Logic between Expressivity and Complexity

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Abstract. Automated deduction is not just application or implementation of logical systems. The field of computational logic also poses deep challenges to our understanding of logic itself. I will discuss some key issues. This text is just an appetizer that will be elaborated in the lecture.

1 Logic and the Balance of Expressive Power and Computational Complexity

Defining and proving/computing are the main faces of logic. But they require a balance. Historically, first-order logic arose from type theory by giving up expressive power in order to gain axiomatizability (and better semantic transfer properties between models). The same move occurred a bit later in going from first-order logic to modal languages: one gives up yet more expressive power, but now one gains decidability (as well as discovering a new nice structural invariance for the modal language: viz. bisimulation).

2 Upward from Modal to Guarded Fragments

What makes the modal move to weaker languages tick? Did we go too fast? Essentially, standard modal operators are local guarded quantifiers of the special first-order form

$$\exists y(G(x, y)\ \&\ \varphi(x, y)),$$

where G is an atomic guard predicate, and the x, y are finite tuples of variables. Restricting quantifiers to only these forms defines the *Guarded Fragment* (GF).

Theorem 1. *GF is* decidable*, with an effective finite model property.*

Up to logical equivalence, GF is also the set of first-order formulas that are invariant for guarded bisimulation, a structural invariance that lies in between bisimulation and (potential) isomorphism.

But the border with complex behaviour lies still a bit higher up inside first-order logic. Decidability continues to hold for the 'Loosely Guarded Fragment' that allows conjunctions of guard atoms $\&G$:

$$\exists y(\&G(x, y)\ \&\ \varphi(x, y)),$$

where any two variables in x, y occur under at least one atom in $\&G$.

J. Giesl and R. Hähnle (Eds.): IJCAR 2010, LNAI 6173, pp. 122–126, 2010.

Beyond this lie the 'cliffs of complexity': quantifiers expressing well-known confluence (grid) properties are not loosely guarded, think of

$$\forall yz((R(x,y) \ \& \ R(x,z)) \rightarrow \exists u(R(y,u) \ \& \ R(z,u))).$$

These can encode Tiling Problems, and so their logic becomes undecidable.

Infinitary	Second-order	Fixed-point logics	highly complex, non-RE
		... ...	
	FOL		RE, undecidable
	GF		decidable, NEXPtime
	ML		decidable, Pspace

Aside (restricting a language versus re-interpretation): modal and related moves in logic have two faces. We either restrict to *fragments*, or we interpret all of FOL in some suitable *generalized semantics*, where not all assignments of objects to variables are available, encoding 'dependencies' in the model.

3 Aside: Downward to 'Poor Man's Logics'

Modal logics tend to be *Pspace*-complete. But this is not rock bottom yet. Going down even fruther to *feasible logics* with *(N)Ptime* satisfiability problems often takes non-Boolean languages.

Open Problem. Find a principled logical analysis for this move.

4 Model Theory in the Small: Lindström Theory

Our style of analysis multiplies logics. So, how can we understand the landscape of possible logical systems in greater generality?

Theorem 2 (Lindström). *FOL is inclusion-maximal with respect to the Compactness and Löwenheim-Skolem properties. For the latter one can also choose: the Karp Property (invariance of all sentences for potential isomorphisms), the RE property (axiomatizability of the valid sentences).*

Traditionally, foundational attention has only been paid to extensions of first-order logic (second-order, infinitary logics). Many characterizations exist, mostly via model-theoretic properties. But what happens if we look down in the landscape *below* FOL, on the idea that 'Small is Beautiful'?

Proof methods in Lindström theory require explicit encoding for back-and-forth properties of partial isomorphisms that capture first-order expressive power. But these are typically non-guarded grid properties. Still, new methods have been developed that work for small languages:

Theorem 3. *ML is maximal with respect to the properties of Compactness and Invariance for Bisimulation.*

Surprisingly, this yields a classical result in the theory of process logics:

Corollary 1. *A first-order formula is definable by a modal formula iff it is invariant for bisimulation.*

Now we can start a general abstract model theory of fragments, with new sorts of result. E.g., FOL is the largest extension of the *3-variable fragment* L_3 with the Compactness and Löwenheim-Skolem properties. (The crux is that 3 variables suffice for the encoding needed in a standard Lindström proof.)

Open Problem. Find a Lindström Theorem for GF.

For a related open problem for modal logic with a universal modality, there is a partial solution by Otto & Piro, but no best result yet [2].

5 Challenge 1: Fixed Point Logics

Logics with fixed-point operators have proved resistant to model-theoretic analysis ever sine the 1970s. (So far, the only things known about LFP(FO) are Karp and strong Löwenheim-Skolem properties.) And yet they are very natural as general logics of induction and recursion.

But modal lightweight logics can carry non-first-order fixed-point structure. Famously, the modal μ-calculus is decidable. And so is the *guarded fixed-point logic* LFP(GF). These logics present a challenge in terms of characterization, since they even improve on standard logics in having uniform interpolation, effective proofs of preservation theorems, etc.

Open Problem. Find a Lindström-type analysis for fixed-point logics (whether first-order or modal).

Perhaps, we have hit a boundary here of the usual model-theoretic stance in logic. We might essentially need further computational properties such as the Effective Finite Model Property, or even more explicitly procedural properties of logics, say, from Automata Theory.

This fits with a general issue in understanding logical systems today. Should we think of them as consisting essentially of *definition plus procedure*? This fits with the current trend to *logic games* that cast basic logical notions (truth, consistency, proof) in terms of interactive procedures. But we have no abstract game theory yet to back this up.

6 Challenge 2: Logic Combinations

As we go down in the landscape, logics get simpler. But in applications to agent theory (and cognitive science) we need to put the simple pieces back together again. Then *logic combinations* become ubiquitous. Can we do this while keeping trees and avoiding grids? Warning example: the *mode of combination* may crucially affect the outcome:

Theorem 4. *Putting together simple modal logics of time/action and knowledge for agents with* Perfect Memory *gives commuting diagrams, and hence the Recurrent Tiling Problem can be encoded, making the logic* Π^1_1*-complete.*

Similar surprises may be in store in the upcoming logical study of games:

Open Problem. What is the complexity for modal logics of action and preference in games for players that satisfy the usual assumption of *Rationality*?

Issue for reflection: What do these results mean? How bad is high complexity for a logic describing agents? Better focus on *complexity of agent tasks*?

7 Aside: Let the Structure Help

There is another traditional source of low complexity in logics: through wealth rather than poverty. Rich structures can also create decidability, even for dangerous languages with grid patterns:

Theorem 5. *Tarski's elementary geometry is decidable.*

The reason is that Euclidean space allows for 'elimination of quantifiers' in arbitrary first-order sentences. It is not so clear, however, whether this line is very helpful to us in computational logic.

8 Discussion: Practical Perspectives on Expressiveness and Complexity

Finally, if time permits, we will discuss a few outrageously general perspectives on the above issues. *Philosophy*: low complexity may not be needed, since a key aspect of rationality is a talent for exercising 'judgment' in using a potentially dangerous tool. *Cognitive science*: we operate in a complex world by learning where our best talents lie, and then selecting the right inputs. In line with these ideas, there are now new richer views of a reasoning system as merging two crucial abilities: logical inference plus memory search for pattern recognition, combining logic with probabilistic features.

Open Problem. Develop a general model theory for combined logics with probabilistic components.

Interestingly, in this area, the probabilistic Zero-One laws for first-order logic were discovered around the same time as Lindström's Theorem, but no similar theory has emerged yet.

But in this setting, we may also want to rethink our traditional view of 'logic and cognition'. Computational logic designs new forms of behaviour that get *inserted* into existing cognitive practice (just as mathematics has done in history). The new challenge may be understanding this insertion and its 'hybrids' of natural and designed behaviour.

Finally, even without resolving all this in depth, there is the practical world of *teaching*: introducing automated logic and a sense of the above considerations into general logic teaching seems rewarding (see the Opencourse project `http://staff.science.uva.nl/~jaspars/OpenCourse/`)

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