Interpolation in Guarded Fragments*

Eva Hoogland & Maarten Marx
{ehooglan, marx}@wins.uva.nl
ILLC, Universiteit van Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

July 4, 2000

Abstract

The guarded fragment (GF) was introduced by Andréka, van Benthem and Németi as a fine-structure of first order logic which combines a great expressive power with nice modal behavior. It consists of relational first order formulas whose quantifiers are relativized by atoms in a certain way. Slightly generalizing the admissible relativizations yields the packed fragment (PF). In this paper we chart the behavior of these fragments with regard to interpolation. While GF and PF have been established as particularly well-behaved fragments of first order logic in many respects, it will be shown that the interpolation property of first order logic fails in restriction to GF and PF. However, each of these fragments turns out to have an alternative interpolation property that closely resembles the interpolation property usually studied in modal logic. These results are strong enough to entail the Beth definability theorem for GF and PF. Even better, every n-variable guarded or packed fragment with up to n-ary relations has the Beth property. Finally it will be demonstrated that any n-variable guarded fragment which contains at most k-ary relations has interpolation if and only if $k \leq 2$ or $n \leq 2$. In particular, the 2-variable guarded fragment has interpolation. The packed fragments have interpolation if and only if $n \leq 2$ or $k \leq 1$.

Contents

*The part of this paper concerning Beth's definability property for the guarded fragment has appeared as the conference paper [HMO99] jointly written with Martin Otto.

1 Introduction

The Guarded Fragment and the Packed Fragment As is well-known, the basic modal logic K can be seen as a fragment of first order logic via the translation t which maps a proposition letter p to the atom Px, which commutes with the Boolean connectives, and which maps formulas of the form $\Diamond\varphi$ to $\exists y(Rxy \wedge \varphi^t(y))$ and $\Box\varphi$ to $\forall y(Rxy \to \varphi^t(y))$. The image of K under this translation is referred to as the *modal fragment.* This fragment turns out to behave excellently. It shares several nice model-theoretic properties with full first order logic (e.g., interpolation, Beth definability, and the Łos-Tarski property), and has in addition good algorithmic qualities: it is decidable and every satisfiable modal formula has a finite model and a tree model (in other words, the modal fragment has the finite model property and the tree model property). Moreover, the decidability of this fragment is robust in the sense that various expansions remain decidable. For example, adding features like counting quantifiers or fixed points to the modal fragment does not affect decidability.

In [ABN98] it is argued that the distinguishing characteristic of the modal fragment is its restriction on quantifier patterns. This brings Andréka, van Benthem and Németi to investigate to what extent these quantifier restrictions can be loosened while retaining the attractive modal behavior. The outcome is the *guarded fragment* (GF) which allows for quantifications of the form $\exists\bar{y}(R\bar{x}\bar{y} \wedge \varphi(\bar{x},\bar{y}))$ and $\forall\bar{y}(R\bar{x}\bar{y} \to \varphi(\bar{x},\bar{y}))$, where $\bar{x},\bar{y}$ are finite sequences of variables and φ is a guarded formula with free variables among $\bar{x},\bar{y}$ which *all* must appear in the atomic formula $R\bar{x}\bar{y}$.

In [ABN98] the (equality free) guarded fragment is shown to be decidable. Grädel [Grä99] improves on this result by classifying the satisfiability problem for GF to be complete for deterministic double exponential time. Satisfiability for the finite variable guarded fragments is even in $\mathsf{Exptime}$, in fact $\mathsf{Exptime}$-complete. [Grä99] also establishes the finite model property for GF, using a technique which involves the so-called Herwig's Theorem which was first employed by Andréka, Hodkinson and Németi to show the finite model property for similar fragments, [AHN99]. What is more, GF has a certain tree model property, [ABN98]. Since the tree model property of the modal fragment can be seen as the main reason behind the robustness of the decidability of that fragment (cf. e.g., [Var98]), this gives hope as to the robustness of GF. And indeed, adding least and greatest fixed points to GF yields a decidable expansion [GW99]. Interpolation and definability are other yardsticks by which to measure the modal behavior of GF. They are investigated in this paper.

We also study a natural expansion of GF, called the *packed fragment* (PF). Roughly speaking, the packed fragment allows for quantifications of the form $\exists\bar{y}(\varphi \wedge \psi)$, where φ is a conjunction of atoms such that for every pair of free variables in ψ there exists some conjunct in which both variables occur. For readers familiar with [Ben97] we note that PF bears close correspondence to the *loosely guarded fragment* introduced and studied there. [Ben97] shows that PF is decidable. More specifically, just as for GF, the satisfiability problem for PF is complete for $\mathsf{2Exptime}$, while for finite variable packed fragments this problem is $\mathsf{Exptime}$-complete, [Grä99]. Moreover, PF has a tree-model property, [Grä99]. The question as to whether PF has the finite model property, has recently been solved by Hodkinson. In [Hod00], the answer is affirmative.

Interpolation and Definability In 1956, W. Craig proved the interpolation theorem for first order logic which states that any pair of first order formulas φ, ψ such that $\varphi \to \psi$ is a validity, has an *interpolant*. That is, there exists a first order formula ϑ in the common language of φ, ψ, such that $\varphi \to \vartheta$ and $\vartheta \to \psi$ are again validities. A logic for which an analogous statement holds, is said to have the *interpolation property*. This property can be seen as a sign for a well-behaved deductive system. Besides its theoretical interest interpolation plays a role in e.g., the field of automatic theorem proving, where it can be used to restrict the search space of the inference algorithm by looking for intermediate lemmas. Logics with the interpolation property include classical (and intuitionistic) propositional calculus, and the modal logics K, $\mathsf{K4}$ and $\mathsf{S5}$. Examples of logics without this property are the finite variable fragments of first order logic, with the exception of the one variable fragment, [Com69].

Another important reason for investigating interpolation, is that it can be seen as an intermediary step in proving the *Beth definability property*. In a slogan, the Beth property states that implicit definability equals explicit definability. Generally, this may be regarded as an indication that there is a good balance between syntax and semantics of a logic: the semantic phenomenon that the meaning of a basic relation is implicitly determined guarantees that there is an explicit syntactic expression for that relation. Examples of logics with the Beth property include the afore-mentioned logics with the interpolation property, but many more. E.g., all normal expansions of the modal logic $\mathsf{K4}$, [Mak92]. For any $n \geq 2$, the n-variable fragments of first order logic do not have the Beth property, cf. [Sai90], [Hod93].

In this paper we show that the interpolation theorem fails for the guarded fragment and the packed fragment. That is, we will exhibit two guarded (resp. packed) formulas φ, ψ such that $\varphi \to \psi$ is a validity, without a guarded (resp. packed) interpolant. The reason for this failure of interpolation becomes apparent if we take seriously the idea of these fragments being generalizations of the modal fragment. For, from this point of view, an interpolation theorem for these fragments is not so much a restriction of the interpolation theorem for first order logic, as a generalization of the interpolation theorem for modal logic. However, in multi-modal logics an interpolant is usually subject to slightly different conditions than in first order logic: it is confined to proposition letters in the common language but may contain non-shared modalities. In this paper we generalize this condition to guarded and packed formulas, and show that GF and PF do have a generalized version of the modal interpolation property. This reveals once more the truly modal character of these fragments. Moreover, this alternative interpolation property turns out to be strong enough to entail the Beth definability property for GF and PF.

Guarded Finite Variable Fragments In first order logic, restricting the number of variables causes failure of interpolation and definability. As mentioned above, for any $n \geq 2$, the n-variable fragment of first order logic fails to have the Beth property. In fact, a counterexample can already be given using only binary relation symbols, [Sai90]. With regard to interpolation the situation is even worse. For any $n \in \omega$, there exist first order formulas φ_n, ψ_n in just two variables and binary relation symbols such that $\varphi_n \to \psi_n$ is valid, but which lack a first order interpolant in n variables. This result is due to Andréka (unpublished), for a stronger result see [MA98].

In this paper, it will be shown that the Beth property holds for the guarded and the packed finite variable fragments. For interpolation such a result is not possible, for the simple reason that GF and PF themselves do not have this property. However, the positive results on interpolation for GF and PF mentioned above do carry over to all their finite variable fragments. Moreover, we show that the strong interpolation property of first order logic does hold for the guarded two variable fragment and the packed two variable fragment. Note the striking contrast with the situation in the full 2-variable fragment. Another result along this line shows the existence of a guarded interpolant in the first order sense for guarded formulas φ, ψ such that $\varphi \to \psi$ is a validity, provided φ, ψ contain at most binary predicates. A similar result for packed formulas turns out to fail. This difference will be accounted for.

The upshot of all these results is twofold. First it shows that guarded and packed finite variable fragments are much better behaved w.r.t. interpolation and definability than the full finite variable fragments of first order logic. Second it completely charts the behavior of interpolation with regard to guarded and packed finite variable fragments. Such a chart is in general not straightforward, as interpolation is a very sensitive property that is in general not preserved under either expanding or restricting the logic under consideration. For example, as we already saw, first order logic has interpolation whereas its two-variable fragment does not and the basic modal logic K, in its turn a fragment of the two variable fragment, again has interpolation. However, for the guarded and packed fragments considered in this paper we have indicated a sharp boundary between failure and success of interpolation. More precisely, denoting the restriction on the arity of the relation symbols by k, and the restriction on the number of variables by n, we have shown that GF^k_n has interpolation if and only if $min(k, n) \leq 2$. Furthermore, PF^k_n has interpolation if and only if $n \leq 2$ or $k \leq 1$.

Outline of Paper First we will give precise definitions of all the notions introduced so far, most notably that of the guarded and packed fragment. These fragments will also be semantically characterized in terms of suitable notions of bisimulations. Readers familiar with the notion of a guarded bisimulation are advised to start reading in Section 3, where counterexamples to interpolation in GF and PF are presented. The reason for the failure of interpolation in these logics is discussed in Section 4, and an alternative interpolation theorem for them is offered. This alternative, it will be argued, is a natural property to investigate in a guarded context which is moreover quite strong. Strong enough at least to entail the Beth definability theorem, cf. Section 5. Finally, we investigate interpolation for guarded and packed finite variable fragments. The main result of this last section is that the guarded two variable fragment has interpolation.

2 Preliminaries

In this section we will collect all the necessary preliminaries. It also serves to fix notation and terminology.

Convention 2.1 By a *language* $\mathcal{L}$ we will henceforth understand a relational first order language without function- or constant symbols. Besides variables, and the parentheses

), (, we consider as *logical symbols* the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, the quantifiers $\exists$, $\forall$ and the identity symbol $=$. $\dashv$

Notation 2.2 Models are denoted by calligraphic letters like $\mathcal{M}, \mathcal{N}$, and their respective universes by M, N, etc. The interpretation of an k-ary predicate R in the model $\mathcal{M}$ (notation: $I^{\mathcal{M}}(R) \subseteq M^k$) is defined as usual. For any formula φ, we write $\varphi(v_1, \ldots, v_n)$ to make explicit that the free variables in φ (notation: $free(\varphi)$) are among $\{v_1, \ldots, v_n\}$. For a formula $\varphi(v_1, \ldots, v_n)$, a model $\mathcal{M}$, and $m_1, \ldots, m_n \in M$, we write $\mathcal{M} \models \varphi[m_1, \ldots, m_n]$ iff each assignment which maps v_i to m_i satisfies φ in $\mathcal{M}$. If Σ is a set of formulas, and ψ a formula, then $\Sigma \models \psi$ denotes the *consequence* relation. That is, $\Sigma \models \psi$ iff any assignment into a model $\mathcal{M}$ which satisfies all formulas in Σ also satisfies ψ. In particular, $\varphi \models \psi$ is the same as to say that $\varphi \rightarrow \psi$ is valid, i.e., $\models \varphi \rightarrow \psi$. $\dashv$

Let $\bar{v} = v_1, \ldots, v_k$. Below, we use *generalized quantifiers* of the form $\exists \bar{v}(\gamma, \psi)$, which denote the first order formula $\exists v_1 \cdots \exists v_k(\gamma \wedge \psi)$. We call γ the *guard* of $\exists \bar{v}(\gamma, \psi)$.

Definition 2.3 (Guarded Fragment) Let $\mathcal{L}$ be a language. As usual, the *atomic $\mathcal{L}$-formulas* (or, $\mathcal{L}$-atoms) are of the form:

1. $v_1 = v_2$, for variables v_1, v_2.
2. $Pv_1 \cdots v_k$, for k-ary $P \in \mathcal{L}$, and variables $v_1, \ldots, v_k$, not necessarily distinct.

The *guarded fragment* of first order logic (GF) is defined by induction as follows.

1. Any atomic formula is in GF.
2. If $\varphi, \psi \in$ GF, then $\varphi \wedge \psi$ and $\neg\varphi$ are in GF.
3. Let $\bar{v}$ be a finite, non-empty sequence of variables, $\psi \in$ GF and G an atomic formula such that $free(\psi) \subseteq free(G)$. Then $\exists \bar{v}(G, \psi) \in$ GF.

$\dashv$

Note that as a dual of guarded existential quantification we also get guarded universal quantification, of the form $\forall \bar{v}(G \rightarrow \psi)$.

A typical example of a formula in GF is $\forall v_1 v_2(Rv_1v_2 \rightarrow Rv_2v_1)$, expressing the symmetry of the relation R. On the other hand, the translation of the tense logical formula $\mathsf{Until}(\varphi, \psi)$, i.e., the formula $\exists v_2(v_1 < v_2 \wedge \psi(v_2) \wedge \forall v_3[(v_1 < v_3 \wedge v_3 < v_2) \rightarrow \varphi(v_3)])$, is not in GF as the guard in the subformula expressing betweenness (i.e., the formula $\forall v_3[(v_1 < v_3 \wedge v_3 < v_2) \rightarrow \varphi(v_3)]$) is not atomic.

Remark 2.4 For readers familiar with [ABN98] we note that contrary to that paper, Definition 2.3 allows for identity atoms as guards. Since this issue does not affect decidability nor interpolation, we decided to concentrate on this slightly more general fragment. This also places us in line with [Grä99]. $\dashv$

Let γ be a formula with $free(\gamma) = \{v_1, \ldots, v_k\}$. We say that γ *packs the set of variables* $\{v_1, \ldots, v_k\}$ if γ is a conjunction of formulas of the form $v_i = v_j$, or $R(v_1, \ldots, v_l)$, or $\exists\bar{v}R(v_1, \ldots, v_l)$, such that every pair $v_i, v_j \in \{v_1, \ldots, v_k\}$ is guarded by γ. That is, for every pair v_i, v_j there exists a conjunct in γ in which v_i and v_j both occur free. A generalized quantifier $\exists\bar{v}(\gamma, \psi)$ is called *packed* if γ packs $free(\gamma)$ and $free(\psi) \subseteq free(\gamma)$. Again, we call γ the *guard* of $\exists\bar{v}(\gamma, \psi)$.

Definition 2.5 (Packed Fragment) The *packed fragment* of first order logic (PF) is defined similarly to GF, by relaxing clause (3) as follows:

> (3'): If $\psi \in$ PF, then any packed generalized quantifier $\exists\bar{v}(\gamma, \psi)$ is in PF.

$\dashv$

Obviously, GF $\subseteq$ PF. The translation of $\mathsf{Until}(\varphi, \psi)$ is not packed itself, as the guard $(v_1 < v_3 \wedge v_3 < v_2)$ does not pack its own free variables $\{v_1, v_2, v_3\}$. However, the translation is equivalent to the packed $\exists v_2(v_1 < v_2 \wedge \psi(v_2) \wedge \forall v_3[(v_1 < v_2 \wedge v_1 < v_3 \wedge v_3 < v_2) \rightarrow \varphi(v_3)])$.

Notation 2.6 Let $k, n \in \omega$. By GF$_n$ we understand the fragment of GF that consists of formulas whose variables (free or bound) are among $v_1, \ldots, v_n$. GF$_n^k$ denotes the collection of formulas in GF$_n$ which are built up from at most k-ary relation symbols. Similar notation will be used for fragments of PF. $\dashv$

Remark 2.7 For readers familiar with [Ben97], we point out the differences between the packed fragment and the so-called *loosely guarded fragment* (LGF) introduced and studied in [Ben97]. We recall from that paper that a generalized quantifier $\exists\bar{v}(\gamma, \psi)$ is *loosely guarded* if γ is a conjunction of relational atoms such that $free(\gamma) = free(\psi) \cup \bar{v}$ and moreover every pair $v \in \bar{v}, u \in free(\gamma)$ is guarded by γ.

On the one hand, PF is more liberal. For, contrary to the case for LGF, formulas of the form $\exists\bar{v}R(v_1, \ldots, v_k)$ and identities may occur in the guard of a packed quantifier. This is a real expansion. For example, the packed formula $\exists v_2 v_3[(\exists v_4 Rv_1v_2v_4 \wedge Sv_2v_3 \wedge Sv_1v_3) \wedge \neg Rv_1v_2v_3]$ has no loosely guarded equivalent.

On the other hand, LGF is more liberal as in a loosely guarded quantifier $\exists\bar{v}(\gamma, \psi)$ a pair of variables from $free(\gamma)\backslash\bar{v}$ does not need to be guarded by γ. For this reason, the translation of $\mathsf{Until}(\varphi, \psi)$ is already in LGF.

For sentences though, PF is an expansion of LGF. For, let χ be a loosely guarded sentence, and let $\chi' = \exists\bar{v}(\gamma, \psi)$ be a subformula of χ which is not packed. Then some $u, u' \in free(\gamma)\backslash\bar{v}$ are not guarded by γ. As χ is a sentence, at some stage u, u' will be quantified over. More precisely, χ has a subformula of the form $\exists\bar{v}'(\gamma', \psi')$, such that χ' is a subformula of ψ' and $\bar{v}' \cap \{u, u'\} \neq \emptyset$ and $\{u, u'\} \subseteq free(\psi')$. By definition of LGF, there exists a conjunct G in γ' which guards u, u'. Replace χ' in χ by $\exists\bar{v}(\gamma \wedge G, \psi)$. The thus obtained sentence is equivalent to χ, and in this replacement u, u' are guarded. Repeating this procedure will eventually yield a packed equivalent of χ.

Viewed as an expressive fragment of first order logic which is still decidable, the choice between PF and LGF seems to be irrelevant. By introducing new predicates, one easily defines simple translations from PF to LGF and vice versa, such that a formula is satisfiable iff its translation is. In this paper we choose to study PF. The reason for this twofold.

First, as we just saw, for sentences PF is a real expansion of LGF. Second, the definition of PF seems to be more straightforward. This is reflected in the fact that PF allows for a semantic characterization which naturally extends the characterization of GF known from [ABN98] (cf. Definition 2.9). $\dashv$

Similar to modal logics, the guarded fragment can be semantically analyzed via a suitable notion of bisimulation. This has been done in [ABN98]. Here we will recapitulate as much of these results as needed for the purposes of the present paper. Moreover, we will introduce the notion of a *packed bisimulation*.

Definition 2.8 (live set, packed set) Let Z be a finite subset of a model $\mathcal{M}$. Z is called *live* in $\mathcal{M}$ if Z is either a singleton, or there exists a relation R and a set X such that $Z \subseteq X$ and the elements of X are R-related, in any order or multiplicity. E.g., if $\langle m_2, m_1, m_1, m_3 \rangle \in I^{\mathcal{M}}(R)$, then $\{m_2, m_3\}$ is live in $\mathcal{M}$. In this case we say that $\{m_2, m_3\}$ is *R-live* (in $\mathcal{M}$). For any language $\mathcal{L}$, $Z \subseteq^l_{\mathcal{L}} \mathcal{M}$ denotes that Z is $\mathcal{L}$-live in $\mathcal{M}$. That is, Z is R-live in $\mathcal{M}$, for some $R \in \mathcal{L}$. Z is called *$\mathcal{L}$-packed* in $\mathcal{M}$ if any pair $m, m' \in Z$ is $\mathcal{L}$-live in $\mathcal{M}$. Notation: $Z \subseteq^p_{\mathcal{L}} \mathcal{M}$. $\dashv$

Below, by a finite *partial $\mathcal{L}$-isomorphism* we mean a finite one-to-one partial map between two models which preserves $\mathcal{L}$-relations in both ways. By the *image* of a map $f : X \longrightarrow Y$ we understand the set $\{f(x) : x \in X\}$, and we refer to X as its *domain*.

Definition 2.9 (Guarded bisimulation, packed bisimulation) A non-empty set F of finite partial $\mathcal{L}$-isomorphisms between two models $\mathcal{M}$ and $\mathcal{N}$ is called a *guarded $\mathcal{L}$-bisimulation* between $\mathcal{M}$ and $\mathcal{N}$ if for any $f : X \longrightarrow Y \in F$ the following holds:

1. For any $Z \subseteq^l_{\mathcal{L}} \mathcal{M}$ there is a $g \in F$ with domain Z such that g and f agree on the intersection of their domains. (The *zig*-condition)
2. For any $W \subseteq^l_{\mathcal{L}} \mathcal{N}$ there is a $g \in F$ with image W such that g^{-1} and f^{-1} agree on the intersection of their domains. (The *zag*-condition)

Stipulating the zig- and zag-condition for $\mathcal{L}$-packed subsets yields the notion of a *packed $\mathcal{L}$-bisimulation* $\dashv$

Notation 2.10 We write $\mathcal{M}, m_1 \cdots m_k \sim_F \mathcal{N}, n_1 \cdots n_k$ to denote that F is a guarded bisimulation between $\mathcal{M}$ and $\mathcal{N}$ with some $f \in F$ such that $f(m_i) = n_i$, for $i \leq k$. $\dashv$

Guarded bisimulations are defined in such a way as to preserve guarded formulas. That is, for a guarded $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_k)$, a guarded $\mathcal{L}$-bisimulation F between models $\mathcal{M}$, $\mathcal{N}$, $f \in F$, and $m_1, \ldots, m_k \in dom(f)$ it is the case that $\mathcal{M} \models \varphi[m_1, \ldots, m_k] \Leftrightarrow \mathcal{N} \models \varphi[f(m_1), \ldots, f(m_k)]$. This can be shown by a straightforward induction on the complexity of φ. The zig- and zag-conditions precisely take care of the induction step for existential quantification. Indeed, preservation under guarded bisimulations is the characteristic feature of GF, in the sense of the following Characterization Theorem from [ABN98]: up to logical equivalence, GF precisely consists of those first order formulas that are preserved under guarded bisimulations. Similarly, preservation under packed bisimulations is the characteristic feature of PF, cf. [Mar99b].

In the definition of a guarded bisimulation that can be found in [ABN98], the role of live sets in Definition 2.9 is taken over by what Andréka, van Benthem and Németi call *guarded* sets. The main difference is that singletons need not be guarded, whereas they are always live. This semantically reflects the fact that we allow for identity atoms as guards. Mutatis mutandis, all arguments in [ABN98] and in particular the characterization theorem also apply to guarded formulas and guarded bisimulations as defined in this paper.

[Mar97] contains a reduction of the satisfiability problem of PF to a particularly nice subfragment of the guarded fragment. This fragment is the first order counterpart of the modal logic of relations *MLR* from [VM99]. We use the same reduction here to obtain our results on interpolation and definability. The reduction is given as follows.

Let $\chi \in \mathrm{PF}^k_k$, and let C be a k-ary predicate not occurring in χ. χ^C denotes the result of replacing each packed quantification in χ of the form $\exists \bar{v}(\gamma, \psi)$, by $\exists \bar{u}\bar{v}(Cv_1 \cdots v_k, \gamma \wedge \psi)$, where $\exists \bar{u}$ binds exactly the variables $\{v_1, \ldots, v_k\} \setminus \mathit{free}(\gamma \wedge \psi)$. Note that $\chi^C \in \mathrm{GF}^k_k$. Moreover, $Cv_1 \cdots v_k$ is the only guard in χ^C and the predicate symbol C solely occurs at guarding position. We call formulas in GF^k_k with this property MLR^C_k*–formulas.*

Define the set $\mathrm{LC} = \{\forall v_1 \cdots v_k(Cv_1 \cdots v_k \rightarrow Cv_{\sigma(1)} \cdots v_{\sigma(k)}) \mid \sigma : k \longrightarrow k\}$. LC axiomatizes the class of models on which C is interpreted as a *local cube*, cf. [VM99].

THEOREM 2.11 ([Mar97]) *Let* $\varphi \in \mathrm{PF}^k_k$. *Then*

$$\models \varphi \;\; \textit{iff} \;\; \mathrm{LC}, Cv_1 \cdots v_k \models \varphi^C.$$

Proof of THEOREM 2.11: First let $\mathcal{M} = (D, I) \not\models \phi$. Expand $\mathcal{M}$ to a model $\mathcal{M}^+$ by setting $I^+(C) = {}^nD$. Obviously $\mathcal{M}^+ \models LC \wedge \forall \bar{v} Cv_1 \cdots v_k$, but $\mathcal{M}^+ \not\models \phi^C$.

For the other direction, we use the fact that guarded formulas can be satisfied in unraveled or tree models. This is established in several papers, e.g., [ABN98, Mar97, Grä99]. We use here the formulation of Grädel. Assume that $LC, Cv_1 \cdots v_k \not\models \phi^C$. Thus the GF^k_k formula $LC \wedge Cv_1 \cdots v_k \wedge \neg\phi^C$ is satisfiable. By Lemma 3.6 and the construction in the proof of Theorem 3.7 in [Grä99] this formula is satisfiable in a tree model $\mathcal{M}$ in which every packed set is C–live. We claim that in this model $\mathcal{M}$, for every assignment $s \in I(C)$ and for every subformula ψ of ϕ, we have

$$\mathcal{M} \models \psi[s] \text{ if and only if } \mathcal{M} \models \psi^C[s].$$

Obviously from this the theorem follows. The only non–trivial case of the inductive proof is the following. Let $\bar{v} = v_1, \ldots, v_n$, $\bar{u} = u_1, \ldots, u_m$, and $\bar{s} = s_1, \ldots, s_n$. Let $\mathcal{M} \models \exists \bar{v}(\phi(\bar{u}, \bar{v}), \psi(\bar{u}, \bar{v}))[\bar{s}]$. Then $\mathcal{M} \models \phi(\bar{u}, \bar{v}) \wedge \psi(\bar{u}, \bar{v})[\bar{s}, \bar{t}]$, for some $\bar{t} = t_1, \ldots, t_m$. As $\phi(\bar{u}, \bar{v})$ packs all its free variables, the set $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ is packed. Thus, by the assumption on the model $\mathcal{M}$, it is C–live. Whence, as $\mathcal{M} \models LC$, every k–tuple with elements from this set is in $I(C)$. Thus the inductive hypothesis applies and $\mathcal{M} \models Cv_1 \cdots v_k \wedge \phi(\bar{u}, \bar{v}) \wedge \psi^C(\bar{u}, \bar{v})[\bar{s}, \bar{t}, r_1, \ldots, r_l]$, for appropriately chosen $r_1, \ldots, r_l$. Thus $\mathcal{M} \models \exists \bar{v}\bar{v'}(Cv_1 \cdots v_k \wedge \phi(\bar{u}, \bar{v}) \wedge \psi^C(\bar{u}, \bar{v}))[s_1, \ldots, s_n]$, which is the C–translation of $\exists \bar{v}(\phi(\bar{u}, \bar{v}), \psi(\bar{u}, \bar{v}))$.
Q.E.D. THEOREM 2.11.

3 Failure of Interpolation in GF and PF

In this section, we present examples showing that interpolation fails both in the guarded fragment and in the packed fragment. More precisely, we define guarded sentences φ, ψ in three variables using at most ternary relations such that $\models \varphi \to \psi$, which lack a guarded interpolant in any number of variables. Next, a similar example will be given for packed sentences built up from three variables and binary relations.

Definition 3.1 (Strong and weak interpolation) A fragment $\mathcal{F}$ of first order logic is said to have the *strong interpolation property* if for any pair of formulas $\varphi, \psi \in \mathcal{F}$ such that $\models \varphi \to \psi$ there exists an interpolant in $\mathcal{F}$. That is, there exists a formula $\vartheta \in \mathcal{F}$ which is built up from relation symbols which occur both in φ and ψ such that $\models \varphi \to \vartheta$ and $\models \vartheta \to \psi$. Restricting this definition to sentences yields the *weak interpolation property.* $\dashv$

THEOREM 3.2 (Failure of interpolation in the guarded fragment) *There exist sentences $\varphi, \psi \in \mathrm{GF}^3_3$ such that $\models \varphi \to \psi$, without a guarded interpolant in any number of variables.*

Proof of THEOREM 3.2: For $i = 1, 2$, define the languages $\mathcal{L}_i$ and the GF^3_3-theories T_i as follows. $\mathcal{L}_1$ consists of one ternary predicate Δ, and the binary predicates R, B, G, P, W. Intuitively, Δxyz holds for nodes x, y and z which together form a triangle, and Rxy (resp. B, G, P, W) holds whenever the edge between the nodes x and y is painted red (resp. blue, green, purple and white). $\mathcal{L}_2$ does not contain the predicate Δ, but instead contains two extra binary predicates R_1, R_2. Theory T_1 is defined by

$$\exists xyz[\Delta xyz \wedge Rxy \wedge Bxz \wedge Gzy \wedge \exists z(\Delta xyz \wedge Pxz \wedge Wzy)]. \tag{1}$$

T_1 demands the existence of a situation as e.g., occurs in model $\mathcal{M}_1$ in Figure 1 below.

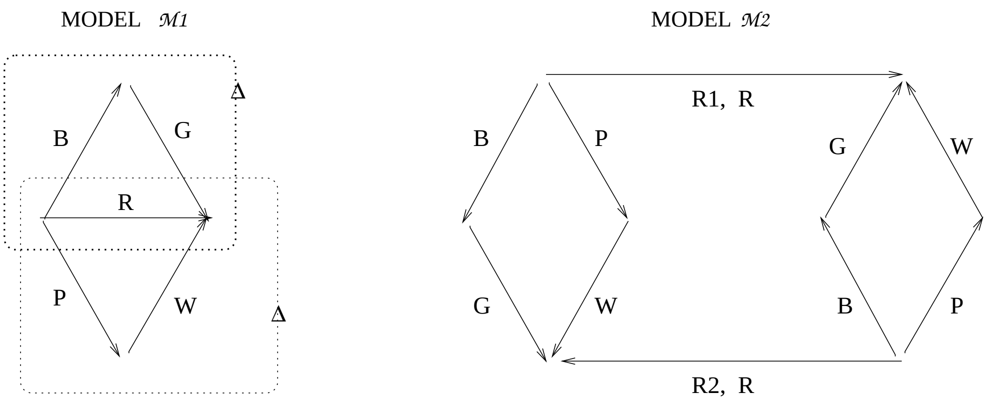

Figure 1

Theory T_2 is the conjunction of

$$\forall xy[R_1xy \to Rxy], \quad \forall xy[R_2xy \to Rxy], \quad \forall xy[Rxy \to (R_1xy \vee R_2xy)]. \tag{2}$$

$$\forall xy[R_1xy \to \forall y(Pxy \to \forall x[Wyx \to \forall y(Ryx \to \neg R_1yx)])]. \tag{3}$$

$$\forall xy[R_2xy \to \forall y(Bxy \to \forall x[Gyx \to \forall y(Ryx \to \neg R_2yx)])]. \tag{4}$$

The formulas numbered (2) can be read as saying that the color red has precisely two shades R_1 and R_2. By (3) and (4) the situations in Figure 2 below are prohibited.

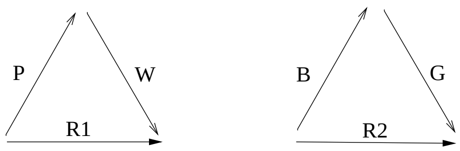

Figure 2

From Figures 1 and 2 it is clear that $\models T_1 \to \neg T_2$. However, T_1 and $\neg T_2$ do not have a guarded interpolant. For, consider the models $\mathcal{M}_1$, $\mathcal{M}_2$ in Figure 1. Obviously, $\mathcal{M}_i$ is a model for T_i. Let F be the set of finite partial maps from $\mathcal{M}_1$ to $\mathcal{M}_2$ which map an $\mathcal{L}_1 \cap \mathcal{L}_2$-live subset of $\mathcal{M}_1$ to a subset of $\mathcal{M}_2$ which is live by the same predicates. The reader easily checks that F is a guarded $\mathcal{L}_1 \cap \mathcal{L}_2$-bisimulation. Let us suppose that there does exist some guarded interpolant ϑ for $T_1, \neg T_2$. As $\mathcal{M}_1 \models T_1$, then $\mathcal{M}_1 \models \vartheta$. Since guarded $\mathcal{L}$-formulas are preserved by guarded $\mathcal{L}$-bisimulation, we obtain that $\mathcal{M}_2 \models \vartheta$. Hence $\mathcal{M}_2 \models \neg T_2$. Contradiction.
Q.E.D. THEOREM 3.2.

Corollary 3.3 GF, GF_3 *and* GF^3 *do not have weak interpolation.*

It is not hard to check that the packed formula $\exists xyz[Rxy \wedge Bxz \wedge Gzy \wedge \exists z(Pxz \wedge Wzy)]$ is an interpolant for the theories T_1 and $\neg T_2$ in the above example. This leaves the question of interpolation for PF open. However, along the same lines as above an example can be given showing that also in PF weak interpolation fails.

THEOREM 3.4 (Failure of interpolation in PF) *There exist sentences $\varphi, \psi \in \mathrm{PF}_3^2$ such that $\models \varphi \to \psi$, without a packed interpolant in any number of variables.*

Proof of THEOREM 3.4: For $i = 1, 2$, define the languages $\mathcal{L}_i$ and the PF_3^2 theories T_i as follows. $\mathcal{L}_1$ contains the binary predicates B, W, G, P, O, Y, R and J. $\mathcal{L}_2$ does not contain J, but has two extra binary predicates R_1, R_2. Set

$$T_1 \stackrel{\text{def}}{=} \exists xyz[Bxy \wedge Gyz \wedge Jxz \wedge \exists y(Rxy \wedge Wzy \wedge \exists z(Pxz \wedge Jzy \wedge \exists x(Ozx \wedge Yxy)))].$$

Model $\mathcal{M}_1$ in Figure 4 is a typical model of T_1. Theory T_2 is the conjunction of (2) together with

$$\forall xy[R_1xy \to \forall y(Bxy \to \forall x[Gyx \to \forall y(Wxy \to \forall x[Rxy \to \neg R_1xy])])],$$

and

$$\forall xy[R_2xy \to \forall y(Pxy \to \forall x[Oyx \to \forall y(Yxy \to \forall x[Rxy \to \neg R_2xy])])].$$

This theory T_2 prohibits the situations in Figure 3 below.

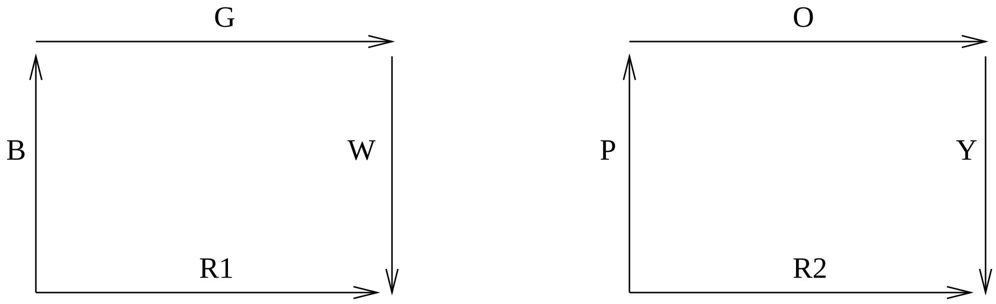

Figure 3

The figures clearly show that $\models T_1 \to \neg T_2$. We will imitate the model-theoretical argument in the proof of Theorem 3.2 to see that T_1 and $\neg T_2$ do not have a packed interpolant. This time consider the models $\mathcal{M}_1$, $\mathcal{M}_2$ in Figure 3.

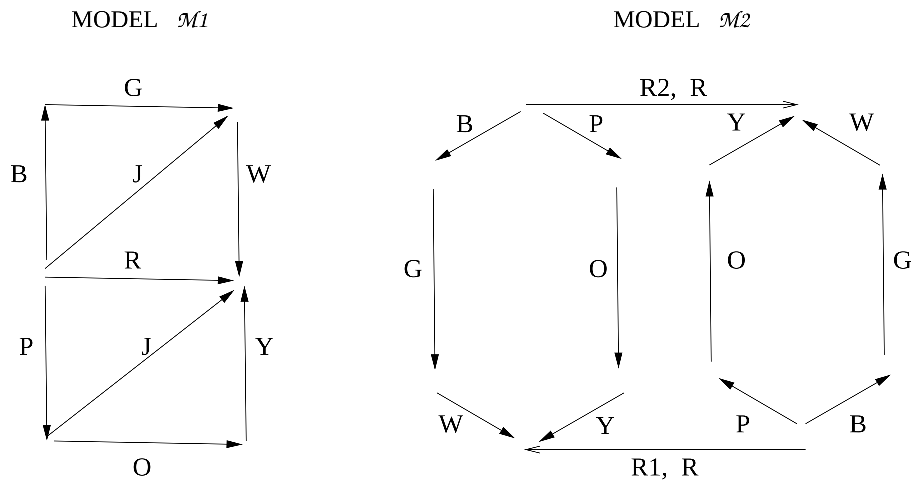

Figure 4

Again, $\mathcal{M}_i$ is a model for T_i. One readily verifies that the models $\mathcal{M}_1$ and $\mathcal{M}_2$ are packed $\mathcal{L}_1 \cap \mathcal{L}_2$-bisimilar. The crucial observation is that all $\mathcal{L}_1 \cap \mathcal{L}_2$-packed sets in $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{L}_1 \cap \mathcal{L}_2$-live. Therefore, the set F of finite partial maps from $\mathcal{M}_1$ to $\mathcal{M}_2$ which map an $\mathcal{L}_1 \cap \mathcal{L}_2$-live subset of $\mathcal{M}_1$ to a subset of $\mathcal{M}_2$ which is live by the same predicates, forms a *packed* $\mathcal{L}_1 \cap \mathcal{L}_2$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. Since packed $\mathcal{L}$-sentences are invariant under packed $\mathcal{L}$-bisimulation, the same argument as in the proof of the previous theorem leads to the desired contradiction.
Q.E.D. THEOREM 3.4.

Corollary 3.5 PF*,* PF_3 *and* PF^2 *do not have weak interpolation.*

Recapitulating the results in this section, we see that interpolation fails in GF and PF if we have at least 3 variables at our disposal. We also note that in the example showing the failure of interpolation in GF we made use of a ternary relation, whereas in the case of PF we could do with binary relations only. In Section 6 it will be shown that these are the smallest counterexamples possible.

Remark 3.6 According to Convention 2.1, in this paper we consider purely relational languages. Without deviating too much from this, let us just observe that treating constants as in [Grä99] and allowing e.g., guarded formulas of the form $T_1 \stackrel{\text{def}}{=} \forall x(Px \to (x = c))$, will yield failure of interpolation even in GF_1^1. For, consider $T_2 \stackrel{\text{def}}{=} \forall x(P_1x \to (Px \wedge \neg P_2x)) \wedge \forall x(P_2x \to Px) \wedge \exists x P_1 x \wedge \exists x P_2 x$. Then $\models T_1 \to \neg T_2$, and similar to the previous examples one verifies that there is no guarded interpolant. $\dashv$

4 Restoring Interpolation

To see why the interpolation property of first order logic fails in restriction to the guarded fragment, it is useful to compare it to the interpolation property studied in modal logic. In modal logic, the interpolant is usually confined to proposition letters in the common language but may contain non-shared modalities. Also in first order logic this kind of interpolation is studied, when we view $\exists x$ as a modality. This is e.g, the case in cylindric algebra theory, cf. [HMT85]. The point is that strengthening the requirement on the common language to include common modalities results in a much stronger interpolation property. [Ben99] shows this property for the basic multi-modal logic K_n. [Mar99a] generalizes this result to Sahlqvist axiomatizable multi-modal logics whose axioms correspond to universal Horn frame conditions which do not specify any interaction between the different accessibility relations (e.g., bi-modal $\mathsf{S5}$). When there is interaction, the stronger interpolation property is easily lost as the following example from [Ben99] shows. Consider the multi-modal logic defined by the axiom $\Diamond_1 p \to \Diamond_2 p$. This logic does not have the stronger interpolation property. For, in this logic $\Diamond_1 True \to \Diamond_2 True$ is a theorem whereas the only formulas in the common language (in the strong sense) are $True$ and $False$, which are obviously not interpolants. However, this logic does have the usual interpolation property.

Thinking of guarded formulas as translations of modal formulas, we see that Definition 3.1 formulates exactly this stronger version of interpolation, where 'common language' means the set of common relation symbols, which includes both the relations which are translated proposition letters and the relations that are obtained in translating the modalities. This suggests to consider an alternative interpolation property for the guarded fragment that more closely resembles the one that is usually studied in modal logic. For this we will distinguish occurrences of relation symbols as guards from other occurrences.

Notation 4.1 For any guarded or packed formula φ we understand by $\mathcal{L}_{G(\varphi)}$ the set of relations that occur in the guard of some quantification in φ. Similarly, $\mathcal{L}_{\bar{G}(\varphi)}$ denotes the set of relations that occur in φ at a non-guard position. We call a formula φ *uniformly guarded* in a formula χ, if χ is the only guard occurring in φ. $\dashv$

The sets $\mathcal{L}_{G(\varphi)}$ and $\mathcal{L}_{\bar{G}(\varphi)}$ are guarded counterparts of the sets of modalities and proposition letters, respectively, occurring in a modal formula. Note however that whereas there is a syntactic distinction between modalities and proposition letters, the sets $\mathcal{L}_{G(\varphi)}$ and $\mathcal{L}_{\bar{G}(\varphi)}$ are not necessarily disjoint. For example, in $\varphi = \exists x(Px \land \forall y(Sxy \to Py))$, the relation P occurs both as a guard and as a non-guard.

Our Main Lemma concerns MLR_k^C-formulas. Recall that these are formulas in GF_k^k which are uniformly guarded in $Cv_1 \cdots v_k$, while C only occurs at guarding position. Recall that $\mathrm{LC} = \{\forall v_1 \cdots v_k(Cv_1 \cdots v_k \to Cv_{\sigma(1)} \cdots v_{\sigma(k)}) \ : \ \sigma : k \longrightarrow k\}$.

Lemma 4.2 (Interpolation for MLR_k^C formulas) *Let $k \in \omega$, and let $\varphi, \psi \in MLR_k^C$ such that $LC, Cv_1 \cdots v_k \models \varphi \to \psi$. Then there exists an interpolant $\vartheta \in MLR_k^C$ such that*

1. *$LC, Cv_1 \cdots v_k \models \varphi \to \vartheta$ and $LC, Cv_1 \cdots v_k \models \vartheta \to \psi$,*
2. *$\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)}$, and*
3. *$free(\theta) \subseteq free(\varphi) \cup free(\psi)$.*

We prove Lemma 4.2 at the end of this section. First we continue our story and show that the guarded fragment indeed has a 'modal' interpolation property. The main result is Theorem 4.5 which states that under some mild assumptions (that e.g., all sentences satisfy) every $\varphi, \psi \in \mathrm{GF}_k^k$ such that $\varphi \to \psi$ is a validity, have an interpolant ϑ in GF_k^k in which relations in the common language occur at non-guard positions, and guards from φ or ψ at guard positions. So the guards in the interpolant need not be in the common language. It is possible to strengthen this result by showing ϑ to be built up from non-guards in the common language which moreover occur both in φ and in ψ as non-guards. However, this takes one extra variable. In both these cases the analogy with the interpolation property for modal logics is clear: an interpolant may use non-shared modalities, but uses shared proposition letters only.

For future use, we observe that for any relation symbol R and $n \in \omega$, the set of R-live n-tuples can be defined by a guarded formula. More precisely, there exists a formula $\mathsf{Live}_R(v_1, \ldots, v_n) \in \mathrm{GF}$ such that for any model $\mathcal{M}$ and $m_1, \ldots, m_n \in M$:

$$\mathcal{M} \models \mathsf{Live}_R(v_1, \ldots, v_n)[m_1, \ldots, m_n] \text{ iff the set } \{m_1, \ldots, m_n\} \text{ is } R\text{-live in } \mathcal{M}.$$

Details on how to obtain this formula can be found in the Appendix. For any *finite* language $\mathcal{L}$ we further obtain a formula $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_n)$ defining the set of $\mathcal{L}$-live n-tuples. If $\mathcal{L}$ contains at most k-ary relations and $n \leq k$, then $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_n) \in \mathrm{GF}_k^k$. Moreover, as the following proposition states, we may use the formula $\mathsf{Live}_{\mathcal{L}}$ as a guard while staying within the guarded fragment, modulo logical equivalence.

Proposition 4.3 *Let $\mathcal{L}$ be a language which contains at most k-ary relation symbols. Any first order formula φ in k variables which is uniformly guarded in $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k)$ is equivalent to some $\varphi' \in \mathrm{GF}_k^k$ such that $\mathcal{L}_{G(\varphi')} = \mathcal{L}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\varphi)}$.*

Proof: See Appendix. Ξ

Theorem 4.5 shows the existence of an interpolant for validities $\varphi \to \psi$ under the conditions that $free(\psi) \subseteq free(\varphi)$ and $\varphi, \neg\psi$ have the following property.

Definition 4.4 (Self-guarded formula) A formula $\varphi(v_1, \ldots, v_n)$ is called *self-guarded* if $\models \varphi \to \mathsf{Live}_{\mathcal{L}_{G(\varphi)}}(v_1, \ldots, v_n)$. $\dashv$

Recall that singletons are live sets. Therefore any formula in at most one free variable is self-guarded. In particular, sentences are self-guarded. Restricting attention to self-guarded formulas brings the GF syntax closer to the semantic intuition as captured by the guarded bisimulation. According to Definition 2.9 the domain of a partial isomorphism which is contained in a guarded bisimulation F need not be live. When we consider self-guarded formulas only, we can change the definition of a bisimulation such that for all $f \in F$, the domain of f is live, and the characterization theorem still goes through.

For this reason, the proof of the interpolation property is much simplified by only considering self-guarded formulas. Note that for decidability this matter does not show up, as a guarded formula $\varphi(\bar{x})$ is satisfiable iff the self-guarded formula $\exists \bar{x}(R\bar{x} \wedge \varphi(\bar{x}))$ is satisfiable, where R is some hitherto unused predicate symbol. Obviously, changing the language is no option when investigating interpolation.

THEOREM 4.5 (GF_k^k **has modal interpolation**) *Let $k \in \omega$. Let $\varphi, \psi \in \mathrm{GF}_k^k$ be such that $free(\psi) \subseteq free(\varphi)$ and $\varphi, \neg\psi$ are self-guarded. If $\models \varphi \to \psi$, then there exists an interpolant $\vartheta \in \mathrm{GF}_k^k$ such that*

1. $\models \varphi \to \vartheta$ *and* $\models \vartheta \to \psi$,
2. $\mathcal{L}_{G(\vartheta)} \subseteq \mathcal{L}_{G(\varphi)} \cup \mathcal{L}_{G(\psi)}$,
3. $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_\varphi \cap \mathcal{L}_\psi$, *and*
4. *ϑ is equivalent to a formula χ uniformly guarded in $\mathsf{Live}_{\mathcal{L}_{G(\varphi)} \cup \mathcal{L}_{G(\psi)}}(v_1, \ldots, v_k)$ with $\mathcal{L}_{\bar{G}(\chi)} \subseteq \mathcal{L}_{\bar{G}(\vartheta)}$.*

The condition that $free(\psi) \subseteq free(\varphi)$ exempts us from the obligation to give an interpolant for implications like $\models \varphi(u) \to \psi(u, v)$. Rightfully so, as this implication is equivalent to $\models \varphi(u) \to \forall v(\psi(u, v))$ which happens to lie outside the guarded context.

Proof of THEOREM 4.5: Let k, φ, ψ be as in the theorem.

By simply renaming the variables we may assume $free(\varphi) = \{v_1, \ldots, v_l\}$, for some $l \leq k$. Moreover, we assume $free(\psi) = free(\varphi)$. For, from the fact that $\models \varphi \to \psi$ we infer that $\models \underbrace{\exists \bar{v}(\mathsf{Live}_{\mathcal{L}_{G(\varphi)}}(v_1, \ldots, v_l) \wedge \varphi)}_{\varphi'} \to \psi$, where $\exists \bar{v}$ binds all variables in $free(\varphi) \setminus free(\psi)$. Then $free(\varphi') = free(\psi)$, and we obtain an interpolant ϑ for φ', ψ. This ϑ is also an interpolant for φ, ψ.

Since $\models \varphi \to \psi$, it follows from Theorem 2.11 that $\mathrm{LC}, Cv_1 \cdots v_k \models \varphi^C \to \psi^C$. Let $\vartheta \in MLR_k^C$ be the interpolant obtained by Lemma 4.2.

Let us write $\mathcal{L}$ for $\mathcal{L}_{G(\varphi)} \cup \mathcal{L}_{G(\psi)}$, and let $\vartheta^L = \vartheta[\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k)/Cv_1 \cdots v_k]$. (That is, we replace the predicate $Cv_1 \cdots v_k$ everywhere in ϑ by the formula $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k)$). Note that $free(\vartheta^L) \subseteq \{v_1, \ldots, v_l\}$.

By Proposition 4.3, ϑ^L has an equivalent $\vartheta' \in \mathrm{GF}_k^k$ such that $\mathcal{L}_{G(\vartheta')} = \mathcal{L}$ and $\mathcal{L}_{\bar{G}(\vartheta')} = \mathcal{L}_{\bar{G}(\vartheta^L)} = \mathcal{L}_\varphi \cap \mathcal{L}_\psi$. Hence if we show that $\models \varphi \to \vartheta^L$ and $\models \vartheta^L \to \psi$, then ϑ' turns out to be an interpolant for φ, ψ which satisfies the conditions in Theorem 4.5.

We first show that $\models \varphi \to \vartheta^L$. Hereto, consider an $\mathcal{L} \cup \mathcal{L}_\varphi$-model $\mathcal{M}$ and $\bar{m} \in M^l$ such that $\mathcal{M} \models \varphi[\bar{m}]$. Since φ is self-guarded, $\bar{m} \in I^{\mathcal{M}}(\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_l))$. Expand the model $\mathcal{M}$ to the model $(\mathcal{M}, C)$ by setting $I^{(\mathcal{M},C)}(C) = I^{\mathcal{M}}(\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k))$. Then

$$(\mathcal{M}, C) \models LC. \tag{5}$$

Extend the $\mathcal{L}$-live l-tuple $\bar{m}$ to an k-tuple $\bar{m}\bar{n}$ by repeating some of its elements. Note that $\bar{m}\bar{n} \in I^{\mathcal{M}}(\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k))$, whence

$$\bar{m}\bar{n} \in I^{(\mathcal{M},C)}(C). \tag{6}$$

By induction on the complexity of χ it can be shown that for all $\chi \in \mathrm{GF}_k^k$ such that $\mathcal{L}_\chi \subseteq \mathcal{L} \cup \mathcal{L}_\varphi$ and $\mathcal{L}_{G(\chi)} \subseteq \mathcal{L}$, and all $\bar{t} \in M^k$, $\mathcal{M} \models \chi[\bar{t}] \Leftrightarrow (\mathcal{M}, C) \models \chi^C[\bar{t}]$. In particular,

$$(\mathcal{M}, C) \models \varphi^C[\bar{m}\bar{n}]. \tag{7}$$

As ϑ is the MLR_k^C interpolant for $LC, Cv_1, \ldots, v_k \models \varphi^C \to \psi^C$, it follows from (5), (6) and (7) that $(\mathcal{M}, C) \models \vartheta[\bar{m}\bar{n}]$.

Recall that ϑ^L is obtained from ϑ by replacing the predicate $Cv_1 \cdots v_k$ by the formula $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k)$). As $I^{(\mathcal{M},C)}(C) = I^{(\mathcal{M},C)}(\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k))$, we see that $(\mathcal{M}, C) \models \vartheta^L[\bar{m}\bar{n}]$. Since $free(\theta^L)$ are among $\{v_1, \ldots, v_l\}$, then ϑ^L is already satisfied in $(\mathcal{M}, C)$ by the l-tuple $\bar{m}$. As C does not occur in ϑ^L, also $\mathcal{M} \models \vartheta^L[\bar{m}]$.

To show that $\models \vartheta^L \to \psi$, one reasons similarly using the fact that $\neg\psi$ is self-guarded.
Q.E.D. THEOREM 4.5.

Remark 4.6 Let us compare the interpolation theorem formulated in Theorem 4.5 with the interpolation property usually studied in modal logics. In the latter, the interpolant is built up from proposition letters that occur both in φ and ψ (as proposition letters). This is not completely similar to condition 3 in Theorem 4.5 according to which an interpolant ϑ for the validity $\models \varphi \to \psi$ is allowed to contain relation symbols from $(\mathcal{L}_\varphi \cap \mathcal{L}_\psi) \setminus (\mathcal{L}_{\bar{G}(\varphi)} \cap$

$\mathcal{L}_{\bar{G}(\psi)}$) at non-guard positions. That is, the non-guards from which the interpolant is built, do not necessarily occur both in φ and ψ *as non-guards*. This allowance is necessary, as is witnessed by the next example.

Let $\varphi = \exists v(Pv, Rv)$ and $\psi = \exists v(Rv, Pv)$. Obviously, $\models \varphi \to \psi$, and both φ and ψ are interpolants satisfying the conditions in Theorem 4.5. However, an interpolant $\vartheta \in \mathrm{GF}_1$ such that $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)} = \emptyset$ can not be found.

In order to obtain a truly modal interpolation property, we should adopt another truly modal feature, namely the syntactic distinction between modalities and proposition letters. Such a distinction between state and action predicates in the guarded fragment has been advocated by van Benthem, [Ben98]. Condition 3 in Theorem 4.5 can be strengthened to $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)}$ if the interpolant ϑ is allowed to contain one extra variable. Hereto, simply replace every non-guard occurrence of a relation $R\bar{v} \in (\mathcal{L}_\varphi \cap \mathcal{L}_\psi) \setminus (\mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)})$ in ϑ by $\exists v_{k+1}(R\bar{v}, \mathit{True})$. In the example above, this gives e.g, the formula $\vartheta = \exists v(Pv, \exists u(Rv, v = v))$ as an interpolant in GF_2 for φ, ψ. $\dashv$

A result similar to Theorem 4.5 holds for the packed fragment. Actually, one of the advantages of proving Theorem 4.5 via Lemma 4.2 is that it provides a uniform method which is also applicable to the packed fragment. We just need a packed formula which can play the role of $\mathsf{Live}_\mathcal{L}$. Hereto we define, for any finite language $\mathcal{L}$ and $n \in \omega$,

$$\mathsf{Pack}_\mathcal{L}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \bigwedge_{i,j \leq n} \mathsf{Live}_\mathcal{L}(v_i, v_j).$$

Then for all $\mathcal{L}$-models $\mathcal{M}$ and $m_1, \ldots, m_n \in M$:

$\mathcal{M} \models \mathsf{Pack}_\mathcal{L}(v_1, \ldots, v_n)[m_1, \ldots, m_n]$ iff the set $\{m_1, \ldots, m_n\}$ is $\mathcal{L}$-packed in $\mathcal{M}$.

Moreover, the packed equivalent of Proposition 4.3 holds.

Proposition 4.7 *Let $\mathcal{L}$ be a language which contains at most k-ary relation symbols. Any first order formula φ in k variables which is uniformly guarded in $\mathsf{Pack}_\mathcal{L}(v_1, \ldots, v_k)$ is equivalent to some $\varphi' \in \mathrm{PF}_k^k$ such that $\mathcal{L}_{G(\varphi')} = \mathcal{L}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\varphi)}$.*

Proof: See Appendix. Ξ

In view of the remarks concerning Theorem 4.5, we restrict ourselves in the interpolation theorem for PF to validities $\varphi \to \psi$ such that $free(\psi) \subseteq free(\varphi)$ and $\varphi, \neg\psi$ have the following property.

Definition 4.8 (Self-packed formula) A formula $\varphi(v_1, \ldots, v_n)$ is called *self-packed* if $\models \varphi \to \mathsf{Pack}_{\mathcal{L}_{G(\varphi)}}(v_1, \ldots, v_n)$. $\dashv$

THEOREM 4.9 (PF_k^k **has modal interpolation)** *Let $k \in \omega$. Let $\varphi, \psi \in \mathrm{PF}_k^k$ be such that $free(\psi) \subseteq free(\varphi)$ and $\varphi, \neg\psi$ are self-packed. If $\models \varphi \to \psi$, then there exists an interpolant $\vartheta \in \mathrm{PF}_k^k$ such that*

1. $\models \varphi \to \vartheta$ *and* $\models \vartheta \to \psi$,

2. $\mathcal{L}_{G(\vartheta)} \subseteq \mathcal{L}_{G(\varphi)} \cup \mathcal{L}_{G(\psi)}$,

3. $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\varphi} \cap \mathcal{L}_{\psi}$, *and*

4. ϑ *is equivalent to a formula* χ *uniformly guarded in* $\mathsf{Pack}_{\mathcal{L}_{G(\varphi)} \cup \mathcal{L}_{G(\psi)}}(v_1, \ldots, v_k)$ *with* $\mathcal{L}_{\bar{G}(\chi)} \subseteq \mathcal{L}_{\bar{G}(\vartheta)}$.

Proof: Similar to the proof of Theorem 4.5. This time, $\mathsf{Pack}_{\mathcal{L}}$ and Proposition 4.7 play the role of $\mathsf{Live}_{\mathcal{L}}$ and Proposition 4.3. Ξ

As observed in Remark 4.6, condition 3 in Theorem 4.9 may be strengthened to $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)}$, if we allow for one extra variable in the interpolant.

The rest of this section is devoted to the proof of the main lemma, i.e., Lemma 4.2. Hereto, we need the following preliminaries which will continue to play a role in future sections.

Definition 4.10 ($\mathcal{L}_1/\mathcal{L}_2$-Bisimulation) A *guarded $\mathcal{L}_1/\mathcal{L}_2$-bisimulation* between models $\mathcal{M}$ and $\mathcal{N}$ is defined as a non-empty set of finite partial $\mathcal{L}_2$-isomorphisms between $\mathcal{M}, \mathcal{N}$ with zig- and zag- condition (cf. Definition 2.9) stipulated for $\mathcal{L}_1$-live sets only. $\dashv$

In the above definition we did not specify any relationship between the languages $\mathcal{L}_1$ and $\mathcal{L}_2$. However, if we want to syntactically characterize the guarded formulas that are preserved under this type of bisimulation, we have to add the requirement that $\mathcal{L}_1 \subseteq \mathcal{L}_2$.

Proposition 4.11 *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and let $\varphi \in \mathrm{GF}$ be such that $\mathcal{L}_{G(\varphi)} \subseteq \mathcal{L}_1$ and $\mathcal{L}_{\bar{G}(\varphi)} \subseteq \mathcal{L}_2$. Then φ is preserved under guarded $\mathcal{L}_1/\mathcal{L}_2$-bisimulations.*

Proof: By a straightforward induction on the complexity of φ. Ξ

Notation 4.12 Let $\mathcal{F}$ be a fragment of first order logic, and $\mathcal{L}$ a language. For models $\mathcal{M}, \mathcal{N}$, and $\bar{m} \in M^k, \bar{n} \in N^k$, write $\mathcal{M}, \bar{m} \equiv^{\mathcal{F}}_{\mathcal{L}} \mathcal{N}, \bar{n}$ to denote that $\mathcal{M} \models \theta[\bar{m}]$ iff $\mathcal{N} \models \theta[\bar{n}]$, for all $\theta \in \mathcal{F}$ in the language $\mathcal{L}$. Similarly, $\mathcal{M}, \bar{m} \equiv^{\mathcal{F}}_{\mathcal{L}_1/\mathcal{L}_2} \mathcal{N}, \bar{n}$ denotes equivalence with respect to formulas ϑ such that $\mathcal{L}_{G(\vartheta)} \subseteq \mathcal{L}_1$ and $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_2$. $\dashv$

Recall that MLR_k^C formulas are uniformly guarded in the k–ary predicate C. Besides that, C only occurs at guarded position. Let ϕ be an MLR_k^C formula with non guard relations in $\mathcal{L}$. Then ϕ is preserved under $\{C\}/\mathcal{L} \cup \{C\}$ bisimulations.
As shown in the proof of the Characterization Theorem for GF in [ABN98, Theorem 4.2.2], the relation of guarded $\mathcal{L}$-equivalence between ω-saturated structures induces a guarded $\mathcal{L}$-bisimulation. Similarly, the following lemma can be shown.

Lemma 4.13 *Let $\mathcal{M}, \mathcal{N}$ be ω-saturated models. Let F be the set of partial maps f from $\mathcal{M}$ to $\mathcal{N}$ such that for any $\bar{x} \in dom(f)$, $\mathcal{M}, \bar{x} \equiv^{\mathrm{GF}_n}_{C/\mathcal{L}} \mathcal{N}, f(\bar{x})$ and $\bar{x} \in I^{\mathcal{M}}(C)$ iff $f(\bar{x}) \in I^{\mathcal{N}}(C)$. If F is non-empty, then F is a guarded $\{C\}/\mathcal{L} \cup \{C\}$-bisimulation between $\mathcal{M}$ and $\mathcal{N}$.*

Proof: Standard. Ξ

Proof of Lemma 4.2: We will show 'amalgamation via bisimulation' in the same spirit as e.g., the proof of interpolation for the basic modal logic K in [ABN98, Theorem 2.5]. Its main construction is a deviation of an amalgamation method as can be found e.g., in [Ném85].

Let k, φ, ψ be as in the lemma, and let $\bar{v} = v_1 \cdots v_k$. Set

$$\Theta \stackrel{\text{def}}{=} \{\vartheta \in MLR_k^C \ : \ \mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)} \text{ and } LC, C\bar{v} \models \varphi \to \vartheta\}.$$

Our aim is to show that

$$LC, C\bar{v}, \Theta \models \psi, \tag{8}$$

for then, by compactness, it follows that ψ is implied by $LC, C\bar{v}$ together with some finite conjunction ϑ of formulas in Θ. This ϑ then is an interpolant for φ, ψ satisfying conditions 1 and 2 of Lemma 4.2. Finally, let $\vartheta' = \forall \bar{u}(Cv_1 \cdots v_k \to \vartheta)$, where $\forall \bar{u}$ quantifies over all variables in $\{v_1, \ldots, v_k\} \setminus (free(\varphi) \cup free(\psi))$. Then ϑ' is an interpolant satisfying all the conditions in the theorem.

To prove (8), consider an arbitrary $\mathcal{L}_\psi$-model $\mathcal{N}$, and $b_1, \ldots, b_k \in N$ such that $\mathcal{N} \models \vartheta[b_1, \ldots, b_k]$, for every $\vartheta \in \Theta \cup LC \cup \{C\bar{v}\}$. Our task is to show that $\mathcal{N} \models \psi[b_1, \ldots, b_k]$.

We first note that there exists some $\mathcal{L}_\varphi$-model $\mathcal{M}$ and $a_1, \ldots, a_k \in M$ such that $\mathcal{M} \models LC \wedge \varphi \wedge C\bar{v}[a_1, \ldots, a_k]$, and $\mathcal{M}, a_1 \cdots a_k \equiv^{MLR_k^C}_{C/\mathcal{L}_{nG(\varphi)} \cap \mathcal{L}_{nG(\psi)}} \mathcal{N}, b_1 \cdots b_k$. For, reasoning by contraposition, assume such $\mathcal{M}$, $a_1, \ldots, a_k$ do not exist. In that case, $\Phi = \{\vartheta \in MLR_k^C : \mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)} \ \& \ \mathcal{N} \models \vartheta[b_1, \ldots, b_k]\} \cup LC \cup \{C\bar{v}\} \models \neg\varphi$. By compactness it follows that $LC, C\bar{v} \models \varphi \to \neg \bigwedge \Phi_0$, for some finite conjunction of formulas in Φ. Therefore $\neg \bigwedge \Phi_0 \in \Theta$, whence $\mathcal{N} \models \neg \bigwedge \Phi_0[b_1, \ldots, b_k]$. Quod non.

By passing to ω-saturated elementary extensions of $\mathcal{M}$ and $\mathcal{N}$, we may w.l.o.g. assume that $\mathcal{M}$, $\mathcal{N}$ are ω-saturated cf. [ABN98][Theorem 2.2.1].

By Lemma 4.13 there exists a guarded $C/(\mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)}) \cup \{C\}$-bisimulation F between $\mathcal{M}$, $\mathcal{N}$ which links $\langle a_1, \ldots, a_k \rangle$ and $\langle b_1, \ldots, b_k \rangle$. As $(\mathcal{L}_{\bar{G}(\varphi)} \cap \mathcal{L}_{\bar{G}(\psi)}) \cup \{C\} = \mathcal{L}_\varphi \cap \mathcal{L}_\psi$, F is a $C/\mathcal{L}_\varphi \cap \mathcal{L}_\psi$-bisimulation.

The aim of the rest of this proof is to amalgamate the models $\mathcal{M}$ and $\mathcal{N}$ in such a way that we can define guarded $C/\mathcal{L}_\varphi$ (resp. $C/\mathcal{L}_\psi$) -bisimulations between the amalgamated model and $\mathcal{M}$ (resp. $\mathcal{N}$) which, when composed, will map $\langle a_1, \ldots, a_k \rangle$ to $\langle b_1, \ldots, b_k \rangle$. Chasing the resulting diagram and using the fact that $LC, C\bar{v} \models \varphi \to \psi$ will yield the desired conclusion that $\mathcal{N} \models \psi[b_1, \ldots, b_k]$. This will be made precise in the sequel.

Define the model $\mathcal{MN}$ by setting

- $MN \stackrel{\text{def}}{=} \{\langle m, n \rangle \in M \times N \ : \ \mathcal{M}, m \ \sim_F \ \mathcal{N}, n\}$.
- For l-ary $R \in \mathcal{L}_\varphi$, set $\langle \langle m_1, n_1 \rangle, \ldots, \langle m_l, n_l \rangle \rangle \in I^{\mathcal{MN}}(R)$ iff
  - $\mathcal{M}, m_1 \cdots m_l \ \sim_F \ \mathcal{N}, n_1 \cdots n_l$,
    (i.e., the m_i and n_i are not only pairwise bisimilar but jointly so), and

– $\langle m_1, \ldots, m_l \rangle \in I^{\mathcal{M}}(R)$.

- The interpretation of relations in $\mathcal{L}_\psi$ is defined similarly.

Note that the universe MN is non-empty, as F is non-empty. Moreover, the interpretation of relations in the common language is well-defined thanks to the requirement on live subsets of $\mathcal{MN}$ to be jointly bisimilar. The upshot of amalgamating our models into a product is that we can take projection functions as building blocks for the desired bisimulations. This is the purport of the following Amalgamation Lemma. Let π_i, $i = 1, 2$, denote the projection function to the i-th coordinate, and define

$$F_{\pi_i} \stackrel{\text{def}}{=} \{\pi_i : X \longrightarrow Y \ : \ X \subseteq^l_C \mathcal{MN}\}.$$

Amalgamation Lemma *The set F_{π_1} is a guarded $C/\mathcal{L}_\varphi$-bisimulation between $\mathcal{MN}$ and $\mathcal{M}$. F_{π_2} is a guarded $C/\mathcal{L}_\psi$-bisimulation between $\mathcal{MN}$ and $\mathcal{N}$.*

Before proving the Amalgamation Lemma, let us first demonstrate its use and finish the proof of Lemma 4.2. Recall that $\mathcal{M}$ and $a_1, \ldots, a_k \in M$ were chosen in such a way that $\mathcal{M} \models \varphi[a_1, \ldots, a_k]$. Note that $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\} \in I^{\mathcal{MN}}(C)$, whence there exists some $f \in F_{\pi_1}$ with domain $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$. By Proposition 4.11, φ is preserved under guarded $C/\mathcal{L}_\varphi$-bisimulations. Therefore, by the amalgamation lemma, $\mathcal{MN} \models \varphi[\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle]$. Note that $\mathcal{MN} \models LC$. By assumption $LC, Cv_1, \ldots, v_k \models \varphi \rightarrow \psi$, whence $\mathcal{MN} \models \psi[\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle]$. As $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ is also the domain of some projection in F_{π_2}, the second part of the amalgamation lemma allows us to conclude that $\mathcal{N} \models \psi[b_1, \ldots, b_k]$, as desired.

Now we turn to the proof of the amalgamation lemma. We will prove the first part of the lemma concerning F_{π_1}. The second statement about F_{π_2} can be shown similarly.

We already observed that F_{π_1} is non-empty. Let $\pi_1 \in F_{\pi_1}$. The domain of π_1 is a set of pairs $X = \{x_1, \ldots, x_l\}$, for some $l \leq k$, such that $\mathcal{M}, \pi_1(x_1) \cdots \pi_1(x_l) \sim_F \mathcal{N}, \pi_2(x_1) \cdots \pi_2(x_l)$. By construction this implies that for any m-ary $R \in \mathcal{L}_\varphi$, and any $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in X^m$ it is the case that $\langle x_{i_1}, \ldots, x_{i_m}\rangle \in I^{\mathcal{MN}}(R)$ iff $\langle \pi_1(x_{i_1}), \ldots, \pi_1(x_{i_m})\rangle \in I^{\mathcal{M}}(R)$. In other words, π_1 preserves $\mathcal{L}_\varphi$-relations in both ways. Moreover, π_1 is 1-1, as F consists of 1-1 maps. We conclude that π_1 is a partial $\mathcal{L}_\varphi$-isomorphism.

The zig-condition is trivially fulfilled. For the zag-condition, consider some $\pi_1 \in F_{\pi_1}$, and $W \subseteq^l_C \mathcal{M}$. As we noted before, the domain of π_1 is a set of pairs $X = \{x_1, \ldots, x_l\}$, for some $l \leq k$, such that $\mathcal{M}, \pi_1(x_1) \cdots \pi_1(x_l) \sim_F \mathcal{N}, \pi_2(x_1) \cdots \pi_2(x_l)$. By the zig-condition for F there exists some $g \in F$ with domain W such that $g(\pi_1(x)) = \pi_2(x)$, for $\pi_1(x) \in \pi_1[X] \cap W$. Let $W^* = \{\langle w, g(w)\rangle : w \in W\}$. Then $W^* \subseteq^l_C \mathcal{MN}$ and W^* is the desired pre-image for W.
Q.E.D. Lemma 4.2.

5 An Excursion into Definability

In general, an important reason to investigate the interpolation property is that it can be seen as an intermediate stage in proving the Beth definability theorem. It will be shown that the limited form of interpolation expressed in Theorem 4.5 and Theorem 4.9 still serves this purpose for GF^k_k and PF^k_k.

Let $\mathcal{L}_0$ be a language and R and R' distinct relation symbols of the same arity that are not in $\mathcal{L}_0$. Let $\mathcal{L} = \mathcal{L}_0 \cup \{R\}$. Let Σ be a set of guarded sentences in the language $\mathcal{L}$, and let Σ' denote the result of renaming R to R' in Σ.

THEOREM 5.1 (Beth Theorem for GF^k_k) *Let $\mathcal{L}_0$, $\mathcal{L}$, R, R', Σ and Σ' be as above. Let $k \in \omega$ be such that $\Sigma \cup \{R\bar{v}\} \subseteq \mathrm{GF}^k_k$. If Σ implicitly defines R, i.e., if*

$$\Sigma, \Sigma' \models \forall \bar{v}(R\bar{v} \leftrightarrow R'\bar{v}),$$

then there exists some $\varphi(\bar{v}) \in \mathrm{GF}^k_k$ in the language $\mathcal{L}_0$ such that

$$\Sigma \models \forall \bar{v}(R\bar{v} \leftrightarrow \varphi(\bar{v})).$$

This formula φ is called an explicit definition *for R relative to Σ. Moreover, φ can be chosen such that $\mathcal{L}_{G(\varphi)} \subseteq \mathcal{L}_{G(\Sigma)} \setminus R$.*

The proof of Theorem 5.1 uses the following lemma, due to Martin Otto.

Lemma 5.2 *Let Σ be a set of guarded sentences that implicitly define R. Let $\mathcal{M}$ be a model of Σ, and let $Y \subseteq^l_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. Then $Y \subseteq^l_{(\mathcal{L}_{G(\Sigma)} \setminus R)} \mathcal{M}$.*

Proof of Lemma 5.2: Let $\mathcal{M}$ be a model of Σ, and let $Y_0 \subseteq^l_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. Reasoning by contraposition, suppose $Y_0 \not\subseteq^l_{\mathcal{L}_{G(\Sigma)} \setminus R} \mathcal{M}$. We will derive a contradiction from this.

Let $\mathcal{M} \times \underline{2}$ denote the model with domain $M \times \{0,1\}$ and the interpretation of the relations defined as follows. For every l-ary $P \in \mathcal{L}$ and every $\bar{s} = \langle s_1, \ldots, s_l \rangle \in \{0,1\}^l$, $\bar{s} \in I^{\mathcal{M} \times \underline{2}}(P)$ iff

1. $\langle \pi_1(s_1), \ldots, \pi_1(s_l) \rangle \in I^{\mathcal{M}}(P)$, and
2. for all $m, m' \in M$, if $\langle m, 0 \rangle = s_i$ and $\langle m', 1 \rangle = s_j$, for some $i, j \leq l$, then $m \neq m'$.

As the reader can easily verify, this implies that $F_1 \stackrel{\text{def}}{=} \{\pi_1 : X \longrightarrow Y \ : \ X \subseteq^l_{\mathcal{L}_{G(\Sigma)}} \mathcal{M} \times \underline{2}\}$ is a guarded $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimulation between $\mathcal{M} \times \underline{2}$ and $\mathcal{M}$. Condition 2 guarantees that every projection in F_1 is 1-1. Since $\mathcal{M} \models \Sigma$, we conclude that $\mathcal{M} \times \underline{2} \models \Sigma$.

Our aim is to modify the interpretation of R on $\mathcal{M} \times \underline{2}$ in such a way that the resulting structure is again a model for Σ, contradicting the fact that Σ implicitly defines R. For this, we consider $X_0 \stackrel{\text{def}}{=} Y_0 \times \{0\}$. Let $(\mathcal{M} \times \underline{2})'$ be the model which differs from $\mathcal{M} \times \underline{2}$ only in that $X_0 \not\subseteq^l_R (\mathcal{M} \times \underline{2})'$. Note that $X_0 \subseteq^l_R \mathcal{M} \times \underline{2}$, whence the two models do really differ. We claim that $F_1' \stackrel{\text{def}}{=} \{\pi_1 : X \longrightarrow Y \ : \ X \subseteq^l_{\mathcal{L}_{G(\Sigma)}} (\mathcal{M} \times \underline{2})'\}$ is a guarded $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimulation between $(\mathcal{M} \times \underline{2})'$ and $\mathcal{M}$.

F_1' is certainly not empty. Consider some $\pi_1 : X \longrightarrow Y$ in F_1'. If $X_0 \not\subseteq X$, then $\mathcal{L}_\Sigma$-relations are obviously preserved by π_1 in both ways. Let us see that it is the case that $X_0 \not\subseteq X$. For, we changed the interpretation of R such that X_0 is not R-live in $(\mathcal{M} \times \underline{2})'$.

As Y_0 is not $(\mathcal{L}_{G(\Sigma)} \setminus R)$-live, it follows that X_0 is not $(\mathcal{L}_{G(\Sigma)} \setminus R)$-live in $(\mathcal{M}\times\underline{2})'$ either. Whence X_0 is not $\mathcal{L}_{G(\Sigma)}$-live in $(\mathcal{M}\times\underline{2})'$. We conclude that no superset of X_0 is the domain of some $\pi_1 \in F_1'$.

The zig-condition needs no comment. For the zag-condition, consider some projection $\pi_1 : X \longrightarrow Y$ in F_1', and $W \subseteq^l_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. If $W \subseteq Y$, the condition is trivially fulfilled. If not, consider $Z \stackrel{\text{def}}{=} \pi_1^{-1}[Y \cap W] \cup \{\langle m, 1\rangle : m \in W \setminus Y\}$. Note that $Z \subseteq^l_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}\times\underline{2}$. As $W \setminus Y \neq \emptyset$ it follows that $\pi_2[Z] \neq \{0\}$. In particular, $Z \neq X_0$ whence $Z \subseteq^l_{\mathcal{L}_{G(\Sigma)}} (\mathcal{M}\times\underline{2})'$. We conclude that Z fulfills the zag-condition for π_1, W.

This shows that $\mathcal{M}$ and $(\mathcal{M}\times\underline{2})'$ are $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimilar. As $\mathcal{M} \models \Sigma$, also $(\mathcal{M}\times\underline{2})' \models \Sigma$. Summarizing, we see that $\mathcal{M}\times\underline{2} \models \Sigma$, $(\mathcal{M}\times\underline{2})' \models \Sigma$, $I^{\mathcal{M}\times\underline{2}}(P) = I^{(\mathcal{M}\times\underline{2})'}(P)$, for every $P \in \mathcal{L}_0$ but $I^{\mathcal{M}\times\underline{2}}(R) \neq I^{(\mathcal{M}\times\underline{2})'}(R)$. This contradicts the fact that Σ implicitly defines R. We conclude that Y_0 is indeed $(\mathcal{L}_{G(\Sigma)} \setminus R)$-live, as was to be shown.
Q.E.D. Lemma 5.2.

Proof of THEOREM 5.1: Let all data be as in the theorem, and assume that Σ implicitly defines R. By compactness we may assume Σ to be a single sentence, and we obtain that

$$\models \underbrace{(\Sigma \wedge R\bar{v})}_{\varphi} \to \underbrace{((\Sigma' \wedge \mathsf{Live}_{\mathcal{L}_{G(\Sigma')}}(\bar{v})) \to R'\bar{v})}_{\psi}. \tag{9}$$

We may assume $R \in \mathcal{L}_{G(\Sigma)}$ (else, add $\forall\bar{v}(R\bar{v} \to R\bar{v})$ to Σ), whence $\varphi, \neg\psi$ are self-guarded. Applying Theorem 4.5 to (9) we obtain an interpolant $\vartheta \in \mathrm{GF}^k_k$ uniformly guarded in $\mathsf{Live}_{\mathcal{L}_{G(\Sigma)} \cup \mathcal{L}_{G(\Sigma')}}$ with $\mathcal{L}_{\bar{G}(\vartheta)} \subseteq \mathcal{L}_0$ such that

$$\models (\Sigma \wedge R\bar{v}) \to \vartheta \text{ and } \models \vartheta \to ((\Sigma' \wedge \mathsf{Live}_{\mathcal{L}_{G(\Sigma')}}(\bar{v})) \to R'\bar{v}). \tag{10}$$

Substituting R for R' in (10) gives

$$\Sigma \models R\bar{v} \leftrightarrow (\vartheta[R/R'] \wedge \mathsf{Live}_{\mathcal{L}_{G(\Sigma)}}(\bar{v})). \tag{11}$$

Note that $\vartheta[R/R']$ is uniformly guarded in $\mathsf{Live}_{\mathcal{L}_{G(\Sigma)}}$. Let ϑ_0 be obtained from $\vartheta[R/R'] \wedge \mathsf{Live}_{\mathcal{L}_{G(\Sigma)}}(\bar{v})$ by replacing $\mathsf{Live}_{\mathcal{L}_{G(\Sigma)}}$ by $\mathsf{Live}_{(\mathcal{L}_{G(\Sigma)} \setminus R)}$. By Lemma 5.2, $\Sigma \models \mathsf{Live}_{\mathcal{L}_{G(\Sigma)}} \leftrightarrow \mathsf{Live}_{(\mathcal{L}_{G(\Sigma)} \setminus R)}$. Whence by (11), $\Sigma \models \forall\bar{v}(R\bar{v} \leftrightarrow \vartheta_0)$. By Proposition 4.3, ϑ_0 is equivalent to some $\vartheta_0' \in \mathrm{GF}^k_k$ uniformly guarded in $\mathcal{L}_{G(\Sigma)} \setminus R$ with the same language for non-guard occurrences, i.e., $\mathcal{L}_{\bar{G}(\vartheta_0')} = \mathcal{L}_{\bar{G}(\vartheta_0)} \subseteq \mathcal{L}_0$. This ϑ_0' provides the desired explicit definition of R relative to Σ.
Q.E.D. THEOREM 5.1.

THEOREM 5.3 (Beth Theorem for PF^k_k) *Let $\mathcal{L}_0$, $\mathcal{L}$, R, R', Σ, Σ' and k be as in Theorem 5.1 with this distinction that Σ and Σ' are sets of* packed *sentences. If Σ* implicitly defines *R, then there exists some $\varphi \in \mathrm{PF}^k_k$ in the language $\mathcal{L}_0$ which is an* explicit definition *for R, relative to Σ. Moreover, φ can be chosen such that $\mathcal{L}_{G(\varphi)} \subseteq \mathcal{L}_{G(\Sigma)} \setminus R$.*

This theorem is proven via the following version of Lemma 5.2 for the packed fragment.

Lemma 5.4 *Let Σ be a set of packed sentences that implicitly define R. Let $\mathcal{M}$ be a model of Σ, and let $Y \subseteq^p_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. Then $Y \subseteq^p_{(\mathcal{L}_{G(\Sigma)} \setminus R)} \mathcal{M}$.*

Proof of Lemma 5.4: We follow the proof of Lemma 5.2, making adaptations where necessary.

Let $\mathcal{M}$ be a model of Σ, and let $Y_0 \subseteq^p_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. Suppose $Y_0 \not\subseteq^p_{\mathcal{L}_{G(\Sigma)} \setminus R} \mathcal{M}$. This implies the existence of $y, y' \in Y_0$ such that $\{y, y'\} \subseteq^l_R \mathcal{M}$ but $\{y, y'\} \not\subseteq^l_{\mathcal{L}_{G(\Sigma)} \setminus R} \mathcal{M}$.

Let $\mathcal{M} \times \underline{2}$ be the model defined in the proof of Lemma 5.2. It is not hard to see that $\{\pi_1 : X \longrightarrow Y \ : \ X \subseteq^p_{\mathcal{L}_{G(\Sigma)}} \mathcal{M} \times \underline{2}\}$ is a packed $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimulation between $\mathcal{M} \times \underline{2}$ and $\mathcal{M}$. As $\mathcal{M} \models \Sigma$, also $\mathcal{M} \times \underline{2} \models \Sigma$.

Let $X_0 \stackrel{\text{def}}{=} \{\langle y, 0\rangle, \langle y', 0\rangle\}$. Then $X_0 \subseteq^l_R \mathcal{M} \times \underline{2}$. Let $(\mathcal{M} \times \underline{2})'$ be the model which differs from $\mathcal{M} \times \underline{2}$ only in that $X_0 \not\subseteq^l_R (\mathcal{M} \times \underline{2})'$. Similar to the proof of Lemma 5.2 it can be verified that $F_1' \stackrel{\text{def}}{=} \{\pi_1 : X \longrightarrow Y \ : \ X \subseteq^p_{\mathcal{L}_{G(\Sigma)}} (\mathcal{M} \times \underline{2})'\}$ is a packed $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimulation between $(\mathcal{M} \times \underline{2})'$ and $\mathcal{M}$. We only check the zag-condition. Let $\pi_1 : X \longrightarrow Y$ in F_1', and $W \subseteq^p_{\mathcal{L}_{G(\Sigma)}} \mathcal{M}$. Define $Z \stackrel{\text{def}}{=} \pi_1^{-1}[Y \cap W] \cup \{\langle m, 1\rangle : m \in W \setminus Y\}$. Note that $Z \subseteq^p_{\mathcal{L}_{G(\Sigma)}} \mathcal{M} \times \underline{2}$. As X is $\mathcal{L}_{G(\Sigma)}$-packed, it follows that $\{\langle y, 0\rangle, \langle y', 0\rangle\} \not\subseteq X$. Therefore, $\{\langle y, 0\rangle, \langle y', 0\rangle\} \not\subseteq Z$. This implies that $Z \subseteq^p_{\mathcal{L}_{G(\Sigma)}} (\mathcal{M} \times \underline{2})'$, and Z fulfills the zag-condition for π_1, W. This shows that $\mathcal{M}$ and $(\mathcal{M} \times \underline{2})'$ are packed $\mathcal{L}_{G(\Sigma)}/\mathcal{L}_\Sigma$-bisimilar. Reasoning as in the proof of Theorem 5.1, this leads to the desired contradiction.
Q.E.D. Lemma 5.4.

Proof of THEOREM 5.3: Similar to the proof of Theorem 5.1, using Theorem 4.9, Lemma 5.4 and Proposition 4.7. Ξ

6 Guarded Finite Variable Fragments with Strong Interpolation

Let us return to the examples in Section 3 which showed that weak interpolation fails in GF^k, GF_n and PF_n if $n, k \geq 3$ and in PF^k if $k \geq 2$. Recall that k denotes the restriction on the arity of the relation symbols and n denotes the restricted number of variables. In this section we establish the reverse of the earlier implication. That is, it will be shown that GF^k_n has strong interpolation if $k \leq 2$ or $n \leq 2$, and PF^k_n has strong interpolation if $k \leq 1$ or $n \leq 2$. All these results follow from the fact that GF^k_2 has strong interpolation. In particular, the result for PF^k_2 follows since the expressive power of GF_2 and PF_2 coincide, see Proposition 6.5. The difference in the behavior of GF^2 and PF^2 lies in the fact that the size of R-live sets, with R binary, is at most 2 whereas R-packed sets may have arbitrary size.

The strong interpolation property has been introduced in Definition 3.1.

THEOREM 6.1 *For any $k, n \in \omega$,*

1. GF^k_2 *has the strong interpolation property.*

2. GF_n^2 *has the strong interpolation property for self-guarded formulas.*

Remark 6.2 This theorem reveals an interesting asymmetry between GF_2 and the full 2-variable fragment of first order logic. For, not only does the latter fail to have interpolation, it even fails in a very strong sense. For any $n \in \omega$, there exist first order formulas φ_n, ψ_n in just two variables and binary relation symbols such that $\varphi_n \to \psi_n$ is valid, but which lack a first order interpolant in n variables. This result is due to Andréka (unpublished). For a stronger result see [MA98]. $\dashv$

Before we turn to the proof of Theorem 6.1, we need some preliminaries. First we observe that the minimum number of variables in a self-guarded formula is determined by the arity of the language.

Proposition 6.3 *Let $k, n \in \omega$, and let φ be a self-guarded formula in GF_n^k. Then φ is equivalent to some $\varphi' \in \mathrm{GF}_k^k$ in the same language.*

Proof: Note that $|free(\varphi)| \leq k$, as all free variables in φ are guarded by some relation in $\mathcal{L}_{G(\varphi)}$ which is at most k-ary. W.l.o.g., assume $free(\varphi) \subseteq \{v_1, \ldots, v_k\}$. Suppose $v_{k+1} \in var(\varphi)$. Then φ has a subformula ϑ of the form $Qv_{k+1}(R\bar{v}, \psi(\bar{v}))$, where $Q \in \{\forall, \exists\}$. If $v_{k+1} \notin \bar{v}$, the quantification is vacuous and can be removed. If $v_{k+1} \in \bar{v}$, then v_i does not appear in $\bar{v}$, for some $i \in 1, \ldots, k$, as R is at most k-ary. In this case replace everywhere in ϑ the variable v_{k+1} by v_i. In both cases we obtain an equivalent of ϑ in at most the variables $v_1, \ldots, v_k$. Repeating this procedure eventually yields the desired equivalent of φ. Ξ

Using the terminology from [ABN98], by an $\mathcal{L}$-*guarded* set we understand a set whose elements are R-related, in any order or multiplicity, by some $R \in \mathcal{L}$. So for any $n \in \omega$, a guarded set of size n is not only live, but moreover this fact can be expressed using n variables. We define an n-*guarded* $\mathcal{L}$-*bisimulation* as a guarded $\mathcal{L}$-bisimulation (cf. Definition 2.9) where the zigzag conditions are restricted to $\mathcal{L}$-*guarded* sets *of size at most* n. The relation $\equiv_{\mathcal{L}}^{\mathrm{GF}_n}$ has been introduced in Notation 4.12.

As before, we have the following results which are essential in the proof of the interpolation theorem. Here n denotes an arbitrary natural number.

1. Between ω-saturated models, the relation $\equiv_{\mathcal{L}}^{\mathrm{GF}_n}$ between n-tuples induces an n-guarded $\mathcal{L}$-bisimulation.

2. Every $\varphi \in \mathrm{GF}_n$ is preserved under n-guarded $\mathcal{L}_\varphi$-bisimulations.

Proof of THEOREM 6.1: The proof concentrates on the first statement. The second statement follows from the first using Proposition 6.3.

Let $k \in \omega$. Let $\varphi, \psi \in \mathrm{GF}_2^k$ be such that $\models \varphi \to \psi$ is valid. Write $\mathcal{L}_{\varphi\psi}$ for $\mathcal{L}_\varphi \cap \mathcal{L}_\psi$, and set $\Theta \stackrel{\text{def}}{=} \{\vartheta \in \mathrm{GF}_2^k : \mathcal{L}_\vartheta \subseteq \mathcal{L}_{\varphi\psi} \ \& \ \models \varphi \to \vartheta\}$. As in the proof of Theorem 4.5, our aim is to show that $\Theta \models \psi$. Consider an arbitrary $\mathcal{L}_\psi$-model $\mathcal{N}$, and $b_1, b_2 \in N$ such that $\mathcal{N} \models \vartheta[b_1, b_2]$, for every $\vartheta \in \Theta$. Our task is to show that $\mathcal{N} \models \psi[b_1, b_2]$.

As before we first note the existence of some $\mathcal{L}_\varphi$-model $\mathcal{M}$ and $a_1, a_2 \in M$ such that $\mathcal{M} \models \varphi[a_1, a_2]$ and $\mathcal{M}, a_1 a_2 \equiv^{\mathrm{GF}_2}_{\mathcal{L}_{\varphi\psi}} \mathcal{N}, b_1 b_2$. W.l.o.g., we assume that $\mathcal{M}, \mathcal{N}$ are ω-saturated.

By 1. above, there exists a 2-guarded $\mathcal{L}_{\varphi\psi}$-bisimulation F between $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}, a_1 a_2 \sim_F \mathcal{N}, b_1 b_2$. Define the $\mathcal{L}_\varphi \cup \mathcal{L}_\psi$-model $\mathcal{MN}$ as follows:

- $MN \stackrel{\mathrm{def}}{=} \{\langle m, n\rangle \in M \times N \ : \ \mathcal{M}, m \sim_F \mathcal{N}, n\}$.
- For binary $R \in \mathcal{L}_{\varphi\psi}$, $\langle\langle m_1, n_1\rangle, \langle m_2, n_2\rangle\rangle \in I^{\mathcal{MN}}(R)$ iff
  - $\mathcal{M}, m_1 m_2 \sim_F \mathcal{N}, n_1 n_2$, and
  - $\langle m_1, m_2\rangle \in I^{\mathcal{M}}(R)$.
- For binary $R \in \mathcal{L}_\varphi \backslash \mathcal{L}_\psi$, $\langle\langle m_1, n_1\rangle, \langle m_2, n_2\rangle\rangle \in I^{\mathcal{MN}}(R)$ iff
  - Either $\mathcal{M}, m_1 m_2 \sim_F \mathcal{N}, n_1 n_2$, or $\{m_1, m_2\}$ is not $\mathcal{L}_{\varphi\psi}$-live, and
  - $\langle m_1, m_2\rangle \in I^{\mathcal{M}}(R)$.
- Similarly, we define the interpretation of binary relations in $\mathcal{L}_\psi \backslash \mathcal{L}_\varphi$. Also the interpretation of unary relations is defined in the same way. For relations of arbitrary arity we stipulate as an important side condition that we will not allow for live sets in $\mathcal{MN}$ with more than 2 elements. For example, for a ternary relation R, we set $\langle\langle m_1, n_1\rangle, \ldots, \langle m_3, n_3\rangle\rangle \in I^{\mathcal{MN}}(R)$ iff $\mid \{\langle m_1, n_1\rangle, \ldots, \langle m_3, n_3\rangle\} \mid \leq 2$ and the above conditions, modified in the obvious way, are fulfilled.

Note that the universe MN is non-empty, as F is non-empty. Also, the interpretation of relations in the common language is well-defined since any $f \in F$ preserves relations in the common language.

Let π_i denote the projection function to the i-th coordinate, and define $F_{\pi_1} = \{\pi_1 : X \longrightarrow Y \ : \ X \subseteq^l_{\mathcal{L}_\varphi} \mathcal{MN}$ or $X = \{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}\}$. F_{π_2} is defined similarly. As before, the key-lemma is the following.

Amalgamation Lemma *F_{π_1} is a 2-guarded $\mathcal{L}_\varphi$-bisimulation from $\mathcal{MN}$ to $\mathcal{M}$. F_{π_2} is a 2-guarded $\mathcal{L}_\psi$-bisimulation from $\mathcal{MN}$ to $\mathcal{N}$.*

Reasoning as in the proof of Lemma 4.2, the theorem follows from the Amalgamation Lemma via result 2. above. We will prove the first part of the Amalgamation Lemma. The second statement follows analogously.

F_{π_1} is obviously non-empty. By definition of the interpretation of relations in $\mathcal{MN}$, every $f_{\pi_1} \in F_{\pi_1}$ preserves $\mathcal{L}_\varphi$-relations in both ways. Let f_{π_1} be a projection in F_{π_1}. By construction, the domain of f_{π_1} is a set of pairs $\{\langle m_1, n_1\rangle, \langle m_2, n_2\rangle\}$ such that either $\mathcal{M}, m_1 m_2 \sim_F \mathcal{N}, n_1 n_2$, or $\{m_1, m_2\}$ is not $\mathcal{L}_{\varphi\psi}$-live. In the first case, f_{π_1} is 1-1 since F consists of 1-1 maps. In the latter case, f_{π_1} is 1-1 since the size of the domain of X is at most 2, and $\{m_1, m_2\}$ is not live, whence in particular not a singleton. We conclude that f_{π_1} is a partial $\mathcal{L}_\varphi$-isomorphism. The zig-condition is trivially fulfilled. To show the zag-condition, consider some $f_{\pi_1} : X \longrightarrow Y$ in F_{π_1} and $W \stackrel{\mathrm{def}}{=} \{m_1, m_2\} \subseteq^l_R \mathcal{M}$, for some

$R \in \mathcal{L}_\varphi$. We distinguish three cases.
(a). $|W \cap Y| = 2$. Then $W = Y$, and there is nothing to be shown.
(b). $|W \cap Y| = 1$. Say, $W \cap Y = \{m_1\}$. We distinguish as to whether or not W is $\mathcal{L}_{\varphi\psi}$-live. If not, consider some $\langle m_2, n\rangle \in MN$. Note that such pair exists, as singletons are live. Then by construction $\{f_{\pi_1}^{-1}(m_1), \langle m_2, n\rangle\}$ is an R-live subset of $\mathcal{MN}$, and the projection from this set fulfills the zag-condition for f_{π_1}, W.

Thus assume W is $\mathcal{L}_{\varphi\psi}$-live. Then $\mathcal{M}, m_1 \sim_F \mathcal{N}, \pi_2(f_{\pi_1}^{-1}(m_1))$ and by the zig-condition for F there exists some $n_2 \in N$ such that $\mathcal{M}, m_1 m_2 \sim_F \mathcal{N}, \pi_2(f_{\pi_1}^{-1}(m_1))n_2$. By construction, $\{f_{\pi_1}^{-1}(m_1), \langle m_2, n_2\rangle\}$ is R-live in $\mathcal{MN}$. Therefore, the projection from this set is in F_{π_1} and fulfills the zag-condition for f_{π_1}, W.
(c). $W \cap Y = \emptyset$. Reason as in case (b).
Q.E.D. THEOREM 6.1.

We mention that the restriction in Theorem 6.1.2 to self-guarded formulas can be removed. That is, we claim that the strong interpolation property holds for GF_n^2. The interested reader is invited to supply a proof along the lines of the proof of Theorem 6.1.1.

Corollary 6.4 *Let $n \in \omega$. The following fragments have the strong interpolation property:*

1. PF_2.
2. GF_n^1 *and* PF_n^1, *for self-guarded formulas.*
3. GF_1 *and* PF_1.

This corollary will be proven via the following proposition.

Proposition 6.5 *Let $n \in \omega$, $i \in \{1, 2\}$. Then*

1. $(\forall \varphi \in \mathrm{PF}_i)(\exists \varphi' \in \mathrm{GF}_i) \models \varphi \leftrightarrow \varphi'$, *and* $\mathcal{L}_\varphi = \mathcal{L}_{\varphi'}$.
2. $(\forall \varphi \in \mathrm{PF}_n^1)(\exists \varphi' \in \mathrm{GF}_n^1) \models \varphi \leftrightarrow \varphi'$, *and* $\mathcal{L}_\varphi = \mathcal{L}_{\varphi'}$.

Proof of Proposition 6.5: Both claims are proven by induction on the complexity of φ. **(1)**. The only non-trivial case is for φ of the form $\exists \bar{v}(\phi, \psi)$. We observe that all (i.e., at most 2) variables in $free(\psi)$ occur free in some conjunct ϕ_0 of ϕ. If ϕ_0 is an atom, then set $\varphi' = \exists \bar{v}(\phi_0, \phi \wedge \psi)$. Else, ϕ_0 is of the form $\exists \bar{u}\chi$, with χ an atom and we let $\varphi' = \exists \bar{v}\bar{u}(\chi, \phi \wedge \psi)$. As $\psi \in \mathrm{GF}_i$ by induction hypothesis, in both cases $\varphi' \in \mathrm{GF}_i$. **(2)**. Again we only consider the case for φ of the form $\exists \bar{v}(\phi, \psi)$. By induction hypothesis, $\psi \in \mathrm{GF}_n^1$. Note that every two variables $u, v \in free(\psi)$ are guarded in ϕ by $u = v$. If $free(\psi) = \emptyset$, φ is equivalent to $\phi \wedge \psi \in \mathrm{GF}_n^1$. Else, fix some variable $v \in free(\psi)$ and replace in $\phi \wedge \psi$ all occurrences (free or bound) of variables from $free(\psi)$ by v. Let $(\phi \wedge \psi)'$ denote the result. Then $\exists v(v = v, (\phi \wedge \psi)')$ is an equivalent of φ in GF_n^1.
Q.E.D. Proposition 6.5.

Proof of Corollary 6.4: (1). By Theorem 6.1.1 and Proposition 6.5.1. **(2)**. Strong interpolation for GF_n^1 for self-guarded formulas follows directly from Theorem 6.1. Together with Proposition 6.5.2, this implies the result for PF_n^1. **(3)**. Recall that in GF_1 *all* quantifications are admissible as they may be regarded as being guarded by an identity.

In other words, every first order formula in one variable is equivalent to a formula in GF_1 in the same language. Therefore, interpolation for GF_1 is equivalent to interpolation for the one variable fragment of first order logic. This latter result is well-known, cf. [Pig72]. Interpolation for PF_1 follows via Proposition 6.5.1.
Q.E.D. Corollary 6.4.

7 Appendix

In Section 4 we claimed, for any relation R and $l \in \omega$, the existence of a guarded formula $\mathsf{Live}_R(v_1, \ldots, v_l)$ that is satisfied in a model $\mathcal{M}$ by $m_1, \ldots, m_l \in M$ if and only if the set $\{m_1, \ldots, m_l\}$ is R-live in $\mathcal{M}$. Let us see what this formula looks like.

Let s be the arity of R. Let e range over all complete equality types in variables $v_1, \ldots, v_l$. We regard e both as a quantifier-free formula $e(v_1, \ldots, v_l)$ in the empty vocabulary and as an equivalence relation on the set $\{1, \ldots, l\}$ according to $(j, i) \in e$ iff $e \models v_j = v_i$. Let $\rho\colon \{1, \ldots, s\} \to \{1, \ldots, l+s\}$ be a mapping that is onto $\{1, \ldots, l\}/e$, i.e., for every $j \in \{1, \ldots, l\}$ there is some $i \in \{1, \ldots, s\}$ such that $\rho(i)$ is in the same e equivalence class with j. Put, for any such pair of e and ρ,

$$\gamma_{e,\rho} = e(v_1, \ldots, v_l) \wedge \exists \bar{v}(Rv_{\rho(1)} \ldots v_{\rho(s)}, \mathit{true}), \tag{12}$$

where $\bar{v}$ consists of those $v_{\rho(i)}$ for which $\rho(i) > l$ (if there are such; else no quantification is necessary and $\gamma_{e,\rho}$ is actually atomic). The desired formula $\mathsf{Live}_R(v_1, \ldots, v_l)$ is obtained as the disjunction over all $\gamma_{e,\rho}$ for matching pairs (e, ρ).

For any *finite* language $\mathcal{L}$ we further obtain a formula $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_l)$ defining the set of $\mathcal{L}$-live l-tuples by putting

$$\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_l) = (\bigwedge_{1 \leq i,j \leq l} v_i = v_j) \vee \bigvee_{R \in \mathcal{L}} \mathsf{Live}_R(v_1, \ldots, v_l), \tag{13}$$

where the first disjunct reflects the fact that all singleton sets are regarded as live (namely, as guarded by equality).

Now we are in a position to prove Proposition 4.3. Let us recall its formulation.

Proposition 4.3 *Let $\mathcal{L}$ be a language which contains at most k-ary relation symbols. Any first order formula φ in k variables which is uniformly guarded in $\mathsf{Live}_{\mathcal{L}}(v_1, \ldots, v_k)$ is equivalent to some $\varphi' \in \mathrm{GF}_k^k$ such that $\mathcal{L}_{G(\varphi')} = \mathcal{L}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\varphi)}$.*

Proof of Proposition 4.3: Proof is by induction on the complexity of φ. The only interesting case is the step for the quantifier. Thus, let us consider the guarded formula φ, uniformly guarded in $\mathsf{Live}_{\mathcal{L}}$, of the form $\exists \bar{v}[\mathsf{Live}_{\mathcal{L}}(\bar{u}, \bar{v}) \wedge \psi(\bar{u}, \bar{v})]$. Unraveling the definition of $\mathsf{Live}_{\mathcal{L}}$, and using distributivity of $\exists$ over $\vee$, we see that φ is equivalent to $\exists \bar{v}[(\bigwedge_{x,y \in \bar{u}\bar{v}} x = y) \wedge \psi] \vee \bigvee_{R \in \mathcal{L}} \bigvee_{e,\rho} (\exists \bar{v}(\gamma_{e,\rho} \wedge \psi))$, where e, ρ range over all matching pairs. The former disjunct can easily be rewritten in the desired format. Our remaining task is to conveniently rewrite the latter disjuncts.

Hereto, fix $R \in \mathcal{L}$ and a matching pair e, ρ. By (12), $\exists \bar{v}(\gamma_{e,\rho}, \psi)$ is of the form

$$\exists \bar{v}(e(\bar{u}, \bar{v}) \wedge \exists \bar{w}(R\bar{u}'\bar{v}'\bar{w}) \wedge \psi(\bar{u}, \bar{v})), \tag{14}$$

for some $\bar{u}' \subseteq \bar{u}, \bar{v}' \subseteq \bar{v}$ such that any variable in $\bar{u}, \bar{v}$ has an e-equivalent among $\bar{u}', \bar{v}'$. Fix, for any e-class V of variables in $\bar{u}, \bar{v}$, a representative v_V such that $v_V \in \bar{u}$, if $V \cap \bar{u} \neq \emptyset$. Now in (14) we subsequently

- bring $e(\bar{u})$ outside the scope of the quantifier.
- replace each variable x (free or bound) in $\bar{u}, \bar{v}$ by the representative of its e-class, denoted by v_{x_e}. Don't do this in $e(\bar{u})$.
- make sure no variables from $\bar{u}$ appear in $\exists \bar{v}$ after this substitution. If they do, remove them from this quantification.

This yields the following equivalent of (14),

$$e(\bar{u}) \wedge \exists v_{\bar{v}_e} \exists \bar{w}[R v_{\bar{u}'_e} v_{\bar{v}'_e} \bar{w}, ((\bigwedge_{v \in \bar{v}, u \in \bar{u}\bar{v}, <u,v> \notin e} v_{v_e} \neq v_{u_e}) \wedge \psi(v_{\bar{u}_e}, v_{\bar{v}_e}))]. \tag{15}$$

Note that the formula φ' displayed in (15) is guarded. For this it is crucial that any variable in $\bar{u}, \bar{v}$ has an e-equivalent among $\bar{u}', \bar{v}'$. Observe that if the arity of the relation symbols in $\mathcal{L}$ is at most k, and our original φ is a formula in at most k variables, then φ' contains at most k variables. Moreover, $\mathcal{L}_{G(\varphi')} = \{R\} \cup \mathcal{L}_{G(\psi)}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\psi)}$. By induction hypothesis we conclude that φ' is an equivalent of the original formula φ which fulfills the conditions of the proposition.
Q.E.D. Proposition 4.3.

Recall that for a finite language $\mathcal{L}$, and $n \in \omega$, we defined

$$\mathsf{Pack}_{\mathcal{L}}(v_1, \ldots, v_n) = \bigwedge_{i,j \leq n} \mathsf{Live}_{\mathcal{L}}(v_i, v_j).$$

According to Proposition 4.7, $\mathsf{Pack}_{\mathcal{L}}$ is allowed at guard positions in packed formulas.

Proposition 4.7 *Let $\mathcal{L}$ be a language which contains at most k-ary relation symbols. Any first order formula φ in k variables which is uniformly guarded in $\mathsf{Pack}_{\mathcal{L}}(v_1, \ldots, v_k)$ is equivalent to some $\varphi' \in \mathrm{PF}^k_k$ such that $\mathcal{L}_{G(\varphi')} = \mathcal{L}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\varphi)}$.*

Proof of Proposition 4.7: Proof is by induction on the complexity of φ. The only interesting case is the step for the quantifiers. Therefore, consider the packed formula φ, uniformly guarded in $\mathsf{Pack}_{\mathcal{L}}$, of the form $\exists v_1 \cdots v_l[\mathsf{Pack}_{\mathcal{L}}(v_1, \ldots, v_n) \wedge \psi(v_1, \ldots, v_n)]$. By definition of $\mathsf{Pack}_{\mathcal{L}}$, φ is equivalent to

$$\exists v_1 \cdots v_l[\underbrace{(\bigwedge_{i,j \leq n} \bigvee_{R \in \mathcal{L}} \bigvee_{\gamma_{\langle e,\rho \rangle}} \gamma_{e,\rho}) \wedge}_{\chi} \psi]. \tag{16}$$

Write χ in disjunctive normal form. This yields a huge disjunction χ' ranging over all possibilities for v_i, v_j, $i, j \leq n$, to be $\mathcal{L}$-live. More precisely, let F be the set of maps

$f : \{v_1, \ldots, v_n\} \times \{v_1, \ldots, v_n\} \longrightarrow \{\gamma_{R,e,\rho} : R \in \mathcal{L}, \langle e, \rho \rangle$ a matching pair for $R\}$. Then χ' is of the form

$$\bigvee_{f \in F} (\bigwedge_{i,j \leq n} f(v_i, v_j)), \tag{17}$$

whence (16) is equivalent to

$$\bigvee_{f \in F} \exists v_1 \cdots v_l (\bigwedge_{i,j \leq n} f(v_i, v_j) \wedge \psi). \tag{18}$$

By definition of $\gamma_{e,\rho}$, (cf. (12)), each $f(v_i, v_j)$ is either the identity $v_i = v_j$ or the conjunction of $v_i \neq v_j$ and a formula of the form $\exists \bar{u}(R v_i v_j \bar{u})$. Moving these inequalities inside ψ, we obtain a formula φ' in PF_n which is equivalent to our original φ, such that $\mathcal{L}_{G(\varphi')} = \mathcal{L} \cup \mathcal{L}_{G(\psi)}$ and $\mathcal{L}_{\bar{G}(\varphi')} = \mathcal{L}_{\bar{G}(\psi)}$. By induction hypothesis we are done.
Q.E.D. Proposition 4.7.

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