Modal Logics for Reasoning about Preferences and Cooperation: Expressivity and Complexity

Cédric Dégremont and Lena Kurzen*

Universiteit van Amsterdam

Abstract. This paper gives a survey of expressivity and complexity of normal modal logics for reasoning about cooperation and preferences. We identify a class of notions expressing local and global properties relevant for reasoning about cooperative situations involving agents that have preferences. Many of these notions correspond to game- and social choice-theoretic concepts. We specify what expressive power is required for expressing these notions. This is done by determining whether they are invariant under certain relevant operations on different classes of Kripke models and frames. A large class of known extended modal languages is specified and we show how the chosen notions can be expressed in fragments of this class. In order to determine how demanding reasoning about cooperation is in terms of computational complexity, we use known complexity results for extended modal logics and obtain for each local notion an upper bound on the complexity of modal logics expressing it.

1 Introduction

Cooperation of agents is a major issue in many fields such as computer science, economics and philosophy. The conditions under which coalitions are formed can occur in various situations involving multiple agents. A single airline company could e.g. not afford the cost of an airport runway whereas a group of companies can. More generally, agents can decide to form groups in order to share complementary resources or because as a group they can achieve better results than individually. Modal logic (ML) frameworks for reasoning about cooperation, mostly focus on what coalitions can achieve. In Coalition Logic (**CL**) [1], this is done by using modalities of the form $[\mathtt{C}]\phi$ saying that "the coalition $\mathtt{C}$ has a joint strategy to ensure that ϕ". The semantics of **CL** is based on neighborhood models but it has recently been shown how it can be simulated on Kripke models [2].

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Another crucial concept for reasoning about interactive situations is that of *preferences*. It has also received considerable attention from modal logicians (see [3] for a survey). Recent works (e.g. [4, 5]) propose different mixtures of cooperation logics and preference logics and argue for it as the right system for reasoning about cooperation. Indeed, in cooperation logics, many concepts from *game theory* (GT) and *social choice theory* are commonly encountered. Depending on the situations to be modelled, different bundles of notions are important. The ability to express these notions – together with good computational behavior – make a logic appropriate for reasoning about the situations under consideration.

Rather than proposing a new logical framework, with specific expressivity and complexity, we identify how social-choice and game-theoretical notions are demanding for MLs in terms of expressivity and computational complexity. We identify notions relevant for describing interactive situations and give satisfiability and validity invariance results as well as definability results for them, identifying the natural (extended) modal languages needed depending on the class of frames actually considered and the particular bundle of notions of interest. We draw some consequences about the complexity of reasoning about cooperation in ML. Our results apply to logics interpreted on Kripke structures using a preference relation for each agent and a relation for each coalition. There are various interpretations for the coalition relation. The pair (x, y) being in the relation for coalition $\mathtt{C}$ can e.g. mean:

1. Coalition $\mathtt{C}$ considers y as being at least as good as x.
2. If the system is in state x, for $\mathtt{C}$ it would be at least as good if it was in y.
3. If she can switch the state of the system from x to y, $\mathtt{C}$ might/would do it.
4. $\mathtt{C}$ can submit a request such that if it is the first one received by the server while the state is in x, then the state of the system will change from x to y.
5. When the system is in state x, $\mathtt{C}$ considers it possible that it is in state y.
6. If the system is in state x, $\mathtt{C}$ would choose y as the next state.

We immediately stress that whenever the basic relation is interpreted as the possibility to bring the system in a different state, we will be dealing with scenarios in which agents take actions sequentially (e.g. with a server treating requests in a first arrived, first served manner) rather than simultaneously (as in ATL or CL e.g.). In some very special cases - e.g. for turn-based [6] or locally dictatorial [1] frames - the two approaches coincide. However we would like to stress that the two approach are first of all complementary. Our main focus in this paper is on concepts bridging both powers and preferences and we think the same analyze should be provided for powers themselves in e.g. ATL style. These two analyses could latter be combined in an interesting way. Finally an important alternative interpretation of the coalition relation is that of group preferences, in which case ATL models can simply be merged with the models we will consider. We return to this issue at the end of the next section.

The paper is structured as follows. Sect. 2 introduces three classes of models of cooperative situations that our results will be about. In Sect. 3, we introduce local and global notions indicating local properties of a system and global properties that characterize classes of frames, respectively. They are motivated by ideas from game theory and social choice theory. We study the expressive power required to express the local notions (resp. to define the global properties) in Sect. 6 (resp. Sect. 7) by giving invariance results for relevant operations and relations between models (resp. frames). Sect. 4 completes this work by introducing a large class of well-known extended modal languages and giving explicit definitions of both local and global notions within fragments of them. Moreover, we present the complexity results model checking and satisfiability for these languages and thereby give upper bounds for the complexity of logics that can express the notions we introduce. Finally, we conclude in Sect. 9.

2 The Models

As mentioned, our aim is to study how demanding certain game- and social choice-theoretical concepts are in terms of expressive power and computational complexity. The answer of course depends on the models we choose. We consider three classes of models. Due to the models' simplicity, there are many suitable interpretations, which gives our results additional significance. In what follows, we refer to a frame as the relational part of a model. For simplicity, we introduce the latter and assume that the domain of the valuation is given by a countable set of propositional letters $\mathtt{PROP}$ and a countable set of nominals $\mathtt{NOM}$. We focus on model theory and postpone the discussion of formal languages to Sect. 4.

Definition 1 ($\mathtt{N}$-$\mathtt{LTS}$). *A* $\mathtt{N}$-$\mathtt{LTS}$ *(Labeled Transition Systems indexed by a finite set of agents* $\mathtt{N}$*) is of the form* $\langle W, \mathtt{N}, \{ \stackrel{\mathtt{C}}{\rightarrow} \mid \mathtt{C} \subseteq \mathtt{N}\}, \{ \leq_i \mid i \in \mathtt{N}\}, V\rangle$, *where* $W \neq \emptyset$, $\mathtt{N} = \{1, \dots, n\}$ *for some* $n \in \mathbb{N}$, $\stackrel{\mathtt{C}}{\rightarrow} \subseteq W \times W$ *for each* $\mathtt{C} \subseteq \mathtt{N}$, $\leq_j \subseteq W \times W$ *for each* $j \in \mathtt{N}$, *and* $V : \mathtt{PROP} \cup \mathtt{NOM} \rightarrow \wp W$, $|V(i)| = 1$ *for each* $i \in \mathtt{NOM}$.

W is the set of states, $\mathtt{N}$ a set of agents and $w \stackrel{\mathtt{C}}{\rightarrow} v$ says that coalition $\mathtt{C}$ can change the state of the system from w into v. As mentioned above, other interpretations are of course possible, and we will sometimes refer to different interpretations. $w \leq_i v$ means that i finds the state v at least as good as the state w. $w \in V(p)$ means that p is true at state w. A standard assumption is that agents' preferences are total pre-orders ($\mathtt{TPO}$). Let $\mathtt{TPO}$-$\mathtt{N}$-$\mathtt{LTS}$ denote the class of $\mathtt{N}-\mathtt{LTS}$s in which for each $i \in \mathtt{N}$, $\leq_i$ is a $\mathtt{TPO}$. Finally, we consider models in which the strict preference relation is an explicit primitive.

Definition 2 ($\mathtt{S/TPO}-\mathtt{N}-\mathtt{LTS}$). *Define* $\mathtt{S/TPO}-\mathtt{N}-\mathtt{LTS}$ *as models of the form* $\langle W, \mathtt{N}, \{ \stackrel{\mathtt{C}}{\rightarrow} \mid \mathtt{C} \subseteq \mathtt{N}\}, \{ \leq_i \mid i \in \mathtt{N}\}, \{ <_i \mid i \in \mathtt{N}\}, V\rangle$,*which extend* $\mathtt{TPO}-\mathtt{N}-\mathtt{LTS}$ *models by an additional relation* $<_i \subseteq W \times W$ *for each* $i \in \mathtt{N}$ *with the constraint that for each* $i \in \mathtt{N}$, $w <_i v$ *iff* $w \leq_i v$ and $v \not\leq_i w$.

Depending on the interpretation one is willing to give to $\stackrel{\mathtt{C}}{\rightarrow}$, it might either be enriched or replaced by either effectivity functions (**CL**), or actions and outcome functions, or more generally transition functions in ATL style. In the latter sense, computing powers of coalitions would different and would not reduce in the general case to relations on the state space. As mentioned before, we leave a full analysis of powers in such settings to future work. In particular, there would be two way to go. Either drawing on the recently well-understood model-theory of neighborhood semantics [7] or on a normal simulation of **CL** [2]. Generally, we can expect the expressive power to depend on wether powers of coalitions are taken as primitives or computed from individual powers.

A last methodological comment: when analyzing the expressivity required by certain local notions, we consider the three class of models to see how the choice of models affects the invariance results. We now turn to the notions playing the central role in the paper.

3 The Notions

First, we consider two components of reasoning about cooperative interaction: what coalitions of agents can achieve together and what individuals prefer. From these two elements, more elaborated notions can be built. We consider natural counterparts of social choice- and game theoretical notions and are interested in local and global notions. Local notions are properties of a particular state in a particular system. Formally, they are properties of pointed models $\mathcal{M}, w$. Global notions are properties of classes of systems: we are interested in the class of frames some property characterizes. W.r.t to content, we distinguish between notions describing coalitional power and those describing preferences. There are also combinations of both kinds of notions, such as stability and effectivity concepts.

3.1 Power of Coalitions. What global constraints about what coalitions can achieve? $w \xrightarrow{\mathtt{C}} v$ means "C can achieve the state v at w". Recall also other possible interpretations.

Local Notions. Considering the power of a coalition at a given state, the most basic expression $PowL1$ says that a group has the ability to achieve a state satisfying some proposition. Interesting properties concerning the power of a group include the relation between the coalitional powers of different groups ($PowL3$) and the contribution of a single agent to a group's power, e.g. an agent can be needed in order to achieve something ($PowL2$).

- $PowL1$. Coalition C can achieve a state where p is true.
  $\exists x(w \xrightarrow{C} x \wedge P(x))$
- $PowL2$. Only coalitions containing i can achieve a p-state.
  $\bigwedge_{\mathtt{C}\subseteq\mathtt{N}\setminus\{i\}}(\forall x(w \xrightarrow{\mathtt{C}} x \rightarrow \neg P(x)))$
- $PowL3$. Coalition C can force every state that coalition D can force.
  $\forall x(w \xrightarrow{\mathtt{C}} x \leftrightarrow w \xrightarrow{\mathtt{D}} x)$

Global Notions. Global properties of coalitional power include general properties such as the property that each coalition can achieve exactly one result ($PowG1$), and coalition monotonicity ($PowG3$), which says that if a coalition can achieve some result, then also every superset of that coalition can achieve it.

In various contexts, the power of a group can depend on its size; the condition that certain results can only be achieved by coalitions containing the majority of agents ($PowG2$) occurs in many situations involving processes of decision making in groups. $PowG4$ and $PowG5$ say that some coalition $\mathtt{C}$ is very powerful compared to other coalitions: Any group of agents not contained in $\mathtt{C}$ cannot achieve anything ($PowG4$) or cannot achieve anything that $\mathtt{C}$ cannot achieve ($PowG5$).

- $PowG1$. Determinacy for coalition-relation, i.e. in any state each coalition can achieve exactly one state.
  $\bigwedge_{C\subseteq \mathtt{N}} \forall x \exists y (x \xrightarrow{C} y \wedge \forall z (x \xrightarrow{C} z \rightarrow z = y))$
- $PowG2$. Only coalitions containing a majority of $\mathtt{N}$ can achieve something.
  $\forall x (\bigwedge_{\mathtt{C}\subseteq \mathtt{N}, |\mathtt{C}| < \frac{|N|}{2}} (\neg \exists y (w \xrightarrow{\mathtt{C}} y)))$
- $PowG3$. Coalition monotonicity, i.e. if for $\mathtt{C}$ and $\mathtt{D}$, $\mathtt{C} \subseteq \mathtt{D}$, then $R_{\mathtt{C}} \subseteq R_{\mathtt{D}}$.
  $\forall x (\bigwedge_{\mathtt{C}\subseteq \mathtt{N}} \bigwedge_{\mathtt{D}\subseteq \mathtt{N}, \mathtt{C}\subseteq \mathtt{D}} (\forall y (x \xrightarrow{\mathtt{C}} y \rightarrow x \xrightarrow{\mathtt{D}} y)))$
- $PowG4$. If $R_{\mathtt{C}}[w] \neq \emptyset$, then for any coalition $\mathtt{D}$ such that $\mathtt{C} \cap \mathtt{D} = \emptyset$, $R_{\mathtt{D}}[w] = \emptyset$.
  $\forall x \bigwedge_{\mathtt{C}\subseteq \mathtt{N}} (((\exists y (x \xrightarrow{\mathtt{C}} y)) \rightarrow \bigwedge_{\mathtt{D}\subseteq \mathtt{N}\setminus \mathtt{C}} \neg \exists z (x \xrightarrow{\mathtt{D}} z)))$
- $PowG5$. If $R_{\mathtt{C}}[w] \neq \emptyset$, then for any $\mathtt{D}$ such that $\mathtt{C} \cap \mathtt{D} = \emptyset$, $R_{\mathtt{D}}[w] \subseteq R_{\mathtt{C}}[w]$.
  $\forall x \bigwedge_{\mathtt{C}\subseteq \mathtt{N}} (((\exists y (x \xrightarrow{\mathtt{C}} y)) \rightarrow \bigwedge_{\mathtt{D}\subseteq \mathtt{N}\setminus \mathtt{C}} \forall z (x \xrightarrow{\mathtt{D}} z \rightarrow x \xrightarrow{\mathtt{C}} z)))$

3.2 Preferences. What do agents prefer? What are suitable global constraints on preferences? $w \leq_i v$ ($w <_i v$) means "i finds v *at least as good* (a.l.a.g.) as w" ("i strictly prefers v over w").

Local Notions. First of all, we can distinguish between strict and nonstrict preferences. The most basic preference relation that we consider is that of being "at least as good". Alternatively, we can also look at the relation "at least as bad" ($PrefL4$). Agents' preferences over states can also be seen as being based on preferences over propositions [8]. $PrefL8$ ($PrefL10$) says the truth of a given proposition is a sufficient (necessary) condition for an agent to prefer some state.

- $PrefL1$. There is a state that i finds at least as good where p is true.
  $\exists x (w \leq_i x \wedge P(x))$
- $PrefL2$. There is a state that i strictly prefers where p is true.
  $\exists x (w \leq_i x \wedge \neg (x \leq_i w) \wedge P(x))$
- $PrefL3$.There is a state that all agents find a.l.a.g and that at least one strictly prefers.
  $\exists x (\bigwedge_{i\in \mathtt{N}} (w \leq_i x) \wedge \bigvee_{j\in \mathtt{N}} \neg (x \leq_j w))$
- $PrefL4$. There is a state that i finds at least as bad where p is true.
  $\exists x (x \leq_i w \wedge P(x))$
- $PrefL5$. There is a state that i finds strictly worse where p is true.
  $\exists x (x \leq_i w \wedge \neg (w \leq_i x) \wedge P(x))$
- $PrefL6$. i finds a state at least as good as the current one iff j does.
  $\forall x (w \leq_i x \leftrightarrow w \leq_j x)$
- $PrefL7$. There is a state that only i finds a.l.a.g. as the current state.
  $\exists x (w \leq_i x \wedge \bigwedge_{j\in \mathtt{N}\setminus\{i\}} \neg (w \leq_j x))$

– $PrefL8$. i finds every p-state at least as good as the current one.
$\forall x(P(x) \rightarrow w \leq_i x)$
– $PrefL9$. i strictly prefers every p-state over the current one.
$\forall x(P(x) \rightarrow (w \leq_i x \wedge \neg(x \leq_i w)))$
– $PrefL10$. i considers only p-states to be as least as good as the current one.
$\forall x(w \leq_i x \rightarrow P(x))$
– $PrefL11$. i strictly prefers only p-states over the current one.
$\forall x((w \leq_i x \wedge \neg(x \leq_i w)) \rightarrow P(x))$

Global Notions. In order to ensure that the intuitive idea of preferences is captured, usually several conditions for the preference relation are required. These conditions include reflexivity for non-strict preferences and additionally transitivity and completeness (trichotomy being its analogue for strict preferences). In some situations, it can also be appropriate to require determinacy and thereby model the property that for each alternative there is exactly one that is at least as good ($PrefG8$).

– $PrefG1$. "at least as good as" is reflexive.
$\forall x(\bigwedge_{i\in \mathtt{N}}(x \leq_i x))$
– $PrefG2$. "at least as good as" is transitive.
$\forall x\forall y\forall z(\bigwedge_{i\in \mathtt{N}}((x \leq_i y \wedge y \leq_i z) \rightarrow x \leq_i z))$
– $PrefG3$. "at least as good as" is complete.
$\forall x\forall y(\bigwedge_{i\in \mathtt{N}}(x \leq_i y \vee y \leq_i x))$
– $PrefG4$. "at least as good as" is a total pre-order.
Conjunction of the two previous formulas.
– $PrefG5$. "strictly better than" is transitive.
$\forall x\forall y\forall z(\bigwedge_{i\in \mathtt{N}}((x \leq_i y \wedge \neg(y \leq_i x) \wedge y \leq_i z \wedge \neg(z \leq_i y)) \rightarrow (x \leq_i z \wedge \neg(z \leq_i x))))$
– $PrefG6$. "strictly better than" is trichotomous [1] .
$\forall x\forall y(\bigwedge_{i\in \mathtt{N}}((x \leq_i y \wedge \neg(y \leq_i x)) \vee (y \leq_i x \wedge \neg(x \leq_i y)) \vee x = y))$
– $PrefG7$. "strictly better than" is a strict total order [2] .
Conjunction of the previous two formulas.
– $PrefG8$. Determinacy for "at least as good as", i.e. exactly one successor.
$\forall x(\bigwedge_{i\in \mathtt{N}}(\exists y(w \leq_i y \wedge \forall z(x \leq_i z \rightarrow z = y))))$

So far, we focussed on preferences of individuals. A natural question in social choice is how to aggregate individual preferences into group preferences. This question can be considered in our framework by interpreting $\xrightarrow{C}$ as a preference relation for each $\mathtt{C} \subseteq \mathtt{N}$. We now consider how group preferences arise from the preferences of individuals:

– $PrefG9$. $\mathtt{C}$ finds a state a.l.a.g. as the current one iff all its members do.
$\forall x\forall y(\bigwedge_{\mathtt{C}\subseteq \mathtt{N}}(x \xrightarrow{\mathtt{C}} y \leftrightarrow \bigwedge_{i\in \mathtt{C}} x \leq_i y))$
– $PrefG10$. — iff at least one of its members does.
$\forall x\forall y(\bigwedge_{\mathtt{C}\subseteq \mathtt{N}}(x \xrightarrow{\mathtt{C}} y \leftrightarrow \bigvee_{i\in \mathtt{C}} x \leq_i y))$

[1] A relation R is trichotomous if for every x, y we have that xRy or yRx or $x = y$
[2] A strict total order is an irreflexive, transitive and trichotomous relation.

- $PrefG11$. — iff a majority of its members do.
  $\forall x \forall y (\bigwedge_{C \subseteq N} (x \xrightarrow{C} y \leftrightarrow \bigvee_{D \subseteq C, |D| > \frac{|C|}{2}} (\bigwedge_{i \in D} x \leq_i y)))$

3.1 Simple combinations of the preceding concepts.

We start with a conceptually and historically important social-choice theoretic notion: that of a *dictator*. An agent d is a dictator of the system under consideration if the preferences of the group exactly mimic the preferences of d. (Note that an even stronger notion of a dictator is obtained by interpreting $\xrightarrow{C}$ as an achievement relation, in which case if there is a dictator d, anybody can only *do* what d likes.) We introduce the notion of *local* dictator as a dictator who controls one state in the system. The usual notion of the dictator is the obvious generalization to all states of the system. A definition follows:

Definition 3 (Local Dictatorship).

- *An agent i is a* weak local dictator *at w iff any coalition* C *prefers v at w only if i thinks that v is at least as good as w.*
- *An agent i is a* strong local dictator *at w iff any coalition* C *prefers v at w only if i thinks that v is strictly better than w.*

We also introduce natural combinations of powers and preferences. The first notion e.g. describes the fact that the coalition C could do something useful for i (in some cases giving i an incentive to participate in it.) and the third notion characterizes the situations (pointed models) in which a unanimously desired state remains unachievable.

Local Notions

- $PPL1$. C can achieve a state that i finds at least as good as the current one.
  $\exists x (w \xrightarrow{C} x \wedge w \leq_i x)$
- $PPL2$. C can achieve a state that all $i \in D$ find a.l.a.g. as the current one.
  $\exists x (w \xrightarrow{C} x \wedge \bigwedge_{i \in D} w \leq_i x)$
- $PPL3$. There is a state that all agent prefers but no coalition can achieve it.
  $\exists x ((\bigwedge_{i \in N} w \leq_i x) \wedge \bigwedge_{C \subseteq N} \neg (w \xrightarrow{C} x))$
- $PPL4$. C can achieve all states that agent i finds a.l.a.g. as the current one.
  $\forall x (w \leq_i x \rightarrow w \xrightarrow{C} x)$
- $PPL5$. C can achieve all states that i strictly prefers over the current one.
  $\forall x ((w \leq_i x \wedge \neg (x \leq_i w)) \rightarrow w \xrightarrow{C} x)$
- $PPL6$. i is a weak local dictator.
  $\forall x (w \xrightarrow{C} x \rightarrow w \leq_i x)$
- $PPL7$. i is a strong local dictator.
  $\forall x (w \xrightarrow{C} x \rightarrow (w \leq_i x \wedge \neg (x \leq_i w)))$

Global Notions. The first notion is a natural constraints on the powers of a coalition: a coalition can achieve a state iff it does something good for all its agents - otherwise they would not take part in the collective action (even so the agent might be indeed needed). $PPG3$ is one of the conditions of Arrow's impossibility theorem.$PPG4$ reflects a notion of individual rationality that says that you would not join a coalition if you don't win something by doing so. Sometimes the requirement is generalized to every sub-coalition and/or weaken to "not joining if you lose something". (See for example the definition of the core of the a coalitional game in [9] (Definition 268.3). $PPG5$ applies to system in which an agent is indispensable to achieving anything: an unique capitalist in a production economy or a unique server can be typical examples. (Note that the agent does not need not be unique.)

- $PPG1$.Coalitions can only achieve states that all its members consider at least as good as the current one.
  $\forall x \forall y \bigwedge_{\mathtt{C} \subseteq \mathtt{N}} (x \stackrel{\mathtt{C}}{\rightarrow} y \rightarrow \bigwedge_{i \in \mathtt{C}} (x \leq_i y))$
- $PPG2$. One agent is a weak local dictator in every state (*dictator*).
  $\bigvee_{i \in \mathtt{N}} \forall x \forall y (x \stackrel{\mathtt{C}}{\rightarrow} y \rightarrow x \leq_i y)$
- $PPG3$. There is no *dictator*.
  $\neg(\bigvee_{i \in \mathtt{N}} \forall x \forall y (x \stackrel{\mathtt{C}}{\rightarrow} y \rightarrow x \leq_i y))$
- $PPG4$. If agent i can achieve some state that he strictly prefers then for any $\mathtt{C}$ containing i it holds that whenever $\mathtt{C} - \{i\}$ cannot achieve some state but $\mathtt{C}$ can, then i strictly prefers that state over the current one.
  $\bigwedge_{i \in \mathtt{N}} \forall x (\exists y (x \stackrel{\{i\}}{\rightarrow} y \wedge x \leq_i y \wedge \neg(x \leq_i y)) \rightarrow \bigwedge_{\mathtt{C} \subseteq \mathtt{N}, i \in \mathtt{C}} (\forall z (x \stackrel{\mathtt{C}}{\rightarrow} z \wedge \neg(x \stackrel{\mathtt{C} \setminus \{i\}}{\rightarrow} z)) \rightarrow (x \leq_i z \wedge \neg(z \leq_i x))))$
- $PPG5$. Only coalitions containing i can achieve something.
  $\forall x \bigwedge_{\mathtt{C} \subseteq \mathtt{N} \setminus \{i\}} \neg \exists y (x \stackrel{\mathtt{C}}{\rightarrow} y)$
- $PPG6$. In each state, there is some i such that any coalition containing i can achieve exactly the same states as it can without i.
  $\forall x (\bigvee_{i \in \mathtt{N}} \bigwedge_{\mathtt{C} \subseteq \mathtt{N}, i \in \mathtt{C}} \forall y (x \stackrel{\mathtt{C}}{\rightarrow} y \leftrightarrow x \stackrel{\mathtt{C} \setminus \{i\}}{\rightarrow} y))$
- $PPG7$. For any agent, there is some state in which coalitions not containing this agent cannot achieve any state.
  $\bigwedge_{i \in \mathtt{N}} \exists x (\bigwedge_{\mathtt{C} \subseteq \mathtt{N}, i \in \mathtt{C}} \neg \exists y (x \stackrel{\mathtt{C}}{\rightarrow} y))$

3.1 Efficiency and Stability Notions

In our setting, it is natural to interpret our state space as possible social states or allocations of goods. In general, an important criterion from welfare economics is to discriminate "good" from "bad" states in terms of *efficiency*. The idea is that if we can change the allocation or social state and make an agent happier without making anyone else less happy then we are using resources more efficiently and it is socially desirable to do so. For example $PrefL3$ in this respect means that the current state is not efficient – we say that there is a state that is a *Pareto-improvement* of it. If no state *Pareto*-improves the current one, we say that it is *Pareto-efficient*. Importing the notion of Pareto-efficiency into our framework is straightforward and we distinguish three different variations of it.

Definition 4 (Pareto-efficiency). *A state will be said to be*

- *weakly* Pareto-*efficient iff there is no state that everyone strictly prefers.*

– Pareto-*efficient iff there is no state such that everyone considers it to be at least as good and at least one agent thinks it is strictly better.*
– *strongly* Pareto-*efficient iff no state is at least as good for everyone.*

Equilibrium concepts in game theory characterize stable states: Considering what the other agents are doing in equilibrium I don't have an incentive to do something that would make us leave this stable state. Generalizing this idea, a state of a system can be thought of as stable if no agent has an incentive to make the system transit to another state, i.e. no agent can make the system transit in a state she prefers. We can think in this way of strategy profiles in a strategic game as assigning roles or task to the different agents. Two profiles $x = (s^*_{-i}, s^*_i), y$ are related by $\stackrel{\{i\}}{\to}$ iff i can unilaterally change role (strategy) to s'_i in the next repetition of the game and $y = (s^*_{-i}, s'_i)$. As an example, the stability of a state where an agent provides the public good on his own depends on whether he cares enough about the public good to provide it on his own. Formally, a state will be said to be stable if there is no other state that an agent can achieve alone and strictly prefers. Given the close correspondence to *Nash* equilibria (see e.g. [9]), we use the names *Nash-stability*, and *Nash-cooperation stability* for its group version.

Definition 5 (Nash-stability). *A state will be said to be*

– Nash-*stable iff there is no state that an agent i strictly prefers and that i can achieve alone.*
– *strongly* Nash-*stable iff there is no state that an agent i finds it a.l.a.g. and that i alone is able to achieve.*
– Nash-*cooperation stable iff there is no state v and coalition such $\mathtt{C}$ that every $i \in \mathtt{C}$ strictly prefers v and $\mathtt{C}$ can achieve v.*
– *strongly* Nash-*cooperation stable iff there is no state v and coalition $\mathtt{C}$ such that every $i \in \mathtt{C}$ finds v a.l.a.g. and $\mathtt{C}$ can achieve v.*

Local Notions

– $EF1$.The current state is weakly *Pareto*-efficient.
  $\neg\exists x(\bigwedge_{i\in\mathtt{N}}(w \leq_i x \wedge \neg(x \leq_i w)))$
– $EF2$. The current state is *Pareto*-efficient.
  $\neg\exists x((\bigwedge_{i\in\mathtt{N}} w \leq_i x) \wedge \bigvee_{j\in\mathtt{N}} \neg(x \leq_i w))$
– $EF3$. The current state is strongly *Pareto*-efficient.
  $\neg\exists x(\bigwedge_{i\in\mathtt{N}} w \leq_i x)$
– $ST1$. The current state is *Nash* stable.
  $\neg\exists x(\bigvee_{i\in\mathtt{N}}(w \stackrel{\{i\}}{\to} x \wedge w \leq_i x \wedge \neg(x \leq_i w)))$
– $ST2$. The current state is strongly *Nash* stable.
  $\neg\exists x(\bigvee_{i\in\mathtt{N}}(w \stackrel{\{i\}}{\to} x \wedge w \leq_i x))$
– $ST3$. The current state is *Nash*-cooperation stable.
  $\neg\exists x(\bigvee_{C\subseteq\mathtt{N}}(w \stackrel{\mathtt{C}}{\to} x \wedge \bigwedge_{i\in\mathtt{C}}(w \leq_i x \wedge \neg(x \leq_i w))))$
– $ST4$. The current state is strongly *Nash*-cooperation stable.
  $\neg\exists x(\bigvee_{C\subseteq\mathtt{N}}(w \stackrel{\mathtt{C}}{\to} x \wedge \bigwedge_{i\in\mathtt{C}} w \leq_i x))$

4 Languages

As it will be clearer from the invariance results in the next section, Basic Modal Language will generally be too weak for reasoning about cooperation. On the other hand any notion expressible in the first-order correspondence language is expressible in the hybrid language $\mathcal{H}(\mathtt{E}, @, \downarrow)$ [10]. Amongst temporal logics, boolean modal logics and the various hybrid logics, there are are well-understood fragments. We introduce these all these **Extended Modal Languages** at once as a "super" logic.

Syntax. The syntax of this "super" logic is recursively defined as follows:

$\alpha ::= \ \leq_j \ | \ \mathtt{C} \ | \ \upsilon \ | \ \alpha^{-1} \ | \ ?\phi \ | \ \alpha;\alpha \ | \ \alpha \cup \alpha \ | \ \alpha \cap \alpha \ | \ \overline{\alpha}$

$\phi ::= \ p \ | \ i \ | \ x \ | \ \neg\phi \ | \ \phi \wedge \phi \ | \ \langle\alpha\rangle\phi \ | \ \mathtt{E}\phi \ | \ @_i\phi \ | \ @_x\phi \ | \ \downarrow x.\phi \ | \ [\![\ \alpha \ ]\!] \ \phi \ |$

where $j \in \mathtt{N}$, $\mathtt{C} \in \wp(\mathtt{N}) - \{\emptyset\}$, p is an element of a countable set of propositional letters $\mathtt{PROP}$, i is an element of the countable set of nominals $\mathtt{NOM}$ and $x \in \mathtt{SVAR}$, for $\mathtt{SVAR}$ being a countable set of variables.

Semantics. A valuation now maps propositional letters to subsets of the domain and nominals to singleton subsets. Given a $\mathtt{N}-\mathtt{LTS}$, a program α is interpreted on a relation as indicated on the left side. Formulas are interpreted together with an assignment $g : \mathtt{SVAR} \to W$ as indicated on the right side.

$R_{\leq_i}$	$= \leq_i$	$\mathcal{M}, w, g \Vdash p$	iff $w \in V(p)$
R_C	$= \overset{C}{\to}$	$\mathcal{M}, w, g \Vdash i$	iff $w \in V(i)$
R_υ	$= W \times W$	$\mathcal{M}, w, g \Vdash x$	iff $w = g(x)$
$R_{\beta^{-1}}$	$= \{(v,w) \mid wR_\beta v\}$	$\mathcal{M}, w, g \Vdash \phi \wedge \psi$	iff $\mathcal{M}, w, g \Vdash \phi$ and $\mathcal{M}, w, g \Vdash \psi$
$R_{?\phi}$	$= \{(w,w) \mid w \Vdash \phi\}$	$\mathcal{M}, w, g \Vdash \neg\phi$	iff $\mathcal{M}, w, g \nVdash \phi$
$R_{\beta;\gamma}$	$= \{(v,w) \mid \exists x \ : vR_\beta x R_\gamma w\}$	$\mathcal{M}, w, g \Vdash \langle\alpha\rangle\phi$	iff $\exists v : wR_\alpha v$ and $\mathcal{M}, v, g \Vdash \phi$
$R_{\beta\cup\gamma}$	$= R_\beta \cup R_\gamma$	$\mathcal{M}, w, g, \Vdash \mathtt{E}\phi$	iff $\exists v \ \in W \ \mathcal{M}, v, g \Vdash \phi$
$R_{\beta\cap\gamma}$	$= R_\beta \cap R_\gamma$	$\mathcal{M}, w, g, \Vdash @_i\phi$	iff $\mathcal{M}, v, g \Vdash \phi$ where $V(i) = \{v\}$
$R_{\overline{\beta}}$	$= (W \times W) - R_\beta$	$\mathcal{M}, w, g, \Vdash @_x\phi$	iff $\mathcal{M}, g(x), g \Vdash \phi$
R_{α^*}	$= \bigcup_{n\in\mathbb{N}^0} (R_\alpha)^n$	$\mathcal{M}, w, g, \Vdash \downarrow x.\phi$	iff $\mathcal{M}, w, g[x := w] \Vdash \phi$
		$\mathcal{M}, w, g \Vdash [\![\ \alpha \ ]\!] \ \phi$	iff $wR_\alpha v$ whenever $\mathcal{M}, v, g \Vdash \phi$

Expressive Power Tree. The least expressive modal language we consider is $\mathcal{ML}(\mathtt{N})$, which is of similarity type $\langle(\mathtt{C})_{\mathtt{C}\subseteq\mathtt{N}}, (\leq_i)_{i\in\mathtt{N}}\rangle$. Its natural extensions go along two lines: adding program constructs and new operators. $\mathcal{ML}(\mathtt{N}, \cap, i)$ e.g. refers to the logic with language:

$\alpha ::= \ \leq_\mathtt{j} \ | \ \mathtt{C} \ | \ \alpha \cap \alpha \qquad \phi ::= \ p \ | \ i \ | \ \neg\phi \ | \ \phi \wedge \phi \ | \ \langle\alpha\rangle\phi.$

As language inclusion implies expressive power inclusion, we only indicate (some) non-obvious facts of inclusions in this space of modal languages. $\mathcal{L} \leq \mathcal{L}'$ says "$\mathcal{L}'$ is at least as expressive as $\mathcal{L}$".

Fact 1. $\mathcal{ML}(\mathtt{N}, \cup, \ ; \ , ?) \leq \mathcal{ML}(\mathtt{N})$.
Fact 2. $\mathcal{ML}(\mathtt{N}, @, i) \leq \mathcal{ML}(\mathtt{N}, \mathtt{E}, i)$.
Fact 3. $\mathcal{ML}(\mathtt{N}, \cap) \leq \mathcal{ML}(\mathtt{N}, \downarrow, @, x)$.
Fact 4. $\mathcal{ML}(\mathtt{N}, [\![\ ]\!]) \leq \mathcal{ML}(\mathtt{N}, \text{-})$.
Fact 5. $\mathcal{ML}(\mathtt{N}, \text{-}) \leq \mathcal{ML}(\mathtt{N}, \downarrow, \mathtt{E}, x)$.
Fact 6. $\mathcal{ML}(\mathtt{N}, \ ^{-1}) \leq \mathcal{ML}(\mathtt{N}, \downarrow, \mathtt{E}, x)$.
Fact 7. $\mathcal{ML}(\mathtt{N}, \mathtt{E}) \leq \mathcal{ML}(\mathtt{N}, \text{-})$.

Expressivity of modal logics is usually characterized by invariance results. The following section gives the basic definitions and background results.

5 Invariance of Modal Languages: Background Results

We start by introducing some relations between models. Let τ be a finite modal similarity type involving only binary relations. Let $\mathcal{M} = \langle W, (R_k)_{k\in\tau}, V\rangle$ and $\mathcal{M}' = \langle W', (R'_k)_{k\in\tau}, V'\rangle$ be two models of similarity type τ.

Definition 6 (Bisimulations). *A bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ is a non-empty binary relation $Z \subseteq W \times W'$ fulfilling the following conditions:*
`AtomicHarmony` *For every $p \in$ `PROP`, wZw' implies $w \in V(p)$ iff $w' \in V'(p)$.*
`Forth` $\forall k \in \tau$ *, if wZw' & R_kwv then $\exists v' \in W'$ s.t. $R'_kw'v'$ & vZv'.*
`Back` $\forall k \in \tau$ *, if wZw' & $R'_kw'v'$ then $\exists v \in W$ s.t. R_kwv & vZv'.*

We briefly describe the other bisimulations (see [10] for details). Intuitively, $\cap$-Bisimulations (resp. CBisimulations) require that `Back` and `Forth` salso hold for the intersection (resp. the converse) of the relations, while $\mathcal{H}$-Bisimulations extend `AtomicHarmony` to nominals. TBisimulations (resp. $\mathcal{H}(@)$-bisimulations) are total [3] bisimulations (resp. total $\mathcal{H}$-Bisimulations), and $\mathcal{H}(\mathtt{E})$-Bisimulations are $\mathcal{H}$-Bisimulations matching states "with the same name". Let us define bounded morphisms, generated subframes and disjoint unions.

Definition 7 (`BM`). *$f : W \to W'$ is a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$ iff:*
`AtomicHarmony` *For every $p \in$ `PROP`, $w \in V(p)$ iff $f(w) \in V'(p)$.*
`R − homomorphism` $\forall k \in \tau$ *, if R_kwv then $R'f(w)f(v)$.*
`Back` $\forall k \in \tau$ *, if $R'_kf(w)v'$ then $\exists v \in W$ s.t. $f(v) = v'$ and R_kwv.*

Definition 8 (Generated Submodel). *We say that that $\mathcal{M}'$ is a generated submodel (`GSM`) of $\mathcal{M}$ iff $W' \subseteq W$, $\forall k \in \tau$, $R'_k = R_k \cap (W' \times W')$, $\forall p \in$ `PROP` , $V'(p) = V(p) \cap (W' \times W')$ and if $w \in W'$ and Rwv then $v \in W'$.*

Definition 9 (Disjoint Unions). *Let $(\mathcal{M}_j)_{j\in J}$ be a collection of models with disjoint domains. Define their disjoint union $\biguplus_j \mathcal{M}_j = \langle W, R, V\rangle$ as the union of their domains and relations, and define for each $p \in$ `PROP`, $V(p) := \bigcup_j V_j(p)$.*

Definition 10 (Invariance). *A property of pointed models $\Phi(X, y)$ is* invariant under λ-Bisimulations *iff whenever there exists a λ-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$, then $\Phi(\mathcal{M}, w')$ holds iff $\Phi(\mathcal{M}', w')$ holds. Invariance for other operations is defined similarly.*

The three following (classical) results gives a good idea of the respective expressive power of certain extended modal languages. We recall them to be selfcontained. For details and examples, the reader is referred to [11, 10].

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[3] $Z \subseteq W \times W'$ is *total* iff $\forall w \in W\ \exists w' \in W'\ .wZw'$ & $\forall w' \in W'\ \exists w \in W\ .wZw'$.

Theorem. *[12] Let $\phi(x)$ be a formula of the first-order correspondence language with at most one free variable. The following are equivalent:*

1. *$\phi(x)$ is invariant under bisimulations*
2. *$\phi(x)$ is equivalent to the standard translation of a modal formula*

Theorem. *[13, 14] Let $\phi(x)$ be a formula of the first-order correspondence language with at most one free variable. The following are equivalent:*

1. *$\phi(x)$ is invariant under taking generated submodels*
2. *$\phi(x)$ is equivalent to the standard translation of a formula of $\mathcal{ML}(\mathtt{N}, \downarrow, @, x)$*

Theorem. *[15] A first-order definable class of frames is modally definable iff it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

The reader might now like to see immediately how the notions can be defined in extended modal languages and go directly to Sect. 8. Of course, the choice of the languages is only justified once we have determined the required expressive power both to express the local notions and to define the class of frames corresponding to the global ones. Thus we start by doing so in the next section.

6 Invariance Results: Satisfiability

Satisfiability invariance results for the three classes of pointed models defined in Sect. 2 follow. A "Y" in a cell means that the row notion is invariant under the column operation.

6.1 Results for the General Case

	Bis	CBis	∩-Bis	TBis	$\mathcal{H}$-Bis	$\mathcal{H}$(@)-Bis	$\mathcal{H}$(E)-Bis	BM	GSM	DU
$[PowL1]$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$[PowL2]$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$[PowL3]$	N	N	N	N	N	N	N	N	Y	Y
$[PrefL1]$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$[PrefL2]$	N	N	Y	N	N	N	N	N	Y	Y
$[PrefL3]$	N	N	N	N	N	N	N	N	Y	Y
$[PrefL4]$	N	Y	N	N	N	N	N	N	N(2)	Y
$[PrefL5]$	N	N	N	N	N	N	N	N	N	Y
$[PrefL6]$	N	N	N	N	N	N	N	N	Y	Y
$[PrefL7]$	N	N	N	N	N	N	N	N	Y	Y
$[PrefL8]$	N	N	N	N	N	N	N	N	N	N
$[PrefL9]$	N	N	N	N	N	N	N	N	N	N
$[PrefL10]$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$[PrefL11]$	N	N	N	N	N	N	N	N	Y	Y
$[PPL1]$	N	N	Y	N	N	N	N	N	Y	Y
$[PPL2]$	N	N	Y	N	N	N	N	N	Y	Y
$[PPL3]$	N	N	N	N	N	N	N	N	Y	Y
$[PPL4]$	N	N	N	N	N	N	N	N	Y	Y
$[PPL5]$	N	N	N	N	N	N	N	N	Y	Y
$[PPL6]$	N	N	N	N	N	N	N	Y	Y	Y
$[PPL7]$	N	N	N	N	N	N	N	N	Y	Y
$[EF1]$	N	N	N	N	N	N	N	N	Y	Y
$[EF2]$	N	N	N	N	N	N	N	N	Y	Y
$[EF3]$	N	N	Y	N	N	N	N	N	Y	Y
$[ST1]$	N	N	N	N	N	N	N	N	Y	Y
$[ST2]$	N	N	N(1)	N	N	N	N	N	Y	Y
$[ST3]$	N	N	N	N	N	N	N	N	Y	Y
$[ST4]$	N	N	Y	N	N	N	N	N	Y	Y

Comments. Most of our notions are not invariant under bisimulations. Thus, the basic modal language [4] is not expressive enough to describe our local notions (without further restrictions on the class of frames). Invariance under BM often fails; some failures are due to intersections of relations, but as ∩-Bis also fails quite often, this cannot be the only reason. Invariance under GSM however holds in many cases; its (rare) failures occur for properties with backward looking features. This is good news for expressivity: we can expect definability in the hybrid language with ↓-binder [5] . However, we cannot expect decidability since the satisfiability problem of the bounded fragment [6] is highly undecidable. Finally, the results are the same for hybrid and basic bisimulations. This is no suprise: roughly speaking, at the level of local satisfaction, to exploit the

[4] of similarity type $\langle\{ \xrightarrow{\mathtt{C}} \mid \mathtt{C} \subseteq \mathtt{N}\}, \{ \leq_i \mid i \in \mathtt{N}\}\rangle$

[5] [14, 13] have proved that all notions definable in the first-order correspondence language that are invariant under GSM are equivalent to a formula of the bounded fragment, i.e. to a formula of the hybrid language with ↓-binder (which are notational variants).

[6] Equivalent to the hybrid language with ↓-binder.

expressive power of nominals, the notions would have to refer explicitly to some state. Here are two representative results.

Representative Proofs for the General Case

Proposition 1. *On the class* $\mathtt{N}-\mathtt{LTS}$*,* $ST2$ *is not invariant under* $\cap$*-bisimulation.*

Proof. Let $\mathcal{M} = \langle \{w, v\}, \{1, 2\}, \{\overset{\mathtt{C}}{\to} |\mathtt{C} \subseteq \{1, 2\}\}, \{\leq_1, \leq_2\}, V\rangle$, where $w \overset{\{1,2\}}{\to} v, w \leq_1 v, v \leq_1 w, w \leq_2 v, V(p) = \{w, v\}$. Let $\mathcal{M}' = \langle \{s, t, u\}, \{1, 2\}, \{\overset{\mathtt{C}}{\to}'|\mathtt{C} \subseteq \{1, 2\}\}, \{\leq'_1, \leq'_2\}, V'\rangle$, where $s \overset{\{1,2\}}{\to}' t, u \overset{\{1,2\}}{\to}' t, s \leq'_1 t, u \leq'_1 t, t \leq'_1 u, s \leq'_2 t, u \leq'_2 t, V'(p) = \{s, t, u\}$. Then, $\mathcal{M}, w \Vdash ST2$ and $\mathcal{M}', s \nVdash ST2$ because $s \overset{\{1,2\}}{\to}' t$ and $s <'_1 t, s <'_2 t$. Moreover, $Z = \{(w, s), (w, u), (v, t)\}$ is a $\cap$-bisimulation. $\square$

Proposition 2. *On* $\mathtt{N}-\mathtt{LTS}$*,* $PrefL4$ *is not invariant under* $\mathtt{GSM}$*.*

Proof. Let $\mathcal{M} = \langle \{w, v\}, \{1\}, \{\overset{\mathtt{C}}{\to} |\mathtt{C} \subseteq \{1\}\} = \emptyset, \{\leq_1\}, V\rangle$, where $v \leq_1 w, V(p) = \{v\}$. Then, $\mathcal{M}, w \Vdash PrefL4$ because $v \leq_1 w$ and $v \in V(p)$. But for the submodel $\mathcal{M}'$ generated by $\{w\}$, $\mathcal{M}', w \nVdash PrefL4$ since v is not contained in $\mathcal{M}'$. $\square$

6.2 Results for the Total Pre-orders (TPO) Case

Overview of the Results. This table shows rows that differ from the table for the general case. The entries that differ are in boldface.

	Bis	CBis	$\cap$-Bis	TBis	$\mathcal{H}$-Bis	$\mathcal{H}(@)$-Bis	$\mathcal{H}(\mathtt{E})$-Bis	BM	GSM	DU
$[PrefL8]$	N	**Y**	N	N	N	N	N	N	N	**Y***
$[PrefL9]$	N	**Y**	N	N	N	N	N	N	N	**Y***
$[ST2]$	N	N	**Y**	N	N	N	N	N	Y	Y*

Comments. Except for disjoint union (DU), the restriction to the TPO case brings only slight benefits. * marks trivial invariance: the only DU of models that is complete is the trivial one: mapping a model to itself.

Proposition 3. *On* $\mathtt{TPO}-\mathtt{N}-\mathtt{LTS}$*,* $EF1$ *is preserved under bounded morphisms.*

Proof. Let f be a bounded morphism from $\mathcal{M} = \langle W, \mathtt{N}, \{\overset{\mathtt{C}}{\to} |\mathtt{C} \subseteq \mathtt{N}\}, \{\leq_i |i \in \mathtt{N}\}, V\rangle$ to $\mathcal{M}' = \langle W', \mathtt{N}, \{\overset{\mathtt{C}}{\to}'|\mathtt{C} \subseteq \mathtt{N}\}, \{\leq'_i |i \in \mathtt{N}\}, V'\rangle$. Assume that $\mathcal{M}, w \Vdash EF1$ and $\mathcal{M}', f(w) \nVdash EF1$. Then, $\exists v'$ such that $f(w) <'_i v', \forall i \in \mathtt{N}$. Then, by the back condition of bounded morphisms, $\exists v_1, \ldots v_n \in W$ such that $w \leq_i v_i$ and $f(v_i) = v', \forall i \in \mathtt{N}$. Then, for all $i \in N : w <_i v_i$ because otherwise $f(v_i) = v' \leq'_i f(w)$, which contradicts $f(w) <_i v'$. Also, for every v_i such that $\exists i$ such that $w <_i v_i$, there is some j such that $w \not<_j v_i$ because $\mathcal{M}, w \Vdash EF1$. But since $\leq_j$ is *total* and $f(w) \neq v' = f(v_i)$, we conclude that $v_i \leq_j w$. Then $f(v_i) = v' \leq'_j f(w)$, which contradicts $f(w) <'_j v'$. Hence, $EF1$ is preserved under bounded morphisms. $\square$

6.3 Results for the TPO Case with Strict Preference Relation

Overview of the Results. The following table contains the rows that differ from the ones in the table for total preorders without strict preference relation.

	Bis	CBis	∩-Bis	TBis	$\mathcal{H}$-Bis	$\mathcal{H}$(@)-Bis	$\mathcal{H}$(E)-Bis	BM	GSM	DU
$[PrefL2]$	**Y**	**Y**	Y	**Y**	**Y**	**Y**	**Y**	**Y**	Y	Y
$[PrefL3]$	N	N	N	N	N	N	N	**Y**	Y	Y
$[PrefL5]$	N	**Y**	N	N	N	N	N	N	N	Y
$[PrefL6]$	N	N	N	N	N	N	N	**Y**	Y	Y
$[PrefL7]$	N	N	N	N	N	N	N	**Y**	Y	Y
$[PrefL11]$	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**Y** (4)	Y	Y
$[PPL7]$	N	N	N	N	N	N	N	**Y**	Y	Y
$[EF1]$	N	N	**Y**	N	N	N	N	N	Y	Y
$[EF2]$	N	N	**Y**	N	N	N	N	**Y**	Y	Y
$[ST1]$	N	N	**Y**	N	N	N	N	N	Y	Y
$[ST3]$	N	N	**Y**	N	N	N	N	N	Y	Y

Comments. The failures of invariance under GSM are still present, reflecting the fact that we do not have converse relations. By contrast, $PrefL11$ and $PrefL2$ are now invariant under bisimulation and a simple boolean modal logic with intersection seems to have the right expressive power to talk about efficiency and stability notions, since all of them are now invariant under ∩-Bisimulations. A representative result follows:

Proposition 4. *On* S/TPO-N-LTS*,* $PrefL11$ *is invariant under* BM*.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ be two $\texttt{S/TPO} - \texttt{N} - \texttt{LTS}$ and assume that f is a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$. Assume that the property $PrefL11$ does not hold for $\mathcal{M}, w$, i.e. there is a state $v \in Dom(\mathcal{M})$ such that $w <_i^{\mathcal{M}} v$ and $v \notin V^{\mathcal{M}}(p)$. But then by $\texttt{R} - \texttt{homomorphism}$, we have $f(w) <_i^{\mathcal{M}'} f(v)$ and by AtomicHarmony, $f(v) \notin V^{\mathcal{M}'}$, and thus $PrefL11$ does not hold for $\mathcal{M}', f(w)$. For the other direction assume that $PrefL11$ is not satisfied at $\mathcal{M}', f(w)$, it follows that there is a state $v' \in Dom(\mathcal{M}')$ such that $v' \notin V^{\mathcal{M}'}(p)$ but then by Back there is a state $v \in Dom(\mathcal{M})$ such that $f(v) = v'$ and $w <_i^{\mathcal{M}} v$. But by AtomicHarmony, $v \notin V^{\mathcal{M}}(p)$ and thus $PrefL11$ is not satisfied at $\mathcal{M}, w$, concluding our proof.

7 Closure Results for global notions: Validity

General Definitions. First, we consider bounded morphic images (BMI) of frames. BM on frames are obtained by dropping AtomicHarmony in Definition 7. A property is preserved under BMI iff it is preserved under *surjective* BM. Moreover, we consider closure under generated subframes (GSF) – the frame-analogue to GSM (cf. Definition 8). We will also check whether properties *reflect* GSF. A property ϕ *reflects* GSF if whenever for every frame $\mathcal{F}$, it holds that every GSF of $\mathcal{F}$ has property ϕ, then so does $\mathcal{F}$. We also consider closure under taking disjoint unions (DU) of frames, which are defined in the obvious way. Moreover, we look at closure under images of bisimulation systems [10], which are families of partial isomorphisms.

Definition 11 (Bisimulation System). *A bisimulation system from a frame* $\mathcal{F}$ *to a frame* $\mathcal{F}'$ *is a function* $\mathcal{Z} : \wp W' \to \wp(W \times W')$ *that assigns to each* $Y \subseteq W'$ *a total bisimulation* $\mathcal{Z}(Y) \subseteq W \times W'$ *such that for each* $y \in Y$*:*

1. *There is exactly one* $w \in W$ *such that* $(w, y) \in \mathcal{Z}(Y)$*.*
2. *If* $(w, y), (w, w') \in \mathcal{Z}(Y)$*, then* $w' = y$*.*

7.1 Validity Preservation Results

	BMI	GSF	DU	refl.GSF	BisSysIm		BMI	GSF	DU	refl.GSF	BisSysIm
$[PowG1]$	Y	Y	Y	Y	Y	$[PrefG8]$	N	Y	Y	Y	Y
$[PowG2]$	Y	Y	Y	Y	Y	$[PrefG9]$	N	Y	Y	Y	Y
$[PowG3]$	Y	Y	Y	Y	Y	$[PrefG10]$	Y	Y	Y	Y	Y
$[PowG4]$	Y	Y	Y	Y	Y	$[PrefG11]$	N(5)	Y	Y	Y	Y
$[PowG5]$	Y	Y	Y	Y	Y	$[PPG1]$	Y	Y	Y	Y	Y
$[PrefG1]$	Y	Y	Y	Y	Y	$[PPG2]$	Y	Y	N	N	Y
$[PrefG2]$	Y	Y	Y	Y	Y	$[PPG3]$	N	N	N	Y	N?
$[PrefG3]$	Y	Y	N	N	Y	$[PPG4]$	N	Y	Y	Y	Y
$[PrefG4]$	Y	Y	N	N	Y	$[PPG5]$	Y	Y	Y	Y	Y
$[PrefG5]$	N	Y	Y	Y	Y	$[PPG6]$	Y	Y	Y	Y	Y
$[PrefG6]$	N	Y	N	N	Y	$[PPG7]$	Y	N	Y	Y	Y
$[PrefG7]$	N	Y	N	N	Y						

Comments. At the frame validity level, modal logic is a fragment of Monadic Second Order Logic. That it does better at this level is thus not only an artifact of the chosen notions.

Proposition 5. *Validity of* $PrefG11$ *is not preserved under* BMI*.*

Proof. Consider the frames $\mathcal{F} = \langle\{u, v, w\}, \{1, 2\}, \{\stackrel{\mathtt{C}}{\to} \mid \mathtt{C} \subseteq \{1, 2\}\}, \{\leq_1, \leq_2\}\rangle$, with $w \stackrel{1}{\to} v, w \stackrel{2}{\to} v, \stackrel{\{1,2\}}{\to} = \emptyset, w \leq_1 v, w \leq_2 u$ and $\mathcal{F}' = \langle\{s, t\}, \{1, 2\}, \{\stackrel{\mathtt{C}}{\to} \mid \mathtt{C} \subseteq \{1, 2\}\}, \{\leq'_1, \leq'_2\}\rangle$, with $s \stackrel{1}{\to} t, s \stackrel{2}{\to} t, \stackrel{\{1,2\}}{\to} = \emptyset, s \leq_1 t, s \leq_2 t$. Then $f : W \to W', f(w) = s, f(v) = f(u) = t$ is a surjective BM. However, $\mathcal{F} \Vdash PrefG11$ and $\mathcal{F}' \nVdash PrefG11$ because $s \leq_1 t, s \leq_2$ and it is not the case that $s \stackrel{\{1,2\}}{\to} t$.

8 Modal Definability

The model-theoretic results of previous sections do give us some information about possible definability results. However, let us be more constructive and give formulas that indeed do the job: be it for local-satisfaction or frame-definability aims. We indicate the“best” (least expressive) language we found still being able to express the property under consideration. Another useful indicator is that of the computational complexity of the logic. More precisely to its satisfiability problem (SAT). and model checking problem (MC). Since we lack the space to discuss these issues in depth here is how we bridge our expressivity and complexity results: for each local (resp. global) notion, find the least expressive logic that is still able to express it locally (resp. define the class of frames corresponding to it) and take the complexity of this logic as an *upper bound*. We assume the reader to be familiar with the the classes PSPACE and EXPTIME (see [16]). Π^0_1 is a notation from the arithmetical hierarchy (see [17]). Problems in Π^0_1 are undecidable but co-recursively enumerable. A typical problem establishing Π^0_1-hardness is $\mathbb{N} \times \mathbb{N}$ tiling (see [11, 18]). We start by giving an overview of the known complexity results of the satisfiability problem for extended modal languages.

8.1 Satisfiability Problem for Extended Modal Languages

A piece of notation: in the following table, a language $\mathcal{L}$ after the lower bound indicates that the lower bound follows from the fact the current language extends $\mathcal{L}$, and $\rho \leftarrow \mathcal{L}$ that there is a polynomial reduction of $\mathcal{L}$ to the current language, while $\sigma \rightarrow \mathcal{L}$ in the Upper Bound part means that there exists a polynomial translation of the current language to the modal language $\mathcal{L}$. The first column shows whether the satisfiability problem is decidable.

	Dec.	Lower Bound for SAT	Upper bound for SAT
$\mathcal{ML}$	Y	PSPACE-hard (Prenex QBF) [19]	PSPACE [19, 20]
$\mathcal{ML}(\cup, \ ; \ )$	Y	PSPACE-hard ($\mathcal{ML}$)	PSPACE ($\sigma \rightarrow \mathcal{ML}$)
$\mathcal{ML}(\ ^{-1})$	Y	PSPACE-hard	PSPACE
$\mathcal{ML}(\cap)$	Y	PSPACE-hard ($\mathcal{ML}$)	PSPACE
$\mathcal{ML}(\cap, \cup)$	Y	PSPACE-hard ($\mathcal{ML}$)	PSPACE [21]
$\mathcal{ML}(i)$	Y	PSPACE-hard ($\mathcal{ML}$)	PSPACE [22]
	Dec.	Lower Bound for SAT	Upper bound for SAT
$\mathcal{ML}(@, i)$	Y	PSPACE-hard ($\mathcal{ML}$)	PSPACE (MCG [7]) [22]
$\mathcal{ML}(\mathtt{E})$	Y	EXPTIME-hard [23, 24]	EXPTIME
$\mathcal{ML}(\mathtt{E}, i)$	Y	EXPTIME-hard ($\mathtt{E}$)	EXPTIME
$\mathcal{ML}(\ ^{-1}, \mathtt{E})$	Y	EXPTIME-hard ($\mathtt{E}$)	EXPTIME
$\mathcal{ML}(\ ^{-1}, i)$	Y	EXPTIME-hard [22] Red. of $\mathcal{M} \Vdash \phi$ [8]	EXPTIME $\sigma \rightarrow$ 2VGF [9] [25]
PDL	Y	EXPTIME-hard (Corridor tiling) [26]	EXPTIME (Elimination of Hintikka Sets) [11]
$\mathcal{ML}(\ ^{-1}, \cup, \ ; \ , *)$	Y	EXPTIME-hard (**PDL**)	EXPTIME
$\mathcal{ML}(\ ^{-1}, \cup, \ ; \ , *, \mathtt{E})$	Y	EXPTIME-hard (**PDL**)	EXPTIME
$\mathcal{ML}(\ ^{-1}, \cup, \ ; \ , *, \mathtt{E}, i)$	Y	EXPTIME-hard (**PDL**)	EXPTIME
$\mathcal{ML}(\text{-})$	Y	EXPTIME-hard $\rho \leftarrow$ fragment of $\mathtt{E}$	EXPTIME [27, sec.5]
$\mathcal{ML}(\cap, \text{-})$	Y	EXPTIME-hard ($\mathcal{ML}(\text{-})$)	EXPTIME [27, sec.5]
$\mathcal{ML}(\cup, \text{-})$	Y	EXPTIME-hard ($\mathcal{ML}(\text{-})$)	EXPTIME [27, sec.5]
$\mathcal{ML}(\cap, \cup, \text{-})$	Y	EXPTIME-hard ($\mathcal{ML}(\text{-})$)	EXPTIME [27, sec.5]
$\mathcal{ML}(\downarrow, @, x)$	N	Π^0_1-hard (reduction class) [10] ($\mathbb{N} \times \mathbb{N}$ tiling) [28]	Π^0_1
$\mathcal{L}^1$	N	Π^0_1-hard (reduction class) [29]	Π^0_1
$\mathcal{ML}(\mathtt{E}, \ ; \ , \cap)$	N	Π^0_1-hard ($\mathbb{N} \times \mathbb{N}$ tiling) [11]	Π^0_1

8.2 Defining Local Notions

	Local Formula	Best Language	SAT	MC
$PowL1$	$\langle \mathtt{C} \rangle p$	$\mathcal{ML}(\mathtt{N})$	PSPACE[19, 20]	P
$PowL2$	$\bigwedge_{\mathtt{C} \not\supseteq} [\mathtt{C}] \neg p$	$\mathcal{ML}(\mathtt{N})$	PSPACE[19, 20]	P
$PowL3$	$\downarrow x.[\mathtt{D}] \downarrow y.@_x \langle \mathtt{C} \rangle y$	$\mathcal{ML}(\mathtt{N}, \downarrow, @, x)$	EXPTIME	PSPACE[30]
$PrefL1$	$\langle \leq_i \rangle$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefL2$	$\downarrow x.\langle \leq_i \rangle (p \wedge [\leq_i] \neg x)$	$\mathcal{ML}(\mathtt{N}, \downarrow, x)$	EXPTIME	PSPACE[30]
$PrefL3$	$\downarrow x.\langle \bigcap_{i \in N} \leq_i \rangle (\bigvee_{j \in \mathtt{N}} [\leq_j] \neg x)$	$\mathcal{ML}(\mathtt{N}, \downarrow, \cap, x)$	Π_1^0	PSPACE
$PrefL4$	$\langle {\leq_i}^{-1} \rangle p$	$\mathcal{ML}(\mathtt{N}, \downarrow, @, x)$	PSPACE	PSPACE[30]
$PrefL5$	$\downarrow x.\langle {\leq_i}^{-1} \rangle (p \wedge [{\leq_i}^{-1}] \neg x)$	$\mathcal{ML}(\mathtt{N}, \downarrow, {}^{-1}, x)$	Π_1^0	PSPACE
$PrefL6$	$[(\leq_i \cap \overline{\leq_j}) \cup (\leq_j \cap \overline{\leq_i})] \bot$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PrefL7$	$\langle \leq_i \cap (\bigcap_{j \in \mathtt{N} - \{i\}} \overline{\leq_j}) \rangle \top$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PrefL8$	$[\![\ \leq_i]\!] p$	$\mathcal{ML}(\mathtt{N}, [\![\ ]\!])$	EXPTIME	P
$PrefL9$	$\downarrow x.\mathtt{A} \downarrow y.(\neg \langle \leq_i \rangle x \wedge @_x \langle \leq_i \rangle y)$	$\mathcal{ML}(\mathtt{N}, \downarrow, @, x, \mathtt{E})$	Π_1^0	PSPACE[32]
$PrefL10$	$[\leq_i] p$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefL11$	$\downarrow x.[\leq_i]([\leq_i] \neg x \rightarrow p)$	$\mathcal{ML}(\downarrow, x)$	EXPTIME	PSPACE[30]
$PPL1$	$\langle \mathtt{C} \cap \leq_i \rangle \top$	$\mathcal{ML}(\mathtt{N}, \cap)$	PSPACE	P[31]
$PPL2$	$\langle \mathtt{C} \cap (\bigcap_{i \in \mathtt{D}} \leq_i) \rangle \top$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PPL3$	$\langle (\bigcap_{i \in \mathtt{N}} \leq_i) \cap (\bigcup_{\mathtt{C} \subseteq \mathtt{N}} \overset{\mathtt{C}}{\rightarrow}) \rangle \top$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PPL4$	$[\overline{\mathtt{C}} \cap \leq_i] \bot$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PPL5$	$\downarrow x.[\overline{\mathtt{C}} \cap \leq_i] \langle \leq_i \rangle x$	$\mathcal{ML}(\mathtt{N} \downarrow, , -, \cap, x)$	Π_1^0	PSPACE
$PPL6$	$\bigvee_{\mathtt{C} \subseteq \mathtt{N}} [\mathtt{C} \cap \overline{\leq_i}] \bot$	$\mathcal{ML}(\mathtt{N}, -, \cap)$	EXPTIME	P[31]
$PPL7$	$\downarrow x.[\mathtt{C}] \downarrow .y.(\neg \langle \leq_i \rangle x \wedge @_x \langle \leq_i \rangle y)$	$\mathcal{ML}(\mathtt{N}, \downarrow, @, x)$	Π_1^0	PSPACE[30]
$EF1$	$\downarrow x.[\bigcap_{i \in \mathtt{N}} \leq_i] \bigvee_{i \in \mathtt{N}} \langle \leq_i \rangle x$	$\mathcal{ML}(\mathtt{N}, \downarrow, \cap)$	Π_1^0	PSPACE
$EF2$	$\neg \downarrow x.\langle \bigcap_{i \in N} \leq_i \rangle (\bigvee_{j \in \mathtt{N}} [\leq_j] \neg x)$	$\mathcal{ML}(\mathtt{N}, \downarrow, \cap)$	Π_1^0	PSPACE
$EF3$	$[\bigcap_{i \in \mathtt{N}} \leq_i] \bot$	$\mathcal{ML}(\mathtt{N}, \cap)$	PSPACE	P[31]
$ST1$	$\bigwedge_{i \in \mathtt{N}} \downarrow x.[i \cap \leq_i] \langle \leq_i \rangle x$	$\mathcal{ML}(\mathtt{N}, \downarrow, \cap)$	Π_1^0	PSPACE
$ST2$	$\bigwedge_{i \in \mathtt{N}} [i \cap \leq_i] \bot$	$\mathcal{ML}(\mathtt{N}, \cap)$	PSPACE	P[31]
$ST3$	$\bigwedge_{\mathtt{C} \subseteq \mathtt{N}} \downarrow x.[\mathtt{C} \cap (\bigcap_{i \in \mathtt{C}} \leq_i)] \bigvee_{j \in \mathtt{C}} \langle \leq_j \rangle x$	$\mathcal{ML}(\mathtt{N}, \downarrow, \cap)$	Π_1^0	PSPACE
$ST4$	$\bigwedge_{\mathtt{C} \subseteq \mathtt{N}} [\mathtt{C} \cap (\bigcap_{i \in \mathtt{C}} \leq_i)] \bot$	$\mathcal{ML}(\mathtt{N}, \cap)$	PSPACE	P[31]

8.3 Defining Global Notions

First of all, we define what it means for a formula to be valid on a class of frames.

Definition 12 (Validity on a class of frames). *We say that a formula ϕ is valid on a class of frames $\mathtt{F}$ iff for any frame $\mathcal{F} \in \mathtt{F}$ and any model $\mathcal{M}$ based on $\mathcal{F}$, at all states w in $Dom(\mathcal{F})$, $\mathcal{M}, w \Vdash \phi$. We write $\mathtt{F} \Vdash \phi$.*

Modal definability has again two sides: We can look for a formula ϕ such that $\mathcal{M}, w \Vdash \phi$ iff $\mathcal{M}, w$ has some property, or such that $\mathtt{F} \Vdash \phi$ iff $\mathtt{F}$ has the property.

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[7] Model Construction Game
[8] global satisfaction
[9] Two-Variable guarded fragment

	Axiom	Best Language	SAT	MC
$PowG1$	$\bigwedge_{\mathtt{C}\subseteq\mathtt{N}}(((\langle\mathtt{C}\rangle\phi \rightarrow [\mathtt{C}]\phi) \wedge \langle\mathtt{C}\rangle\top)$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PowG2$	$\bigwedge_{\mathtt{C}:\|\mathtt{C}\|<\|\mathtt{N}\|/2}[\mathtt{C}]\bot$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PowG3$	$\bigwedge_{\mathtt{C}\subseteq\mathtt{N}}(\langle\mathtt{C}\rangle\phi \rightarrow [\mathtt{C}]\phi)$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PowG4$	$\bigwedge_{\mathtt{C}\subseteq\mathtt{N}}\bigwedge_{\mathtt{D}\supseteq\mathtt{C}}(\langle\mathtt{C}\rangle\phi \rightarrow \langle\mathtt{D}\rangle\phi)$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PowG5$	$\langle\mathtt{C}\rangle\top \rightarrow \bigwedge_{\mathtt{D}:\mathtt{C}\cap\mathtt{D}=\emptyset}(\langle\mathtt{D}\rangle\phi \rightarrow \langle\mathtt{C}\rangle\phi)$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG1$	$\phi \rightarrow \langle\leq_i\rangle\phi$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG2$	$\langle\leq_i\rangle\langle\leq_i\rangle\phi \rightarrow \langle\leq_i\rangle\phi$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG3$	$p \wedge \mathtt{E}q \;\rightarrow\; (\mathtt{E}(p \wedge \langle\leq_i\rangle q) \vee \mathtt{E}(q \wedge \langle\leq_i\rangle p))$	$\mathcal{ML}(\mathtt{N},\mathtt{E})$	EXPTIME	P[31]
$PrefG4$	Conjunction of the 3 previous axioms	$\mathcal{ML}(\mathtt{N},\mathtt{E})$	EXPTIME	P[31]
$PrefG5$	see below	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG6$	$\bigwedge_{i\in\mathtt{N}}(@_j\langle\leq_i\rangle k \vee @_k j \vee @_k\langle\leq_i\rangle j)$	$\mathcal{ML}(\mathtt{N},@,i)$	PSPACE	P[30]
$PrefG7$	$[PrefG5] \wedge [PrefG6] \wedge (\bigwedge_{i\in\mathtt{N}}(j \rightarrow \neg\langle\leq_j\rangle j))$	$\mathcal{ML}(\mathtt{N},@,i)$	PSPACE	P[30]
$PrefG8$	$\bigwedge_{i\in\mathtt{N}}\;(((\langle\leq_i\rangle\phi \rightarrow [\leq_i]\phi) \wedge \langle\leq_i\rangle\top)$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG9$	$\langle\mathtt{C}\rangle i \leftrightarrow \bigwedge_{i\in\mathtt{C}}\langle\leq_i\rangle i$	$\mathcal{ML}(\mathtt{N},i)$	PSPACE	P[30]
$PrefG10$	$\langle\mathtt{C}\rangle p \leftrightarrow \bigvee_{i\in\mathtt{C}}\langle\leq_i\rangle p$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PrefG11$	$\langle\mathtt{C}\rangle i \leftrightarrow \bigvee_{\mathtt{D}\subseteq\mathtt{C}\&\|\mathtt{D}\|>\frac{\|\mathtt{C}\|}{2}}(\bigwedge_{i\in\mathtt{D}}\langle\leq_i\rangle i)$	$\mathcal{ML}(\mathtt{N},i)$	PSPACE	P[30]
$PPG1$	$\langle\mathtt{C}\rangle\phi \rightarrow \bigwedge_{i\in\mathtt{N}}\langle\leq_i\rangle p$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PPG2$	$\bigvee_{i\in\mathtt{N}}\mathtt{A}\bigwedge_{\mathtt{C}\subseteq\mathtt{N}}(\langle\mathtt{C}\rangle\phi \rightarrow \langle\leq_i\rangle\phi)$	$\mathcal{ML}(\mathtt{N},\mathtt{E})$	EXPTIME	P
$PPG3$	$\bigwedge_{i\in\mathtt{N}}\bigvee_{\mathtt{C}\subseteq\mathtt{N}}\langle\overline{\leq_i}\cup\leq_i\rangle\langle\overline{\leq_i}\cap\mathtt{C}\rangle\top$	$\mathcal{ML}(\text{-},\cap,\cup)$	EXPTIME	P[31]
$PPG4$	see below	$\mathcal{ML}(\mathtt{N},i)$	PSPACE	P[30]
$PPG5$	$\bigwedge_{\mathtt{C}\not\supseteq\{i\}}[\xrightarrow{\mathtt{C}}]\bot$	$\mathcal{ML}(\mathtt{N})$	PPSPACE	P
$PPG6$	$\langle\mathtt{C}\rangle\phi \rightarrow \bigvee_{\mathtt{D}\subset\mathtt{C}}\langle\mathtt{D}\rangle\phi$	$\mathcal{ML}(\mathtt{N})$	PSPACE	P
$PPG7$	$\bigwedge_{i\in\mathtt{N}}\mathtt{E}\bigwedge_{\mathtt{C}\not\supseteq\{i\}}[\mathtt{C}]\bot$	$\mathcal{ML}(\mathtt{N},\mathtt{E},i)$	EXPTIME	P[30]

The two missing axioms of the previous table.

$$\bigwedge_{i\in\mathtt{N}} (p \wedge \langle\leq_i\rangle(q \wedge \neg\langle\leq_i\rangle p \wedge \langle\leq_i\rangle(r \wedge \neg\langle\leq_i\rangle q))) \rightarrow p \wedge \langle\leq_i\rangle(r \wedge \neg\langle\leq_i\rangle p) \qquad (AxPrefG5)$$

$$[p \wedge \langle\{i\}\rangle q \wedge \langle\leq_i\rangle(q \wedge \langle\leq_i\rangle\neg p)] \rightarrow \bigwedge_{\{i\}\subseteq\mathtt{C}\subseteq\mathtt{N}}[(\langle\mathtt{C}\rangle r \wedge \bigwedge_{\mathtt{D}\subseteq\mathtt{C}-\{i\}}\neg\langle\mathtt{D}\rangle r) \rightarrow \langle\leq_i\rangle(r \wedge \neg\langle\leq_i\rangle p)] \qquad (AxPPG4)$$

Representative definability results.

Proposition 6. *$PowL3$ is true of $\mathcal{M}, w$ iff $\mathcal{M}, w, g \Vdash \downarrow x.[\mathtt{D}] \downarrow y.@_x\langle\mathtt{C}\rangle y$.*

Proof. From right to left: Assume that $\mathcal{M}, w, g \Vdash \downarrow x.[\mathtt{D}] \downarrow y.@_x\langle\mathtt{C}\rangle y$. Then $\mathcal{M}, w, g[x := w], \Vdash [\mathtt{D}] \downarrow y.@_x\langle\mathtt{C}\rangle y$. But now assume there is a state v that coalition $\mathtt{D}$ can force from w. By definition, $w \xrightarrow{\mathtt{D}} v$ (1). But by (1) and semantics of $[\mathtt{D}]$ then we have $\mathcal{M}, v, g[x := w], \Vdash \downarrow y.@_x\langle\mathtt{C}\rangle y$ (2). (2) and semantics of $\downarrow$ gives us $\mathcal{M}, v, g[x := w, y := v] \Vdash @_x\langle\mathtt{C}\rangle y$ (3). From (3) and semantics of $@_x$ and the fact that $g(x) = w$ we have $\mathcal{M}, w, g[x := w, y := v] \Vdash \langle \mathtt{C} \rangle y$ (4). But by semantics of $\langle \mathtt{C} \rangle$ and the fact that $g(y) = v$, (4) really means that $w \overset{\mathtt{C}}{\to} v$ (5). Since the v was arbitrary, it follows from (5) that at w for any state v, if D can achieve it, then C can do so, too. But this precisely means that $PowL3$ is true of $\mathcal{M}, w$. $\square$

Theorem 1 (ten Cate [28]). *The satisfiability problems for formulas in* $\mathcal{ML}(\mathtt{N}, \downarrow, @, x) - \square \downarrow \square$ *with bounded width is* EXPTIME*-complete.*

Proposition 7. *$PowL3$ is expressible in an extended modal language with a satisfiability problem in* EXPTIME*.*

Proof. By the previous proposition, we have $PowL3$ is defined by $\downarrow x.[\mathtt{D}] \downarrow y.@_x \langle \mathtt{C} \rangle y$. But $\downarrow x.[\mathtt{D}] \downarrow y.@_x \langle \mathtt{C} \rangle y$ does contain the $\square \downarrow \square$ scheme. Thus, $PowL3$ is defined by a formula in $\mathcal{ML}(\mathtt{N}, \downarrow, @, x) - \square \downarrow \square$ (1). But by Theorem 1 the satisfiability problem of $\mathcal{ML}(\mathtt{N}, \downarrow, @, x) - \square \downarrow \square$ is in EXPTIME. $\square$

9 Conclusion

We identified a set of natural a set of natural notions for reasoning about cooperation: local notions giving properties of a state of a given system and global notions defining a class of frames. We provided satisfiability (resp. validity) invariance results for these notions for a large class of operations and relations between models (resp. frames). We also gave explicit definability results and observed that defining frames for cooperation logics does not seem too demanding in terms of expressive power, as most of the notions considered are definable in the basic modal language. On the other hand, our results show that local notions call for modal logics for which satisfaction is not *invariant* under bounded morphisms. However, as long as we avoid converse modalities, interesting reasoning about cooperation can be done within GSM-invariant modal languages. Though this fact does not directly lead to a nice upper bound on the complexity of the logic's SAT(nor to its decidability), our definability results show that most of the considered notions can (invidually) be expressed in MLs in EXPTIME. Based on our current work, the following lines seem worth exploring:

- Since dealing with real coalitional powers is probably more natural using neighborhood semantics, it will be useful to do the same work for modal logics of the **CL**-type or of the type of one of its normal simulations [2].
- It would be interesting to obtain similar invariance results and upper bounds on the complexity of the logics needed to encode *concrete arguments* from social choice theory and (cooperative) GT, thus addressing the complexity of *actual reasoning* about cooperative situations.
- In order to obtain a complete picture of the complexity of reasoning about cooperation, we need a procedure to assess the LB of the complexity of modal logics that can express some notion. Moreover, the complexity of some aspects of cooperation may be captured rather by the complexity of model checking than by that of SAT. Thus, future work also includes determining the model checking complexity for notions that are of interest in reasoning about cooperation.

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