On Definability in Dependence Logic

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Abstract

We study the expressive power of open formulas of Dependence Logic introduced in [9]. In particular, we answer a question raised by Wilfrid Hodges: how to characterize the sets of teams definable by means of identity only in dependence logic, or equivalently in independence friendly logic.

1 Introduction

The Independence Friendly (IF) Logic, incorporating explicit dependence of quantifiers from each other, was introduced in [4, 3]. By the method of [2] and [10] it can be seen that every sentence of IF logic has a definition in Σ^1_1, and vice versa. Hodges gave in [5] a compositional semantics for IF logic in terms of, what he calls trumps. He showed in [6] that every formula of IF logic can be represented in an equivalent form in Σ^1_1 with an extra predicate interpreting the trump. Hodges went on to ask about the converse: what sets of subsets of an infinite domain M are expressible as the set of trumps of a formula of the logic IF by means of identity only. We show in this paper that the answer is: exactly those that can be defined in Σ^1_1 with an extra predicate, occurring only negatively, for the trump.

We use the framework of [9] and accordingly talk about dependence logic rather than IF logic. At the end of the paper we state our results also for IF logic.

2 Preliminaries

In this section we define Dependence Logic ($\mathcal{D}$) and recall some of its properties.

Definition 2.1 ([9])**.** The syntax of $\mathcal{D}$ extends the syntax of FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by new atomic (dependence) formulas of the form

$$=\!(t_1, \ldots, t_n), \tag{1}$$

where $t_1, \ldots, t_n$ are terms. If L is a vocabulary, we use $\mathcal{D}[L]$ to denote the set of formulas of $\mathcal{D}$ based on L.

The intuitive meaning of the dependence formula (1) is that the value of the term t_n is determined by the values of the terms $t_1, \ldots, t_{n-1}$. As singular cases we have

$$=\!(),$$

which we take to be universally true, and

$$=\!(t),$$

which declares that the value of the term t depends on nothing, i.e., is constant. In order to define the semantics of $\mathcal{D}$, we first need to define the concept of a *team*.

Let $\mathfrak{A}$ be a model with domain A. *Assignments* of $\mathfrak{A}$ are finite mappings from variables to A. The value of term t in the assignment s is denoted by $t_1^{\mathfrak{A}}\langle s\rangle$. If s is an assignment, x a variable in the domain of s and $a \in A$, then $s(a/x)$ denotes the assignment obtained from s by changing the value of s at x to a.

Let A be a set and $\{x_1, \ldots, x_k\}$ a finite set of variables. A *team* X of A with domain $\{x_1, \ldots, x_k\}$ is any set of assignments from the variables $\{x_1, \ldots, x_k\}$ into the set A. We denote by $\mathrm{rel}(X)$ the k-ary relation of A corresponding to X $\mathrm{rel}(X) = \{(s(x_1), \ldots, s(x_k)) \mid s \in X\}$. If X is a team of A, and $F\colon X \to A$, we use $X(F/x_n)$ to denote the team $\{s(F(s)/x_n) \ : s \in X\}$ and $X(A/x_n)$ the team $\{s(a/x_n) \ : s \in X \text{ and } a \in A\}$.

We are now ready to define the semantics of $\mathcal{D}$. We restrict attention to formulas in negation normal form, i.e., negation is assumed to appear only in front of atomic formulas.

Definition 2.2 ([9])**.** Let $\mathfrak{A}$ be a model and X a team of A. The satisfaction relation $\mathfrak{A} \models_X \varphi$ is defined as follows:

1. $\mathfrak{A} \models_X t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle = t_2^{\mathfrak{A}}\langle s\rangle$.

2. $\mathfrak{A} \models_X \neg t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle \neq t_2^{\mathfrak{A}}\langle s\rangle$.

3. $\mathfrak{A} \models_X =\!(t_1, ..., t_n)$ iff for all $s, s' \in X$ such that $t_1^{\mathfrak{A}}\langle s\rangle = t_1^{\mathfrak{A}}\langle s'\rangle, \ldots, t_{n-1}^{\mathfrak{A}}\langle s\rangle = t_{n-1}^{\mathfrak{A}}\langle s'\rangle$, we have $t_n^{\mathfrak{A}}\langle s\rangle = t_n^{\mathfrak{A}}\langle s'\rangle$.

4. $\mathfrak{A} \models_X \neg =\!(t_1, ..., t_n)$ iff $X = \emptyset$.

5. $\mathfrak{A} \models_X R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \in R^{\mathfrak{A}}$.

6. $\mathfrak{A} \models_X \neg R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \notin R^{\mathfrak{A}}$.

7. $\mathfrak{A} \models_X \psi \wedge \phi$ iff $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \phi$.

8. $\mathfrak{A} \models_X \psi \vee \phi$ iff $X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \phi$.

9. $\mathfrak{A} \models_X \exists x_n \psi$ iff $\mathfrak{A} \models_{X(F/x_n)} \models \psi$ for some $F\colon X \to A$.

10. $\mathfrak{A} \models_X \forall x_n \psi$ iff $\mathfrak{A} \models_{X(A/x_n)} \psi$.

Finally, a sentence φ is true in a model $\mathfrak{A}$ if $\mathfrak{A} \models_{\{\emptyset\}} \varphi$.

Our goal in this paper is to characterize *definable sets of teams*, i.e. sets of the form

$$\{X : \mathfrak{A} \models_X \phi\}, \tag{2}$$

where $\mathfrak{A}$ is a fixed model and $\phi \in \mathcal{D}$. For reasons that we discuss in the next section we attempt to characterize the set (2) in the special case that the vocabulary of $\mathfrak{A}$ is empty. Note that this case is still non-trivial. For example, if the domain of $\mathfrak{A}$ is infinite, the set of ϕ such that $\{\emptyset\}$ is in the set (2), is non-recursive (in fact Π^0_1-complete, by Theorem 2.4) even if the vocabulary of $\mathfrak{A}$ is empty. The following fact [5] is very basic:

Proposition 2.3 (Downward closure)**.** *Suppose $Y \subseteq X$. Then $\mathfrak{A} \models_X \varphi$ implies $\mathfrak{A} \models_Y \varphi$.*

Another basic fact is the result that the expressive power of sentences of $\mathcal{D}$ coincides with that of existential second-order sentences (Σ^1_1):

Theorem 2.4 ([10, 2])**.** *For every sentence ϕ of $\mathcal{D}$ there is a sentence Φ of Σ^1_1 such that*

$$\text{For all models } \mathfrak{A}\text{: } \mathfrak{A} \models_{\{\emptyset\}} \phi \iff \mathfrak{A} \models \Phi. \tag{3}$$

Conversely, for every sentence Φ of Σ^1_1 there is a sentence ϕ of $\mathcal{D}$ such that (3) holds.

However, Theorem 2.4 does not – a priori – tell us anything about definable sets of teams. In our main result below (Theorem 4.9) we generalize Theorem 2.4 from sentences to formulas. Since formulas of $\mathcal{D}$ define sets of teams and formulas of Σ_1^1 define sets of assignments, the two concepts cannot be directly compared. To remedy this we compare definability by a formula of $\mathcal{D}$ to definability by a sentence of Σ_1^1 with an extra predicate.

3 Two examples

The two examples of this section demonstrate the difficulties in characterizing all definable properties of teams. The first example is from [5]. It shows that, over a fixed model, the family of teams satisfying a formula can be extremely complex.

Example 3.1. Let A be a set, n a positive integer, and F a family of sets of n-tuples of A which is closed under taking subsets. Suppose that there happens to be an $n+1$-ary relation R on A such that for every set $T \subseteq A^n$,

$$T \in F \Leftrightarrow \text{ there is } b \in A \text{ such that } R(\overline{a}b) \text{ for all } \overline{a} \in T.$$

Let $\varphi(\overline{x})$ be the formula $\exists y(=(y) \wedge R(\overline{x}, y))$, then

$$(A, R) \models_X \varphi(\overline{x}) \Leftrightarrow \mathrm{rel}(X) \in F.$$

As emphasized in [5], this shows that it is very difficult to say anything more about definable properties of teams on arbitrary structures except that they are closed downwards. This example is elaborated in [1].

The previous example used in an essential way the predicate R. In the next example, we construct formulas defining certain downward closed properties of teams over the empty vocabulary.

Proposition 3.2. *Let $k \in \mathbb{N}$ and let $P(x)$ be a polynomial with positive integer coefficients. Then there is a formula $\varphi(\overline{x}) \in \mathcal{D}$ such that for all finite sets A and teams X over $\{x_1, \ldots, x_k\}$*

$$A \models_X \varphi \Leftrightarrow |X| \leq P(|A|).$$

Proof. Suppose first that $P(x) = c \in \mathbb{N}$. Note that $|X| \leq 1$ can be defined by the formula ψ:

$$=(x_1) \wedge \cdots \wedge =(x_k).$$

Therefore, $|X| \leq c$ can be expressed as

$$\psi \vee \psi \cdots \vee \psi,$$

where the disjunction is taken c times. Suppose then that $P(x) = x^c$. Now the following formula can be used

$$\exists y_1 \ldots \exists y_c (\bigwedge_{1 \le i \le k} =(y_1, \ldots, y_c, x_i)).$$

This formula declares that there is a function from the set X to the set A^c which is one-to-one. Finally, note that $|X| \le (P_1 + P_2)(|A|)$ can be expressed as $\psi_1 \vee \psi_2$ assuming that ψ_i defines the property $|X| \le P_i(|A|)$. □

4 Characterizing definable properties of teams

In this section we restrict attention to properties of teams definable over the empty vocabulary. We show that, over the empty vocabulary, definable team properties correspond exactly to the downwards closed quantifiers of Σ^1_1.

Definition 4.1. Let $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ and R a k-ary predicate. We denote by Q_φ the following class of $\{R\}$-structures

$$Q_\varphi = \{(A, \mathrm{rel}(X)) \mid A \models_X \varphi\}.$$

Lemma 4.2. *For every formula $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$, the class Q_φ is closed under isomorphisms.*

Since satisfiability is preserved in subteams, the quantifier Q_φ is always monotone downwards. The question we are studying can be formulated as follows.

Question 1. *For which downwards monotone quantifiers Q we can find a formula $\varphi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\varphi$.*

Denote by $\Sigma^1_1[\{R\}]$ existential second-order sentences of vocabulary $\{R\}$. It is easy to see that Σ^1_1-definability is an upper bound for the solution.

Proposition 4.3. *For every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ the quantifier Q_φ is definable in $\Sigma^1_1[\{R\}]$.*

Proof. By [6], for every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$, there is a sentence $\psi \in \Sigma^1_1[\{R\}]$ such that for all sets A and teams X over $\{y_1, \ldots, y_k\}$ it holds that

$$A \models_X \varphi \Leftrightarrow (A, \mathrm{rel}(X)) \models \psi.$$

□

Corollary 4.4. *Let $k \in \mathbb{N}$. There is no formula $\varphi(x_1, \dots, x_k) \in \mathcal{D}[\emptyset]$ such that for all A and teams X*

$$A \models_X \varphi \Leftrightarrow |X| \text{ is finite.}$$

Proof. This follows by Proposition 4.3 and the Compactness Theorem of Σ^1_1. □

Since, e.g., transitivity is not a downward monotone property, the family of quantifiers we are looking for will be a proper subclass of $\Sigma^1_1[\{R\}]$. We shall next show that there is a syntactic criterion for a $\Sigma^1_1[\{R\}]$ sentence to be monotone downwards.

Definition 4.5. Let R be a k-ary relation symbol and $\varphi \in \Sigma^1_1[\{R\}]$ a sentence. We say that φ is downwards monotone with respect to R if for all A and $B' \subseteq B \subseteq A^n$

$$(A, B) \models \varphi \Rightarrow (A, B') \models \varphi.$$

Definition 4.6. An occurence of a relation symbol R in a formula φ is called positive (negative) if it is in the scope of an even (odd) number of nested negation symbols.

Proposition 4.7. *A sentence $\varphi \in \Sigma^1_1[\{R\}]$ is downwards monotone with respect to R iff there is $\psi \in \Sigma^1_1[\{R\}]$ such that*

$$\models \varphi \leftrightarrow \psi,$$

and R appears only negatively in ψ.

Proof. Assume that $\varphi \in \Sigma^1_1[\{R\}]$ is monotone downwards. Let φ^* be a formula acquired by replacing all the occurences of R in φ by a new predicate variable R'. Using the downwards monotonicity of φ, it is straightforward to verify that

$$\models \varphi \leftrightarrow \exists R'(\varphi^* \wedge \forall \overline{x}(R(\overline{x}) \rightarrow R'(\overline{x}))).$$

Note that, on the right hand side, the predicate R appears only negatively.

For the other direction, we may assume that negation appears in φ only in front of atomic formulas. Now the claim follows by induction on the construction of φ (case $\varphi = \neg R(\overline{t})$ being the only non-trivial one). □

In the following, we shall be using the fact that Σ^1_1 formulas can be transformed to the so-called Skolem Normal Form [7] (see [8]).

Theorem 4.8 (Skolem Normal Form Theorem)**.** *Every Σ^1_1 formula is equivalent to a formula of the form*

$$\exists f_1 \dots \exists f_n \forall x_1 \dots \forall x_m \psi,$$

where ψ is a quantifier-free formula.

We are now ready to prove the main result of this paper.

Theorem 4.9. *Let Q be a downwards monotone quantifier. Then there is a formula $\varphi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\varphi$ if and only if Q is $\Sigma^1_1[\{R\}]$-definable.*

Proof. Note that Proposition 4.3 already gives the other half of the claim. Assume that Q is a downwards monotone Σ^1_1-quantifier. We need to find a formula $\varphi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\varphi$. By Theorem 4.8, there is a sentence λ of the form

$$\exists f_1 \dots \exists f_n \forall x_1 \dots \forall x_m \psi \tag{4}$$

defining Q. We may assume that ψ is in conjunctive normal form and that for all the function symbols appearing in ψ there are unique pairwise distinct variables $z_1, \dots, z_s$ ($(z_1, \dots, z_s)$ a subsequence of $(x_1, \dots, x_m)$) such that all occurences of f are of the form $f(z_1, \dots, z_s)$ (see [9] for details). As in the proof of Proposition 4.7, we then pass on to the equivalent formula

$$\exists R'(\lambda^* \wedge \forall \overline{x}(R(\overline{x}) \rightarrow R'(\overline{x})))$$

and translate it again to Skolem normal form

$$\exists f_1 \dots \exists f_n \exists f_{n+1} \exists f_{n+2} \forall x_1 \dots \forall x_{m'} (\psi' \wedge (\neg R(\overline{x}) \vee f_{n+1}(\overline{x}) = f_{n+2}(\overline{x}))),$$

i.e., we replace all subformulas of the form $R'(t_1, \dots, t_k)$ by the formula $f_{n+1}(t_1, \dots, t_k) = f_{n+2}(t_1, \dots, t_k)$ and place the universal quantifiers in front by changing bound variables if necessary. We still need to make sure that all the occurences of the new function symbols f_{n+1} and f_{n+2} are of the form $f(z_1, \dots, z_s)$ for some pairwise distinct variables $z_1, \dots, z_s$ ($(z_1, \dots, z_s)$ a subsequence of $(x_1, \dots, x_m)$). This requires some transformations on the quantifier-free part since we want it to maintain conjunctive normal form. These transformations might add a new conjunct (a disjunction of identities) to

$$(\psi' \wedge (\neg R(\overline{x}) \vee f_{n+1}(\overline{x}) = f_{n+2}(\overline{x}))$$

or add new disjuncts (identity atoms) to all the conjucts via the equivalence $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. However, after these trasformations, only one of the conjuncts has a literal of the form $\neg R(\overline{x})$. In other words, the predicate R has in total only one occurence in the formula and it is negative.

Let us now assume that the formula in (4) defines Q and satisfies all the conditions required above. The formula $\chi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ defining Q is now defined as

$$\forall x_1 \ldots \forall x_m \exists x_{m+1} \ldots \exists x_{m+n} (\theta_1 \wedge \theta_2),$$

where θ_1 is the formula

$$\bigwedge_{1 \le i \le n} =\!(z_1^i, \ldots, z_{s_i}^i, x_{m+i}),$$

and $(z_1^i, \ldots, z_{s_i}^i)$ is the unique tuple of variables to which f_i is applied in ψ. The formula θ_2 is acquired from ψ by first replacing the terms $f_i(z_1^i, \ldots, z_{s_i}^i)$ by the corresponding variables x_{m+i} in ψ. Note that the assumptions on the way function terms can occur guarantee that the variable x_{m+i} always denotes the same element as the term $f_i(z_1^i, \ldots, z_{s_i}^i)$ in the translation. Finally, we replace the subformula $\neg R(x_1, \ldots, x_k)$ in ψ by the formula

$$\bigvee_{1 \le i \le k} y_i \neq x_i.$$

We shall next show that the translation works as intended, i.e., that for all A and teams X over $\{y_1, \ldots, y_k\}$

$$A \models_X \chi(y_1, \ldots, y_k) \Leftrightarrow (A, \mathrm{rel}(X)) \models \varphi.$$

Clearly, it suffices to show that for all functions $\overline{f}$ of the appropriate arity

$$A \models_{X^*} \theta_2 \Leftrightarrow (A, \mathrm{rel}(X), \overline{f}) \models \forall x_1 \ldots \forall x_m \psi,$$

where

$$X^* = \{ s\overline{a} f_1(\overline{a}) \cdots f_n(\overline{a}) \mid s \in X \text{ and } \overline{a} \in A^k \}$$

and $f_i(\overline{a})$ denotes the result of applying function f_i to the appropriate subsequence of $\overline{a}$ determined by the way $z_1^i, \ldots, z_{s_i}^i$ reside in $x_1, ..., x_m$. Recall that ψ is assumed to be in conjunctive normal form

$$\psi = \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha_{j_i}.$$

Hence, the formula θ_2 can be written as

$$\psi = \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha_{j_i}^*,$$

where $\alpha_{j_i}^*$ arises from α_{j_i} by replacing the terms $f_i(z_1^i, \ldots, z_{s_i}^i)$ by the variables x_{m+i} and $\neg R(x_1, \ldots, x_k)$ by $\bigvee_{1 \le i \le k} y_i \neq x_i$.

Let us assume first that the claim holds for all the conjuncts of ψ. Suppose that

$$(A, \mathrm{rel}(X), \overline{f}) \models \forall x_1 \dots \forall x_m \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha_{j_i}.$$

Then, for all j we have that

$$(A, \mathrm{rel}(X), \overline{f}) \models \forall x_1 \dots \forall x_m \bigvee_{1 \le i \le r_j} \alpha_{j_i}.$$

By the assumption, it holds that

$$A \models_{X^*} \bigvee_{1 \le i \le r_j} \alpha^*_{j_i}$$

for all j, and thus

$$A \models_{X^*} \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha^*_{j_i}.$$

The other direction is analogous. Therefore, it suffices to show the claim for disjunctions of atomic formulas. Suppose that $\vee_{1 \le i \le r} \alpha_i$ is a disjunction of atomic formulas in which R appears only negatively. Assume that

$$(A, \mathrm{rel}(X), \overline{f}) \models \forall x_1 \dots \forall x_m \bigvee_{1 \le i \le r} \alpha_i.$$

Then, for each $\overline{a} \in A^k$, some α_i is satisfied. Define a partition $Y_1, \dots Y_r$ of X^* as follows: $s\overline{a}f_1(\overline{a}) \cdots f_n(\overline{a})$ is put to Y_v iff v is the least index j for which

$$(A, \mathrm{rel}(X), \overline{f}) \models \alpha_j(\overline{a}).$$

It is easy to verify that $X^* = \cup_{1 \le i \le r} Y_i$ and that

$$A \models_{Y_i} \alpha^*_i.$$

For the other direction, (here we need the assumption that at most one α_j is of the form $\neg R(t_1, \dots, t_k)$), suppose that

$$A \models_{X^*} \bigvee_{1 \le i \le r} \alpha^*_i. \tag{5}$$

By definition, there is a partition of X^* to sets $Y_1, \dots, Y_r$ such that

$$A \models_{Y_i} \alpha^*_i.$$

We may assume that α_1 is the formula $\neg R(x_1, \ldots, x_k)$. We next define a new partition of X^* in the following way. In the natural order, starting with Y_2, we inflate Y_2 to the maximal $W_2 \subseteq X^*$ satisfying

$$A \models_{W_2} \alpha_2^*.$$

Then, we keep W_2 fixed and replace Y_3 with the maximal subset of $X^* \setminus W_2$ satisfying α_3^*. Finally, we define $W_1 = Y_1 \setminus (W_2 \cup \cdots \cup W_r)$. Since $W_1 \subseteq Y_1$, this new partition also witnesses (5) by the downward closure. If, in the new partition, some tuple $s\overline{a}f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$, then we must have

$$s'\overline{a}f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$$

for all $s' \in X$. This follows from the maximality of the sets $W_2, \ldots, W_r$ and the fact that the variables $y_1, \ldots, y_k$ do not appear in any of the formulas α_i^* for $i > 1$. Therefore,

$$A \models_{W_1} \bigvee_{1 \leq i \leq k} y_i \neq x_i$$

implies that

$$(A, \mathrm{rel}(X), \overline{f}) \models \neg R(\overline{t})(\overline{a})$$

for all $\overline{a} \in A^k$ such that, for some s, we have $s\overline{a}f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$.

We may conclude that

$$(A, \mathrm{rel}(X), \overline{f}) \models \forall x_1 \ldots \forall x_m \bigvee_{1 \leq i \leq r} \alpha_i.$$

$\square$

Note that the defining formula $\chi(y_1, \ldots, y_k)$ in Theorem 4.9 can be translated to a formula of Independence Friendly Logic as

$$\forall x_1 \ldots \forall x_m (\exists x_{m+1}/W_1) \ldots (\exists x_{m+n}/W_n)\theta_2,$$

where $W_i = (\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_k\}) \setminus \{z_{i_1}, \ldots, z_{i_s}\}$. Therefore, the analogue of Theorem 4.9 also holds for Independence Friendly Logic.

Corollary 4.10. *The properties of trumps over variables $\{y_1, \ldots, y_k\}$ definable in Independence Friendly Logic over the empty vocabulary are exactly the downwards monotone $\Sigma_1^1[\{R\}]$ quantifiers.*

Recall that the existential quantifier of $\mathcal{D}$ is defined by

$$\mathfrak{A} \models_X \exists x_n \psi \text{ iff } \mathfrak{A} \models_{X(F/x_n)} \models \psi \text{ for some } F\colon X \to A.$$

Denote by $\exists^1$ the following variant of the existential quantifier

$$\mathfrak{A} \models_X \exists^1 x_n \psi \text{ iff there is } a \in A \text{ such that } \mathfrak{A} \models_{X(a/x_n)} \models \psi.$$

It is easy to see that $\exists^1 x\psi$ can be expressed in a "uniform" way as

$$\exists x(=(x) \wedge \psi).$$

The analogue of $\exists^1$ for the universal quantifier is

$$\mathfrak{A} \models_X \forall^1 x_n \psi \text{ iff for all } a \in A \text{ it holds that } \mathfrak{A} \models_{X(a/x_n)} \models \psi.$$

It is an open question whether the quantifier $\forall^1$ can be given a uniform definition in the logic $\mathcal{D}$. It is easy to verify that extending the syntax of $\mathcal{D}$ by $\forall^1$ does not increase the expressive power of $\mathcal{D}$. This follows from the fact that Theorem 68 in [9] generalizes to cover also the case of $\forall^1$. More interestingly, Theorem 4.9, and the fact that $\forall^1$ is downwards monotone, shows that the quantifier $\forall^1$ does not increase the expressive power of $\mathcal{D}$ with respect to open formulas either. It remains open whether the quantifier $\forall^1$ is "uniformly" definable in the logic $\mathcal{D}$.

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