Formalizability and Knowledge Ascriptions in Mathematical Practice

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Résumé

Nous examinons les conditions de vérité pour des attributions de savoir dans le cas des connaissances mathématiques. La disposition d'une démonstration formalisable semble être un critère naturel :

> ($*$) X sait que p exactement quand si X en principe dispose d'une démonstration formalisable pour p.

La formalisabilité pourtant ne joue pas un grand rôle dans la pratique mathématique effective. Nous présentons des résultats d'une recherche empirique qui indiquent que les mathématiciens *n'employent pas* de certaines spécifications des critères comme ($*$) quand ils attribuent du savoir. En plus, nous montrons que le concept du savoir mathématique qui est la base de l'emploi effectif du ≪ savoir ≫ de la pratique mathématique est très bien compatible avec certaines intuitions philosophiques mais qui apparaissent comme *différentes* des concepts scientifiques philosophiques formant la base de ($*$).

Abstract

We investigate the truth conditions of knowledge ascriptions for the case of mathematical knowledge. The availability of a formalizable mathematical proof appears to be a natural criterion:

> ($*$) X knows that p iff X has available a formalizable proof of p.

Yet, formalizability plays no major role in actual mathematical practice. We present results of an empirical study, which suggest that certain readings of formalizability criteria like ($*$) are *not* necessarily applied by mathematicians when ascribing knowledge. Further, we argue that the concept of mathematical knowledge underlying the actual use of "to know" in mathematical practice is compatible with certain philosophical intuitions, but seems to *differ* from philosophical knowledge conceptions underlying ($*$).

1 Introduction

In this paper, we will be concerned with the role of formalizability for an epistemology of mathematics.

Formalizablity is a feature of informal mathematical proofs: A formalizable proof is a proof that can be transferred into a formal proof, that means it can be transferred into a formal derivation with respect to a formal axiomatic system with consistent axioms. However, the notion of formalizability is not a fixed notion like the notion of formal proof. What has still to be specified is the meaning of "can be transferred" in this context. We may choose the semantics of the phrase "formalizable proof" from a spectrum spread between two extremes. One extreme in the definition spectrum would be the definition "a proof of p is formalizable iff the informally proven mathematical theorem is also formally derivable in a consistent formal axiomatic system" (weak transfer relation); the other extreme is the definition "a proof of p is formalizable iff it can be translated step by step into a formal proof"[1] (strong transfer relation).[2]

Formalizablity is an important feature of mathematical proofs, regarding foundational issues in the philosophy of mathematics: The thesis that every informal mathematical proof can be transferred into a formal proof has been suggested to be named *Hilbert's thesis*,[3] as it is linked to the so called *Hilbert programme*.[4]

But is formalizability also essential for a philosophical understanding of *mathematical knowledge*?

This question is open, and several answers have been proposed. For one thing, Yehuda Rav points out in an article on the epistemic role of mathematical proof [Rav 1999] that due to the popular opinion about mathematics, a mathematician's job consists more or less in manipulating formulas to render a decision "true" or "false" about certain theorems. Facing the increasing use of computer tools, like computer algebra systems, simulation software or proof checkers, up to the famous examples of computer proofs (e.g., the proof of the *Four-Colour-Theorem*), the question arises whether mathematicians, in so far as they are working on *proofs*, might even be completely replacable by computers without loss.

[1]The original proof may for example be written in some semi-formal language.

[2]Note that the transfer relations include the case that a formalizable proof is already a formal proof. Therefore, all formal proofs are formalizable.

[3]Cf. [Rav 1999, 11]

[4]In its final version, *Hilbert's programme* towards a new foundation of mathematics has been proposed by David Hilbert in 1921. It aims at a formalization of mathematics *and* at a proof that the axioms of the formal system that is used are consistent. *Hilbert's thesis* is actually weaker than *Hilbert's programme* itself, which, as is known, failed as a result of Gödel's theorems about the completeness and incompleteness of formal systems. Whether *Hilbert's thesis* is true or not may strongly depend on which reading of "can be transferred" is chosen, and therefore, on which meaning of "formalizable proof" is used.

Which effects does it have on an epistemology of mathematics if we take *Hilbert's thesis* as an exhaustive characterization of the epistemic role of mathematical proof? In the following, we will examine the epistemological impact of criteria for the truth of "X knows that the mathematical statement p is true" which are based on the availability of formalizable proof.

This paper is a first report on work in progress concerning a socio-empirical investigation of the role of formalizability in actual mathematical practice. After some terminological and methodological preliminaries, we will first sketch an analysis of different readings of formalizability criteria for mathematical knowledge. Then we will turn to the results of our recently conducted empirical project on formalizability and knowledge ascriptions in mathematical practice.

2 Preliminaries

A *formalizability criterion* for "X knows that p" is a criterion of the following form:

$$(*) \quad X \text{ has available a formalizable proof of } p,$$

where the notion of availability may be weaker than the notion of current cognitive access.

My main research question is:

> Is there any specification of "formalizable" and "available" such that a formalizability criterion provides an adequate criterion for the truth of "X knows that the mathematical statement p is true"?

As we will discuss below (section 3.2), from the philosophical point of view, there is a natural way to answer 'yes' to this question by staying very close to the naive interpretations of "formalizable" and "available".

However, it turns out that in actual mathematical practice, formalizability seems to play no major role. Employing the results of an empirical study, we will suggest that, at least for common readings of $(*)$, mathematicians do not necessarily use formalizability criteria when ascribing knowledge. Knowledge ascriptions in mathematical practice work in a way that differs systematically from how they should work if they were based on these readings of formalizability criteria.

This points to the conclusion that, in spite of their *prima facie* attractiveness, formalizability criteria may be inadequate for the concept of mathematical knowledge that is used in actual mathematical practice.

We will argue that for an adequate investigation of the epistemic role of formalizability, one needs to distinguish pure philosophical accounts of mathematical knowledge from the concept of knowledge employed in mathematical practice. Both contribute to an adequate understanding of mathematical knowledge. Hence, our investigation will proceed in three main steps:

1. Identify those philosophical conceptions of knowledge that are consistent with a formalizability criterion – this presupposes an appropriate classification of knowledge conceptions, due to the different possible specifications of "formalizable" and "available".
2. Examine and analyze the concept of mathematical knowledge developed in actual mathematical practice. Which of the philosophical conceptions of knowledge captures it?
3. Combine the results of both analyses. Are there differences between the results of step 1 and step 2? Are these differences due to different aspects of knowledge? Can we develop a conceptual epistemological framework which is able to cope with these different aspects of mathematical knowledge?

Such an approach will also shed light on the nature and epistemic role of mathematical proof in general, which is of great interest for modern philosophy of mathematics.[5]

This paper focusses on the second step of the agenda. We will first sketch part of the analysis from step 1, and then turn to step 2 with two main concerns: On the one hand, we will present results of our recently conducted empirical study about the conception of knowledge in mathematical practice. One the other hand, we will discuss a reasonable embedding of these empirical results into the overall philosophical framework, which is also necessary for step 3.

3 Step 1 – knowledge conceptions behind formalizability criteria

Mathematical knowledge is alleged to have an exceptional epistemic status, due to the fact that its theorems can be *proven*, which is exceptional among the sciences, and due to its historical stability and the high degree of systematic unity and uniformity even across different branches of mathematics.

We stated above that *prima facie*, the adequacy of an appropriately specified formalizability criterion for mathematical knowledge appears to be quite obvious. This stems from the claim that there seems to be a certain specification of formalizability criteria which can easily account for an exceptional epistemic status of mathematical knowledge.

In this section, we will first examine this claim by discussing the involved specification of formalizability criteria and the way it accounts for an exceptional epistemic status of mathematical knowledge. We will then investigate the

[5] See for example [Detlefsen 1992].

question whether this specification really yields an adequate criterion for mathematical knowledge, in view of step 2 and step 3 of our agenda. This question will lead us to the empirical part of the paper.

3.1 The classical conception of knowledge

Classical epistemology defines knowledge as *justified true belief* via the following three individually necessary and jointly sufficient conditions for "X knows that p":

- X believes that p
- p is true
- X has good reasons to believe that p is true with respect to some fixed epistemic standards

This definition of knowledge has a high intuitive appeal. Yet, it has been seriously challenged by the famous *Gettier examples*, and by scepticism.[6] The Gettier examples, as well as the sceptical challenge, are essentially based on the fact that there is a gap between justification and truth. As the classical conception of knowledge cannot fill this gap for the general case, this conception has to be regarded as inadequate for a general theory of knowledge, despite its intuitive appeal.

By employing the distinction between *internalistic* and *externalistic* conceptions, and the distinction between *invariantist* and *context sensitive* conceptions of knowledge, the classical conception of knowledge can be systematically arranged among the different conceptions of knowledge that have been proposed as a reaction to Gettier and scepticism. Following John MacFarlane, we will understand invariantism and contextualism as two competing views, concerning the question whether the epistemic standards a subject X must meet in order to know that p are fixed, or sensitive to contextual changes (cf. [MacFarlane 2005, 198-199]).[7] Each of these views is compatible with both internalistic and externalistic theories of knowledge. An internalist claims that X has to meet the relevant epistemic standards *and* has to have itself cognitive access to this fact, whereas an externalist claims that X has to meet the standards objectively, but

[6]The Gettier examples show that in certain cases, justified true belief is not sufficient for knowledge. Gettier published his examples in 1963 [Gettier 1963]. The so called *sceptical challenge* derives from the possibility of sceptical scenarios like Descartes' *Evil Demon* or Putnam's *brain in a vat* scenario [Putnam 1992].

[7]In this section, we will not distinguish between contextualism and a sensitive form of invariantism, namely a *(subject) sensitive invariantism* that allows the truth of "X knows that p" to depend on the circumstances of evaluation. Here, the main argument will be *against strict invariantist readings* of formalizability criteria, and will not be an argument in favor of either sensitive invariantism or contextualism.

does not need to have access to that fact. Accordingly, the classical conception of knowledge falls into the category of *internalistic strict invariantism.*

3.2 A strictly invariantist internalistic reading of formalizability criteria

A criterion for "X knows that p" that satisfies the following conditions:

(Int) It yields an internalistic and

(Inv) strictly invariantist conception of knowledge, *and*

(T) links epistemic justification reliably to truth.

apparently characterizes some exceptional epistemic status, due to what has been said in section 3.1.

The following specification of ($*$) satisfies the conditions (Int) and (Inv), due to the employed adjustments of the two parameters "available" and "formalizable":

X has current cognitive access to a formal proof of p.

Moreover, it seems to exemplify an ideally reliable relation between truth and justification: Having access to a formal proof guarantees the existence of a formal derivation in a consistent axiomatic system, and thus the truth of the derived theorem relative to the truth of the axioms. Hence, also (T) appears to be fulfilled. So, "X has current cognitive access to a formal proof of p" as a criterion for "X knows that the mathematical statement p is true" may seem to characterize some exceptional epistemic status of mathematical knowledge.

But is "X has current cognitive access to a formal proof of p" really an adequate criterion for mathematical knowledge? In a paper on a context sensitive account of mathematical knowledge (Löwe & Müller [2007]), Benedikt Löwe and Thomas Müller point out that the adequacy of this kind of internalistic invariantist specification of formalizability criteria would lead to the conclusion that mathematicians have nearly zero nontrivial mathematical knowledge. With regard to our agenda for an adequate analysis of mathematical knowledge, respecting both its exceptional nature and the concept of knowledge used in actual mathematical practice, "X has current cognitive access to a formal proof of p" should thus be rejected as an adequate criterion for mathematical knowledge.

Still, there may be other specifications of ($*$) that satisfy (Inv), (Int), and (T). And there may also be different ways to account for the exceptional epistemic status of mathematical knowledge *instead* of an analysis via (Inv), (Int) and (T). This leads to the following questions:

1. Are there other specifications of formalizability criteria that both satisfy (Int), (Inv) and (T), and capture the concept of knowledge employed in mathematical practice?
2. Are there other, e.g. context sensitive or externalistic, specifications of formalizability criteria that can account both for the exceptional nature of mathematical knowledge and for the concept of knowledge employed in mathematical practice?[8]

We hold the view that providing answers to these questions necessarily requires some empirical investigation of mathematical practice (step 2 of our agenda). In the remaining part of this paper, we will present the results of a socio-empirical study on the conditions for knowledge ascriptions in mathematical practice. The results shall be employed in further work to answer the question whether and how internalistic formalizability criteria can capture the knowledge conception used in mathematical practice. The question whether and how externalistic formalizability criteria can capture the knowledge conception used in mathematical practice calls for a different methodological approach, as will be discussed below.

3.3 A methodological turning point

Externalist knowledge conceptions do not claim that the epistemic subject has to have access to a justification for p in order to know that p. For them, it is sufficient that some appropriate external relation between X's belief that p and the fact that p is true is instantiated. This relation is characterized in different ways. For example, it may be a causal relation, or the objective reliability of the process that generated the belief that p (as a state of mind). Therefore, fruitful empirical input to an externalist analysis of the concept of knowledge employed in mathematical practice may be provided by cognitive science and psychology.

An *internalist* conception of knowledge claims that a necessary condition for the truth of "X knows that p" is that X is able to justify her belief that p. Therefore, an internalist criterion for knowledge has to specify what it means for some proposition p to be justified *by the epistemic subject's argumentative practice* (cf. [Brendel 1999, 236]). At the same time, the epistemic subject's argumentative practice is the basis on which mathematicians usually assess knowledge ascriptions, namely *based on a claimed proof* that is given by the epistemic subject. An empirical investigation of conditions on which mathematicians accept knowledge ascriptions as appropriate based on a claimed proof by the epistemic subject, may thus support an internalistic analysis of the concept of knowledge employed in mathematical practice. We regard it as fruitful to investigate

[8] Löwe and Müller suggest a context sensitive account for the concept of knowledge that fits with mathematical practice [Löwe & Müller 2007].

the conditions for knowledge ascriptions employed in mathematical practice by means of empirical sociology: First of all, knowledge ascriptions are *actions*. Some ascriber Y acts in a certain way by saying "X knows that p is true" (cf. [Kompa 2001, 16-17]). Whether an action counts as justified or appropriate (in a certain community) is rather determined by social mechanisms than by explicit verbal rules. Tools for an empirical investigation of social actions and social mechanisms are provided by sociology, in the form of quantitative and qualitative research techniques. Under the methodological assumption that mathematicians ascribe knowledge systematically, the results of a socio-empirical investigation of knowledge ascriptions in mathematical practice will point to certain standards for knowledge ascriptions that are taken to be appropriate by mathematicians.

In this paper, we will restrict ourselves to a socio-empirical investigation, and therefore to a rather internalist analysis of the concept of knowledge used in mathematical practice. Empirical questions concerning a rather externalist analysis will be adressed in future work.

In what follows, we will present a mainly quantitative web-based survey on standards for knowledge ascriptions in actual mathematical practice. The analysis of the empirical standards for knowledge ascriptions may also help to provide an answer to the questions whether and how an internalist formalizability criterion could be adjusted to the knowledge conception used in mathematical practice.

4 Step 2 – a socio-empirical study on the concept of knowledge in mathematical practice

4.1 What is out there?

A lot of authors emphasize the role of sociology for an epistemology of mathematics, for instance, the late Wittgenstein, Philip Kitcher [Kitcher 1984], Paul Ernest [Ernest 1998], or David Bloor [Bloor 1996, 2004].

Yet, there has hardly been done any *socio-empirical* research about actual mathematical research practice: The first socio-empirical study published was Bettina Heintz's work about the culture and practice of mathematics as a scientific discipline [Heintz 2000]. In her study, Heintz used qualitative methods. Her work is based on a detailed field study (at the Max-Planck-Institute for Mathematics in Bonn, Germany) and a series of qualitative interviews with mathematicians.[9]

[9]There is a PhD Thesis [Markowitsch 1997] which refers to results from interviews with mathematicians.

In our survey on the conditions for knowledge ascriptions in mathematical practice, we employed mainly quantitative methods. We used an online questionnaire which contains mostly multiple choice questions, and some space for qualitative comments. The survey was announced together with a link to the online questionnaire via postings in different scientific newsgroups. It was opened in August 2006 and closed in October 2006.

4.2 Methodology and project data

4.2.1 Quantitative vs. qualitative methods

Our study may be seen as a methodological attempt to employ quantitative methods for a socio-empirical investigation of actual mathematical research practice, though we do not aim at using all state-of-the-art statistical evaluation tools from empirical sociology. In the first place, we try to explore if and how the empirical results may generally bear on an epistemology of mathematics. The use of quantitative methods is due to the assumption that the way mathematicians *think* about mathematics and mathematical knowledge is heavily influenced by individual factors. It might even be a matter of personal style. In contrast, mathematical *practice* appears to be very homogenous and uniform. It is a well known fact in sociology that the way people think about certain issues does not necessarily coincide with how they act, it may only point into the right direction.[10] By the use of quantitative methods, we hope to get more significant results about the conditions for appropriate knowledge ascriptions on which mathematicians *agree*.

The results from the free-text part are supposed to deepen the hypotheses that are tested in the quantitative part, which is a usual practice in sociology.

4.2.2 Target group and adequacy of the sample

A great methodological obstacle for anyone who wants to investigate mathematical practice by surveys is the apparent unwillingness of its protagonists to participate. To avoid this obstacle, we used an online questionnaire in our project that was posted in three scientific internet newsgroups. Newsgroup readers are supposed to be less averse to surveys, but as a result, the sample of the study is limited to mathematicians *who read newsgroups*. Yet, we do not regard the loss of adequacy of the sample as a serious limitation. A significant correlation between the habit of reading newsgroups and a certain attitude towards formalizable proof does not seem very likely.

The link to the online questionnaire was posted two times. The first posting was a test-run in the newsgroup `sci.math`. For the second time, the link

[10] Cf. [Klammer 2005, 220].

was posted in `sci.math.research` and `de.sci.mathematik`, with 108 valid responses out of 250 responses in total. A response was considered as valid if the personal data part and at least one question in one of the three parts of the questionnaire (cf. below) was completed.[11] The personal data part served to decide whether a participant belonged to the target group.

The target group of our survey consists of international research or teaching mathematicians from all branches of mathematics. We got 76 valid responses from the target group. Most of these responses came from the US, Germany, and the Netherlands, from mathematicians with both research and teaching experiences at university level from 1 up to 35 years. 13.2% of the participants that belong to the target group have received a B.A. (or an equivalent degree), 19.7% a M.Sc. (or an equivalent degree), and 46.1% a Ph.D. (or an equivalent degree) in mathematics.

What we will present below are the results from the second posting, exclusively based on valid answers from the target group.

4.3 Empirical hypotheses

The questionnaire we used falls into three parts:

- **Part I** deals with the abstract concept of knowledge and proof mathematicians use,
- **Part II** focusses on knowledge ascriptions "in action", and
- **Part III** is on mathematical beauty.

In what follows, we will only consider results from the first and the second part, because the results from part III do not bear on the hypotheses we want to discuss within the scope of this paper.

The results from part I and II are supposed to complement each other. The questions in part I also serve as control questions for part II.

The empirical hypotheses that have been tested in the part II derive from two sources. The first source is the analysis of the concept of knowledge underlying formalizability criteria, which means, the results of step 1. The other source are experiences from mathematical practice itself. I will discuss three of these hypotheses here:

(H1) *Whether mathematicians ascribe knowledge depends systematically on contextual changes.*

The prediction that knowledge ascriptions in mathematical practice are

[11] In the presentation of the results in section 4.4, we will give the total count of valid answers for each reported multiple choice question.

context dependent has been made by Löwe and Müller in their paper discussed in section 3.2 [Löwe & Müller 2007]. The apparent uniformity of mathematical practice suggests that this dependency should be accessable by empirical means.

(H2) *The actual possession of a formal proof by the epistemic subject X is sufficient for ascribing knowledge to X.*

In section 3.2, it is argued that truth criteria for "X knows that p" which demand the actual possession of a formal proof of p are too restrictive, because as a consequence nearly no mathematician would have much more than non-trivial mathematical knowledge. Yet, for any current philosophical conception of knowledge, the actual possession of a formal proof of p is *sufficient* to know that p. Therefore, I also expect to find in mathematical practice that the possession of a formal proof of p is sufficient to count as a knower of p. Otherwise, it would appear questionable whether the endeavor of combining the analytical, philosophical conception of mathematical knowledge with the concept of knowledge used in mathematical practice might be successful.

(H3) *Mathematicans do not necessarily demand a formal proof in order to ascribe knowledge.*

In [Löwe & Müller 2007] it is argued that regarding mathematical practice, the actual possession of a formal proof of p is no reasonable criterion for knowing that p, because formal proofs are rarely used there. Accordingly, the standards for knowledge ascriptions actually used in mathematical practice should not include the constraint that X must actually possess a formal proof of p in order to know that p.

4.4 Selected results

In the following, we will present selected results of part I and II by giving an excerpt of examples from the questions and directly connecting them with the corresponding results. The questionnaire has 74 questions in total, so we will only present those questions and results that have been significant for the interpretation regarding our hypotheses (H1), (H2) and (H3).

Part I These are two examples of questions people were asked in the first part of the questionnaire:

- "*Is mathematical knowledge objective?*"

response	**frequency**	**count** (Σ 74)
yes	82.4 %	61
no	17.6%	13

- "*Please select to which degree you accept the following statement:*

  One can precisely define what a mathematical proof is."

response	**frequency**	**count** (Σ 74)
strongly agree	28.4%	21
agree	60.8%	45
disagree	9.5%	7
strongly disagree	1.4%	1

Participants who gave a positive answer to the latter question, either "strongly agree" oder "agree", were then asked in a free-text question to give a definition of "mathematical proof". We'll give selected quotes from the answers. The majority of answers gave a formal definition of mathematical proof. The quotes below show some less frequently given answers. Note that the second and fourth answer still state that the informally proven theorem has to be formally derivable:

- "Formally: from a given set of deductive rules, and a set of axioms, a sequence of statements (machine-verifiable in the correctness of application of the rules), starting with hypothesis and ending with conclusion. Ideally, anyway."
- "I would defined a proof fundamentally as argument that convinces mathematicians, less fundmentally as an argument that can be formalised and proven mechanically."
- "A convincing argument that instills belief that it is possible to construct a sequence of formal logical steps leading from generally accepted axioms to the given assertion."
- "A finite sequence of statements following logically from each other, that begin with a given set of axioms and have the statement to be proved as a conclusion. In actual mathematical practice, this sequence tends to be shortened and written in some human language rather than pure symbolic logic, so the only difficulty that may arise in defining what a 'real-life' mathematical proof is, is to decide what constitutes an acceptable abbreviation of the hypothetical, full-length logical proof."

Part II In the second part, people were led through four scenarios. Each screen of the online questionnaire contained a piece of the story, and at the end of each screen the participants were repeatedly asked whether they would ascribe knowledge to the protagonist of the scenario or not. In the following, we will give excerpts from scenario 1 and scenario 3. The ongoing story of each scenario will be shortly summarized when parts are left out:

- In scenario 1, the protagonist is a PhD student named John:

  > "***Scenario 1***
  > *John is a graduate student, and Jane Jones, a world famous expert on holomorphic functions, is his supervisor. One evening, John is working on the Jones conjecture and seems to have made a break-through. He produces scribbled notes on yellow sheets of paper and convinces himself that these notes constitute a proof of his theorem.*"

  Then participants were asked to answer the following question:

  > "*Does John know that the Jones conjecture is true?*"

  On the following screens, the story continues, and the question "Does John know that the Jones conjecture is true?" is repeated several times. Finally, John and his supervisor jointly prove the Jones Conjecture, and publish their proof in a mathematical journal of high reputation:

  > "*Eighteen months later, the editor accepts the paper for publication, based on a positive referee report.*"

  At this point of the story, the majority (84.9 %) of the participants gave a positive answer on the question:

  > "*Does John know that the Jones conjecture is true?*"

response	**frequency**	**count** (Σ 66)
yes	28.8%	19
almost surely yes	56.1%	37
almost surely no	3.0%	2
no	4.5%	3
can't tell	7.6%	5

  On the next screen, which is the last of scenario 1, the story ends with:

  > "*After his Ph.D., John continues his mathematical career. Five years after the paper was published, he listens to a talk on anti-Jones functions. That evening, he discovers that based on these functions, one can construct a counterexample to the Jones conjecture. He is shocked, and so is professor Jones.*"

  Now, 61.3 % of the participants gave a positive answer on the question whether John knows that the Jones conjecture is *false*:

  > "*Does John know that the Jones conjecture is false?*"

response	frequency	count (Σ 62)
yes	14.5%	9
almost surely yes	46.8%	29
almost surely no	6.5%	4
no	8.1%	5
can't tell	24.2%	15

On the same screen, the participants were also queried:[12]

> "*Did John know that the Jones conjecture was true on the morning before the talk?*"

The result is somehow astonishing, because still, 71 % gave a positive answer to this question:

response	frequency	count (Σ 62)
yes	24.2%	15
almost surely yes	46.8%	29
almost surely no	4.8%	3
no	14.5%	9
can't tell	9.7%	6

Note that when answering these last two questions, participants *already got the information* that John has discovered a counterexample to the Jones conjecture.

After finishing scenario 1, we asked the participants to give comments on the scenario in a free-text field. These are some selected quotes, emphasizing different factors that influenced the answers in scenario 1:

- "How important is the Jones conjecture? How large is the community?"
- "My answers would have been very different with different time frames mentioned."
- "I don't know John or Bob, so I don't have a good feel for how rigorously they work."

• In the third scenario, the protagonist is a maths student named Tom, and the setting is an oral examination at the end of the semester:

> "***Scenario 3:***
> *Tom Jenkins is a student of mathematics and has to pass an oral exam at the end of the algebra lecture held by his professor Robin Smith. Tom did some oral exams before, so he is not too nervous, and is able to pay concentrated attention to the professor's questions during the whole exam. At*

[12]Participants were anyway able, though not requested, to switch back and forth between the different screens.

some point of the exam, Smith asks Tom for the proof of a certain algebraic theorem T1. The proof consists mainly of a tricky application of the fundamental theorem on homomorphisms and was conducted in one lecture on the blackboard. Tom is able to give a rather technical, but absolutely correct step-by-step proof in full detail."

83.3 % of the participants gave a positive answer to the question:

"*Does Tom know that T1 is true?*"

response	**frequency**	**count** (Σ 54)
yes	46.3%	25
almost surely yes	37%	20
almost surely no	1.9%	1
no	1.9%	1
can't tell	13%	7

In the proceeding story, the professor Smith challenges Tom with questions concerning the ideas behind the proof of T1, and further applications of these ideas. Tom fails at all these questions:

"*The exam continues with some questions about definitions, and after some minutes Smith asks Tom to explain why the general idea of how to apply the fundamental theorem on homomorphisms in the proof of the former theorem is also fruitful to prove a second algebraic theorem T2. Tom completely fails in his answer.*"

Still, 83.3 % gave a positive answer to the question whether Tom knows that T1 is true. There is only a slight adjustment from "yes" towards "almost surely yes":

"*Does Tom know that T1 is true?*"

response	**frequency**	**count** (Σ 54)
yes	40.7%	22
almost surely yes	42.6%	23
almost surely no	3.7%	2
no	1.9%	1
can't tell	11.1%	6

On the same screen, participants were asked:

"*Did Tom know that the first theorem T1 was true before he failed in answering Professor Smith's last question correctly?*"

Again, there is nearly no quantitative effect on the positive knowlegde ascriptions, 83 % gave a positive answer. Note the slight adjustment from "almost surely yes" towards "yes" compared with the answers corresponding to the preceding screen:

response	**frequency**	**count** (Σ 53)
yes	47.2%	25
almost surely yes	35.8%	19
almost surely no	1.9%	1
no	1.9%	1
can't tell	13.2%	7

The story continues:

> "*Smith asks Tom to formulate the general idea behind the proof of the first theorem T1. Tom fails in his answer.*"

Note that although there is a clear shift from positive to negative answers (13.2 %) to the next question, there is no effect on the results concerning the qualitative behavior of positive and negative knowledge ascriptions:

> "*Does Tom know that the first theorem T1 is true?*"

response	**frequency**	**count** (Σ 53)
yes	32.1%	17
almost surely yes	37.7%	20
almost surely no	11.3%	6
no	5.7%	3
can't tell	13.2%	7

After finishing scenario 3, we asked the participants again to give comments on the scenario in a free-text field. These are some selected quotes. The quotes support the quantitative result that a correct step-by-step proof, at least when it is given by the epistemic subject, is sufficient for knowledge ascriptions or knowledge claims:

– "He knows the truth of the theorems, because these are well known and proved theorems."
– "Once you are sure that you gave a correct proof of theorem T1, you need not revise your opinion on the truth of T1."
– "Tom knew all the time that T1 was true. He could give a complete and correct proof, after all!"
– "Well you don't need do unterstand the idea behind the prove of some theorem [...] and to some point the idea is not important at all – just as i said before: symbol processing."

4.5 Summary and interpretation of the selected results regarding (H1), (H2) and (H3)

In the following, we will propose some interpretation of the above presented results.

First of all, there appears to be a certain tension between the results of part I and II:

- In part I, the majority of the free-text answers on the question for a definition of "mathematical proof" refer to the definition of formal proof. In contrast, the quantitative results from scenario 1 in part II affirm (H3): Formal proof is not necessary for knowledge ascriptions in mathematical practice (cf. below).
- In part I, 82.4 % of the participants answered that mathematical knowledge is objective. In contrast, the free-text comments on the scenarios in part II show that the standards for knowledge ascriptions that are taken to be appropriate seem to depend on less objective factors like the size of the community, the importance of the proven theorems, or on time frames.

Regarding the three hypotheses (H1), (H2) and (H3) formulated in section 4.3, the following can be observed:

(H1) *Whether mathematicians ascribe knowledge depends systematically on contextual changes.*

The free-text comments in part II show that whether subjects ascribe knowledge or not depends on

(a) the size of the community of the corresponding branch of mathematics, or the number of referees.

(b) what is at stake, on how important the proven theorem is.

(c) the epistemic subject (e.g. working habits).

(d) time frames.

This points to an interdependency of the standards for knowledge ascriptions that are taken to be appropriate in mathematical practice, and contextual features.

The quantitative results from scenario 1 point to the same conclusion: *After the participants got the information* that John has discovered a counterexample to the Jones conjecture, they were asked both if he does *now* know that the Jones conjecture is *false*, and if he did know *before* his discovery that the Jones conjecture was *true*. Assumed that the appropriateness of the ascription "John knows that the Jones conjecture is false" implies the appropriateness of the ascriptions "John does not know that the Jones conjecture is true", the results suggest that the standards for knowledge ascriptions that are taken to be appropriate in mathematical practice are not strictly invariant, but depend on different epistemic standards in play at different contexts:

- *Does John know that the Jones conjecture is false?*
  61.3 % 'yes' or 'almost surely yes'

- *Did John know that the Jones conjecture was true on the morning before the talk?*
  71 % 'yes' or 'almost surely yes'

If the standards were strictly invariant, the result for "Did John know that the Jones conjecture is true on the morning before the talk?" should be a majority of negative answers.

(H2) *The actual possession of a formal proof by the epistemic subject X is sufficient for ascribing knowledge to X.*

The quantitive results as well as the free-text comments from scenario 3 affirm this hypothesis.

(H3) *Mathematicans do not necessarily demand a formal proof in order to ascribe knowledge.*

This hypothesis appears to be affirmed, again by the results from scenario 1 in part II:

61.3 % of the participants gave a positive answer on the question whether John knows that the Jones conjecture is false, *after they got the information that he has discovered the counterexample*. 71 % also gave a positive answer to the question whether he still knew that the Jones conjecture was true before he discovered the counterexample. If a formal proof would have been demanded by these participants, they would have commited themselves rather consciously to the claim that a formal proof of both the Jones conjecture and its negation would be possible, and thus to an apparent contradiction: Formal derivability of both the Jones conjecture and its negation yields the inconsistency of the axioms of the formal system that is used, but the notion of formal proof includes the consistency of the axioms.

The results even suggest a stronger conclusion: For readings of formalizability criteria that necessarily entail the logical possibility to prove p formally (with respect to a fixed formal system), the same contradiction holds. So, it seems that mathematicians do not necessarily employ at least these readings of formalizability criteria for knowledge ascriptions.

5 Towards step 3 – outlook

The proposed interpretation of our empirical results points to the following issues for further work on step 1 and 2 towards step 3 of our agenda:

- The tension between the empirical results concerning mathematician's abstract concepts of mathematical knowledge and proof, and the empirical results concerning knowledge ascriptions in mathematical practice, has to be investigated further. Are the concepts of knowledge and proof used in mathematical practice inconsistent, or is the tension between the results due to different, but compatible aspects of these concepts?

- It has to be clarified on which contextual features the standards for knowledge ascriptions that are taken to be appropriate in mathematical practice depend. For example, they may depend on the context of use, the context of assessment, or the circumstances of evaluation of the knowledge ascription.[13] The outcome will be important for an epistemological analysis of the empirical results. The reported results from scenario 1 point to a dependency on the circumstances of evaluation:

  - *Does John know that the Jones conjecture is false?*
    61.3 % 'yes' or 'almost surely yes'

context of use	after counterexample
context of assessment	after counterexample
circumstances of evaluation	after counterexample

  - *Did John know that the Jones conjecture was true on the morning before the talk?*
    71 % 'yes' or 'almost surely yes'

context of use	after counterexample
context of assessment	after counterexample
circumstances of evaluation	before counterexample

- An internalistic invariantist conception of mathematical knowledge does not seem to fit the concept of knowledge used in mathematical practice. If the concept of knowledge employed in actual mathematical practice is supposed to be internalistic, it has to be analyzed in terms of context sensitivity. Are there any context sensitive specifications of an internalist reading of formalizability criteria?

[13] Also see [MacFarlane 2005] for a detailed taxonomy.

- If mathematicians do not necessarily demand a formal proof for appropriate knowledge ascriptions, which other features of a proof, given by the epistemic subject, are demanded? Some of the quotes from free-text answers on the definition of mathematical proof given in section 4.4 suggest that there are other distinct features, because people who gave these answers agreed on the statement that one can *precisely* define what a mathematical proof is.

- Are there reasonable specifications of formalizability criteria that fit the reported empirical results from scenario 1 concerning (H3)?

Part of these questions will be pursued empirically in a follow-up project with a refined questionnaire and a series of qualitative interviews at two university's math departments.[14]

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[14]This will also serve to close the methodological gap concerning the adequacy of the sample in the study here presented (cf. section 4.2.2).

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