset. Then we claim that (Φ, τ_Φ) is a topological space, where $\tau_\Phi = \{U \cap \Phi \mid U \in \tau\}$. We will call τ_Φ the Subspace Topology.

Thus, we can now present the definition of the qualitative evidence frame and of the correspondent evidential topology, drawing inspiration from Definition 29.

Definition 31 (Conditionalization on the qualitative evidence frame and evidential topology)**.** Given a qualitative evidence frame $(X, \mathcal{E})$, the associated topology $\tau_\mathcal{E}$ and a proposition $\Phi \subseteq X$, we define conditionalization on the qualitative evidence frame and on the evidential topology, respectively, as follows:

- $\left(X^{!\Phi}, \mathcal{E}^{!\Phi}\right)$ where $X^{!\Phi} = \Phi$ and $\mathcal{E}^{!\Phi} = \{E \cap \Phi \mid E \in \mathcal{E} \text{ and } E \cap \Phi \neq \varnothing\}$
- $\tau_\mathcal{E}^{!\Phi} = \{U \cap \Phi \mid U \in \tau_\mathcal{E}\}$

We notice that as proven in Proposition 3 $\left(X^{!\Phi}, \tau_\mathcal{E}^{!\Phi}\right)$ is indeed a topological space.

Now we present the algorithms to compute the conditional update on the qualitative evidence frame and on the evidential topology, presenting also their running-time analysis and correctness according to Definition 31. The algorithms can be found respectively in algorithms 1 and 2.

Algorithm 1: Conditionalization on the set of evidence

Input : A set of evidence $\mathcal{E}$ A proposition Φ
Output : An updated set of evidence $\mathcal{E}^{!\Phi}$

$\mathcal{E}^{!\Phi} \leftarrow \varnothing$;
forall $E \in \mathcal{E}$ **do**
  if $E \cap \Phi \neq \varnothing$ **then**
    $\mathcal{E}^{!\Phi} \leftarrow \mathcal{E}^{!\Phi} \cup \{E \cap \Phi\}$;
  end
end
return $\mathcal{E}^{!\Phi}$

Proposition 4. Algorithm 1 and Algorithm 2 are both polynomial in the size of the respective input.

Proof. We proceed by analyzing Algorithm 1. In Algorithm 1, we have the following steps:

- We first initialize an empty set, denoted by $\mathcal{E}^{!\Phi}$, which will contain the updated evidence at the end of the algorithm. This initialization requires $\mathcal{O}(1)$ time.
- Next, the algorithm iterates over each evidence set $E \in \mathcal{E}$, which requires $\mathcal{O}(|\mathcal{E}|)$ iterations. For each E, we check whether $E \cap \Phi = \varnothing$. Computing this intersection requires at most $\mathcal{O}(\min(|\max \mathcal{E}|, |\Phi|))$ time.

Algorithm 2: Conditionalization on the evidential topology

Input : An evidential topology $\tau_{\mathcal{E}}$
A proposition Φ

Output: An updated evidential topology $\tau_{\mathcal{E}}^{!\Phi}$

```
τ_ℰ^{!Φ} ← ∅
forall U ∈ τ_ℰ do
    U^{!Φ} ← U ∩ Φ
    τ_ℰ^{!Φ} ← τ_ℰ^{!Φ} ∪ U^{!Φ}
end
return τ_ℰ^{!Φ}
```

- If the intersection is non-empty, we add $E \cap \Phi$ to $\mathcal{E}^{!\Phi}$; otherwise, the algorithm immediately returns $\mathcal{E}^{!\Phi}$. Each of these operations requires $\mathcal{O}(1)$ time.

Putting everything together, the overall time complexity of the algorithm is $\mathcal{O}(|\mathcal{E}| \cdot \min(|\max \mathcal{E}|, |\Phi|))$. Therefore, Algorithm 1 runs in polynomial time with respect to the size of the set of evidence $\mathcal{E}$ and the size of the proposition Φ.

By a similar reasoning as for Algorithm 1, also Algorithm 2 is polynomial in the size of the topology and the proposition with running time $\mathcal{O}\left(|\tau_{\mathcal{E}}| \cdot \min\left(|\max \tau_{\mathcal{E}}|, |\Phi|\right)\right)$.

Hence, both Algorithm 1 and Algorithm 2 run in polynomial time with respect to the size of their respective inputs, as claimed. $\square$

Now we define the characteristic functions of the set of evidence after conditionalization and the evidential topology after conditionalization in order to simplify the correctness proof of Algorithm 1 and Algorithm 2.

Definition 32 (Characteristic functions of the set of evidence and the evidential topology after conditionalization)**.** Let $\mathcal{E}$ be a set of evidence, $\tau_{\mathcal{E}}$ the evidential topology generated by $\mathcal{E}$ and $\Phi \subseteq X$ any proposition. Then we can define the following functions $\chi_{\mathcal{E}^{!\Phi}} : \mathcal{E} \times \Phi \longrightarrow \{0, 1\}$ and $\chi_{\tau_{\mathcal{E}}^{!\Phi}} : \tau_{\mathcal{E}} \times \Phi \longrightarrow \{0, 1\}$ such that:

$$\chi_{\mathcal{E}^{!\Phi}}(E, \Phi) = \begin{cases} 1 & \text{if } E \cap \Phi \neq \varnothing, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \chi_{\tau_{\mathcal{E}}^{!\Phi}}(U, \Phi) = 1.$$

The correctness of these functions is trivial from Definition 31. Then we claim that Algorithm 1 and Algorithm 2 are both correct with respect to the functions in Definition 32. To prove this claim consider the following proposition.

Proposition 5. Let $(X, \mathcal{E})$ be a qualitative evidence frame, and let $\tau_{\mathcal{E}}$ be the evidential topology generated by $\mathcal{E}$. Let us write e for Algorithm 1 and t for Algorithm 2. Then, given a proposition $\Phi \subseteq X$, for all evidence $E \in \mathcal{E}$ and all open sets $U \in \tau_{\mathcal{E}}$, we have the following equivalences:

$$E \cap \Phi \in e(\mathcal{E}, \Phi) \iff \chi_{\mathcal{E}^{!\Phi}}(E, \Phi) = 1, \tag{$\clubsuit_1$}$$

$$U \cap \Phi \in t(\tau_{\mathcal{E}}, \Phi) \iff \chi_{\tau_{\mathcal{E}}^{!\Phi}}(U, \Phi) = 1. \tag{$\clubsuit_2$}$$

Proof. Let $\Phi \subseteq X$ be a proposition. Then we will prove separately the two statements:

($\clubsuit_1$) Let $E \in \mathcal{E}$ be an arbitrary piece of evidence. Now we show the two directions of our claim:

($\Rightarrow$) Assume that $E \cap \Phi \in e(\mathcal{E}, \Phi)$ then it means that $E \cap \Phi \neq \varnothing$, by the definition of e. Since $E \cap \Phi \neq \varnothing$, then it holds that $\chi_{\mathcal{E}^{!\Phi}}(E, \Phi) = 1$.

($\Leftarrow$) Assume that $\chi_{\mathcal{E}^{!\Phi}}(E, \Phi) = 1$, then it means that $E \cap \Phi \neq \varnothing$. If that is the case then for $E \in \mathcal{E}$, the if statements in Lines 3-6 will update $\mathcal{E}^{!\Phi}$ adding $E \cap \Phi$ to it. Then it means that $E \cap \Phi \in e(\mathcal{E}, \Phi)$.

Since both directions hold, we can claim that for an arbitrary $E \in \mathcal{E}$ the claim holds. Since $E \in \mathcal{E}$ was arbitrary it holds for all evidence.

($\clubsuit_2$) Let $U \in \tau_{\mathcal{E}}$ be an arbitrary open set. The left-to-right direction holds trivially, since $\chi_{\tau_{\mathcal{E}}^{!\Phi}}(U, \Phi)$ is a constant function. It remains to prove the right-to-left direction.

Assume that $\chi_{\tau_{\mathcal{E}}^{!\Phi}}(U, \Phi) = 1$. Then this holds for all $U \in \tau_{\mathcal{E}}$. By the construction of t, for each open set in the iteration (Lines 2–5), we update it by taking its intersection with Φ (Line 3) and then adding this intersection to the updated evidential topology (Line 4). Consequently, for the arbitrary open set U, we have $U \cap \Phi \in t(\tau_{\mathcal{E}}, \Phi)$.

Since both directions hold, we conclude that the claim is true for an arbitrary $U \in \tau_{\mathcal{E}}$. Because U was chosen arbitrarily, the claim holds for all evidence.

□

A core element of the qualitative layer is the frame of justification. As we established at the beginning of this section, we do not assume any precise frame of justification, but we consider any admissible one according to Definition 18. Thus, establishing a unique way to update this object is not clearly definable, since any update will directly depend on the definition, and because the definition itself makes it very difficult if not impossible to give a unique procedure to update.

Furthermore, notice that even if some possible frames of justification might be very easy to update by taking non-empty intersections between the open sets in the frame of justification and the proposition on which we are conditioning (since every open set of the topology is updated by the intersection with the proposition), this is not the case for all possible frames of justification. Here we present an example where this does not occur using $\mathcal{J}_{\mathrm{SD}}$ as the frame of justification.

Example 1. Consider the qualitative evidence frame $(X, \mathcal{E})$, with the space $X = \{1, 2, 3, 4\}$ and the set of evidence $\mathcal{E} = \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Therefore the topology $\tau_{\mathcal{E}}$ generated by $\mathcal{E}$ as a subbase is the following:

$$\tau_{\mathcal{E}} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}.$$

Consider now a proposition $\Phi \subseteq X$, such that $\Phi = \{3, 4\}$. Thus applying Definition 31 we get that the new space is given by Φ and the updated set of evidence and evidential topology are the following:

- $\mathcal{E}^{!\Phi} = \{\{3\}, \{4\}\}$
- $\tau_{\mathcal{E}}^{!\Phi} = \{\varnothing, \{3\}, \{4\}, \{3, 4\}\}$

Let us now compute the frame of justification $\mathcal{J}_{\mathrm{SD}}$ with respect to the original topology and with respect to the updated topology. To differentiate them we use the notation $\mathcal{J}_{\mathrm{SD}}^{!\Phi}$ to denote the latter.

- The original frame of justification is the following:

$$\mathcal{J}_{\mathrm{SD}} = \{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}.$$

- The updated frame of justification is the following:

$$\mathcal{J}^{!\Phi}_{\mathrm{SD}} = \{\{3,4\}\}.$$

Notice now that considering the set $\{U \cap \Phi \mid U \in \mathcal{J}_{\mathrm{SD}} \text{ and } U \cap \Phi \neq \varnothing\}$ to update $\mathcal{J}_{\mathrm{SD}}$ to $\mathcal{J}^{!\Phi}_{\mathrm{SD}}$ does not give the desired result. Indeed

$$\mathcal{J}^{!\Phi}_{\mathrm{SD}} = \{\{3,4\}\} \neq \{\{3\},\{4\},\{3,4\}\} = \{U \cap \Phi \mid U \in \mathcal{J}_{\mathrm{SD}} \text{ and } U \cap \Phi \neq \varnothing\}.$$

Thus, we define the conditional update on the frame of justification, just by considering it with respect to the updated topology, according to Definition 31.

Definition 33 (Conditionalization on the frame of justification)**.** Let $(X, \mathcal{E})$ be a qualitative evidence frame, let $\tau_{\mathcal{E}}$ be the evidential topology generated by $\mathcal{E}$ as a subbase, and let $\mathcal{J}$ be a frame of justification defined with respect to $(X, \mathcal{E}, \tau_{\mathcal{E}})$. Given a proposition $\Phi \subseteq X$, we apply the conditional update according to Definition 31 to obtain a new qualitative evidence frame $(\Phi, \mathcal{E}^{!\Phi})$ and a new evidential topology $\tau^{!\Phi}_{\mathcal{E}}$. We then denote by $\mathcal{J}^{!\Phi}$ the frame of justification recomputed with respect to the updated structure $(\Phi, \mathcal{E}^{!\Phi}, \tau^{!\Phi}_{\mathcal{E}})$. In other words, $\mathcal{J}^{!\Phi}$ is the frame of justification of the updated qualitative evidence frame after conditioning on Φ.

Notice that this definition assumes that the frame of justification is given by a functional characterization. We adopt this assumption because it enables our construction and provides a principled way to update the object in question. Moreover, this aligns with the work in (Pinto Prieto et al., 2023; Pinto Prieto, 2024), where the frames of justification considered are always defined via such a functional characterization. However, we acknowledge that this may restrict the full generality of the MLBM framework. Nevertheless, the procedure still covers the two canonical cases, $\mathcal{J}_{\mathrm{DS}}$ and $\mathcal{J}_{\mathrm{SD}}$, which are of primary interest in our comparison with Dempster-Shafer theory and topological evidence models.[4]

Moreover, we assume that the considered frame of justification, whose characteristic function takes as input the qualitative evidence frame and the evidential topology, can be computed in polynomial time with respect to the size of its input.

2.3 *Conditionalization on the quantitative layer*

Let us now turn to the quantitative layer. In this layer, we assume a quantitative evidence frame $(X, \mathcal{E}_{\mathrm{Q}})$, where $\mathcal{E}_{\mathrm{Q}}$ is a set of evidence, where each piece of evidence is associated with a value p in the real interval $(0, 1)$. From this frame, we compute a δ function that yields a mass function over the quantitative evidence set $\mathcal{E}_{\mathrm{Q}}$. Our goal here is to determine how the weights should be updated, using only the tools provided by our model.

As noted in Section 2.1, we cannot directly apply any of the standard update rules discussed there. Doing so would implicitly revert our model to the very theoretical frameworks we aim to abstract away from. Therefore,

4 We thank Dr. Aybüke Özgün for pointing out this limitation during the defence of this thesis.

the update must be derived in a way that remains faithful to the original motivations and constraints of our model, without borrowing unjustified assumptions from external theories.

Furthermore, notice that we want to obtain an update that matches the result of the update defined in Section 2.2 for the qualitative layer. This means that after the conditional update, all the pieces of evidence in $\mathcal{E}^{!\Phi}$ should receive a unique updated weight.

A first possibility would be to employ our δ function on the quantitative set of evidence. This function provides a mechanism for aggregating evidence over the same set of states when it originates from different sources. A natural idea, then, is to incorporate a new certain piece of evidence by applying the δ function to the union of the existing quantitative set of evidence and the newly acquired certain information, i.e. $(\Phi, 1)$.

At first glance, this procedure may appear to be a way of updating consistent with our model. However, notice that according to Definition 17, for any $(E, p) \in \mathcal{E}_Q$ we have that $p \in (0, 1)$. So we first need to allow our quantitative set to include certain piece of evidence, which means we have to allow $p = 1$, so $p \in (0, 1]$. Thus, consider the following proposition which shows that under reasonable assumptions[5], we can get that applying the δ function to combine the new certain information will give positive weights only to the pieces of evidence in $\mathcal{E}_Q$ which contain Φ.

Proposition 6. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and $\mathcal{E}$ the corresponding set of evidence, such that $\mathcal{E}_Q \subseteq \mathcal{E} \times (0, 1]$. Assume that

- there is an evidence $E \in \mathcal{E}$ such that $(E, 1) \in \mathcal{E}_Q$,
- and for no other $E_i \in \mathcal{E}$ it holds that neither $(E_i, 1) \in \mathcal{E}_Q$.

Then we claim that for all $U \in \wp(\mathcal{E})$, holds that

$$\delta(U) = 0 \iff E \notin U,$$

where δ is defined as in Definition 21.

Proof. We will prove the statement showing the two directions of the implication.

($\Rightarrow$) Let $U \in \wp(\mathcal{E})$ be arbitrary evidence such that $\delta(U) = 0$. Then by Definition 21 we can claim that

$$\delta(U) = \prod_{E_i \in U} p_i \prod_{E_i \notin U} (1 - p_i) = 0.$$

Then it must hold that

$$\prod_{E_i \in U} p_i = 0 \quad \text{or} \quad \prod_{E_i \notin U} (1 - p_i) = 0$$

This is equivalent to the following two cases:

1. for some $E_i \in U$ we have that there must be $(E_i, 0) \in \mathcal{E}_Q$

5 We claim that these are reasonable assumptions, since even in (Pinto Prieto, 2024, pp. 22-23) it is assumed that for each $(E, p) \in \mathcal{E}_Q$ it holds that $p \neq 1$ and $p \neq 0$. In fact, it is stated there that each weight can be read as the value of a non-dogmatic (no focal element has weight 1) and non-vacuous (the total space has weight less than 1) simple (there is only one focal element other than the total space) basic probability assignment. The reader might notice that these assumptions match our requirements, in the sense that we require that only one piece of evidence has to be certain.

2. for some $E_i \notin U$ we have that there must be $(E_i, p_i) \in \mathcal{E}_Q$ such that $1 - p_i = 0$, i.e. $(E_i, 1) \in \mathcal{E}_Q$.

First, notice that by construction there is no evidence in $E_i \in \mathcal{E}$ such that $(E_i, 0) \in \mathcal{E}_Q$, since $p \in (0, 1]$. Thus, the first case can never happen. Similarly, we know by assumption that the only evidence $E_i \in \mathcal{E}$ such that $(E_i, 1) \in \mathcal{E}_Q$ is E. Therefore, we can conclude that

$$\prod_{E_i \in U} p_i > 0.$$

Then it must be that

$$\prod_{E_i \in U} p_i = 0.$$

Thus, for some $(E_i, 1) \in \mathcal{E}_Q$ we have that $E_i \notin U$. But as we notice above the only candidate can be E. Hence, we can conclude that $E \notin U$. Since $U \in \wp(\mathcal{E})$ was arbitrary it holds for all $U \in \wp(\mathcal{E})$, establishing in this way the left to right direction.

($\Leftarrow$) Let $U \in \wp(\mathcal{E})$ be an arbitrary evidence such that $E \notin U$, then

$$\delta(U) = \prod_{E_i \in U} p_i \prod_{E_i \notin U} (1 - p_i).$$

Since $(E, p_E) = (E, 1) \in \mathcal{E}_Q$. Then it means that

$$\begin{aligned}
\delta(U) &= \prod_{E_i \in U} p_i \prod_{E_i \notin U} (1 - p_i) \\
&= \prod_{E_i \in U} p_i \prod_{\substack{E_i \notin U \\ E_i \neq E}} (1 - p_i) \cdot (1 - p_E) \\
&= \prod_{E_i \in U} p_i \prod_{\substack{E_i \notin U \\ E_i \neq E}} (1 - p_i) \cdot (1 - 1) \\
&= \prod_{E_i \in U} p_i \prod_{\substack{E_i \notin U \\ E_i \neq E}} (1 - p_i) \cdot 0 \\
&= 0
\end{aligned}$$

Since $U \in \wp(\mathcal{E})$ was an arbitrary set such that $E_i \notin U$, then it holds for all U.

Since both direction of the claim hold separately, we can conclude the claim holds as well. □

This method, however, does not work, since we can show that when we employ the function in this way, we do not match the desired result of replicating the qualitative update. Furthermore, we obtain the undesired outcome of having sets intersecting our proposition which nonetheless contain pieces of evidence that should be deleted.

Example 2. Consider the quantitative evidence frame $(X, \mathcal{E}_Q)$, where

- $X = \{1, 2, 3\}$ and
- $\mathcal{E}_Q = \{(\{1\}, 0.2), (\{1, 3\}, 0.6), (\{3\}, 0.4)\}$.

Consider that we get from a reliable source that $\Phi = \{1, 2\}$ is certain. Then we want to update our model according to the proposition that we believe to be certain. First we try to apply the δ function on the quantitative set of evidence $\mathcal{E}_Q \cup \{(\Phi, 1)\}$. Since the conditions of Proposition 6 are met, we can directly compute that all the subsets of $\mathcal{E}_Q \cup \{(\Phi, 1)\}$ which do not contain Φ will get a mass of 0.

For the sake of exposition, we rename the elements of $\mathcal{E}_Q$ as E_1, E_2, and E_3, following the order introduced above. Since the quantitative set of evidence on which we compute δ contains 4 elements, the function δ is defined on $2^4 = 16$ subsets, i.e., on all elements of the power set of the quantitative set of evidence.

However, we only need to compute δ on half of these subsets. This follows from a simple combinatorial argument: given a fixed piece of certain evidence, exactly half of the subsets of the set of evidence contain it, while the other half do not. More precisely, if the set of evidence has cardinality n, then the number of subsets containing a given element is 2^{n-1}. Hence, in our case, we only need to evaluate δ on $2^{4-1} = 8$ subsets, as the remaining ones are assigned value 0.

From the computation, we get the following values:

$$
\begin{aligned}
\delta(\{\varnothing\}) &= 0 \\
\delta(\{E_1\}) &= 0 \\
\delta(\{E_2\}) &= 0 \\
\delta(\{E_3\}) &= 0 \\
\delta(\{\Phi\}) &= (1-p_1)\cdot(1-p_2)\cdot(1-p_3)\cdot p_\Phi && = 0.8\cdot 0.4\cdot 0.6\cdot 1 = 0.192 \\
\delta(\{E_1,\Phi\}) &= p_1\cdot(1-p_2)\cdot(1-p_3)\cdot p_\Phi && = 0.2\cdot 0.4\cdot 0.6\cdot 1 = 0.048 \\
\delta(\{E_2,\Phi\}) &= (1-p_1)\cdot p_2\cdot(1-p_3)\cdot p_\Phi && = 0.8\cdot 0.6\cdot 0.6\cdot 1 = 0.288 \\
\delta(\{E_3,\Phi\}) &= (1-p_1)\cdot(1-p_2)\cdot p_3\cdot p_\Phi && = 0.8\cdot 0.4\cdot 0.4\cdot 1 = 0.128 \\
\delta(\{E_1,E_2\}) &= 0 \\
\delta(\{E_1,E_3\}) &= 0 \\
\delta(\{E_2,E_3\}) &= 0 \\
\delta(\{E_1,E_2,\Phi\}) &= p_1\cdot p_2\cdot(1-p_3)\cdot p_\Phi && = 0.2\cdot 0.6\cdot 0.6\cdot 1 = 0.072 \\
\delta(\{E_1,E_3,\Phi\}) &= p_1\cdot(1-p_2)\cdot p_3\cdot p_\Phi && = 0.2\cdot 0.4\cdot 0.4\cdot 1 = 0.032 \\
\delta(\{E_2,E_3,\Phi\}) &= (1-p_1)\cdot p_2\cdot p_3\cdot p_\Phi && = 0.8\cdot 0.6\cdot 0.4\cdot 1 = 0.192 \\
\delta(\{E_2,E_3,E_3\}) &= 0 \\
\delta(\{E_1,E_2,E_3,\Phi\}) &= p_1\cdot p_2\cdot p_3\cdot p_\Phi && = 0.2\cdot 0.6\cdot 0.4\cdot 1 = 0.048
\end{aligned}
$$

From the results presented in Example 2, we can observe two very problematic features of this method.

First, certain combinations receive a positive value even though we would like to discard them in the update. Consider, for instance, the set $\{E_3, \Phi\} = \{\{3\}, \{1, 2\}\}$. In the computation, this set receives a mass greater than 0, despite the fact that we would like to discard the evidence E_3 entirely, since it conflicts with the certainty of $\Phi = \{1, 2\}$.

Furthermore, this example does not match the update at the qualitative level. According to the qualitative update defined in Section 2.2, the updated set of evidence would be $\mathcal{E}^{!\Phi} = \{\{1\}\}$. Thus, the qualitative conditional update should output a clear single weight to associate with $\{1\}$. However, the δ function applied to $\mathcal{E}_Q \cup \{(\Phi, 1)\}$ produces a complex mass function with multiple positive values distributed across different subsets (e.g., $\delta(\{E_1, \Phi\}) = 0.048$, $\delta(\{E_2, \Phi\}) = 0.288$, $\delta(\{E_1, E_2, \Phi\}) = 0.072$, etc.), none of which directly correspond to a single updated weight for $\{1\}$. Hence, this method fails to replicate the qualitative update result.

One might try to resolve this issue by simply transferring the mass from such combinations to the intersection of each evidence item. However, this approach leads to a second problem: multiple distinct combinations can produce mass for the same intersection set, causing the resulting function to be ill-defined. For example, the intersection of E_1 and Φ is $\{1\} \cap \{1, 2\} = \{1\}$, but the intersection of E_2 and Φ is $\{1, 3\} \cap \{1, 2\} = \{1\}$ as well. Both $\delta(\{E_1, \Phi\}) = 0.048$ and $\delta(\{E_2, \Phi\}) = 0.288$ would then assign mass to the same intersection $\{1\}$, leading to conflicting assignments and violating the well-definedness of the function.

Thus, we move to analyze another method, namely applying the function δ to combine each piece of evidence in $\mathcal{E}_Q$ individually with the certain evidence on Φ. More precisely, for each $(E_i, p_i) \in \mathcal{E}_Q$, we consider the combination of $\{E_i, \Phi\}$ via the δ function.

By the same reasoning as in Proposition 6, but now applied to a pair $\{E_i, \Phi\}$ instead of the full set, we can claim that the only subsets receiving positive mass are those that contain Φ. In fact, for the pair $\{E_i, \Phi\}$, the δ function yields:

$$\begin{aligned}
\delta(\{E_i, \Phi\}) &= p_i \cdot 1 = p_i, \\
\delta(\{\Phi\}) &= (1 - p_i) \cdot 1 = 1 - p_i, \\
\delta(\{E_i\}) &= 0, \\
\delta(\emptyset) &= 0.
\end{aligned}$$

This outcome is, in principle, desirable: the evidence E_i survives with weight p_i only when combined with Φ, that we could associate with the intersection when non empty, while the certain evidence Φ alone receives the complementary weight $1 - p_i$.

However, this method fails when we have multiple pieces of evidence that interact in more complex ways. In particular, we cannot handle cases where two different evidence items where the same intersection receives contributions from multiple sources, leading to an ill-defined combined mass function. We illustrate this problem with two examples.

Example 3. Consider again the quantitative evidence frame from Example 2:

$$\mathcal{E}_Q = \{(\{1\}, 0.2),\ (\{1, 3\}, 0.6),\ (\{3\}, 0.4)\}, \qquad \Phi = \{1, 2\}.$$

First, we apply the pairwise δ combination for each E_i with Φ and transfer the mass to the intersections:

- For $E_1 = \{1\}$: $\delta(\{E_1, \Phi\}) = 0.2$ transferred to $\{1\} \cap \{1, 2\} = \{1\}$.
- For $E_2 = \{1, 3\}$: $\delta(\{E_2, \Phi\}) = 0.6$ transferred to $\{1, 3\} \cap \{1, 2\} = \{1\}$.
- For $E_3 = \{3\}$: $\delta(\{E_3, \Phi\}) = 0.4$ transferred to $\{3\} \cap \{1, 2\} = \emptyset$ (discarded).

After this transfer, we obtain a new quantitative evidence set:

$$\mathcal{E}'_Q = \{(\{1\}, 0.2),\ (\{1\}, 0.6)\}.$$

Now, we apply the δ function to $\mathcal{E}'_Q$. Let us denote $F_1 = \{1\}$ with weight 0.2, and $F_2 = \{1\}$ with weight 0.6. The power set of $\{F_1, F_2\}$ has $2^2 = 4$ subsets. The full computation yields:

$$\begin{aligned}
\delta(\emptyset) &= 0 \\
\delta(\{F_1\}) &= p_1 \cdot (1 - p_2) && = 0.2 \cdot (1 - 0.6) = 0.2 \cdot 0.4 = 0.08 \\
\delta(\{F_2\}) &= (1 - p_1) \cdot p_2 && = (1 - 0.2) \cdot 0.6 = 0.8 \cdot 0.6 = 0.48 \\
\delta(\{F_1, F_2\}) &= p_1 \cdot p_2 && = 0.2 \cdot 0.6 = 0.12
\end{aligned}$$

The problem is now evident. All three non-empty subsets $\{F_1\}$, $\{F_2\}$, and $\{F_1, F_2\}$ produce positive masses. However, since F_1 and F_2 refer to the exact same set $\{1\}$, these masses should all be assigned to $\{1\}$ in the final mass function. Yet the δ function provides no rule for merging them. The set $\{1\}$ would receive three different contributions (0.08, 0.48, and 0.12) that cannot be combined into a single mass without violating the definition of a mass function. Hence, this method fails to produce a well-defined updated mass function.

What we need, therefore, is a function capable of handling both problems simultaneously.

On the one hand, as shown in Example 2, when we apply the δ function to the entire set $\mathcal{E}_Q \cup \{(\Phi, 1)\}$, we obtain positive mass for combinations that contain evidence conflicting with Φ (such as $E_3 = \{3\}$). This is unacceptable, because evidence that does not intersect the certain proposition Φ should be discarded entirely in the update. The δ function treats all evidence symmetrically and does not give priority to the certain evidence $(\Phi, 1)$ in a way that eliminates conflicting items.

On the other hand, as shown in Example 3, when we try to circumvent this problem by applying the δ function to each evidence, and then transferring masses to intersections, we can end up with a new quantitative evidence set where multiple distinct weights are associated to the same intersection set. Applying the δ function to this new set then produces multiple positive masses that all should be assigned to the same set $\{1\}$, but the δ function provides no principled way to merge them into a single mass value.

Thus, what we require for our quantiatative update can be summed in what follows:

1. When a piece of evidence has an intersection with Φ which is unique among all pieces of evidence, its weight remains unchanged, replicating the behavior of the δ function applied to the single pair $\{E_i, \Phi\}$.
2. If several pieces of evidence $E_i, E_j, \ldots$ all have the same intersection with Φ, i.e., $E_i \cap \Phi = E_j \cap \Phi$, then the function must combine their respective weights into a single, unique mass for $E_i \cap \Phi$.

Thus consider the following function

Definition 34 (Merging Function)**.** Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and let $\Phi \subseteq X$ be a proposition, which we treat as certain evidence. Define a relation $\sim_\Phi$ on $\mathcal{E}_Q$ by:

$$(E_1, p_1) \sim_\Phi (E_2, p_2) \quad \Longleftrightarrow \quad E_1 \cap \Phi = E_2 \cap \Phi.$$

This is an equivalence relation. For each $(E, p) \in \mathcal{E}_Q$, denote its equivalence class under $\sim_\Phi$ by $[E]_\Phi$ and let $|[E]_\Phi| = n$ be its cardinality. Then we define the *merging function* $\delta^t : \mathcal{E}_Q/_{\sim_\Phi} \longrightarrow [0, 1]$ as:

$$\delta^t([E]_\Phi) = 1 - \prod_{i=1}^{n}(1 - p_i),$$

where p_i are the weights of the evidence items in $[E]_\Phi$.

Intuitively, the reader might think of this function as subtracting from the certainty in a proposition the level of uncertainty derived from each piece of evidence, assuming they all refer to the same intersection. In fact, as noted in (Pinto Prieto, 2024, pg. 22), given a weighted piece of evidence $(E, p) \in \mathcal{E}_Q$, the value $1 - p$ represents the uncertainty associated with that piece of evidence. Since all sources are assumed to be independent, we can encode the combined uncertainty for a specific class $[E]_\Phi \in \mathcal{E}_Q/_{\sim_\Phi}$ by taking the product of all the uncertainties associated with the pieces of evidence in $[E]_\Phi$. The resulting certainty is then given by the difference between total certainty (i.e., 1) and the uncertainty associated with the equivalence class, which represents the updated evidence.

Now, we want to show that this function respects the requirements listed above.

Proposition 7 (Properties of the Merging Function)**.** Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and $\Phi \subseteq X$ a proposition. Define the equivalence relation $\sim_\Phi$ and the merging function δ^t as in Definition 34. Then δ^t satisfies the following properties:

(1) If $E_i \in \mathcal{E}_Q$ is such that for all other $E_j \in \mathcal{E}_Q$ with $j \neq i$ we have $E_i \cap \Phi \neq E_j \cap \Phi$, then $\delta^t([E_i]_\Phi) = p_i$.

(2) If several pieces of evidence $E_i, E_j, \ldots$ all have the same intersection with Φ, i.e., $E_i \cap \Phi = E_j \cap \Phi$, then δ^t combines their respective weights into a single value for the equivalence class $[E]_\Phi$, given by

$$\delta^t([E]_\Phi) = 1 - \prod_{i=1}^{m}(1 - p_i)$$

where $m = |[E]_\Phi|$.

Proof. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and $\Phi \subseteq X$ a proposition. Now we aim to prove the two properties separately

(1) Assume that $(E_i, p_i) \in \mathcal{E}_Q$ is such that for all other $(E_j, p_j) \in \mathcal{E}_Q$ with $j \neq i$ we have $E_i \cap \Phi \neq E_j \cap \Phi$. Consider the equivalence class $[E_i]_\Phi$ under $\sim_\Phi$. By definition of $\sim_\Phi$, two evidence items belong to the same class if and only if their intersections with Φ are equal. Since E_i has a unique intersection with Φ (i.e., no other evidence shares this intersection), the class $[E_i]_\Phi$ contains exactly one element: (E_i, p_i) itself. Therefore, $n = |\,[E_i]_\Phi\,| = 1$. Substituting into the definition of δ^t:

$$\delta^t([E_i]_\Phi) = 1 - \prod_{k=1}^{1}(1 - p_k) = 1 - (1 - p_i) = p_i.$$

Hence, $\delta^t([E_i]_\Phi) = p_i$. Since $(E_i, p_i) \in \mathcal{E}_Q$ such that for all other $(E_j, p_j) \in \mathcal{E}_Q$ with $j \neq i$ we have $E_i \cap \Phi \neq E_j \cap \Phi$ was arbitary, the property holds for all such $(E_i, p_i) \in \mathcal{E}_Q$. Thus, 1. holds.

(2) Assume that several pieces of evidence $(E_1, p_1), (E_2, p_2), \ldots, (E_n, p_n) \in \mathcal{E}_Q$ all have the same intersection with Φ, i.e., $E_1 \cap \Phi = E_2 \cap \Phi = \cdots = E_n \cap \Phi = E$. By definition of $\sim_\Phi$, all these evidence items belong to the same equivalence class $[E]_\Phi$. Let this class contain exactly these n items. Then $n \geqslant 1$ and the class has cardinality at least n. The merging function δ^t applied to this class yields:

$$\delta^t([E]_\Phi) = 1 - \prod_{k=1}^{m} (1 - p_k),$$

where $m = |[E]_\Phi|$ and p_k are the weights of all evidence items in the class. Since the product runs over all items in the class, the weights of $E_1, \ldots, E_n$ are included. Thus, δ^t produces a single combined value for the entire class, which by construction corresponds to the unique intersection S. This establishes the second property. We have discharged the assumption that the evidence items share the same intersection.

Since both properties hold under their respective assumptions, then we can conclude that the claim holds. □

Now we can define the conditional update on the quantitative layer as follows:

Definition 35 (Conditionalization on the quantitative evidence frame)**.** Given a quantitative evidence frame $(X, \mathcal{E}_Q)$ and a proposition $\Phi \subseteq X$, define a relation $\sim_\Phi$ on $\mathcal{E}_Q$ by:

$$(E_1, p_1) \sim_\Phi (E_2, p_2) \iff E_1 \cap \Phi = E_2 \cap \Phi.$$

Let $[E]_\Phi$ denote the equivalence class of $(E, p) \in \mathcal{E}_Q$ under $\sim_\Phi$, and let $|[E]_\Phi| = n$ be its cardinality. The conditionalization of the quantitative evidence frame on Φ is the pair $(X^{!\Phi}, \mathcal{E}_Q^{!\Phi})$, where:

- $X^{!\Phi} = \Phi$,
- $\mathcal{E}_Q^{!\Phi} = \left\{ \left(E \cap \Phi,\ \delta^t([E]_\Phi)\right) \;\middle|\; (E, p) \in \mathcal{E}_Q \text{ and } E \cap \Phi \neq \varnothing \right\}$,

and δ^t is the merging function defined in Definition 34. That is, to each equivalence class $[E]_\Phi$ we associate the weight:

$$\delta^t([E]_\Phi) = 1 - \prod_{i=1}^{n} (1 - p_i),$$

where p_i are the weights of the evidence items in $[E]_\Phi$.

Now, we proceed presenting the algorithm to compute the update on the qualitative set of evidence, presented in Algorithm 3. Furthermore, we will show also its running-time analysis and its correctness.

Proposition 8. Algorithm 3 is polynomial in the size of it input.

Proof. In Algorithm 3 we have the following steps.

- We first initialize an empty set, denoted by $\mathcal{E}_Q^{!\Phi}$, which will contain the updated evidence at the end of the algorithm. This initialization requires $\mathcal{O}(1)$ time.
- Next, the algorithm iterates over each evidence set $(E_i, p_i) \in \mathcal{E}_Q$, which requires $\mathcal{O}\left(\left|\mathcal{E}_Q\right|\right)$ iterations. For each E_i, we check whether $E_i \cap \Phi = \varnothing$. Computing this intersection requires at most $\mathcal{O}\left(\min\left(\left|\max_E \mathcal{E}_Q\right|, |\Phi|\right)\right)$ time.

Algorithm 3: Conditionalization on the quantitative set of evidence

Input : A quantitative set of evidence $\mathcal{E}_{\mathrm{Q}}$
A proposition Φ

Output: An updated quantitative set of evidence $\mathcal{E}^{!\Phi}_{\mathrm{Q}}$

$\mathcal{E}^{!\Phi}_{\mathrm{Q}} \leftarrow \varnothing$
forall $(E_i, p_i) \in \mathcal{E}_{\mathrm{Q}}$ **do**
  if $E_i \cap \Phi \neq \varnothing$ **then**
    $\mathcal{E}^{!\Phi}_{\mathrm{Q}} \leftarrow \mathcal{E}^{!\Phi}_{\mathrm{Q}} \cup \{(E \cap \Phi), p_i\}$
  end
end
while $(E_1, p_1), (E_2, p_2) \in \mathcal{E}^{!\Phi}_{\mathrm{Q}}$ *with* $E_1 = E_2$ **do**
  $p \leftarrow p_1 + p_2 - (p_1 \cdot p_2)$
  $\mathcal{E}^{!\Phi}_{\mathrm{Q}} \leftarrow \mathcal{E}^{!\Phi}_{\mathrm{Q}} \setminus \{(E_1, p_1), (E_2, p_2)\} \cup \{(E_1, p)\}$
end
return $\mathcal{E}^{!\Phi}_{\mathrm{Q}}$

- If the intersection is non-empty, we add $(E_i \cap \Phi, p_i)$ to $\mathcal{E}^{!\Phi}_{\mathrm{Q}}$; otherwise, no element is added. Each insertion requires $\mathcal{O}(1)$ time. Hence, this entire step runs in $\mathcal{O}\left(\left|\mathcal{E}_{\mathrm{Q}}\right| \cdot \min\left(\left|\max \mathcal{E}_{\mathrm{Q}}\right|, |\Phi|\right)\right)$ time.
- Next, the algorithm repeatedly searches for pairs $(E_1, p_1), (E_2, p_2) \in \mathcal{E}^{!\Phi}_{\mathrm{Q}}$ such that $E_1 = E_2$. In the worst case, $\mathcal{E}^{!\Phi}_{\mathrm{Q}}$ contains at most $\left|\mathcal{E}_{\mathrm{Q}}\right|$ elements, and each iteration of the loop reduces its size by at least one. Hence, the loop executes at most $\mathcal{O}\left(\left|\mathcal{E}_{\mathrm{Q}}\right|\right)$ times.
- In each iteration, identifying two elements with identical evidence sets may require scanning $\mathcal{E}^{!\Phi}_{\mathrm{Q}}$, which takes $\mathcal{O}\left(\left|\mathcal{E}_{\mathrm{Q}}\right|\right)$ time in the worst case. The subsequent update and replacement operations require $\mathcal{O}(1)$ time.
- Therefore, the total running time of the while-loop is bounded by $\mathcal{O}\left(\left|\mathcal{E}_{\mathrm{Q}}\right|^2\right)$.

Putting together the running times of the two iterations, we obtain

$$\mathcal{O}\Big(\left|\mathcal{E}_{\mathrm{Q}}\right| \cdot \min\left(\left|\max \mathcal{E}_{\mathrm{Q}}\right|, |\Phi|\right) + \left|\mathcal{E}_{\mathrm{Q}}\right|^2\Big).$$

Now observe that

$$\min\left(\left|\max \mathcal{E}_{\mathrm{Q}}\right|, |\Phi|\right) \leqslant \left|\max \mathcal{E}_{\mathrm{Q}}\right|.$$

In the worst case, the size of each evidence set is bounded by $\left|\mathcal{E}_{\mathrm{Q}}\right|$. Therefore,

$$\min\left(\left|\max \mathcal{E}_{\mathrm{Q}}\right|, |\Phi|\right) = \mathcal{O}\left(\left|\mathcal{E}_{\mathrm{Q}}\right|\right).$$

Substituting this bound give us as final result.

$$\mathcal{O}\Big(\left|\mathcal{E}_{\mathrm{Q}}\right| \cdot \left|\mathcal{E}_{\mathrm{Q}}\right| + \left|\mathcal{E}_{\mathrm{Q}}\right|^2\Big) = \mathcal{O}\Big(\left|\mathcal{E}_{\mathrm{Q}}\right|^2\Big).$$

Thus, we can conclude that Algorithm 3 is polynomial in the size of the input. □

As we did in Definition 32 for the updates of the elements of qualitative frame, we define the characteristic function for the conditional qualitative evidence set, according to Definition 35, in order to prove the correctness of Algorithm 3, following the same technique we used in Proposition 5.

Definition 36 (Characteristic function of $\mathcal{E}_{\mathrm{Q}}^{!\Phi}$)**.**

$$\chi_{\mathcal{E}_{\mathrm{Q}}^{!\Phi}}(E,p) = \begin{cases} 1 & \text{if } (E,p) \in \mathcal{E}_{\mathrm{Q}}^{!\Phi} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 9. Let $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ be a quantitative evidence frame and $\Phi \subseteq X$ a proposition. Let us call $q\left(\mathcal{E}_{\mathrm{Q}}, \Phi\right)$ the output of Algorithm 3 on input $\mathcal{E}_{\mathrm{Q}}$ and Φ. Then we claim that for all E, p it holds that

$$(E,p) \in q\left(\mathcal{E}_{\mathrm{Q}}, \Phi\right) \iff \chi_{\mathcal{E}_{\mathrm{Q}}^{!\Phi}}(E,p) = 1$$

Proof. Let $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ be a quantitative evidence frame and $\Phi \subseteq X$ a proposition. We will prove the statement showing the two directions of the implication.

($\Rightarrow$) Let $(E,p) \in q\left(\mathcal{E}_{\mathrm{Q}}, \Phi\right)$ be arbitrary. Then by Lines 2-6 we know that $E = E_i \cap \Phi$ for some $(E_i, p_i) \in \mathcal{E}_{\mathrm{Q}}$. Let $\{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|}$ be the quantitative set of evidence such that for all $(E_i, p_i) \in \{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|}$ it holds that $E_i \cap \Phi = E$. We notice that this set is exactly the equivalence class under the relation $\sim_\Phi$ as described in Definition 35. We will proceed proving our claim by induction on the cardinality of $\{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|}$, that we will denote with n.

Base case If $n = 1$, then it means that $\{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|} = \{(E_1, p_1)\}$. Then it means that after the execution of the first iteration there is no pair $(E_1, p_1), (E_2, p_2)$ such that $E_1 = E_2$. Thus, the second iteration will never begin since the condition required is always false. Thus, $p = p_1$. Then we are left to show that

$$\begin{aligned} \delta^t\left(\{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|}\right) &= \delta^t\left(\{(E_1, p_1)\}\right) \\ &= p && \text{(Proposition 7 (1))} \end{aligned}$$

Thus, we can conclude that $(E,p) = \left(E_1 \cap \Phi, \delta^t\left([E_1]_\Phi\right)\right)$. Therefore, we claim that $\chi_{\mathcal{E}_{\mathrm{Q}}^{!\Phi}}(E,p) = 1$

If $n = 2$, then it means that $\{(E_i, p_i)\}_{i \leqslant |\mathcal{E}_{\mathrm{Q}}|} = \{(E_1, p_1), (E_2, p_2)\}$. Then it means that after the execution of the first iteration, we have a pair $(E_1, p_1), (E_2, p_2)$ such that $E_1 = E_2$. Thus, the second iteration will start in order to merge the same pieces of evidence. Then by Line 11

$$p = p_1 + p_2 - (p_1 \cdot p_2).$$

Then we have that

$$\begin{aligned} p &= p_1 + p_2 - (p_1 \cdot p_2) \\ &= 1 - (1 - p_1) + p_2 - ((1 - (1 - p_1)) \cdot p_2) && (1 - (1 - p_1) = p_1) \\ &= 1 - (1 - p_1) + p_2 - (p_2 - p_2(1 - p_1)) \\ &= 1 - (1 - p_1) + p_2 - p_2 + p_2(1 - p_1) \\ &= 1 - (1 - p_1) + p_2(1 - p_1) \\ &= 1 - (1 - p_1)(1 - p_2) \end{aligned}$$

$$
\begin{aligned}
&= 1 - \prod_{i=1}^{2}(1-p_i) \\
&= \delta^t\left(\{(E_1,p_1),(E_2,p_2)\}\right)
\end{aligned}
$$

Thus, we can conclude that $(E,p) = (E_1 \cap \Phi, \delta^t([E_1]_\Phi))$. Therefore, we claim that $\chi_{\mathcal{E}_Q^{!\Phi}}(E,p) = 1$.

Induction step Assume, as induction hypothesis (IH), that the claim holds for $n \in \mathbb{N}$, i.e., for every equivalence class $\{(E_i,p_i)\}_{i\leqslant n}$ such that all $E_i \cap \Phi = E$, the algorithm returns

$$(E,p) = \left(E, \delta^t([E_i]_\Phi)\right),$$

and hence $\chi_{\mathcal{E}_Q^{!\Phi}}(E,p) = 1$.

Now consider a class of size $n+1$, namely $\{(E_i,p_i)\}_{i\leqslant n+1}$, such that for all i we have $E_i \cap \Phi = E$. After the first loop of the algorithm, all these elements appear in $\mathcal{E}_Q^{!\Phi}$ as pairs of the form (E,p_i).

By the definition of the while-loop, the algorithm repeatedly selects two elements with identical evidence sets and merges them. By the IH, after merging the first n elements, we obtain a single pair (E,e) such that

$$e = \delta^t\left(\{(E_i,p_i)\}_{i\leqslant n}\right) = 1 - \prod_{i=1}^{n}(1-p_i). \qquad (☆)$$

It remains to merge (E,e) with $(E_{n+1} \cap \Phi, p_{n+1})$, which also satisfies $E_{n+1} \cap \Phi = E$.

Then by Line 11 of the algorithm we obtain

$$p = e + p_{n+1} - (e \cdot p_{n+1}).$$

Then we are left to show what follows:

$$
\begin{aligned}
p &= e + p_{n+1} - (e \cdot p_{n+1}) \\
&= 1 - \prod_{i=1}^{n}(1-p_i) + p_{n+1} - \left(1 - \prod_{i=1}^{n}(1-p_i)\right) \cdot p_{n+1} && \text{(Equation (☆))} \\
&= 1 - \prod_{i=1}^{n}(1-p_i) + p_{n+1} - p_{n+1} + p_{n+1}\prod_{i=1}^{n}(1-p_i) \\
&= 1 - \prod_{i=1}^{n}(1-p_i) + p_{n+1}\prod_{i=1}^{n}(1-p_i) \\
&= 1 - \prod_{i=1}^{n}(1-p_i)(1-p_{n+1}) \\
&= 1 - \prod_{i=1}^{n+1}(1-p_i)
\end{aligned}
$$

Therefore, $(E,p) = (E, \delta^t([E_1]_\Phi))$. Therefore, we claim that $\chi_{\mathcal{E}_Q^{!\Phi}}(E,p) = 1$. Hence, we can claim that it holds also for equivalence classes of size $n+1$.

Thus, by induction we can conclude that the claim holds for all $n \in \mathbb{N}$. Furthermore, since (E,q) was arbitrary we can conclude that the left to right direction holds.

($\Leftarrow$) Suppose that for some arbitrary (E, q) we have that $\chi_{\mathcal{E}_Q^{!\Phi}}(E, p) = 1$. Then, by definition of the characteristic function, $(E, p) \in \mathcal{E}_Q^{!\Phi}$. Hence, by Definition 35, there exists an equivalence class $[E]_\Phi$ under $\sim_\Phi$ such that

$$p = \delta^t([E]_\Phi) \quad \text{and} \quad E = E_i \cap \Phi \text{ for all } (E_i, p_i) \in [E]_\Phi.$$

By Lines 2–6 of Algorithm 3, the algorithm inserts all pairs $(E_i \cap \Phi, p_i)$ into $\mathcal{E}_Q^{!\Phi}$. Hence, all elements of $[E]_\Phi$ appear as pairs with identical first component E.

Now we proceed showing by induction on the cardinality of $[E]_\Phi = n$, that for each $n \in \mathbb{N}$ that

$$p = \delta^t([E]_\Phi).$$

Base case Assume $n = 1$. Then $[E]_\Phi = \{(E_1, p_1)\}$, which implies that $E = E_1 \cap \Phi$. Moreover,

$$\begin{aligned} p = \ & \delta^t([E]_\Phi) \\ = \ & \delta^t\left(\{(E_1, p_1)\}\right) \\ = \ & p_1. && \text{(Proposition 7 (1))} \end{aligned}$$

Since $|[E]_\Phi| = 1$, the condition of the while-loop in Algorithm 3 is never satisfied. Consequently, the algorithm proceeds only executing the for-loop, where it adds $(E_1 \cap \Phi, p_1)$, which is equal to (E, p) as we showed above, and makes no further modifications. Therefore, we conclude that $(E, p) \in q\left(\mathcal{E}_Q, \Phi\right)$.

Assume now that $n = 2$. Then $[E]_\Phi = \{(E_1, p_1), (E_2, p_2)\}$, which implies that $E = E_1 \cap \Phi = E_2 \cap \Phi$. Moreover,

$$\begin{aligned} p \quad & = \delta^t([E]_\Phi) \\ & = \delta^t\left(\{(E_1, p_1), (E_1, p_2)\}\right) \\ & = 1 - \prod_{i=1}^{2}(1 - p_1) \\ & = 1 - (1 - p_1)(1 - p_2) \\ & = 1 + (-1 + p_1)(1 - p_2) \\ & = 1 - 1 + p_1 + p_2 - p_1 \cdot p_2 \\ & = p_1 + p_2 - (p_1 \cdot p_2) \end{aligned}$$

Since $|[E]_\Phi| = 2$, the condition of the while-loop in Algorithm 3 is satisfied. The algorithm therefore proceeds to the for-loop, where it adds $(E_1 \cap \Phi, p_1)$ and $(E_2 \cap \Phi, p_2)$ to $\mathcal{E}_Q^{!\Phi}$. These are then merged into a single piece of evidence $(E_1 \cap \Phi, p_n)$, where

$$p_n = p_1 + p_2 - (p_1 \cdot p_2).$$

As shown above, $(E_1 \cap \Phi, p_n) = (E, p)$. Hence, we conclude that $(E, p) \in q\left(\mathcal{E}_Q, \Phi\right)$.

Induction step Assume, as induction hypothesis (IH), that the claim holds for $n \in \mathbb{N}$, i.e., for every equivalence class $[E]_\Phi$ such that $E_i \cap \Phi = E$ for all $(E_i, p_i) \in [E]_\Phi$ with $|[E]_\Phi| = n$, we have that $(E, p) \in q\left(\mathcal{E}_Q, \Phi\right)$, where

$$p = \delta^t([E]_\Phi).$$

Now let $|[E]_\Phi| = n + 1$, and write

$$[E]_\Phi = \{(E_1, p_1), \dots, (E_n, p_n), (E_{n+1}, p_{n+1})\},$$

where $E_i \cap \Phi = E$ for all $1 \leqslant i \leqslant n + 1$. Then

$$p = 1 - \prod_{i=1}^{n+1}(1 - p_i).$$

By Lines 2–6 of Algorithm 3, the algorithm first inserts all pairs $(E_i \cap \Phi, p_i) = (E, p_i)$ into $\mathcal{E}_Q^{!\Phi}$. Since $|[E]_\Phi| = n + 1 > 1$, the condition of the while-loop is satisfied, and the algorithm proceeds by merging pairs that share the same first component.

Consider the subset $\{(E_1, p_1), \dots, (E_n, p_n)\}$. Since the cardinality of the latter set is n, by the induction hypothesis, these are merged into a single pair (E, p'), where

$$p' = 1 - \prod_{i=1}^{n}(1 - p_i).$$

This leaves two pairs with identical first component E, namely (E, p') and $(E_{n+1} \cap \Phi, p_{n+1})$.

Applying the merging step once more, the algorithm combines these into a single pair (E, p), where $E = E_i \cap \Phi$ for all i such that $1 \leqslant i \leqslant n + 1$ and

$$\begin{aligned}
p'' &= p' + p_{n+1} - (p' \cdot p_{n+1}) \\
&= 1 - \prod_{i=1}^{n+1}(1 - p_i) && \text{(As shown above)} \\
&= \delta([E]_\Phi) \\
&= p
\end{aligned}$$

Thus, the final merging step produces a pair $(E_1 \cap \Phi, p'')$, that as we showed is equal to (E, p). Hence, $(E, p) \in q\left(\mathcal{E}_Q, \Phi\right)$.

Thus, by induction we can conclude that the claim holds for all $n \in \mathbb{N}$. Furthermore, since (E, q) was arbitrary we can conclude that the right to left direction holds.

Since both direction separately hold, we can conclude that Algorithm 3 is correct with respect to Definition 35. □

As we analyzed at the beginning of this section, in the quantitative layer a core component is represented by the δ function, defined as in Definition 21. After the conditional update of the quantitative evidence frame $\left(X, \mathcal{E}_Q\right)$ on the proposition $\Phi \subseteq X$, as defined in Definition 35, we obtain a new quantitative evidence frame, namely $\left(\Phi, \mathcal{E}_Q^{!\Phi}\right)$. We are now left to understand how to define the δ function on this new frame. However, notice that since we already have at our disposal a new space restricted to our certain evidence and a new conditionalized quantitative evidence set, we define the conditional update on the δ function simply by recomputing the δ function with respect to the new quantitative evidence frame. In other words, the new δ function is $\delta^{!\Phi}$, which is formally the same function as in Definition 21 but evaluated on the updated frame $\left(\Phi, \mathcal{E}_{\mathrm{Q}}^{!\Phi}\right)$.

Definition 37 (Conditionalized δ function)**.** Let $\left(\Phi, \mathcal{E}_{\mathrm{Q}}^{!\Phi}\right)$ be the quantitative evidence frame obtained from $(X, \mathcal{E}_{\mathrm{Q}})$ by conditioning on $\Phi \subseteq X$ as in Definition 35. Suppose $\mathcal{E}_{\mathrm{Q}}^{!\Phi} = \left\{\left(E_i^{!\Phi}, p_i^{!\Phi}\right)\right\}_{1\leqslant i\leqslant m}$, and let $\mathcal{E}^{!\Phi}$ be its underlying set of evidence. We define the conditionalized δ function, $\delta^{!\Phi} : \wp(\mathcal{E}^{!\Phi}) \longrightarrow [0,1]$ by

$$\delta^{!\Phi}(E') \;=\; \prod_{E_i^{!\Phi}\in E'} p_i^{!\Phi} \prod_{E_i^{!\Phi}\notin E'} \left(1 - p_i^{!\Phi}\right)$$

for any $E' \subseteq \mathcal{E}^{!\Phi}$.

2.4 *Preservation of the Evidence Structure under Conditionalization*

We now establish a fundamental correspondence between the qualitative and quantitative evidence frames under conditionalization updates. This result ensures that the relationship between the two layers of the MLBM is preserved when we update with new information.

Definition 38 (Underlying set of evidence)**.** Let be $(X, \mathcal{E}_{\mathrm{Q}})$ a quantitative evidence frame. Then we call $\mathcal{E} = \{E \mid (E, p) \in \mathcal{E}_{\mathrm{Q}}\}$ the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}$. Notice that $(X, \mathcal{E})$ is a qualitative evidence frame.

Proposition 10 (Preservation of evidence structure under conditionalization)**.** Let $(X, \mathcal{E})$ be a qualitative evidence frame and $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ a quantitative evidence frame such that $\mathcal{E}$ is the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}$, according to Definition 38. Let $\Phi \subseteq X$ be any proposition. Then, after updating both frames according to Definition 31 and Definition 35 respectively, we have that for all $E \subseteq X$

$$E \in \mathcal{E}^{!\Phi} \iff (E, p) \in \mathcal{E}_{\mathrm{Q}}^{!\Phi},$$

where $p \in [0,1]$ such that $p = \delta^t([E]_\Phi)$ and $[E]_\Phi$ is the equivalence class defined in Definition 35. This is equivalent to state that the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}^{!\Phi}$ is exactly $\mathcal{E}^{!\Phi}$.

Proof. Let $(X, \mathcal{E})$ be a qualitative evidence frame and $\left(X, \mathcal{E}_{\mathrm{Q}}\right)$ a quantitative evidence frame such that $\mathcal{E}$ is the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}$, according to Definition 38. Let $\Phi \subseteq X$ be any proposition. We prove the equivalence by establishing both directions separately.

($\Rightarrow$) Let $E \in \mathcal{E}^{!\Phi}$ be arbitrary. By Definition 31, this means there exists some $E' \in \mathcal{E}$ such that $E = E' \cap \Phi$ and $E \neq \varnothing$.

Since $\mathcal{E}$ is the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}$, for the evidence $E' \in \mathcal{E}$ there exists at least one weighted evidence $(E', p') \in \mathcal{E}_{\mathrm{Q}}$ with $p' \in [0,1]$. Consider the equivalence class of (E', p') under $\sim_\Phi$:

$$[E']_\Phi = \{(E_i, p_i) \in \mathcal{E}_{\mathrm{Q}} \mid E_i \cap \Phi = E' \cap \Phi = E\}.$$

By Definition 35, the updated quantitative set of evidence contains the element (E, p) where

$$p = \delta^t([E']_\Phi).$$

Hence, we have $(E, p) \in \mathcal{E}_Q^{!\Phi}$. Since $E \in \mathcal{E}^{!\Phi}$ was arbitrary, the forward direction holds.

($\Longleftarrow$) Assume $(E, p) \in \mathcal{E}_Q^{!\Phi}$ for some $p \in [0, 1]$. By Definition 35, there exists an equivalence class $[E']_\Phi$ for some $(E', p') \in \mathcal{E}_Q$ such that:

- $E = E' \cap \Phi$,
- $E \cap \Phi \neq \varnothing$ (by construction, since otherwise the element would not be included in $\mathcal{E}_Q^{!\Phi}$),
- $p = \delta^t([E']_\Phi)$.

Since $\mathcal{E}$ is the underlying set of $\mathcal{E}_Q$, from $(E', p') \in \mathcal{E}_Q$ we obtain $E' \in \mathcal{E}$. As $E = E' \cap \Phi$ and $E' \cap \Phi \neq \varnothing$ (because $E \neq \varnothing$), by Definition 31 we conclude that $E \in \mathcal{E}^{!\Phi}$.

Having established both directions, we have shown that for every $E \subseteq X$,

$$E \in \mathcal{E}^{!\Phi} \iff (E, p) \in \mathcal{E}_Q^{!\Phi}.$$

where $p \in [0, 1]$. Consequently, the underlying set of evidence of $\mathcal{E}_Q^{!\Phi}$ is precisely $\mathcal{E}^{!\Phi}$, which completes the proof. □

Proposition 10 establishes that the relationship between the qualitative and quantitative layers is preserved under conditionalization updates. This result is crucial for the coherence of the MLBM framework, as it ensures that the evidence structure computed by Algorithm 1 and Algorithm 3 remains consistent across both layers of the model.

2.5 *Conditionalization on the bridging layer*

Having defined how to conditionalize both the qualitative layer (Section 2.2) and the quantitative layer (Section 2.3), we now turn to the bridging layer. Recall from Section 1.3 that the bridging layer connects the qualitative and quantitative layers through evidence allocation functions and frames of justification, ultimately producing a basic probability assignment $\delta_{\mathcal{J}}(f, \cdot)$ and a multi-layer belief function $\mathrm{Bel}_{\mathcal{J}}(f, \cdot)$.

After conditioning on a proposition $\Phi \subseteq X$, we obtain: a new qualitative evidence frame $(\Phi, \mathcal{E}^{!\Phi})$ with its associated evidential topology $\tau_{\mathcal{E}}^{!\Phi}$, according to Definition 31; a new frame of justification $\mathcal{J}^{!\Phi}$ recomputed with respect to the updated topology, as in Definition 33; a new quantitative evidence frame $(\Phi, \mathcal{E}_Q^{!\Phi})$, as described in Definition 35; and a new δ function $\delta^{!\Phi}$ defined on Φ, according to Definition 37.

The bridging layer must now recompute its components using these updated structures. A natural question is: how should the evidence allocation functions be applied to the updated frame? Since the domain of these functions is $\wp(\mathcal{E})$, but after conditioning we have a new set of evidence $\mathcal{E}^{!\Phi}$, we need to determine how to evaluate f on subsets of $\mathcal{E}^{!\Phi}$. One might propose to simply take the original evidence allocation function f, apply it to the pre-image of $E^{!\Phi} \subseteq \mathcal{E}^{!\Phi}$ in $\mathcal{E}$, and then intersect the result with Φ. To do so we introduce first a way to map the subsets of the set of evidence to the subsets of the conditionalized set of evidence, according to Definition 31.

Definition 39. Let $(X, \mathcal{E})$ be a qualitative evidence frame and $\Phi \subseteq X$ a proposition. Then consider the qualitative evidence frame conditionalized on Φ, according to Definition 31, which is $(\Phi, \mathcal{E}^{!\Phi})$. Then we define the function $\rho_\Phi : \wp(\mathcal{E}) \longrightarrow \wp(\mathcal{E}^\Phi)$ such that

$$\rho_\Phi(E) = \{E_i \cap \Phi \mid E_i \in E \text{ and } E_i \cap \Phi \neq \varnothing\}.$$

This would mean that the conditional evidence allocation function $f^{!\Phi} : \wp\left(E^{!\Phi}\right) \to \tau_{\mathcal{E}}^{!\Phi}$ would be defined, as follows:

$$f^{!\Phi}(E^{!\Phi}) = f(E) \cap \Phi,$$

where E is such that $\rho_\Phi(E) = E^{!\Phi}$.

The fundamental problem is that ρ_Φ might be not injective. Consequently, there may exist distinct $E, E' \in \wp(\mathcal{E})$ such that $\rho_\Phi(E) = \rho_\Phi(E') = E^{!\Phi}$, but $f(E) \cap \Phi \neq f(E') \cap \Phi$. In such a case, the value of $f^{!\Phi}(E^{!\Phi})$ is not uniquely determined, rendering the definition ill-posed.

One might try to salvage the definition by restricting attention to cases where ρ_Φ is injective. This occurs, for instance, when all evidence propositions have distinct, non-empty intersections with Φ. In such a scenario, each $E^{!\Phi}$ corresponds to exactly one E, so the definition appears well-defined. However, this does not solve the deeper conceptual issue.

Now, consider the following evidence allocation function, which was proven to be such in (Pinto Prieto, 2024, Proposition 3.10)

Definition 40. Let $(X, \mathcal{E})$ be a qualitative evidence frame and let $\tau_{\mathcal{E}}$ be the topology generated by $\mathcal{E}$. Define the function $t : \wp(\mathcal{E}) \longrightarrow \tau_{\mathcal{E}}$ by

$$t(E) = \begin{cases} \bigcap E & \text{if } E \neq \varnothing \text{ and } \bigcap E \neq \varnothing, \\ X & \text{otherwise,} \end{cases}$$

with $t(\varnothing) = X$. Then for any set $\mathfrak{F}$ of evidence allocation functions on $(X, \mathcal{E})$, the set $\mathfrak{F} \cup \{t\}$ is also a set of evidence allocation functions.

Now, we can analyze a deeper issue of this approach by the following example.

Example 4. Let $X = \{1, 2, 3\}$ and $\Phi = \{1, 2\}$. Let the evidence set be $\mathcal{E} = \{\{1, 3\},\ \{2, 3\}\}$. Then the conditionalized set of evidence is

$$\mathcal{E}^{!\Phi} = \{\{1\},\ \{2\}\}.$$

Notice that in this case ρ_Φ is a bijection between $\wp(\mathcal{E})$ and $\wp\left(\mathcal{E}^{!\Phi}\right)$, since every basic piece of evidence updated have a distinct, non-empty intersections with Φ.

Now, consider $\{1, 3\}$, then we have that

$$\bigcap E = \{1, 3\} \cap \{2, 3\} = \{3\} \neq \varnothing,$$

so $t(E) = \{3\}$. Since $3 \notin \Phi = \{1, 2\}$, we have

$$t(E) \cap \Phi = \{3\} \cap \{1, 2\} = \varnothing.$$

Now consider $E^{!\Phi} = \rho_\Phi(E) = \{\{1\}, \{2\}\}$. The intersection of $\{\{1\}, \{2\}\}$ is $\{1\} \cap \{2\} = \varnothing$. Thus, by definition of t,

$$t\left(E^{!\Phi}\right) = \Phi = \{1, 2\}.$$

Therefore, we can conclude that

$$t(E) \cap \Phi = \varnothing \neq \{1, 2\} = t\left(E^{!\Phi}\right).$$

This example shows that even when ρ_Φ is a bijection, the naive definition $f^{!\Phi}(E^{!\Phi}) = f(E) \cap \Phi$ fails to capture the correct behavior of the evidence allocation function after conditioning. The function t evaluated on the original evidence set yields $\{3\}$, whose intersection with Φ is empty, while the same functional rule applied directly to the updated evidence set yields Φ.

Consequently, we adopt the simpler and more principled approach: after conditioning, we recompute the evidence allocation function from scratch using the same definition on the updated frame $(\Phi, \mathcal{E}^{!\Phi})$. This ensures that the epistemic role of the evidence allocation function is preserved across updates, avoiding the arbitrary transformations and ill-defined cases that plague the naïve definition.

Definition 41. Let $(X, \mathcal{E})$ be a qualitative evidence frame, $\tau_\mathcal{E}$ the topology generated by $\mathcal{E}$ and let $f : \wp(\mathcal{E}) \to \tau_\mathcal{E}$ be an evidence allocation function satisfying Definition 22. Given a proposition $\Phi \subseteq X$, let $\left(\Phi, \mathcal{E}^{!\Phi}\right)$ and $\tau_\mathcal{E}^{!\Phi}$ be the updated qualitative evidence frame obtained via Definition 31. Then the conditionalized evidence allocation function $f^{!\Phi} : \wp\left(\mathcal{E}^{!\Phi}\right) \to \tau_\mathcal{E}^{!\Phi}$ is defined by f, applied to the updated frame.

In other words, $f^{!\Phi}$ is the evidence allocation function obtained by recomputing f on the conditionalized frame according to the same epistemic rule. Furthermore, since we assume that f is polynomial in the size of its input, then also $f^{!\Phi}$ is polynomial in its computation.

Having defined how to conditionalize each component of the model, we now adapt the definitions of δ_τ, $\delta_\mathfrak{J}$, and the multi-layer belief function to the conditionalized setting.

Definition 42. Let $(\Phi, \mathcal{E}_Q^{!\Phi})$ be the quantitative evidence frame obtained after conditioning on Φ according to Definition 35, and let $\mathfrak{F}^{!\Phi}$ be a set of evidence allocation functions on $(\Phi, \mathcal{E}^{!\Phi})$. Then we define the function $\delta_\tau^{!\Phi} : \mathfrak{F}^{!\Phi} \times \wp(\Phi) \to [0, 1]$ such that

$$\delta_\tau^{!\Phi}(f^{!\Phi}, T) = \begin{cases} \displaystyle\sum_{E' : f^{!\Phi}(E') = T} \delta^{!\Phi}(E') & \text{if } T \in \tau_\mathcal{E}^{!\Phi}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta^{!\Phi}$ is defined as in Definition 37.

Definition 43. Let $(\Phi, \mathcal{E}_Q^{!\Phi})$ be a quantitative evidence frame, let $\mathfrak{J}^{!\Phi}$ be a frame of justification over $\tau_\mathcal{E}^{!\Phi}$ (obtained via Definition 33), and let $\mathfrak{F}^{!\Phi}$ be a set of evidence allocation functions. Then we define the function $\delta_{\mathfrak{J}^{!\Phi}} : \mathfrak{F}^{!\Phi} \times \wp(\Phi) \to [0, 1]$ such that

$$\delta_{\mathfrak{J}^{!\Phi}}(f^{!\Phi}, A) = \begin{cases} \dfrac{\delta_\tau^{!\Phi}(f^{!\Phi}, A)}{\displaystyle\sum_{T \in \mathfrak{J}^{!\Phi}} \delta_\tau^{!\Phi}(f^{!\Phi}, T)} & \text{if } A \in \mathfrak{J}^{!\Phi}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 44 (Conditional multi-layer belief function)**.** Given a quantitative evidence frame $(\Phi, \mathcal{E}_Q^{!\Phi})$, a frame of justification $\mathcal{J}^{!\Phi}$, and a set of evidence allocation functions $\mathfrak{F}^{!\Phi}$, for any $P \subseteq \Phi$ we call the function

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, P) = \sum_{A \subseteq P} \delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, A)$$

a *conditional multi-layer belief function*. When $\mathcal{J}^{!\Phi}$ and $f^{!\Phi}$ are clear from the context, we simply write $\mathsf{Bel}^{!\Phi}(P)$.

Analogously to the unconditional case, $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ is a basic probability assignment on Φ, and therefore $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ is a belief function on Φ according to Definition 13.

Recall that the bridging layer produces a belief function $\mathsf{Bel}_{\mathcal{J}}(f, \cdot)$ based on two key components: a frame of justification $\mathcal{J}$, which encodes which subsets of the evidential topology are considered admissible justifications for belief, and an evidence allocation function f, which determines how subsets of evidence are mapped into the topology.

After conditioning on Φ, we obtain a new frame of justification $\mathcal{J}^{!\Phi}$ by recomputing it with respect to the updated topology $\tau_{\mathcal{E}}^{!\Phi}$ according to Definition 33. Since $\mathcal{J}^{!\Phi}$ is defined by the same rule that defined $\mathcal{J}$ on the original frame, it follows that $\mathcal{J}^{!\Phi}$ respects the same doxastic requirements as $\mathcal{J}$. In other words, the agent's standard for what counts as a justification remains unchanged; only the underlying evidential topology has been updated.

Similarly, the conditionalized evidence allocation function $f^{!\Phi}$ is obtained by recomputing f on the updated frame $(\Phi, \mathcal{E}^{!\Phi})$ according to the same functional rule (see Definition 41). Since f satisfies the three conditions of an evidence allocation function on the original frame (see Definition 22), and since $f^{!\Phi}$ is defined by the same rule applied to the updated structure, it follows that $f^{!\Phi}$ also satisfies these three conditions on $(\Phi, \mathcal{E}^{!\Phi})$. Therefore, $f^{!\Phi}$ is an evidence allocation function in the sense of Definition 22, and the set $\mathfrak{F}^{!\Phi}$ of such functions is constructed analogously to $\mathfrak{F}$.

Consequently, the conditionalized multi-layer belief function $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ defined in Definition 44 inherits the properties of the original belief function. In particular, $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ is a basic probability assignment on Φ, and therefore $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ is a belief function on Φ satisfying the axioms of Definition 13.

Thus, the conditionalization produces a new belief function that represents the same type of agent as before, who adheres to the same standards of justification and the same evidential allocation rule, but now operating under the updated information represented by Φ. This consideration confirms that the doxastic requirements established in the original model are preserved throughout the conditionalization process.

2.6 *Satisfaction of the Desiderata*

Having defined the conditionalization procedure for each layer of the MLBM, we want to check that this procedure satisfies the *desiderata* enumerated in Section 2.1. Now, for each *desideratum*, we either provide a formal proof or argue directly from the constructions presented above. To simplify the notation we first introduce the definition of a MLBM structure and of the conditionalization procedure over it as follows:

Definition 45 (MLBM structure)**.** Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame, $\mathcal{J}$ a frame of justification, δ the standard δ-function and f an evidence allocation function respectively according to Definitions 17, 18, 21 and 22. Then we call $\mathfrak{B} = (X, \mathcal{E}_Q, \mathcal{J}, \delta, f)$ a MLBM structure.

Definition 46 (Conditionalization on a MLBM structure)**.** Let $\mathfrak{B} = \left(X, \mathcal{E}_{\mathrm{Q}}, \mathcal{J}, \delta, f\right)$ be a MLBM structure and let $\Phi \subseteq X$ be a proposition. We define the conditionalization of $\mathfrak{B}$ on Φ, denoted $\mathfrak{B}^{!\Phi}$, as the MLBM structure

$$\mathfrak{B}^{!\Phi} = \left(\Phi, \mathcal{E}_{\mathrm{Q}}^{!\Phi}, \mathcal{J}^{!\Phi}, \delta^{!\Phi}, f^{!\Phi}\right),$$

where:

- $\mathcal{E}_{\mathrm{Q}}^{!\Phi}$ is the conditionalized quantitative evidence set defined in Definition 35;
- $\mathcal{J}^{!\Phi}$ is the conditionalized frame of justification defined in Definition 33;
- $\delta^{!\Phi}$ is the conditionalized δ function defined in Definition 37;
- $f^{!\Phi}$ is the conditionalized evidence allocation function defined in Definition 41.

The underlying set of evidence $\mathcal{E}^{!\Phi}$ and the evidential topology $\tau_{\mathcal{E}}^{!\Phi}$ are recovered from $\mathcal{E}_{\mathrm{Q}}^{!\Phi}$ according to Definition 38 and Definition 31, respectively.

Notice that these elements are sufficient to define the MLBM, since all the components of the qualitative layer can be recovered by considering the underlying set of evidence of $\mathcal{E}_{\mathrm{Q}}$, as in Definition 38, and the evidential topology $\tau_{\mathcal{E}}$ generated by it, according to Proposition 1. The frame of justification $\mathcal{J}$ provides the justification base for our belief, and together with the evidence allocation function f and the δ function, it determines the bridging layer and the resulting multi-layer belief function as defined in Definition 26.

The first *desideratum* requires that upon learning a proposition $\Phi \subseteq X$, the new set of possible states be restricted to Φ. This is satisfied immediately by our construction. In Definition 31, we explicitly set $X^{!\Phi} = \Phi$ for the updated qualitative evidence frame. Since all subsequent layers operate on this updated frame, the entire conditionalized model MLBM$^{!\Phi}$ is defined over Φ as its state space. No further proof is required.

The second *desideratum* requires that the mass function $\delta_{\mathcal{J}}(f, \cdot)$ be conditioned on Φ by discarding all mass assigned to subsets not contained in Φ and renormalizing the remaining mass.

Proposition 11. Let $\mathfrak{B} = \left(X, \mathcal{E}_{\mathrm{Q}}, \mathcal{J}, \delta, f\right)$ be an MLBM structure and let $\Phi \subseteq X$ be a proposition. Consider the conditionalized basic probability assignment $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ defined in Definition 43. Then:

1. For any $\Psi \not\subseteq \Phi$, we have $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = 0$.
2. $\displaystyle\sum_{A \subseteq \Phi} \delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, A) = 1.$

Proof. We prove the two statements separately.

1. Let $\Psi \not\subseteq \Phi$. By definition, $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) \neq 0$ if only if $\Psi \in \mathcal{J}^{!\Phi} \subseteq \tau_{\mathcal{E}}^{!\Phi}$ (see Definition 43). Recall from Definition 31 that $\tau_{\mathcal{E}}^{!\Phi} = \{U \cap \Phi \mid U \in \tau_{\mathcal{E}}\}$. If $\Psi \not\subseteq \Phi$, then there exists no $U \in \tau_{\mathcal{E}}$ such that $\Psi = U \cap \Phi$, since any set of the form $U \cap \Phi$ is always a subset of Φ. Hence, $\Psi \notin \tau_{\mathcal{E}}^{!\Phi}$, and consequently $\Psi \notin \mathcal{J}^{!\Phi}$. Therefore, $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = 0$.
2. The conditionalized basic probability assignment $\delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \cdot)$ is defined in Definition 43 as a mass function over the space Φ, normalized with respect to the frame of justification $\mathcal{J}^{!\Phi}$. By definition of a basic probability assignment, the sum of the masses over all subsets of the domain equals 1. Thus, $\displaystyle\sum_{A \subseteq \Phi} \delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, A) = 1$.

Hence, the conditionalization procedure satisfies the mass renormalization *desideratum*. □

The third *desideratum* requires that the frame of justification $\mathfrak{J}$ and the evidence allocation function f be renormalized on a topological space and a set of evidence that has Φ as total space, while preserving the epistemic demands they encode.

As argued in Section 2.2 and Section 2.5, this *desideratum* is satisfied by recomputing $\mathfrak{J}^{!\Phi}$ and $f^{!\Phi}$ from scratch on the updated structure $(\Phi, \mathcal{E}^{!\Phi}, \tau_{\mathcal{E}}^{!\Phi})$. The frame of justification is recomputed according to the same rule that defined $\mathfrak{J}$, defined in Definition 33, and the evidence allocation function is recomputed according to the same functional rule that defined f, as in Definition 41. The alternative approach of simply taking intersections, as shown in Examples 1 and 4, fails to preserve the epistemic demands. Therefore, recomputation from scratch on the updated topological space with total space Φ is the correct method, and it preserves the epistemic demands by construction, since the same defining rules are applied to the updated structure.

The fourth *desideratum* requires that the conditionalization operation produce a new MLBM of the same type as the original, allowing for iterated conditioning.

Proposition 12 (Iterability)**.** Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathfrak{J}, \delta, f)$ be an MLBM structure and let $\Phi, \Psi \subseteq X$ be propositions. Then

$$\mathfrak{B}^{!\Phi\cap\Psi} = \left(\mathfrak{B}^{!\Phi}\right)^{!\Psi|_\Phi},$$

where $\Psi|_\Phi = \Psi \cap \Phi$.

Proof. We prove the equality by showing that each component of the MLBM structure coincides under the two conditioning procedures.

Set of possible states The first component follows directly from the fact that

$$X^{!\Phi\cap\Psi} = \Phi \cap \Psi = \Psi|_\Phi \quad \text{and} \quad \left(X^{!\Phi}\right)^{!\Psi|_\Phi} = (\Phi)^{!\Psi\cap\Phi} = \Phi \cap (\Psi \cap \Phi) = \Phi \cap \Psi = \Psi|_\Phi.$$

Thus, $X^{!\Phi\cap\Psi} = \left(X^{!\Phi}\right)^{!\Psi|_\Phi}$.

Quantitative evidence frame We show that $\mathcal{E}_Q^{!\Phi\cap\Psi} = \left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$.

Let $\mathcal{E}_Q = \{(E_i, p_i)\}_{i\leqslant n}$ for some $n \in \mathbb{N}$.

($\subseteq$) Let $(E, p) \in \mathcal{E}_Q^{!\Phi\cap\Psi}$. By Definition 35, there exists $(E_i, p_i) \in \mathcal{E}_Q$ such that:

- $E = E_i \cap (\Phi \cap \Psi) = E_i \cap \Phi \cap \Psi \neq \varnothing$,
- $p = \delta^t([E_i]_{\Phi\cap\Psi}) = 1 - \prod_{(E_j,p_j)\in[E_i]_{\Phi\cap\Psi}}(1 - p_j)$,

where $[E_i]_{\Phi\cap\Psi} = \{(E_j, p_j) \in \mathcal{E}_Q \mid E_j \cap (\Phi \cap \Psi) = E_i \cap (\Phi \cap \Psi)\}$.

Since $E_i \cap \Phi \cap \Psi \neq \varnothing$, we have $E_i \cap \Phi \neq \varnothing$, so $(E_i \cap \Phi, \delta^t([E_i]_\Phi)) \in \mathcal{E}_Q^{!\Phi}$. When conditioning $\mathcal{E}_Q^{!\Phi}$ on $\Psi|_\Phi = \Psi\cap\Phi$, we group together all $(E_j\cap\Phi, \delta^t([E_j]_\Phi))$ such that $(E_j\cap\Phi)\cap(\Psi\cap\Phi) = (E_i\cap\Phi)\cap(\Psi\cap\Phi)$. This condition simplifies to $E_j \cap \Phi \cap \Psi = E_i \cap \Phi \cap \Psi$, i.e., $E_j \sim_{\Phi\cap\Psi} E_i$. Hence, the equivalence class $[[E_i]_\Phi]_{\Psi|_\Phi}$ in $\mathcal{E}_Q^{!\Phi}$ corresponds to $\{(E_j \cap \Phi, \delta^t([E_j]_\Phi)) \mid E_j \in [E_i]_{\Phi\cap\Psi}\}$. The merged weight is:

$$\delta^t\left([[E_i]_\Phi]_{\Psi|_\Phi}\right) = 1 - \prod_{E_j\in[E_i]_{\Phi\cap\Psi}} \left(1 - \delta^t([E_j]_\Phi)\right).$$

Since $\delta^t([E_j]_\Phi) = 1 - \prod_{(E_k,p_k)\in[E_j]_\Phi}(1 - p_k)$, we have $1 - \delta^t([E_j]_\Phi) = \prod_{(E_k,p_k)\in[E_j]_\Phi}(1 - p_k)$.

Substituting:

$$\prod_{E_j \in [E_i]_{\Phi\cap\Psi}} \prod_{(E_k,p_k)\in[E_j]_\Phi} (1-p_k) = \prod_{(E_k,p_k)\in[E_i]_{\Phi\cap\Psi}} (1-p_k),$$

because the double product runs exactly once over each evidence item in $[E_i]_{\Phi\cap\Psi}$. Therefore,

$$\delta^t\left([[E_i]_\Phi]_{\Psi|_\Phi}\right) = 1 - \prod_{(E_k,p_k)\in[E_i]_{\Phi\cap\Psi}} (1-p_k) = p.$$

Moreover, $(E_i \cap \Phi) \cap (\Psi \cap \Phi) = E_i \cap \Phi \cap \Psi = E$. Hence, $(E,p) \in \left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$.

($\supseteq$) Let $(E,p) \in \left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$. By Definition 35, there exists $(E_i \cap \Phi, \delta^t([E_i]_\Phi)) \in \mathcal{E}_Q^{!\Phi}$ such that:
- $E = (E_i \cap \Phi) \cap (\Psi \cap \Phi) = E_i \cap \Phi \cap \Psi \neq \varnothing$,
- $p = \delta^t\left([[E_i]_\Phi]_{\Psi|_\Phi}\right) = 1 - \prod_{(E_j\cap\Phi,\delta^t([E_j]_\Phi))\in[[E_i]_\Phi]_{\Psi|_\Phi}} \left(1-\delta^t([E_j]_\Phi)\right)$,

where $[[E_i]_\Phi]_{\Psi|_\Phi} = \{(E_j \cap \Phi, \delta^t([E_j]_\Phi)) \in \mathcal{E}_Q^{!\Phi} \mid (E_j \cap \Phi) \cap (\Psi \cap \Phi) = (E_i \cap \Phi) \cap (\Psi \cap \Phi)\}$.

The condition simplifies to $E_j \cap \Phi \cap \Psi = E_i \cap \Phi \cap \Psi$, i.e., $E_j \sim_{\Phi\cap\Psi} E_i$. Thus, $[[E_i]_\Phi]_{\Psi|_\Phi} = \{(E_j \cap \Phi, \delta^t([E_j]_\Phi)) \mid E_j \in [E_i]_{\Phi\cap\Psi}\}$. Now:

$$\prod_{(E_j\cap\Phi,\delta^t([E_j]_\Phi))\in[[E_i]_\Phi]_{\Psi|_\Phi}} \left(1-\delta^t([E_j]_\Phi)\right) = \prod_{E_j\in[E_i]_{\Phi\cap\Psi}} \left(1-\delta^t([E_j]_\Phi)\right).$$

Since $1-\delta^t([E_j]_\Phi) = \prod_{(E_k,p_k)\in[E_j]_\Phi}(1-p_k)$, we obtain:

$$\prod_{E_j \in [E_i]_{\Phi\cap\Psi}} \prod_{(E_k,p_k)\in[E_j]_\Phi} (1-p_k) = \prod_{(E_k,p_k)\in[E_i]_{\Phi\cap\Psi}} (1-p_k).$$

Hence, $p = 1 - \prod_{(E_k,p_k)\in[E_i]_{\Phi\cap\Psi}}(1-p_k) = \delta^t([E_i]_{\Phi\cap\Psi})$, and therefore $(E,p) \in \mathcal{E}_Q^{!\Phi\cap\Psi}$.

Thus, $\mathcal{E}_Q^{!\Phi\cap\Psi} = \left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$.

Frame of justification We show that $\mathcal{J}^{!\Phi\cap\Psi} = \left(\mathcal{J}^{!\Phi}\right)^{!\Psi|_\Phi}$.

($\subseteq$) Let $U \in \mathcal{J}^{!\Phi\cap\Psi}$. By Definition 33, $\mathcal{J}^{!\Phi\cap\Psi}$ is the frame of justification recomputed on $\left(\Phi\cap\Psi, \mathcal{E}^{!\Phi\cap\Psi}, \tau_\mathcal{E}^{!\Phi\cap\Psi}\right)$. Since $\tau_\mathcal{E}^{!\Phi\cap\Psi} = \{V\cap\Phi\cap\Psi \mid V \in \tau_\mathcal{E}\}$ by Definition 31, we have $U = V\cap\Phi\cap\Psi$ for some $V \in \tau_\mathcal{E}$ such that U satisfies the defining rule of $\mathcal{J}$. Now consider $V\cap\Phi \in \tau_\mathcal{E}^{!\Phi}$. Then $(V\cap\Phi)\cap(\Psi\cap\Phi) = V\cap\Phi\cap\Psi = U$. Since $\mathcal{J}^{!\Phi}$ is recomputed on $(\Phi, \mathcal{E}^{!\Phi}, \tau_\mathcal{E}^{!\Phi})$, and $\left(\mathcal{J}^{!\Phi}\right)^{!\Psi|_\Phi}$ is recomputed on the subspace, U satisfies the same defining rule. Hence, $U \in \left(\mathcal{J}^{!\Phi}\right)^{!\Psi|_\Phi}$.

($\supseteq$) Let $U \in \left(\mathcal{J}^{!\Phi}\right)^{!\Psi|_\Phi}$. Then $U = W \cap (\Psi\cap\Phi)$ for some $W \in \mathcal{J}^{!\Phi}$ such that U satisfies the defining rule of $\mathcal{J}$. Since $W \in \mathcal{J}^{!\Phi}$, we have $W = V \cap \Phi$ for some $V \in \tau_\mathcal{E}$ satisfying the defining rule. Then $U = (V\cap\Phi)\cap(\Psi\cap\Phi) = V\cap\Phi\cap\Psi \in \tau_\mathcal{E}^{!\Phi\cap\Psi}$. Because U satisfies the same defining rule on the restricted space, we have $U \in \mathcal{J}^{!\Phi\cap\Psi}$.

Thus, $\mathcal{J}^{!\Phi\cap\Psi} = \left(\mathcal{J}^{!\Phi}\right)^{!\Psi|_\Phi}$.

δ function We show that $\delta^{!\Phi\cap\Psi} = \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$.

- ($\subseteq$) Let $(E', p) \in \delta^{!\Phi\cap\Psi}$, i.e., $E' \subseteq \mathcal{E}^{!\Phi\cap\Psi}$ and $p = \delta^{!\Phi\cap\Psi}(E')$. By definition, $\delta^{!\Phi\cap\Psi}$ is computed from $\mathcal{E}_Q^{!\Phi\cap\Psi}$ using the product formula. Since we have already shown $\mathcal{E}_Q^{!\Phi\cap\Psi} = \left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$, and $\left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$ is computed from $\left(\mathcal{E}_Q^{!\Phi}\right)^{!\Psi|_\Phi}$ using the same product formula, it follows that $p = \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}(E')$. Hence, $(E', p) \in \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$.
- ($\supseteq$) Let $(E', p) \in \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$. Then $E' \subseteq \left(\mathcal{E}^{!\Phi}\right)^{!\Psi|_\Phi} = \mathcal{E}^{!\Phi\cap\Psi}$ and $p = \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}(E')$. Since both $\delta^{!\Phi\cap\Psi}$ and $\left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$ are defined by the same product rule on identical quantitative evidence frames, we have $p = \delta^{!\Phi\cap\Psi}(E')$. Thus, $(E', p) \in \delta^{!\Phi\cap\Psi}$.

Therefore, $\delta^{!\Phi\cap\Psi} = \left(\delta^{!\Phi}\right)^{!\Psi|_\Phi}$.

Evidence allocation function We show that $f^{!\Phi\cap\Psi} = \left(f^{!\Phi}\right)^{!\Psi|_\Phi}$.

- ($\subseteq$) Let $(E', U) \in f^{!\Phi\cap\Psi}$, i.e., $E' \subseteq \mathcal{E}^{!\Phi\cap\Psi}$ and $U = f^{!\Phi\cap\Psi}(E')$. By Definition 41, $f^{!\Phi\cap\Psi}$ is recomputed on $(\Phi \cap \Psi, \mathcal{E}^{!\Phi\cap\Psi})$ according to the same rule that defined f. Since $\mathcal{E}^{!\Phi\cap\Psi} = \left(\mathcal{E}^{!\Phi}\right)^{!\Psi|_\Phi}$ and $\left(f^{!\Phi}\right)^{!\Psi|_\Phi}$ is recomputed on $\left(\Psi|_\Phi, \left(\mathcal{E}^{!\Phi}\right)^{!\Psi|_\Phi}\right)$ according to the same rule, we have $U = \left(f^{!\Phi}\right)^{!\Psi|_\Phi}(E')$. Hence, $(E', U) \in \left(f^{!\Phi}\right)^{!\Psi|_\Phi}$.
- ($\supseteq$) Let $(E', U) \in \left(f^{!\Phi}\right)^{!\Psi|_\Phi}$. Then $E' \subseteq \left(\mathcal{E}^{!\Phi}\right)^{!\Psi|_\Phi} = \mathcal{E}^{!\Phi\cap\Psi}$ and $U = \left(f^{!\Phi}\right)^{!\Psi|_\Phi}(E')$. Since both $f^{!\Phi\cap\Psi}$ and $\left(f^{!\Phi}\right)^{!\Psi|_\Phi}$ are defined by the same functional rule on identical frames, we have $U = f^{!\Phi\cap\Psi}(E')$. Thus, $(E', U) \in f^{!\Phi\cap\Psi}$.

Hence, $f^{!\Phi\cap\Psi} = \left(f^{!\Phi}\right)^{!\Psi|_\Phi}$.

Since all five components of the MLBM structure are identical under the two conditioning procedures, we conclude:

$$\mathfrak{B}^{!\Phi\cap\Psi} = \left(\mathfrak{B}^{!\Phi}\right)^{!\Psi|_\Phi}.$$

$\square$

With this last proof, we have established that our conditionalization procedure for MLBM respects the four *desiderata*. This means that our procedure not only behaves as a conditionalization should, but since it was constructed using only the tools and elements of our model, it is a genuine construction with no further or arbitrary reference to any of the theoretical bases.

The fact that our model extensively uses the elements introduced in Definition 29 is not problematic, since the restriction of the space to the proposition is a common trend in all conditional models; therefore, it is justifiable to adopt the same qualitative structure as TME, as it is shared among all reasonable definitions.

Furthermore, the choice that may seem arbitrary of adopting a new δ^t function is justified by the construction itself. As we saw in Definition 35, the problems introduced in Examples 2 and 3 required a new function to do the job, since the original model does not provide one. This choice is motivated both by technical and epistemic reasons. Indeed, the faction is constructed in such a way that the properties in Proposition 7 are met. Moreover, as we noticed, the function can be read as recomputing the weight of the evidence by re-evaluating the uncertainty, taking the product of the uncertainties of each independent source, and thereby recomputing the actual certainty value of our evidence.

Moreover, our conditionalization preserves the doxastic requirements of our agent by preserving the definitions of the frame of justification and the evidence allocation function during the update. This is important because changing them would mean changing our agent and thus our model at its core, which is not only technically problematic but also epistemically. Technically, altering these components would break the iterability of the conditionalization procedure, since each update would potentially redefine the agent's evidential standards, making it impossible to reason about sequences of updates in a coherent way. Epistemically, modifying the frame of justification or the evidence allocation function would amount to changing the agent's criteria for what counts as a justification and how evidence is interpreted. An agent who updates on new information should not thereby change their fundamental epistemic standards; rather, they should apply the same standards to the new information. Thus, preserving these components across updates is essential for modeling a stable doxastic agent.

2.7 *Conclusion*

In this chapter we have developed a full conditionalization procedure for the *Multi-Layer Belief Model* (MLBM). We began by analyzing the motivations and the different treatments of conditional belief in the two background frameworks, i.e. *Topological Models of Evidence* (TME) and *Dempster-Shafer Theory* (DST). From those we distilled four *desiderata* that a proper conditionalization operation for MLBM should satisfy: restriction of the state space, renormalization of the mass function, preservation of the epistemic standards encoded in the justification frame and the evidence allocation function, and iterability.

We then addressed each layer of the model separately. For the qualitative layer we defined updates for the space, the evidence set, and the evidential topology, directly inspired by the public announcement operation in TME according to Definition 31. We introduced a known general topological results as stated in Proposition 3 which shows that our definition computes in fact a topological space. Then we presented the corresponding algorithms, shown in Algorithms 1 and 2, which have been proven to be polynomial in the size of their inputs, as shown in Proposition 5. For the frame of justification we argued that it cannot be updated by a simple intersection; instead it must be recomputed from scratch on the updated topology following Definition 33. This ensures that the agent's evidential demands remain unchanged even when the underlying topological space is restricted, a phenomenon illustrated by Example 1.

For the quantitative layer we examined two natural but undesirable approaches. The first consists of applying the δ function to the whole augmented evidence set, when we add to it the certain evidence, while the second was the pairwise combination of each singular evidence with the certain evidence. Both were shown to produce undesirable results, as demonstrated in Examples 2 and 3. To overcome these problems we introduced a new merging function δ^t defined in Definition 34, which groups all pieces of evidence that have the same intersection with the conditioning proposition and merges their weights using the rule $p_{\text{new}} = 1 - \prod(1 - p_i)$. This function respects the requirements of uniqueness and consistency, and we proved that it satisfies the desired properties according to Proposition 7. Moreover, we showed that it can be computed by a polynomial-time algorithm, namely Algorithm 3, and that the algorithm is correct by Proposition 9. The resulting quantitative evidence frame is then used to recompute the δ function on the conditioned space as described in Definition 37.

The bridging layer was handled by recomputing the evidence allocation function on the updated qualitative frame following Definition 41 and then redefining δ_τ, $\delta_{\mathcal{J}}$, and the final belief function exactly as in the original model, but now over the updated components as shown in Definitions 42 to 44. We proved that this conditionalization procedure satisfies all four *desiderata*. In particular, we showed that it restricts the space to the proposition, that the conditional mass function is properly normalized and discards mass on subsets outside the proposition, by Proposition 11, that the justification frame and allocation function are recomputed in a way that preserves the agent's epistemic standards, and that the operation is iterable according to Proposition 12.

All definitions and algorithms presented in this chapter have been implemented in Python. The implementation builds on the existing code-base of the MLBM as implemented in the repositories linked to (Pinto Prieto et al., 2023; Compagno et al., 2026). The complete code with the details and modifications applied to the code-base, together with a detailed Jupyter notebook that walks through a concrete and simple example, is available in the accompanying repository (Manes, 2026). In that notebook the reader can the whole computation of the conditionalization procedure applied to a simple example. We present at first the original model and then we condition it to a certain piece of evidence according to our procedure, with all intermediate numerical results displayed. The code is fully documented and can be adapted to any qualitative evidence frame and any quantitative evidence set.

In the next chapter we will analyze how our conditional procedure relates to the type of conditional beliefs analyzed in Section 2.1.

3 Relationship with conditionalization in other belief models

The *God-Intoxicated Man*, in his *Ethica, Ordine Geometrico Demonstrata*, Part 1, Proposition 27 claims that

> Res quæ a Deo ad aliquid operandum determinata est, se ipsam indeterminatam reddere non potest.[1]

What Spinoza called *determinatum* is a concept very similar to what we call in Section 2.1 conditionalization. In fact the claim states that once the belief system has been determined by a certain evidence to adopt a certain doxastic state, it cannot, by a mere act of will or self-reflection, return itself to an unconditioned or prior state without new evidence. The determination, once in place, is irreversible from within the agent's own epistemic resources.

Nevertheless, different evidential theories disagree on what this *determinatum* consists of, and consequently on whether, or to what extent, conditionalization changes the total *epistemic horizon* of our agent.

As we saw in Section 2.1 each of the models we analyzed has different notions of conditionalization that can be really different. However, from the intuition borrowed from these methods we were able to develop a conditionalization procedure that respects the *desiderata* common to all the notion that we considered.

Our model, however, does not directly apply any of the conditionalizations examined in Section 2.1, even though it borrows intuitions from each. Our aim was to construct a procedure capable of conditioning our belief system on new evidence while remaining *faithful* to the specific architecture of our model. Hence, one question naturally arises: *Can our procedure simulate the behavior of the other conditionalizations, and if so, under which conditions?*

Thus, in what follows we intend to study the relationship between our conditionalization and the ones defined in Section 2.1. More precisely, we shall investigate how our update rule captures the notion of *determinatum* and how our notion relates to the ones captured by the update rule in Section 2.1.

3.1 *Conditionalization on* MLBM *and conditionalization on* DST

In Definition 46 we define what a conditionalization in our model is. The belief computed after conditionalization on the model defined by the MLBM structure in Definition 46 is a MLBM conditional belief and it will be defined as in Definition 44.

Now, we want to relate this belief function to the way of conditioning within the DST framework. As we saw in Section 2.1, to condition a belief inside the DST framework, as presented in Definition 30, we employ Dempster's rule of combination as in Definition 16. We do this combining the certain basic probability assignment

1 Personal translation: "A thing that has been determined by *God* to act in some way cannot render itself undetermined."
Please note that *God* here can be read as *Nature*. Thus the sentence can also be read with no religious underlying meaning, even if this is not what Spinoza intended. There is a great deal of discussion regarding what the author meant with *God*, but such themes go far beyond the scope of this work, where the sentence is used only as a mere stylistic choice; we do not engage in this discussion.

associated with the given piece of evidence with the basic probability assignment from which the belief function was computed, according to Definition 15. In this sense, conditioning is performed directly on the underlying basic probability assignment, and the conditional belief function is then derived from the conditional basic probability assignment.

We now proceed on analyzing under which conditions the belief function computed from a conditioned MLBM structure behaves exactly like the two kinds of measures described above. In order to analyze how the belief computed after the conditionalization procedure defined in Chapter 2 relates to the model conditioned on a proposition, as defined in Definition 30, we can take two different approaches. The first approach, which is more faithful to our construction, conditions the model at the level of the evidence frame, updating each piece of evidence individually before aggregating them. The second approach conditions at the level of the final basic probability assignment $\delta_{\mathcal{J}}(f, \cdot)$, treating the entire belief state as a single mass function to be conditioned directly.

3.1.1 Relating our conditionalization to Dempster-Shafer conditioning via Dempster's rule of combination applied to the individual source

In what follows, we focus on the first kind of conditioning: conditioning each singular piece of evidence on the certain information gained. This approach aligns closely with our update procedure as developed in Section 2.3. There, we update each piece of evidence by combining it with a newly defined function δ^t that emulates the behavior of δ when only one piece of evidence is mapped to a particular updated piece of evidence. The procedure then combines all pieces of evidence under the same update. Now, we aim to analyze how this procedure relates to Dempster's rule of combination, as defined in Definition 16. Indeed, both methods operate by working on the intersection between pieces of evidence and combining the weights in such a way that, after computation, the combined mass is transferred to the intersection of the focal elements.

More precisely, two distinct but related functions are at play here. First, Dempster's rule of combination, as defined in Definition 16, takes two basic probability assignments and produces a new one by redistributing mass to the intersections of their focal elements, after normalizing by the total conflict. Second, the merging function δ^t defined in Definition 34 aggregates multiple weighted pieces of evidence that share the same restriction after conditioning on a proposition Φ, combining their weights. Thus, we want to study the relationship between these two ways of performing an evidence combination.

To make this connection precise, we first analyze how Dempster's rule behaves when combining the pieces of evidence in $\mathcal{E}_Q$. Thus we need to understand how to extract basic probability assignments from each $(E_i, p_i) \in \mathcal{E}_Q$. To do so, as described in (Pinto Prieto, 2024, pp. 22-23), notice that with each piece of evidence (E_i, p_i) we can associate a basic probability assignment which is *non-dogmatic*, *non-vacuous*, and *simple*. This means that each piece of evidence can be associated with a basic probability assignment whose only focal elements are the evidence itself and the total space, where the level of certainty is associated with the evidence and the level of uncertainty with the total space, and none of these values can be either 0 or 1. This can be formally defined as follows.

Definition 47 (Evidence Basic Probability Assignment)**.** Given a quantitative evidence frame $(X, \mathcal{E}_Q)$, we define for each $(E_i, p_i) \in \mathcal{E}_Q$ the following associated basic probability assignment $m_i : \wp(X) \to [0, 1]$:

$$m_i(A) = \begin{cases} p_i & \text{if } A = E_i \\ 1 - p_i & \text{if } A = X \\ 0 & \text{otherwise} \end{cases}$$

Now we introduce two propositions. The first one studies the behavior of Dempster's rule of combination when applied to a certain basic probability assignment associated with a proposition $\Phi \subseteq X$ and an evidence basic probability assignment, defined according to Definition 47. Notice that this is exactly equivalent to conditioning a single piece of evidence on Φ, according to Definition 30. In fact, if we compute the belief function associated with the basic probability assignment derived after this combination, we would obtain a function of the type $\mathrm{Bel}|_\Phi$, as defined in Definition 30.

Proposition 13. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and m_i the basic probability assignment associated with each $(E_i, p_i) \in \mathcal{E}_Q$, according to Definition 47. Let $\Phi \subseteq X$ be a proposition and let $m_\Phi : \wp(X) \to [0, 1]$ be the corresponding certain basic probability assignment defined by

$$m_\Phi(A) = \begin{cases} 1 & \text{if } A = \Phi \\ 0 & \text{otherwise} \end{cases}.$$

Then for all $(E_i, p_i) \in \mathcal{E}_Q$ the following claims hold:

1. If $E_i \cap \Phi = \varnothing$, then $m_i \oplus m_\Phi(\Phi) = 1$ and for all other $C \in \wp(X)$ we have $m_i \oplus m_\Phi(C) = 0$.
2. If $E_i \cap \Phi \neq \varnothing$, then $m_i \oplus m_\Phi(E_i \cap \Phi) = p_i$, $m_i \oplus m_\Phi(\Phi) = 1 - p_i$, and for all other $C \in \wp(X)$ we have $m_i \oplus m_\Phi(C) = 0$.

Proof. Let $(E_i, p_i) \in \mathcal{E}_Q$ be arbitrary and let $\Phi \subseteq X$ be any proposition with associated basic probability assignment m_Φ as defined above.

1. Suppose $E_i \cap \Phi = \varnothing$. By the definitions of m_i and m_Φ, the sets assigned positive mass are E_i and X for m_i, and Φ for m_Φ. Since $E_i \cap \Phi = \varnothing$ by assumption, the only non-empty intersection among focal elements is $X \cap \Phi = \Phi$. Hence
$$\sum_{A \cap B = \Phi} m_i(A) m_\Phi(B) = m_i(X) m_\Phi(\Phi) = (1 - p_i) \cdot 1 = 1 - p_i.$$
The conflict is $\sum_{A \cap B = \varnothing} m_i(A) m_\Phi(B) = m_i(E_i) m_\Phi(\Phi) = p_i \cdot 1 = p_i$. Applying Dempster's rule:
$$m_i \oplus m_\Phi(\Phi) = \frac{1 - p_i}{1 - p_i} = 1,$$
and $m_i \oplus m_\Phi(C) = 0$ for all $C \neq \Phi$. This proves claim 1.
2. Suppose $E_i \cap \Phi \neq \varnothing$. The focal elements are again E_i, X for m_i and Φ for m_Φ. All intersections are non-empty

($E_i \cap \Phi$ and $X \cap \Phi = \Phi$), so total conflict is 0. Thus the denominator is 1, and we obtain

$$m_i \oplus m_\Phi(E_i \cap \Phi) = m_i(E_i)m_\Phi(\Phi) = p_i,$$

$$m_i \oplus m_\Phi(\Phi) = m_i(X)m_\Phi(\Phi) = 1 - p_i,$$

and $m_i \oplus m_\Phi(C) = 0$ for all other C. This proves claim 2.

□

This proposition tells us that combining a piece of evidence with a certain proposition yields exactly the same result as our δ^t function when applied to a single piece of evidence. However, our function goes beyond this, since it is able to effectively merge the weights associated with pieces of evidence that have the same intersection with the certain proposition Φ. Therefore, in order to see whether Dempster's rule of combination can replicate this behavior as well, we will introduce a proposition that aims to study what the basic probability assignment obtained by the combination of two evidence basic probability assignments looks like when they are associated with the same piece of evidence.

This can be read as combining the evidence basic probability assignments of two pieces of evidence after they have both been conditioned on Φ and have the same intersection. Indeed, as we just saw in Proposition 13, the weight associated with a piece of evidence after being conditioned on a certain proposition, assuming it has non-empty intersection, is transferred to the intersection between the piece of evidence and the proposition. Thus, we are left with the case of two pieces of evidence with the same intersection, each having an evidence basic probability assignment.

Proposition 14. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and suppose that for some $E \subseteq X$ it holds that $(E, p), (E, q) \in \mathcal{E}_Q$. Let m_p and m_q be the evidence basic probability assignments corresponding to (E, p) and (E, q), respectively, according to Definition 47. Then there exists an evidence basic probability assignment obtained by combining m_p and m_q via Dempster's rule.

Proof. Let $(E, p), (E, q) \in \mathcal{E}_Q$ be arbitrary. Their assignments are:

$$m_p(A) = \begin{cases} p & \text{if } A = E \\ 1 - p & \text{if } A = X \\ 0 & \text{otherwise} \end{cases}, \qquad m_q(A) = \begin{cases} q & \text{if } A = E \\ 1 - q & \text{if } A = X \\ 0 & \text{otherwise} \end{cases}.$$

The focal elements of both are E and X. Since all intersections are non-empty ($E \cap E = E$, $E \cap X = E$, $X \cap E = E$, $X \cap X = X$), the total conflict is 0 and the denominator equals 1. Therefore

$$m_p \oplus m_q(E) = m_p(E)m_q(E) + m_p(X)m_q(E) + m_p(E)m_q(X) = pq + (1 - p)q + p(1 - q) = q + p - pq.$$

Let $e = p + q - pq$. Then

$$m_p \oplus m_q(X) = m_p(X)m_q(X) = (1 - p)(1 - q) = 1 - e.$$

For all other $A \subseteq X$, $m_p \oplus m_q(A) = 0$. Since $0 \leqslant e \leqslant 1$ (because $e = 1 - (1-p)(1-q)$ and $(1-p)(1-q) \in [0,1]$), the combined assignment is itself an evidence basic probability assignment with weight e on E and $1-e$ on X. $\square$

As we saw, after combining via Dempster's rule of combination two evidence basic probability assignments associated with the same evidence, we are left with a new evidence basic probability assignment that has the same focal elements as the previous functions. Notice that we did not introduce any proposition Φ, but the same proposition holds in the case where we have a proposition $\Phi \subseteq X$ and the evidence E in Proposition 14 is $E = E_j \cap \Phi = E_i \cap \Phi$ for some $E_j, E_i \subseteq X$.

Now, we proceed by introducing an operator that applies Dempster's rule of combination recursively. When the operator is defined on a single piece of evidence, it simply conditions that evidence on the proposition Φ, producing a conditioned basic probability assignment. When the operator is defined on multiple pieces of evidence, it first conditions each individual piece of evidence on Φ and then combines the resulting conditioned basic probability assignments recursively using Dempster's rule of combination.

Definition 48 (Merging Operator)**.** Given a quantitative evidence frame $(X, \mathcal{E}_Q)$ and a proposition $\Phi \subseteq X$ with associated certain basic probability assignment

$$m_\Phi(A) = \begin{cases} 1 & \text{if } A = \Phi \\ 0 & \text{otherwise} \end{cases},$$

we define the merging operator recursively for all $A \subseteq X$ as follows:

$$\begin{aligned} \bigoplus(E,p)(A) &= (m_\Phi \oplus m_E)(A), \\ \bigoplus((E_1,p_1),(E_2,p_2))(A) &= \bigoplus(E_1,p_1) \oplus \bigoplus(E_2,p_2)(A) = (m_\Phi \oplus m_{E_1}) \oplus (m_\Phi \oplus m_{E_2})(A), \\ \bigoplus_{i=1}^{n}(E_i,p_i)(A) &= \bigoplus\left(\bigoplus_{i=1}^{n-1}(E_i,p_i),(E_n,p_n)\right)(A). \end{aligned}$$

Now, we introduce a final proposition that aims to relate our merging function δ^t, as defined in Definition 34, with the merging operator, just introduced.

Proposition 15. Let $(X, \mathcal{E}_Q)$ be a quantitative evidence frame and let $\Phi \subseteq X$ be a proposition. Consider an equivalence class $[E]_\Phi = \{(E_1,p_1),\dots,(E_n,p_n)\}$ under the relation $\sim_\Phi$ defined in Definition 35, where for each i we have $E_i \cap \Phi = E \neq \varnothing$. According to Definition 34 the merged weight p_E assigned to $E^{!\Phi}$ in the updated quantitative evidence frame $\mathcal{E}_Q^{!\Phi}$ is given by:

$$p_E = \delta^t\left([E]_\Phi\right) = 1 - \prod_{i=1}^{n}(1-p_i).$$

Then, we claim that

$$p_E = \bigoplus_{i=1}^{n}(E_i,p_i)\,(E)$$

where $\bigoplus_{i=1}^{n}(E_i,p_i)$ is the merging operator as defined in Definition 48.

Proof. We prove this proposition by induction on the number n of weighted pieces of evidence in the equivalence class $[E]_\Phi$.

Base case Consider an arbitatry equivalence class which contains a single weighted evidence (E_1, p_1). By Definition 35

$$p_E = \delta^t\left([E]_\Phi\right) = 1 - (1 - p_1) = p_1.$$

Consider now the evidence basic probability assignments m_1 associated with (E_1, p_1) and m_Φ associated with $(\Phi, 1)$. Then, by Proposition 13, we know that

$$\begin{aligned}
\bigoplus_{i=1}^{n}(E_i, p_i)\,(E) &= \bigoplus(E_1, p_1)\,(E) && \text{(Definition 48)}\\
&= (m_1 \oplus m_\Phi)\,(E) \\
&= (m_1 \oplus m_\Phi)(E_1 \cap \Phi) && (E = E_1 \cap \Phi)\\
&= p_1 && \text{(Proposition 13, 2)}\\
&= p_E
\end{aligned}$$

Thus, for $[E]_\Phi = \{(E_1, p_1)\}$ holds that

$$p_E = \bigoplus_{i=1}^{n}(E_i, p_i)\,(E)\,.$$

Furthermore, since $[E]_\Phi$ was an arbitrary class with one weighted piece of evidence, we can conlcude that the claim holds for all such equivalence classes. Therefore, the base case holds.

Inductive step Assume that for any equivalence class containing n weighted pieces of evidence (with $n \geqslant 1$), the merged weight satisfies:

$$p_E^{(n)} = \bigoplus_{i=1}^{n}(E_i, p_i)\,(E) = 1 - \prod_{i=1}^{n}(1 - p_i). \tag{IH}$$

Consider an arbitrary equivalence class $[E]_\Phi = \{(E_1, p_1), \ldots, (E_n, p_n), (E_{n+1}, p_{n+1})\}$ containing $n+1$ weighted pieces of evidence . By Definition 34, we know that

$$\begin{aligned}
p_E &= 1 - \prod_{i=1}^{n+1}(1 - p_i)\\
&= 1 - \prod_{i=1}^{n}(1 - p_i)(1 - p_{n+1})\\
&= 1 - \left(\prod_{i=1}^{n}(1 - p_i) - \left(\prod_{i=1}^{n}(1 - p_i)\right) \cdot p_{n+1}\right)\\
&= 1 - \prod_{i=1}^{n}(1 - p_i) + p_{n+1}\left(\prod_{i=1}^{n}(1 - p_i)\right)\\
&= 1 - \prod_{i=1}^{n}(1 - p_i) + p_{n+1}\left(\prod_{i=1}^{n}(1 - p_i)\right) - p_{n+1} + p_{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \prod_{i=1}^{n}(1-p_i) - p_{n+1}\left(1 - \prod_{i=1}^{n}(1-p_i)\right) + p_{n+1} \\
&= p_E^{(n)} - p_{n+1} \cdot p_E^{(n)} + p_{n+1} && \text{(IH)} \\
&= \bigoplus_{i=1}^{n}(E_i, p_i) \oplus m_{n+1}\,(E) && \text{(Proposition 14)} \\
&= \bigoplus\left(\bigoplus_{i=1}^{n}(E_i, p_i), (E_{n+1}, p_{n+1})\right)(E) && \text{(Definition 48)} \\
&= \bigoplus_{i=1}^{n+1}(E_i, p_i)\,(E)
\end{aligned}
$$

Thus, we can conclude that for $[E]_\Phi$ holds that

$$p_E = \bigoplus_{i=1}^{n+1}(E_i, p_i)\,(E)\,.$$

Furthermore, since $[E]_\Phi$ was an arbitrary class with $n+1$ weighted pieces of evidence, we can conlcude that the claim holds for all such equivalence classes

By mathematical induction, the formula

$$p_E = \bigoplus_{i=1}^{n}(E_i, p_i)\,(E)$$

holds for all $n \geqslant 1$. Therefore, we can conclude that the claim holds □

Thus, the δ^t function exactly captures the combined weight of multiple pieces of evidence that share the same restriction after conditioning on Φ. Moreover, Proposition 15 establishes a formal equivalence between our merging function δ^t and the recursive application of Dempster's rule of combination via the merging operator. This demonstrates that at the level of evidence aggregation i.e., when we combine the weights of pieces of evidence that have the same intersection with Φ, our conditionalization procedure behaves identically to conditioning via Dempster's rule.

3.1.2 Relating our conditionalization to Dempster-Shafer conditioning via Dempster's rule of combination applied to the final basic probability assignment

Having established this correspondence at the level of evidence aggregation, we are now in a position to compare the belief functions produced by the two frameworks. Recall that in our MLBM model, the final multi-layer belief function $\mathrm{Bel}_{\mathcal{J}}(f,\cdot)$ is obtained by applying the bridging layer to the aggregated evidence weights via the evidence allocation function f and the frame of justification $\mathcal{J}$. In the Dempster-Shafer framework, by contrast, the conditional belief function $\mathrm{Bel}|_\Phi$ is obtained by applying Dempster's rule directly to the basic probability assignment derived from the original belief function.

The question that now arises is whether, under suitable choices of the evidence allocation function f and the frame of justification $\mathcal{J}$, the conditional multi-layer belief function $\mathrm{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi},\cdot)$ coincides with the Dempster-

Shafer conditional belief function $\mathrm{Bel}|_\Phi$. Thus, we will analyze this question for the two types of Dempster-Shafer conditioning introduced above, identifying the conditions under which the two approaches yield equivalent results.

To do so we will first introduce some auxiliary lemmas in order to be able to relate the two functions.

The identities in these lemmas are immediate algebraic consequences of the principle of inclusion-exclusion (Hall, 1998, Chapter 2), which determines, for a given set of objects and collection of properties, the number of objects satisfying none of those properties. This is formally expressed as follows:

Definition 49 (Inclusion-exclusion principle)**.** Let N be a set and $\mathcal{P} = \{P_1, \ldots, P_k\}$ a set of properties. Let be n_i the number of objects in N with the property P_i, with $1 \leqslant i \leqslant k$, and more generally $n_{i_1,i_2,i_3,\ldots,i_\ell}$ the number of objects in N with the properties $P_{i_1}, P_{i_2}, P_{i_3}, \ldots, P_{i_\ell}$, where $1 \leqslant i_1 < i_2 < i_3 < \cdots < i_\ell \leqslant k$. Then the number of objects with none of the properties n_0, is given by the following formula:

$$n(0) = |N| - \sum_{1\leqslant i\leqslant k} n_i + \sum_{1\leqslant i_1<i_2\leqslant k} n_{i_1,i_2} - \sum_{1\leqslant i_1<i_2<i_3\leqslant k} n_{i_1,i_2,i_3} + \ldots$$
$$+ (-1)^\ell \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant k} n_{i_1,\ldots,i_\ell} + \cdots + (-1)^k n_{1,2,\ldots,k}.$$

Now we want to connect this formula to our δ function. To see the connection intuitively, consider a quantitative evidence frame with m pieces of evidence. For each subset $E \subseteq \mathcal{E}$ of the evidence, we can compute the δ function, as in Definition 21, as follows:

$$\delta(E) = \prod_{E_i\in E} p_i \prod_{E_i\notin E} (1 - p_i).$$

One can interpret these 2^m subsets as *objects* and assign to each object $E \subseteq \mathcal{E}$ the property P_i meaning "$E_i \in E$". Then:

- The total mass of all objects is $n = \sum_{E\subseteq\mathcal{E}} \delta(E) = 1$, being δ a mass function over a set of evidence according to Definition 20.
- The weight of objects satisfying properties $P_{i_1}, \ldots, P_{i_\ell}$ is $n_{i_1\ldots i_\ell} = \sum_{E\supseteq\{E_{i_1},\ldots,E_{i_\ell}\}} \delta(E) = \prod_{j=1}^{\ell} p_{i_j}$.

Applying the inclusion-exclusion formula to count the weight of objects with *none* of the properties (i.e., only the empty set $S = \varnothing$) yields:

$$\prod_{i=1}^{m}(1 - p_i) = 1 - \sum_i p_i + \sum_{i<j} p_i p_j - \cdots + (-1)^m \prod_{i=1}^{m} p_i.$$

The other identity, which expresses $1 - \prod(1 - p_i)$ as a sum over non-empty subsets with alternating signs, follows immediately by rearranging terms. We state them formally in the two propositions below.

Lemma 1. Given a set $\{p_1, \ldots, p_m\} \subseteq [0, 1]^m$, then we claim that

$$1 - \prod_{i=1}^{m}(1 - p_i) = \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S\neq\varnothing}} (-1)^{|S|+1} \prod_{i\in S} p_i,$$

Lemma 2. Given a set $\{p_1, \ldots, p_m\} \subseteq [0,1]^m$, then we claim that

$$\prod_{i=1}^{m}(1-p_i) = \sum_{S\subseteq\{1,\ldots,m\}} (-1)^{|S|} \prod_{i\in S} p_i.$$

Their derivation, while straightforward, involves a series of algebraic manipulations. Furthermore, these identities mainly serve to simplify the main results. Accordingly, we omit the full proof here, as it offers little additional insight. However, a detailed derivation can be found in the Appendix *Proof of Lemma 1* and Appendix *Proof of Lemma 2* for interested readers.

The last lemma we prove is the crucial identity for the main proof: it explicitly characterizes how the δ function behaves when applied to the updated evidence set, as defined in Definition 31, in terms of its behavior on the original evidence set before conditionalization.

Lemma 3. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathfrak{J}, \delta, f)$ be an MLBM structure as in Definition 45, and let $\mathcal{E}$ be the underlying set of evidence of $\mathcal{E}_Q$ according to Definition 38. Let $\Phi \subseteq X$ be a proposition, and let $\mathfrak{B}^{!\Phi} = \left(\Phi, \mathcal{E}_Q^{!\Phi}, \mathfrak{J}^{!\Phi}, \delta^{!\Phi}, f^{!\Phi}\right)$ be the conditionalized MLBM structure as in Definition 46. Let ρ_Φ be the function defined in Definition 39. Then for any $E \subseteq \mathcal{E}^{!\Phi}$,

$$\delta^{!\Phi}(E) = \frac{\displaystyle\sum_{\substack{E'\subseteq\mathcal{E}\\ \rho_\Phi(E')=E}} \delta(E')}{\displaystyle\sum_{\substack{F\subseteq\mathcal{E}\\ \rho_\Phi(F)\neq\varnothing}} \delta(F)},$$

where $\delta^{!\Phi}$ is the conditionalized δ function defined on the updated quantitative evidence frame $\left(\Phi, \mathcal{E}_Q^{!\Phi}\right)$ according to Definition 37.

Proof. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathfrak{J}, \delta, f)$ be an MLBM structure and $\mathcal{E}$ its underlying set of evidence. Let $\Phi \subseteq X$ be an arbitrary proposition. Then $\mathfrak{B}^{!\Phi} = \left(\Phi, \mathcal{E}_Q^{!\Phi}, \mathfrak{J}^{!\Phi}, \delta^{!\Phi}, f^{!\Phi}\right)$ be the conditionalized MLBM structure as in Definition 46, and $\mathcal{E}^{!\Phi}$ is the underlying set of evidence of $\mathcal{E}_Q^{!\Phi}$. By Proposition 10 we know that $\mathcal{E}^{!\Phi}$ is also the result of conditioning the set of evidence $\mathcal{E}$ on Φ according to Definition 31. Then, according to Definition 37, $\delta^{!\Phi} : \wp(\mathcal{E}^{!\Phi}) \longrightarrow [0,1]$ satisfies for all $E \subseteq \mathcal{E}^{!\Phi}$:

$$\delta^{!\Phi}(E) = \prod_{E_i\in E} p_i \prod_{E_i\notin E}(1-p_i). \tag{$\star_1$}$$

Consider an arbitrary $E \subseteq \mathcal{E}^{!\Phi}$. Then $E = \{E_1^{!\Phi}, \ldots, E_k^{!\Phi}\}$, where for every $i \in \{1, \ldots, k\}$, it follows from Definition 31 that $E_i^{!\Phi} = E_i \cap \Phi$ for some $E_i \in \mathcal{E}$.

Let us denote with $p_i^{!\Phi}$ the updated weight after applying Definition 35 to $(X, \mathcal{E}_Q)$ with proposition Φ. By Definition 35, $p_i^{!\Phi}$ is obtained by merging all weighted evidence pieces in the equivalence class $[E_i]_\Phi = \{(E,p) \in \mathcal{E}_Q \mid E \cap \Phi = E_i^{!\Phi}\}$:

$$p_i^{!\Phi} = \delta^t([E_i]_\Phi),$$

where $[E_i]_\Phi$ denotes the equivalence class of weighted evidence pieces that update to $E_i^{!\Phi}$, and δ^t is the merging function defined in Definition 34. Denote the cardinality of $[E_i]_\Phi$ by m_i for each $i \in \{1, \ldots, k\}$. Applying Equation ($\star_1$)

to E we obtain:

$$\delta^{!\Phi}(E) = \prod_{E_i^{!\Phi} \in E} p_i^{!\Phi} \prod_{E_i^{!\Phi} \notin E} \left(1 - p_i^{!\Phi}\right) = \prod_{i=1}^{k} p_i^{!\Phi} \prod_{E_i^{!\Phi} \notin E} \left(1 - p_i^{!\Phi}\right),$$

for all $E \subseteq \mathcal{E}^{!\Phi}$. By Definition 35, for every $i \in \{1, \ldots, k\}$,

$$p_i^{!\Phi} = \delta^t([E_i]_\Phi) = 1 - \prod_{j=1}^{m_i}(1 - p_j).$$

Thus,

$$\begin{aligned}
\delta^{!\Phi}(E) &= \prod_{i=1}^{k} p_i^{!\Phi} \prod_{E_i^{!\Phi} \notin E} \left(1 - p_i^{!\Phi}\right) \\
&= \prod_{i=1}^{k} \left(1 - \prod_{j=1}^{m_i}(1 - p_j)\right) \prod_{E_i^{!\Phi} \notin E} \left(1 - \left(1 - \prod_{j=1}^{m_i}(1 - p_j)\right)\right) \\
&= \prod_{i=1}^{k} \left(1 - \prod_{j=1}^{m_i}(1 - p_j)\right) \prod_{E_i^{!\Phi} \notin E} \prod_{j=1}^{m_i}(1 - p_j).
\end{aligned}$$

Let $\{F_1^{!\Phi}, \ldots, F_\ell^{!\Phi}\} = \mathcal{E}^{!\Phi} \setminus E$ be the updated evidence pieces not in E, and for each $t \leqslant \ell$ let n_t be the cardinality of its equivalence class $[F_t]_\Phi$. Then

$$\delta^{!\Phi}(E) = \prod_{i=1}^{k} \left(1 - \prod_{j=1}^{m_i}(1 - p_j)\right) \prod_{t=1}^{\ell} \prod_{j=1}^{n_t}(1 - p_j).$$

By Lemmas 1 and 2,

$$\delta^{!\Phi}(E) = \prod_{i=1}^{k} \left(\sum_{\substack{S \subseteq \{1,\ldots,m_i\} \\ S \neq \varnothing}} (-1)^{|S|+1} \prod_{j \in S} p_j \right) \prod_{t=1}^{\ell} \left(\sum_{S \subseteq \{1,\ldots,n_t\}} (-1)^{|S|} \prod_{j \in S} p_j \right). \tag{$\star_2$}$$

We now apply the distributive law. Indeed, we can state a general property of distributivity of products over sums for any finite index sets $I_1, \ldots, I_\ell$ and complex numbers $a_{i_1}^{(1)}, \ldots, a_{i_\ell}^{(\ell)}$,

$$\prod_{u=1}^{\ell} \left(\sum_{i_u \in I_u} a_{i_u}^{(u)} \right) = \sum_{i_1 \in I_1} \cdots \sum_{i_r \ell \in I_\ell} \prod_{u=1}^{r} a_{i_u}^{(u)}. \tag{Distributive Law}$$

Since we are working with finite index sets and real numbers, we can apply the latter to Equation ($\star_2$), getting as a result:

$$\delta^{!\Phi}(E) = \sum_{\substack{S_1 \subseteq \{1,\ldots,m_1\} \\ S_1 \neq \varnothing}} \cdots \sum_{\substack{S_k \subseteq \{1,\ldots,m_k\} \\ S_k \neq \varnothing}} \sum_{T_1 \subseteq \{1,\ldots,n_1\}} \cdots \sum_{T_\ell \subseteq \{1,\ldots,n_\ell\}}$$

$$\left(\prod_{i=1}^{k}(-1)^{|S_i|+1}\right)\left(\prod_{t=1}^{\ell}(-1)^{|T_t|}\right)\left(\prod_{i=1}^{k}\prod_{j\in S_i}p_j\right)\left(\prod_{t=1}^{\ell}\prod_{j\in T_t}p_j\right). \tag{$\star_3$}$$

Consider now any tuple of the form $(S_1,\ldots,S_k,T_1,\ldots,T_\ell)$, then we can define

$$U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)=\bigcup_{i=1}^{k}\left\{E_i^j \mid j\in S_i\right\}\cup\bigcup_{t=1}^{\ell}\left\{F_t^j \mid j\in T_t\right\}. \tag{$\spadesuit$}$$

Clearly $U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)\subseteq\mathcal{E}$, as each E_i^j and F_t^j belongs to $\mathcal{E}$. Moreover, the sets $\{E_i^j \mid j\in S_i\}$ and $\{F_t^j \mid j\in T_t\}$ are pairwise disjoint (they belong to distinct equivalence classes), so

$$|U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)|=\sum_{i=1}^{k}|S_i|+\sum_{t=1}^{\ell}|T_t|. \tag{1}$$

Hence,

$$\left(\prod_{i=1}^{k}(-1)^{|S_i|+1}\right)\left(\prod_{t=1}^{\ell}(-1)^{|T_t|}\right)=(-1)^{\sum_{i=1}^{k}|S_i|+k+\sum_{t=1}^{\ell}|T_t|}=(-1)^{|U|+k},$$

and

$$\left(\prod_{i=1}^{k}\prod_{j\in S_i}p_j\right)\left(\prod_{t=1}^{\ell}\prod_{j\in T_t}p_j\right)=\prod_{E\in U}p_x.$$

Let

$$\mathcal{U}=\{U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)\mid \text{for all } 1\leqslant i\leqslant k,\ U\cap[E_i]_\Phi\neq\emptyset\}.$$

In other words, $\mathcal{U}$ consists of all subsets $U\subseteq\mathcal{E}$ that contain at least one element from each equivalence class $[E_i]_\Phi$ for $i\in\{1,\ldots,k\}$.

We now prove that we cam map every tuple to an element of $\mathcal{U}$ by showing two directions.

($\Rightarrow$) Let $(S_1,\ldots,S_k,T_1,\ldots,T_\ell)$ be an arbitrary tuple with $S_i\neq\emptyset$ for all i. For any $i\in\{1,\ldots,k\}$, and let $j\in S_i$ (such j exists because $S_i\neq\emptyset$). Then $E_i^j\in U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)$ by definition of $U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)$, and $E_i^j\in[E_i]_\Phi$ by construction. Hence $U\cap[E_i]_\Phi\neq\emptyset$, so $U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)\in\mathcal{U}$. Since the tuple was arbitrary, every tuple maps to an element of $\mathcal{U}$.

($\Leftarrow$) Conversely, let $U\in\mathcal{U}$ be arbitrary. For each $i\in\{1,\ldots,k\}$, since $U\cap[E_i]_\Phi\neq\emptyset$, we define

$$S_i=\{j\mid 1\leqslant j\leqslant m_i \text{ and } E_i^j\in U\cap[E_i]_\Phi\}.$$

Similarly, for each $t\leqslant\ell$, we define

$$T_t=\{j\mid 1\leqslant j\leqslant n_t \text{ and } F_t^j\in U\cap[F_t]_\Phi\}.$$

First notice that for all i, S_i is non-empty because $U\in\mathcal{U}$ implies $U\cap[E_i]_\Phi\neq\emptyset$. Let $E\in U\cap[E_i]_\Phi$. Then there exists a unique $j\in\{1,\ldots,m_i\}$ such that $E=E_i^j$. By definition of S_i, this j belongs to S_i. Hence $S_i\neq\emptyset$. Now we show that $U(S_1,\ldots,S_k,T_1,\ldots,T_\ell)=U$.

Let $E \in U$ be arbitrary. Since $U \subseteq \mathcal{E}$ and $\mathcal{E}$ is partitioned into the equivalence classes $[E_1]_\Phi, \ldots, [E_k]_\Phi$, $[F_1]_\Phi, \ldots, [F_\ell]_\Phi$, E belongs to exactly one of these classes. We have two possibilities:

- If $E \in [E_i]_\Phi$ for some $i \in \{1, \ldots, k\}$, then $E = E_i^j$ for a unique j. By definition of S_i, we have $j \in S_i$. Hence $E \in U(S_1, \ldots, S_k, T_1, \ldots, T_\ell)$.
- If $E \in [F_t]_\Phi$ for some $t \in [\ell]$, then $E = F_t^j$ for a unique j. By definition of T_t, we have $j \in T_t$. Hence $E \in U(S_1, \ldots, S_k, T_1, \ldots, T_\ell)$.

Thus $U \subseteq U(S_1, \ldots, S_k, T_1, \ldots, T_\ell)$. The reverse inclusion $U(S_1, \ldots, S_k, T_1, \ldots, T_\ell) \subseteq U$ holds by construction, since each E_i^j with $j \in S_i$ belongs to $U \cap [E_i]_\Phi \subseteq U$, and each F_t^j with $j \in T_t$ belongs to $U \cap [F_t]_\Phi \subseteq U$. Therefore $U(S_1, \ldots, S_k, T_1, \ldots, T_\ell) = U$, and the tuple maps back to U. Since $U \in \mathcal{U}$ was arbitrary, every element of $\mathcal{U}$ is obtained a tuple.

Thus, we get a mapping between the set of tuples and $\mathcal{U}$.
Therefore,

$$\delta^{!\Phi}(E) = \sum_{U \in \mathcal{U}} (-1)^{|U|+k} \prod_{x \in U} p_x. \tag{$\star_4$}$$

Now consider ρ_Φ as defined in Definition 39. For any $U \subseteq \mathcal{E}$ and $i \in \{1, \ldots, k\}$,

$$U \cap [E_i]_\Phi \neq \emptyset \iff E_i^{!\Phi} \in \rho_\Phi(U).$$

Thus, combining the previous result with the latter, we can conclude that

$$U \in \mathcal{U} \iff E_i^{!\Phi} \in \rho_\Phi(U)$$

for all $i \in \{1, \ldots, k\}$, i.e., $E \subseteq \rho_\Phi(U)$. Hence,

$$\delta^{!\Phi}(E) = \sum_{\substack{U \subseteq \mathcal{E} \\ E \subseteq \rho_\Phi(U)}} (-1)^{|U|+k} \prod_{x \in U} p_x = \sum_{\substack{U \subseteq \mathcal{E} \\ E \subseteq \rho_\Phi(U)}} (-1)^{|U|+|E|} \prod_{x \in U} p_x. \tag{$\star_5$}$$

We now connect this to the original δ. For any $E' \subseteq \mathcal{E}$,

$$\delta(E') = \prod_{E_x \in E'} p_x \prod_{E_x \notin E'} (1 - p_x). \tag{$\bigstar_1$}$$

Expanding $\prod_{E_x \notin E'}(1 - p_x)$ by Lemmas 1 and 2,

$$\delta(E') = \sum_{V \subseteq \mathcal{E} \setminus E'} (-1)^{|V|} \prod_{x \in E' \cup V} p_x. \tag{$\bigstar_2$}$$

Set $U = E' \cup V$. Then $V = U \setminus E'$ and $|V| = |U| - |E'|$, so

$$\delta(E') = \sum_{U \supseteq E'} (-1)^{|U|-|E'|} \prod_{x \in U} p_x. \tag{$\bigstar_3$}$$

Summing over E' with $\rho_\Phi(E') = E$,

$$\sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \delta(E') = \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \sum_{U \supseteq E'} (-1)^{|U|-|E'|} \prod_{x \in U} p_x$$
$$= \sum_{U \subseteq \mathcal{E}} (-1)^{|U|} \left(\sum_{\substack{E' \subseteq U \\ \rho_\Phi(E')=E}} (-1)^{|E'|} \right) \prod_{x \in U} p_x. \qquad (\bigstar_4)$$

Fix $U \subseteq \mathcal{E}$. The inner sum $\sum_{\substack{E' \subseteq U \\ \rho_\Phi(E')=E}} (-1)^{|E'|}$ is non-zero only if $U \cap [E_i]_\Phi \neq \varnothing$ for all $i \in \{1, \dots, k\}$ (otherwise no E' exists). When this holds, the sum factors over the disjoint classes:

$$\sum_{\substack{E' \subseteq U \\ \rho_\Phi(E')=E}} (-1)^{|E'|} = \prod_{i=1}^{k} \sum_{\substack{S \subseteq U \cap [E_i]_\Phi \\ S \neq \varnothing}} (-1)^{|S|} \cdot \prod_{j=1}^{\ell} 1.$$

For each i, since $U \cap [E_i]_\Phi \neq \varnothing$,

$$\sum_{\substack{S \subseteq A_i \\ S \neq \varnothing}} (-1)^{|S|} = \left(\sum_{S \subseteq A_i} (-1)^{|S|} \right) - 1 = 0 - 1 = -1,$$

where $A_i = U \cap [E_i]_\Phi$. Hence the product over i gives $(-1)^k = (-1)^{|E|}$. Thus

$$\sum_{\substack{E' \subseteq U \\ \rho_\Phi(E')=E}} (-1)^{|E'|} = \begin{cases} (-1)^{|E|} & \text{if } U \cap [E_i]_\Phi \neq \varnothing \text{ for all } i \in \{1, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \delta(E') = \sum_{\substack{U \subseteq \mathcal{E} \\ U \cap [E_i]_\Phi \neq \varnothing \ \forall i}} (-1)^{|U|} \cdot (-1)^{|E|} \prod_{x \in U} p_x$$
$$= \sum_{\substack{U \subseteq \mathcal{E} \\ E \subseteq \rho_\Phi(U)}} (-1)^{|U|+|E|} \prod_{x \in U} p_x. \qquad (\bigstar_5)$$

Comparing Equation ($\star_5$) and Equation ($\bigstar_5$), the right-hand sides are identical. Hence

$$\delta^{!\Phi}(E) = \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \delta(E'). \qquad (\star_6)$$

However, this expression is not yet the correct definition of $\delta^{!\Phi}$ because the right-hand side does not necessarily sum to 1 over all $E \subseteq \mathcal{E}^{!\Phi}$. Indeed,

$$\sum_{E \subseteq \mathcal{E}^{!\Phi}} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \delta(E') = \sum_{\substack{F \subseteq \mathcal{E} \\ \rho_\Phi(F) \neq \varnothing}} \delta(F),$$

since every $F \subseteq \mathcal{E}$ with $\rho_\Phi(F) \neq \varnothing$ is counted exactly once (for $E = \rho_\Phi(F)$), while those with $\rho_\Phi(F) = \varnothing$ are not counted at all. The sum of δ over all $F \subseteq \mathcal{E}$ is 1, so the right-hand side above is $1 - \sum_{\substack{F \subseteq \mathcal{E} \\ \rho_\Phi(F)=\varnothing}} \delta(F) < 1$ whenever there exists F with $\rho_\Phi(F) = \varnothing$ and $\delta(F) > 0$, which is the typical case.

But $\delta^{!\Phi}$ is defined as a mass function on $\mathcal{E}^{!\Phi}$, meaning it must satisfy $\sum_{E \subseteq \mathcal{E}^{!\Phi}} \delta^{!\Phi}(E) = 1$. Therefore, the expression in Equation ($\star_6$) gives the correct relative proportions but lacks the proper normalization. To obtain a genuine mass function, we must divide by the total mass of the right-hand side, i.e.,

$$\delta^{!\Phi}(E) = \frac{\sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E')=E}} \delta(E')}{\sum_{\substack{F \subseteq \mathcal{E} \\ \rho_\Phi(F) \neq \varnothing}} \delta(F)}.$$

This division ensures that $\sum_{E \subseteq \mathcal{E}^{!\Phi}} \delta^{!\Phi}(E) = 1$, as required.

Furthermore, since $\mathfrak{B} = (X, \mathcal{E}_Q, \mathcal{J}, \delta, f)$ was an arbitrary MLBM structure, $\Phi \subseteq X$ an arbitrary proposition, and $E \subseteq \mathcal{E}^{!\Phi}$ an arbitrary subset, we can conclude that the claim holds for all MLBM structures, for all propositions $\Phi \subseteq X$, and for all $E \subseteq \mathcal{E}^{!\Phi}$. Thus we can conclude that our claim holds. $\square$

Equipped with the lemmas established above, we are finally in a position to connect the belief function resulting from our conditional update to the standard Dempster-Shafer conditional belief function introduced in Definition 30, when applied to the belief defined in Definition 26.

Proposition 16. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathcal{J}, \delta, f)$ be an MLBM structure as in Definition 45, and let $\mathcal{E}$ be its underlying set of evidence of $\mathcal{E}_Q$, as in Definition 38. Let $\Phi \subseteq X$ be a proposition such that $m_\Phi(\Phi) = 1$ and assume that for all $A \subseteq \mathcal{E}$ it holds that:

$$\sum_{A \cap \Phi \neq \varnothing} \sum_{E : f(E)=A} \delta(E) > 0. \qquad (\spadesuit)$$

Let $\mathfrak{B}^{!\Phi} = \left(\Phi, \mathcal{E}_Q^{!\Phi}, \mathcal{J}^{!\Phi}, \delta^{!\Phi}, f^{!\Phi}\right)$ be the conditionalized MLBM structure as in Definition 46. Recall that $\rho_\Phi : \wp(\mathcal{E}) \to \wp\left(\mathcal{E}^{!\Phi}\right)$, defined in Definition 39, is defined by

$$\rho_\Phi(E') = \{E \cap \Phi \mid E \in E',\ E \cap \Phi \neq \varnothing\}.$$

Then for any proposition $\Psi \subseteq X$,

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$$

if and only if it holds that for every $E' \subseteq \mathcal{E}$ with $\delta(E') > 0$

1. $f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi$;
2. $f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}$ if and only if $f(E') \cap \Phi \neq \emptyset$.

Outline of the proof of Proposition 16. We will proof the claim, unravelling the definitions of $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi)$ and $\mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$ to arrive at the level of the respective δ-functions, i.e. the functions defined respectively in Definitions 21 and 37. Then we will study separately the two directions of our claim.

First, we apply the result in Lemma 3 to translate the function $\delta^{!\Phi}$ in the function $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi)$ to the original δ function employed in $\mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$. Thanks to this, we are able to compare the two functions on the same terms, and applying condition (1) and (2), respectively to the numerator and denominator of $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi)$, we can then translate the latter function into the $\mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$. Proving the first direction.

Second, we assume the equality $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$ to hold, and prove that conditions (1) and (2) must hold conjunctively.

Under the assumption that the two belief functions are equal for all propositions, we first prove that condition (2) implies condition (1). We proceed by induction on the cardinality of evidence subsets. For the base case $|E'| = 0$, condition (1) holds trivially. Assuming it holds for all subsets of size less than m, we consider a subset E' with $|E'| = m$ where condition (1) might fail. Using the equality of the belief functions and the induction hypothesis, we isolate the contribution of E' in the numerators. If condition (1) failed for E', we would be forced to generate an infinite sequence of distinct subsets to balance the sums, which is impossible because $\wp(\mathcal{E})$ is finite. Hence condition (1) must hold for E', completing the induction.

Next, we prove that condition (1) implies condition (2). We establish the two directions of the bi-conditional separately. The left-to-right direction is straightforward: if $f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}$, then it must be non-empty, so by condition (1) we have $f(E') \cap \Phi \neq \emptyset$. For the right-to-left direction, we assume $f(E') \cap \Phi \neq \emptyset$ and need to show $f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}$. From the overall equality of the belief functions and condition (1), we deduce that the two normalization denominators are equal. If $f^{!\Phi}(\rho_\Phi(E')) \notin \mathcal{J}^{!\Phi}$, then E' would contribute its mass $\delta(E')$ to the right denominator but not to the left denominator, making the denominators unequal, getting i this way a contradiction. Thus the condition holds.

Finally, we show that the equality of belief functions forces the conjunction of (1) and (2), not merely their bi-implication. We construct a concrete example using the strong denseness frame $\mathcal{J}_{\mathrm{SD}}$, defined in Definition 19, and the t evidence allocation function, defined in Definition 40, where both conditions fail. In this example, $\mathsf{Bel}_{\mathcal{J}^{!\Phi}_{\mathrm{SD}}}(t^{!\Phi}, \Psi) = 3/4$ while $\mathsf{Bel}_{\mathcal{J}_{\mathrm{SD}}}(t, \Psi \mid \Phi) = 3/7$, so the equality does not hold. This demonstrates that the bi-conditional between (1) and (2) is insufficient; both must be true together for the equality to hold, completing the proof of the second direction.

Proof. Let $\Psi \subseteq X$ be an arbitrary proposition such that Equation (♠) holds.[2] We begin by unraveling the two sides of our equality, starting with $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi)$.

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \sum_{U \subseteq \Psi} \delta_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, U) \qquad \text{(Definition 44)}$$

2 Note that condition in Equation (♠) ensures that the denominators appearing in the Dempster-Shafer combination rule and in the conditionalized belief functions are non-zero, so we can safely apply these operations without encountering division by zero. In particular, $\sum_{A \cap \Phi \neq \emptyset} \sum_{E: f(E) = A} \delta(E) > 0$ guarantees that the denominator in Equation ($\star_2$) is non-zero, and similarly for the denominators in the conditionalized definitions.

$$= \sum_{U \subseteq \Psi} \frac{\delta_{\tau}^{!\Phi}(f^{!\Phi}, U)}{\sum_{T \in \mathcal{J}^{!\Phi}} \delta_{\tau}^{!\Phi}(f^{!\Phi}, T)} \qquad \text{(Definition 43)}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{E : f^{!\Phi}(E) = U} \delta^{!\Phi}(E)}{\sum_{T \in \mathcal{J}^{!\Phi}} \sum_{E : f^{!\Phi}(E) = T} \delta^{!\Phi}(E)} \qquad \text{(Definition 42)}$$

Thus, we consider from now on, as follows:

$$\mathrm{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \sum_{U \subseteq \Psi} \frac{\sum_{E : f^{!\Phi}(E) = U} \delta^{!\Phi}(E)}{\sum_{T \in \mathcal{J}^{!\Phi}} \sum_{E : f^{!\Phi}(E) = T} \delta^{!\Phi}(E)} \qquad (\star_1)$$

Now, we proceed by unpacking the definition of $\mathrm{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$, according to Definition 30.

$$\mathrm{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi) = \sum_{U \subseteq \Psi} (\delta_{\mathcal{J}}(f, \cdot) \oplus m_{\Phi})(U) \qquad \text{(Definition 30)}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap B = U} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(B)}{1 - \sum_{A \cap B = \emptyset} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(B)} \qquad \text{(Definition 16)}$$

Since $m_{\Phi}(\Phi) = 1$, the only focal element of m_{Φ} is Φ itself; consequently, in Dempster's rule the only contributions come from intersections of the form $A \cap \Phi$, while any term involving $A \cap B$ with $B \neq \Phi$ receives zero mass from m_{Φ} and thus we can substitute B with Φ.

$$\mathrm{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi) = \sum_{U \subseteq \Psi} \frac{\sum_{A \cap B = U} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(B)}{1 - \sum_{A \cap B = \emptyset} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(B)}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap B = U} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(\Phi)}{1 - \sum_{A \cap B = \emptyset} \delta_{\mathcal{J}}(f, A) \cdot m_{\Phi}(\Phi)}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap \Phi = U} \delta_{\mathcal{J}}(f, A) \cdot 1}{1 - \sum_{A \cap \Phi = \emptyset} \delta_{\mathcal{J}}(f, A) \cdot 1}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap \Phi = U} \delta_{\mathcal{J}}(f, A)}{1 - \sum_{A \cap \Phi = \emptyset} \delta_{\mathcal{J}}(f, A)}$$

We know that $\delta_{\mathfrak{J}}(f, \cdot)$ is a basic probability assignment, hence $\sum_{A \subseteq X} \delta_{\mathfrak{J}}(f, A) = 1$. Now consider the collections of subsets of X defined by those that have empty intersection with Φ and those that have non-empty intersection with Φ. These two collections form a partition of $\wp(X)$. Thus, we have that

$$\sum_{A \subseteq X} \delta_{\mathfrak{J}}(f, A) = \sum_{\substack{A \subseteq X \\ A \cap \Phi = \varnothing}} \delta_{\mathfrak{J}}(f, A) + \sum_{\substack{A \subseteq X \\ A \cap \Phi \neq \varnothing}} \delta_{\mathfrak{J}}(f, A) = 1 \tag{$\spadesuit_2$}$$

Furthermore, notice that the inner sum requires $U = A \cap \Phi \subseteq \Phi$, while the outer sum requires $U \subseteq \Psi$. Combining these two constraints gives $U \subseteq \Psi \cap \Phi$, so the summation index may be restricted accordingly. In light of these observations, we may rewrite the denominator as follows:

$$\begin{aligned}
\mathrm{Bel}_{\mathfrak{J}}(f, \Psi \mid \Phi) &= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap \Phi = U} \delta_{\mathfrak{J}}(f, A)}{1 - \sum_{A \cap \Phi = \varnothing} \delta_{\mathfrak{J}}(f, A)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \delta_{\mathfrak{J}}(f, A)}{\sum_{A \cap \Phi \neq \varnothing} \delta_{\mathfrak{J}}(f, A)} && \text{(Equation ($\spadesuit_2$) and } U = A \cap \Phi \subseteq \Psi\text{)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \frac{\delta_\tau(f, A)}{\sum_{T \in \mathfrak{J}} \delta_\tau(f, T)}}{\sum_{A \cap \Phi \neq \varnothing} \frac{\delta_\tau(f, A)}{\sum_{T \in \mathfrak{J}} \delta_\tau(f, T)}} && \text{(Definition 25)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\frac{1}{\sum_{T \in \mathfrak{J}} \delta_\tau(f, T)} \sum_{A \cap \Phi = U} \delta_\tau(f, A)}{\frac{1}{\sum_{T \in \mathfrak{J}} \delta_\tau(f, T)} \sum_{A \cap \Phi \neq \varnothing} \delta_\tau(f, A)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \delta_\tau(f, A)}{\sum_{A \cap \Phi \neq \varnothing} \delta_\tau(f, A)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \sum_{E : f(E) = A} \delta(E)}{\sum_{A \cap \Phi \neq \varnothing} \sum_{E : f(E) = A} \delta(E)}. && \text{(Definition 24)}
\end{aligned}$$

Thus, we consider from now on, as follows:

$$\mathrm{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi) = \sum_{U \subseteq \Psi \cap \Phi} \frac{\displaystyle\sum_{A \cap \Phi = U} \sum_{E : f(E) = A} \delta(E)}{\displaystyle\sum_{A \cap \Phi \neq \varnothing} \sum_{E : f(E) = A} \delta(E)} \tag{$\star_2$}$$

Now, to prove the claim we will show that the equivalence between Equations ($\star_1$) and ($\star_2$) holds if and only if conditions (1) and (2) hold. Thus we have as follows:

($\Rightarrow$) Assume conditions (1) and (2) hold. By Lemma 3, we know that for any $E \subseteq \mathcal{E}^{!\Phi}$,

$$\delta^{!\Phi}(E) = \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E'),$$

where

$$M = \sum_{\substack{F \subseteq \mathcal{E} \\ \rho_\Phi(F) \neq \varnothing}} \delta(F).$$

Thus, we can substitute $\delta^{!\Phi}(E)$ with $1/M \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E')$ in Equation ($\star_1$).

Consider an arbitrary $U \subseteq \Psi$. Then we will analyze separately the numerator and denominator of the fraction in Equation ($\star_1$). We start by the numerator, as follows:

$$\begin{aligned} \sum_{E : f^{!\Phi}(E) = U} \delta^{!\Phi}(E) &= \frac{1}{M} \sum_{E : f^{!\Phi}(E) = U} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E') && \text{(Lemma 3)} \\ &= \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) = U}} \delta(E'). \end{aligned}$$

By condition (1), for every E' with $\delta(E') > 0$, $f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi$. Hence,

$$\begin{aligned} \sum_{E : f^{!\Phi}(E) = U} \delta^{!\Phi}(E) &= \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ f(E') \cap \Phi = U}} \delta(E') \\ &= \frac{1}{M} \sum_{A \cap \Phi = U} \sum_{E' : f(E') = A} \delta(E'). \end{aligned} \tag{$\clubsuit_1$}$$

Now consider the denominator in Equation ($\star_1$). Then

$$\begin{aligned} \sum_{T \in \mathcal{J}^{!\Phi}} \sum_{E : f^{!\Phi}(E) = T} \delta^{!\Phi}(E) &= \frac{1}{M} \sum_{T \in \mathcal{J}^{!\Phi}} \sum_{\substack{E' \subseteq \mathcal{E} \\ f(E') \cap \Phi = T}} \delta(E') && \text{(Lemma 3)} \\ &= \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ f(E') \cap \Phi \in \mathcal{J}^{!\Phi}}} \delta(E'). \end{aligned}$$

By condition (2), $f(E') \cap \Phi \in \mathcal{J}^{!\Phi}$ if and only if $f(E') \cap \Phi \neq \varnothing$. Therefore,

$$\frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ f(E') \cap \Phi \in \mathcal{J}^{!\Phi}}} \delta(E') = \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ f(E') \cap \Phi \neq \varnothing}} \delta(E')$$

$$= \frac{1}{M} \sum_{A \cap \Phi \neq \varnothing} \sum_{E' : f(E') = A} \delta(E'). \qquad (♣_2)$$

Since U was arbitrary, then it holds for all $U \subseteq \Psi$.

Now, substituting Equations ($♣_1$) and ($♣_2$) into Equation ($☆_1$), we obtain

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \sum_{U \subseteq \Psi} \frac{\frac{1}{M} \sum_{A \cap \Phi = U} \sum_{E' : f(E') = A} \delta(E')}{\frac{1}{M} \sum_{A \cap \Phi \neq \varnothing} \sum_{E' : f(E') = A} \delta(E')}$$

$$= \sum_{U \subseteq \Psi} \frac{\sum_{A \cap \Phi = U} \sum_{E' : f(E') = A} \delta(E')}{\sum_{A \cap \Phi \neq \varnothing} \sum_{E' : f(E') = A} \delta(E')}.$$

Since $U = A \cap \Phi \subseteq \Phi$, the sum over $U \subseteq \Psi$ reduces to $U \subseteq \Psi \cap \Phi$, yielding

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \sum_{E' : f(E') = A} \delta(E')}{\sum_{A \cap \Phi \neq \varnothing} \sum_{E' : f(E') = A} \delta(E')}. \qquad (☆_1')$$

The right-hand sides of Equations ($☆_2$) and ($☆_1'$) are identical. Thus, since $\Psi \subseteq X$ was arbitrary, it holds for any $\Psi \subseteq X$, and, therefore we can conclude that

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi).$$

($\Leftarrow$) Assume that for all $\Psi \subseteq X$,

$$\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi).$$

Substituting the Equations ($☆_1$) and ($☆_2$) into this equality, we have for all $\Psi \subseteq X$:

$$\sum_{U \subseteq \Psi} \frac{\sum_{E : f^{!\Phi}(E) = U} \delta^{!\Phi}(E)}{\sum_{T \in \mathcal{J}^{!\Phi}} \sum_{E : f^{!\Phi}(E) = T} \delta^{!\Phi}(E)} = \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \sum_{E : f(E) = A} \delta(E)}{\sum_{A \cap \Phi \neq \varnothing} \sum_{E : f(E) = A} \delta(E)}.$$

Now apply Lemma 3 to the left-hand side. For any $E \subseteq \mathcal{E}^{!\Phi}$,

$$\delta^{!\Phi}(E) = \frac{1}{M} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E'),$$

where $M = \sum_{\substack{F \subseteq \mathcal{E} \\ \rho_\Phi(F) \neq \varnothing}} \delta(F)$. Substituting this into the left-hand side, we obtain

$$\begin{aligned}
&\sum_{U \subseteq \Psi} \frac{\frac{1}{M} \sum_{E : f^{!\Phi}(E) = U} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E')}{\frac{1}{M} \sum_{T \in \mathcal{J}^{!\Phi}} \sum_{E : f^{!\Phi}(E) = T} \sum_{\substack{E' \subseteq \mathcal{E} \\ \rho_\Phi(E') = E}} \delta(E')} \\
&= \sum_{U \subseteq \Psi} \frac{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) = U}} \delta(E')}{\sum_{T \in \mathcal{J}^{!\Phi}} \sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) = T}} \delta(E')}. \\
&= \sum_{U \subseteq \Psi} \frac{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) = U}} \delta(E')}{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}}} \delta(E')}.
\end{aligned}$$

Now consider the right-hand side, then we have that

$$\begin{aligned}
&\sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{A \cap \Phi = U} \sum_{E : f(E) = A} \delta(E)}{\sum_{A \cap \Phi \neq \varnothing} \sum_{E : f(E) = A} \delta(E)} \\
&= \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi = U}} \delta(E)}{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \neq \varnothing}} \delta(E)}
\end{aligned}$$

Thus, since Ψ was arbitrary we have that for all $\Psi \subseteq X$,

$$\sum_{U \subseteq \Psi} \frac{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E'))=U}} \delta(E')}{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}}} \delta(E')} = \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi = U}} \delta(E)}{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \neq \varnothing}} \delta(E)} \tag{$\star_3$}$$

Now, we will proceed showing that if Equation ($\star_3$) holds then we can conclude that conditions (1) and (2) hold.

(2) $\Rightarrow$ (1) Now, assume that condition (2) holds, i.e., for all $E' \subseteq \mathcal{E}$ with $\delta > 0$, we have that

$$f(E') \cap \Phi \in \mathcal{J}^{!\Phi} \iff f(E') \cap \Phi \neq \varnothing$$

We want to show that condition (1) must also hold. First, notice that condition (2) implies that the denominators of Equation ($\star_3$) are the same, since we are summing over the same $E' \subseteq \mathcal{E}$. Furthermore, observe that the denominators in Equation ($\star_3$) are constants that do not depend on the summation index U. Therefore, we can factor them out of the sums.

$$\begin{aligned}
& \sum_{U \subseteq \Psi} \frac{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E'))=U}} \delta(E')}{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}}} \delta(E')} = \sum_{U \subseteq \Psi \cap \Phi} \frac{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi = U}} \delta(E)}{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \neq \varnothing}} \delta(E)} \\
\iff & \frac{\sum_{U \subseteq \Psi} \sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E'))=U}} \delta(E')}{\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi}}} \delta(E')} = \frac{\sum_{U \subseteq \Psi \cap \Phi} \sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi = U}} \delta(E)}{\sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \neq \varnothing}} \delta(E)} \\
\iff & \sum_{U \subseteq \Psi} \sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E'))=U}} \delta(E') = \sum_{U \subseteq \Psi \cap \Phi} \sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi = U}} \delta(E) \\
\iff & \sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \subseteq \Psi}} \delta(E') = \sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \subseteq \Psi}} \delta(E)
\end{aligned}$$

Thus, for all $\Psi \subseteq X$ we have

$$\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \subseteq \Psi}} \delta(E') = \sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \subseteq \Psi}} \delta(E). \tag{$\star_4$}$$

Now we prove condition (1) by induction on the cardinality of E'. Thus, we want to show by induction on $n = |E'|$ that for every $E' \in \wp(\mathcal{E})$,

$$f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi.$$

Base case Let $n = 0$, then we know that for no $E' \in \mathcal{E}$ it holds that $E' = \varnothing$, then the only element of $\wp(\mathcal{E})$ such that $n = 0$ is the $\varnothing$. Thus, $\rho_\Phi(\varnothing) = \varnothing$, so $f^{!\Phi}(\rho_\Phi(\varnothing)) = f^{!\Phi}(\varnothing)$. By Definition 22, $f^{!\Phi}(\varnothing) = \Phi$. Similarely, $f(\varnothing) \cap \Phi = X \cap \Phi = \Phi$. Hence,

$$f^{!\Phi}(\rho_\Phi(\varnothing)) = \Phi = f(\varnothing) \cap \Phi.$$

So the base case holds.

Inductive step Assume that for all $X \subseteq \mathcal{E}$ with $|X| < m$, we have

$$f^{!\Phi}(\rho_\Phi(X)) = f(X) \cap \Phi. \tag{IH}$$

Let $E' \subseteq \mathcal{E}$ with $|E'| = m$.

Consider Equation ($\star_4$) for an arbitrary $\Psi \subseteq \Phi$. Now we will split the sums into subsets of size less than m and subsets of size $\geqslant m$, getting as follows

$$\begin{aligned}
&\sum_{\substack{E' \subseteq \mathcal{E} \\ f^{!\Phi}(\rho_\Phi(E')) \subseteq \Psi}} \delta(E') = \sum_{\substack{E \subseteq \mathcal{E} \\ f(E) \cap \Phi \subseteq \Psi}} \delta(E) \\
\iff &\sum_{\substack{X \subseteq \mathcal{E} \\ |X| < m \\ f^{!\Phi}(\rho_\Phi(X)) \subseteq \Psi}} \delta(X) + \sum_{\substack{X \subseteq \mathcal{E} \\ |X| \geqslant m \\ f^{!\Phi}(\rho_\Phi(X)) \subseteq \Psi}} \delta(X) = \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| < m \\ f(Y) \cap \Phi \subseteq \Psi}} \delta(Y) + \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| \geqslant m \\ f(Y) \cap \Phi \subseteq \Psi}} \delta(Y).
\end{aligned}$$

By the induction hypothesis, for any X with $|X| < m$, we have $f^{!\Phi}(\rho_\Phi(X)) = f(X) \cap \Phi$. Hence the condition $f^{!\Phi}(\rho_\Phi(X)) \subseteq \Psi$ is equivalent to $f(X) \cap \Phi \subseteq \Psi$. Therefore, the first terms on both sides are equal. Let

$$S(\Psi) = \sum_{\substack{X \subseteq \mathcal{E} \\ |X| < m \\ f^{!\Phi}(\rho_\Phi(X)) \subseteq \Psi}} \delta(X) = \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| < m \\ f(Y) \cap \Phi \subseteq \Psi}} \delta(Y).$$

Subtracting $S(\Psi)$ from both sides, we obtain for all $\Psi \subseteq \Phi$:

$$\sum_{\substack{X \subseteq \mathcal{E} \\ |X| \geqslant m \\ f^{!\Phi}(\rho_\Phi(X)) \subseteq \Psi}} \delta(X) = \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| \geqslant m \\ f(Y) \cap \Phi \subseteq \Psi}} \delta(Y). \tag{$\star_4'$}$$

Now we define two functions on $\wp(\Phi)$. For each $U \subseteq \Phi$, let

$$a(U) = \sum_{\substack{X \subseteq \mathcal{E} \\ |X| \geqslant m \\ f^{!\Phi}(\rho_\Phi(X)) = U}} \delta(X), \qquad b(U) = \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| \geqslant m \\ f(Y) \cap \Phi = U}} \delta(Y).$$

For any $\Psi \subseteq \Phi$, we have

$$\sum_{\substack{X\subseteq\mathcal{E}\\|X|\geqslant m\\f^{!\Phi}(\rho_\Phi(X))\subseteq\Psi}} \delta(X) = \sum_{U\subseteq\Psi} a(U), \qquad \sum_{\substack{Y\subseteq\mathcal{E}\\|Y|\geqslant m\\f(Y)\cap\Phi\subseteq\Psi}} \delta(Y) = \sum_{U\subseteq\Psi} b(U).$$

Thus, equation Equation ($\star'_4$) becomes

$$\sum_{U\subseteq\Psi} a(U) = \sum_{U\subseteq\Psi} b(U) \qquad \text{for all } \Psi \subseteq \Phi. \tag{$\star_5$}$$

We now prove by induction on $|\Psi|$ that $a(\Psi) = b(\Psi)$ for all $\Psi \subseteq \Phi$.

Base case The base case simply follows by taking $\Psi = \varnothing$ in Equation ($\star_5$), which gives $a(\varnothing) = b(\varnothing)$, directly.

Inductive step Assume that for all $W \subsetneq \Psi$, we have $a(W) = b(W)$. Then from Equation ($\star_5$) we have

$$\sum_{U\subseteq\Psi} a(U) = \sum_{U\subseteq\Psi} b(U).$$

Separate the term $U = \Psi$ from the sum over proper subsets:

$$a(\Psi) + \sum_{\substack{U\subseteq\Psi\\U\neq\Psi}} a(U) = b(\Psi) + \sum_{\substack{U\subseteq\Psi\\U\neq\Psi}} b(U).$$

By the induction hypothesis, the sums over proper subsets are equal. Hence $a(\Psi) = b(\Psi)$.Thus, $a(U) = b(U)$ for every $U \subseteq \Phi$.
Now we prove that for every $X \subseteq \mathcal{E}$ with $|X| \geqslant m$, we have $f^{!\Phi}(\rho_\Phi(X)) = f(X) \cap \Phi$.
Suppose, for contradiction, that there exists $X_1 \subseteq \mathcal{E}$ with $|X_1| \geqslant m$ such that $f^{!\Phi}(\rho_\Phi(X_1)) \neq f(X_1) \cap \Phi$. Define

$$U_1 = f^{!\Phi}(\rho_\Phi(X_1)), \qquad V_1 = f(X_1) \cap \Phi,$$

with $U_1 \neq V_1$.
Consider equation Equation ($\star'_4$) for $U = U_1$:

$$\sum_{\substack{X\subseteq\mathcal{E}\\|X|\geqslant m\\f^{!\Phi}(\rho_\Phi(X))=U_1}} \delta(X) = \sum_{\substack{Y\subseteq\mathcal{E}\\|Y|\geqslant m\\f(Y)\cap\Phi=U_1}} \delta(Y).$$

The left-hand side contains the term $\delta(X_1) > 0$ because $f^{!\Phi}(\rho_\Phi(X_1)) = U_1$. The right-hand side sums over all Y with $f(Y) \cap \Phi = U_1$. Since $f(X_1) \cap \Phi = V_1 \neq U_1$, the term $\delta(X_1)$ does not appear on the right-hand side. Therefore, there must exist some $X_2 \neq X_1$ with $|X_2| \geqslant m$ such that $f(X_2) \cap \Phi = U_1$ and $\delta(X_2) > 0$.

Now consider Equation ($\star'_4$) for $U = V_1$:

$$\sum_{\substack{X \subseteq \mathcal{E} \\ |X| \geqslant m \\ f^{!\Phi}(\rho_\Phi(X)) = V_1}} \delta(X) = \sum_{\substack{Y \subseteq \mathcal{E} \\ |Y| \geqslant m \\ f(Y) \cap \Phi = V_1}} \delta(Y).$$

The right-hand side contains the term $\delta(X_1) > 0$ because $f(X_1) \cap \Phi = V_1$. The left-hand side sums over all Z with $f^{!\Phi}(\rho_\Phi(Z)) = V_1$. Since $f^{!\Phi}(\rho_\Phi(X_1)) = U_1 \neq V_1$, the term $\delta(X_1)$ does not appear on the left-hand side. Therefore, there must exist some $X_3 \neq X_1$ with $|X_3| \geqslant m$ such that $f^{!\Phi}(\rho_\Phi(X_3)) = V_1$ and $\delta(X_3) > 0$.

Now we have three distinct subsets: X_1, X_2, X_3, with the following properties:

* $f^{!\Phi}(\rho_\Phi(X_1)) = U_1, f(X_1) \cap \Phi = V_1$,
* $f(X_2) \cap \Phi = U_1$,
* $f^{!\Phi}(\rho_\Phi(X_3)) = V_1$.

Now consider X_2. Let $U_2 = f^{!\Phi}(\rho_\Phi(X_2))$. If $U_2 \neq U_1$, then we have a new mismatch. Apply the same argument to X_2: there exists $X_4 \neq X_2$ with $f(X_4) \cap \Phi = U_2$ and $\delta(X_4) > 0$. Similarly, consider X_3. Let $V_2 = f(X_3) \cap \Phi$. If $V_2 \neq V_1$, then there exists $X_5 \neq X_3$ with $f^{!\Phi}(\rho_\Phi(X_5)) = V_2$ and $\delta(X_5) > 0$.

This process generates an infinite sequence of distinct subsets $X_1, X_2, X_3, X_4, \ldots$ because each step produces a new subset not previously seen (since if a subset repeats, we would have a cycle that forces $U_1 = V_1$). However, $\wp(\mathcal{E})$ is finite, so an infinite sequence of distinct subsets is impossible. Therefore, our initial assumption was false, and we must have $f^{!\Phi}(\rho_\Phi(X)) = f(X) \cap \Phi$ for every $X \subseteq \mathcal{E}$ with $|X| \geqslant m$.

In particular, for our E' with $|E'| = m$, we obtain $f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi$. This completes the inductive step.

(1) $\Rightarrow$ (2) Suppose now that condition (1) holds. This means that for all $E' \in \mathcal{E}$ such that $\delta(E') > 0$ it holds that,

$$f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi.$$

Now, we want to show that (2) must hold as well.

We aim to prove the statement by showing the two directions of the inclusion.

($\Rightarrow$) Let $E' \subseteq \mathcal{E}$ be arbitrary such that $f^{!\Phi}(\rho_\Phi(E')) \in \mathfrak{J}^{!\Phi}$ and $\delta(E') > 0$.
By (1), we can conclude that $f(E') \cap \Phi \in \mathfrak{J}^{!\Phi}$. By Definition 18 we know that $\varnothing \notin \mathfrak{J}^{!\Phi}$ and therefore we can conclude that $f(E') \cap \Phi \neq \varnothing$.
Since $E' \subseteq \mathcal{E}$ was arbitrary then we can conclude that it holds for all $E' \subseteq \mathcal{E}$ such that $\delta(E') > 0$, and therefore the left to right direction holds. In particular notice that this direction holds for any $E' \subseteq \mathcal{E}$ by the same argument.

($\Leftarrow$) Let $E' \subseteq \mathcal{E}$ be arbitrary such that $f(E') \cap \Phi \neq \varnothing$ and $\delta(E') > 0$
Then, consider Equation ($\star_3$) as done before we can isolate the denominator and by (1) conclude

that the denominators are equal as follows:

$$\begin{aligned}
&\sum_{U\subseteq\Psi}\frac{\sum\limits_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))=U}}\delta(E')}{\sum\limits_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))\in\mathfrak{J}^{!\Phi}}}\delta(E')}=\sum_{U\subseteq\Psi\cap\Phi}\frac{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi=U}}\delta(E)}{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E)}\\
\iff&\frac{\sum\limits_{U\subseteq\Psi}\sum\limits_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))=U}}\delta(E')}{\sum\limits_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))\in\mathfrak{J}^{!\Phi}}}\delta(E')}=\frac{\sum\limits_{U\subseteq\Psi\cap\Phi}\sum\limits_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi=U}}\delta(E)}{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E)}\\
\iff&\frac{1}{\sum\limits_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))\in\mathfrak{J}^{!\Phi}}}\delta(E')}=\frac{1}{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E)} && \text{(Condition (1))}\\
\iff&\sum_{\substack{E'\subseteq\mathcal{E}\\ f^{!\Phi}(\rho_\Phi(E'))\in\mathfrak{J}^{!\Phi}}}\delta(E')=\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E)\\
\iff&\sum_{\substack{E'\subseteq\mathcal{E}\\ f(E')\cap\Phi\in\mathfrak{J}^{!\Phi}}}\delta(E')=\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E) && \text{(Condition (1))}
\end{aligned}$$

Therefore we get that the following equality holds

$$\sum_{\substack{E'\subseteq\mathcal{E}\\ f(E')\cap\Phi\in\mathfrak{J}^{!\Phi}}}\delta(E')=\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset}}\delta(E) \tag{$\star_5$}$$

Since we already show that the other direction holds for every $E'\subseteq\mathcal{E}$, then we can conclude that for every $E'\subseteq\mathcal{E}$ such that $\delta(E')$ appears in the left-side sum then it must be in the right side sum. Then we have that:

$$\begin{aligned}
&\sum_{\substack{E'\subseteq\mathcal{E}\\ f(E')\cap\Phi\in\mathfrak{J}^{!\Phi}}}\delta(E')=\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset\\ \delta(E)>0}}\delta(E)+\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset\\ \delta(E)=0}}\delta(E)\\
\iff&\sum_{\substack{E'\subseteq\mathcal{E}\\ f(E')\cap\Phi\in\mathfrak{J}^{!\Phi}}}\delta(E')=\sum_{\substack{E\subseteq\mathcal{E}\\ f(E)\cap\Phi\neq\emptyset\\ \delta(E)>0}}\delta(E) && (\star_5')
\end{aligned}$$

Suppose now for sake of contradiction that for E' holds that $f(E')\cap\Phi\notin\mathfrak{J}^{!\Phi}$. Then by assumption, we have that $\delta(E')>0$, thus it will appear on the right side of Equation ($\star_5'$) but not in the left since we know by assumption that $f(E')\cap\Phi\notin\mathfrak{J}^{!\Phi}$. But then we will have an addend on the right side that does not appear on the left side.
However, since the left to right direction holds for all $E\subseteq\mathcal{E}$ such that $\delta(E)>0$, then in

particular it holds also for $E = E'$. Therefore, we get that in the right-side of Equation ($\star_5$) we have an addend that does not appear on the left side and that is positive. Therefore, this will imply that the two sums are not equal and therefore, bring a contradiction. Hence, we conclude that $f(E') \cap \Phi \in \mathcal{J}^{!\Phi}$

Since $E' \subseteq \mathcal{E}$ with $\delta(E') > 0$ was arbitrary, we can conclude that the right to left direction holds for all $E' \subseteq \mathcal{E}$ with $\delta(E') > 0$

Since both directions hold, we can claim that assuming Equation ($\star_3$) and (1), we can conclude that (2) must hold.

We have shown that, assuming equality, condition (1) holds if and only if condition (2) holds. However, this would mean the implication could still hold even when both conditions are false, which is not permissible if we require the conditions to hold conjunctively. Therefore, we proceed by contradiction: we assume that both condition (1) and condition (2) do not hold, and show that Equation ($\star_3$) cannot hold. This contradiction forces condition (1) and condition (2) to hold together, thereby establishing the right-to-left direction. Thus, consider the following MLBM structure $(X, \mathcal{E}_{\mathrm{Q}}, \delta, \mathcal{J}_{\mathrm{SD}}, t)$, where

- $X = \{1, 2, 3, 4\}$,
- $\mathcal{E}_{\mathrm{Q}} = \{(\{1, 4\}, 0.5), (\{3, 4\}, 0.5), (\{2, 3\}, 0.5)\}$, and
- $\mathcal{J}_{\mathrm{SD}}$ is the strong-densness frame of justification, defined in Definition 19 and the function t is the evidence allocation function defined in Definition 40.

Let $\mathcal{E}$ denote the underlying set of evidence with respect to $\mathcal{E}_{\mathrm{Q}}$. Furthermore, consider a proposition $\Phi \subseteq X$ such that $\Phi = \{1, 2, 3\}$. Now we will evaluate the two objects using the set of possible states and the pieces of evidence defined above. First, we compute the evidential topology $\tau_{\mathcal{E}}$ generated by $\mathcal{E}$. Thus we have that:

$$\tau_{\mathcal{E}} = \{\emptyset, \{3\}, \{4\}, \{1, 4\}, \{3, 4\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

Then we have that

$$\mathcal{J}_{\mathrm{SD}} = \{\{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

and for all $E \subseteq \mathcal{E}$, $t(E)$ can be evaluated like in the following table

Values of t	
E	$t(E)$
$\emptyset$	X
$\{1, 4\}$	$\{1, 4\}$
$\{3, 4\}$	$\{3, 4\}$
$\{2, 3\}$	$\{2, 3\}$
$\{\{1, 4\}, \{3, 4\}\}$	$\{4\}$
$\{\{1, 4\}, \{2, 3\}\}$	X
$\{\{3, 4\}, \{2, 3\}\}$	$\{3\}$
$\{\{1, 4\}, \{3, 4\}, \{2, 3\}\}$	X

Furthermore, notice that for every $E_i \in \mathcal{E}$, we have that $(E_i, 0.5) \in \mathcal{E}_{\mathrm{Q}}$, then for all $E_i \in \mathcal{E}$ holds that

$p_i = 1 - p_i$. Therefore, we can conclude that for all $E \subseteq \mathcal{E}$, it holds that

$$\delta(E) = \prod_{E_i \in E} p_i \prod_{E_i \notin E} (1 - p_i) = 0.5^3.$$

Consider now the result of conditioning $\mathfrak{B}$ on $\Phi = \{1, 2, 3\}$. Thus, we firstly update the various elements of the model applying the respective algorithms. Thus, we get that

- applying Algorithm 1 we get the following updated set of evidence

$$\mathcal{E}^{!\Phi} = \{\{1\}, \{3\}, \{2, 3\}\}$$

- applying Algorithm 2 we get the following updated evidential topology

$$\tau_{\mathcal{E}}^{!\Phi} = \{\varnothing, \{3\}, \{1\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

- applying Algorithm 3 we get the following updated quantitative set of evidence

$$\mathcal{E}_{\mathrm{Q}}^{!\Phi} = \{(\{1\}, 0.5), (\{3\}, 0.5), (\{2, 3\}, 0.5)\}$$

We proceed evaluating our frame of justification and our evidence allocation function of the updated model, according to Definitions 33 and 41. Hence, we have that

$$\mathcal{J}_{\mathrm{SD}}^{!\Phi} = \{\{1, 3\}, \{1, 2, 3\}\}$$

and for all $E' \subseteq \mathcal{E}^{!\Phi}$, $t^{!\Phi}(E')$ can be evaluated like in the following table

Values of $t^{!\Phi}$	
E'	$t^{!\Phi}(E')$
$\varnothing$	Φ
$\{1\}$	$\{1\}$
$\{3\}$	$\{3\}$
$\{2, 3\}$	$\{2, 3\}$
$\{\{1\}, \{3\}\}$	Φ
$\{\{1\}, \{2, 3\}\}$	Φ
$\{\{3\}, \{2, 3\}\}$	$\{3\}$
$\{\{1\}, \{3\}, \{2, 3\}\}$	Φ

Remind that

$$\rho_\Phi(E') = \{E \cap \Phi \mid E \in E' \text{ and } E \cap \Phi \neq \varnothing\}.$$

Thus, we can compute for all the $E \subseteq \mathcal{E}$ the correspondent $\rho_\Phi(E) \subseteq \mathcal{E}^{!\Phi}$. Calculating $\rho_\Phi(E)$ for all $E \subseteq \mathcal{E}$ we get the sets in the following table:

Values of ρ_Φ	
E	$\rho_\Phi(E)$
$\emptyset$	$\emptyset$
$\{1,4\}$	$\{1\}$
$\{3,4\}$	$\{3\}$
$\{2,3\}$	$\{2,3\}$
$\{\{1,4\},\{3,4\}\}$	$\{\{1\},\{3\}\}$
$\{\{1,4\},\{2,3\}\}$	$\{\{1\},\{2,3\}\}$
$\{\{3,4\},\{2,3\}\}$	$\{\{3\},\{2,3\}\}$
$\{\{1,4\},\{3,4\},\{2,3\}\}$	$\{\{1\},\{3\},\{2,3\}\}$

Now, notice that $\delta(\{\{1,4\},\{3,4\}\}) > 0$ and

$$\begin{aligned} t^{!\Phi}(\rho_\Phi(\{\{1,4\},\{3,4\}\})) &= t^{!\Phi}(\{\{1\},\{3\}\}) \\ &= \Phi \neq \emptyset \\ &= \{4\} \cap \Phi = t(\{\{1,4\},\{3,4\}\}) \cap \Phi \end{aligned}$$

Thus, we can conclude that condition (1) does not hold. Similarly, we can check that condition (2) does not hold, in fact we have that $\delta(\{1,4\}) > 0$, and it holds that

- $t(\{1,4\}) \cap \Phi = \{1,4\} \cap \{1,2,3\} = \{1\} \neq \emptyset$, and
- $t^{!\Phi}(\rho_\Phi(\{1,4\})) = t^{!\Phi}(\{1\}) = \{1\} \notin \mathcal{J}^{!\Phi}_{\mathrm{SD}}$

Thus, we have that one of the two directions of the bi-conditional statement in condition (2) fails, and so we can claim that (2) does not hold.
Now, we are left to show that also Equation ($\star_3$) fails. Consider $\Psi = \{1,3\}$, then we have that the numerator of the left side of Equation ($\star_3$) can be evaluated as follows:

$$\begin{aligned} \sum_{U \subseteq \Psi} \sum_{\substack{E \subseteq \mathcal{E} \\ t^{!\Phi}(\rho_\Phi(E)) = U}} \delta(E) &= \sum_{\substack{E \subseteq \mathcal{E} \\ t^{!\Phi}(\rho_\Phi(E)) \subseteq \{1,3\}}} \delta(E) \\ &= \sum_{\substack{E \subseteq \mathcal{E} \\ t^{!\Phi}(\rho_\Phi(E)) = \{1\}}} \delta(E) + \sum_{\substack{E \subseteq \mathcal{E} \\ t^{!\Phi}(\rho_\Phi(E)) = \{3\}}} \delta(E) \\ &= \delta(\{1,4\}) + \delta(\{\{3,4\}\}) + \delta\{\{2,3\},\{3,4\}\} = 3 \cdot 0.5^3 \end{aligned} \tag{$\heartsuit$}$$

Moreover, notice that the only $E \subseteq \mathcal{E}$ such that $t^{!\Phi}(\rho_\Phi(E)) \in \mathcal{J}^{!\Phi}_{\mathrm{SD}}$ are the E such that $t^{!\Phi}(\rho_\Phi(E)) = \Phi$. This means that

$$\begin{aligned} \sum_{\substack{E \subseteq \mathcal{E} \\ c^{!\Phi}(\rho_\Phi(E)) \in \mathcal{J}^{!\Phi}_{\mathrm{SD}}}} \delta(E) &= \sum_{\substack{E \subseteq \mathcal{E} \\ t^{!\Phi}(\rho_\Phi(E)) = \Phi}} \delta(E) \\ &= \delta(\{\emptyset\}) + \delta(\{\{1,4\},\{3,4\}\}) + \delta(\{\{1,4\},\{2,3\}\}) + \delta(\{\{1,4\},\{3,4\},\{2,3\}\}) \\ &= 4 \cdot 0.5^3 \end{aligned}$$

Therefore, we have that the left-side of Equation ($\star_3$) corresponding to the belief of the proposition Ψ after our conditionalization procedure on the proposition Φ is

$$\mathrm{Bel}_{\mathfrak{J}^{!\Phi}_{\mathrm{SD}}}(t^{!\Phi},\Psi)=\sum_{U\subseteq\Psi}\frac{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ t^{!\Phi}(\rho_\Phi(E))=U}}\delta(E)}{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ t^{!\Phi}(\rho_\Phi(E))\in\mathfrak{J}^{!\Phi}_{\mathrm{SD}}}}\delta(E)}=\frac{3\cdot 0.5^3}{4\cdot 0.5^3}=\frac{3}{4}. \tag{$\star_3'$}$$

Now, we proceed evaluating the right-hand side of Equation ($\star_3$).
First notice that the $E\subseteq\mathcal{E}$ such that $t(E)\cap\Phi\subseteq\Psi$ are the same as in Equation ($\heartsuit$). Therefore, the numerator of the right-hand side of Equation ($\star_3$) becomes as follows:

$$\sum_{U\subseteq\Psi\cap\Phi}\ \sum_{\substack{E\subseteq\mathcal{E}\\ t(E)\cap\Phi=U}}\delta(E)=\sum_{\substack{E\subseteq\mathcal{E}\\ t(E)\cap\Phi\subseteq\{1,3\}}}\delta(E)=3\cdot 0.5^3$$

Now, notice that the only $E\subseteq\mathcal{E}$ such that it holds that $t(E)\cap\Phi=\varnothing$, is $\{\{1,4\},\{3,4\}\}$ since

$$t(\{\{1,4\},\{3,4\}\})\cap\Phi=\{4\}\cap\Phi=\varnothing.$$

Thus, we have that

$$\sum_{\substack{E\subseteq\mathcal{E}\\ t(E)\cap\Phi\neq\varnothing}}\delta(E)=\sum_{\substack{E\subseteq\mathcal{E}\\ E\neq\{\{1,4\},\{3,4\}\}}}\delta(E)=7\cdot 0.5^3.$$

Hence, we have that the left-side of Equation ($\star_3$)corresponding to the belief of the proposition Ψ combining it with the certain evidence on Φ is

$$\mathrm{Bel}_{\mathfrak{J}_{\mathrm{SD}}}(t,\Psi\mid\Phi)=\sum_{U\subseteq\Psi\cap\Phi}\frac{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ t(E)\cap\Phi=U}}\delta(E)}{\sum\limits_{\substack{E\subseteq\mathcal{E}\\ t(E)\cap\Phi\neq\varnothing}}\delta(E)}=\frac{3\cdot 0.5^3}{7\cdot 0.5^3}=\frac{3}{7} \tag{$\star_3''$}$$

Therefore, by Equations ($\star_3'$) and ($\star_3''$), we see that Equation ($\star_3$) cannot hold. This contradiction shows that it is impossible for (1) and (2) to both fail while Equation ($\star_3$) is true. Hence, (1) and (2) must hold together, which finally establishes the right-to-left direction.

Since $\mathfrak{B}=(X,\mathcal{E}_Q,\mathfrak{J},\delta,f)$ was an arbitrary MLBM structure, $\Phi\subseteq X$ an arbitrary proposition, and $\Psi\subseteq X$ an arbitrary proposition, we can conclude that the claim holds for all MLBM structures, for all propositions $\Phi\subseteq X$, and for all propositions $\Psi\subseteq X$. □

The proofs presented in this section show that conditions (1) and (2) in Proposition 16 are both necessary and sufficient for the conditionalization procedure outlined in Chapter 2 on an arbitrary MLBM structure $\mathfrak{B}$ to behave as a conditioning operation in the sense of Definition 30. In other words, when these conditions are satisfied, updating the MLBM structure via our procedure yields a belief function that coincides with the Dempster-Shafer conditional belief.

We now turn to analyzing a particular case in which these two conditions actually hold. To do so, we first establish two auxiliary propositions that lay the groundwork for the main result.

Proposition 17. Let $(X, \mathcal{E})$ be a qualitative evidence frame, $\tau_{\mathcal{E}}$ the evidential topology generated by $\mathcal{E}$ and consider the evidence allocation function i, as defined in Definition 23. Let $\Phi \subseteq X$ be proposition. Furthermore, suppose that for all $E_i \in \mathcal{E}$ it holds that $E_i \cap \Phi \neq \emptyset$. Then we claim that for all $E \subseteq \mathcal{E}$, it holds that

$$i(E) \cap \Phi = i^{!\Phi}(\rho_\Phi(E))$$

where $i^{!\Phi}$ is the function after conditional update as in Definition 41, and ρ_Φ defined as in Definition 39.

Proof. Let $(X, \mathcal{E})$ be a qualitative evidence frame, $\tau_{\mathcal{E}}$ the evidential topology generated by $\mathcal{E}$ and $\Phi \subseteq X$ an arbitrary proposition. Suppose that for all $E_i \in \mathcal{E}$ it holds that $E_i \cap \Phi \neq \emptyset$.

Furthermore consider the updated qualitative evidence frame and evidential topology, after conditioning on $\Phi \subseteq X$, according to Definition 31, which will be respectively $(\Phi, \mathcal{E}^{!\Phi})$ and $\tau_{\mathcal{E}}^{!\Phi}$.

Consider now the function i as defined in Definition 23, and its conditional update $i^{!\Phi} : \wp(\mathcal{E}^{!\Phi}) \to \tau_{\mathcal{E}}^{!\Phi}$, as in Definition 41. Then consider some arbitrary $E \subseteq \mathcal{E}$, then we have two cases.

1. If $E = \emptyset$ then by the definition of ρ_Φ, we have that $\rho_\Phi(E) = \rho_\Phi(\emptyset) = \emptyset$. Then it holds that:

$$\begin{aligned} i(E) \cap \Phi &= i(\emptyset) \cap \Phi = X \cap \Phi = \Phi \\ &= i^{!\Phi}(\emptyset) = i^{!\Phi}(\rho_\Phi(\emptyset))i = i^{!\Phi}(\rho_\Phi(E)). \end{aligned}$$

2. If $E \neq \emptyset$, then by assumption we know that for all $E_i \in \mathcal{E}$, it holds that $E_i \cap \Phi \neq \emptyset$. Then we have that $\rho_\Phi(E) = \{E_i \cap \Phi \mid E_i \in E\}$. Therefore, we can conclude that

$$\begin{aligned} i(E) \cap \Phi &= \bigcap_{E_i \in E} E_i \cap \Phi = \bigcap_{E_i \in E} (E_i \cap \Phi) \\ &= \bigcap \{E_i \cap \Phi \mid E_i \in E\} = \bigcap \rho_\Phi(E) = i^{!\Phi}(\rho_\Phi(E)). \end{aligned}$$

Since in both cases the equality holds then we can claim that $i(E) \cap \Phi = i^{!\Phi}(\rho_\Phi(E))$. Furthermore, since $E \in \mathcal{E}$ was arbitrary, then we can conclude that it holds for all $E \subseteq \mathcal{E}$. Thus, we can conclude that the claim holds. □

Proposition 18. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathfrak{J}_{DS}, \delta, i)$ be an MLBM structure according to Definition 45, and let $\mathcal{E}$ be its underlying set of evidence. Let $\Phi \subseteq X$ be a proposition. Furthermore, suppose that for all $E \in \mathcal{E}$ it holds that $E \cap \Phi \neq \emptyset$. Let $\mathfrak{B}^{!\Phi} = (\Phi, \mathcal{E}_Q^{!\Phi}, \mathfrak{J}_{DS}^{!\Phi}, \delta^{!\Phi}, i^{!\Phi})$ be the conditionalized MLBM structure according to Definition 46. Then for every $E \subseteq \mathcal{E}$ such that $E \neq \emptyset$ it holds that $\bigcap_{E_i \in E} E_i \neq \emptyset$, we have that

$$i^{!\Phi}(\rho_\Phi(E)) \in \mathfrak{J}_{DS}^{!\Phi} \iff i(E) \cap \Phi \neq \emptyset,$$

where ρ_Φ is defined as in Definition 39.

Proof. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathfrak{J}_{DS}, \delta, i)$ be an MLBM structure and $\Phi \subseteq X$ an arbitrary proposition. Suppose that for all $E_i \in \mathcal{E}$ it holds that $E_i \cap \Phi \neq \emptyset$.

Furthermore, consider the conditionalized MLBM structure $\mathfrak{B}^{!\Phi} = (\Phi, \mathcal{E}_Q^{!\Phi}, \mathcal{J}_{DS}^{!\Phi}, \delta^{!\Phi}, i^{!\Phi})$ according to Definition 46. Recall that $\mathcal{J}_{DS}^{!\Phi} = \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$.
Notice that since it holds that for all $E_i \in \mathcal{E}$ we have $E_i \cap \Phi \neq \varnothing$, then in particular for all $E_i \in E$ we have $E_i \neq \varnothing$. Therefore, by Proposition 17 it holds that

$$i(E) \cap \Phi = i^{!\Phi}(\rho_\Phi(E)). \tag{$\star$}$$

Now consider some arbitrary $E \subseteq \mathcal{E}$ such that $E \neq \varnothing$ it holds that $\bigcap_{E_i \in E} E_i \neq \varnothing$ and show the two directions of our conditional.

($\Rightarrow$) Let $i^{!\Phi}(\rho_\Phi(E)) \in \mathcal{J}_{DS}^{!\Phi}$. Then it means that $i^{!\Phi}(\rho_\Phi(E)) \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$. By ($\star$) we can claim that $i(E) \cap \Phi \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$. Therefore, we get that $i(E) \cap \Phi \neq \varnothing$, and the left-to-right direction holds.

($\Leftarrow$) Let $i(E) \cap \Phi \neq \varnothing$. Then by ($\star$) we get that $i^{!\Phi}(\rho_\Phi(E)) \neq \varnothing$. Now we have two cases:

1. If $E = \varnothing$, then $\rho_\Phi(E) = \varnothing$. Thus we have:

$$i^{!\Phi}(\rho_\Phi(E)) = i^{!\Phi}(\varnothing) = \Phi.$$

By Definition 31, $\Phi \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\} = \mathcal{J}_{DS}^{!\Phi}$.

2. If $E \neq \varnothing$, then by ($\star$) we have $i^{!\Phi}(\rho_\Phi(E)) = i(E) \cap \Phi \neq \varnothing$. Now we aim to show that $i^{!\Phi}(\rho_\Phi(E)) \in \tau_{\mathcal{E}}^{!\Phi}$. First, notice that for all $E_i \in E$, it holds that $E_i \cap \Phi \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$, since $\tau_{\mathcal{E}}^{!\Phi}$ is the topology generated by $\mathcal{E}^{!\Phi}$ and we know by assumption that $E_i \cap \Phi \neq \varnothing$. Furthermore, $\rho_\Phi(E) = \{E_i \cap \Phi \mid E_i \in E\}$. Thus we conclude as follows:

$$i^{!\Phi}(\rho_\Phi(E)) = i^{!\Phi}(\{E_i \cap \Phi \mid E_i \in E\}) = \bigcap \{E_i \cap \Phi \mid E_i \in E\} = \bigcap_{E_i \in E} E_i \cap \Phi.$$

Now, since we assume that $\bigcap_{E_i \in E} E_i \neq \varnothing$, by condition 2 of Definition 4 we know that $\bigcap_{E_i \in E} E_i \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$. By condition 1 of Definition 4 we know that $\Phi \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\varnothing\}$. Finally, again by condition 2 of Definition 4, we can claim that $\bigcap_{E_i \in E} E_i \cap \Phi \in \tau_{\mathcal{E}}^{!\Phi}$.

Since the statement holds in both cases, the right-to-left direction holds.

Since both directions hold, we can claim that the bi-conditional statement holds for $E \subseteq \mathcal{E}$ such that $\bigcap_{E_i \in E} E_i \neq \varnothing$. Furthermore, since such an $E \subseteq \mathcal{E}$ was arbitrary, the claim holds for all such E. $\square$

With Propositions 17 and 18 in place, we are now ready to state and prove a result that follows the path of (Pinto Prieto, 2024, Proposition 3.8). As recalled at the end of Section 1.3, the combination of the Dempster-Shafer frame of justification $\mathcal{J}^{DS}$ with the evidence allocation function i enabled the connection between the MLBM framework and standard Dempster-Shafer theory. We now extend that connection to the conditional setting. This is formally emphasized in the following proposition.

Proposition 19. Let $\mathfrak{B} = (X, \mathcal{E}_Q, \mathcal{J}_{DS}, \delta, i)$ be an MLBM structure, according to Definition 45, and let $\mathcal{E}$ be its underlying set of evidence, and $\tau_{\mathcal{E}}$ be the evidential topology generated by $\mathcal{E}$. Let $\Phi \subseteq X$ be a proposition. Suppose that:

1. for every $E \in \mathcal{E}$, $E \cap \Phi \neq \varnothing$,

2. for every $E \subseteq \mathcal{E}$ such that $E \neq \emptyset$ it holds that $\bigcap_{E_i \in E} E_i \neq \emptyset$ and

3. $\displaystyle\sum_{A \cap \Phi \neq \emptyset} \sum_{E : i(E) = A} \delta(E) > 0.$

Let $\mathfrak{B}^{!\Phi} = \left(\Phi, \mathcal{E}_{\mathrm{Q}}^{!\Phi}, \mathcal{J}_{\mathrm{DS}}^{!\Phi}, \delta^{!\Phi}, i^{!\Phi}\right)$ be the conditionalized MLBM structure, according to Definition 46. Then for any proposition $\Psi \subseteq X$,

$$\mathsf{Bel}_{\mathcal{J}_{\mathrm{DS}}^{!\Phi}}(i^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}_{\mathrm{DS}}}(i, \Psi \mid \Phi),$$

where ρ_Φ is as defined in Definition 39.

Proof. By Propositions 17 and 18 we know that the following properties hold:

1. $i(E) \cap \Phi = i^{!\Phi}(\rho_\Phi(E))$,
2. $i^{!\Phi}(\rho_\Phi(E)) \in \mathcal{J}_{\mathrm{DS}}^{!\Phi} \iff i(E) \cap \Phi \neq \emptyset$.

Note that the latter are equivalent respectively to condition (1) and (2) in Proposition 16 when the frame of justification is $\mathcal{J}_{\mathrm{DS}}$ and the evidence allocation function i. Therefore we can conclude that for any $\Psi \subseteq X$, it holds that

$$\mathsf{Bel}_{\mathcal{J}_{\mathrm{DS}}^{!\Phi}}(i^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}_{\mathrm{DS}}}(i, \Psi \mid \Phi),$$

proving our claim. □

Therefore, we can now claim to have a precise correspondence between the conditionalization procedure on the MLBM framework and the standard Dempster-Shafer conditioning operation applied directly to the final basic probability assignment. Proposition 16 provides necessary and sufficient conditions for the equality $\mathsf{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$ to hold for all propositions $\Psi \subseteq X$. This result replicates and extends the correspondence established in (Pinto Prieto, 2024, Proposition 3.8) from the static to the dynamic setting, showing that the relationship between MLBM and Dempster-Shafer theory is preserved under our conditionalization procedure.

Furthermore, analyzing the conditions in Proposition 16 we can draw the following conclusions:

- The first condition,

  $$f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi,$$

  requires that recomputing the evidence allocation function on the updated frame yields the same result as restricting the original output to Φ. This means that interpreting a collection of evidence pieces after updating on Φ yields the same result as first interpreting them and then intersecting with the certain evidence Φ. Interestingly, this is exactly the naive definition we considered in Section 2.5 for updating evidence allocation functions, but here it is required only on the support of δ, that is, only for those evidence subsets that actually carry positive mass.

- The second condition,

  $$f^{!\Phi}(\rho_\Phi(E')) \in \mathcal{J}^{!\Phi} \iff f(E') \cap \Phi \neq \emptyset,$$

  states that a proposition derived from evidence counts as a justification in the updated model precisely when its restriction to Φ is nonempty. This captures the epistemic principle that justifications must be truth-conducive with respect to the certain evidence Φ: a justification that has empty intersection with Φ

would be completely inconsistent with what the agent now knows to be true, and therefore cannot serve as a basis for belief. Conversely, any proposition that does have nonempty intersection with Φ is at least partially compatible with the certain evidence and therefore qualifies as a potential justification.

Together, these conditions ensure that the agent's epistemic standards for justification and evidence interpretation are preserved across the update, bridging the static and dynamic perspectives on belief revision within the MLBM framework.

3.2 *Conditionalization on* MLBM *and conditioning in* TME

Now, we move forward to study the relation between the conditional procedure as defined in Chapter 2 and the ways of conditionalize our belief on a new evidence that we have in the context of TME.

In Section 2.1 we analyzed two possible ways of conditioning our belief on a given evidence in TME. The first is a static notion, the conditional belief operator $\mathsf{B}^{\Phi}\psi$, as defined in Definition 28, which captures the agent's disposition to believe ψ upon learning Φ without modifying the underlying model. The second is a dynamic operation, the public announcement model $\mathfrak{M}^{!\Phi}$, as in Definition 29, which produces a new epistemic state by restricting the original model to the proposition Φ.

In this section, we want to analyze under which conditions our model can replicate their behavior. For this purpose we remember that, as noticed in Section 1.3 in line with (Pinto Prieto, 2024, Proposition 3.9), the combination of the frame of justification $\mathcal{J}_{\mathrm{SD}}$ and evidence allocation function d, defined respectively in Definitions 19 and 23, defines the agent that replicates the behavior of the belief operator in TME, with a semantics defined accordingly to Definition 11. Thus, in the following section we consider for our analysis this type of agent.

3.2.1 Relating the conditionalization on MLBM to public announcement models

We begin by examining the relationship between our conditionalization procedure and the dynamic public announcement operation. In Definition 29, we find the definition of a public announcement model for TME, from which we took inspiration to define the update at the qualitative level of our model in Definition 31. Now, we aim to see whether the same correspondence in (Pinto Prieto, 2024, Proposition 3.9) can be achieved after updating our belief model conditioning on a proposition Φ and after updating the topo-e-model (build using the same space and set of evidence) via public announcement.

Now, we want formalize our claim in the following proposition:

Proposition 20. Let $\mathfrak{B} = \left(X, \mathcal{E}_{\mathrm{Q}}, \mathcal{J}_{\mathrm{SD}}, \delta, d\right)$ be a MLBM structure, $\mathcal{E}$ its underlying set of evidence and $\tau_{\mathcal{E}}$ the evidential topology generated by $\mathcal{E}$. Consider a topo-e-model $\mathfrak{M} = (X, \mathcal{E}, \tau_{\mathcal{E}}, V)$, where V is an arbitrary valuation function and all the other components are the same as before. Let $\Phi \subseteq X$ be any proposition. Furthermore consider the belief function on the updated MLBM structure, according to Definitions 44 and 46, and the topo-e-model via public announcement, as defined in Definition 29. Then we claim that for all $\psi \in \mathcal{L}$, it holds that

$$\mathfrak{M}^{!\Phi}, x \models \mathsf{B}\psi \iff \mathsf{Bel}_{\mathcal{J}^{\Phi}_{\mathrm{SD}}}\left(d^{!\Phi}, \Psi\right) > 0.$$

Proof. Let $\psi \in \mathcal{L}$ be an arbitrary formula and let $\Psi = \|\psi\|$ be its truth set. Furthermore let $\mathfrak{B} = \left(X, \mathcal{E}_{\mathrm{Q}}, \mathcal{J}_{\mathrm{SD}}, \delta, d\right)$ be an arbitrary MLBM structure, $\mathcal{E}$ its underlying set of evidence and $\tau_{\mathcal{E}}$ the evidential topology generated by

$\mathcal{E}$. Consider a topo-e-model $\mathfrak{M} = (X, \mathcal{E}, \tau_{\mathcal{E}}, V)$, where V is an arbitrary valuation function and all the other components are the same as before. Let $\Phi \subseteq X$ be any proposition. Then consider the belief function on the updated MLBM structure, according to Definition 46, and the topo-e-model via public announcement, as defined in Definition 29. We prove the two directions of the equivalence.

($\Rightarrow$) Assume that $\mathfrak{M}^{!\Phi}, x \models \mathrm{B}\psi$. By the semantics in Definition 11, this means that there exists an open set $U \in \tau_{\mathcal{E}}^{!\Phi}$ such that $U \subseteq \Psi$ and $\mathsf{cl}(U) = \Phi$, i.e. U is dense in Φ with respect to the subspace topology on Φ. Now consider the function $d^{!\Phi}(\mathcal{E}^{!\Phi}) = \min(\mathrm{dense}(\mathcal{E}^{!\Phi}), \subseteq)$, which returns the minimal dense open set in the topology generated by $\mathcal{E}^{!\Phi}$. By construction, $d^{!\Phi}(\mathcal{E}^{!\Phi})$ is contained in every dense open set of $\tau_{\mathcal{E}}^{!\Phi}$, and in particular $d^{!\Phi}(\mathcal{E}^{!\Phi}) \subseteq U \subseteq \Psi$. Moreover, since $d^{!\Phi}(\mathcal{E}^{!\Phi})$ is dense in Φ, it is non-empty and therefore belongs to $\mathcal{J}_{\mathrm{SD}}^{!\Phi}$ (as $\mathcal{J}_{\mathrm{SD}}^{!\Phi}$ consists of all dense open sets in $\tau_{\mathcal{E}}^{!\Phi}$). Hence,

$$\delta^{!\Phi}_{\mathcal{J}_{\mathrm{SD}}^{!\Phi}}(d^{!\Phi}, d^{!\Phi}(\mathcal{E}^{!\Phi})) > 0,$$

because $\delta_{\tau}^{!\Phi}(d^{!\Phi}, d^{!\Phi}(\mathcal{E}^{!\Phi})) = \sum_{E : d^{!\Phi}(E) = d^{!\Phi}(\mathcal{E}^{!\Phi})} \delta^{!\Phi}(E) \geqslant \delta^{!\Phi}(\mathcal{E}^{!\Phi}) = \prod_{E_i \in \mathcal{E}^{!\Phi}} p_i > 0$. Consequently,

$$\mathrm{Bel}_{\mathcal{J}_{\mathrm{SD}}^{!\Phi}}(d^{!\Phi}, \Psi) \geqslant \delta_{\mathcal{J}_{\mathrm{SD}}^{!\Phi}}(d^{!\Phi}, d^{!\Phi}(\mathcal{E}^{!\Phi})) > 0.$$

($\Leftarrow$) Assume that $\mathrm{Bel}_{\mathcal{J}_{\mathrm{SD}}^{!\Phi}}(d^{!\Phi}, \Psi) > 0$. Then there exists a set $E \subseteq \mathcal{E}^{!\Phi}$ such that $d^{!\Phi}(E) \subseteq \Psi$ and $d^{!\Phi}(E) \in \mathcal{J}_{\mathrm{SD}}^{!\Phi}$. By definition of $\mathcal{J}_{\mathrm{SD}}^{!\Phi}$, $d^{!\Phi}(E)$ is a dense open set in $\tau_{\mathcal{E}}^{!\Phi}$. Let $U = d^{!\Phi}(E)$. Then $U \subseteq \Psi$ and U is dense. By the semantics in Definition 11, for any $x \in X^{!\Phi} = \Phi$, we have $\mathfrak{M}^{!\Phi}, x \models \mathrm{B}\psi$. Therefore, $\mathfrak{M}^{!\Phi}, x \models \mathrm{B}\psi$ holds for all $x \in \Phi$.

Since both direction hold we can claim that the bi-conditional statement holds for $\psi \in \mathcal{L}$. Furthermore, since $\psi \in \mathcal{L}$, $\mathfrak{B}$ and $\mathfrak{M}$ were arbitrary it holds for any formula in $\mathcal{L}$, proving our claim. □

Notice that this result is equivalent to applying (Pinto Prieto, 2024, Proposition 3.9) to our updated frames. In fact, after conditioning on Φ, we recompute the evidential topology $\tau_{\mathcal{E}}^{!\Phi}$, the frame of justification $\mathcal{J}_{\mathrm{SD}}^{!\Phi}$, and the evidence allocation function $d^{!\Phi}$ entirely from the updated evidence set $\mathcal{E}^{!\Phi}$. Since the construction of the Multi-Layer Belief Model and the topological semantics are uniform, the same reasoning as in Proposition 3.9 applies directly to the updated components. Thus, the equivalence between positive belief and the existence of a dense open justification is preserved under conditionalization.

3.2.2 Relating the conditionalization on MLBM to conditional belief operator in TME.

Now we want to expand this notion and, to do so, we study the relationship between our conditionalization procedure and conditional belief in topological models of evidence. As discussed in Section 2.1 and as we can see in the discussion in (Özgün, 2017, Section 5.3.3), the TME conditional belief $\mathrm{B}^{\Phi}\psi$ is a *static* notion: it captures the agent's disposition to believe ψ upon learning Φ without actually updating the model. Consequently, to properly evaluate its behavior in relation to our conditionalization procedure, as defined in Chapter 2, we must assess it within the *original* model before conditioning, while comparing it to the output of our procedure after conditioning.

The following proposition establishes under what conditions our topological agent i.e., the combination of the strong denseness frame of justification $\mathcal{J}_{\mathrm{SD}}$, defined in Definition 19, and the evidence allocation function d, defined according to Definition 23, can replicate the TME conditional belief after our conditionalization procedure has been applied.

Proposition 21. Let $\mathfrak{B} = \left(X, \mathcal{E}_Q, \mathcal{J}_{SD}, \delta, d\right)$ be a MLBM structure, $\mathcal{E}$ its underlying set of evidence, and $\tau_{\mathcal{E}}$ the evidential topology. Let $\mathfrak{M} = (X, \mathcal{E}, \tau_{\mathcal{E}}, V)$ be topo-e-model, where the elements of the frame are the same as above and V is an arbitrary valuation. Let $\Phi \subseteq X$ be a proposition such that $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Phi) > 0$. Then for any formula $\psi \in \mathcal{L}$ with $\Psi = \|\psi\|$ such that $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Psi) > 0$

$$\mathfrak{M}, x \models \mathsf{B}^{\Phi}\psi \quad \Longleftrightarrow \quad \mathsf{Bel}_{\mathcal{J}^{!\Phi}_{SD}}(d^{!\Phi}, \Psi) > 0.$$

where $\mathsf{B}^{\Phi}\psi$, is defined according to Definition 28.

Proof. Consider a MLBM structure $\mathfrak{B} = \left(X, \mathcal{E}_Q, \mathcal{J}_{SD}, \delta, d\right)$, $\mathcal{E}$ the underlying set of evidence, $\tau_{\mathcal{E}}$ the evidential topology and a proposition $\Phi \subseteq X$ such that $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Phi) > 0$. Let V be an arbitrary valuation, $\mathfrak{M} = (X, \mathcal{E}, \tau_{\mathcal{E}}, V)$ be the topo-e-model whose frame has the same elements as above, and $\psi \in \mathcal{L}$ be an arbitrary formula and Ψ its truth set, such that $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Psi) > 0$.
Then by Proposition 20 we know that $\mathsf{Bel}_{\mathcal{J}^{!\Phi}_{SD}}(d^{!\Phi}, \Psi) > 0$ holds if and only if $\mathfrak{M}^{!\Phi}, x \models \mathsf{B}\psi$, where $\mathfrak{M}^{!\Phi}$ is the public announcement model corresponding to $\mathfrak{M}$, according to Definition 29. Then we can reduce the statement to proving that

$$\mathfrak{M}^{!\Phi}, x \models \mathsf{B}\psi \iff \mathfrak{M}, x \models \mathsf{B}^{\Phi}\psi \tag{$\spadesuit$}$$

Furthermore, notice that according to (Pinto Prieto, 2024, Proposition 3.9) we can reduce the assumptions $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Phi) > 0$ and $\mathsf{Bel}_{\mathcal{J}_{SD}}(d, \Psi) > 0$, to the claims that $\mathfrak{M}, x \models \psi$, and $\mathfrak{M}, x \models \varphi$, where φ is any formula such that $\|\varphi\| = \Phi$.
Now we will proceed proving the two directions of Equation ($\spadesuit$).

($\Rightarrow$) Suppose that $\mathfrak{M}^{!\Phi}, x \models \mathsf{B}\psi$. Then, by Definition 11, we know that for any $U \in \tau^{!\Phi}_{\mathcal{E}} \setminus \{\varnothing\}$ there exists $U' \in \tau^{!\Phi}_{\mathcal{E}} \setminus \{\varnothing\}$ such that $U' \subseteq U \cap \Psi$. Thus consider any $U \in \tau^{!\Phi}_{\mathcal{E}} \setminus \{\varnothing\}$. This means that there must be some $V \in \tau_{\mathcal{E}} \setminus \{\varnothing\}$ such that $U = V \cap \Phi \neq \varnothing$. Now we aim to show that there exists a set $V' \in \tau_{\mathcal{E}}$ such that $V' \cap \Phi \neq \varnothing$ and $V' \subseteq V \cap (\Phi \to \Psi)$. By assumption, we know that $\mathfrak{M}, x \models \mathsf{B}\psi$, and $\mathfrak{M}, x \models \mathsf{B}\varphi$. Now, recall that the semantics defined in Definition 11 is sound and complete with respect to the systems KD45. This means that for all formulae $\chi, \gamma \in \mathcal{L}$ holds that

$$\mathsf{B}\chi \wedge \mathsf{B}\gamma \leftrightarrow \mathsf{B}(\chi \wedge \gamma). \tag{K}$$

Applying now Equation (K) to the formulae ψ and φ for which hold that $\mathfrak{M}, x \models \mathsf{B}\psi$, and $\mathfrak{M}, x \models \mathsf{B}\varphi$. Then we can deduce that $\mathfrak{M}, x \models \mathsf{B}\psi \wedge \mathsf{B}\varphi$, and so $\mathfrak{M}, x \models \mathsf{B}(\psi \wedge \varphi)$.
This means, according to Definition 11, that for all $O \in \tau_{\mathcal{E}} \setminus \{\varnothing\}$ there must exist an open set $T \in \tau_{\mathcal{E}} \setminus \{\varnothing\}$ such that $T \subseteq O \cap (\Phi \cap \Psi)$. Since $V \in \tau_{\mathcal{E}} \setminus \{\varnothing\}$ then we can conclude that $T \subseteq V \cap \Phi \cap \Psi$. Therefore, we can directly conclude that $T \cap \Phi \neq \varnothing$, since $T \subseteq \Phi$ and $T \in \tau_{\mathcal{E}} \setminus \{\varnothing\}$. Furthermore, since $T \subseteq V \cap \Phi \cap \Psi$, then it holds that

$$\begin{aligned} & T \subseteq V \cap \Phi \cap \Psi \\ \Rightarrow \quad & T \subseteq V \cap \Psi \end{aligned}$$

$$\Rightarrow \quad T \subseteq V \cap (\overline{\Phi} \cup \Psi)$$
$$\Rightarrow \quad T \subseteq V \cap (\Phi \to \Psi)$$

Therefore we can conclude that there exists a set $T \in \tau_{\mathcal{E}}$ such that $T \cap \Phi \neq \emptyset$ and $T \subseteq V \cap (\Phi \to \Psi)$. Furthermore, being $U \in \tau_{\mathcal{E}}^{!\Phi}$ arbitrary then also $V \in \tau_{\mathcal{E}}$ such that $U = V \cap \Phi$ is arbitrary. Therefore we can conclude that it holds for all $V \in \tau_{\mathcal{E}}$. Therefore we get that $\mathfrak{M}, x \models \mathrm{B}^{\Phi}\psi$, according to Definition 28.

($\Leftarrow$) Suppose that $\mathfrak{M}, x \models \mathrm{B}^{\Phi}\psi$. Consider now an arbitrary $V \in \tau_{\mathcal{E}}$ such that $V \cap \Phi \neq \emptyset$. Then by Definition 28 it means that there exists an open set $V' \in \tau_{\mathcal{E}}$ such that $V' \cap \Phi \neq \emptyset$ and $V' \subseteq V \cap (\Phi \to \Psi)$. Then according to Definition 29, this means that there exist $U, U' \in \tau_{\mathcal{E}}^{!\Phi}$ such that $U = V \cap \Phi \neq \emptyset$ and $U' = V' \cap \Phi \neq \emptyset$. Now, we want to show that $U' \subseteq U \cap \Psi$. We know that $V' \subseteq V \cap (\Phi \to \Psi)$, then by monotonicity of intersection with respect to subset we can conclude that $V' \cap \Phi \subseteq (V \cap (\Phi \to \Psi)) \cap \Phi$. Then we have that

$$
\begin{aligned}
& V' \cap \Phi \subseteq (V \cap (\Phi \to \Psi)) \cap \Phi \\
\Rightarrow \quad & V' \cap \Phi \subseteq (V \cap (\overline{\Phi} \cup \Psi)) \cap \Phi \\
\Rightarrow \quad & V' \cap \Phi \subseteq ((V \cap \overline{\Phi}) \cup (V \cap \Psi)) \cap \Phi \\
\Rightarrow \quad & V' \cap \Phi \subseteq ((V \cap \overline{\Phi} \cap \Phi) \cup ((V \cap \Psi)) \cap \Phi) \\
\Rightarrow \quad & V' \cap \Phi \subseteq (V \cap \Psi) \cap \Phi \\
\Rightarrow \quad & V' \cap \Phi \subseteq (V \cap \Phi) \cap \Psi \\
\Rightarrow \quad & U' \subseteq U \cap \Psi
\end{aligned}
$$

Thus, we got that exists a non empty open set in the updated topology $U \in \tau_{\mathcal{E}}^{!\Phi}$ for which exists a non empty open set $U' \in \tau_{\mathcal{E}}^{!\Phi}$ such that $U' \subseteq U \cap \Psi$. Thus, according to (Özgün, 2013, Proposition 5.3.4), this is equivalent to the definition of B in Definition 11. Furthermore, since $V \in \tau_{\mathcal{E}}$ such that $V \cap \Phi \neq \emptyset$ was arbitrary then also $U \in \tau_{\mathcal{E}}^{!\Phi} \setminus \{\emptyset\}$ is arbitrary. Thus we can conclude that $\mathfrak{M}^{!\Phi}, x \models \mathrm{B}\psi$.

Since under our assumptions both direction of the claim hold we can conclude the the bi-conditional claim does as well. Furthermore, since both the formula $\psi \in \mathcal{L}$, the proposition $\Phi \subseteq X$, the MLBM structure $\mathfrak{B}$ and the valuation V were arbitrary, we can claim it holds for any such elements that respect the assumptions in our claim. Therefore, we can conclude the claim holds. □

This result is conceptually coherent with the nature of conditional statements. In classical logic, a conditional $\varphi \to \psi$ is false only when the antecedent φ is true and the consequent ψ is false. In our setting, the conditional belief $\mathrm{B}^{\Phi}\psi$ is evaluated in the original model, while $\mathrm{Bel}_{\mathcal{J}_{\mathrm{SD}}^{!\Phi}}(d^{!\Phi}, \Psi) > 0$ is evaluated after conditioning on Φ. By Proposition 11, which establishes the *desiderata* 1 and 2, introduced at the end of Section 2.1, we know that after our conditionalization procedure the antecedent φ is guaranteed to hold in the updated model. Thus, the only way the equivalence could fail is if the conditional belief were false in the original model while the updated belief were positive, or vice versa. The assumptions $\mathrm{Bel}_{\mathcal{J}_{\mathrm{SD}}}(d, \Phi) > 0$ and $\mathrm{Bel}_{\mathcal{J}_{\mathrm{SD}}}(d, \Psi) > 0$ ensure that the relevant antecedents are satisfied, making the comparison meaningful. Proposition 21 therefore confirms that our conditionalization procedure faithfully replicates the static conditional belief of the topological evidence model.

3.3 *Conclusion*

In this chapter we studied the relationship between the conditionalization procedure developed in Chapter 2 and the conditioning operations in the two source theories, i.e. *Dempster-Shafer Theory* (DST) and *Topological Models of Evidence* (TME), explained in Section 2.1. We began by noting that our procedure does not directly apply the conditioning rules of those frameworks, but borrows intuitions from both. The natural question we addressed was whether, and under what conditions, our procedure can simulate the behavior of those conditioning rules.

For the Dempster-Shafer side, we distinguished two approaches: conditioning at the level of individual evidence sources and conditioning at the level of the final basic probability assignment. We proved that the merging function δ^t introduced in Definition 34 exactly corresponds to applying Dempster's rule of combination recursively to all pieces of evidence that share the same restriction after conditioning, as shown in Proposition 15. We then established a general necessary and sufficient condition according to Proposition 16 for the equality $\mathsf{Bel}_{\mathfrak{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathsf{Bel}_{\mathfrak{J}}(f, \Psi \mid \Phi)$ to hold for all propositions Ψ. This condition involves two properties of the evidence allocation function f and the frame of justification $\mathfrak{J}$:

$$f^{!\Phi}(\rho_\Phi(E')) = f(E') \cap \Phi \qquad \text{and} \qquad f^{!\Phi}(\rho_\Phi(E')) \in \mathfrak{J}^{!\Phi} \iff f(E') \cap \Phi \neq \varnothing.$$

We showed that for the combination of the Dempster-Shafer frame $\mathfrak{J}_{\mathrm{DS}}$ and the intersection allocation function i, these properties are satisfied under mild assumptions, as proved in Propositions 17 and 18. Consequently, for that combination the conditionalization procedure recovers the standard Dempster-Shafer conditioning. This extends the static correspondence of (Pinto Prieto, 2024, Proposition 3.8) to the dynamic, conditional setting.

For the topological side, we compared our procedure with both the dynamic public announcement operation and the static conditional belief operator of TME. We proved that after conditioning on a proposition Φ, the updated multi-layer belief model with the strong denseness frame $\mathfrak{J}_{\mathrm{SD}}$ and the allocation function d yields a belief that is positive exactly for those formulae that are believed in the public announcement model $\mathfrak{M}^{!\Phi}$, as shown in Proposition 20. As for the aforementioned case, this results extends the relation between MLBM and TME, extending th result in (Pinto Prieto, 2024, Proposition 3.9), to the public announcement update. Moreover, under the extra assumptions that the original model already gives positive belief to both Φ and Ψ, we demonstrated that positive belief in the conditioned model is equivalent to the static conditional belief $\mathsf{B}^{\Phi}\psi$ in the original topological model, as shown in Proposition 21. Thus our procedure faithfully replicates the conditioning behavior of TME as well.

The main theoretical correspondences established in this chapter have been implemented in the accompanying Python code-base. The implementation includes explicit functions to compute the conditionalized belief for the Dempster-Shafer combination (using i and $\mathfrak{J}_{\mathrm{DS}}$) and for the topological case (using d and $\mathfrak{J}_{\mathrm{SD}}$). The Jupyter notebook provided in the repository in (Manes, 2026) runs a colored-balls example and verifies numerically that the results of our conditionalization match the standard Dempster-Shafer conditioning, since in our example the conditions of Proposition 19 are met, and the belief of the topological agent after the public announcement update, established in Proposition 20. Thus the code demonstrates, through concrete computation, that the conditions of Proposition 19, thus also the ones of Proposition 16, hold for the DST case and that the equivalence of Proposition 20 holds for the TME case. The reader is invited to experiment with the code to confirm the correspondences for other inputs.

Conclusion

In this work we studied how to model the *effectus* from a *causa determinata* in the context of the *epistemic horizon*, set by the agent's collection of evidence. More precisely, we analyzed how the landscape of epistemic possibilities that we can deduce from a collection of evidence is determined by a new certain evidence.

Consider now this assertion from *Ethica, Ordine Geometrico Demonstrata*, Part 1, Axiom 4 by Spinoza:

> Effectus cognitio a cognitione causæ dependet et eandem involvit.[1]

Here, we are enquiring about a kind of *cognitio* involving evidence that are partial and uncertain, thus *cognitio ab experientia vaga*, as the *God-Intoxicated Man* calls it in (Spinoza, 1677, Part 2, Proposition 40, Scholium 2). This kind of *cognitio* is what we called during this work *belief*, and more precisely belief from uncertain pieces of evidence. Applying the sentence just introduced above to this precise type of *cognitio*, it is possible to derive that *the belief on the effect depends and involves the belief on its cause.*

In this work we proceed to establish exactly this statement in the context of MLBM, i.e. we define a way of modeling how the belief in a certain evidence, i.e. our *causa determinata*, implies the belief in the *effectus* that it entails. In other words, how our certainty about a piece of evidence changes the model when we become certain about it.

To do so we first presented in Chapter 1 the doxastic frameworks we are analyzing. We introduced the Topological Models of Evidence (Özgün, 2017; Baltag, Bezhanishvili, Özgün, et al., 2018) and Dempster-Shafer Theory (Shafer, 1976; Halpern, 2003), respectively our qualitative and quantitative theoretical base. In order to develop a common language that could allow us to work and move easily between frameworks, we established basic concepts like *set of states*, *proposition*, and *collection of pieces of evidence*, as in Definitions 1 to 3 defining them in the context of a state-representation of the *epistemic horizon* of our agent. Then, we exposed the basic definitions and constructions underlying the two evidential frameworks, focusing on the representation of the doxastic attitude that an agent adopts given a collection of evidence. To do so, we also recalled some basic definitions in modal logic and topology for TME and probability for DST, which are the two traditions on which this two frameworks ascribe. Then, we proceeded on defining the basic construction of MLBM explaining both the technical aspects and the epistemic intuitions behind each layer of the model, and how the model can define different types of doxastic agent based on the definitions given of frame of justification and evidence allocation function, defined in Definitions 18 and 22. We also recalled some results from (Pinto Prieto et al., 2023; Pinto Prieto, 2024), where the MLBM under some particular definitions of the frame of justification and evidence allocation function can reconstruct

1 Personal transalation: The knowledge of an effect depends on the knowledge of the cause, and involves the same.

- the behavior of the Dempster's rule of combination, as in Definition 16, on the set of evidence, using the combination of $\mathcal{J}_{\mathrm{DS}}$ ad frame of justification and i as evidence allocation function,
- the belief operator in doxastic logic for justified belief, as defined in Definition 11, , using the combination of $\mathcal{J}_{\mathrm{SD}}$ ad frame of justification and d as evidence allocation function.

From this results, we got the two type of doxastic attitudes that we targeted in our work for the results in Chapter 3, i.e. the Dempster-Shafer agent, given by the combination of $\mathcal{J}_{\mathrm{DS}}$ and i, and the topological agent, given by the combination of $\mathcal{J}_{\mathrm{SD}}$ and d.

Established this theoretical base, we could proceed to define in Chapter 2 the conditional procedure. First, we explored how this type of learning, that we address as *conditionalization*, is defined in the various theoretical bases. We start from exploring the most well known way to define this type of learning in a model: *conditional probability*. Then we tried to target specific dynamics that were modeling the same kind of update in our theoretical base. For TME, we recalled the definition of a static operator and of a dynamic update, which aim to condition the belief on a given certain evidence. These are respectively *conditional belief*, introduced in Definition 28, and *public announcement models*, as defined in Definition 29. Then we considered DST, *conditional belief defined via Dempster's rule of combination*, according to Definition 30. From these definitions and the ways they condition the doxastic attitude of an agent on a specific proposition, we defined four *desiderata* that a conditionalization for MLBM should have: (1) restriction of the set of states to the learned certain proposition; (2) normalization of the basic probability assignment on the new space; (3) preservation of the agent's doxastic attitudes, i.e. the frame of justification and the evidence allocation function should be renormalized to the new space; and (4) the conditionalization should return a new MLBM structure that allows iteration, meaning we are able to perform any sequential update in a single step.

In Chapter 2 we then developed a conditionalization procedure that satisfies these *desiderata*. We updated the qualitative layer by applying the public announcement frame construction to it. This is not problematic in terms of dependencies of our framework to the topological one, since this is the first layer and the elements here are purely qualitative, and we are updating this at a frame level, so no assumption on the behavior of operators in the specific case is brought inside our model. Furthermore, we argued that the frame of justification must be recomputed from scratch on the updated topology, as in Definition 33, showing via an argument that applying the same method of considering the intersection might bring undesired results changing the type of agent we are considering, as shown in Example 1. For the quantitative layer, after showing via some examples that we cannot combine the new evidence using the δ function built into the model, neither at the level of the whole set of evidence (as in Example 2) nor at the level of a single piece of evidence (as in Example 3), we introduced a new merging function δ^t, as defined in Definition 34, that correctly combines weights of evidence pieces having the same intersection with the conditioning proposition. Furthermore, we showed that the qualitative layer and the quantitative layer, which are updated separately, after update return the same pieces of evidence, as shown in Proposition 10. The bridging layer was then recomputed on the updated components, as described in Definitions 42 to 44. Moreover, we argued that updating the evidence allocation function by taking the intersection of the previous output might bring undesired results since it can be ill-defined, and even if it were well defined this procedure might change our agent, as argued in Example 4.

For each update we designed explicit algorithms, presented in Algorithms 1 to 3 and proved in Propositions 5 and 9 their correctness. Furthermore, all our algorithms are polynomial time computable with respect to the size of the input. The new updated MLBM structure can be retrieved by composing these singular updates in the natural order: first update the qualitative frame, then the quantitative frame, then recompute the bridging layer. The overall complexity is therefore the sum of the complexities of the individual steps, each of which is polynomial, so the whole conditionalization procedure runs in polynomial time, more precisely its complexity is $\mathcal{O}\left(|\mathcal{E}|^2 + |\tau_{\mathcal{E}}| \cdot \min(|\max \tau_{\mathcal{E}}|, |\Phi|) + |\mathcal{E}_Q|^2\right)$. All definitions and algorithms have been implemented in Python, building on the existing MLBM code-base by (Pinto Prieto et al., 2023; Compagno et al., 2026). The code and a detailed Jupyter notebook illustrating the procedure on a concrete example are available in the accompanying repository that can be found in (Manes, 2026).

Having defined the conditionalization, we studied in Chapter 3 its relationship with the conditioning operations of the two source theories, presented in Section 2.1. For DST we proved in Proposition 15 that the merging function δ^t exactly corresponds to recursively applying Dempster's rule to all pieces of evidence that share the same restriction after conditioning. We then gave a necessary and sufficient condition in Proposition 16 under which our procedure is equivalent to conditioning the original belief function via Dempster's rule of combination, as defined in Definition 30, i.e. $\mathrm{Bel}_{\mathcal{J}^{!\Phi}}(f^{!\Phi}, \Psi) = \mathrm{Bel}_{\mathcal{J}}(f, \Psi \mid \Phi)$ for all Ψ. For the specific combination of the Dempster-Shafer frame of justification $\mathcal{J}_{\mathrm{DS}}$ and the intersection allocation function i, these conditions are satisfied under mild assumptions, presented in Propositions 17 and 18, so our procedure recovers standard Dempster-Shafer conditioning under the aforementioned combination, as shown in Proposition 19. For the topological side we showed that after conditioning, the updated MLBM with $\mathcal{J}_{\mathrm{SD}}$ and d yields a positive belief exactly for those formulae that are believed in the public announcement model, as shown in Proposition 20. Moreover, under the extra assumptions that the original model already gave positive belief to both Φ and Ψ, positive belief in the conditioned model is equivalent to the static conditional belief $\mathrm{B}^{\Phi}\psi$ in the original topological model, as proved in Proposition 21. Thus our conditionalization procedure faithfully replicates the conditioning behavior of both DST and TME under the appropriate agent configurations. Furthermore, the notebook presented in (Manes, 2026) also verifies numerically the correspondences established in Chapter 3 for the Dempster-Shafer and topological agents.

Overall this work, in the spirit of Spinoza's insight that the belief of an effect depends on and involves the belief of its cause, showed within the Multi-Layer Belief Model that conditioning on certain evidence, i.e. our *causa determinata*, indeed determines the agent's subsequent beliefs in a principled, computationally feasible, and epistemically well-motivated way, thereby illuminating the path from *cognitio ab experientia vaga*. Our procedure, which is built without assumptions from other theories, has been shown to satisfy natural *desiderata*, to be iterable, and under suitable choices of the agent's epistemic attitudes to reproduce the conditioning rules of the two foundational theories on which the MLBM is built. By doing so, it extends the expressive power of the MLBM to handle dynamic information, bringing the model closer to practical applications in reasoning with uncertain, partial, and possibly inconsistent evidence.

Future work includes several directions. A first line of research would aim to bridge the MLBM more tightly to the imprecise probabilistic setting. We plan to study the correspondence between our conditionalization procedure and conditional inner measures, and hence conditional belief defined via conditional inner measures, as explained in (Halpern, 2003; Halpern and Fagin, 1992). Our preliminary attempts in this direction produced only minor results, and we would like to work further on this connection.

A second line, which we believe should be implemented, is a multi-agent setting. Extending the MLBM to handle multiple interacting agents would allow us to model scenarios with distributed evidence, information sharing, and strategic communication. As we can see both in DST and TME the multi-agent setting have been studied for example the reader might consider (Baltag, Bezhanishvili, and Fernández González, 2022; Halpern, 2003), however the MLBM is still missing such explorations. This type of work would involve defining how each agent's belief state updates upon learning not only from the environment but also from other agents' announcements or actions, and studying the resulting epistemic and doxastic dynamics in a multi-agent framework.

A third line, relevant to dynamic epistemic logic, would be to embed new types of dynamics into the model, such as updating with uncertain evidence, evidence addition, evidence upgrade, or evidence removal, as studied in (Baltag, Bezhanishvili, Özgün, et al., 2016; Özgün, 2017). Each of these operations changes the epistemic state in a different manner, and adapting our conditionalization framework to handle them would provide a more complete picture of belief revision.

On the same line of the previous ones a more general goal, that we we think would be profitable in order to get a more comprehensive theory, would be to define a logical language that can describe these models and their dynamics, extending the results in (Pinto Prieto, 2024, Chapter 4). This would involve first axiomatizing the static logic presented there and proving its soundness and completeness with respect to the MLBM semantics, then axiomatizing the new dynamic operators corresponding to conditionalization, evidence addition, and other operations, and proving soundness and completeness with respect to the MLBM semantics. A general logical apparatus that unifies the models and these dynamics would greatly enhance the model's applicability and theoretical clarity.

Moreover, to deepen the quantitative side, we think it would be relevant to introduce new measures within the system, such as expected value and variance of belief, as we can see in (Halpern, 2003), to enable a richer quantitative analysis and to continue bridging the MLBM to Dempster-Shafer theory and imprecise probabilities overall.

Finally, we aim to extend the model to handle larger (possibly infinite) sets of states, moving beyond the finiteness assumption adopted in (Pinto Prieto, 2024). Removing this restriction would yield a more general framework that can serve as a powerful bridge between qualitative and quantitative aspects of reasoning under uncertainty, opening the door to applications in continuous domains and infinite-state systems.

We believe the Multi-Layer Belief Model offers a new and promising framework in the field of reasoning under uncertainty, and a new step in bridging between the worlds of qualitative and quantitative aspects of symbolic reasoning. We hope that our work even if little could be a step in the development of such unification.

Appendix

Proof of Lemma 1. We prove this lemma by induction on the number m.

Base case Suppose that $m = 1$, then $\{1, \dots, m\} = \{1\}$. Therefore, it holds that

$$1 - \prod_{i=1}^{m} (1 - p_i) = 1 - (1 - p_1) = p_1.$$

Moreover, since $S \neq \emptyset$ and $S \subseteq \{1, \dots, m\} = \{1\}$, we have that $S = \{1\}$. Thus, we get that

$$\sum_{\substack{S \subseteq \{1,\dots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i = (-1)^{|\{1\}|-1} \prod_{i \in \{1\}} p_i = (-1)^2 \cdot p_1 = p_1.$$

Hence, the base case holds.

Induction step Assume for induction hypothesis that for m it holds that

$$1 - \prod_{i=1}^{m} (1 - p_i) = \sum_{\substack{S \subseteq \{1,\dots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i. \tag{IH}$$

We want to show that the claim holds for $m + 1$.

Thus, we have that

$$\begin{aligned}
1 - \prod_{i=1}^{m+1} (1 - p_i) &= 1 - \prod_{i=1}^{m} (1 - p_i) \cdot (1 - p_{m+1}) \\
&= 1 - \prod_{i=1}^{m} (1 - p_i) + \prod_{i=1}^{m} (1 - p_i) \cdot p_{m+1} \\
&= 1 - \prod_{i=1}^{m} (1 - p_i) + \left(1 - \left(1 - \prod_{i=1}^{m} (1 - p_i)\right)\right) \cdot p_{m+1} \\
&= \sum_{\substack{S \subseteq \{1,\dots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i + \left(1 - \left(\sum_{\substack{S \subseteq \{1,\dots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i\right)\right) \cdot p_{m+1} && \text{(IH)}
\end{aligned}$$

$$= \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i + p_{m+1} - \left(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} p_{m+1} \prod_{i \in S} p_i \right)$$

Analogously to the base case, we write p_{m+1} as a sum over a singleton, as follows:

$$\begin{aligned} p_{m+1} &= (-1)^2 \cdot p_{m+1} \\ &= (-1)^2 \cdot \prod_{i \in \{m+1\}} p_i \\ &= \sum_{U = \{m+1\}} (-1)^{|U|+1} \prod_{i \in U} p_i \end{aligned}$$

Therefore, returning to the previous expression, we can rewrite it as follows:

$$\begin{aligned} 1 - \prod_{i=1}^{m+1} (1 - p_i) &= \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i + p_{m+1} - \left(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} p_{m+1} \prod_{i \in S} p_i \right) \\ &= \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i + \sum_{U = \{m+1\}} (-1)^{|U|+1} \prod_{i \in U} p_i \\ &\quad - \left(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \sum_{U = \{m+1\}} (-1)^{|U|+1} \prod_{i \in U} p_i \prod_{i \in S} p_i \right) \end{aligned}$$

Now we turn to the third summation, which carries a negative sign. To begin, observe that:

$$\begin{aligned} &- \left(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \sum_{U = \{m+1\}} (-1)^{|U|+1} \prod_{i \in U} p_i \prod_{i \in S} p_i \right) \\ &= - \sum_{U = \{m+1\}} (-1)^{|U|+1} \prod_{i \in U} p_i \left(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} p_i \right) \end{aligned}$$

Since $|U| = 1$, we have $(-1)^{|U|+1} = (-1)^2 = 1$. Hence, we can adjust the indices to absorb the negative sign into the summation by replacing $(-1)^{|U|+1}$ with $(-1)^{|U|}$. Moreover, we may rewrite the sum as ranging over sets of the form $S \subseteq \{1, \ldots, m\}$ with $U = m + 1$, and thereby combine the products of identical terms.

Therefore, we have:

$$
\begin{aligned}
&-\sum_{U=\{m+1\}}(-1)^{|U|+1}\prod_{i\in U}p_i\left(\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\prod_{i\in S}p_i\right)\\
&=\sum_{U=\{m+1\}}(-1)^{|U|}\prod_{i\in U}p_i\left(\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\prod_{i\in S}p_i\right)\\
&=\sum_{\substack{T=S\cup\{m+1\}\\ S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\cdot(-1)^{|\{m+1\}|}\prod_{i\in T}p_i\\
&=\sum_{\substack{T=S\cup\{m+1\}\\ S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|T|+1}\prod_{i\in T}p_i
\end{aligned}
$$

Now, we can go back to the original equivalence:

$$
\begin{aligned}
1-\prod_{i=1}^{m+1}(1-p_i)&=\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\prod_{i\in S}p_i+\sum_{U=\{m+1\}}(-1)^{|U|+1}\prod_{i\in U}p_i\\
&\quad-\left(\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\sum_{U=\{m+1\}}(-1)^{|U|+1}\prod_{i\in U}p_i\prod_{i\in S}p_i\right)\\
&=\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\prod_{i\in S}p_i+\sum_{U=\{m+1\}}(-1)^{|U|+1}\prod_{i\in U}p_i+\sum_{\substack{T=S\cup\{m+1\}\\ S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|T|+1}\prod_{i\in T}p_i
\end{aligned}
$$

Notice first that all three summations involve terms of the same form, namely products of the p_i weighted by alternating signs. Moreover, the second and third summations in the first line together range over all subsets of $\{1,\dots,m+1\}$ that contain the element $m+1$. The second summation accounts for the singleton $m+1$, while the third runs over all subsets obtained by adjoining $m+1$ to a nonempty subset $S\subseteq\{1,\dots,m\}$. Since S is required to be nonempty in the third summation, there is no overlap with the singleton case, and thus no double counting occurs. Consequently, these two sums can be merged into a single summation over all subsets of $\{1,\dots,m+1\}$ containing $m+1$. Combining this with the first summation, which ranges over nonempty subsets of $\{1,\dots,m\}$, gives back a single sum over all nonempty subsets of $\{1,\dots,m+1\}$, as follows:

$$
1-\prod_{i=1}^{m+1}(1-p_i)=\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|S|+1}\prod_{i\in S}p_i+\sum_{U=\{m+1\}}(-1)^{|U|+1}\prod_{i\in U}p_i+\sum_{\substack{T=S\cup\{m+1\}\\ S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}}(-1)^{|T|+1}\prod_{i\in T}p_i
$$

$$
\begin{aligned}
&= \sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|+1}\prod_{i\in S} p_i + \sum_{\substack{T=S\cup\{m+1\}\\ S\subseteq\{1,\dots,m\}}} (-1)^{|T|+1}\prod_{i\in T} p_i\\
&= \sum_{\substack{S\subseteq\{1,\dots,m+1\}\\ S\neq\varnothing}} (-1)^{|S|+1}\prod_{i\in S} p_i
\end{aligned}
$$

This completes the inductive step.

Therefore, by mathematical induction, we can conclude that the identity holds for all m. □

Proof of Lemma 2. By the result of Lemma 1 we know that

$$
1-\prod_{i=1}^{m}(1-p_i) = \sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|+1}\prod_{i\in S} p_i.
$$

We aim to employ this result in order to get the *desideratum*. Thus, notice that

$$
\begin{aligned}
\prod_{i=1}^{m}(1-p_i) &= 1-\left(1-\prod_{i=1}^{m}(1-p_i)\right)\\
&= 1-\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|+1}\prod_{i\in S} p_i && \text{(Lemma 1)}
\end{aligned}
$$

Notice that employing basic arithmetical properties of the product and exponentiation, we can claim that

$$
-(-1)^{|S|+1} = (-1)\cdot(-1)^{|S|+1} = (-1)^{|S|+2} = (-1)^{|S|}. \tag{$\spadesuit$}
$$

Thus,

$$
\begin{aligned}
\prod_{i=1}^{m}(1-p_i) &= 1-\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|+1}\prod_{i\in S} p_i && \text{(Lemma 1)}\\
&= 1+\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|}\prod_{i\in S} p_i && \text{(by Equation ($\spadesuit$))}
\end{aligned}
$$

Notice that for $T=\varnothing$ it holds that

1. $|T|=0$, so $(-1)^{|T|}=(-1)^0=1$ and
2. $\prod_{i\in T} p_i = 1$.

Therefore, we have that

$$
\begin{aligned}
\prod_{i=1}^{m}(1-p_i) &= 1+\sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|}\prod_{i\in S} p_i && \text{(by Equation ($\spadesuit$))}\\
&= \sum_{T=\varnothing}(-1)^{|T|}\prod_{i\in T} p_i + \sum_{\substack{S\subseteq\{1,\dots,m\}\\ S\neq\varnothing}} (-1)^{|S|}\prod_{i\in S} p_i
\end{aligned}
$$

$$= \sum_{S \subseteq \{1,\ldots,m\}} (-1)^{|S|} \prod_{i \in S} p_i$$

Therefore, we can conclude that our claim holds. $\square$

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