CATEGORIAL VERSUS MODAL INFORMATION THEORY

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Abstract

In this very brief note, I raise a few worries about interpreting the Lambek Calculus, admired and cherished as it may be by all connoisseurs, as a base logic of information flow.

1 The Lambek Calculus at a crossroads of interpretations

The Lambek Calculus is a wonderfully elegant little system that captures the essence of an abstract composition operation and its left- and right-inverses. Its associative version *LC* manipulates sequents $X \Rightarrow A$, with X a finite sequence of terms and A a single term, in a language having primitive type symbols, and closed under the three operations $\bullet$, $\rightarrow$, $\leftarrow$. The key inference rules are those introducing product conjunction and directed implication:

if $X, A, B, Y \Rightarrow C$,	then	$X, A\bullet B, Y \Rightarrow C$
if $X \Rightarrow A$ and $Y \Rightarrow B$,	then	$X, Y \Rightarrow A\bullet B$
if $X, A \Rightarrow B$,	then	$X \Rightarrow B \leftarrow A$
if $X \Rightarrow A$ and $Y, B, Z \Rightarrow C$,	then	$Y, B \leftarrow A, X, Z \Rightarrow C$
if $A, X \Rightarrow B$,	then	$X \Rightarrow A \rightarrow B$
if $X \Rightarrow A$ and $Y, B, Z \Rightarrow C$,	then	$Y, X, A \rightarrow B, Z \Rightarrow C$

These are designed to reflect combination of ordered 'resources' in an associative manner, with concatenation of syntactic strings in formal or natural languages as a typical example.

This system sits at an interface. Its extension *LPC* with a structural rule of *permutation*

if $X, A, B, Y \Rightarrow C$,	then	$X, B, A, Y \Rightarrow C$

that manipulates unordered multi-sets of objects (their multiplicity remains crucial), has been used widely in the semantics of natural language (van Benthem 1991, 2005). Especially, it seems to provide just the right amount of 'glue' for composing linguistic meanings via the Curry-Howard isomorphism, via single-bind 'linear lambda terms'.

On the other hand, its weaker *non-associative* variant *NLC,* that can be cast as manipulating single algebraic terms in the above language, gets still closer to the actual syntax of languages, where a branching tree pattern for $(A \bullet B) \bullet C$ is not the same as $A \bullet (B \bullet C)$.

One virtue of Lambek's system is its wealth of different interpretations. We already hinted at *language models*, where primitive types a denote sets L_a of strings in some alphabet, while product involves straightforward concatenation. More precisely,

> the language $L_{A\bullet B}$ is the set of strings $\{xy \mid x \in L_A \ \& \ y \in L_B\}$,
> $L_{A\rightarrow B} = \{x \mid \forall y \in L_A\colon yx \in L_B\}$, $L_{B\leftarrow A} = \{x \mid \forall y \in L_A\colon xy \in L_B\}$.

At the end of a long history starting with Buszkowski 1986 and Dosen 1985, Pentus 1995 showed that *LC* is complete for this interpretation. But equally interesting are *relational models*, where primitive types a denote binary relations R_a over some state space, while

> $R_{A\bullet B}$ is the standard relational composition $R_A\,;\,R_B$ of relational set algebra,
> $R_{A\rightarrow B} = \{(x, y) \mid \forall (z, x) \in R_A\colon (z, y) \in R_B\}$, $R_{B\leftarrow A} = \{(x, y) \mid \forall (y, z) \in R_A\colon (x, z) \in R_B\}$.

Completeness for this interpretation was shown in Andréka & Mikulas 1994. Thus, the Lambek Calculus models both static syntactic structure and dynamic action. But the list can be continued. *LPC* is also the 'multiplicative fragment' of *linear logic*: providing a proof-theoretic computational interpretation, and further extensions of *LPC* with Weakening turn out to be *relevant logics*, linking up with the philosophical tradition (Kurtonina 1995).

Indeed, new interpretations are still emerging today, including the following appealing geometrical one. Aiello & van Benthem 2003 point out that *LPC* is also the logic of the 'Minkowski operations' in mathematical morphology (we refer to this paper for a concrete motivation in image processing). Here objects are vectors in a vector space, primitive types a denote arbitrary subsets V_a (not necessarily linear subspaces). The key operations are

> $V_{A\bullet B}$ is the set of sum vectors $\{x + y \mid x \in V_A \ \& \ y \in V_B\}$, (Minkowski addition)
> $V_{A\rightarrow B}$ is $\{x \mid \forall y \in V_A\colon x+y \in V_B\}$ (Minkowski subtraction)

Again, typical features of the Lambek Calculus return, such as the non-equivalence of terms A with products $A\bullet A$, since the figures studied in mathematical morphology are not linear subspaces. Completeness for this geometrical interpretation is still an open problem.

2 Does the Lambek Calculus also describe the basics of information flow?

Now, here is my question, inspired by a correspondence with Sebastian Sequoiah-Grayson: who also has a paper on this topic in the present volume (cf. also Sequoiah-Grayson 2009). The Lambek Calculus captures essentials of resource combination. But then, thinking of one of the most ubiquitous resources of all, is it also a base calculus of *information flow*? [1] Some answers to the above question are around. The relevant logic tradition has voted for a resounding "Yes" (cf. Mares 1996). But frankly, I have not been satisfied with the stated reasons, since the information channel metaphor seemed mostly a thin spin on abstract ternary models, whatever their intrinsic merits in systematizing substructural logics. [2]

As a test, I will first confront the Lambek Calculus with the main calculus of information in my current world, the dynamic logic of public announcements. We will see that the fit is very problematic. Analyzing the issue further, I then distinguish different logical aspects of information, and explain why unification under one simple heading seems unlikely. There are no new results, but I would be happy if more people are led to think about these issues.

3 Dynamic epistemic logic as a test case

Here is a lightning sketch of the paradigm of information flow in my current world, the logic *PAL* of public announcements. It embodies what is perhaps the most intuitive notion of information update, in line with Carnap's notion of 'semantic information', and also found in probability theory or game theory. Our information 'statics' works with spaces of epistemic options $(\boldsymbol{M}, s)$ (usually a model for a single- or multi-agent epistemic language, with a distinguished world s), representing the current information of one or more agents. Dynamically, an event $!\varphi$ of incoming new public 'hard information' that φ is the case then *restricts this space to the sub-model* $(\boldsymbol{M}|\varphi, s)$ keeping just those worlds where φ is true.

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[1] Stating the issue more personally, I have worked on the foundations of information in both modal and categorial frameworks (van Benthem 1989, van Benthem 1991, van Benthem & Martinez 2007): so, is there a common thread among these approaches, or just a hopelessly split personality?

[2] I may have to reconsider, though, since the recent paper Mares, Seligman & Restall 2009 provides a much deeper analysis of the logic underlying modern developments in Channel Theory.

There are many mathematical subtleties in the paradigm, including construction of complex new models triggered by delicate informational events (cf. Baltag, Moss & Solecki 1998, van Benthem to appear). But for present purposes, the simplest version will do.

Consider the universe of all epistemic models linked by informational actions of public announcement. Clearly, announcements can be made one after the other, so we have longer 'conversations' or informational scenarios with sequential composition, which looks like Lambek-style product. Indeed, there is a promising initial analogy with resource calculus:

> announcing the same formula twice need not have the same effect
> as announcing it once: $!\varphi ; !\varphi = !\varphi$ is not a law of logical dynamics.

The reason is this. While announcing purely *factual* formulas φ is idempotent, things change with epistemic assertions about information. In famous puzzles like 'Muddy Children', round after round, a public announcement is made of 'everyone's ignorance of the true state of affairs'. But after several repetitions of this epistemic formula, there comes a crucial stage where its true announcement produces knowledge, switching its truth value. The general reason is that information change may change the truth values of epistemic formulas at each step, by giving the assertion a new meaning every time it is uttered.

But how far does this occurrence-oriented analogy go? The intuitions behind 'resource-conscious logics' seem very different from this dynamic failure of idempotence. Indeed, I feel that awareness of the dynamic subtleties of *higher-order epistemic* 'information about information' is largely absent in the area of sub-structural logic, and its analyses of what structural rules should be valid a priori for basic styles of reasoning. It also seems to play no role that I can see in the relevant logic take on the informational nature of the Lambek Calculus. But even if we take the above analogy at face value, I see a larger obstacle, having to do with the *natural repertoire of operations* that drive informational frameworks.

The crucial feature of *LC* is that product comes with two natural inverses. Do these make sense for dynamic-epistemic information flow? What would be the meaning of, say,

> a left-looking implication $!p \rightarrow !q$, even for factual formulas p, q?

This would be an action that, given a prior announcement of p on some initial model $(\boldsymbol{M}, s)$ now turns the restricted model $(\boldsymbol{M}|p, s)$ into the model $(\boldsymbol{M}|q, s)$. But this makes little sense.

First, no announcement action *!φ* for any formula *φ* will have this effect, since $\boldsymbol{M}|q$ may have $\neg p$-worlds left, while these have already disappeared in $\boldsymbol{M}|p$. In particular, $!(p\supset q)$ announcing a material implication between the two facts will not do the right job, since it results in conveying the information that both p and q. [3] And the more radical alternative of *adding new formulas* of this sort to the dynamic-epistemic language lacks motivation: the categorial implication would first have to *undo* the effect of the antecedent announcement, partially or wholly – going totally against the intuition for a system of information flow. [4] [5]

Here is the other possible conclusion from the preceding observations: dynamic-epistemic logic has its own fundamental account of information flow, and it is *sui generis*. This shows already in the set of structural rules for public announcement actions, where we read a sequent $\boldsymbol{P} \Rightarrow C$ as saying that successive updates with the premises in their given order always result in a model where updating with the conclusion is a fixed-point (in particular, the conclusion is already common knowledge). Reflexivity and Cut fail, but a natural style of reasoning remains, differing sharply from that encoded by *LC* (van Benthem 1996):

if $\boldsymbol{P} \Rightarrow C$, then $A, \boldsymbol{P} \Rightarrow C$	*Left-Monotonicity*
if $\boldsymbol{P} \Rightarrow A$ and $\boldsymbol{P}, A, \boldsymbol{Q} \Rightarrow C$, then $\boldsymbol{P}, \boldsymbol{Q} \Rightarrow C$	*Left-Cut*
if $\boldsymbol{P} \Rightarrow A$ and $\boldsymbol{P}, \boldsymbol{Q} \Rightarrow C$, then $\boldsymbol{P}, A, \boldsymbol{Q} \Rightarrow C$	*Cautious Monotonicity*

Van Benthem 2008 shows that this set is indeed complete for public announcement logic.

Moreover, on top of this, the natural repertoire forming complex informational actions is the regular algebra of *sequential composition*, *choice*, and *iteration* (parallel composition also makes sense). Thus, dynamic epistemic logic has its own natural algebra of operations over informational actions: and it does not seem to be that of the Lambek Calculus.

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[3] One might think that this points at an unclarity in the usual motivation of *LC*: does an implication $A \rightarrow B$ say that performing it gives us *only B*, or *both A and B*? We will not pursue this line here.

[4] One might defend this feature as a hidden form of 'belief revision', but that seems far-fetched.

[5] Another lack of fit is the following. Lambek implications $A \rightarrow B$ look at past A-actions that ook us to the current model. While this temporal extension is now a serious research theme in the field, cf. Renne, Sacks & Yap 2009, van Benthem 2006 shows how assertions about past epistemic actions need not be *bisimulation-invariant*, therefore leaving the area of standard epistemic modeling.

4 More distinctions instead of more unification?

What might this diversity suggest? I am now inclined to think that perhaps we need more distinctions, rather than more unification. The chapter van Benthem & Martinez 2008 ('The Logical Stories of Information', *Handbook of the Philosophy of Information*) points out three apparently mutually irreducible conceptions of information, even within such a small field as logic, each combining static structure with paradigmatic dynamic processes. They are semantic *information-as-range* as produced by observational update (this is like in epistemic logic), semantic *information-as-correlation* (dependence between situations) with channel transmission, and *information-as-code*, produced by dynamic 'elucidation' in proof or computation. Whether there is one base logic underlying all three is not intuitively obvious. Indeed, the Lambek calculus, with its genesis in proof-theoretic considerations, may be closer to the third kind of logical information than to the first or second. [6] And then, why force it to be something it clearly was not intended to be?

These differences also show up in the imagery found in discussions of the 'information' in logical systems. For instance, the Lambek operations are often read (I do it myself when teaching) in terms of 'information pieces' or 'evidence' that we have for propositions. But frankly, I find it hard to say what the image of 'pieces' of information means concretely, and often it just seems a way of dressing up a formal system that was already chosen beforehand, perhaps for its proof-theoretic elegance and simplicity. What *is* a 'piece of information', and how does one combine them? In which sense does a categorial implication $A \rightarrow B$ take a piece of information for A and then 'turn it into' one for B? Normal information processes do not destroy code, right (unless one is careless)?

Indeed, the discussion of intuitionistic semantics [7] in van Benthem 2009 points out how two notions are often conflated in discussions of information in logic, viz. *evidence* and *information stages*. These are clearly very different entities. Perhaps we need *both* to get at a better picture of information flow, but then, a single-level system like the Lambek

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[6] This proof-theoretic emphasis is also clear in Sequoiah Grayson 2009, who analyzes the Lambek Calculus as an engine for 'reasoning' with its appropriate notion of information, not for update.

[7] In intuitionistic logic, semantic and proof-theoretic conceptions of information meet naturally.

Calculus does not suffice. Van Benthem 1989 gives a temporal modeling with stages where some information is available. But in that setting, there is no preference for the past over the future, and the repertoire of natural informational actions would involve both *forward* update and *backward* contraction, and forward 'suprema' as well as backward 'infima' of information states. We will not develop this view here, [8] but maybe its very wealth of basic operations will suffice to show how we will just see too little when we get obsessed with merely interpreting some elegant categorial base repertoire •, →, ←.

5 Critical discussion and conclusion

Information dynamics seems to have two major paradigms, categorial (proof-theoretic) and modal-dynamic (semantic). I have long wrestled with their relationship, for instance, in van Benthem 1998, when comparing modal and categorial themes in the work of Dov Gabbay. I fear that my brief discussion in this paper is as inconclusive as the earlier one.

But maybe, there was also too much of a fatal attraction to the Lambek Calculus in the first place. The very fact that it fits so many interpretations already may just mean that it *says so very little*. Here is one example. As we have seen, *LC* is an elegant decidable complete logic for a very small fragment of Relational Algebra. But this fact can also be highly misleading as to the true structure of action. We know nowadays that the main source of high complexity in Relational Algebra is the apparently 'harmless' assumption of *associativity* built into the very Lambek sequent notation: cf. Németi 1995. So, the true behaviour of logics of action requires stepping away from the suggestion implicit in *LC*. [9]

And as a further critical point, the little one can say in *LC* or *NLC,* even if they were the right base logic for information, may merely reflect the 'unification' proposed in van Benthem 1998, close to the semantics of relevant logic: we can cast both categorial and model paradigms in terms of one base logic of a ternary predicate of 'abstract composition'

[8] I intend to develop this two-level 'piece'/'stage' information model in a subsequent publication.

[9] Admittedly, Sequoiah-Grayson 2009 tacklers just this point, and argues for the *non-associative* Lambek Calculus *NLC* as the base logic of information. But my other critical points remain.

Rx, yz saying that *x* is a combination of *y* and *z*. But this most general unifier may have little of interest left, except for the very thin smile of some Cheshire Cat. [10]

A third critical point is this. All earlier-mentioned interpretations that meet in *LC* come with their own natural repertoire of *further operations*. Van Benthem 2005 discusses some of these, and finds wide divergences. Often, the natural companions of the basic product operation are not any sort of categorial inverses, but Booleans and other logical operations, including the *regular algebra* of dynamic-epistemic logic, or even modal fixed-point operators like in propositional dynamic logic or the modal μ–calculus. And then, the proper study of the relevant phenomena of language, information, or action crucially involves these additional operators. Their agreement on a tiny subset may not mean all that much.

The Lambek Calculus is a wonderfully elegant little system that captures the essence of an abstract composition operation and its left- and right-inverses. But maybe we should not strain its role beyond what it can reasonably tell us in its simple language.

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[10] A much more detailed study of Lambek Calculus, Relational Algebra in its abstract 'Arrow Logic' format, Relevant Logic, and Labeled Deductive Systems is the dissertation Kurtonina 1995.

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