# On Compatible Multi-issue Group Decisions

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**Abstract**

A crucial problem in the design of multi-issue group decisions is the definition of rules that select outputs that are *consistent* with existing correlations between multiple issues. A less known problem arises when the collective outcome is supported by none or by the fewest individuals, bringing into question the *compatibility* of a collective decision with respect to individual choices. The aim of this paper is to make a first step into providing a definition of compatible outcome for binary aggregation procedures. We provide several definitions of compatibility, both for complete binary ballots and for the more general case of allowing abstentions in the individual judgments. We define a number of rules that draw inspiration from the literature on argumentation theory, social choice theory and belief merging, and for each of these rules we investigate their behaviour with respect to compatibility and consistency, and we study their social choice theoretic properties.

## 1 Introduction

The problem of collective decision making received considerable attention in the literature on artificial intelligence in recent years, with a strong focus on problems arising from the use of multi-issue domains, in which the structure of the alternatives has a combinatorial structure (Chevaleyre et al., 2008). We identify two questions in the design of collective procedures for multi-issue domains: the problem of *consistency* and that of *compatibility*. In this paper we focus on the latter, which received less attention so far.

The difficulty of defining consistent collective outcomes for group decisions was clear since the work of Condorcet at the end of the XVIII century. Condorcet observed that pairwise majority voting, perhaps the simplest example of a collective decision process, may give rise to cyclic collective preferences. Such a situation is known in the literature as the *Condorcet paradox*. Suppose that there are three possible candidates $a$, $b$, and $c$ and three voters $V_1$, $V_2$ and $V_3$, who express their preferences in the following way: $V_1 = \{a < b, b < c\}$, $V_2 = \{b < c, c < a\}$ and $V_3 = \{c < a, a < b\}$. According to the Condorcet's method, $a < b$ has the majority of the voters' preferences, so does $b < c$, and also $c < a$. This leads to a cyclic (and thus untenable) group preference $a < b < c < a$.

Not only social choice theory, i.e., the discipline that studies methods for collective decision making, could not make such difficulties disappear, but the situation even worsened when Kenneth Arrow proved his famous impossibility theorem (Arrow, 1963), which states that *no* aggregation procedure that satisfies a few desirable conditions exists. Impossibility theorems characterise also more recent disciplines inside social choice theory,

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like *judgment aggregation* (List and Puppe, 2009; Grossi and Pigozzi, to appear), where individuals express opinions in the form of 'yes' or 'no' choices on a given agenda of logically related propositions. In similarity with the Condorcet paradox, even if individuals submit judgments that are logically consistent by respecting the rules that connect the issues, issue-by-issue majority voting cannot guarantee a logically consistent outcome (a situation that comes under the name of *discursive dilemma*). Both the Condorcet and the discursive dilemma show that an aggregation function that takes a set of individual inputs and maps it to a group outcome cannot guarantee a consistent position, even though the individuals submitted consistent opinions. This clearly represents a serious problem to be taken into consideration when defining group decision methods.

However, a guarantee of a consistent outcome is not the only obstacle to group decision-making. In the *multiple election paradox*, introduced by Brams et al. (1998), individuals vote on issues that are logically independent and no external constraint is imposed. In other words, both at the individual and collective level, any combination of 'yes' and 'no' is acceptable. Thus, no consistency concern may arise here. Yet, even in these situations, the group may face a problem. This happens when issue-by-issue aggregation returns a combination that *none* or the smallest number of individuals submitted. Brams, Kilgour and Zwicker provided several empirical examples of this paradox that appeared in federal elections in the US. We illustrate the multiple election paradox with an example by Brams et al. (1998). $Y, N, A$ denotes respectively a favourable/unfavourable vote to the issue and an abstention.

> Suppose there are 52 voters who cast the following numbers of votes for the 27 combinations:
>
> YYY:0 YYN:4 YNY:4 NYY:4 YYA:4 YAY:4 AYY:4 YNN:1 NYN:1 NNY:1 YAA:1 AYA:1 AAY:1 NAA:1 NAN:1 NNA:1 NYA:1 ANY:1 YAN:1 AAA:5 ANN:1 ANA:1 AAN:1 AYN:1 NAY:1 YNA:1 NNN:5
>
> There is a complete-reversal paradox, because YYY wins under proposition voting by 20 votes to 16 on each proposition but, as a combination, it has the fewest (in this case 0) votes. On the other hand, the other two "pure" combinations, AAA and NNN (the latter might be considered the opposite of YYY), have the most votes (i.e., 5). The foregoing examples illustrate a range of discrepancies between aggregating votes by proposition and aggregating them by combination. (Brams et al., 1998, p.217)

A multiple election paradox may arise when the correlation between issues is not shared by all the individuals, or when voters have different preferential dependencies between issues (Benoit and Kornhauser, 1991; Lacy and Niou, 2000). We illustrate this second case with an example. Consider a group of individuals who needs to elect a 3-member committee for which there are 10 candidates. It is understood that people will submit their most preferred committee composition. Let us assume that $a_1, a_2, \ldots, a_{10}$ are the 10 candidates and that voter $i$ casts $(a_1, a_2, a_3)$. It is perfectly reasonable for $i$ to have $(a_1, a_2, a_3)$ as top preference and $(a_1, a_2, a_4)$ as bottom preference. This happens, for example, if $i$ believes that $a_1, a_2$ and $a_3$ would constitute a stable and efficient committee, but at the same time she believes that $a_1$ and $a_2$ would not be able to work together with $a_4$, making $(a_1, a_2, a_4)$ the worst possible outcome. Since individuals submit only their most preferred ballot, the information about $(a_1, a_2, a_4)$ being $i$'s worst possible outcome gets lost. As a consequence, a candidate-based aggregation procedure is vulnerable to return the outcome that was not submitted by any of the voters, or even the one that was ranked last by all voters.

As in these situations any combination is admissible, we will say that the outcome appears to be lacking of *compatibility* (to distinguish it from the problem of consistency). One way out of the problem is to increase the expressive power of the individuals, enabling them, for instance, to compactly represent their full or partial preferences or to state explicitly their preferential dependencies (Boutilier et al., 2004). This, however, is often not practical or even feasible, and individuals may not be aware of their preferential dependencies (and so they are unable to provide them). In those cases, voters will express only their top candidate. Hence, a more general perspective on compatible aggregation seems to be necessary.

Aim of this paper is to make a first step towards providing a definition of compatibility for aggregation procedures. Whereas the notion of consistent outcome has been extensively investigated, this is not the case for the notion of compatibility. We claim that these are two separate problems and that, in order to ensure fully rational group decision processes, we need procedures that ensure not only consistent but also compatible decisions. If the outcome is inconsistent, no group decision can be taken. If the agents do not accept the outcome, they might resist to perform the actions that are implied by the decision or might not accept to be deemed responsible for the group decision. Both problems are of the utmost importance in the design of well-behaved mechanisms for collective decisions.

The structure of the paper is as follows. In Section 2 we define the setting and we give basic definitions for aggregation procedures and axiomatic properties. In Section 3 we put forward five notions of compatibility. In Section 4 and 5 we define four rules and we study their behavior with respect to compatibility and consistency, and we investigate their axiomatic properties. Section 6 gives an overview of related work and concludes the paper.

# 2 The Framework

In this section we introduce the basic definitions of our framework for aggregation with multiple issues. We build our setting on binary aggregation with integrity constraints (Grandi and Endriss, 2011), which constitutes a general framework for modeling aggregation problems on interconnected issues. In this setting, a group of agents has to make decisions on a set of binary issues. Each agent expresses its acceptance or rejection for each issue, and these choices are then aggregated into a collective one. We generalise the original setting to allow individuals to abstain on single issues, and aggregation procedures to select more than one possible collective outcome. We also introduce axiomatic properties for the study of aggregation procedures, providing novel formulations of classical axioms for non-resolute aggregators.

## 2.1 Basic Definitions

Let $\mathcal{N} = \{1, \dots, n\}$ be a finite set of *individuals* and let $\mathcal{I} = \{1, \dots, m\}$ be a finite set of issues on which a decision needs to be taken.[1] We consider two options: a first in which individuals submit a yes/no choice about every issue in the agenda, and a second in which abstention is possible. In the first case, define a *complete ballot* $B$ as the choice of acceptance/rejection for each issue in $\mathcal{I}$. Formally, a complete ballot is an element of $\mathcal{D} = \{0, 1\}^{\mathcal{I}}$. A *complete profile* $\boldsymbol{B}$ is a vector of complete ballots, one for each individual in $\mathcal{N}$. In the second case, define an *incomplete ballot* $B$ as an element of $\mathcal{D}_A = \{0, 1, A\}^{\mathcal{I}}$, i.e. the choice of acceptance, rejection or abstention for each of the issues. An *incomplete profile* $\boldsymbol{B}$ consists of the choice of

[1] We occasionally refer to the set $\mathcal{I}$ as the *agenda*.

an incomplete ballot for each individual. We write $b_j$ to denote the $j$th element of a ballot $B$, and $b_{i,j}$ for the $j$th element of ballot $B_i$ within a profile $\boldsymbol{B} = (B_1, \ldots, B_n)$.

Given the choice of a ballot by each individual, *aggregation procedures* have the task to merge this information into a collective choice for each of the issues. In our setting we focus on *non-resolute* aggregation procedures, i.e., functions that associate with every profile a set of possible collective ballots. We give two definitions, one for complete profiles and one for incomplete profiles. Call *complete (non-resolute) aggregation procedure* a function $F : \mathcal{D}^{\mathcal{N}} \to 2^{\mathcal{D}}$, mapping each profile to a set of complete ballots. An *incomplete (non-resolute) aggregation procedure* is a function $F : \mathcal{D}_A^{\mathcal{N}} \to 2^{\mathcal{D}_A}$, mapping each profile to a set of incomplete ballots. $F(\boldsymbol{B})_j$ denotes the result of the aggregation on issue $j$, in case this is unique.

It is important to observe that we include the notion of universal domain directly in our definition, i.e., aggregation procedures are defined for every possible profile. Examples of standard aggregation procedures studied in social choice theory are the majority rule, the unanimity rule, rules giving acceptance quotas for every issue, etc.

## 2.2 Axiomatic Properties

In the literature on social choice theory there are several standard axiomatic properties for resolute aggregation procedures (i.e., procedures whose outcome is a single ballot) that have been proposed (see, e.g., List and Puppe, 2009; Grossi and Pigozzi, to appear). In this section, we adapt their standard formulation to the case of incomplete ballots and we generalise such properties to the case of non-resolute voting procedures.

The first property, *unanimity*, requires that if all individuals submit the same opinion on one issue in the agenda, then the collective opinion on that issue should agree with the individuals. This does not apply to abstentions, since, in case issues are logically connected, an individual abstention may be turned into acceptance or rejection by using the logical connections between issues. The second property is *anonymity*, which requires that all agents count the same in determining the outcome.

**Unanimity** (U): For any profile $\boldsymbol{B}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = x$.

**Anonymity** (A): For any profile $\boldsymbol{B}$ and permutation $\sigma : \mathcal{N} \to \mathcal{N}$, $F(B_1, \ldots, B_n) = F(B_{\sigma(1)}, \ldots, B_{\sigma(n)})$.

A crucial property is *independence*, which requires that the collective decision on a certain issue depend only on the individual inputs on that issue (and not on other "independent" issues). *Monotonicity* requires that if one agent switches from rejecting an issue to accepting it, all other judgments remaining the same, then if the collective outcome accepted that issue in the first profile so should be in the second.

**Independence** (I): For all issues $j \in \mathcal{I}$ and any two profiles $\boldsymbol{B}, \boldsymbol{B}'$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = F(\boldsymbol{B}')_j$.

**Monotonicity** (M): For any $j \in \mathcal{I}$ and any two profiles $\boldsymbol{B}$ and $\boldsymbol{B}'$, if $b_{i,j} = 1$ entails $b'_{i,j} = 1$ for all $i \in \mathcal{N}$, and for some $s \in \mathcal{N}$ we have that $b_{s,j} = 0$ and $b'_{s,j} = 1$, then $F(\boldsymbol{B})_j = 1$ entails $F(\boldsymbol{B}')_j = 1$.

We now extend these axioms to the case of non-resolute aggregation procedures, in which the outcome is a set of possible ballots. While the axiom of anonymity remains the same, we need to adapt the remaining ones:[2]

[2]Similar versions of the axioms of unanimity and monotonicity for non-resolute aggregation procedures have been proposed by Lang et al. (2011) in the context of judgment aggregation.

**Unanimity**$^\star$ (U$^\star$): For any profile $\boldsymbol{B}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $b_j = x$ for all $B \in F(\boldsymbol{B})$.

**Independence**$^\star$ (I$^\star$): For any issue $j \in \mathcal{I}$, $x \in \{0, 1\}$ and profiles $\boldsymbol{B}, \boldsymbol{B}'$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then there exists a $B \in F(\boldsymbol{B})$ such that $b_j = x$ iff there exists $B' \in F(\boldsymbol{B}')$ such that $b'_j = x$.

Consider the following notion of monotonicity: if there exists a ballot $B$ in the collective outcome such that $b_j = 1$ and we increase acceptance for issue $j$ while keeping fixed the individual judgments about *all other issues*, then there still exists a winning ballot $B'$ such that $b'_j = 1$. This requirement can be formalised in the following axiom:

**Weak Monotonicity**$^\star$ (M$^\star$): For any $j \in \mathcal{I}$ and profile $\boldsymbol{B}$, if $F(\boldsymbol{B})$ contains a ballot $B$ such that $b_j = 1$ and $\boldsymbol{B}'$ is obtained from $\boldsymbol{B}$ by increasing acceptance of issue $j$ keeping everything else fixed, then there also exists a ballot $B'$ in $F(\boldsymbol{B}')$ such that $b_j = 1$.[3]

## 2.3 Consistency

The paradoxical examples presented in the introduction highlight a crucial problem that can arise whenever issues are correlated and the choice of individuals is bounded by a rationality assumption. Numerous studies of aggregation problems focused on how to guarantee that the outcome of the aggregation satisfies the same rationality constraint as the individuals. This condition is traditionally referred to as *collective rationality* (Arrow, 1963; List and Puppe, 2009). In the case of preference aggregation, for instance, individuals are required to submit a weak order over a set of alternatives, and collective rationality requires that the outcome should also be a weak order. In the case of judgment aggregation, collective rationality requires that the outcome be a consistent and complete judgment set.

Frameworks like preference aggregation and judgment aggregation can be viewed as particular instances of the framework for binary aggregation we presented, and requirements of collective rationality can be translated accordingly. An ordering over three alternatives, for instance, can be represented in binary aggregation with a binary ballot over three issues, one for each pair of alternatives. If $P_{a>b}$, $P_{b>c}$ and $P_{a>c}$ are three binary issues, with their natural interpretation, we can represent the rationality assumption of transitivity of a preference relation with a propositional formula like the following: $P_{a>b} \wedge P_{b>c} \rightarrow P_{a>c}$. The embedding is less straightforward for the case of judgment aggregation, and can be achieved by explicitly representing the logical correlations between the propositional formulas that constitute the object of judgment (Grandi and Endriss, 2011).

As shown in the previous paragraph for the specific case of preferences, the approach taken by Grandi and Endriss (2010, 2011) is that of specifying with a propositional formula a subset of the domain $\mathcal{D}$ as the set of rational ballots, and use this formula as a variable in the definition of collective rationality. Thus, collective rationality ceases to be included as an axiom but becomes a property that an aggregation procedure can satisfy depending on the application at hand.

Here, we generalise this approach to cover the case of incomplete ballots and non-resolute procedures. While working with propositional formulas guarantees succinteness and a direct potential implementation, its

[3] A similar property is presented in the context of judgment aggregation by Lang et al. (2011), under the name of *insensitivity to reinforcement of collective judgments*. Our property is the existential version of it, for we do not require issue $j$ to be accepted by all outcome ballots but rather by at least one of the collective outcomes.

generalisation to the case of incomplete ballots is not straightforward. We therefore provide our definition by explicitly introducing a set of rational ballots, rather than having it defined by a propositional formula. Thus, we propose the following definition that applies to both complete and incomplete aggregation procedures:

**Definition 1.** Given a subset $\mathcal{R} \subseteq \mathcal{D}$, respectively $\mathcal{R} \subseteq \mathcal{D}_A$, an aggregation procedure $F$ is *collectively rational* with respect to $\mathcal{R}$ if $F(\boldsymbol{B}) \in \mathcal{R}$ for all profiles $\boldsymbol{B} \in \mathcal{R}^{\mathcal{N}}$.

The literature on social choice theory is plagued with results showing that it is impossible to combine collective rationality (on given specific domains) with other natural axiomatic conditions (Arrow, 1963; List and Puppe, 2009; Grossi and Pigozzi, to appear). Similar results have been provided also for the case of judgment aggregation with abstentions (Dietrich and List, 2008; Dokow and Holzman, 2010).

# 3 Compatibility

In this section we propose five definitions of compatibility both for complete and for incomplete aggregation procedures. Our aim is not to prescribe a unique notion of compatibility, but rather to explore several alternatives. We introduce and discuss several definitions whose aim is to formalise the notion of a decision being more or less connected with the individual opinions on a set of issues. Some of these notions are intended to be applied to situations in which the set of individuals is small, even though they can be also applied to cases of large groups (the empirical examples of the multiple election paradox given by Brams et al. (1998) concerns, for instance, the electorate of an American state).

As shown by the introductory examples, the problem of compatibility arises when the collective choice does not mirror the submitted individual views, and can occur in many different situations. A conceptual clarification is necessary: in our model, individuals submit a binary ballot, their preferred one, without unveiling any information concerning their preference about the remaining ballots. This models real world situations like multiple referenda, in which the only observable part of an individual's preference is the ballot she cast. Thus, when measuring compatibility, the definitions we put forward can only refer to those ballots submitted by the individuals.

A second interpretation is although possible. Binary ballots, as mentioned in Section 2.3, can be employed to represent preferences, judgments and sets of alternatives. Thus, under this second interpretation, a binary ballot does not only represent the observable fragment of an individual's preferences, but rather encodes an entire individual expression, such as a judgment or a preference relation. The definitions of compatibility that we put forward in this section become very strong requirements for aggregation procedures when interpreted under this view. While we will consider the former interpretation as the primary one, we do not want to disregard this second possibility, leaving a more detailed discussion of the differences between these two cases for future work.

## 3.1 Complete Procedures

We first give two definitions that are intended for complete aggregation procedures. The first definition that we propose rules out situations in which the group outcome is a ballot that no individual has submitted:

**Definition 2.** An aggregation procedure satisfies *strong compatibility* if the outcome $F(\boldsymbol{B})$ on every profile is supported by at least one individual, i.e., all $F(\boldsymbol{B}) \subseteq \{B_1, \ldots, B_n\}$ for all profiles $\boldsymbol{B} = (B_1, \ldots, B_n)$.

Strong compatibility captures the basic intuition of a compatible outcome, that is, to avoid instances of a multiple election paradox where the group decision is an outcome that no member submitted. Strong compatibility can be generalised to a notion inspired by quota rules (although it refers to outcomes and not to issues, as is customary for quota rules). This captures a notion of compatibility in which several individuals must submit the ballot selected as group outcome for the group decision to be legitimate.

**Definition 3.** An aggregation procedure satisfies *$k$-strong compatibility* ($k \leqslant |\mathcal{N}|$) if, for every profile, all ballots in the outcome $F(\boldsymbol{B})$ are supported by at least $k$ individuals.

The notion of $k$-strong compatibility can be seen as the second natural way out from instances of a multiple election paradox, in which the selected outcome is a ballot submitted by the fewest individuals. Note that when $k = 1$, $k$-strong compatibility coincides with strong compatibility, the case of $k = \lceil \frac{|\mathcal{N}|+1}{2} \rceil$ is majority-consistency on combinations, and the case of $k = |\mathcal{N}|$ requires the outcome to be the same as the individuals' one on unanimous decisions on the whole ballot. Both strong compatibility and $k$-strong compatibility are demanding notions that are not likely to be satisfied, especially in the case of small groups of individuals.

## 3.2 Incomplete Procedures

Let us now turn to the case of *incomplete ballots*. The first notion we put forward draws inspiration from a notion that was first introduced by Caminada and Pigozzi (2011). Call two incomplete ballots $B$ and $B'$ *compatible* if they do not disagree on any issue $j$, i.e., it is never the case that $B_j = 1 - B'_j$ while it is always possible that $B_j = 1$ (resp. $B_j = 0$) and $B'_j = A$. The intuition is that there is disagreement only when one or more agents accept and one or more agents reject a certain issue. If one agent abstains on an issue, then her judgment is compatible both with an acceptance or a rejection. So, for example, if there are three issues and agent 1 submits ballot $(1, 0, A)$ and agent 2 submits $(A, 0, 1)$, the two ballots are compatible because every time agent 1 accepts (or rejects) one issue, agent 2 either agrees with the first agent (for example on the second issue of the ballot we are considering here), or abstains, and *vice-versa*.

**Definition 4.** An incomplete aggregation procedure satisfies *weak-compatibility* if for every profile the outcome $F(\boldsymbol{B})$ is compatible with all the individual ballots $B_1, \ldots, B_n$.

This notion can be generalised by requiring that the outcome is compatible with at least $k$ individuals:

**Definition 5.** An incomplete aggregation procedure satisfies *$k$-weak compatibility* ($k \leqslant |\mathcal{N}|$) if the outcome $F(\boldsymbol{B})$ is compatible with at least $k$ individual ballots in $B_1, \ldots, B_n$.

The last definition of compatibility we introduce focuses on single issues rather than on the whole ballots:

**Definition 6.** An aggregation procedure (complete or incomplete) satisfies *$k$-compatibility over issues* ($k \leqslant |\mathcal{N}|$) if, for every profile and every issue, the outcome $F(\boldsymbol{B})_j$ is compatible with at least $k$ individuals.

The definition of compatibility as $k$-compatibility over issues captures the intuition that a decision *over each issue* is legitimate when that decision was supported by at least $k$ agents. Observe that for $k = |\mathcal{N}|$ all the last three notions of compatibility are equivalent.

# 4 The Average Voter Rule

In this section we provide the definition of an aggregation procedure that proves to be both collectively rational and satisfies a strong notion of compatibility. When defining the notion of strong compatibility, we required the outcome to be equal to at least one of the individual ballots. If we take this as the notion of compatibility we want to enforce, then every selection function that selects an individual ballot as the collective outcome is a compatible rule. An interesting example of such a procedure is the *average voter rule* proposed by Grandi and Endriss (2011) in the context of binary aggregation, inspired by work on belief merging (Konieczny and Pino Pérez, 2011). Note that this rule is only defined for complete ballots, but can be easily generalised to the incomplete case.

**Definition 7.** The *average voter rule* (AVR) chooses the individual ballot that minimises the sum of the Hamming distance (i.e. the number of issues on which two agents disagree) to all other individual ballots:

$$\text{AVR}(\boldsymbol{B}) \quad = \quad \underset{\{B_i | i \in \mathcal{N}\}}{\text{argmin}} \sum_{i' \in \mathcal{N}} H(B_i, B_{i'}),$$

where $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ is the Hamming distance.

This rule is a non-resolute aggregation procedure. Even if it shares many properties with the classical issue-by-issue majority aggregation, the AVR does not coincide with it. This can be seen in Table 1, in which the outcome of the majority rule is different from all individual ballots.

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $B_1$ | 1 | 1 | 0 | 1 | 1 |
| $B_2$ | 0 | 1 | 1 | 0 | 1 |
| $B_3$ | 1 | 0 | 1 | 1 | 0 |
| Maj | 1 | 1 | 1 | 1 | 1 |
| AVR | 1 | 1 | 0 | 1 | 1 |

Table 1: Majority differs from AVR

According to the AVR, the first individual ballot is selected as the group outcome. The first ballot differs on three issues from each of the remaining two ballots, having a sum of the Hamming distances of 6. The last two ballots differ on 4 issues from each other, having a total distance of 7. The first ballot is therefore selected by minimisation.

As previously remarked, it is straightforward to see that the AVR satisfies the first notion of compatibility. So we can state the following proposition:

**Proposition 1.** *The* AVR *satisfies strong compatibility.*

This rule also satisfies the following axiomatic properties:

**Proposition 2.** *The* AVR *satisfies* U$^\star$, A *and* M$^\star$*, and it does not satisfy* I$^\star$.

*Proof.* The AVR is clearly anonymous and, as the outcome is composed by individual ballots, it is also unanimous. To see that the AVR satisfies the monotonicity condition M$^\star$, assume that $B_i$ is the ballot of one of the average voters in profile $\boldsymbol{B}$ and that $b_{ij} = 1$, i.e., issue $j$ is accepted. If we now increase acceptance of $j$ by modifying other individual ballots, then the Hamming distance of $B_i$ from other individual ballots decreases, hence leaving $B_i$ in the collective outcome.

To see that the independence condition I$^\star$ does not hold, consider the profile in Table 1. According to the AVR, the first individual ballot is selected as the group outcome. If we call $\boldsymbol{B}$ the profile described in Table 1, then we have that AVR$(\boldsymbol{B})_{j_3} = 0$. We now construct a different profile $\boldsymbol{B}'$ such that AVR$(\boldsymbol{B}')_{j_3} = 1$, while keeping the individual judgments about the same issue $j_3$ fixed, contradicting the axiom of independence I$^\star$.

Let therefore $\boldsymbol{B}'$ be obtained from $\boldsymbol{B}$ by changing the third individual ballot to $B'_3 = (1,1,1,1,1)$. We have that $H(B_1, B_2) = 3$, $H(B_1, B'_3) = 1$ and $H(B_2, B'_3) = 2$. The third ballot $B'_3$ has now the lowest total Hamming distance with a value of 3, and is the only ballot that gets selected. This contradicts independence I$^\star$, as there is no ballot in the outcome of $\boldsymbol{B}'$ which rejects the third issue $j_3$. □

A straightforward generalisation of the definition of the AVR rule enables us to define a rule that complies with the notion of $k$-strong compatibility introduced in Section 3. It is sufficient to define a rule by means of selection functions that choose between the ballots submitted by at least $k$ individuals, outputting a general abstention in case no combination was supported by at least $k$ individuals.

Having showed that the AVR satisfies a strong notion of compatibility, we now turn to investigate its behavior with respect to consistency. It is sufficient to observe that, since all individuals are submitting rational ballots, it is impossible for the AVR to choose an irrational outcome, as the minimisation is performed over the individual ballots.[4] Therefore, we can state the following proposition:

**Proposition 3.** *The* AVR *rule is collectively rational for all rationality assumptions* $\mathcal{R} \subseteq \mathcal{D}$.

We therefore conclude that the average voter rule (AVR) is a good candidate for a compatible *and* consistent aggregation procedure (and the only rule in this paper that satisfies both notions).

# 5 Compatible Aggregation Rules

In this section we define three rules for incomplete profiles, corresponding to the three notions of compatibility we have introduced in Section 3 for incomplete procedures.

The first rule we consider is the *conflict-free rule* (CFR). CFR selects as collective outcome the ballot that is compatible with all ballots of the profile. In particular, if an agent accepts (*resp.* rejects) an issue $i$ and no agent rejects (*resp.* accepts) it, the issue $i$ will be accepted at the collective level. In all the other cases, the group will abstain on $i$.

**Definition 8** (Conflict-free rule (CFR))**.** For every $j \in \mathcal{I}$, let $b^c_j$ be the $j$th element of the collective outcome $\mathrm{CFR}(\boldsymbol{B})$:

[4] We can observe that this is a common property of aggregation procedures that are restricted to the individual ballots in the selection of the collective outcome. This fact is formalised by Grandi and Endriss (2010) for the case of resolute aggregation procedures, defining the class of *generalised dictatorships* as the class of procedures that copy the ballot of a (possibly different) individual in every profile.

$$b_j^c = \begin{cases} 1 & \exists i \in \mathcal{N} \text{ s.t. } b_{i,j} = 1 \text{ and } \nexists l \in \mathcal{N} \text{ s.t. } b_{l,j} = 0 \\ 0 & \exists i \in \mathcal{N} \text{ s.t. } b_{i,j} = 0 \text{ and } \nexists l \in \mathcal{N} \text{ s.t. } b_{l,j} = 1 \\ A & \text{otherwise} \end{cases}$$

This rule shares similarities with the *credulous* aggregation procedure introduced by Caminada and Pigozzi (2011) in an abstract argumentation framework.

**Example 1.** *To illustrate the* CFR*, consider two profiles* $\boldsymbol{B}$ *and* $\boldsymbol{B}'$ *in Table 2.*

| | $p$ | $q$ | $r$ |
|---|---|---|---|
| $B_1$ | 1 | 0 | A |
| $B_2$ | A | A | 1 |
| $B_3$ | A | 0 | 1 |
| CFR $(\boldsymbol{B})$ | 1 | 0 | 1 |

| | $p$ | $q$ | $r$ |
|---|---|---|---|
| $B'_1$ | 1 | 1 | A |
| $B'_2$ | 1 | 0 | 0 |
| $B'_3$ | A | 0 | 1 |
| CFR $(\boldsymbol{B}')$ | 1 | A | A |

Table 2: The CFR

*The collective decision on* $p$ *is 1 in both profiles because, for an issue to be accepted (resp. rejected) at the group level, it suffices that there is one individual that accepts (resp. rejects) it and no one who disagrees with her (abstention does not count as a disagreement in* CFR*). For the same reason,* $q$ *is rejected under the first profile. However, the group abstains on* $q$ *under the second profile because there is a disagreement among the individuals about* $q$*.*

The example above illustrates an interesting feature of the CFR. In the first profile, all individuals are undecided on at least one issue. However, the collective outcome does not contain any abstention. In this case, we have that the group is compatible with the individual positions and, at the same time, is *more committed* than any of the group's members. On the other hand, it may seem disturbing that in the second profile two agents are undecided about two different issues and yet, the group abstains on two issues. Is this really something negative? Well, no. The reason is that the CFR selects the *most committed* among the compatible outcomes. To see that, it is enough to observe that for the second profile, $(1, A, A)$ is the most committed compatible outcome. None of the individual ballots can be selected to be the group outcome since they are all incompatible with each other. The only compatible options are $(A, A, A)$ and $(1, A, A)$. Since the second one contains fewer abstentions (*i.e.* is more committed) than the first, $(1, A, A)$ is the group's position selected by CFR.

The second rule we consider is the $k$*-conflict-free rule* ($k$-CFR). We first need to define a *subprofile* of $\boldsymbol{B}$.

**Definition 9** (Subprofile of $\boldsymbol{B}$)**.** Given a profile $\boldsymbol{B} = (B_1, \ldots, B_n)$ and a subset of agents $K \subseteq N$, the restriction of $\boldsymbol{B}$ to $K$ is $\boldsymbol{B}_K = (B_k, k \in K)$ and is called a *subprofile* of $\boldsymbol{B}$.

**Definition 10** ($k$-conflict-free rule ($k$-CFR))**.** The $k$*-conflict-free rule* maps $\boldsymbol{B}$ to $k$-CFR($\boldsymbol{B}$)={CFR($\boldsymbol{B}_K$) | $K \subseteq \mathcal{N}$}, where CFR($\boldsymbol{B}_K$) is the CFR over the subprofile $\boldsymbol{B}_K$.

Note that $k$-CFR does not guarantee a unique result, i.e. it is a non-resolute rule, as illustrated in the example below.

**Example 2.** *Let the profile be the same as that of Example 1. Let us assume that $k$=2. $F$ refers to the $k$-CFR. The two examples in Table 3 show that for $k = 2$ we have three different outcomes on both profiles, depending*

| | $p$ | $q$ | $r$ |
|---|---|---|---|
| $B_1$ | 1 | 0 | A |
| $B_2$ | A | A | 1 |
| $B_3$ | A | 0 | 1 |
| $F(\boldsymbol{B}_{2,3})$ | A | 0 | 1 |
| $F(\boldsymbol{B}_{1,2})$ | 1 | 0 | 1 |
| $F(\boldsymbol{B}_{1,3})$ | 1 | 0 | 1 |

| | $p$ | $q$ | $r$ |
|---|---|---|---|
| $B'_1$ | 1 | 1 | A |
| $B'_2$ | 1 | 0 | 0 |
| $B'_3$ | A | 0 | 1 |
| $F(\boldsymbol{B}'_{2,3})$ | 1 | 0 | A |
| $F(\boldsymbol{B}'_{1,2})$ | 1 | A | 0 |
| $F(\boldsymbol{B}'_{1,3})$ | 1 | A | 1 |

Table 3: The k-CFR

*on which subprofiles we consider. For example, $k$- CFR$(\boldsymbol{B}_{2,3})$ is the outcome obtained by the $k$-conflict-free aggregation rule when the subprofile considered consists of agents 2 and 3.*

Note that, when $k$ is bigger or equal to the number of individuals, the $k$-CFR coincides with the CFR.

It is now sufficient to observe that the CFR selects the only ballot that is compatible with all individual ballots to prove the following proposition:

**Proposition 4.** *The* CFR *satisfies weak compatibility. The $k$-*CFR *rule satisfies $k$-weak compatibility.*

We now turn to study the behavior of the two rules with respect to axiomatic properties (recall that the CFR is a resolute procedure while the $k$-CFR is not).

**Proposition 5.** *The* CFR *satisfies U, I, A and M.*

*Proof.* Let us start with unanimity: in case all individuals agree on a single issue on accepting it, rejecting it or abstaining on it, then there obviously is no conflict and the CFR selects the same judgment as outcome. Independence is verified since the rule is defined issue-by-issue, and, as the definition does not contain any reference to specific individuals, the CFR is clearly anonymous. To prove that monotonicity holds, note that whenever $b_j^c = 1$ for a certain $c$, then all individuals must either abstain or accept the same issue $j$. Therefore, increasing support for this same issue cannot create any conflict between individual judgments, and the result does not change. □

The $k$-CFR satisfies the same set of axioms adapted for non-resolute procedures. The proof of the following proposition is a direct generalisation of the previous one:

**Proposition 6.** *The $k$-*CFR *satisfies U$^\star$, I$^\star$, A and M$^\star$.*

The last rule we consider is an adaptation from quota rules (where a proposition is accepted if and only if that proposition is accepted by a number of individuals greater than a prefixed threshold), as defined by Dietrich and List (2007) in judgment aggregation.

**Definition 11** ($k$-quota rule ($k$-QR)). Let $1 \leqslant k \leqslant n$ and $b^c_j$ be the $j$th element of the collective outcome $k-\mathrm{QR}(\boldsymbol{B})$:

$$b^c_j = \begin{cases} 1 & \text{iff } \exists M \subseteq \mathcal{N}, |M| \geqslant k \text{ s.t. } \forall i \in M, b_{i,j} = 1 \\ 0 & \text{iff } \exists M' \subseteq \mathcal{N}, |M'| \geqslant k \text{ s.t. } \forall i \in M', b_{i,j} = 0 \\ A & \text{iff } \exists M'' \subseteq \mathcal{N}, |M''| \geqslant k \text{ s.t. } \forall i \in M'', b_{i,j} = A \\ A & \text{otherwise} \end{cases}$$

Note that $k$-QR guarantees a unique result only when $k \geqslant \frac{|\mathcal{N}|}{2}$. This rule selects as outcome on a certain issue all the outcomes that are supported by at least $k$ individuals. An abstention on an issue in the outcome means either that the initial profile contained too many abstentions to take a definitive stand on the issue, or that there was not enough support for either its acceptance or its rejection. Outputting $A$ whenever there is no $M$ of cardinality $k$ such that all the individuals in $M$ accepted, rejected or abstained on an issue, ensures that $k$-QR is a well-defined aggregation function, i.e. it always returns an outcome. Suppose, for example, that $k$ is set to be equal to $n$. Then, if we do not add the last condition in Definition 11, and if there is no unanimous vote on the issues, the rule would not be able to assign a group outcome to the given profile.

The $k$-QR satisfies the last notion of compatibility and several axiomatic properties:

**Proposition 7.** *The $k$-QR satisfies $k$-compatibility over issues.*

**Proposition 8.** *The $k$-QR satisfies $U^\star$, $I^\star$, $M^\star$, A unless $k = |\mathcal{N}|$ or $k = 0$.*

*Proof.* For what observed in previous proofs, the $k$-QR is independent and anonymous since it is defined issue-by-issue and it does not make any reference to a specific individual in its definition. Unless $k = |\mathcal{N}|$ or $k = 0$, in which case the rule outputs a complete abstention or outputs all possible values on every issue, this rule is also unanimous. To see this, suppose for instance that $b_i = 0$ for all individuals $i$, then there cannot be any $M \subseteq \mathcal{N}$ such that $b_{i,j} = 1$ or $b_{i,j} = A$ for all $i \in M$. With similar arguments it can easily be shown that the $k$-QR is monotonic. □

We now turn to analyse the behavior of the rules defined in this section with respect to consistency. Unfortunately, we can easily conclude that the result is negative and we explain this with a very simple example. Consider the profile in Table 4, and assume that the rationality constraint rules out $(0, 0)$ as inconsistent.

| | $i$ | $j$ |
|---|---|---|
| $B_1$ | 0 | A |
| $B_2$ | A | 0 |
| $B_3$ | 0 | A |
| CFR($\boldsymbol{B}$) | 0 | 0 |

Table 4: The CFR is not collectively rational.

This example shows that the CFR is not collectively rational with respect to the rationality constraint in question. Similar results can be obtained for the $k$-conflict-free rule. Even more radically, a recent result by Dokow and Holzman (2010) rules out the possibility of obtaining consistent, unanimous and independent aggregators for the case of incomplete ballots. The authors proved that any such rule (including our CFR) is oligarchic on non-trivial agendas, i.e., the outcome is obtained by copying the unanimous decision of a subset of individuals in every profile, or by abstaining on the issue.

# 6 Related work and conclusions

While the literature on preference and judgment aggregation widely investigated the problem of defining aggregation rules that can guarantee consistent outcomes, the issue of ensuring a compatible group decision received little attention. There is, however, some work that needs to be mentioned and with respect to which our contribution has to be positioned.

Recently, Caminada and Pigozzi (2011) used an abstract argumentation framework to propose three procedures that ensure consistent and compatible collective positions. In their approach, given an argumentation framework and a group of agents, the individuals may have divergent opinions on the status of the arguments. The question is then how to combine individual evaluations of an argumentation framework into a collective one. In particular, they introduced and studied three aggregation rules that ensure outcomes that are not in conflict with the individuals' positions. There are several important differences between the approach of this paper and the one in Caminada and Pigozzi (2011). Other than the different frameworks being used, the main difference resides in the fact that in Caminada and Pigozzi (2011) cardinality considerations did not play a role in the aggregation procedures. Those had to yield to outcomes that were not in conflict with *any* of the agents. Instead, we have provided definitions of compatibility on a wider spectrum, going from compatible outcomes with respect to all the individuals, to some of the individuals or to only one. We indeed believe that different applications require different *graded concepts of compatibility*.

Our definition of compatibility is related to the work of Meskanen and Nurmi (2008) and Elkind et al. (2009) on distance rationalisability of voting rules. A voting rule associates a set of candidates to a profile of preference orders. Work on distance rationalisability of voting rules seeks justification for a collective decision in its distance from a notion of consensus accepted by all individuals. Our concept of compatibility, although being formulated in the different framework of multi-issue decisions, is strongly related, and a comparison between the two lines of research constitutes a very promising direction for future work.

In this paper we have claimed that, when a group makes decisions, it is not only important that the outcome is consistent in order to guarantee that the decision is transformable into actions, but also that it is perceived as compatible by the group's members, to ensure that the individuals agree to perform the action implied by the collective decision. The literature in social choice theory focused on the first aspect, but seems to have neglected considerations regarding the notion of compatibility.

The multiple-election paradox explained in the introduction highlights the arbitrariness of an aggregation procedure that selects an outcome that was submitted by none or by the fewest agents. In this paper we started exploring the concept of compatibility of an aggregation rule. Our aim is not to impose one single notion of compatibility, but rather to investigate possible declinations of such a notion. We have given several definitions to capture the compatibility of a collective decision in binary aggregation, both for complete and incomplete aggregation procedures. We then introduced five definitions of aggregation procedures, and we have studied their behaviour with respect to our notions of compatibility. Of each of these rules we have also studied the axiomatic properties that are satisfied. We do not claim to have exhausted all possible formulations of compatibility. Other notions can be defined, and classical and new aggregation rules (see e.g. Lang et al., 2011) should be tested with respect to these properties. An interesting feature is that our notions of compatibility do not necessarily lead to undecided collective decisions. Indeed, we have seen that CFR selects the most committed (*i.e.* containing fewest abstentions) among the compatible outcomes. Another issue that remains to be explored is the relationship between compatible aggregation procedures and strategic voting. Our intuition is that, if the collective outcome is ensured to be compatible with all the individuals, agents have less incentive to submit insincere ballots. This, however, needs to be formalized and investigated.

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