Unifiability in extensions of **K4**

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Abstract

We give a semantic characterization for unifiability and non-unifiability in the extensions of **K4**. We apply this in particular to extensions of **KD4**, **GL** and **K4.3** to obtain a syntacic characterization and give a concrete decision procedure for unifiability for those logics. For that purpose we use universal models.

Keywords: Unification, unifier, provability logic, closed formula, universal model

1 Introduction

In [5] a uniform syntactic characterization was given for unifiability of formulas in **KD4** and its extensions. (This includes of course many of the best known modal logics like **S4** and **S5**.) The characterization implies decidability of unifiability (for decidable logics).

Theorem 1 [5] For any modal logic λ extending **KD4** and any modal formula α, α is not unifiable in λ iff $\vdash_\lambda \Box\alpha \wedge \alpha \rightarrow \bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$.

We prove similar results for **GL** and its extensions. The situation is more complicated than for **KD4**, not all extensions of **GL** behave in the same way. For **K4** itself and its extensions we have only been able to give a semantic characterization, and the decidability of unifiability remains open. For **K4.3** the semantic characterization does lead to decidability. A rather awkward syntactic characterization can be given.

Characterization of non-unifiability of formulas in a logic brings with it a characterization of its passive admissible rules (see [5]) and in the case of the logic studied in this paper give a concrete decision procedure for those rules. Conversely, decidability of the passive admissible rules for a logic implies decidability of unifiability for the logic.

To obtain the main result of the paper on **K4**, **GL** and other logics we use 0-universal models of these logics (see for more details on their construction than are given here, e.g. [1]).

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2 Preliminaries

Definition 1 *The* language of modal propositional logic *consists of the propositional variables: p, q, r,$\dots$, connectives: $\vee, \wedge, \rightarrow, \leftrightarrow, \neg, \top, \bot$ and a unary modal operator $\Box$.*
The modal logic **K** *is axiomatized by the following schemes:*

- *All propositional tautologies in the modal language,*
- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

The modal logic **K4** *is axiomatized by adding the scheme 4 to* **K***:*

- *4:* $\Box\alpha \rightarrow \Box\Box\alpha$.

The modal logic **K4.3** *is axiomatized by adding the scheme 3 to* **K4***:*

- *3:* $\Box(\boxdot\alpha \rightarrow \beta) \vee \Box(\boxdot\beta \rightarrow \alpha)$ *where* $\boxdot\alpha = \alpha \wedge \Box\alpha$.

The modal logic **S4** *is axiomatized by adding the scheme T to* **K4***:*

- *T:* $\Box\alpha \rightarrow \alpha$.

The modal logic **KD4** *is axiomatized by adding the scheme D to* **K4***:*

- *D:* $\Box\bot \rightarrow \bot$.

The modal logic **GL** *is axiomatized by adding the scheme L to* **K4** (*or equivalently to* **K**)*:*

- *L:* $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$.

Inference rules for these logics are modus ponens $\frac{\alpha, \alpha\rightarrow\beta}{\beta}$ *and* necessitation $\frac{\alpha}{\Box\alpha}$.

The scheme L plays an essential role in **GL** where $\Box\phi$ is read as "it is provable that ϕ". It is named after Löb, who proved L as a theorem of the provability logic of **PA**.

Definition 2

1. *A* Kripke frame *for* **K** *is a pair* $< W, R >$ *with* W *a nonempty set of so-called* worlds *or* nodes, *and* R *a binary relation, the so-called* accessibility relation.

2. *A* Kripke frame *for* **K4** *is a pair* $< W, R >$ *with* R *transitive.*

3. *A* Kripke frame *for* **K4.3** *is a pair* $< W, R >$ *with* R *transitive, upwards linear.*

4. *A* Kripke frame *for* **GL** *is a pair* $< W, R >$ *with* R *a transitive relation such that the converse of* R *is well-founded (there is no infinite sequence* $x_0 R x_1 R x_2 R \dots$*). (This excludes cycles and loops, and in the finite case comes down to irreflexivity.)*

Definition 3

1. *A* Kripke model *for* **K (K4, K4.3, GL)** *is a triple* $< W, R, \Vdash >$ *with* $< W, R >$ *a Kripke frame for* **K(K4, K4.3, GL)** *together with a satisfaction relation* $\Vdash$ *between worlds and propositional variables. We usually write* $w \Vdash p$ *for* $\mathfrak{M}, w \Vdash p$*, etc. The relation* $\Vdash$ *is extended to a relation between worlds and all formulas by the stipulations* $w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$*,* $w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$*, and similarly for the other connectives,* $w \Vdash \Box\alpha$ *iff for all* w' *such that* wRw'*,* $w' \Vdash \alpha$*.*

2. *If* $\mathfrak{M} =< W, R, \Vdash >$*, and* $\mathfrak{M}, w \Vdash \alpha$ *for each* $w \in W$*, and we write* $\mathfrak{M} \Vdash \alpha$ *and we say that* α *is* valid in $\mathfrak{M}$.

Henceforth we restrict attention to transitive frames.

Definition 4

1. *A* root *is a node* w *such that* wRw' *for all* $w \neq w'$ *in the frame.*

2. *The* depth m *of a node* w *is the lenght of the longest chain* $w = w_0 R w_1 R \dots R w_{m-1}$*. If this is not finite we just call* w *a node of* infinite depth.

3. *The* depth *of a model is the maximum of the depth of its nodes.*

Note that the depth of an *end point* (a node without successors) is 1. If there are cycles in the model the definition should be adapted so that a whole cycle (or all the nodes in it) should get the same depth, but we will not need to discuss models with cycles.

Definition 5 *Let* $< W, R >$ *be a frame.* $A \subseteq W$ *is called an* antichain *if* $|A| > 1$ *and for each* $w, v \in A$*,* $w \neq v$ *implies* $\neg(wRv)$ *and* $\neg(vRw)$*. We say that a set* $A \subseteq W$ totally covers *a point* v ($v \prec A$) *if* A *is the set of all immediate successors of* v*. In case* A *consists of a single element* w*, we write* $v \prec w$*.*

3 Closed Formulas and 0-Universal Models

In this section we give a semantic characterization for unifiability of formulas in the extensions of **K4** using 0-universal models of these extensions. These models are really useful only in case the logics do have the finite model property for closed formulas.

Definition 6 *A formula* $\alpha(p_1, ..., p_n)$ *is* unifiable *in a logic* λ *iff there is a tuple of formulas* $\delta_1, ..., \delta_n$ *such that* $\vdash_\lambda \alpha(\delta_1, ..., \delta_n)$*. The formulas* $\delta_1, ..., \delta_n$ *are called* unifiers *for the formula* α*.*

Definition 7 *A formula is called a* closed *formula if it is built up from the formulas* $\top$, $\bot$ *by Boolean connectives and* $\Box$.

The following lemma was obvious in [5], even if it was not stated as such.

Lemma 1 *If a formula* $\alpha(p_1, ..., p_n)$ *is* unifiable *in a logic* λ*, then it has a sequence of closed unifiers* $\delta_1, ..., \delta_n$.

Proof. Just substitute $\bot$ for all the propositional variables in a sequence of unifiers for α. $\dashv$

An immediate corollary is:

Corollary 1

1. *If* λ_1 *and* λ_2 *prove the same closed formulas, then the sets of unifiable formulas of* λ_1 *and* λ_2 *are the same.*

2. *For each* λ *the set of its unifiable formulas is uniquely determined by its closed fragment.*

This means that to determine the set of unifiable formulas of extensions of **K4** it is sufficient to determine the set of unifiers of logics extending **K4** by closed formulas only. For the study of such fragments so-called 0-universal models are very useful. We will now introduce them. One way to see 0-universal models is as the the part of the 0-*canonical model* (the part of the canonical model that is constructed using closed formulas only) consisting of its nodes of finite depth.

Definition 8 *The* 0-Universal model $\mathcal{U}_{\mathbf{K4}}(0)$ *of* **K4** *is constructed as follows: It contains two maximal elements; a reflexive and an irreflexive element. Under any finite anti-chain* A *in* $\mathcal{U}_{\mathbf{K4}}(0)$ *we put a new reflexive element that is covered by* A*, and a new irreflexive element that is covered by* A*. Under each irreflexive element* w *we put a reflexive* v_1 *such that* $v_1 \prec w$ *and an irreflexive* v_2 *such that* $v_2 \prec w$. $\mathcal{U}_{\mathbf{K4}}(0)$ *is the result of iterating this procedure.*

An extensive discussion of universal models is given in [1]. Note that a 0-universal model is a frame because there is no valuation. In this case there is no distinction between universal model and universal frame. The most important facts about this universal model are stated in the next theorem.

Theorem 2

1. *Each finite Kripke frame for* **K4** *can be mapped p-morphically onto a generated submodel of* $\mathcal{U}_{\mathbf{K4}}(0)$ *in a unique manner.*

2. *For each closed formula* α, $\mathbf{K4} \vdash \alpha$ *iff* $\mathcal{U}_{\mathbf{K4}}(0) \Vdash \alpha$.

3. *For each node* w *of* $\mathcal{U}_{\mathbf{K4}}(0)$ *there exists a formula* φ_w *such that* $v \Vdash \varphi_w$ *iff* $v = w$.

Proof. See [3].

⊣

The important fact that we do not need cycles in $\mathcal{U}_{\mathbf{K4}}(0)$ is connected to clause (1) in the above theorem: any cycle in a **K4**-frame can p-morphically be replaced by a reflexive node. Clause (1) of Theorem 2 then shows that we do not want/need to introduce such cycles in $\mathcal{U}_{\mathbf{K4}}(0)$; the same holds for single predecessors of reflexive nodes.

Theorem 3 *The modal logic* **K4** *has the finite model property (fmp).*
Proof. See [3].

⊣

Let us say that a logic λ has the 0-*fmp property* if λ has the finite model property with respect to closed formulas. For all we know each extension of **K4** by closed formulas (or even any formulas) may have this property, but no proof is known to us.

Theorem 4 *There is a 1-1 correspondence between* 0*-fmp extensions of* **K4** *by closed formulas and upsets in* $\mathcal{U}_{\mathbf{K4}}(0)$*.*
Proof. Straightforward.

⊣

Corollary 2 *There are uncountably many* 0*-fmp extensions of* **K4** *by closed formulas.*
Proof. For this it is sufficient to note that there are three incomparable elements in $\mathcal{U}_{\mathbf{K4}}(0)$, which is easy to check.

⊣

Definition 9

1. *For a* 0*-fmp logic* λ *extending* **K4** *the* 0-universal model *and* frame $\mathcal{U}_\lambda(0)$ *is the restriction of the* 0-universal model $\mathcal{U}_{\mathbf{K4}}(0)$ *to those nodes* w *for which the upward closed set generated by* w *is a* λ*-frame.*
2. *A subset* $A \subseteq \mathcal{U}_\lambda(0)$ *is called* definable *or* admissible *in* $\mathcal{U}_\lambda(0)$ *iff there exists a (closed) formula* α *such that* $\{x | x \in \mathcal{U}_\lambda(0), x \Vdash_v \alpha\}$*. A valuation* v *on* $\mathcal{U}_\lambda(0)$ *is called* admissible *iff, for any propositional variable* p_i *from the domain of* v*,* $v(p_i)$ *is admissible.*
3. *The restriction to the elements of depth* n *or less of the* 0*-universal model* $\mathcal{U}_\lambda(0)$ *is written* $(\mathcal{U}_\lambda(0))_n$*.*

For 0-fmp λ extending **K4** a theorem analogous to Theorem 2 applies.

Theorem 5 *For each* 0*-fmp extension of* **K4***,*

1. *Each finite Kripke frame for λ can be mapped p-morphically onto a generated submodel of $\mathcal{U}_\lambda(0)$ in a unique manner.*

2. *For each closed formula α, $\lambda \vdash \alpha$ iff $\mathcal{U}_\lambda(0) \Vdash \alpha$.*

3. *For each node w of $\mathcal{U}_\lambda(0)$ there exists a formula φ_w such that $v \Vdash \varphi_w$ iff $v = w$.*

Proof. See [3].

$\dashv$

It is also obvious that

Theorem 6 *Let λ be a 0-fmp logic extending* **K4***, and $\gamma_1, ..., \gamma_n$ be closed formulas. Then, for any $\alpha(p_1, \ldots, p_n)$, $\vdash_\lambda \alpha(\gamma_1, ..., \gamma_n)$ iff $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$.*

Proof. Just note that $\alpha(\gamma_1, ..., \gamma_n)$ is closed if $\gamma_1, ..., \gamma_n$ are and apply Theorem 5(2).

$\dashv$

We can now formulate the following general theorem.

Theorem 7 *For each 0-fmp λ extending* **K4** *and each $\alpha(p_1, ..., p_n)$, α is unifiable in λ iff there exists an admissible valuation v on the 0-universal frame $\mathcal{U}_\lambda(0)$ such that $\mathcal{U}_\lambda(0) \Vdash_v \alpha(p_1, ..., p_n)$.*

Proof. ($\Rightarrow$): If $\alpha(p_1, ..., p_n)$ is unifiable then there are closed formulas $\gamma_1, ..., \gamma_n$ such that $\vdash_L \alpha(\gamma_1, ..., \gamma_n)$. So, $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$, by Theorem 6. Take $v(p_i) = v(\gamma_i)$ then $\mathcal{U}_\lambda(0) \Vdash \alpha(p_1, ..., p_n)$.
($\Leftarrow$): Suppose there is an admissible valuation v on $\mathcal{U}_\lambda(0)$. Since v is admissible $v(p_i) = v(\gamma_i)$ for some closed γ_i, for each i. So $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$ and hence $\vdash_\lambda \alpha(\gamma_1, ..., \gamma_n)$ by Theorem 6. Therefore α is unifiable.

$\dashv$

This theorem by itself does in general not lead to decidability of unifiability for a logic, but, if one succeeds in exhibiting an effective procedure that provides for each formula α an n such that the existence of an admissible valuation on $\mathcal{U}_\lambda(0)$ is guaranteed by the existence of such a valuation on $(\mathcal{U}_\lambda(0))_n$, then decidability follows. Of course, this decidability was known by the decidability of the admissible rules for these logics, but the decision procedure is much more concrete. We have succeeded in the calculation of such an n for the logics **K4.3** and **GL** but not for **K4** itself.

4 Semantic results on Unifiability in KD4 and GL and their extensions, and in K4.3

In this section we give semantic results for the unifiability and non-unifiability of a formula in various logics. We start with **KD4**.

Theorem 8 *The* 0*-universal model* $\mathcal{U}_{\mathbf{KD4}}(0)$ *of* **KD4** *and all extensions of* **KD4** *consists of a single reflexive point.*

Proof. Obvious.

⊣

To obtain results for **GL** and its extensions, we use 0-universal models as planned. In addition, to obtain non-unifiability results, we consider α-soundness of **GL**-models and validity of boxed subformulas of formulas in these models.

Lemma 2 [2]. *Let* w *be node in a* **GL***-model.* $w \Vdash \Box^n \bot$ *iff* $depth(w) \leq n$.

Proof. By induction on n.

⊣

The following is the *normal form theorem* for closed formulas in **GL**.

Theorem 9 [2]. *Any closed formula* α *in* **GL** *is equivalent to a Boolean combination of some* $\Box^n \bot$.

Proof. See [2].

⊣

Corollary 3 [2]. *For each closed formula* α *of* **GL** *there exists a finite or cofinite subset* F_α *of* $\mathbb{N}$ *such that for each node* w *of finite depth,* $w \Vdash \alpha$ *iff* $depth(\alpha) \in F_\alpha$.

Theorem 10 *The* 0*-universal model* $\mathcal{U}_{\mathbf{GL}}(0)$ *of* **GL** *consists of the set of irreflexive worlds* $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ *where* $w_i R w_j$ *iff* $j < i$.

Proof. Obvious.

⊣

By Theorem 7 we then have immediately:

Theorem 11 *For each* $\alpha(p_1, ..., p_n)$*,* α *is unifiable in* **GL** *iff there exists an admissible valuation* v *on* $\mathcal{U}_{\mathbf{GL}}(0)$ *such that* $\mathcal{U}_{\mathbf{GL}}(0) \Vdash_v \alpha(p_1, ..., p_n)$.

We will now show how we can restrict this universal model to an upper part of it that is sufficient for our purposes.

Definition 10 *A Kripke model* K *is* α-sound *if* K *is rooted and in its root* w*,* $\Vdash \Box\beta \to \beta$ *holds for each subformula* $\Box\beta$ *of* α.

The following lemma is a slight generalization (to models containing reflexive nodes) of a lemma in [6].

Lemma 3 *Let* K *be* α*-sound, and let* K' *be defined by adding a new root* u *below* K *with its satisfaction relation identical to the one at* w *for all atoms. Then* $u \Vdash \beta$ *iff* $w \Vdash \beta$*, for all subformulas* β *of* α.

Proof. Let K be α-sound, and K' be defined by adding a new root u below K with the forcing identical to w for all the atoms. We prove by induction on the length of α that for all subformulas β of α that $u \Vdash \beta$ iff $w \Vdash \beta$. This is trivial for atoms and Boolean combinations.

Let $\beta = \Box\delta$ and the theorem hold for the formula δ. If $u \Vdash \Box\delta$ then $w \Vdash \Box\delta$ since uRw and R is transitive. If $w \Vdash \Box\delta$ then, not only for all v such that wRv, $v \Vdash \delta$, but also, by the α-soundness of K, $w \Vdash \delta$. By the induction hypothesis, $u \Vdash \delta$ as well. But then, irregardless of whether u is reflexive or irreflexive, for all v such that uRv, $v \Vdash \delta$, i.e., $u \Vdash \Box\delta$. Therefore, for every subformula β of α, $u \Vdash \beta$ iff $w \Vdash \beta$. $\dashv$

Theorem 12 *Let m be the number of subformulas of the form $\Box\beta$ in α plus one. Then, for each $\alpha(p_1, ..., p_n)$, α is unifiable in* **GL** *iff there exists a valuation v on $(\mathcal{U}_{\mathbf{GL}}(0))_m$ such that $(\mathcal{U}_{\mathbf{GL}}(0))_m \Vdash_v \alpha(p_1, ..., p_n)$.*
Proof.
($\Rightarrow$): Follows from Theorem 7.
($\Leftarrow$): Assume v is a valuation on $(\mathcal{U}_{\mathbf{GL}}(0))_m$ such that $(\mathcal{U}_{\mathbf{GL}}(0))_m \Vdash_v \alpha(p_1, ..., p_n)$. $(\mathcal{U}_{\mathbf{GL}}(0))_m$ is simply a chain of depth m. By the pigeon hole principle there is a $k < m$ such that the set of subformulas $\Box\beta$ of α that are forced at w of depth k and u of depth $k+1$ are the same because going up the number of such formulas can only increase or stay equal. Let K^*_k be the submodel of $(\mathcal{U}_{\mathbf{GL}}(0))_m$ generated by w. For each subformula $\Box\beta$ of α, $w \Vdash \Box\beta \to \beta$ holds because, if $w \Vdash \Box\beta$, then $u \Vdash \Box\beta$ and hence $w \Vdash \beta$. Therefore K^*_k is α-sound. Moreover, K^*_k is a model of $\alpha \wedge \Box\alpha$. By Lemma 3 we can conclude that by adding a new root w' to K^*_k with the same valuation as w we obtain a model K' that again satisfies $\alpha \wedge \Box\alpha$. Continuing by similarly adding w'' to obtain K'', w''' to obtain K''', etc. we get an infinite linear model for $\alpha \wedge \Box\alpha$. The special property of this model is that the valuation of p_i, is constant from depth k downwards for $1 \leq i \leq l$. That is because we kept the valuation constant each time we added a new root.

This means that p_i is equivalent to a closed formula γ_i on this model for each i, $1 \leq i \leq l$. The infinite linear frame of the model is of course nothing but $\mathcal{U}_{\mathbf{GL}}(0)$. The valuation v is determined by the formulas γ_i and is therefore admissible. Since $v(\alpha) = 1$ everywhere on the model, by Theorem 7, α is unifiable in **GL**. $\dashv$

Now consider a logic λ extending **GL**. To determine the set of unifiers of extensions of **GL** it is, by Corollary 1, sufficient to determine the set of unifiers of logics extending **GL** by closed formulas only. It is well-known that extensions of **GL** are 0-fmp (see e.g. [3]). But we have a more precise description of the extensions by closed formulas only.

Theorem 13

1. *The closed fragments of extensions of* **GL** *are the logics axiomatized by $\Box^n\bot$ for some n over the closed fragment of* **GL**.

2. An extension λ of **GL** *has the same closed fragment as* **GL** *iff, for no n, $\lambda \vdash \Box^n \bot$.*

Proof. See [3].

$\dashv$

This enables us to extend the characterization of the unifiable formulas for **GL** to its extensions.

Definition 11 *For a logic λ extending* **GL** *the* 0-universal model *and* frame *$\mathcal{U}_\lambda(0)$ consists of the set of irreflexive worlds $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ where $w_i R w_j$ iff $j < i$, if for no n, $\Box^n \bot$ is provable in λ, and of the set of irreflexive worlds $\{w_1, \ldots, w_n\}$ ordered in the same way if n is the smallest number for which $\Box^n \bot$ is provable in λ.*

Theorem 14 *Let λ be a logic extending* **GL**. *The formula α is unifiable in L iff, for some valuation, α is valid in $(\mathcal{U}_\lambda(0))_n$ where n is the number of boxed subformulas of α.*

Proof. Proof is the same as the proof of Theorem 12.

$\dashv$

Now consider the logic **K4.3**.

Definition 12 *The 0-universal model of* **K4.3** *is constructed as follows: The set of worlds consists of a set of irreflexive worlds $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ and a set of reflexive worlds $\{\bar{w}_i \mid i \in \mathbb{N}\backslash\{0\}\}$ where*

$w_i R w_j$ *iff* $j < i$,

$\bar{w}_i R w_j$ *iff* $j < i$,

$\bar{w}_i R \bar{w}_j$ *iff* $i = j$,

not $w_i R \bar{w}_j$.

Theorem 15 *The formula α is unifiable in* **K4.3** *iff, for some valuation, α is valid in $(\mathcal{U}_{\mathbf{K4.3}}(0))_{n+1}$ where n is the number of boxed subformulas of α.*

Proof. Proof is done similarly to proof of Theorem 12. Just note that $\mathcal{U}_{\mathbf{K4.3}}(0)$ is upwards linear, and apply the pigeon hole principle to the chain of irreflexive elements in $(\mathcal{U}_{\mathbf{K4.3}}(0))_{m+1}$.

$\dashv$

The theorem of course applies to the extensions of **K4.3** as well. Because the set of reflexive elements in $\mathcal{U}_{\mathbf{K4.3}}(0)$ is an infinite antichain there are uncountably many of such extensions among which many undecidable ones.

5 Syntactic Results on Unifiability

Proof of Theorem 1.
($\Leftarrow$): Assume α is unifiable in λ extending **KD4**. By Theorem 7 there exists a valuation on $\mathcal{U}_\lambda(0)$ validating α and hence also $\Box\alpha$. By Theorem 8, $\mathcal{U}_\lambda(0)$ is a single reflexive node. On that node $\bigvee_{p\in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ will always be falsified. Hence, $\Box\alpha \wedge \alpha \rightarrow \bigvee_{p\in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ is not provable in λ.
($\Rightarrow$): Assume $\Box\alpha \wedge \alpha \rightarrow \bigvee_{p\in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ is not provable in λ. Then there exists a **KD4**-model $\mathfrak{M}$ with a node w verifying $\Box\alpha \wedge \alpha$ and falsifying $\bigvee_{p\in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$. Thus, all nodes accessible from w (including possibly w itself), verify the same atoms. Consider a successor u of w (guaranteed to exist by the axiom D) and the submodel $\mathfrak{M}_u$ generated by u. Since each node has a successor in this model, and each node satisfies the same atoms, a p-morphism from $\mathfrak{M}_u$ onto a model on a single reflexive node exists. But this is a model on $\mathcal{U}_\lambda(0)$ and it still validates $\Box\alpha \wedge \alpha$. So, again applying Theorem 7, α is unifiable in λ.

$\dashv$

Definition 13 *For $n \geq 0$, D_n denotes the formula $\Box^n\bot \wedge \neg\Box^{n-1}\bot$ for some n where $\Box^0\bot \equiv \bot$.*

Note that $F_{D_n} = \{n\}$.

Theorem 16 *For each formula $\alpha(p_1, ..., p_l)$, α is not unifiable in* **GL** *iff $\alpha \wedge \Box\alpha \rightarrow (D_n \rightarrow \bigvee_{p_i\in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in* **GL** *for some n.*

In the proof we will see that the number of $\Box$-subformulas of α is a bound on the n, thereby again providing a concrte decision procedure for non-unifiability in **GL**.

Proof.
($\Leftarrow$): Assume α is unifiable and the formula

$$\alpha \wedge \Box\alpha \rightarrow (D_n \rightarrow \bigvee_{p_i\in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$$

is provable in **GL**. We have to obtain a contradiction. By the fact that α is unifiable there is a substitution g of unifiers in place of the variables of α such that $g(\alpha) \in$ **GL** (and hence $g(\Box\alpha) \in$ **GL**):

$$g(\alpha \wedge \Box\alpha \rightarrow (D_n \rightarrow \bigvee_{p_i\in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])) \in \textbf{GL}.$$

Take a linear frame of depth n. Its root w_n validates $g(\alpha)$, $g(\Box\alpha)$ and D_n and hence, for some p_i,

$$w_n \Vdash \bigvee_{k<n} [\Diamond(D_k \wedge g(p_i)) \wedge \Diamond(D_k \wedge \neg g(p_i))].$$

At some depth below n there should be two nodes of that depth satisfying the contradictory formulas $g(p_i)$ and $\neg g(p_i)$. This is impossible on a linear frame.
($\Rightarrow$): Assume $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \mathbf{GL}$ for all n. We have to show that α is unifiable.

Since the formula $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is not provable in **GL** there is, for each n, a **GL**-model $\mathfrak{M}_n$ of depth n that invalidates this formula in its root w_n, i.e.,

$$w_n \Vdash \alpha \wedge \Box\alpha \wedge D_n,\ w_n \nVdash \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)].$$

Therefore, $\mathfrak{M}_n$ has depth n. Because all nodes of each depth $k < n$ have the same valuation we can apply a p-morphism onto a linear model of depth n by mapping all nodes of depth $k < n$ to a node of depth k with the same valuation. So w.l.o.g. we can assume $\mathfrak{M}_n$ to be linear. Also $\alpha \wedge \Box\alpha$ is forced everywhere in the model $\mathfrak{M}_n$. Of course $\mathfrak{M}_n$ is a model on $(\mathcal{U}_{\mathbf{GL}}(0))_n$ so that by Theorem 12, from the case that n is the number of $\Box$-subformulas of α, the theorem follows. $\dashv$

Theorem 17 *If λ is an extension of* **GL**, *then*

1. *if, for no n, $\lambda \vdash \Box^n\bot$, then α is not unifiable in λ if $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ for some n.*
2. *if m is the smallest number for which $\lambda \vdash \Box^m\bot$, then α is not unifiable in λ if $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ for some $n \leqslant m$.*

Proof. (1) Let $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \lambda$ and $\lambda \vdash \Box^n\bot$, for no n. We have to show that α is unifiable.

In this case λ has the same closed fragment as **GL**, by Theorem 13. Since then λ and **GL** have the same finite linear models, the proof is given as for Theorem 16.
(2) Let $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \lambda$ for every $n > m$ and m is the smallest number for which $\lambda \vdash \Box^m\bot$. We have to show that α is unifiable. In this case, we consider the smallest m for which $\lambda \vdash \Box^m\bot$. All models with larger depth are excluded and proof is given as in the proof of Theorem 16 considering finitely many formulas that use D_n for $n \leqslant m$.

⊣

Though true, this theorem is somewhat misleading in that the logic λ may have only upward linear models (e.g. if λ is **GL.3**). Then clearly,

$$\vdash_\lambda \neg \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)],$$

so the condition reduces to $\vdash_\lambda \alpha \wedge \Box\alpha \to \neg D_n$ for some n.

For **K4.3** upward linearity is of course in force as well. Nevertheless, the syntactic conditions are rather complicated. Let us name the formulas guaranteed to exist for the reflexive worlds $\bar{w}_i$ by Theorem 5(3), $\bar{D}_i$. Then, if m is the number of $\Box$-subformulas of α, $\nvdash \alpha \wedge \Box\alpha \to \neg\bar{D}_i$ for each $i \leq m$ is not sufficient to guarantee a model for α on $(\mathcal{U}_{K4.3}(0))_m$ because the valuations on the different counter-models with reflexive nodes as roots obtained may not be the same on the irreflexive nodes.

What we have to do is to look at all the valuations individually, and check whether there are models for all reflexive roots with the same valuation on the irreflexive elements. Let v be a valuation on the irreflexive elements, i.e., v is defined for $p_1, \ldots, p_n$ for each $w_i, 1 \leq i \leq m$ (we write v_i). Let us define $p_j^{v_i}$ to be p_j if $v_i(p_j) = 1$ and $\neg p_j$ if $v_i(p_j) = 0$. Moreover, let us write θ_v for

$$\bigwedge_{i=1}^{m} \Diamond(\Box^i \bot \wedge p_1^{v_i} \wedge \cdots \wedge p_n^{v_i})$$

and $\theta_v^k, 0 < k \leq m$ for

$$\bigwedge_{i=1}^{k} \Diamond(\Box^i \bot \wedge p_1^{v_i} \wedge \cdots \wedge p_n^{v_i})$$

and $\theta_v^0 = \top$. Then truth of θ_v^k in $\bar{w}_{k+1}$ expresses that the valuation v holds in the irreflexive nodes above $\bar{w}_{k+1}$. We can then state the following theorem.

Theorem 18 *For each formula $\alpha(p_1, ..., p_n)$ with a number m of $\Box$-subformulas, α is not unifiable in* **K4.3** *iff, there is a multiple valuation $v = v_1, \ldots, v_{m+1}$ such that, for no $i \leq m+1$, on $\alpha \wedge \Box\alpha \to (\bar{D}_i \to \neg\theta_v^i)$ is provable in* **K4.3**.

Proof. If all these formulas are not provable in **K4.3**, then we can glue the counter-models with roots verifying $\alpha \wedge \Box\alpha \wedge \bar{D}_i$ together to one model of $\alpha \wedge \Box\alpha$ on $(\mathcal{U}_{K4.3}(0))_m$. Note that to keep matters more uniform we have used a root with $\bar{D}_{m+2}$ instead of D_{m+1}.

⊣

It is not difficult to transform this theorem into one for the extensions of **K4.3**. We leave this to the reader.

References

[1] N. Bezhanishvili, *Lattices of Intermediate and Cylindric Modal Logics*, PhD-Thesis, ILLC Dissertation Series DS-2006-02, University of Amsterdam, 2006.

[2] G. Boolos, *The Logic of Provability*, Cambridge University Press, 1993.

[3] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Clarendon Press, Oxford, 1997.

[4] G. Japaridze, D. de Jongh, The Logic of Provability, *Handbook of Proof Theory*, 360-422, Elsevier, 1995.

[5] V. V. Rybakov, M. Terziler, Ç. Gencer, An Essay on Unification and Inference Rules For Modal Logics, *Bulletin of the Section of Logic*, Vol. 28/3, 145-157, 1999.

[6] R. M. Solovay, Provability interpretations of modal logic, *Israel Journal of Mathematics*, 25, pp. 287-304, 1976.