

THE AXIOM OF REAL BLACKWELL DETERMINACY

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ABSTRACT. In the theory of infinite games with slightly imperfect information, the main question (whether the axiom of determinacy and the axiom of Blackwell determinacy are equivalent) is still open. In this paper, we shift the discussion to games with uncountably many possible moves, introducing the axiom of real Blackwell determinacy $\text{Bl-AD}_{\mathbb{R}}$ (as an analogue of the axiom of real determinacy $\text{AD}_{\mathbb{R}}$). We prove that the consistency strength of $\text{Bl-AD}_{\mathbb{R}}$ is strictly greater than that of AD .

1. INTRODUCTION & BACKGROUND

Infinite perfect information games (called *Gale-Stewart games* after [GS53]) play a central rôle in the foundations of mathematics via the investigation of so-called *determinacy axioms*, among them the *Axiom of Determinacy* AD and the *Axiom of Real Determinacy* $\text{AD}_{\mathbb{R}}$. In spite of the fact that these two axioms contradict the axiom of choice, their foundational significance can hardly be overestimated.

Blackwell games are the analogue of Gale-Stewart games without perfect information. Full imperfect information games have so far proved intractable, so in these games, we are restricting the lack of perfect information to an infinite sequence of simultaneously made moves. They were introduced by Blackwell in 1969 [Bla69], and dubbed “games with slightly imperfect information” by him in his [Bla97]. These games allowed new proofs of known consequences of determinacy (e.g., Vervoort’s alternative proof of Lebesgue measurability from determinacy [Ver96, Theorem 4.3] inspired Martin’s derived proof in [Mar03]). Due to the fact that perfect information games can be seen as one of the extreme cases of games of partial information, it was natural to assume

2000 *Mathematics Subject Classification.* 03E60, 03E15, 03E35, 91A05, 91A44.

Key words and phrases. Real Blackwell Determinacy, Sharps, Consistency Strength, Real Determinacy, Axiom of Choice.

The research of the first author was supported by a GLoRiClass fellowship funded by the European Commission (Early Stage Research Training Mono-Host Fellowship MEST-CT-2005-020841); the work of the second author was done while he was at the *Vrije Universiteit Amsterdam*. The authors should like to thank Vincent Kieftenbeld (Denton TX) for discussions in the early phase of this project.

that *Blackwell determinacy axioms* would be stronger than regular determinacy axioms. However, this was not the case: in [Mar98], Martin proved that in most cases, Blackwell determinacy axioms follow from the corresponding determinacy axiom. Martin conjectured that they are equivalent, and many instances of equivalence have been shown (e.g., [Löw04, Corollary 3.9] and [MNV03]). However, the general question, and in particular the most intriguing instance, viz. whether AD and the axiom of Blackwell determinacy BI-AD are equivalent, remain open.

In this paper, we turn to the other mentioned determinacy axiom, the stronger $\text{AD}_{\mathbb{R}}$ and its Blackwell analogue. We shall introduce the *Axiom of Real Blackwell Determinacy* $\text{BI-AD}_{\mathbb{R}}$ and investigate its relationship to $\text{AD}_{\mathbb{R}}$. The axiom of real determinacy has been studied by Solovay in his masterful analysis [Sol78]. Its Blackwell analogue was introduced in [dK05] in two variants, the *countable support* variant and the *Euclidean* variant. While we give the definition of the Euclidean variant below, we will only be concerned with the countable support variant here, and denote it by $\text{BI-AD}_{\mathbb{R}}$. We follow Solovay's lead and provide the results analogous to [Sol78] for $\text{BI-AD}_{\mathbb{R}}$. Our main result is:

Main Theorem 1. Assume $\text{BI-AD}_{\mathbb{R}}$. Then there is a fine normal measure on $\wp_{\omega_1}(\mathbb{R})$, and hence \aleph_1 is $<\Theta$ -supercompact and $\mathbb{R}^{\#}$ exists. In particular, the consistency strength of $\text{BI-AD}_{\mathbb{R}}$ is strictly greater than that of AD.

We would like to point the reader interested in more background to the survey paper [Löw05] written by the third author. It contains a more detailed discussion of the various versions of Blackwell determinacy axioms. It also contains a discussion of $\text{BI-AD}_{\mathbb{R}}$, a proof of the fact that $\text{BI-AD}_{\mathbb{R}}$ does not follow from BI-AD, and a cautious mention of the main result of this paper.

In the following three sections, we first give all necessary definitions to make the proof of the Main Theorem self-contained (§2) and then discuss Solovay's analysis of $\text{AD}_{\mathbb{R}}$ and reduce the Main Theorem to the existence of a fine normal measure (§3), and finally (§4) prove the existence of a fine normal measure.

2. DEFINITIONS

2.1. Blackwell Determinacy. We are using standard notation from set theory and assume familiarity with descriptive set theory throughout the paper. As usual in set theory, we shall be working on Baire space ${}^{\omega}\omega$ instead of the ordinary real numbers. Throughout we shall work in the theory $\text{ZF} + \text{AC}_{\omega}(\mathbb{R})$. This small fragment of the axiom of

choice is necessary for the definition of axioms of Blackwell determinacy. Using $\text{AC}_\omega(\mathbb{R})$, we can develop the basics of measure theory. If we need more than $\text{ZF} + \text{AC}_\omega(\mathbb{R})$ for some definitions and statements, we explicitly mention the additional axioms.

Let X be a set with more than one elements and assume $\text{AC}_\omega({}^\omega X)$. The case most interesting for us is $X = \mathbb{R}$. Since there is a bijection between ${}^\omega \mathbb{R}$ and \mathbb{R} , the axioms $\text{AC}_\omega({}^\omega \mathbb{R})$ and $\text{AC}_\omega(\mathbb{R})$ are equivalent. By $\text{Prob}(X)$, we denote the set of all Borel probability measures on X with a countable support, i.e., the set of all Borel probability measures p such that there is a countable set $C \subseteq X$ with $p(C) = 1$.¹ From now on, we regard X as a discrete topological space and topologize ${}^\omega X$ as the product space. For any finite sequence s of elements in X , let $[s]$ be the basic open set generated by s , i.e. $[s] = \{x \in {}^\omega X; s \subseteq x\}$.

Let X^{Even} (X^{Odd}) be the set of finite sequences in X with even (odd) length. We call a function $\sigma: X^{\text{Even}} \rightarrow \text{Prob}(X)$ a *mixed strategy for player I* and a function $\tau: X^{\text{Odd}} \rightarrow \text{Prob}(X)$ a *mixed strategy for player II*. Given mixed strategies σ, τ for players I and II, respectively, let $\nu(\sigma, \tau): {}^{<\omega} X \rightarrow \text{Prob}(X)$ as follows: for each finite sequence s of elements in X ,

$$\nu(\sigma, \tau)(s) = \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \text{lh}(s) \text{ is odd,} \end{cases}$$

where $\text{lh}(s)$ is the length of s . Since some of the calculations in this paper require a lot of parentheses, let us reduce their number by convention. If (x_0, \dots, x_n) is a finite sequence, we write $[x_0, \dots, x_n]$ for the basic open set $[(x_0, \dots, x_n)]$. Similarly, if $x \in X$ and $\mu \in \text{Prob}(X)$, we write $\mu(x)$ for $\mu(\{x\})$. Now, for each finite sequence s of elements in X , define

$$\mu_{\sigma, \tau}([s]) = \prod_{i=0}^{\text{lh}(s)-1} \nu(\sigma, \tau)(s \upharpoonright i)(s(i)).$$

By using $\text{AC}_\omega(\mathbb{R} \times {}^\omega X)$ (which follows from $\text{AC}_\omega({}^\omega X)$), we can uniquely extend $\mu_{\sigma, \tau}$ to a Borel probability measure on ${}^\omega X$, i.e., the probability measure whose domain is the set of all Borel sets in ${}^\omega X$. Let us also use $\mu_{\sigma, \tau}$ for denoting this Borel probability measure.

Let A be a subset of ${}^\omega X$. A mixed strategy σ for player I is *optimal in A* if for any mixed strategy τ for player II, A is $\mu_{\sigma, \tau}$ -measurable and $\mu_{\sigma, \tau}(A) = 1$. Similarly, a mixed strategy τ for player II is *optimal in*

¹We are going to amalgamate a sequence of such measures to produce a product measure on ${}^\omega X$ as we construct the Lebesgue measure on ${}^\omega \omega$. For this purpose, the condition of having a countable support is essential.

A if for any mixed strategy σ for player I, A is $\mu_{\sigma,\tau}$ -measurable and $\mu_{\sigma,\tau}(A) = 0$. We say that A is *Blackwell determined* if either player I or II has an optimal strategy in A . Finally, BI-AD_X is the statement “for any subset A of ${}^\omega X$, A is Blackwell determined.”²

Remark 2. For any X , AD_X implies BI-AD_X . In particular, $\text{AD}_{\mathbb{R}}$ implies $\text{BI-AD}_{\mathbb{R}}$. If there is an injective map from X to Y , then BI-AD_Y implies BI-AD_X . In particular, $\text{BI-AD}_{\mathbb{R}}$ implies $\text{BI-AD} := \text{BI-AD}_{\omega}$. Furthermore, if DC holds and BI-AD implies AD , then $\text{BI-AD}_{\mathbb{R}}$ implies $\text{AD}_{\mathbb{R}}$.

2.2. Blackwell Determinacy and Choice. The third author proved in 2005 that $\text{BI-AD}_{\mathbb{R}}$ proves fragments of the axiom of choice that allow us to separate it (in terms of implication, not yet in consistency strength) from BI-AD .

Theorem 3. If $X := Y \cup Z$ is linearly ordered and $\text{AC}_{\omega}({}^\omega X)$ and BI-AD_X hold, then $\text{AC}_Y(Z)$ holds.

Proof. [Löw05, Theorem 9.3]. □

Corollary 4. Therefore, $\text{BI-AD}_{\mathbb{R}}$ implies $\text{AC}_{\mathbb{R}}(\mathbb{R})$, and BI-AD cannot prove $\text{BI-AD}_{\mathbb{R}}$.

Proof. The first claim is an immediate consequence of Theorem 3. It is well-known that if \mathbb{R} is not wellordered, then $\text{AC}_{\mathbb{R}}(\mathbb{R})$ is false in $\mathbf{L}(\mathbb{R})$. But if $\text{BI-AD}_{\mathbb{R}}$ is true in \mathbf{V} , then $\mathbf{L}(\mathbb{R}) \models \text{BI-AD}$ and thus we have a model of $\text{BI-AD} \wedge \neg \text{BI-AD}_{\mathbb{R}}$. □

2.3. Measures and Supercompactness. As usual, $\Theta := \sup\{\alpha$; there is a surjection from \mathbb{R} onto $\alpha\}$. Let X be a set and κ be an uncountable cardinal. As usual, we denote by $\wp_{\kappa}(X)$ the set of all subsets of X with cardinality less than κ , i.e., subsets A such that there is an $\alpha < \kappa$ and a surjection from α to A . Let U be a set of subsets of $\wp_{\kappa}(X)$. We say that U is κ -complete if U is closed under intersections with $< \kappa$ -many elements; we say it is *fine* if for any $x \in X$, $\{a \in \wp_{\kappa}(X); x \in a\} \in U$; we say that U is *normal* if for any family $\{A_x \in U; x \in X\}$, the diagonal intersection $\Delta_{x \in X} A_x$ is in U (where $\Delta_{x \in X} A_x = \{a \in \wp_{\kappa}(X); (\forall x \in a) a \in A_x\}$). We say that U is a *fine measure* if it is a fine κ -complete ultrafilter, and we say that it is *fine*

²This formulation of Blackwell determinacy axioms will make it rather difficult for the reader to see why these games should be considered imperfect information games. The (somewhat surprising) answer is: they should not. The original formulation due to Blackwell involved imperfect information games, but in a surprising development connected to Martin’s theorem [Mar98], these axioms turned out to be equivalent to the version we defined here which could be described as “perfect information determinacy with mixed strategies”. For more details, cf. [Löw05, § 5].

normal measure if it is a fine normal κ -complete ultrafilter. It is easy to check that if there is a surjection from X to Y , and there is a fine (normal) measure on $\wp_\kappa(X)$, then there is one on $\wp_\kappa(Y)$.

In the choice-less context, we can define a cardinal κ to be λ -*supercompact* if there is a fine normal measure on $\wp_\kappa(\lambda)$ and to be λ -*strongly compact* if there is a fine measure on $\wp_\kappa(\lambda)$. In the ZFC-context, this is equivalent to the usual definition (cf. [Kan94, Theorem 22.7]). It is well-known that AD implies that \aleph_1 is \aleph_2 -supercompact [Bec81].

2.4. An alternative definition of the axiom of real Blackwell determinacy. In [dK05], the second author started the investigation of the axiom of real Blackwell determinacy, and gave two alternative definitions of $\mathbf{BI-AD}_{\mathbb{R}}$. The second definition was the source of the definitions of long Blackwell games in [Löw05, §9.2]. For the sake of completeness, we give this definition here.

Instead of considering \mathbb{R} as discretely topologized, we use the usual topology on \mathbb{R} . In this setting, we do not require strategies to have countable support. A function assigning an arbitrary Borel probability measure on \mathbb{R} to each finite sequence s of reals is called an *E-mixed strategy*. If σ and τ are E-mixed strategies for players I and II, respectively, we define

$$\nu(\sigma, \tau)(s) = \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \text{lh}(s) \text{ is odd,} \end{cases}$$

as before. If B is a $k + 1$ -dimensional Borel set and $s \in \mathbb{R}^k$, we let $B_s := \{x; s \frown \langle x \rangle\}$. For $k \in \omega$, we define a Borel probability measure on \mathbb{R}^{k+1} . Define

$$\begin{aligned} \mu_{\sigma, \tau}^0(B) &:= \nu(\sigma, \tau)(\emptyset)(B), \text{ and} \\ \mu_{\sigma, \tau}^{k+1}(B) &:= \int_{s \in \mathbb{R}^k} \nu(\sigma, \tau)(s)(B_s) d\mu_{\sigma, \tau}^k. \end{aligned}$$

Clearly, the sequence of measures $\langle \mu_{\sigma, \tau}^k; k \in \omega \rangle$ coheres, *i.e.*,

$$\mu_{\sigma, \tau}^{k+1}(B \times \mathbb{R}) = \mu_{\sigma, \tau}^k(B),$$

and thus generate a Borel measure $\mu_{\sigma, \tau}^E$ on ${}^\omega\mathbb{R}$ by the Kolmogorov consistency theorem.

For a subset $A \subseteq {}^\omega\mathbb{R}$, we define the other notions as before: An E-mixed strategy σ for player I is *E-optimal in A* if for any E-mixed strategy τ for player II, A is $\mu_{\sigma, \tau}^E$ -measurable and $\mu_{\sigma, \tau}^E(A) = 1$. Similarly, an E-mixed strategy τ for player II is *E-optimal in A* if for any E-mixed strategy σ for player I, A is $\mu_{\sigma, \tau}^E$ -measurable and $\mu_{\sigma, \tau}^E(A) = 0$. We say that A is *Euclidean Blackwell determined* if either player I or

II has an E-optimal strategy in A . Finally, $\text{EBI-AD}_{\mathbb{R}}$ is the statement “for any subset A of ${}^{\omega}\mathbb{R}$, A is Euclidean Blackwell determined.”

It is easy to see that $\text{EBI-AD}_{\mathbb{R}}$ implies BI-AD . The proofs of § 2.2 all go through under the assumption of $\text{EBI-AD}_{\mathbb{R}}$, so it is strictly stronger than BI-AD . We do not know what the exact relationship between $\text{EBI-AD}_{\mathbb{R}}$ and $\text{BI-AD}_{\mathbb{R}}$ is.

3. SOLOVAY’S ANALYSIS OF $\text{AD}_{\mathbb{R}}$

In [Sol78], Solovay provided the foundations of the theory of $\text{AD}_{\mathbb{R}}$ while at the same time gaining some understanding of the relationship between determinacy axioms and fragments of the axiom of choice. The set $\mathbb{R}^{\#}$ encodes a truth definition of $\mathbf{L}(\mathbb{R})$ in the same sense that $0^{\#}$ encodes a truth definition of \mathbf{L} (for details, cf. [Sol78, § 4]). In particular, if $\mathbf{L}(\mathbb{R}) \models T$, then the existence of $\mathbb{R}^{\#}$ implies $\text{Cons}(T)$, and thus by Gödel’s incompleteness theorem, $T \not\vdash$ “ $\mathbb{R}^{\#}$ exists”. The following two theorems are the core of Solovay’s analysis:

Theorem 5 (Solovay). The axiom $\text{AD}_{\mathbb{R}}$ implies that there is a fine normal measure on $\wp_{\omega_1}(\mathbb{R})$, where $\wp_{\omega_1}(\mathbb{R})$ is the set of all countable subsets of \mathbb{R} .

Proof. [Sol78, Lemma 3.1]. □

Theorem 6 (Solovay). Suppose there is a fine normal measure on $\wp_{\omega_1}(\mathbb{R})$ and every real has its sharp. Then $\mathbb{R}^{\#}$ exists.

Proof. [Sol78, Lemma 4.1 & Theorem 4.4]. □

From Theorems 5 and 6, he can deduce that $\text{AD}_{\mathbb{R}}$ is strictly stronger (in terms of consistency strength) than AD (as $\mathbf{L}(\mathbb{R}) \models \text{AD}$). In the § 4, we shall prove the conclusion of Theorem 5 from $\text{BI-AD}_{\mathbb{R}}$, i.e., we prove that there is a fine normal measure on $\wp_{\omega_1}(\mathbb{R})$. We’ll now explain how this implies all claims listed in the Main Theorem.

If $\kappa < \Theta$, then there is a surjection from \mathbb{R} onto κ witnessing this. This surjection allows us to pull back the fine normal measure on $\wp_{\omega_1}(\mathbb{R})$ to $\wp_{\omega_1}(\kappa)$, and so ω_1 is κ -supercompact for every $\kappa < \Theta$.

Theorem 7 (Martin, Neeman, Vervoort). If $\mathbf{V}=\mathbf{L}(\mathbb{R})$ and BI-AD holds, then AD holds.

Assume that $\text{BI-AD}_{\mathbb{R}}$ holds. By Remark 2, we also have BI-AD which pulls back to $\mathbf{L}(\mathbb{R})$, so $\mathbf{L}(\mathbb{R}) \models \text{BI-AD}$, and hence $\mathbf{L}(\mathbb{R}) \models \text{AD}$ by Theorem 7. But by Theorem 6, we get $\mathbb{R}^{\#}$, and thus $\text{Cons}(\text{AD})$.

Let us close this section by raising questions raised by the results in Solovay’s paper. Note that Solovay’s proof of the existence of a

fine normal measure on $\wp_{\omega_1}(\mathbb{R})$ under $\text{AD}_{\mathbb{R}}$ can be modified to work under AD by a simple coding argument (cf. [Kie06, §4.5]) if you give up normality, thus giving a proof of the following theorem:

Theorem 8. If AD holds, then ω_1 is κ -strongly compact for every $\kappa < \Theta$.

Our proof in §4 does not seem to allow us to prove the Blackwell analogue of Theorem 8.

Furthermore, Solovay proved that the consistency strength of $\text{AD}_{\mathbb{R}} + \text{cf}(\Theta) > \omega$ is strictly bigger than that of $\text{AD}_{\mathbb{R}}$. This proof uses model constructions based on the Wadge hierarchy (cf. [Sol78, Theorems 2.5 & 5.7]). In order to prove the appropriate analogues in the Blackwell context, we would need to make use of the Blackwell analogue of the Wadge hierarchy, the *Blackwell Wadge hierarchy* (cf. [Löw05, §7.3]). Unfortunately, we do not know how to prove that this hierarchy is wellfounded (not even under the assumption of $\text{BI-AD}_{\mathbb{R}}$).

4. EXISTENCE OF A FINE NORMAL MEASURE

Finally, we prove the main claim of our Main Theorem: assuming $\text{BI-AD}_{\mathbb{R}}$, we construct a fine normal measure on $\wp_{\omega_1}(\mathbb{R})$. We shall be closely following Solovay's original idea. We define a family $U \subseteq \wp(\wp_{\omega_1}(\mathbb{R}))$ as follows: Fix $A \subseteq \wp_{\omega_1}(\mathbb{R})$ and consider the following game \mathcal{G}_A : players alternately play finite subsets of the reals; say that they produce an infinite sequence $\vec{a} = (a_i; i \in \omega)$. Then player II wins the game \mathcal{G}_A if $\bigcup\{a_n; n \in \omega\} \in A$, otherwise player I wins. We also write $\bigcup \vec{a} := \bigcup\{a_n; n \in \omega\}$. We say that $A \in U$ if and only if player II has an optimal strategy in \mathcal{G}_A . The Solovay object U is the obvious Blackwell analogue of Solovay's normal measure from [Sol78], and we shall show that it is also a fine normal measure under the assumption of $\text{BI-AD}_{\mathbb{R}}$, thus finishing the proof.

A few properties of U are obvious: for instance, we see readily that $\emptyset \notin U$ and that $\wp_{\omega_1}(\mathbb{R}) \in U$, as well as the fact that U is closed under taking supersets. In order to see that U is a fine family, fix a real x , and let player II play $\{x\}$ with probability 1 in her first move: this is an optimal strategy for $\mathcal{G}_{\{a; x \in a\}}$. Note that by Remark 2, all of the games \mathcal{G}_A are Blackwell determined.

In order to prove the other required properties of a normal measure, we need to develop the appropriate *transfer technique* (as discussed and applied in [Löw02]) for the present context. Let $\pi \subseteq \omega$ be an infinite and co-infinite set. We think of π as the set of rounds in which player I moves. We identify π with the increasing enumeration of its members, i.e., $\pi = \{\pi_i; i \in \omega\}$. Similarly, we write $\bar{\pi}$ for the increasing

enumeration of $\omega \setminus \pi$, i.e., $\omega \setminus \pi = \{\bar{\pi}_i; i \in \omega\}$. For notational ease, we call π a **I-coding** if no two consecutive numbers are in π and a **II-coding** if no two consecutive numbers are in $\omega \setminus \pi$.

Fix $A \subseteq \wp_{\omega_1}(\mathbb{R})$ and define two variants of \mathcal{G}_A with alternative orders of play as determined by π . If π is a I-coding, the game $\mathcal{G}_A^{\pi, \text{I}}$ is played as follows:

$$\begin{array}{llll} \text{I} & & a_{\pi_0} & a_{\pi_1} \cdots \\ \text{II} & a_0, \cdots, a_{\pi_0-1} & & a_{\pi_0+1}, \cdots, a_{\pi_1-1} \cdots \end{array}$$

If π is a II-coding, then we play the game $\mathcal{G}_A^{\pi, \text{II}}$ as follows:

$$\begin{array}{llll} \text{I} & a_0, \cdots, a_{\bar{\pi}_0-1} & & a_{\bar{\pi}_0+1}, \cdots, a_{\bar{\pi}_1-1} \cdots \\ \text{II} & & a_{\bar{\pi}_0} & & a_{\bar{\pi}_1} \cdots \end{array}$$

In both cases, player II wins the game if $\bigcup_{n \in \omega} a_n \in A$. Obviously, we have

$$\mathcal{G}_A = \mathcal{G}_A^{\text{Even}, \text{II}}$$

where Even is the set of even numbers.

Lemma 9. Let A be a subset of $\wp_{\omega_1}(\mathbb{R})$ and π be a I-coding. Then there is a translation $\sigma \mapsto \sigma_\pi$ of mixed strategies for player I such that if σ is an optimal strategy for player I in \mathcal{G}_A , then σ_π is an optimal strategy for player I in $\mathcal{G}_A^{\pi, \text{I}}$.

Similarly, if π is a II-coding, there is a translation $\tau \mapsto \tau_\pi$ of mixed strategies for player II such that if τ is an optimal strategy for player II in \mathcal{G}_A , then τ_π is an optimal strategy for player II in $\mathcal{G}_A^{\pi, \text{II}}$.

Proof. We prove only the claim for the games $\mathcal{G}_A^{\pi, \text{I}}$, the other proof being similar. If $\vec{a} = \langle a_i; i \in \omega \rangle$ is an infinite sequence of finite sets of reals, we define

$$b_i^{\vec{a}} = \begin{cases} a_0 \cup \cdots \cup a_{\pi_0-1} \cup a_{\pi_0+1} \cup \cdots \cup a_{\pi_1-1} & \text{if } i = 0, \\ a_{\pi_i+1} \cup \cdots \cup a_{\pi_{i+1}-1} & \text{otherwise.} \end{cases}$$

Note that in order to compute $b_i^{\vec{a}}$, you only need the first π_{i+1} bits of \vec{a} . The idea is that now the \mathcal{G}_A -run

$$\begin{array}{llllll} \text{I} & a_{\pi_0} & a_{\pi_1} & a_{\pi_2} & \cdots & \\ \text{II} & & b_0^{\vec{a}} & b_1^{\vec{a}} & b_2^{\vec{a}} & \cdots \end{array} \quad (*)$$

yields the same output in terms of the union of all played finite sets as the run \vec{a} in the game $\mathcal{G}_A^{\pi, \text{I}}$. We can define a map π^* on infinite sequences of finite sets of reals by

$$(\pi^*(\vec{a}))_i := \begin{cases} a_{\pi_k} & \text{if } i = 2k, \\ b_k^{\vec{a}} & \text{if } i = 2k + 1, \end{cases}$$

and see that $\bigcup\{a_i; i \in \omega\} = \bigcup\{(\pi^*(\vec{a}))_i; i \in \omega\}$.

Now, given a mixed strategy σ for player I in \mathcal{G}_A and a run \vec{a} of the game \mathcal{G}_A , we define σ_π via π^* as follows:

$$\begin{aligned}\sigma_\pi(a_0, \dots, a_{\pi_0-1}) &:= \sigma(\emptyset), \\ \sigma_\pi(a_0, \dots, a_{\pi_m-1}) &:= \sigma(a_{\pi_0}, b_0^{\vec{a}}, \dots, a_{\pi_i}, b_i^{\vec{a}}, \dots, a_{\pi_{m-1}}, b_{m-1}^{\vec{a}}).\end{aligned}$$

Assume that σ is an optimal strategy for player I in \mathcal{G}_A and fix an arbitrary mixed strategy τ in the game $\mathcal{G}_A^{\pi, I}$. We show that the payoff set for A is $\mu_{\sigma_\pi, \tau}$ -measurable and of $\mu_{\sigma_\pi, \tau}$ -measure one. In order to do so, we construct a mixed strategy $\tau_{\pi-1}$ for player II in \mathcal{G}_A so that the game played by σ_π, τ is essentially the same as the game played by $\sigma, \tau_{\pi-1}$.

Given a sequence \vec{b} of moves in $\mathcal{G}_A^{\pi, I}$, we need to unravel it into a sequence of moves in \mathcal{G}_A in an inverse of the maps $\vec{a} \mapsto b_i^{\vec{a}}$ according to (*), i.e., $b_{2i+1} = b_i^{\vec{a}}$. Thus, we define

$$\begin{aligned}A_{2i+1}^{\vec{b}} &:= \{\vec{a}; b_i^{\vec{a}} = b_{2i+1}\}, \\ A_{\leq 2i+1}^{\vec{b}} &:= \bigcap_{j \leq i} A_{2j+1}^{\vec{b}}.\end{aligned}$$

Note that only a finite fragment of \vec{a} is needed to check whether $b_i^{\vec{a}} = b_{2i+1}$, and thus we think of $A_{\leq 2i+1}^{\vec{b}}$ as a set of $(\pi_{i+1} - (i + 1))$ -tuples of finite sets of reals. In the following, when we quantify over all “ $\vec{a} \in A_{\leq i}^{\vec{b}}$ ”, we think of this as a collection of finite strings of finite sets of reals. In order to pad the moves made in $\mathcal{G}_A^{\pi, I}$, we define the following notation: for infinite sequences \vec{a} and \vec{b} , we write

$$\begin{aligned}x_0^{\vec{a}, \vec{b}} &:= (a_0, \dots, a_{\pi_0-1}, b_0, a_{\pi_0+1}, \dots, a_{\pi_1-1}), \\ x_i^{\vec{a}, \vec{b}} &:= (b_{2i}, a_{\pi_i+1}, \dots, a_{\pi_{i+1}-1}) \text{ (if } i > 0).\end{aligned}$$

Compare (*) to see that if \vec{a} corresponds to moves in $\mathcal{G}_A^{\pi, I}$ and \vec{b} to the moves in \mathcal{G}_A , then these are exactly the finite sequences that player II will have to respond to. Moreover, for a given sequence \vec{z} of finite sets of reals, we let

$$P_\tau(z_0, \dots, z_n) := \prod_{i \leq n, i \notin \pi} \tau(z_0, \dots, z_{i-1})(z_i).$$

Fix a sequence \vec{b} of finite sets of reals as moves for player I and define $\tau_{\pi^{-1}}$ as follows:

$$\begin{aligned}\tau_{\pi^{-1}}(b_0)(b_1) &:= \sum_{\vec{a} \in A_1^{\vec{b}}} P_{\tau}(x_0^{\vec{a}, \vec{b}}), \text{ and} \\ \tau_{\pi^{-1}}(b_0, \dots, b_{2m})(b_{2m+1}) &:= \frac{\sum_{\vec{a} \in A_{\leq 2m+1}^{\vec{b}}} P_{\tau}(x_0^{\vec{a}, \vec{b}} \frown \dots \frown x_m^{\vec{a}, \vec{b}})}{\prod_{i=1}^m \tau_{\pi^{-1}}(b_0, \dots, b_{2i-2})(b_{2i-1})}.\end{aligned}$$

Using the two operations $\sigma \mapsto \sigma_{\pi}$ and $\tau \mapsto \tau_{\pi^{-1}}$, since the payoff set for \mathcal{G}_A is invariant under π^* , it now suffices to prove for all basic open sets $[s]$ induced by a finite sequence $s = (b_0, \dots, b_{\text{lh}(s)-1})$ that $\mu_{\sigma, \tau_{\pi^{-1}}}([s]) = \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([s]))$. We prove this by induction on the length of s , and have to consider five different cases:

Case 1. $\text{lh}(s) = 0$: This is immediate.

Case 2. $\text{lh}(s) = 1$. Then $\mu_{\sigma, \tau_{\pi^{-1}}}([b_0]) = \sigma(\emptyset)(b_0) = \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([b_0]))$.

Case 3. $\text{lh}(s) = 2$.

$$\begin{aligned}\mu_{\sigma, \tau_{\pi^{-1}}}([b_0, b_1]) &= \sigma(\emptyset)(b_0) \cdot \tau_{\pi^{-1}}(b_0)(b_1) \\ &= \sigma(\emptyset)(b_0) \cdot \sum_{\vec{a} \in A_1^{\vec{b}}} P_{\tau}(x_0^{\vec{a}, \vec{b}}) \\ &= \sum_{\vec{a} \in A_1^{\vec{b}}} \mu_{\sigma_{\pi}, \tau}([x_0^{\vec{a}, \vec{b}}]) \\ &= \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([b_0, b_1])).\end{aligned}$$

Case 4. $\text{lh}(s) = 2m+1$ with $m \geq 1$. By induction hypothesis, we have that $X := \mu_{\sigma, \tau_{\pi^{-1}}}([b_0, \dots, b_{2m-1}]) = \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([b_0, \dots, b_{2m-1}]))$. Thus,

$$\begin{aligned}\mu_{\sigma, \tau_{\pi^{-1}}}([b_0, \dots, b_{2m}]) &= X \cdot \sigma(b_0, \dots, b_{2m-1})(b_{2m}) \\ &= \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([b_0, \dots, b_{2m}])).\end{aligned}$$

Case 5. $\text{lh}(s) = 2m+2$ with $m \geq 1$.

$$\begin{aligned}\mu_{\sigma, \tau_{\pi^{-1}}}(s) &= \prod_{i=0}^m \sigma(b_0, \dots, b_{2i+1}) \cdot \sum_{\vec{a} \in A_{\leq 2m+1}^{\vec{b}}} P_{\tau}(x_0^{\vec{a}, \vec{b}} \frown \dots \frown x_m^{\vec{a}, \vec{b}}) \\ &= \mu_{\sigma_{\pi}, \tau}((\pi^*)^{-1}([b_0, \dots, b_{2m+1}])).\end{aligned}$$

This calculation finishes the proof of our key lemma. \square

Based on Lemma 9, we can now finish the proof of the main theorem.

Claim 10. If $A \notin U$, then $\wp_{\omega_1}(\mathbb{R}) \setminus A \in U$.

Proof. If player II does not have an optimal strategy in \mathcal{G}_A , then player I does. Let $\pi := \text{Odd}$, the set of odd numbers. Then in $\mathcal{G}_A^{\pi, \text{I}}$, the rôles of players I and II are switched. By Lemma 9, there is an optimal strategy σ_π for player I in the game $\mathcal{G}_A^{\pi, \text{I}}$, but this is optimal for player II in $\mathcal{G}_{\wp\omega_1(\mathbb{R}) \setminus A}$. \square

Claim 11. If $A_1, A_2 \in U$, then $A_1 \cap A_2 \in U$.

Proof. Since $A_1, A_2 \in U$, there are optimal strategies τ_1 and τ_2 for player II in \mathcal{G}_{A_1} and \mathcal{G}_{A_2} , respectively. Let $\pi_1 := \{n; n \not\equiv 1 \pmod{4}\}$ and $\pi_2 := \{n; n \not\equiv 3 \pmod{4}\}$. Both of these sets are II-codings and correspond to the following game diagrams:

$$\begin{array}{l} \mathcal{G}^{\pi_1, \text{II}} \\ \text{I} \quad a_0 \quad \quad a_2, a_3, a_4 \quad \quad a_6, a_7, a_8 \quad \quad \cdots \\ \text{II} \quad \quad a_1 \quad \quad \quad a_5 \quad \quad \quad a_9 \quad \cdots \\ \\ \mathcal{G}^{\pi_2, \text{II}} \\ \text{I} \quad a_0, a_1, a_2 \quad \quad a_4, a_5, a_6 \quad \quad a_8, a_9, a_{10} \quad \quad \cdots \\ \text{II} \quad \quad a_3 \quad \quad \quad a_7 \quad \quad \quad a_{11} \quad \cdots \end{array}$$

By Lemma 9, there are optimal strategies $(\tau_1)_{\pi_1}, (\tau_2)_{\pi_2}$ for player II in $\mathcal{G}_{A_1}^{\pi_1, \text{II}}$ and $\mathcal{G}_{A_2}^{\pi_2, \text{II}}$, respectively. To reduce notation, we write $\tau_1^* := (\tau_1)_{\pi_1}$ and $\tau_2^* := (\tau_2)_{\pi_2}$. We combine these strategies into an optimal strategy τ in $\mathcal{G}_{A_1 \cap A_2}$ by

$$\begin{aligned} \tau(a_0, \dots, a_{4n}) &:= \tau_1^*(a_0, \dots, a_{4n}), \\ \tau(a_0, \dots, a_{4n+2}) &:= \tau_2^*(a_0, \dots, a_{4n+2}). \end{aligned}$$

We show that τ is optimal, by letting σ be arbitrary for player I in $\mathcal{G}_{A_1 \cap A_2}$, and define strategies σ_1 and σ_2 in $\mathcal{G}_{A_1}^{\pi_1, \text{II}}$ and $\mathcal{G}_{A_2}^{\pi_2, \text{II}}$, respectively:

$$\begin{aligned} \sigma_1(\emptyset) &:= \sigma(\emptyset), \\ \sigma_1(a_0, \dots, a_{2n+1}) &:= \sigma(a_0, \dots, a_{2n+1}), \\ \sigma_1(a_0, \dots, a_{4n+2}) &:= \tau_2^*(a_0, \dots, a_{4n+2}), \\ \\ \sigma_2(a_0, \dots, a_{2n+1}) &:= \sigma(a_0, \dots, a_{2n+1}), \\ \sigma_2(a_0, \dots, a_{4n}) &:= \tau_1^*(a_0, \dots, a_{4n}), \end{aligned}$$

Then it is easy to check that $\mu_{\sigma, \tau} = \mu_{\sigma_1, \tau_1^*} = \mu_{\sigma_2, \tau_2^*}$ and thus

$$\mu_{\sigma, \tau}(\{\vec{a}; \bigcup \vec{a} \in A_1\}) = \mu_{\sigma_1, \tau_1^*}(\{\vec{a}; \bigcup \vec{a} \in A_1\}) = 1,$$

and similarly for A_2 , and thus τ is optimal. \square

Claim 12. For a family $\{A_x \in U; x \in \mathbb{R}\}$, we have that the diagonal intersection $\Delta_{x \in \mathbb{R}} A_x$ is in U .

Proof. As in [Sol78], this is the most intricate part of the proof. For each finite set of reals a , let $A_a = \bigcap_{x \in a} A_x$ (if $a = \emptyset$, then we let $A_a := \wp_{\omega_1}(\mathbb{R})$). By Claim 11, player II has an optimal strategy in each game \mathcal{G}_{A_a} .

Subclaim 13. There is a choice function $a \mapsto \tau_a$ picking an optimal strategy for \mathcal{G}_{A_a} .

Proof. Consider the following game \mathcal{G}^* : player I plays a finite set of reals a and player II passes once; after that, they play \mathcal{G}_{A_a} . By BI-AD $_{\mathbb{R}}$, either player I or II has an optimal strategy for \mathcal{G}^* . But if σ is any strategy for player I such that $\sigma(\emptyset)(a) > 0$, then σ cannot be optimal (as σ will lose with non-zero probability against an optimal strategy for player II in \mathcal{G}_{A_a}). Thus player II has an optimal strategy τ in \mathcal{G}^* . Now define $\tau_a(x_0, \dots, x_n) := \tau(a, \emptyset, x_0, \dots, x_n)$. Clearly, τ_a is optimal in \mathcal{G}_{A_a} .

q.e.d. (Subclaim 13)

Fix a bookkeeping bijection ρ from $\omega \times \omega$ to ω such that $\rho(n, m) < \rho(n, m+1)$ and $\rho(n, 0) \geq n$. We are playing infinitely many games in a diagram where the first coordinate is for the index of the game we are playing, and the second coordinate is for the number of moves. Hence the pair (n, m) stands for “ m -th move in the n -th game”. Define a II-coding $\pi_n := \omega \setminus \{2\rho(n, i) + 1; i \in \omega\}$ corresponding to the following game diagram:

$$\begin{array}{l} \text{I} \quad a_0, \dots, a_{2\rho(n,0)} \qquad \qquad \qquad a_{2\rho(n,0)+2}, \dots, a_{2\rho(n,1)} \qquad \qquad \dots \\ \text{II} \qquad \qquad \qquad a_{2\rho(n,0)+1} \qquad \qquad \qquad \qquad \qquad \qquad a_{2\rho(n,1)+1} \qquad \dots \end{array}$$

By Lemma 9 and Subclaim 13, we know that for each a , we get an optimal strategy $(\tau_a)_{\pi_n}$ for the game $\mathcal{G}_{A_a}^{\pi_n, \text{II}}$. Let τ be the following mixed strategy

$$\tau(a_0, \dots, a_{2\rho(n,m)}) := (\tau_{a_n})_{\pi_n}(a_0, \dots, a_{2\rho(n,m)}).$$

The properties of ρ make sure that this strategy is well-defined; we shall now prove that τ is an optimal strategy for player II in $\mathcal{G}_{\Delta_{x \in \mathbb{R}} A_x}$.

Pick any mixed strategy σ for player I in $\mathcal{G}_{\Delta_{x \in \mathbb{R}} A_x}$, and define strategies σ_n for $\mathcal{G}^{\pi_n, \text{II}}$. Let $m = \rho(k, \ell)$, then

$$\begin{aligned} \sigma_n(a_0, \dots, a_{2m-1}) &:= \sigma(a_0, \dots, a_{2m-1}), \text{ and} \\ \sigma_n(a_0, \dots, a_{2m}) &:= (\tau_{a_k})_{\pi_k}(a_0, \dots, a_{2m}) \text{ (if } k \neq n). \end{aligned}$$

Note that for each $n \in \omega$ and $s \in {}^{n+1}([\mathbb{R}]^{<\omega})$, $\mu_{\sigma,\tau}$ and $\mu_{\sigma_n,(\tau_{s(n)})\pi_n}$ agree below $[s]$, i.e., for any set $A \subseteq [s]$, $\mu_{\sigma,\tau}(A) = \mu_{\sigma_n,(\tau_{s(n)})\pi_n}(A)$.

The payoff set (for player II) for $\mathcal{G}_{\Delta_{x \in \mathbb{R}} A_x}$ is $A := \{\vec{a}; \bigcup \vec{a} \in \Delta A_x\}$. We show that $\mu_{\sigma,\tau}(A) = 1$. Since

$$\begin{aligned} \{\vec{a}; \forall x \in \bigcup \vec{a} (\bigcup \vec{a} \in A_x)\} &= \{\vec{a}; \forall n \in \omega (\forall x \in a_n (\bigcup \vec{a} \in A_x))\} \\ &= \bigcap_{n \in \omega} \{\vec{a}; \bigcup \vec{a} \in A_{a_n}\}, \end{aligned}$$

it suffices to check that the sets $A_n := \{\vec{a}; \bigcup \vec{a} \in A_{a_n}\}$ has $\mu_{\sigma,\tau}$ -measure 1. But

$$A_n = \bigcup_{s \in {}^{n+1}([\mathbb{R}]^{<\omega})} ([s] \cap A_n)$$

and for all s , we have $\mu_{\sigma_n,(\tau_{s(n)})\pi_n}([s] \cap A_n) = \mu_{\sigma_n,(\tau_{s(n)})\pi_n}([s])$. Then, using the fact that the measures have countable support, we get

$$\begin{aligned} \mu_{\sigma,\tau}(A_n) &= \sum_{s \in {}^{n+1}([\mathbb{R}]^{<\omega})} \mu_{\sigma,\tau}([s] \cap A_n) \\ &= \sum_{s \in {}^{n+1}([\mathbb{R}]^{<\omega})} \mu_{\sigma_n,(\tau_{s(n)})\pi_n}([s] \cap A_n) \\ &= \sum_{s \in {}^{n+1}([\mathbb{R}]^{<\omega})} \mu_{\sigma_n,(\tau_{s(n)})\pi_n}([s]) \\ &= \sum_{s \in {}^{n+1}([\mathbb{R}]^{<\omega})} \mu_{\sigma,\tau}([s]) \\ &= 1. \end{aligned}$$

q.e.d. (Claim 12)

Note that together with the trivial properties of U mentioned at the beginning of this section, Claims 10, 11, and 12 are all we need to show: the non-principality of U follows from the fineness of U , and the σ -completeness follows from the fact that every set of reals is Lebesgue measurable (an ultrafilter failing σ -completeness defines a non-principal ultrafilter on ω and hence a non-Lebesgue measurable set). Thus, we have proved the main theorem.

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