Of the Hennessy-Milner Property and other Demons

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Abstract

We relate three different, but equivalent, ways to characterise behavioural equivalence for set coalgebras. These are: using final coalgebras, using coalgebraic languages that have the Hennessy- Milner property and using coalgebraic languages that have "logical congruences". On the technical side the main result of our paper is a straightforward construction of the final T-coalgebra of a set functor using a given logical language that has the Hennessy-Milner property with respect to the class of T-coalgebras.

1 Introduction

Characterising behavioral equivalence between coalgebras is an important issue of universal coalgebra and coalgebraic logic. Rutten [15] shows that behavioral equivalence can be *structurally characterized* by final coalgebras. Moss [12] and Pattinson [14] provided two different ways for generalizing modal logic to arbitrary coalgebras. These generalizations are called coalgebraic (modal) logics. Moss showed in [12] that his language provides a *logical characterization* of behavioural equivalence: two pointed coalgebras are behaviourally equivalent iff they satisfy the same formulas of the logic. In modal logic terminology this logical characterisation of behavioural equivalence is usually called the "Hennessy-Milner property" ([5]). For the language of coalgebraic modal logic from [14] Lutz Schröder [16] shows how the Hennessy-Milner property can be obtained using certain congruences of coalgebras, which we will call *logical congruences*.

The main contribution of this paper is to introduce a systematic study of the relationship between these three methods: the structural and logical characterisation of behavioural equivalence and the characterisation that uses logical congruences. We work in a general framework that covers all known logics for set coalgebras and easily generalizes to base categories different from Set. Our main theorem (Theorem 4.23) can be stated as follows: Given a set functor T,

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a final T-coalgebra exists iff there exists a language for T-coalgebras with the Hennessy-Milner property iff there exists a language for T-coalgebras that has logical congruences.

The first equivalence was proven by Goldblatt in [6], the second equivalence was discussed by Schröder in [16] for the case of coalgebraic logics. We provide relatively simple proofs for these equivalences in order to obtain our main theorem. In particular, 1) we simplify Goldblatt's proof, 2) generalize Schröder's argument, and in addition to that, 3) we use our framework to construct canonical models and characterize simple coalgebras. Furthermore we demonstrate that our proofs allow for straightforward generalizations to base categories that are different from Set.

The structure of the paper is as follows: In the Section 2, we introduce some preliminaries and mention the relation between final coalgebras and behavioral equivalence. In Section 3 we introduce abstract coalgebraic languages. Using these we present the connection between behavioral equivalence and the Hennessy-Milner property and we construct canonical models. In Section 4 we discuss the connection between coalgebraic congruences and languages with the Hennessy-Milner property and characterize simple coalgebras. Finally in Section 5 we explore generalizations of our work to other categories. For this last part a bit more of knowledge of category theory is assumed.

2 Behaviour & Final Coalgebras (Preliminaries)

Before we start let us stress the fact that one major feature of our work is its simplicity. Therefore we do not require much knowledge of category theory. When possible, we avoid the use of general categorical notions and use set theoretic terminology. The basic notion of this paper is that of coalgebras for a set endofunctor. However, we introduce the notion of coalgebra for an endofunctor on any category, and not just Set, as we will discuss generalisations of our results.

Definition 2.1. The category of coalgebras $\mathsf{Coalg}(T)$ for an endofunctor on a category $\mathbb{C}$ has as objects arrows $\xi : X \longrightarrow TX$ and morphisms $f : (X, \xi) \longrightarrow (Y, \gamma)$ are arrows $f : X \longrightarrow Y$ such that $T(f)\xi = \gamma f$. The object X is called the states of ξ. A *pointed coalgebra* is a pair (ξ, x) with x a point of the states of ξ.We often write $f : \xi \longrightarrow \gamma$ for a morphism $f : (X, \xi) \longrightarrow (Y, \gamma)$.We call the arrow ξ, in a coalgebra (X, ξ), the *structural map* of the coalgebra. We reserve the letter ξ for coalgebras with a carrier object X, the letter γ for coalgebras over an object Y, and the letter ζ for coalgebras based on an object Z.

The crucial notion that we want to investigate is *behavioural equivalence*. For now, we only consider set coalgebras. One of the important issues towards a generalisation of the results presented in this paper is to find a notion of behavioural equivalence that can be interpreted in other categories different from Set.

Definition 2.2. Let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a functor. Two states x_1 and x_2 on T-coalgebras ξ_1 and ξ_2, respectively, are *behavioural equivalent*, written $x_1 \sim x_2$, if there exists a coalgebra γ and morphisms $g_i : \xi_i \longrightarrow \gamma$ such that $g_1(x_1) = g_2(x_2)$.

It was noted by Rutten [15] that behavioural equivalence could be characterised using final systems, also called final coalgebras.

Definition 2.3. A final coalgebra for an endofunctor T is a terminal object in $\mathsf{Coalg}(T)$. Explicitly, a final coalgebra is a coalgebra $\zeta : Z \longrightarrow TZ$ such that for any other coalgebra $\xi : X \longrightarrow TX$ there exists a unique morphism $f_\xi : \xi \longrightarrow \zeta$. This morphism is called the *final map of* ξ.

Final coalgebras are to coalgebra what initial algebras or term algebras are to algebra (cf. e.g. [9]). In this paper we are interested in final coalgebras mainly because they can be used to characterise behavioural equivalence between states.

Proposition 2.4 ([15])**.** *If a final coalgebra for a set functor T exists, two states x_i in coalgebras ξ_i are behavioural equivalent if and only if they are mapped into the same state of the final coalgebra, ie., if $f_{\xi_1}(x_1) = f_{\xi_2}(x_2)$.*

3 The Hennessy-Milner Property & Behaviour

In this section we introduce languages to describe coalgebras. We will show how languages with the Hennessy-Milner property relate to final coalgebras. We will illustrate this interaction constructing canonical models for those languages.

3.1 Abstract Coalgebraic Languages

We begin by showing that the existence of a final coalgebra is equivalent to the existence of a language with the Hennessy-Milner property. Let us first clarify what is meant by an *abstract coalgebraic language*. In the sequel, unless stated otherwise, T will always denote an arbitrary functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$.

Definition 3.1. An *abstract coalgebraic language* for T, or simply a *language for T-coalgebras*, is a set $\mathcal{L}$ together with a function $\Phi_\xi : X \longrightarrow \mathcal{P}\mathcal{L}$ for each T-coalgebra $\xi : X \longrightarrow TX$. The function Φ_ξ will be called *the theory map of* ξ, elements of $\mathcal{P}\mathcal{L}$ will be called *($\mathcal{L}$-)theories*.

An abstract coalgebraic language is precisely what Goldblatt calls a "small logic" for T ([6]). A first example of a coalgebraic language is the language of basic modal logic.

Example 3.2. Let $T = \mathcal{P}$ be the covariant power set functor. It is well-known that the category of $\mathcal{P}$-coalgebras is isomorphic to the category of Kripke frames and bounded morphisms. Let $\mathcal{L}_{\mathbf{K}}$ be the set of closed modal formulas of the basic similarity type (see [5] for details). For an arbitrary $\mathcal{P}$-coalgebra $\xi : X \longrightarrow \mathcal{P}X$ we define $\Phi_\xi : X \longrightarrow \mathcal{P}\mathcal{L}_{\mathbf{K}}$ to be the "modal theory map", i.e., the function that maps a state $x \in X$ to the set of formulas $\varphi \in \mathcal{L}_{\mathbf{K}}$ such that

$\xi, x \models \varphi$. The set $\mathcal{L}_{\mathbf{K}}$ together with the family $\{\Phi_\xi\}_{\xi \in \mathsf{Coalg}(\mathcal{P})}$ is an abstract coalgebraic language for $\mathcal{P}$. We have a similar example if we take $T = \mathcal{P}_\omega$, the finite power set functor. Recall that coalgebras for $\mathcal{P}_\omega$ are image finite Kripke frames.

In fact modal logic is a particular instance of a more general class of coalgebraic languages. Namely coalgebraic modal logics with predicate liftings, or *languages of predicate liftings* (see [14, 16] for details).

Example 3.3 ([16])**.** Given a set endofunctor, an n-ary predicate lifting is a natural transformation $\mathcal{Q}^n \longrightarrow \mathcal{Q}T$. We can associate an abstract coalgebraic language to each set of predicate liftings, for T, Λ. The language of predicate liftings $\mathcal{L}^\kappa(\Lambda)$ with κ conjunctions[1] associated with Λ is defined by the usual boolean grammar for κ conjunctions together with the following clause for predicate liftings $\lambda \in \Lambda$: If λ is an n-ary predicate lifting, and $(\varphi_i)_{i \in n}$ is a n-sequence of formulas in $\mathcal{L}^\kappa(\Lambda)$ then $\lambda(\varphi_i)_{i\in n} \in \mathcal{L}^\kappa(\Lambda)$. where $|I| < \kappa$ and $\lambda \in \Lambda$. Formulas in $\mathcal{L}^\kappa(\Lambda)$ can be interpreted on set coalgebras as follows: For each coalgebra ξ we define a function $[\![-]\!]^\xi : \mathcal{L}^\kappa(\Lambda) \longrightarrow \mathcal{P}X$, this function is defined by induction on the complexity of the formula. The case of boolean formulas is done as usual. In the case of formulas involving predicate liftings we define $[\![\lambda(\varphi_i)]\!]^\xi = \xi^{-1}\lambda_X([\![\varphi_i]\!]^\xi)_{i\in n}$. The intuition is that predicate liftings are "modalities". The transpose of $[\![-]\!]^\xi$, which is a function $\Phi_\xi : X \longrightarrow \mathcal{P}\mathcal{L}^\kappa(\Lambda)$, is the theory map of ξ; The set $\mathcal{L}^\kappa(\Lambda)$ together with the maps Φ_{xi} is and abstract coalgebraic language for T. Using this we could say that a state x in a coalgebra ξ satisfies a formula $\lambda(\varphi_i)_{i\in n}$, written $\xi, x \vDash \lambda(\varphi_i)_{i\in n}$, iff $\xi(x) \in \lambda_X([\![\varphi_i]\!]^\xi)_{i\in n}$.

The fact that abstract coalgebraic languages are just sets and do not carry by definition any further structure could be seen as a weakness. However, this has that the advantage we are covering any logical language for Set-coalgebras that one can imagine[2]. This means that the results presented in this section hold for any language. For example, for languages with fixpoint operators.

As mentioned before, we are interested in describing behavioural equivalence. To do this we have two requirements on the language, which together lead to what sometimes is called expressivity: 1) *Adequacy:* formulas must be invariant under coalgebra morphisms. 2) *Hennessy-Milner property:* formulas must distinguish states that are not behavioural equivalent.

Definition 3.4. An abstract coalgebraic language $\mathcal{L}$ is said to be *adequate* if for every pair of pointed T-coalgebras (ξ_1, x_1) and (ξ_2, x_2),

$$x_1 \sim x_2 \text{ implies } \Phi_{\xi_1}(x_1) = \Phi_{\xi_2}(x_2).$$

The language $\mathcal{L}$ is said to have the *Hennessy-Milner property* if for every pair of pointed T-coalgebras (ξ_1, x_1) and (ξ_2, x_2),

$$\Phi_{\xi_1}(x_1) = \Phi_{\xi_2}(x_2) \text{ implies } x_1 \sim x_2.$$

[1] Here κ should be an inaccessible cardinal (Definition 4.20)

[2] We have of course the assumption that formulas are interpreted at a state and not in the whole coalgebra (unlike, e.g. first-order logic).

Other names used for adequacy and the Hennessy-Milner property are *soundness* and *completeness*, respectively. So we can talk of languages that are sound and complete with respect to behavioural equivalence.

Example 3.5. The basic modal language is an adequate language for $\mathcal{P}$-coalgebras, but it does not have the Hennessy-Milner property. Nevertheless, the basic modal language has the Hennessy-Milner property with respect to $\mathcal{P}_\omega$-coalgebras (image-finite Kripke frames). All languages of predicate liftings are adequate. However, not all of them have the Hennessy-Milner property. Sufficient conditions for this can be found in [16].

3.2 An Elementary Construction of Final Coalgebras

One advantage of using abstract coalgebraic languages is that we can easily show how a final coalgebra induces an adequate language that has the Hennessy-Milner property.

Theorem 3.6. *For any functor $T : Set \longrightarrow Set$, if there exists a final coalgebra then there exists an adequate language for T coalgebras with the Hennessy-Milner property.*

Proof. Let (Z, ζ) be a final coalgebra, and let f_ξ be the final map for each coalgebra ζ. Take $\mathcal{L} = Z$ and for each coalgebra (X, ξ) define $\Phi_\xi(x) = \{f_\xi(x)\}$. Since Z is a final coalgebra this language together with the maps Φ_ξ is adequate and has the Hennessy-Milner property. This concludes the proof. □

At first glance it might seem that the previous construction is too abstract and that the language we obtain from a final coalgebra is not interesting. However, this is not the case. We consider the following example from [9].

Example 3.7. Consider the set endofunctor $T = 1 + (-)$. Coalgebras for this functor can be considered as a black-box machine with one (external) button and one light. The machine performs a certain action only if the button is pressed. And the light goes one only if the machine stops operating. A final coalgebra for T is given by the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ together with a function $p : \overline{\mathbb{N}} \longrightarrow 1 + \overline{\mathbb{N}}$ defined as follows $p(0) = *$; $p(n + 1) = n$; $p(\infty) = \infty$, where $*$ is the only element of 1. This presentation of the final coalgebra for T contains all the information about the observable behaviour of a state in T-coalgebra as a state can only either lead the machine to stop after n-steps or let the machine run forever.

Other examples demonstrating that final coalgebras are useful for describing the coalgebraic behaviour can be found in [17, 18, 2]. Now we will illustrate how to construct a final T-coalgebra from an adequate language which has the Hennessy-Milner property. Our exposition is a slight generalisation and simplification of the construction introduced in [6].

Theorem 3.8. *For any functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, if there exists an adequate language for T-coalgebras with the Hennessy-Milner property then there exists a final coalgebra.*

We will provide a proof of this theorem after we have made some observations. Our construction has three main points: The first key idea is to notice that if we have a language for T-coalgebras we can identify a concrete set (object) Z which is a natural candidate for the carrier of a final coalgebra. The second observation is that for each coalgebra (X, ξ) there is a natural map $X \longrightarrow TZ$; should the language be adequate and have the Hennessy-Milner property then we can combine these functions into a function $\zeta : Z \longrightarrow TZ$ which decorates Z with the structure of a final T-coalgebra. Moreover, using this approach we can show that the function ζ exists if and only if the language has the Hennessy-Milner property. A natural candidate for the carrier of a final coalgebra is the set of satisfiable theories of the language.

Definition 3.9. Given a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and an abstract coalgebraic language $\mathcal{L}$ for T-coalgebras, the set $Z_{\mathcal{L}}$ of **$\mathcal{L}$-satisfiable theories** is the set $Z_{\mathcal{L}} := \{\Phi \subseteq \mathcal{L} \mid (\exists \xi)(\exists x \in \xi)(\Phi_\xi(x) = \Phi)\}$. We often drop the subindex $\mathcal{L}$.

Remark 3.10. The reader might worry that in the definition of $Z_{\mathcal{L}}$ we quantify over all coalgebras and all states on them and then we might not be defining a set but a proper class. This is not an issue as we required the language $\mathcal{L}$ to be a set and obviously $Z_{\mathcal{L}} \subseteq \mathcal{L}$.

By definition of $Z_{\mathcal{L}}$ it is clear that for each coalgebra $\xi : X \longrightarrow TX$ there is a canonical map $f_\xi : X \longrightarrow Z_{\mathcal{L}}$ that is obtained from $\Phi_\xi : X \longrightarrow \mathcal{P}\mathcal{L}$ by restricting the codomain. This restriction is possible as the range of Φ_ξ is clearly contained in $Z_{\mathcal{L}}$. Using the functions f_ξ we can see that for each coalgebra (X, ξ) there is a natural function from X to TZ, namely the lower edge of the following square

$$\begin{array}{ccc} X & \xrightarrow{f_\xi} & Z \\ {\scriptstyle \xi}\big\downarrow & & \\ TX & \xrightarrow[T(f_\xi)]{} & TZ \end{array} \tag{1}$$

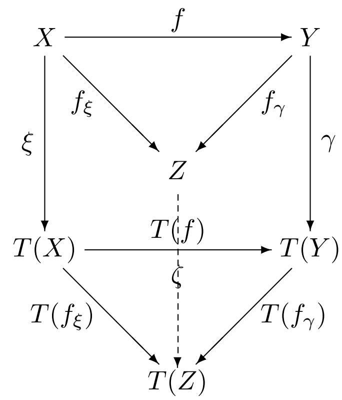
This square suggests the following assignment ζ: a theory $f_\xi(x) = \Phi_\xi(x) \in Z$ is assigned to

$$\zeta(\Phi_\xi(x)) = T(f_\xi)\xi(x). \tag{2}$$

Since in general we will have $\Phi = \Phi_{\xi_1}(x_1) = \Phi_{\xi_2}(x_2)$ for different pointed coalgebras (ξ_1, x_1) and (ξ_2, x_2) it is not clear that (2) defines a function. We are now going to show that (2) defines a function if the language is adequate and has the Hennessy-Milner property. We will prove this in two steps which illustrate that both conditions are really needed.

Lemma 3.11. *Let $\mathcal{L}$ be an adequate language for T-coalgebras. For any morphism $f : \xi \longrightarrow \gamma$ we have: $(Tf_\xi)\xi = (Tf_\gamma)\gamma f$, where f_ξ and f_γ are obtained from the respective theory maps by restricting the domain.*

Proof. The situation is depicted in the following diagram



We want to show that the pentagon in the back commutes. Since $\mathcal{L}$ is adequate, the upper triangle commutes. Using this, since T is a functor, we conclude that the lower triangle commutes, i.e. $T(f_\gamma)T(f) = T(f_\xi)$. Now notice that the back rectangle commutes because f is a morphism of T-coalgebras. Chasing around the diagram we obtain:

$$T(Th_\gamma)\gamma f(x) = T(Th_\gamma)T(f)\xi(x) = T(\Phi_\xi)\xi(x)$$

This concludes the proof. $\square$

If we assume the language $\mathcal{L}$ to be adequate and we have a morphism $f : \xi \longrightarrow \gamma$, then $\Phi_\xi(x) = \Phi_\gamma(f(x))$. The previous lemma implies $\zeta(\Phi_\xi(x)) = \zeta(\Phi_\gamma)(f(x))$. We can show that if in addition to adequacy $\mathcal{L}$ has the Hennessy-Milner property, equation (2) defines a function ζ. In fact these two conditions are equivalent.

Theorem 3.12. *Let T be a set functor, let $\mathcal{L}$ be an adequate language for T-coalgebras, let $Z_\mathcal{L}$ be the set of $\mathcal{L}$-satisfiable theories, and let f_ξ be the function obtained from a theory map Φ_ξ by restricting the domain; the following are equivalent: 1) The language $\mathcal{L}$ has the Hennessy-Milner property. 2) The assignment ζ from (2) which takes an $\mathcal{L}$-theory $\Phi = \Phi_\xi(x) \in Z_\mathcal{L}$ to $(Tf_\xi)\xi(x)$ does not depend on the choice of (ξ, x), i.e., (2) defines a function $\zeta : Z \longrightarrow TZ$.*

Proof. **Form top to bottom:** Assume we have $\Phi_{\xi_1}(x_1) = \Phi_{\xi_2}(x_2)$. Since $\mathcal{L}$ has the Hennessy-Milner property there exists a coalgebra (Y, γ) and morphisms $f_1 : \xi_1 \longrightarrow \gamma$ and $f_2 : \xi_2 \longrightarrow \gamma$ such that $f_1(x_1) = f_2(x_2)$. This combined with the adequacy of $\mathcal{L}$ and the previous lemma implies

$$\begin{aligned}\zeta(\Phi_{\xi_1}(x_1)) = T(f_{\xi_1})\xi_1(x_1) &= T(f_\gamma)\gamma f_1(x_1)\\ &= T(f_\gamma)\gamma f_2(x_2) = T(f_{\xi_2})\xi_2(x_2) = \zeta(\Phi_{\xi_2}(x_2),\end{aligned}$$

which precisely states that ζ does not depend on the choice of (ξ, x).

From bottom to top: Assume the assignation ζ does not depend on the representant (ξ, x). This implies that we have a function $\zeta : Z_\mathcal{L} \longrightarrow TZ_\mathcal{L}$. We have to show that $\mathcal{L}$ has the Hennessy-Milner property. Since each function f_ξ is obtained from the theory map Φ_ξ by restricting the domain this is almost immediate, as it is easy to check that for each coalgebra ξ the map f_ξ is a coalgebra morphism from ξ to ζ. Any two coalgebra states that are logically equivalent will be identified by the corresponding theory maps and are therefore behaviourally equivalent.

□

It is almost immediate by definition of ζ that for each coalgebra ξ the function $f_\xi : X \longrightarrow Z_{\mathcal{L}}$ is a morphism between the coalgebras ξ and ζ. We make this explicit as we will use it in the proof of Theorem 3.8.

Corollary 3.13. *Under the conditions of the previous theorem; for any coalgebra ξ, the function $f_\xi : \xi \longrightarrow \zeta$ is a morphism of coalgebras.*

This previous result already implies that a final coalgebra exists but we can do better by showing that (Z, ζ) is already a final object. The next lemma will be useful in several occasions, in particular in the proof of Theorem 3.8 and in our application to canonical models.

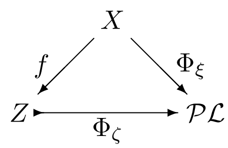
Lemma 3.14. *For a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and a language $\mathcal{L}$ that is adequate and has the Hennessy-Milner property, the theory map $\Phi_\zeta : Z \longrightarrow \mathcal{PL}$ is the inclusion, where Z is the set of satisfiable $\mathcal{L}$-theories and ζ is defined as in equation* (2)*.*

Proof. We will prove that Φ_ζ is the inclusion by showing that Z is the image of a single theory map. In other words, we will show that there exists a coalgebra (Y, γ) such that the function $f_\gamma : Y \longrightarrow Z$ is onto. From this, using that $\mathcal{L}$ is adequate and the previous corollary, it will follow that Φ_ζ is the inclusion; because we would have $i_Z f_\gamma = \Phi_\gamma = \Phi_\zeta f_\gamma$.

For each element $\Phi \in Z$ choose a representant, i.e. and coalgebra ξ_Φ and a state x in it such that $\Phi_{\xi_\Phi}(x) = \Phi$. Let $(Y, \gamma) = \coprod_Z (X_\Phi, \Phi)$ in $\mathsf{Coalg}(T)$. Since $\mathcal{L}$ is adequate and each of the coproduct inclusions is a morphism of coalgebras, we conclude that the image of $\Phi_\gamma : Y \longrightarrow \mathcal{PL}$ is Z. This concludes the proof. □

Now we have all the material to prove Theorem 3.8.

Proof of Theorem 3.8. Let Z be the set of $\mathcal{L}$-satisfiable theories, let f_ξ be the function obtained from a theory map Φ_ξ by restricting the codomain; Theorem 3.12 implies that the assignment ζ which takes a theory $\Phi_\xi(x) \in Z$ to $T(f_\xi)\xi(x)$ does not depend on the choice of (ξ, x), i.e., it is a function $\zeta : Z \longrightarrow TZ$. Corollary 3.13 implies that for each coalgebra ξ the function $f_\xi : \xi \longrightarrow \zeta$ is a morphism of coalgebras. It is only left to prove that $f_\xi : \xi \longrightarrow \zeta$ is the only morphism of coalgebras. Since the language is adequate, this will follow because any morphism of coalgebras $f : \xi_1 \longrightarrow \zeta$ makes the following diagram



commute and Lemma 3.14 tells us that the function Φ_ζ is injective. QED. $\square$

Gathering Theorem 3.6 and Theorem 3.8 we have:

Proposition 3.15. *The following are equivalent: 1) There exists a final T-coalgebra. 2) There exists an adequate language for T-coalgebras with the Hennessy-Milner property.*

The contrapositive of previous result clearly shows the power of our abstract approach as it tells us that if a functor T fails to have a final coalgebra there is no way to completely describe the behavior of T-coalgebras using an abstract coalgebraic language This is particularly relevant for applications. Also notice that the proof of Theorem 3.8 tells us a bit more about the relation of final coalgebras and abstract coalgebraic languages; we can improve Theorem 3.12 into an equivalence as follows.

Theorem 3.16. *Let $\mathcal{L}$ be a language for T-coalgebras, let Z be the set of $\mathcal{L}$-satisfiable theories and let f_ξ be the function obtained from a theory map Φ_ξ by restricting the codomain; the following are equivalent: 1) The language $\mathcal{L}$ is adequate and has the Hennessy-Milner property. 2) There exists a function $\zeta : Z \longrightarrow TZ$ which furnishes Z with a final coalgebra structure in such a way that for each coalgebra (X, ξ) the function $f_\xi : X \longrightarrow Z$ is the final map.*

Theorem 3.16. From bottom to top: The Henessy-Milner property follows because for each coalgebra ξ the map f_ξ is a morphism of coalgebras. Adequacy follows because f_ξ is the only coalgebra map $\xi \longrightarrow \zeta$. The implication from top to bottom follows from the proof of Theorem 3.8. $\square$

3.2.1 An application: Canonical Models

Until here we have illustrated that for set endofunctors, there exists a final coalgebra iff there exists an adequate abstract coalgebraic language with the Hennessy-Milner property. As mentioned before (Example 3.7), the work in [15, 9] presents interesting examples of the implication from left to right. The work on coequational logic in [17, 2] is another instantiation of this. The work on the category $Meas$ in [13, 18], which can be instantiated in Set, shows another non trivial illustration of this implication. We illustrate the implication from right to left with a construction of canonical models. In Lemma 3.14 we showed that the theory map of (Z, ζ) is the inclusion. Since the states of (Z, ζ) are the satisfiable theories of $\mathcal{L}$ we can rewrite Lemma 3.14 into a well known theorem of Modal logic.

Lemma 3.17 (Truth Lemma)**.** *Let $\mathcal{L}$ be an adequate language for T-coalgebras with the Hennessy-Milner property. Let Z be the set of $\mathcal{L}$-satisfiable theories (Definition 3.9) and let $\zeta : Z \longrightarrow TZ$ be defined as in Equation* (2)*. For any $\Phi \in Z$ and any $\varphi \in \mathcal{L}$ we have $\Phi \models_\zeta \varphi$ iff $\varphi \in \Phi$, where $\Phi \models_\zeta \varphi$ means $\varphi \in \Phi_\zeta(\Phi)$*

The previous lemma illustrates that our construction is similar to the canonical model construction from modal logic (see [5]). A natural step is to investigate completeness results for different abstract coalgebraic languages. Assuming that $\mathcal{L}$ has some notion of consistency we can ask: are maximally consistent sets satisfiable? We do not peruse this question in this paper, but notice the following result:

Proposition 3.18. *Let $\mathcal{L}$ be an adequate language for T-coalgebras, let Z be the set of $\mathcal{L}$-satisfiable theories. The set Z is the largest subset of $\mathcal{PL}$ for which we can 1) define a T-coalgebra structure ζ such that the Truth Lemma is satisfied, i.e. the theory map is the inclusion, and 2) such that for each coalgebra the codomain restiricions of the theory maps are morphisms of T-coalgebras.*

Proposition 3.18. Let $Z' \subseteq \mathcal{PL}$ be a set for which conditions 1) and 2) are satisfied, let ζ' be the mentioned coalgebraic structure. Condition 2) implies that $\mathcal{L}$ has the Hennessy-Milner property. From this, using Theorem 3.16, we conclude that there is final coalgebraic structure over Z, the set of $\mathcal{L}$-satisfiable theories, moreover, our construction tells us that the final map $f_{\zeta'} : \zeta' \longrightarrow \zeta$ is obtained by restricting the codomain of the theory map $\Phi_{\zeta'}$. This together with condition 1) implies that $Z' \subseteq Z$; the other inclusion follows from condition 2). □

As corollary we have a well known result from modal logic (see [5]).

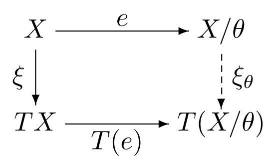
Corollary 3.19. *Let $\mathcal{L}$ be basic modal language, and let (M, μ) be the canonical model for the logic K; there exists a Kripke frame (X, ξ) for which the modal theory map $\Phi_\xi : X \longrightarrow M$ is not a bounded morphism.*

4 Behaviour & Congruences

The Hennessy-Milner property states that if two states are logically equivalent then they are identified in some coalgebra. However, this coalgebra is not made explicit. The work in the previous section provides a canonical coalgebra where logically equivalent states are identified, namely the final coalgebra. In this section we investigate another construction in order to identify logically equivalent states: taking *logical congruences.* Let us first recall the notion of a coalgebraic congruence and its equivalent characterisations.

Definition 4.1. Let (X, ξ) be a T-coalgebra. An equivalence relation θ on the set X is a *congruence of T-coalgebras* iff there exists a coalgebraic structure

$\xi_\theta : X/\theta \longrightarrow T(X/\theta)$ such that the following diagram



commutes. Here e is the canonical quotient map.

Example 4.2. If T is the covariant power set functor, two states are related by a congruence iff they are related by some bisimulation (see [5]). In fact this is the case for any functor that weakly preserves kernels (see [15] for details)

In the category Set, it can be shown that the notion of a congruence for coalgebra can be characterised as the kernel of coalgebra morphisms. In other words, it behaves like the notion of a congruence in universal algebra [8, 15].

Fact 4.3. *Let (X, ξ) be a T-coalgebra for a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, For an equivalence relation θ, on X, the following conditions are equivalent: 1) θ is a congruence of coalgebras. 2) $\theta \subseteq \mathrm{Ker}(T(e)\xi)$. 3) θ is the kernel of some morphism of T-coalgebras with domain ξ.*

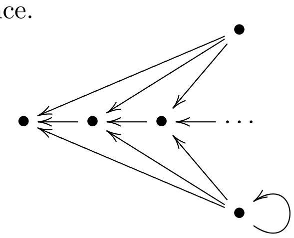
We remark that the previous characterisation of congruences depends on the fact that set functors preserve monomorphisms with non-empty domain [8], something that is not generally true in categories different from Set.

4.1 Simple Coalgebras

Before describing how behavioural equivalence relates to congruences, we discuss the notion of a *simple coalgebra*. As it is the case in algebra, the set of all congruences on a coalgebra (X, ξ) is a complete lattice under the partial ordering of set inclusion. In particular, there is a smallest congruence Δ_X (the identity relation on X) and a largest congruence. However, unlike the universal algebra case, the largest congruence may be smaller than the universal relation. Simple coalgebras are then defined as coalgebras with only one congruence.

Definition 4.4. A coalgebra $\xi : X \longrightarrow TX$ is *simple* if its largest (and hence only) congruence is the identity relation Δ_X on X.

Example 4.5. The following Kripke frame is a simple $\mathcal{P}$-coalgebra. In order to see this we have to use that, in this case, Kripke bisimilarity ([5]) coincides with the largest congruence.



Using Fact 4.3 the following holds from the definition of simple coalgebra.

Lemma 4.6. *A T-coalgebra ξ is simple iff every morphism of coalgebras with domain ξ is injective (monomorphism).*

Corollary 4.7. *If a final T-coalgebra exists, then a T-coalgebra is simple iff it is isomorphic to a subcoalgebra of the final coalgebra.*

Using coalgebraic languages we can give a more concrete characterisation of simple coalgebras; a first step is given by the following result.

Proposition 4.8. *Let T be a set functor and let $\mathcal{L}$ be an adequate language for T-coalgebras. A T-coalgebra ξ is simple if the theory map Φ_ξ is injective.*

Proposition 4.8. Assume Φ_ξ to be injective. since the language is adequate, for any morphisms $f : \xi \longrightarrow \gamma$ we have $\Phi_\xi = \Phi_\gamma f$, which implies that f is injective. Lemma 4.6 implies that ξ is simple. □

The converse of the previous proposition is not true; the Kripke frame in Example 4.5 is a counterexample. In order to obtain an equivalence in the previous proposition the Hennessy-Milner property is needed. This motivates the introduction of logical congruences.

4.2 Logical congruences

Logical congruences are congruences obtained using logical equivalence of states.

Definition 4.9. Given an abstract coalgebraic language $\mathcal{L}$, we say that two pointed coalgebras (ξ_i, x_i) are *logically equivalent*, written $(\xi_1, x_1) \leftrightsquigarrow_{\mathcal{L}} (\xi_2, x_2)$, iff $\Phi_{\xi_1}(x_1) = \Phi_{\xi_2}(x_2)$. We call $\leftrightsquigarrow_{\mathcal{L}}$ the *logical equivalence relation of states.* Given a coalgebra ξ, we write $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ for the relation $\leftrightsquigarrow_{\mathcal{L}}$ restricted to the states of ξ.

Our interest into these equivalence relations has two main reasons. The first one was an attempt to make Proposition 4.8 into an equivalence and then obtain a concrete characterisation of simple coalgebras. The second and most important motivation was to generalise Proposition 3.15 and Theorem 3.16 to arbitrary categories. To our surprise logical congruences proved to be remarkably useful to simplify our constructions. In our opinion, logical congruences provide the appropriate categorical generalisation of the Hennessy-Milner property.

Definition 4.10. Let $\mathcal{L}$ be an abstract coalgebraic language for T. For each T-coalgebra (X, ξ) we identify the *set of satisfiable theories in* ξ, $X/\leftrightsquigarrow^{\xi}_{\mathcal{L}}$, with the set $Z_\xi := \{\Phi \subseteq \mathcal{L} \,|\, (\exists x \in X)(\Phi_\xi(x) = \Phi)\}$. We use $e_\xi : X \longrightarrow Z_\xi$ for the canonical (quotient) map.

If our language happens to be adequate and have the Hennessy-Milner property we can show that logical equivalence of states is a congruence of coalgebras.

Lemma 4.11. *Let $\mathcal{L}$ be a language for T-coalgebras. If $\mathcal{L}$ is adequate and has the Hennessy-Milner property, then for each coalgebra (X, ξ) the relation $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ is a congruence of coalgebras. Moreover, $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ is the largest congruence on (X, ξ).*

Proof. The main idea is to follow the proof of Theorem 3.8, on page 8, relativized to the set Z_ξ. Explicitly this is: we define a function $\zeta_\xi : Z_\xi \longrightarrow T(Z_\xi)$ such that the canonical map e_ξ is a morphism of coalgebras. See the appendix for more details.

The function ζ_ξ is defined as follows: an element $\Phi_\xi(x) \in Z_\xi$ is mapped to $\zeta_\xi(\Phi) := T(e_\xi)(\xi(x))$. Following the argument from the proof of Theorem 3.8 it is not difficult to see that ζ_ξ is well-defined because $\mathcal{L}$ is adequate and has the Hennessy-Milner property. It is then a direct consequence of the definition of ζ_ξ that e_ξ is a coalgebra morphism from ξ to ζ_ξ. Hence $\mathrm{Ker}(e_\xi)$, which is equal to $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$, is a congruence on ξ. In order to see that $\mathrm{Ker}(e_\xi)$ is the largest congruence, one has to observe that for any $x, x' \in \xi$ with $e_\xi(x) \neq e_\xi(x')$ we have $\Phi_\xi(x) \neq \Phi_\xi(x')$ and thus, by adequacy of $\mathcal{L}$, there can be no coalgebra morphism f with $f(x) = f(x')$. □

Now we can easily make Proposition 4.8 into an equivalence.

Theorem 4.12. *Let T be a set endofunctor, and let $\mathcal{L}$ be an adequate language for T-coalgebras with the Hennessy-Milner property. A T-coalgebra ξ is simple iff the theory map Φ_ξ is injective.*

Theorem 4.12. The implication from right to left is proposition 4.8. The implication from left to right: the previous Lemma tells us that $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ is a congruence of coalgebras, hence since ξ is simple $\leftrightsquigarrow^{\xi}_{\mathcal{L}} = \Delta_X$, the identity. Notice that $Ker(\Phi_\xi) = \leftrightsquigarrow^{\xi}_{\mathcal{L}} = \Delta_X$, this concludes the proof. □

Notice that the construction used in the proof of Lemma 4.11 generalises the construction of final coalgebras of the previous section. This leads us to the following definition.

Definition 4.13. An abstract coalgebraic language $\mathcal{L}$ for T-coalgebras is said *to have logical congruences* iff for each coalgebra ξ the equivalence relation $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$, is a congruence of T-coalgebras. The quotient of ξ using $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ is called *the logical quotient of* ξ and we write (Z_ξ, ζ_ξ) for this coalgebra.

In [16] Lutz Schröder noticed that languages of predicate liftings that have logical congruences have the Hennessy-Milner property. We turn his observation into a general theorem for abstract coalgebraic languages.

Theorem 4.14. *If a language $\mathcal{L}$ for T-coalgebras is adequate, the following are equivalent: 1) $\mathcal{L}$ has the Hennessy-Milner property. 2) $\mathcal{L}$ has logical congruences.*

Proof. The implication from (4.14) to (4.14) is an immediate consequence of Lemma 4.11. Conversely suppose that $\mathcal{L}$ has logical congruences and let ξ_1, ξ_2 be T-coalgebras with logically equivalent states $x_1 \in \xi_1$ and $x_2 \in \xi_2$. Let $\xi_1 + \xi_2$

be the coproduct of ξ_1 and ξ_2 in $\mathsf{Coalg}(T)$ and let $\kappa_1(x_1), \kappa_2(x_2) \in \xi_1 + \xi_2$ be the image of x_1 and x_2 under the canonical embeddings. By adequacy of $\mathcal{L}$ it is clear that $\kappa_1(x_1)$ and $\kappa_2(x_2)$ are logically equivalent. Since $\mathcal{L}$ has logical congruences, by assumption, we can make the quotient, in $\mathsf{Coalg}(T)$, using $\leftrightsquigarrow_{\mathcal{L}}^{\xi_1+\xi_2}$; the canonical quotient map e will identify $\kappa_1(x_1)$ and $\kappa_2(x_2)$. In other words, x_1 and x_2 are identified by the morphisms $e \circ \kappa_1$ and $e \circ \kappa_2$ and are thus behaviourally equivalent. As x_1 and x_2 where arbitrary logically equivalent coalgebra states, we proved that $\mathcal{L}$ has the Hennessy-Milner property as required. □

4.2.1 An application: A Concrete Characterization of Simple Coalgebras

As mentioned before, in the work of Schröder [16] we have non trivial use of logical congruences to establish the Hennessy-Milner property for a language. Theorem 4.14 tells us that in fact the two properties are equivalent. In this section, we illustrate the construction of logical congruences (cf. proof of Lemma 4.11) giving a concrete characterization of simple coalgebras. We first make a remark concerning the theory maps of logical quotients (Definition 4.13).

Proposition 4.15. *Let $\mathcal{L}$ be an adequate language for T-coalgebras that has logical congruences and let $\xi : X \longrightarrow TX$ be a T-coalgebra. The theory map $\Phi_{\zeta_\xi} : Z_\xi \longrightarrow \mathcal{P}\mathcal{L}$ of the logical quotient of ξ (Definition 4.13) is equal to the inclusion.*

Proof. Let $e : X \longrightarrow Z_\xi$ the quotient map. By adequacy of $\mathcal{L}$ we have $\Phi_\xi(x) = \Phi_{\zeta_\xi}(e(x))$ and by definition of e we have $e(x) = \Phi_\xi(x)$ for all $x \in X$. Therefore Φ_{ζ_ξ} has to be the inclusion map. □

Now we can use logical congruences in order to characterize simple coalgebras as logical quotients, i.e. quotients using the relations $\leftrightsquigarrow_{\mathcal{L}}^{\xi}$.

Theorem 4.16. *Let $\mathcal{L}$ be an adequate language for T-coalgebras that has logical congruences. Any logical quotient (Z_ξ, ζ_ξ) is simple and any simple T-coalgebra γ is isomorphic to the logical quotient (Z_ξ, ζ_ξ) of some coalgebra ξ.*

Theorem 4.16. Theorem 4.14 tells that if $\mathcal{L}$ has logical congruences, hence the logical quotient (Z_ξ, ζ_ξ) of any T-coalgebra ξ exists; Proposition 4.15 together with Theorem 4.12 imply that each of these quotients is simple. Now we show that every simple coalgebra is isomorphic to a logical quotient of some T-coalgebra. Let $\xi : X \longrightarrow TX$ be a simple coalgebra. Since (X, ξ) is simple and $\leftrightsquigarrow_{\mathcal{L}}^{\xi}$ is a congruence, we conclude $\leftrightsquigarrow_{\mathcal{L}}^{\xi} = \Delta_X$. Therefore $(X, \xi) \cong (Z_\xi, \zeta_\xi)$. □

Using this characterization of simple coalgebras we can easily prove that truth-preserving functions with simple codomain must be coalgebra morphisms. This was a key result used by Goldblatt in [6] to construct final coalgebras.

Corollary 4.17. *Let $\mathcal{L}$ be an adequate language for T-coalgebras with the Hennessy-Milner property. Let $f : X \longrightarrow Y$ be a function with $(X, \xi), (Y, \zeta) \in$ $\mathsf{Coalg}(T)$ and ζ simple such that $\Phi_\xi(x) = \Phi_\zeta(f(x))$ for all $x \in X$. Then $f : \xi \longrightarrow \zeta$ is a coalgebra morphism.*

Proof. Let f be a truth invariant morphism whose codomain is simple. The previous Theorem implies that we can assume the codomain of f to be the logical quotient (Z_γ, ζ_γ) for some coalgebra γ; say $f : (X, \xi) \longrightarrow (Z_\gamma, \zeta_\gamma)$. Since f is truth invariant we have that $Ker(f) \subseteq \leftrightsquigarrow^\xi_\mathcal{L}$, this implies $Z_\xi \subseteq Z_\gamma$. Now using the fact that $\mathcal{L}$ is adequate and has the Hennessy-Milner property, one can prove either directly or using the construction of Theorem 3.8 that the inclusion map $i : Z_\xi \longrightarrow Z_\gamma$ is a morphism of coalgebras. This exhibits f as the composition of the quotient map e_ξ and the inclusion i, since both maps are coalgebra morphism we conclude that so is f. □

4.3 Logical Congruences & Weak Finality

The results of the previous sections already imply that the existence of logical congruences is equivalent to existence of final coalgebras. Nevertheless, we will do a direct proof of this fact because we can provide a categorical proof that can be reused in several other examples. Our main categorical tool to produce final coalgebras is Freyd's existence Theorem of a final object [11]:

Theorem 4.18. *A cocomplete category $\mathbb{C}$ has a final object iff it has a small set of objects S which is weakly final, i.e. for every object $c \in \mathbb{C}$ there exists a $s \in S$ and and arrow $c \longrightarrow s$; a final object is a colimit of the diagram induced by S[3].*

The set S, mentioned in the previous theorem, is called a **solution set**. Freyd's Theorem is strongly related to the Adjoint Functor theorem. Recall that in the category of sets every object only has a set of subobjects (subsets). In Proposition 4.11 we proved that if a language $\mathcal{L}$ is adequate and has logical congruences each coalgebra (X, ξ) can be mapped to some coalgebra of the form (Z_ξ, ζ_ξ) (Definition 4.10). This tells us that the coalgebras that are based on subsets of $\mathcal{PL}$ form a solution set, which by Freyd's theorem implies the existence of a final object. Therefore the following holds true.

Proposition 4.19. *If a language $\mathcal{L}$ is adequate and has logical congruences for T-coalgebras then there exists a final T-coalgebra which is obtained as a colimit of diagram induced by the T-coalgebras (Z_ξ, ζ_ξ) (Definitions 4.10 & 4.13).*

This proposition supplies us with another description of the final coalgebra. Moreover, following the path: Hennessy-Milner ⇒ logical congruences ⇒ final coalgebras we have another proof of Goldblatt's Theorem. This alternative proof is not as simple as the construction presented in Theorem 3.8 but illustrates the importance of adequacy. This can be restated saying that the Hennessy-Milner property is a solution set condition to obtain final coalgebras.

[3] Recall that the diagram induced by a set of objects is the inclusion functor of the full subcategory of $\mathbb{C}$ generated by S

4.3.1 Barr-Aczel-Melender Theorem

We finish our discussion on set coalgebras with the famous final coalgebra theorem. Peter Aczel and Nax Mendler [1] proved a final coalgebra theorem for set endofunctors. They showed that every set endofunctor has a final coalgebra. This final coalgebra might have, however, a proper class as carrier set. Michael Barr noticed, in [4], that

> "such result for an endofunctor on the category of sets do not, for the main part of the results, require looking at functors on the category of (possibly proper) classes. We will see here that the main results are valid for sets up to some regular cardinal. Should that cardinal be inaccessible, then Aczel and Mendler's results are derived."

We will do just as Barr described with his own result using logical congruences and languages of predicate liftings. We consider that our construction is more accessible to readers not familiar with category theory. The following definition is needed for the formulation of the theorem.

Definition 4.20. A cardinal number κ is said to be *weakly inaccessible*if κ is an uncountable, regular cardinal such that for all cardinals λ we have $\lambda < \kappa$ implies $2^{\lambda} \leq \kappa$. The cardinal κ is said inaccessible if in addition these conditions the last inequality is strict.

Intuitively κ is weakly inaccessible if one can construct κ from smaller sets only if one uses all of κ itself. One of the key results used by Barr in [4] was the following.

Lemma 4.21. *Let κ be a weakly inaccessible cardinal and let T be a κ-accessible functor. Every T-coalgebra is a colimit of coalgebras with carrier set of size strictly less than κ*

In fact this previous result already implies the existence of a final T-coalgebra. Barr's proof uses this result which implies the existence of a set of generators and then produces a final coalgebra. We can also ignore generators and use the Hennessy-Milner property to produce a solution set which by Freyds Theorem will conceive a final coalgebra.

Theorem 4.22. *Let κ be a weakly inaccessible cardinal and let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a κ-accessible functor. Suppose, in addition, that for all sets X we have $|X| < \kappa$ implies $|TX| < \kappa$. Then $Coalg(T)$ has a final coalgebra of cardinality no larger than κ.*

Proof. Let $\mathcal{L}^{\kappa}(\Lambda)$ be the language with all predicate liftings, for T, of arity less than κ and conjunctions bounded by κ. As we have mentioned before, this language is adequate and has the Hennessy-Milner property. Theorem 3.8 implies that there exists a final T-coalgebra (Z, ζ). Notice that $|\mathcal{L}^{\kappa}(\Lambda)| = \kappa$, therefore $|Z| \leq 2^{\kappa}$. We will use logical congruences to show $|Z| \leq \kappa$.

Lemma 4.19 states that we can obtain a final coalgebra using logical congruences as a colimit of the coalgebras $Z_\xi = (X/\leftrightsquigarrow_{\mathcal{L}}, \zeta_\xi)$. The previous lemma tells us that it is enough to consider coalgebras with a carrier set such that $|X| < \kappa$. This implies that $|Z_\xi| < \kappa$ for each coalgebra ξ.

Using that $|X| < \kappa$ implies $|TX| < \kappa$ and κ is weakly inaccessible, we can conclude that there are, up to isomorphisms, κ-any coalgebras with carrier set $|X| < \kappa$. Therefore there at most κ-many coalgebras of the form Z_ξ.

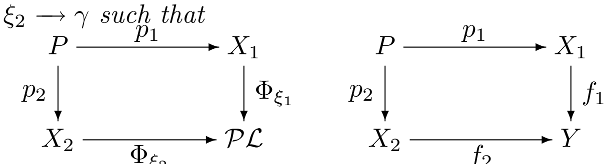
Gathering the previous two paragraphs we conclude that a final T-coalgebra can be obtained as a colimit of at most κ-many sets of cardinality less that κ. Since κ is weakly inaccessible we conclude $|Z| \leq \kappa$. □

4.4 Summary

In summary, in the category of sets and functions, under the assumption of adequacy we have the following equivalents of the Hennessy-Milner property.

Theorem 4.23. *Let $\mathcal{L}$ be an adequate language for T-coalgebras. The following conditions are equivalent:*

1. *$\mathcal{L}$ has the Hennessy-Milner property.*
2. *The function $\zeta : Z_{\mathcal{L}} \longrightarrow TZ_{\mathcal{L}}$ from equation (2), page 6, on the set*
$$Z_{\mathcal{L}} = \{\Phi \subseteq \mathcal{L} \,|\, (\exists \xi)(\exists x \in \xi)(\Phi_\xi(x) = \Phi\}$$
of satisfiable $\mathcal{L}$-theories is well-defined.
3. *The set $Z_{\mathcal{L}}$ admits a coalgebraic structure, for T, such that for each coalgebra ξ the function $f_\xi : X \longrightarrow Z_{\mathcal{L}}$, i.e. the restriction of the codomain of the theory map Φ_ξ, is a morphism of coalgebras.*
4. *For each coalgebra ξ the relation $\leftrightsquigarrow^{\xi}_{\mathcal{L}}$ is a congruence of coalgebras.*
5. *For each coalgebra ξ the set of satisfiable theories in ξ*
$$Z_\xi = \{\Phi \subseteq \mathcal{L} \,|\, (\exists x \in \xi)(\Phi_\xi(x) = \Phi\}$$
admits a coalgebraic structure $\zeta_\xi : Z_\xi \longrightarrow TZ_\xi$, such that the function $e_\xi : X \longrightarrow Z_\xi$ mapping a state $x \in \xi$ to its $\mathcal{L}$-theory $e_\xi(x) = \Phi \in Z_\xi$ is a morphism of coalgebras.
6. *Let (X_1, ξ_1) and (X_2, ξ_2) be T-coalgebras. If the diagram on the left is a pullback (in Set), there exists a coalgebra (Y, γ) and morphisms $f_1 : \xi_1 \longrightarrow \gamma; f_2 : \xi_2 \longrightarrow \gamma$ such that*



the diagram on the right commutes (in Set).

Proof. The equivalence between 1) and 2) is the content of Theorem 3.12. The implication from 2) to 3) is obvious and the converse direction is a consequence of the definition of ζ: any map $\zeta' : Z_{\mathcal{L}} \longrightarrow TZ_{\mathcal{L}}$ that turns the theory maps f_ξ into coalgebra morphisms must be obviously equal to ζ. The equivalence between 1) and 4) follows from Theorem 4.14. Item 4) is equivalent to item 5) because $\leftrightsquigarrow^{\xi}_{\mathcal{L}} = \mathrm{Ker}(\Phi_\xi) = \mathrm{Ker}(f_\xi)$. Finally the equivalence between 6) and 1) is immediate from the canonical characterisation of pullbacks in Set. □

5 Generalization to other categories

In this section we show how the result for coalgebras on Set can be generalised to coalgebras over other base categories. The first part of the section discusses how to generalise the notion of a language to a functor $T : \mathbb{C} \longrightarrow \mathbb{C}$ on an arbitrary category $\mathbb{C}$. After that we focus on a special class of categories, those that are regularly algebraic over Set, and show that the results from the previous section generalise smoothly to these categories. Due to space limitations we cannot provide the necessary categorical definitions. Instead we refer the reader to [3] where all our terminology is explained.

When generalising the notion of an abstract coalgebraic language to categories other than Set we face the problem that we do not know much about the structure of the given base category $\mathbb{C}$. In particular, unlike in the case $\mathbb{C} = \mathsf{Set}$, we do not know how to move freely from an object $\mathcal{L}$ representing the formulas to an object $\mathcal{PL}$ that represents the theories of a given language. This leads us to the following definition of an adequate object for T-coalgebras.

Definition 5.1. Let T be a functor $T : \mathbb{C} \longrightarrow \mathbb{C}$. An object $\mathcal{L}$, in $\mathbb{C}$ is an *adequate object* for T-coalgebras if there exists a natural transformation $\Phi : U \longrightarrow \Delta_{\mathcal{L}}$, where $U : \mathsf{Coalg}(T) \longrightarrow \mathbb{C}$ is the forgetful functor and $\Delta_{\mathcal{L}} : \mathsf{Coalg}(T) \longrightarrow \mathbb{C}$ is the constant functor with value $\mathcal{L}$. We call the components of Φ *theory morphisms*.

At first sight it is not completely clear why our definition of an adequate object for T-coalgebras gives in general a good formalization of what a language for T-coalgebras is. Under the additional assumption, that we are looking at a category $\mathbb{C}$ that is dual to some category $\mathbb{A}$, our notion seems to be quite natural.

Example 5.2.

- For $\mathbb{C} = \mathsf{Set}$ we have $\mathcal{L}$ is an adequate abstract coalgebraic language (Definition 3.1) for T with theory maps $\{\Phi_\xi\}_{\xi \in \mathsf{Coalg}(T)}$ iff $\mathcal{PL}$ together with $\{\Phi_\xi\}_{\xi \in \mathsf{Coalg}(T)}$ is an adequate object for T-coalgebras.
- Let $\mathbb{C} = \mathsf{Stone}$ the category of Stone spaces and continuous functions and let $T : \mathsf{Stone} \longrightarrow \mathsf{Stone}$ be a functor. We can use the duality between Stone and the category BA of Boolean algebras to see that an adequate object $\mathcal{L}$ for T-coalgebras corresponds to some Boolean algebra $A_{\mathcal{L}}$. Hence $\mathcal{L}$ is, again by duality, isomorphic to the collection of ultrafilters (=theories) over $A_{\mathcal{L}}$.

In order to arrive at a generalisation of a language for T-coalgebras which has the Hennessy-Milner property, we use the results that we obtained in Section 4. Theorem 4.23 shows that there are at least three ways to obtain this generalisation.

Definition 5.3. Let $\mathcal{L}$ be an adequate object for T-coalgebras for a functor $T : \mathbb{C} \longrightarrow \mathbb{C}$. We say $\mathcal{L}$ is *almost final* if $\mathcal{L}$ has a subobject $m : Z \longrightarrow \mathcal{L}$ that can be uniquely lifted to a final T-coalgebra (Z, ζ) such that $m = \Phi_\zeta$. If the base category $\mathbb{C}$ has pullbacks we say $\mathcal{L}$ has the *Hennessy-Milner property* if every pullback (P, p_1, p_2) (in $\mathbb{C}$) of theory morphisms Φ_{ξ_1} and Φ_{ξ_2} can be factored (in $\mathbb{C}$) using a pair of coalgebra morphisms. Finally, if the base category $\mathbb{C}$ is $(RegEpi, Mono)$-structured, we say $\mathcal{L}$ has *logical congruences* if for each theory morphism Φ_ξ and each $(RegEpi, Mono)$-factorization (e, Z_ξ, m) of Φ_ξ, there exists a coalgebraic structure $\zeta_\xi : Z_\xi \longrightarrow T(Z_\xi)$ such that e is a coalgebra morphism from ξ to ζ_ξ.

As we proved in Theorem 4.23, all of the three notions from the previous definition are equivalent if our base category $\mathbb{C}$ is Set. How do they relate in other categories?

Proposition 5.4. *Let $\mathbb{C}$ be a cocomplete and* (RegEpi, Mono)*-structured category with pullbacks. Let T be an endofunctor on $\mathbb{C}$ and let $\mathcal{L}$ be an adequate object for T-coalgebras which is wellpowered. We have: $\mathcal{L}$ has logical congruences $\Rightarrow$ $\mathcal{L}$ is almost final $\Rightarrow$ $\mathcal{L}$ has the Hennessy-Milner property. Furthermore, if T preserves monomorphisms, the converse implications are true as well and thus all three notions are equivalent.*

Proposition 5.4. Let $\mathbb{C}$ be a category that satisfies the conditions of the proposition let $T : \mathbb{C} \longrightarrow \mathbb{C}$ be a functor and let $\mathcal{L}$ be an adequate object for T-coalgebras. Suppose that $\mathcal{L}$ has logical congruences, let $\xi : X \longrightarrow TX$ be an arbitrary T-coalgebra and let (e_ξ, Z_ξ, m_ξ) be the (RegEpi, Mono)-factorisation of Φ_ξ, ie., $e_\xi : X \longrightarrow Z_\xi$ is a regular epi, $m_\xi : Z_\xi \longrightarrow \mathcal{L}$ is a monomorphism and $\Phi_\xi = m_\xi e_\xi$. By our assumption that $\mathcal{L}$ has logical congruences, there exists some morphism $\zeta_\xi : Z_\xi \longrightarrow TZ_\xi$ such that $e_\xi : \xi \longrightarrow \zeta_\xi$ is a coalgebra morphism. As ξ was arbitrary this shows that for any T-coalgebra ξ there is a subobject Z_ξ of $\mathcal{L}$ that carries some coalgebra structure $\zeta_\xi : Z_\xi \longrightarrow TZ_\xi$ such that there exists a coalgebra morphism $e : \xi : X \longrightarrow Z_\xi$. By wellpoweredness of $\mathcal{L}$ the collection of subobjects Z_ξ of $\mathcal{L}$, and hence also the collection S of T-coalgebras based on subobjects of $\mathcal{L}$, forms a set. Therefore we proved S is a weakly final in $\mathsf{Coalg}(T)$ in the sense of Theorem 4.18 and an application of this theorem proves the existence of a final T-coalgebra (Z, ζ). It is not difficult to see that $\Phi_\zeta : Z \longrightarrow \mathcal{L}$ is injective which finishes that proof of the fact that $\mathcal{L}$ is almost final.

Let us now assume that $\mathcal{L}$ is almost final. We have to show that $\mathcal{L}$ has the Hennessy-Milner property. To this aim consider two T-coalgebras (X_1, ξ_1) and (X_2, ξ_2) and their respective theory maps Φ_{ξ_1} and Φ_{ξ_2}. Furthermore let (P, p_1, p_2) be the pullback of Φ_{ξ_1} and Φ_{ξ_2}, ie., $\Phi_{\xi_1} p_1 = \Phi_{\xi_2} p_2$. As $\mathcal{L}$ is almost final, there exists some subobject $m : Z \longrightarrow \mathcal{L}$ of $\mathcal{L}$ and some T-coalgebra structure $\zeta : Z \longrightarrow TZ$ such that ζ is the final T-coalgebra and such that $m = \Phi_\zeta$. Let $f_{\xi_1} : \xi_1 \longrightarrow \zeta$ and $f_{\xi_2} : \xi_2 \longrightarrow \zeta$ be the unique coalgebra morphisms from ξ_1 and ξ_2 into ζ. We have

$$\begin{aligned} m f_{\xi_1} p_1 &= \Phi_\zeta f_{\xi_1} p_1 \stackrel{\text{(adequacy)}}{=} \Phi_{\xi_1} p_1 \\ &= \Phi_{\xi_2} p_2 \stackrel{\text{(adequacy)}}{=} \Phi_\zeta f_{\xi_2} p_2 = m f_{\xi_2} p_2, \end{aligned}$$

and as m is a monomorphism this implies $f_{\xi_1} p_1 = f_{\xi_2} p_2$, ie., the pullback "factors" through f_{ξ_1} and f_{ξ_2} as required. This demonstrates that $\mathcal{L}$ has the Hennessy-Milner property.

Consider now some functor T that preserves monomorphisms and assume $\mathcal{L}$ is an adequate object for T-coalgebras that has the Hennessy-Milner property. We have to prove that $\mathcal{L}$ has logical congruences. Let (X, ξ) be a T-coalgebra and let (e_ξ, Z_ξ, m_ξ) be a $(\text{RegEpi}, \text{Mono})$-factorization of Φ_ξ. As e_ξ is a regular epimorphism there exist two morphisms $q_1 : Y \longrightarrow X$ and $q_2 : Y \longrightarrow X$ such that e_ξ is the coequalizer of q_1 and q_2. Using the Hennessy-Milner property of $\mathcal{L}$ one can show that there is a T-coalgebra (X', ξ') and a T-coalgebra morphism $e' : \xi \longrightarrow \xi'$ such that $e' q_1 = e' q_2$. By adequacy of $\mathcal{L}$ we obtain

$$m_\xi e_\xi = \Phi_\xi = \Phi_{\xi'} e'. \tag{3}$$

Furthermore we have the following:

$$\begin{aligned} (Tm_\xi)(Te_\xi)\xi q_1 &= T(m_\xi e_\xi)\xi q_1 \\ &\stackrel{(3)}{=} T(\Phi_{\xi'} e')\xi q_1 = T(\Phi_{\xi'})(Te')\xi q_1 \\ &\stackrel{(e' \text{ co.morph.})}{=} T(\Phi_{\xi'})\xi' e' q_1 = T(\Phi_{\xi'})\xi' e' q_2 \\ &\vdots \quad \text{(use argument backwards)} \\ &= (Tm_\xi)(Te_\xi)\xi q_2 \end{aligned}$$

As m_ξ was a monomorphism and because T preserves monomorphisms we have that Tm_ξ is a monomorphism as well. Therefore we obtain $(Te_\xi)\xi q_1 = (Te_\xi)\xi q_2$, ie., $(Te_\xi)\xi$ is a "competitor" of the coequalizer e_ξ of q_1 and q_2. This implies that there exists a unique morphism $\zeta_\xi : Z_\xi \longrightarrow TZ_\xi$ such that $\zeta_\xi e_\xi = (Te_\xi)\xi$, ie., such that e_ξ is a coalgebra morphism from ξ to ζ_ξ. Moreover adequacy of $\mathcal{L}$ implies that $m_\xi e_\xi = \Phi_\xi = \Phi_{\zeta_\xi} e_\xi$ which implies that $m_\xi = \Phi_{\zeta_\xi}$ because e_ξ is an epimorphism. This shows that ζ_ξ is the logical quotient of ξ as required. □

We can lift regular factorisations of morphisms from Set to $\mathbb{A}$ as the following lemma shows. This lemma is needed for the proof of Theorem 5.6.

Lemma 5.5. *Let $\mathbb{A}$ be a category that is regularly algebraic over Set, let $f : A \longrightarrow C$ be an $\mathbb{A}$-morphism and let $(e : VA \longrightarrow X, m : X \longrightarrow VC)$ be a $(\text{RegEpi}, \text{Mono})$-factorization of Vf in Set. Then there exists a $B \in \mathbb{A}$ and morphisms $e^{\mathbb{A}} : A \longrightarrow B$, $m^{\mathbb{A}} : B \longrightarrow C$ such that $VB = X$, $(e^{\mathbb{A}} : A \longrightarrow B, m^{\mathbb{A}} : B \longrightarrow C)$ is a regular factorization of f in $\mathbb{A}$ and such that $Ve^{\mathbb{A}} = e$, $Vm^{\mathbb{A}} = m$.*

Proof. Suppose that $(e : VA \longrightarrow X, m : X \longrightarrow VC)$ is a (RegEpi, Mono)-factorization of Vf in Set. Furthermore let $(e' : A \longrightarrow B', m' : B' \longrightarrow C)$ be a (RegEpi, Mono)-factorization of f in $\mathbb{A}$. The forgetful functor V preserves regular epimorphisms and V is right adjoint and thus preserves monos. Hence it is easy to see that (Ve', Vm') is a factorization of Vf in Set. Therefore there exists an isomorphism $i : VB' \longrightarrow X$ in Set. By unique transportability of $\mathbb{A}$ there exists a unique object $B \in \mathbb{A}$ and an isomorphism $i' : B' \longrightarrow B$ such that $VB = X$ and $Vi' = i$. Obviously $V(i' \circ e') = e$, $V(m' \circ i^{-1}) = m$. It is now easy to see that $e^{\mathbb{A}} := i' \circ e'$ and $m^{\mathbb{A}} := m' \circ i^{-1}$ fulfil the requirements of the lemma. □

In particular, the previous proposition demonstrates that under mild assumptions on our base category we can establish the existence of a final coalgebra for a functor T by proving that there exists some adequate object for T-coalgebras that has logical congruences. We use this fact in order to prove the following theorem.

Theorem 5.6. *Let $\mathbb{A}$ be a category that is regularly algebraic over* Set *with forgetful functor $V : \mathbb{A} \longrightarrow$* Set *and let $T : \mathbb{A} \longrightarrow \mathbb{A}$ be a functor. The functor T has a final coalgebra iff there exists an adequate object $\mathcal{L}$ for T-coalgebras that has the Hennessy-Milner property.*

Sketch Theorem 5.6. The fact that the existence of a final T-coalgebra implies the existence of an adequate object that has the Hennessy-Milner property is easy. For the converse direction of the theorem, suppose that there is an adequate object $\mathcal{L}$ that has the Hennessy-Milner property. We note first that any regularly algebraic category $\mathbb{A}$ is (RegEpi, Mono)-structured is cocomplete, wellpowered and has pullbacks (cf. [3, Sec. 23]). Therefore by Proposition 5.4 for proving the existence of a final T-coalgebra it suffices to prove that $\mathcal{L}$ has logical congruences. Let (X, ξ) be a T-coalgebra with theory map $\Phi_\xi : X \longrightarrow \mathcal{L}$. Now consider the canonical factorization (e'_ξ, Z'_ξ, m'_ξ) of $V\Phi_\xi$ via the set

$$Z'_\xi = \{\Phi \mid (\exists x \in X)(\Phi = V\Phi_\xi(x))\}.$$

Since $\mathbb{A}$ is regularly algebraic over Set, this factorization can be lifted to a factorization (e_ξ, Z_ξ, m_ξ) in $\mathbb{A}$ of Φ_ξ with $VZ_\xi = Z'_\xi$ (cf. Lemma 5.5). Using the argument from the proof of Theorem 3.8 we define a function $\zeta'_\xi : Z'_\xi \longrightarrow VTZ_\xi$ such that the following diagram

$$\begin{array}{ccc} VX & \xrightarrow{Ve_\xi} & VZ_\xi \\ {\scriptstyle V\xi}\downarrow & & \downarrow{\scriptstyle \zeta'_\xi} \\ VTX & \xrightarrow[VT(e_\xi)]{} & VTZ_\xi \end{array}$$

commutes in Set. It is only left to proof that ζ'_ξ can be lifted to an $\mathbb{A}$-morphism. In order to show this notice that because e'_ξ is onto it is a coequalizer in Set.

Since forgetful functors from regular algebraic categories reflect coequalizers we have that e_ξ is a coequalizer in $\mathbb{A}$. Say that e_ξ coequalizes $\mathbb{A}$-morphisms p and q. Using the square above it is easy to show that $V(T(e_\xi)\xi p) = V(T(e_\xi)\xi q)$, which implies $T(e_\xi)\xi p = T(e_\xi)\xi q$, because V is faithful. Now using the universal property of coequalizers, we conclude that there is an $\mathbb{A}$-morphism $\zeta_\xi : Z_\xi \longrightarrow TZ_\xi$ which makes the square above commute in $\mathbb{A}$. Using on more time the faithfulness of V we conclude that this morphism must have ζ'_ξ as underlying function.

□

Our results apply to any category of algebras such as the category BA of Boolean algebras and the category DL of distributive lattices, but also to categories like the category Stone of Stone spaces. We hope to be able to extend the scope of Theorem 5.6 to categories that are topological over Set such as the category Meas of measurable spaces.

6 Conclusions

In this paper, we have studied three ways to express behavioural equivalence of coalgebra states: using final coalgebras, using coalgebraic languages that have the Hennessy-Milner property and using coalgebraic languages that have logical congruences. We provided a simple proof for the fact that these three different methods are equivalent when used to express behavioural equivalence between set coalgebras. As by-products of our proof we obtained a straightforward construction of final coalgebras as canonical models of coalgebraic logics and a concrete characterisation of simple coalgebras as logical quotients.

A main topic for further research is that of abstract coalgebraic languages for functors on categories different from Set. Section 5 illustrates how abstract coalgebraic languages can be generalised to arbitrary categories. The main result of this section states that for a functor on any category that is regularly algebraic over Set an adequate object with the Hennessy-Milner property exists iff there exists a final coalgebra (Theorem 5.6). The proof of this theorem demonstrates that logical congruences are useful in order to prove the existence of a final coalgebra. A crucial ingredient for the proof is Freyd's Theorem (Thm 4.18). Our hope is that the scope of Theorem 5.6 can be extended to a larger class of categories that satisfy the conditions of Freyd's theorem.

Another gain of using logical congruences is that they revealed that the Hennessy-Milner property is related to the description of a particular factorisation structure (cf. Def. 5.3, Prop. 5.4). In our paper we considered $(RegEpi, Mono)$-structured categories, but it is quite natural to generalise the results here to other factorisation structures. We believe that a study of these factorisations will lead to a coalgebraic understanding of non standard bisimulations as, for example, discussed in [7]. A step in this direction has already been made in [10]. There it was shown that using logically invariant morphisms between coalgebras, that are not necessarily coalgebra morphisms, canonical models for abstract coalgebraic languages can be presented as final objects.

References

[1] Peter Aczel and Nax Paul Mendler, *A final coalgebra theorem*, Category Theory and Computer Science, 1989, pp. 357–365.

[2] J. Adámek, *A logic of coequations*, CSL (C.-H. Luke Ong, ed.), Lecture Notes in Computer Science, vol. 3634, Springer, 2005, pp. 70–86.

[3] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories: The joy of cats*, John Wiley & Sons, 1990.

[4] M. Barr, *Terminal Coalgebras in Well-Founded Set Theory*, Theoretical Computer Science **114** (1993), no. 2, 299–315.

[5] P. Blackburn, M. de Rijke, and Y. Venema, *Modal logic*, Cambridge Tracts in Theoretical Computer Science, vol. 53, Cambridge University Press, 2002.

[6] R. Goldblatt, *Final coalgebras and the Hennessy-Milner property.*, Annals of Pure and Applied Logic **138** (2006), no. 1-3, 77–93.

[7] V. Goranko and M. Otto, *Handbook of Modal Logic*, ch. Model Theory of Modal Logic, pp. 255–325, Elsevier, 2006.

[8] H.P. Gumm, *Elements of the General Theory of Coalgebras*, Tech. report, Rand Africaans University, Johannesburg, South Africa, 1999, Preliminary version.

[9] B. Jacobs and J. Rutten, *A tutorial on (co)algebras and (co)induction*, EATCS Bulletin **62** (1997), 62–222.

[10] A. Kurz and D. Pattinson, *Coalgebraic Modal Logic of Finite Rank*, Mathematical Structures in Computer Science **15** (2005), no. 3, 453–473.

[11] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971, Graduate Texts in Mathematics, Vol. 5.

[12] L. S. Moss, *Coalgebraic Logic.*, Annals of Pure and Applied Logic **96** (1999), no. 1-3, 277–317, Erratum published Ann.P.Appl.Log 99:241-259, 1999.

[13] L.S. Moss and I.D. Viglizzo, *Final coalgebras for functors on measurable spaces*, Information and Computation **204** (2006), no. 4, 610–636.

[14] D. Pattinson, *Coalgebraic Modal Logic: Soundness, Completeness and Decidability of Local Consequence*, Theoretical Computer Science **309** (2003), no. 1–3, 177–193.

[15] J. J. M. M. Rutten, *Universal coalgebra: a theory of systems*, Theoretical Computer Science **249** (2000), no. 1, 3–80.

[16] L. Schröder, *Expressivity of Coalgebraic Modal Logic: The Limits and Beyond*, Theoretical Computer Science **390** (2008), 230–247.

[17] D. Schwencke, *Coequational Logic for Finitary Functors*, Electronic Notes in Theoretical Computer Science **203** (2008), no. 5, 243–262.

[18] I. D. Viglizzo, *Final sequences and final coalgebras for measurable spaces.*, CALCO 2005 (J. L. Fiadeiro et al., ed.), Lecture Notes in Computer Science, vol. 3629, Springer, 2005, pp. 395–407.