Questions and Answers in an Orthoalgebraic Approach

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Abstract

Taking the lead from orthodox quantum theory, I will introduce a handy generalization of the Boolean approach to propositions and questions: the ortho-algebraic framework. I will demonstrate that this formalism relates to a formal theory of questions (or 'observables' in the physicist's jargon). This theory allows to formulate conditioned questions such as "if electron 1 has spin ↑ what is the spin of electron 2?", and thus gives it the semantic power of inquisitive semantics. In the case of commuting observables, there are close similarities between the ortho-algebraic approach to questions and the Jäger/Hulstijn approach to inquisitive semantics. However, there are also differences between the two approaches even in case of commuting observables. The main difference is that the Jäger/Hulstijn approach relates to a partition theory of questions whereas the ortho-algebraic approach relates to a 'decorated' partition theory (i.e. the elements of the partition are decorated by certain semantic values). Surprisingly, the ortho-algebraic approach is able to overcome most of the difficulties of the Jäger/Hulstijn approach. It will be shown that the present decorated partition theory is fully compatible with the structured meaning approach to questions assuming the latter can be extended to include conditioned questions. Concluding, I will suggest that an active dialogue between the traditional model-theoretic approaches to semantics and the ortho-algebraic paradigm is mandatory.

1 Introduction

There is a close analogy between the semantics of questions as developed by Groenendijk & Stokhof (e.g. Groenendijk & Stokhof, 1984a, 1997) and the formal treatment of observables in Quantum Physics (e.g. Birkhoff & von Neumann, 1936; Von Neumann, 1932). In both cases a question (observable) partitions the state space in equivalent classes where two states are equivalent if they give the same answer (the same result of measuring the observable).

Quantum physics reduces to classical physics if all observables are commuting. It could be expected that in this borderline case the treatment of observables coincides in quantum physics and Groenendijk/Stokhof's (GS for short) partition semantics. However this is not the case. The main reason is that in quantum theory the partitions are decorated (with the relevant eigenvalues of the projecting eigenspaces). The GS partition semantics does not know any decorations. Hence, questions such as in (1a) an (1b) are considered equivalent assuming that ‘open’ and ‘not closed’ are semantically equivalent.[1]

(1) a. Is the door open?
b. Is the door closed?
c. Peter knows if the door is open
d. Peter knows if the door is closed

Consequently, the equivalence of (1c) and (1d) comes out automatically as a result of the equivalence between (1a) and (1b). Though not doubting the equivalence between (1c) and (1d) there are some doubts about the semantic equivalence of the two questions (1a) and (1b) (see Krifka, 2001).

Another difference between the two treatments has to do with the analysis of conditional questions such as in (2)[2]:

(2) If electron 1 has spin ↑ what is the spin of electron 2?

Using the operator formalism of quantum theory it isn’t difficult to formalize the content of (2). However, the standard GS partition theory does not introduce a conditional operator general enough to express conditional questions. To be sure, the GS partition theory makes a strict distinction between questions and answers. Semantically, questions are described by an equivalence relation and answers are described by propositions (sets of possible worlds). Surprisingly, such a distinction is not made in quantum physics where both questions (observables) and answers (projection spaces) can be treated as particular linear operators. It is exactly this uniform treatment that allows a straightforward treatment of conditional questions.

Recently, Groenendijk (2008) started a programme to unify the treatment of question and answers. Unfortunately, this programme eliminates the partition theory and is not longer compatible with the quantum theoretic tradition of treating observables. However, there is also one treatment that extends the GS question theory to conditional questions without giving up the partition idea. This is the Jäger/Hulstijn (JH for short) approach to inquisitive semantics (Hulstijn, 1997; Jäger, 1996). In the JH approach a new operator is designed for modelling conditional questions.

In the present paper a new solution to unify the analysis of questions and answers will be proposed based on a decorated partition theory. This solution bears a close resemblance to the JH theory. However, there are also some important differences, and I will demonstrate how several shortcomings of the JH approach can be overcome by using the decorated partition theory including the operator formalism known from quantum theory.

The quantum approach to observables/questions includes its classical (partition-semantic) counterpart but it cannot be reduced to it. It’s much more general. The point is that an observable/question has definite values only in the eigenstates of the question/observable under discussion. If a system is in a certain state that is not an eigenstate of the question/observable, then the system first moves into a corresponding eigenstate where the question gets a definite answer. This change of the state – silently influenced by the question asked (or by the measurement that is performed) is characteristic for the micro-world of electrons, photons and other so-called elementary particles. Recently, it has been argued that the very same phenomenon is observed in the cognitive realm. The mind is very sensitive to

[1] If you don’t like the examples with ‘open’ and ‘closed’ because you feel the relevant constructions aren’t really semantically equivalent, you can construct similar examples using ‘even’ and ‘odd’.

[2] The construction of such examples is very common in the literature that interprets the famous EPR thought experiment (Einstein, Podolsky, & Rosen, 1935).

context. Asking a question is enough to change the state of the mind such that the system's answer to a particular question may depend on other questions asked before. This is a common observation in the context of opinion building and diagnostic questions (e.g. Aerts, Czachor, & D'Hooghe, 2005). Only recently interest was generated in this new and fascinating approach to understanding cognition based on quantum information processing principles (Aerts, Broekaert, & Gabora, 2006; Aerts & Gabora, 2005a, 2005b; Busemeyer, Wang, & Townsend, 2006; Franco, 2007a, 2007c; Geissler, Klix, & Scheidereiter, 1978; Graben, 2004; Graben & Atmanspacher, 2006; Khrennikov, 2003a, 2003b; Primas, 2007). Even though this is an attractive research topic and – in principle – the developed framework is able to deal with the relevant observations, it goes beyond the present squib and must be left for another occasion.

Hence, in the present paper I restrict myself to commuting observables/questions. This allows to focus on the similarities and differences between the question theories developed by students of formal semantics and those developed independently (and much earlier) by physicists.

The organization of this paper is as follows. In the next section I give a concise introduction to inquisitive semantics as developed in the JH approach. Section 3 introduces the basic ortho-algebraic framework as it has been used in quantum information theory. In section 4 I develop my ortho-algebraic semantics for question and answer. The model deviates in some respects from the inquisitive semantics treated in section 3; both the differences and the strict similarities are discussed. A comparison with alternative approaches is made in section 5. Surprisingly, there is a close similarity between the so-called *structured meaning approach* and the present theory of decorated partitions. This suggests extending the former approach to conditional questions along the lines suggested by the latter theory. Section 6 finally draws some general conclusions.

2. Inquisitive Semantics: An Introduction

Many recent analyses of the meaning of questions start with three assumptions: (i) to understand a question is to understand what counts as an answer to that question; (ii) an answer to a question is an assertion or a statement; (iii) an assertion is identical with its propositional content (cf. Groenendijk & Stokhof, 1997, p. 1066). Different approaches that fill into this scheme are (a) Hamblin (1973) who identifies a question with the set of propositional contents of its possible answers, (b) Karttunen (1977) for whom it is the smaller set of its true answers, and (c) the GS partition theory (cf. Groenendijk & Stokhof, 1984b, 1997) defining the meaning of a question as the set of its complete answers. Krifka (2001) categorises these theories under the label *proposition set approach* and contrasts it with the so-called *structured meaning approach* (e.g., Hausser, 1983; Loeser, 1968; for more references see Krifka, 2001). In this latter approach, the answers to *wh*-questions are identified with the senses of noun phrases rather than those of sentences. Accordingly, the meanings of questions are constructed as functions that yield a proposition when applied to the semantic value of the answer (see section 5 for more discussion).

Inquisitive semantics is a version of update semantics which takes into account that sentences not only provide data, but also raise issues. In the GS theory these two tasks are strictly divided over two syntactic categories: declarative sentences provide data and interrogative sentences raise issues. This strategy has its limitations, e.g., it does not allow us to represent conditional questions.[3] Recent developments of inquisitive semantics deviate

[3] There are two other potential shortcomings, but the discussion of them is beyond the scope of the present paper: (i) a proper treatment of hybrid expressions such as disjunctions which act as questions and assertions; (ii) the account for certain typological facts that demand a unification of question and declarative semantics (cf. Groenendijk 2008).

from the classical picture in different ways. Some writers claim it is sufficient to *modify* classical partition theory in order to adapt it for the purpose of conditioned questions (Hulstijn, 1997; Jäger, 1996). Others claim classical partition semantics has to be given up for the same purpose (Velissaratou, 2000).

Obviously, the simplest way to unify questions and answers is to adapt partition semantics by saying that not the whole domain of possible worlds has to be partitioned but only a subpart of it. A proposition then can be seen as partitioning the set of all worlds that make the proposition true into a partition consisting just of one element: the set of worlds that make the proposition true. A conditional question then partitions the set of all worlds where the antecedent of the conditional is true. As we will see immediately, such a version conforms to the JH approach. Surprisingly, this is the variant of partition theory that most naturally results from the ortho-algebraic approach as used in quantum theory.

Following the JH approach (Hulstijn, 1997; Jäger, 1996) we can formulate the following clauses:

(3) a. $\sigma«p» = \sigma \cap \{(u,v) \in W^2: u \in \omega(p) \text{ and } v \in \omega(p)\}$

b. $\sigma«\neg\varphi» = \sigma \cap \{(u,v) \in W^2: (u,u) \notin «\varphi» \text{ and } (v,v) \notin «\varphi»\}$

c. $\sigma«\varphi \wedge \psi» = \sigma«\varphi»«\psi»$

d. $\sigma«?\varphi» = \{(u,v) \in \sigma: (u,u) \in \sigma«\varphi» \text{ iff } (v,v) \in \sigma«\psi»\}$

e. $\sigma«\varphi \Rightarrow \psi» = \{(u,v) \in \sigma«?\varphi»: \text{if } (u,v) \in \sigma«\varphi» \text{ then } (u,v) \in \sigma«\varphi»«\psi»$

For atomic formulas *p*, the first clause expresses the elimination of all possibilities incompatible with *p*. Negation is modelled in (3b) by set complement. The use of the intersection operator in the definition makes sure that negation is a so-called declarative update (cf. Hulstijn, 1997). In (3c) conjunction is modelled by function composition on updates leading to a sequential notion of conjunction. The standard definition (3d) defines question by equivalent relations where two worlds are considered equivalent if they give the same answer to question ?φ.

In the JH framework, the standard definition for $\vee$ is used: $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$. In order to model conditional questions the standard implication $\varphi \rightarrow \psi \equiv \neg(\varphi \wedge \neg\psi)$ cannot be used. The reason is that the clause for negation is declarative, i.e. no structure can be induced under the scope of negation. But conditional questions give an interesting structure and for this reason JH have proposed an alternative definition for conditionals, $\Rightarrow$, as shown in definition (3e). In this definition the restriction $(u,v) \in \sigma«?\varphi»$ is required. That means the antecedent of the conditional must become an issue. Leaving out this restriction it is not longer guaranteed hat the result is an equivalence relation (cf. Hulstijn, 1997, footnote 10).

For an illustration we consider a fragment with two atoms p and q. Identifying possible worlds with functions assigning the truth values 1 (true) and 0 (false) to the atoms, we get four possible worlds abbreviated by 10, 11, 01, 00. Interpreting atoms by sets of worlds in which the atoms are true gives the obvious assignments ω(p) = {10, 11} and ω(q) = {01, 11}. Figure 1 shows the meaning of p in inquisitive semantics. Here we are concerned with a single equivalence class that captures the logical space of p (set of worlds that make p true).

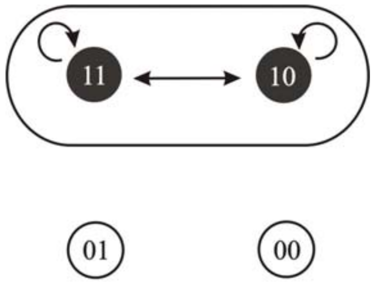

Figure 1: Picture of meaning p (assertion) in inquisitive semantics

Figure 2 pictures the meaning of ?p in inquisitive semantics. It is constituted by two equivalence classes which partition the space W of possible worlds.

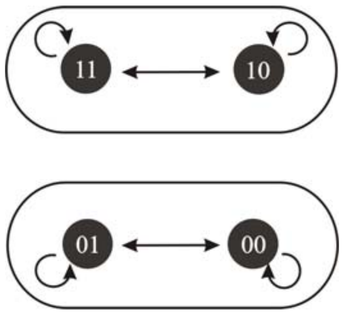

Figure 2: Picture of meaning ?p in inquisitive semantics

The meaning of the conditional interrogative $p \Rightarrow ?q$ is pictured in figure 3. It is the partition of the logical space consisting of three blocks. The blocks of the partition correspond to the propositions expressed by $p \wedge q$, $p \wedge \neg q$ and $\neg p$.

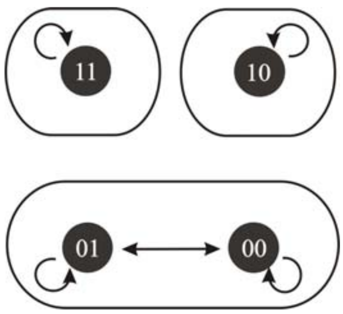

Figure 3: Picture of meaning $p \Rightarrow ?q$ in inquisitive semantics

There is a controversy about this result, mainly concentrated on examples of the following kind (Velissaratou, 2000):

(4) A: If Mary reads this book, will she recommend it to Peter?
B: Mary does not read this book.

According to the Jäger/Hulstijn approach the answer given by (4B) should count as a (complete) answer, having the same status as the two other possible answer, namely "yes, he will" and "no, he won't".[4] However, there is a problem with this analysis. As pointed out by Isaacs and Rawlins (2005), responses like (3B) are not answers in the technical sense; i.e., they do not resolve the issue raised by the question. Instead, they indicate a species of presupposition failure. To say it in another way, the question in (2A) is about whether Mary will recommend the book to Peter. Denying the antecedent addresses the ground on which the question stands, not the question itself. The partition semantics taken by JH fails in giving any indication about the different status of the three blocks of the partition. There are further problems with this approach, to mention only one: $p \Rightarrow ?p$ comes out as semantically equivalent with $?p$ which is rather counterintuitive.

Concluding, we have seen some conceptual and empirical problems of the JH approach. The conceptual flaws are mainly related to the need of two different definitions of conditionals, one relating to the usual material implication, the other to the interrogative conditional. The empirical problems are due to the uniformity of the classical partition semantics which gives all blocks of the partition the same status.

3. Ortho-Algebras

A Hilbert space $\mathcal{H}$ is a complete complex vector space upon which an inner product (= scalar product) is defined. The scalar product of two vectors u, v in $\mathcal{H}$ is written in the form $\langle u|v\rangle$. I assume some familiarity with the notion of a vector space and an inner product. For details, the reader is referred to introducing textbooks in quantum information science, e.g. Vedral (2006).

In the following we will make use of finite Hilbert spaces, i.e. Hilbert spaces which are spanned by a finite system **S** of linearly independent vectors, which can be assumed to be pairwise orthogonal, i.e. the scalar product of two vectors in **S** is zero. A linear operator **a** in $\mathcal{H}$ is called a 'normal operator' if it satisfy the following condition:

(5) $\mathbf{a}^{+}\mathbf{a} = \mathbf{a}\mathbf{a}^{+}$

Hereby the conjugate transpose $\mathbf{a}^{+}$ is defined by the following clause:

(6) $\langle u|\mathbf{a}^{+}|v\rangle = \langle v|\mathbf{a}|u\rangle$ for all vectors u, v in $\mathcal{H}$.

Special types of normal operators are projection operators in $\mathcal{H}$. Projection operators are simply defined by the property that their eigenvalues are 1 or 0. An equivalent definition states that they are idempotent, i.e. $\mathbf{aa}=\mathbf{a}$.

The concept of normal operators is very useful because normal operators have a spectral decomposition. That is, if **a** is a normal operator then **a** can be written as

(7) $\mathbf{a} = \sum_i \lambda_i \cdot \mathbf{a}_i$ where $\mathbf{a}_i$ denotes a projection operator that projects the eigenvectors of **a** with eigenvalue λ_i. The projection operators $\mathbf{a}_i$ relate to distinct eigenvalues (i.e., $\lambda_i = \lambda_j \rightarrow i=j$)

In the spectral decomposition of a normal operator **a**, we can consider the projection operators $\mathbf{a}_i$ as resulting from the *application* of the observable **a** to a possible value λ_i of the observable. We will write this in the following form:

(8) $@(\mathbf{a}, \lambda_i) = \mathbf{a}_i$ (where $\mathbf{a}_i$ is the corresponding term of he spectral decomposition in (7))

[4] This possibility was also suggested by Groenendijk and Stokhof 1997, fn 29.

The application device @ will be a useful instrument for formalizing the idea of a possible (full) answer to a question (see section 4 & 5).

Physical observables of a Hilbert space $\mathcal{H}$ are linear operators with real eigenvalues. They can be represented by Hermitian operators, i.e. operators satisfying the condition $\mathbf{a}^+ = \mathbf{a}$. Projection operators are always observables (detecting whether a certain vector projects into a specified subspace). Using the spectral theorem, each Hermitian operator **a** can be decomposed into a sum of projection operators weighted by real numbers: $\mathbf{a} = \sum_i \lambda_i \cdot \mathbf{a}_i$, with real eigenvalue λ_i. In the present context it is useful also to consider non-Hermitian normal operators which make use of so-called quaternions. Quaternions are a non-commutative extension of complex numbers.[5] In the following we will use the quaternions *y* and *n* to represent the answers 'yes' and 'no', respectively. The point is that that conjunction of *yes/no* questions leads to complex answers such as $y \cdot n$ and $n \cdot y$ and we have to make sure that these answers can be different, i.e. the commutative law $y \cdot n = n \cdot y$ is disobeyed.[6]

The spectral theorem allows partitioning the Hilbert space $\mathcal{H}$ into projection spaces. These subspaces are spanned by eigenvectors with a fixed eigenvalue. In quantum theory, these eigenvalues are necessary to reconstruct the observable from the partition. Hence, in quantum theory observables can be seen as decorated partitions (decorated by the eigenvalue). Hence, the spectral theorem (7) can be seen as generating a decorated partition of the system of eigenvectors. In the next section, I will investigate the idea of decorated partitions further, and I will use this idea for defining a new version of inquisitive semantics. In particular, I will show that most shortcomings of the JH approach can be resolved by using decorated partitions.

An algebraic system of projection operators can be defined straightforwardly. With the projection operators **a** and **b** also the projection operators $\mathbf{a}\wedge\mathbf{b}$, $\mathbf{a}\vee\mathbf{b}$ and $\mathbf{a}^\perp$ are defined using the operations of intersection, union, and ortho-complement of the corresponding Hilbert subspaces. If we assume (= classical case) that all the considered projection operators are commuting we get the following facts:

(9) a. $\mathbf{a}\wedge\mathbf{b} = \mathbf{ab}$
  b. $\mathbf{a}\vee\mathbf{b} = \mathbf{a}+\mathbf{b} - \mathbf{ab}$
  c. $\mathbf{a}^\perp = \mathbf{1}-\mathbf{a}$

The notion of inference can be defined in the following way, where **a** and **b** are observables:[7]

(10) $\mathbf{a} \models \mathbf{b}$ iff $\mathbf{ab} = \mathbf{a}$

In the present framework, states of the system are represented by vectors in the Hilbert space, and observables (representing propositions and questions) are represented by normal operators. In the simplest case the Hilbert space has two dimensions. An arbitrary orthogonal and normalized base of it is labelled $\{|1\rangle, |0\rangle\}$ using Dirac's notation. A physical system realized in this Hilbert space is called a qubit. Each pure qubit state is a linear superposition of $|1\rangle$ and $|0\rangle$: $|u\rangle = \alpha|0\rangle + \beta|1\rangle$. Each of the nontrivial observables in the two-dimensional Hilbert space has a discrete spectrum with two non-degenerate eigenvalues.[8]

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[5] The standard definition sees every quaternion as a unique and real linear combination of the basis quaternions *1*, *i*, *j*, and *k*: $x = x_0 + x_1 i + x_2 j + x_3 k$, with the set of equations $i^2 = j^2 = k^2 = ijk = -1$.

[6] I thank Peter beim Graben who referred me to quaternions.

[7] It is exactly the relation $\models$ that defines the lattice-theoretic properties of our ortho-algebra.

[8] Famous are the so-called Pauli matrices which provide base operators in terms of which every other operator can be defined.

Pure states $|u\rangle$ are uniquely related to certain projection operators written as $|u\rangle\langle u|$ using the Dirac notation:

(11) $|u\rangle\langle u|\,(|v\rangle) = |u\rangle \cdot \langle u|v\rangle$ for each state $|v\rangle$ of the Hilbert space

Obviously these operators can have only the eigenvalues 0 and 1 and, thus, are projection operators. As an abbreviating notation we write bold **u** for the projection operator $|u\rangle\langle u|$. Two commuting operators that can be formulated in the two dimensional Hilbert space are **0** and **1**, realizing the projections of the two base states $|0\rangle$ and $|1\rangle$. Further, we use the notation **I** for the identity operator and ∅ for the zero operator. It is simple to check that $\mathbf{I} = \mathbf{0} + \mathbf{1}$. All the operators **0**, **1**, **I** and ∅ are commuting with each other and they can be seen as realizing the classical bit as subpart of the qubit.

In quantum theory complex systems are built by using tensor products ⊗. This operation applies both to vectors of the Hilbert space $|u\rangle \otimes |v\rangle$ and to linear operators $\mathbf{u} \otimes \mathbf{v}$. If the context excludes misunderstandings, it is convenient to miss out the ⊗. Hence we will write $|011\rangle$ instead of $|0\rangle\otimes|1\rangle\otimes|1\rangle$ and **011** instead of $\mathbf{0}\otimes\mathbf{1}\otimes\mathbf{1}$. All projection operators that are built from summing up pure projections in the case of a 2^n-dimensional Hilbert space (such as **000, 001, 010**, … in case of a 3 qubits) are pairwise commuting and realize a Boolean algebra.

4. Ortho-Algebraic Semantics

We can exclude the *zero part* of an observable (i.e. the vector space corresponding to eigenvectors with eigenvalue zero) from the partition since it doesn't contribute to the sum in (4). Concluding, an observable **a** can effectively be represented by a decorated partition of the Hilbert space $\mathcal{H}$ minus the zero part of observable **a**.

For defining semantics, we start from a standard declarative language *L*, with a set of propositional variables *p*, *q*, r, …, negation $\neg\varphi$, conjunction $\varphi\wedge\psi$, disjunction $\varphi\vee\psi$, declarative $!\varphi$ and question $?\varphi$. The semantics is defined relatively to a Hilbert space $\mathcal{H}$ (and the ortho-lattice defined on it). Further, we assume an assignment function π that assigns projection operators in $\mathcal{H}$ to the propositional variables. Then the semantic values for the formulas of *L* are defined as follows:

(12) a. $«p» = \pi(p)$

b. $«\neg\varphi» = \sum_i \lambda_i\, \mathbf{a}_i^{\perp}$ where $«\varphi» = \sum_i \lambda_i \cdot \mathbf{a}_i$ (the spectral decomposition of «φ»)

c. $«\varphi\wedge\psi» = «\varphi»\, «\psi»$ [assuming «φ» and «ψ» commute]

d. $«!\varphi» = (\sum_{«\varphi»|w\rangle = 0} |w\rangle\langle w|\,)^{\perp}$

e. $«?\varphi» = y \cdot «\varphi» + n \cdot «\neg\varphi»$ ('*y*', '*n*' are quaternions for 'yes', 'no')

Notice that the semantic value of $\neg\varphi$ is a projection operator again if φ is a projection operator. In this case, $«\neg\varphi» = \mathbf{1} - «\varphi»$. If φ is a yes/no question, then $\neg\varphi$ turns the *yes*-answer into the *no*-answer and the *no*-answer into the *yes*-answer. Further, it should be noticed that the value of the conjunction is a projection operator only if the semantic values of the two conjuncts commute. In this case, the ordering of the conjuncts does not matter.

Now two further definitions are required:

(13) a. $\varphi\vee\psi = \neg(\neg\varphi\wedge\neg\psi)$

b. $\varphi\rightarrow\psi = \varphi^{\perp}\vee(\varphi\wedge\psi)$

These definitions look very classical and they correspond to the classical Boolean operations if φ and ψ are declaratives. The implication defined in (13b) is the Sasaki implication well-known from quantum logic.[9] Interestingly, these definitions also apply in case one or both of these expressions are questions. This leads to surprising results, which will be discussed immediately after introducing some basic semantic concepts.

We consider a propositional formula φ (semantically represented by a projection operator) 'true' in a situation $|u\rangle$ iff $|u\rangle$ is an eigenvector of «φ» with eigenvalues 1; formally:

(14) $|u\rangle \vdash \varphi$ iff «φ»$|u\rangle = |u\rangle$

Further, we can define the 'span' of a formula operator φ as the relation between the eigen-states of «φ» that have the same (non-zero) eigenvalues:

(15) $|u, v\rangle \models \varphi$ iff «φ»$|u\rangle = \lambda|u\rangle$ and «φ»$|v\rangle = \lambda|v\rangle$ for some $\lambda \neq 0$

Read $|u, v\rangle \models \varphi$ as 'the pair of states spans φ in a given model'. The span itself could be defined as the set of pairs of states that span φ. For expressions that are interpreted by projection operators (declaratives) the span doesn't give more information than that provided by the truth condition; hence we have the following facts for declaratives φ:

(16) a. $|u, v\rangle \models \varphi$ iff $|u\rangle \vdash \varphi$ and $|v\rangle \vdash \varphi$
b. $|u\rangle \vdash \varphi$ iff $|u, u\rangle \models \varphi$

Of course, the situation is different when we consider other observables than projection operators. These observables could be constructed by using the operations of sum, complementation and composition even in the classical case where all considered operators are commuting. If φ and ψ are declaratives, then the span of $\neg\varphi$, $?\varphi$, $\varphi\wedge\psi$, $\varphi\vee\psi$, and $\varphi\rightarrow\psi$ can be calculated as follows given that he relevant operators commute with each other:

(17) a. $|u, v\rangle \models \neg\varphi$ iff $|u, u\rangle \not\models \varphi$ & $|v, v\rangle \not\models \varphi$
b. $|u, v\rangle \models ?\varphi$ iff $|u, u\rangle \models \varphi \Leftrightarrow |v, v\rangle \models \varphi$
c. $|u, v\rangle \models \varphi\rightarrow ?\psi$ iff $|u, v\rangle \models ?\varphi$ and $|u, v\rangle \models \varphi \Rightarrow |u, v\rangle \models ?\psi$

Interestingly, these clauses conform to the inquisitive semantics as proposed by HJ. However, the present system is more structured than the JH system because it considers decorated partitions instead of standard partitions. To see the important differences let's consider some simple examples.

In figure 2 the meaning of ?p in the JH inquisitive semantics was pictured. We see two equivalence classes which partition the space W of possible worlds. Figure 4 pictures the meaning of ?p in ortho-algebraic semantics. We see the same equivalence classes, but now the two blocks are decorated by the quaternions *y* and *n*, respectively, corresponding to the two possible answers *yes* and *no*.

[9] It is the only implication satisfying a proper deduction theorem (cf. Roman & Zuazua, 1999).

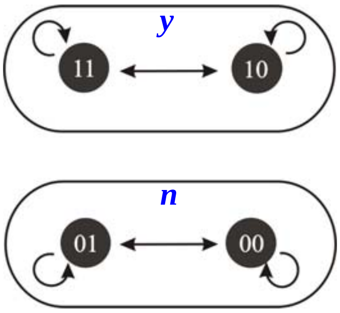

Figure 4: Picture of meaning ?p in ortho-algebraic semantics

The meaning of the conditional interrogative p⇒?q was pictured in figure 3 for the JH semantics. Figure 5 shows the meaning of the related expression in the ortho-algebraic framework.

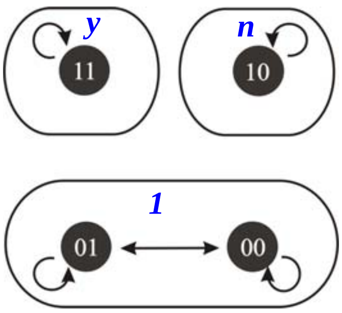

Figure 5: Picture of meaning p→?q in ortho-algebraic semantics

The derivation of the relevant partition is as follows:

$$\text{«}p\rightarrow ?q\text{»}$$
$$= \neg(\mathbf{10}+\mathbf{11})+(\mathbf{10}+\mathbf{11})(y(\mathbf{01}+\mathbf{11})+n(\mathbf{10}+\mathbf{00}))$$
$$= (\mathbf{00}+\mathbf{01})+(y\ \mathbf{11}+n\ \mathbf{10})$$

In both cases the partition of the logical space consists of three blocks, corresponding to the propositions expressed by p∧q, p∧¬q and ¬p. In the latter case these propositions are decorated: p∧q by *y*, p∧¬q by *n*, and ¬p by *1*. As before, we can take the first two decorations as indicating the traditional answer types *yes* and *no*; and we can take the decoration *1* as indicating the condition for a supposition failure.

The following example shows the composition of two questions ?p and ?q forming the composed question ?p ∧?q. Figure 6 pictures the corresponding meaning where the composed decorations *yy*, *yn*, *ny*, and *nn* are used.

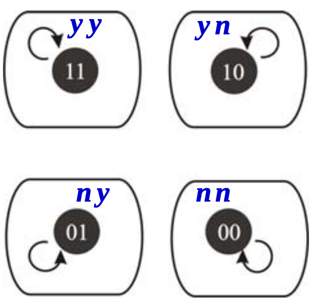

Figure 6: Picture of meaning ?p ∧?q in ortho-algebraic semantics

Obviously, the parts of the complex quaternions are referring to the corresponding subquestions, e.g. in the world 10 the question ?p is answered by *y* and the question ?q is answered by *n*.

An important empirical problem, which any theory of questions and answer has to solve, relates to the proper characterization of congruent answers (e.g. Groenendijk & Stokhof, 1997; Krifka, 2001). In the simplest case of constituent questions a congruent question is just an answer that fills in a constituent for the *wh*-expression in the question. And a congruent full answer is just the question meaning applied to the term answer.

More formally, this can be expressed in the following way making use of the application device @ defined in (8):

(18) φ is a *congruent full answer* to a question ψ iff @(«ψ», t) = «φ» for some element t of the spectrum of «ψ».

A simple example is in order. Clearly, the assertion p is a congruent answer to ?p. This derives from the observation that @(«?p», *y*) = «p». Similarly, ¬p is a proper answer to ?p since @(«?p», *n*) = «¬p».

Consider now the following utterance of a question (19a) made by a competent speaker A. Congruent answers of a speaker B are the conditional answers presented in (19b,c). Intuitively, conjoined answers such as in (19d,e) don't count as congruent answers and are not very appropriate.

(19) a. If Mary reads this book will she recommend it to Peter?
b. Yes. If Mary reads this book, she will recommend it to Peter
c. No. If Mary reads this book she will not recommend it to Peter
d. *Yes. Mary reads this book, and she will recommend it to Peter
e. *No. Mary reads this book, and she will not recommend it to Peter

Interestingly, the JH approach does not predict the proper conditional answers but the conjoined answers. How to handle this problem in ortho-algebraic semantics? Does the definition given in (18) generalize to the idea of congruent answers in case of conditional questions?

Unfortunately, this doesn't work in the case of conditional questions. However, a simple adjustment is possible and provides the proper generalization. The proposal is to change the definition by taking the proposition with the decoration *1* into account. As mentioned above this proposition is expressing the condition for a supposition failure (see figure 5 for an example).

(20) φ is a *congruent full answer* to a question ψ iff @(«ψ», t) + @(«ψ», 1) = «φ» for some element t of the spectrum of «ψ».

A consequence of this definition is that congruency is possible only with conditional answers for conditional questions. For instance, «p→q» comes out as a congruent full answer to «p→?q». We can derive this fact from the equivalences «p→?q» = «p»$^\perp$ + «p»«?q» = 1«p»$^\perp$ + *y*«p»«q» + *n*«p»«q»$^\perp$. The application @(«p→?q», *y*) results in «p»$^\perp$ and the application @(«p→?q», *1*) results in «p»«q». Consequently, the sum gives «p»$^\perp$ + «p»«q», which is nothing else than «p→q». Hence, according to definition (20), «p→q» comes out as a congruent answer to «p→?q».

Similarly, «p→¬q» can be shown to be a congruent full answer to «p→?q»:

$$@(«p\rightarrow?q», n) + @(«p\rightarrow?q», 1) \\ = «p»^{\perp} + «p»«q»^{\perp} \\ = «p\rightarrow\neg q».$$

Further, since the two full answers given before are the only congruent answers to the question «p→?q», «p∧q» cannot be a proper answer to «p→?q».

Concluding this section we claim that the present approach explains why informationally equivalent questions like "is the door open?" and "is the door closed?" have different meanings. Further it overcomes the conceptual imperfection of the JH approach: only one definition of the conditional is required in order to capture both the usual material implication of declaratives and the interrogative conditional connecting a declarative antecedent with a question. It also overcomes the main empirical problems of the JH approach due to the uniformity of the classical partition semantics which gives all partitions the same status. In the case of the interrogative conditional it indicates when an 'answer' counts as a species of presupposition failure. Moreover, it is simple to show that in ortho-algebraic semantics the equivalence between p→p? and ?p is not longer valid. And it resolves perhaps the biggest puzzle of the JH approach, which counter-intuitively predicts conjunctive answers for conditional questions.

5. Comparison with the structured meaning approach

In a seminal paper, Krifka (2001) argued for a structured meaning account of questions and answers (see also Krifka, 2004). He demonstrated that the GS partition theory (and related approaches summarized as proposition set approaches by Krifka) runs into three problems:

> "It does not always predict the right focus structure in answers, it is unable to distinguish between polarity (yes/no) and a certain type of alternative questions, and it does not allow to formulate an important condition for a type of multiple constituent questions" (Krifka, 2001, p. 287).

Further, Krifka made clear in the same paper that the structured meaning approach can handle all three problems. Without going into a detailed discussion, I will illustrate here only the close correspondence of the structured meaning account and the present decorated partition theory. Krifka summarizes the basic idea underlying the structured meaning approach as follows:

> "Question meanings are functions that, when applied to the meaning of the answer, yield a proposition." (Krifka, 2001, p. 288)

When we use the application device proposed in the previous section (instead of the operation of functional application in a λ-categorical language as proposed by Krifka), then we see immediately that our decorated partition semantics shares a basic trait with the structured meaning approach: question meanings can be *applied* to the meaning of (term) answers yielding a proposition. Interestingly, the presented application device @ and the definition of congruent answers given in (20) is also valid for conditional questions. As far as I can see, the structured meaning approach was not yet applied to conditional questions. Fortunately, it is a simple task to extend this approach in order to include conditional questions – a task that can be achieved by implementing the idea underlying the definition (20) into the structured meaning approach. However, I prefer to continue using the operator framework although it might be somewhat unfamiliar for traditional formal semanticists. I have two reasons for that. First, this formalism gives a very concise and elegant description of a theory of question and answers in case the reader is familiar with the orthoalgebraic approach. Second, this formalism straightforwardly generalizes to the non-commutative case that can be used to model phenomena of opinion building including questions as used in personality diagnostics (Blutner & Hochnadel, 2008).

6. Conclusions

Taking the lead from orthodox von Neumann quantum theory (Von Neumann, 1932), I have introduced a handy generalization of the Boolean approach to propositions and questions: the ortho-algebraic framework. I have demonstrated that this formalism relates to a formal theory of questions (or 'observables' in the physicist's jargon). Surprisingly, this theory allows formulating conditional questions, and thus it provides the semantic power for managing inquisitive semantics. In the case of commuting observables, there are close similarities between the ortho-algebraic approach to questions and the Jäger/Hulstijn approach to inquisitive semantics. However, the present approach is able to overcome most of the difficulties of the Jäger/Hulstijn approach. I have further demonstrated that the present approach can be seen as a decorated partition theory of questions and as such it is fully compatible with the structured meaning approach to questions.

An important methodological issue relates to the descriptive power of a theory and its explanatory value. It could be argued that the present formalism is surely adequate when it comes to describe quantum phenomena in physics, but much too powerful when applied to the semantics of natural language. In fact, we have mainly discussed the case of commuting observables/question, i.e. we have restricted ourselves to the classical case of Boolean algebras. Why then use such a powerful formalism?

There are several aspects that have to be discussed in this regard. First, there is the historical interest to relate the formal semantics of questions as developed by Groenendijk & Stokhof (1984a, 1997), Krifka (2001) and others with the formal treatment of observables in quantum physics (e.g. Birkhoff & von Neumann, 1936; Von Neumann, 1932). One result of this comparison is the observation that quantum physics relates to a decorated partition theory, which has much more in common with a structured meaning approach than with the GS partition theory of questions.

Second, the operator formalism of quantum mechanism allows a straightforward formulation of conditional questions by making use of the standard instruments of orthoalgebraic semantics. Of course, we can implement the relevant ideas also in the more traditional theory of structured meanings. However, the operator formalism seems to be very natural for a uniform treatment of questions, answers, and propositions.

Third, the full orthoalgebraic framework – without the restriction to commuting observables – can be useful for understanding how quantum-like features are generated on the macro-level of cognition. I have stressed this point in section 1 already and I have to stress it again at the end of this squib. The phenomena of opinion building and the proper treatment of diagnostic questions as used in personality diagnostics are typical cases in point (for more discussion, see Blutner & Hochnadel, 2008).

Another but related problem is the proper distinction between uncertainty and ignorance, a problem that is highly relevant for any advanced theory of questions and answers. In general, ignorance is reasonable when the expected benefits of information are too small relatively to the costs. A typical situation are elections where people in general choose to remain uninformed (Downs, 1957). In contrast, uncertainty refers to situations where people use statistical information to optimize their decision. Recently, it has been argued by Franco (2007b, 2007c) that the behaviour of people under rational ignorance can be described best within the quantum mechanics formalism when the states of the system are described by vectors in the Hilbert space. Alternatively, a stochastic mixture of the eigenstates of the operators under discussion relate to people that reason under uncertainty. In rational-ignorance regime an uncertainty principle holds, which states that the product of the variances relevant to at least two questions has a non-trivial lower bound.

If there is a bit of truth in the supposition that the abstract formalism of quantum mechanics will find useful applications in the domain of cognition, then this suggests that an active dialogue between the traditional model-theoretic approaches to semantics and the ortho-algebraic paradigm is mandatory.

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