TRACKING INFORMATION

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Abstract Depending on a task at hand, information can be represented at different levels, less or more detailed, each with their own appropriate logical languages. We discuss a few of these levels, and their connections, and how information growth at one level can be tracked at another. The resulting view has two intertwined dynamics for informational agents: one of update and one of representation. [3]

1 Introduction: information and logic

Logic and information has been a lifelong interest of Michael Dunn's, witness his seminal contributions highlighted in this volume in his honor. I have long been intrigued by this interface, but my offering concerns just one special topic: the dynamics of information-driven agency.

A first major issue in this setting is one that every logician studying the area encounters sooner or later: information is not one single notion, but a content that can be *represented* at different levels of detail, rougher or finer (van Benthem & Martinez 2008). Each level supports natural 'attitudes' agents can have, not in any concrete psychological sense, but in the sense of different attunements to information. In this paper, I will start from perhaps the roughest level, that of semantic information as a range of possibilities, and then discuss richer views that are prominent in the recent literature: plausibility order and belief, evidence, and eventually also some levels higher up, such as prioritized evidence or probability.

A second major issue is the pervasive phenomenon of information *dynamics* (van Benthem 2011). Agents do not have fixed information: it changes all the time when receiving new informational signals, hard or soft. Moreover, their attitudes tag along in a systematic manner, leading to dynamic knowledge update, belief revision, or whatever term fits the current representation level. This second issue is not unrelated to the first, since in any serious study of information, the choice of representation cannot be made in isolation from the dynamic actions that one wants to understand (Adriaans & van Benthem, eds., 2007).

However, my main theme in this paper is not to propose new representation levels or new update actions. Rather, I want to investigate how the two directions interact, resulting in the problem of *'tracking'* information dynamics at different levels. I will state a few new results, and on that basis, raise some general problems for the logical analysis of information. [4]

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[2] *Note. This version of 15 March 2016 corrects a mistake in an excursion in Section 7.2 that slipped into the published version of this paper, and Sections 8, 9 add some new remarks on tracking and definability.*

[3] This is a contribution to a volume dedicated to Mike Dunn in Springer's Outstanding Logicians series. Mike has been a lifelong pioneer in the area of relevant and resource logics (and informatics generally), with the semantic, algebraic and proof-theoretic structure of information as his special interest. I offer these thoughts as an old academic fellow-traveler, if not in his line, then at least in his spirit.

[4] This is not a self-contained paper. Reviews of standard material will be brief, making reference to the literature. New notions and new observations are marked as definitions, facts, or theorems. Also, I will be thinking mostly of *finite models* in what follows, not as a point of principle, but to avoid the usual additional complexities in lifting simple intuitions to more complex infinite settings.

2 Semantic information

2.1 Basic epistemic logic Perhaps the roughest form of representing information is the common-sense picture of a set of still live possible candidates for the actual world, an 'epistemic range' that shrinks when new information comes in. These sets are models for the language of epistemic logic whose key operator $K_i\varphi$ says φ holds in all accessible alternative worlds for agent *i*, or in other terminology: the agent has the 'semantic information' that φ is true. [5]

When such models are used to describe some informational scenario taking place in reality, one assumes there is a unique true state of affairs, that can be marked as the 'actual world' in the model – even though no agent needs to know which possibility is in fact the actual one.

The valid principles of reasoning with semantic information are those of the well-known modal logic S5 for each separate agent *i*. Note that this setting has no non-trivial valid laws that relate the information of different agents: any significant dependencies must come from information channels or other forms of alignment between agents.

It is easy to criticize this setting for its extreme simplicity, but a range of options represents a well-chosen mathematical abstraction that occurs across the sciences and even philosophy.

In this paper, we consider just one agent, and so indices *i* will be dropped [6] – although many informational scenarios essentially involve many agents. Another issue that we leave aside is the connection between semantic information in the sense of epistemic logic and the usual philosophical notions of knowledge: see Holliday 2012 for a sophisticated modern treatment.

2.2 Dynamics of hard information The simplest informational events in this setting are 'hard announcements' or observations $!\varphi$ of the fact that the proposition φ is true in the actual world. What this new information does is retain those worlds in the given epistemic model that currently satisfy φ, while eliminating those worlds that do not satisfy φ. This is arguably the simplest common-sense picture of obtaining new information, and it can be modeled technically as taking the current model $\boldsymbol{M}, s$ to the new model $\boldsymbol{M}|\varphi, s$.

Updates of this sort can be tricky if the new information φ is not just factual (being just a Boolean combination of atomic facts), but contains epistemic modalities. [7] However, there is a complete logic for this dynamics of semantic information, which can be brought out in a suitable two-tiered syntax. We introduce *announcement actions* $!\varphi$ for each formula: one can also think of these as public observations, or other ways of receiving hard information. In constructing formulas, we allow Boolean operations, the *K*-modality, and also a new dynamic modality $[!\varphi]\psi$ saying that after a truthful update with φ, ψ is the case in the updated model.

Theorem The dynamic epistemic logic of update with hard information is axiomatizable.

Proof This completeness result is easy to prove, and the heart of the axiomatization is a 'recursion law' describing what new knowledge obtains after information is received:

$$[!\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[!\varphi]\psi)$$

Such laws are the basic principles describing the dynamics of information change. ■

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[5] Each agent *i* gets an equivalence relation $\sim_i$ whose clusters encode its possible information ranges.

[6] In this case, we identify the *K*-operator with the 'universal modality' over all worlds in the model.

[7] A well-known example: if the agent lacks information if *p*, but *p* is the case (that is, $\neg Kp \wedge p$ is true), then announcing this fact will make $\neg Kp \wedge p$ false, as Kp has become true in the updated model $\boldsymbol{M}|p$.

This may suffice as a first level of information. Knowledge grows as agents receive successive new inputs and make matching updates. In some scenarios, this may zoom in to just the actual world, representing a state of common knowledge for the agents. However, one can also study infinite processes of endless learning. For further information on this dynamic-epistemic methodology, including delicate private informational actions with more complex updates, one can consult van Ditmarsch, van der Hoek & Kooi 2007 or van Benthem 2011.

3 Plausibility and belief

3.1 Plausibility ordering Mere semantic ranges ignore the fact that often, not all candidates for the actual world are on a par. Say, in planning my upcoming trip, I may consider some possibilities more plausible than others, and may well confine attention to the most plausible cases. This can be modeled in an enrichment of the earlier models with additional structure. We now expand epistemic models $(W, \sim, V)$ to *epistemic plausibility models* $(W, \sim, \leq, V)$ where $\leq$ is an ordering of 'relative plausibility'. For the purposes of this paper, we assume that $\leq$ is a reflexive transitive order, and nothing else, leaving room for genuinely incomparable worlds. Moreover, to avoid technicalities that are not germane to our main theme, we assume that the plausibility order is the same at each world. I.e., in the sense of semantic information, the agent knows her own plausibility ordering, and the beliefs that she has, based on it.

This fine-structure of information at once suggests a richer repertoire of attitudes for agents. In particular, it makes sense to look only at the most plausible worlds in the current range, and define *belief* $B\varphi$ as truth of φ in all of these worlds. This is a sort of less-demanding semantic information 'to the best of one's knowledge'. As for a logic of belief construed in this way, it is better to also introduce a notion of *conditional belief* $B^{\psi}\varphi$ in φ conditional on already being in the set of ψ–worlds. This logic satisfies exactly the principles of conditional logic over Lewis-style models, with the proviso that the ordering need not be connected.

Plausibility order may look like a technical device, but there are interesting issues in interpreting it. At the start of a process of inquiry, a plausibility order may be viewed as a 'prior', a set of expectations, including conditional beliefs that say what we would believe were certain new information to arrive. [8] But over time, plausibility order can be modified by informational events (see below), and hence we can also think of the current order as a sort of rough record of past experience. Finally, it is this order we tend to use for deciding on new actions, so plausibility models are at the same time a record of the past and a guide to the future.

3.2 Plausibility dynamics What sort of informational events can affect epistemic plausibility models? The earlier public announcements still make sense, and validate a complete logic.

Theorem The logic of belief change under hard information is axiomatizable.

Proof In particular, the key recursion laws for announcement are as follows:

$$[!\varphi]B\psi \leftrightarrow (\varphi \rightarrow B^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]B^{\alpha}\psi \leftrightarrow (\varphi \rightarrow B^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi)$$

Note how conditional beliefs serve here to describe beliefs formed after new information. ■

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[8] In other settings, this prior ordering amounts to a 'learning method' telling the agent how to respond to new information (Baltag, Gierasimczuk & Smets 2011).

Events of hard information can interact with beliefs in surprising ways.

Example Misleading with the truth
Consider a model with an actual world 1 plus two more possible worlds, ordered as follows qua plausibility: $1 \leq 2 \leq 3$. Let proposition p be true in 1 and 3, but not in 2, and let q be true at 3 only. In this model, the agent believes that p, because p is true in the most plausible world 3. Now announce the true fact that $\neg q$. This will eliminate world 3, leaving an ordering $1 \leq 2$ where the agent falsely believes that not p. Despite this potentially surprising feature of update, a logic with the above laws will keep reasoning about such scenarios straight. ■

However, when we have more structure, there is often a richer repertoire of relevant actions. In particular, in addition to the 'hard information' in events $!\varphi$, there is 'soft information' that does not rule out any possibility, but changes plausibility order for the existing possibilities.

A well-known example is *radical upgrade* $\Uparrow\varphi$, whose action is as follows. We put all φ–worlds on top in the ordering, above all $\neg\varphi$-worlds, while inside these two zones, the old plausibility order is retained. Reasoning gets more complex, but yields to techniques like before.

Theorem The logic of belief change under radical upgrade is completely axiomatizable.

Proof This time, the key recursion laws are more involved. Perhaps the most complex principle is the valid equivalence for conditional belief under radical upgrade. [9] ■

But softness comes in different kinds. A less radical way of taking new information is the *suggestion* $\#\varphi$, whose effect is merely to remove any links that run from a φ–world to some more plausible $\neg\varphi$-world. This forces us to take best worlds inside the φ–zone seriously in our beliefs, though formerly best $\neg\varphi$–worlds are still in play, too. Again, the dynamic logic of belief change is completely axiomatizable, but we will not state its key recursion law here.

There is a wide spectrum of order-changing operations behind these three specific examples of update actions, that can be found in many places: not just as plausibility change, but also as operations that change preferences (Liu 2011) or relevance (van Benthem 2015). [10] Indeed, there exist general methods for defining updates and deriving recursion laws in all these cases, for which we refer to van Baltag & Smets 2006, van Benthem & Smets 2015.

Remark Factual or epistemic-doxastic propositions
One reason why the above recursion laws get complex syntactically is having to deal with the possible occurrence of knowledge and belief operators inside incoming new information, which can lead to surprising truth value changes under update. While this sort of 'higher information' is realistic in actual communication, it is also something of a technical side issue in this paper. Therefore, in what follows, we make a sweeping simplification:

we will restrict attention to factual propositions only

– or stated differently, we decontextualize propositions denoted by our epistemic-doxastic syntax, by thinking of them merely as absolute subsets of the model. This huge simplification

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[9] This law reads as follows: $[\Uparrow\varphi]B^{\alpha}\psi \leftrightarrow (\Diamond(\varphi \wedge [\Uparrow\varphi]\alpha) \wedge B^{\varphi \wedge [\Uparrow\varphi]\alpha}[\Uparrow\varphi]\psi) \vee (\neg\Diamond(\varphi \wedge [\Uparrow\varphi]\alpha) \wedge B^{[\Uparrow\varphi]\alpha}[\Uparrow\varphi]\psi)$, where $\Diamond$ stands for the existential modality 'in some world in the current epistemic range'.

[10] Indeed, even hard information can be taken as an order change operation, witness the 'link cutting' versions of public announcement where the φ– and $\neg\varphi$–zones are made disjoint.

will play everywhere in what follows, and the reader should remain aware of it. Of course, a final version of our account should, and can, deal with the more sophisticated full version.

3.3 Interplay of statics and dynamics It may look as if dynamic events of information flow and their matching update acts are just additions to a given static base logic of attitudes, that is usually on the shelf already in philosophical logic. But the dynamic component can also affect the design of the static base. Here are two examples.

Our earlier examples of misleading with the truth suggests a notion of robust or *safe belief* $SB\varphi$ that remains stable under whatever true new information comes in. It is easy to see that this amounts to the following notion, at least in *connected* plausibility orders:

$$\boldsymbol{M}, s \models SB\varphi \text{ iff } \textit{for all } t \textit{ with } s \leq t, \boldsymbol{M}, t \models \varphi$$ [11]

On connected plausibility orders, safe belief is a new attitude for agents to have, that lies in between ordinary belief and knowledge qua informational strength.

In addition to its intuitive attraction, it also has the technical advantage of allowing us to define absolute and conditional belief. This is shown in the following two valid equivalences, that also work on arbitrary pre-orders, where *K* is the earlier knowledge modality over the whole epistemic range, and *<SB>* is the existential dual modality of *SB*:

Fact The following laws hold for knowledge, belief, and safe belief:

$$B\varphi \leftrightarrow K\langle SB\rangle SB\varphi$$

$$B^{\psi}\varphi \leftrightarrow K(\psi \rightarrow \langle SB\rangle(\psi \wedge SB(\psi \rightarrow \varphi))$$

Given these definitions, it is often easier to state recursion laws for informational actions and safe belief, since others will be derivable. As an illustration, here are the laws for the three operations that we discussed in the above:

Fact The following recursion laws hold for factual propositions:

$$[!\varphi]SB\psi \leftrightarrow (\varphi \rightarrow SB(\varphi \rightarrow \psi))$$

$$[\Uparrow\varphi]SB\psi \leftrightarrow (\varphi \wedge SB(\varphi \rightarrow \psi) \vee (\neg\varphi \wedge \Diamond\varphi \wedge SB(\neg\varphi \rightarrow \psi) \wedge K(\varphi \rightarrow \psi)) \vee (\neg\Diamond\varphi \wedge SB\psi)$$

$$[\#\varphi]SB\psi \leftrightarrow (\varphi \wedge SB(\varphi \rightarrow \psi)) \vee (\neg\varphi \wedge SB\psi)$$

Another example of influence from the dynamic component to design of the static base language of attitudes is the notion of a *conditional*, so central in much of Mike Dunn's work:

When we say "if φ, then ψ", there is an issue about the force of the "if". How are we to imagine the hypothetical situation that is introduced? The standard view of making a hypothesis fits public announcement $!\varphi$: we restrict attention to the φ–worlds and work inside that restricted space. But sensitized to the above dynamic distinctions, we can also make the hypothesizer "if" weaker in force, letting it just promote the φ–worlds to top position (as in radical upgrade $\Uparrow\varphi$), or even making all of them relevant cases for inspection, as in a suggestion $\#\varphi$.

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[11] This is of course just the standard universal modality over a binary order. However, on pre-orders that allow incomparable worlds, safe belief in our dynamic sense refers to all worlds that do not strictly precede the current world in the ordering (van Benthem & Pacuit 2011). We will ignore this technicality in what follows in this paper, as it does not affect our main concerns.

This gives us three conditionals that we will read as saying that after the relevant way of assuming φ, the agent believes that the conclusion ψ is true:

$$\varphi \rightarrow^{!} \psi, \qquad \varphi \rightarrow^{\Uparrow} \psi, \qquad \varphi \rightarrow^{\#} \psi$$

These three notions give us more refined ways of thinking about conditional reasoning, and more generally, of generalized notions of consequence. It is not our aim to develop this line in depth here, but a few observations may help illustrate the situation.[12]

Fact The three conditionals validate different laws.

Proof We have that $\varphi \rightarrow^{!} K\varphi$, but not for the other two conditionals. We have that $\varphi \rightarrow^{\Uparrow} B\varphi$, but the latter principle does not hold for the suggestion conditional. ■

Fact The given three conditionals validate the same structural rules.

Proof It is easy to see that the first two conditionals refer to the same worlds, since the maximal worlds within the φ–zone are the same as the maximal worlds overall when the φ–zone lies on top in the model. To see the third equality, note that the conditional $\varphi \rightarrow^{\#} \psi$ refers to maximal worlds of two kinds: maximal within the φ–zone, or maximal worlds overall that are $\neg\varphi$. It says that all of these satisfy ψ. But this can be stated equivalently as follows:

$$\varphi \rightarrow^{!} \psi \wedge T \rightarrow^{!} \psi$$

It is easy to check that this notion satisfies the structural laws of the basic conditional logic. ■

4 Connecting the two levels

Epistemic models as mere semantic ranges and plausibility models are two different ways of representing information, one richer than the other. Although this extension is clear, we give a brief discussion of some general issues that will return later on in this paper.

First, there are systematic connections between the two levels. Moving from poorer to richer, we can *embed* epistemic models into the realm of plausibility models as special cases where all worlds are equiplausible, or the same for this purpose: incomparable qua plausibility. Let us call this the functor *equi(**M**)* that takes epistemic models to plausibility models. Going in the opposite direction, there is also a natural notion of *projection*, by a functor *forg(**M**)* that forgets the plausibility ordering and just returns the bare epistemic domain.

The embedding and projection functors have this obvious connection:

$$forg(equi(\boldsymbol{M})) = \boldsymbol{M}$$

What does not hold is, for plausibility models ***N***, that *equi(forg(**N**)) = **N***. Structure that has been lost cannot be retrieved faithfully by some uniform stipulation.

Corresponding to these maps, there are *translations* between the two levels for representing information. These satisfy the typical 'adjointness' scheme for translations in logic:

$$\boldsymbol{M} \models trans(\varphi) \text{ iff } F(\boldsymbol{M}) \models \varphi, \qquad \text{for all relevant models } \boldsymbol{M} \text{ and formulas } \varphi$$

where *F* is a model transformation, and *trans* a matching translation between languages.

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[12] In what follows, again, we only consider factual propositions. Otherwise, things get more complex.

In our specific case, the translation corresponding to the functor *equi* is as follows:

> we leave atomic propositions, Boolean operations, ad epistemic K the same,
> and replace B by K.

It is easy to show that this yields the above equivalence, since all worlds in plausibility models of the form *equi(**M**)* are maximal. But what this really says of course is that belief does not mean anything new in such special models. The same trivialization extends to safe belief: it, too, collapses into knowledge when the ordering is uniform equiplausibility.

The extra richness of the plausibility level shows, e.g., in the dynamics. It is possible to also translate informational actions from the epistemic to the plausibility level, since we can use public announcements on plausibility models just as on epistemic models, as we have seen. Thus, the translation also extends to modalities $[!\varphi]$.

Things are more complex on the other side: not every natural transformation on plausibility models has an exact counterpart on bare epistemic models. Operations like $\Uparrow\varphi$, $\#\varphi$ will normally turn models of the form *equi(**M**)* into models that are not of this form. Stated in other terms, we cannot faithfully translate formulas $[\Uparrow\varphi]\psi$ into purely epistemic formulas.

But then, the most important direction for us is projection from the richer to the poorer level. What translation takes the epistemic language into the doxastic one according to the above scheme? As fits a case of straightforward extension of models, this translation is just the *identity of formulas* in the dynamic-epistemic language of public announcement.

What is of slightly greater interest to see is that this cannot be extended further. For instance,

Fact There is no translation for belief via the projection functor *forg*.

Proof Consider the same world in two plausibility models that have the same worlds and atomic valuation but differ in their plausibility structure, making Bp true in one model and false in the other. The functor *forg* will take these models to the same epistemic model, and so, the translations of Bp and $\neg Bp$ cannot differ in truth value, whereas they should. ■

Next consider dynamics at the level of plausibility models, say, a soft informational change such as a radical upgrade $\Uparrow\varphi$. It is easy to see that, this time, there is a matching transformation at the epistemic level, namely, just the identity map on models. But this trivial harmony shows precisely what is going on: the epistemic level cannot detect mere plausibility changes. Thus, the latter reordering acts are genuine 'internal' operations in the doxastic realm, leaving no significant 'external' traces that can be tracked epistemically.

None of the preceding observations are deep at all, but they set the scene for more interesting comparisons across informational levels to be made later.

5 Evidence

Plausibility order between worlds does not record which reasons determined these relative differences among the available candidates. A more fine-structured approach is that of van Benthem & Pacuit 2011, where evidence is modeled as a family of subsets of the domain, viewed as information obtained from various sources that may be consistent, but could also contradict each other. Such an array of evidence allows agents to form beliefs, but it also gives them more structure to work with when having to supply reasons, or give up beliefs. In this section, we survey evidence modeling as our third richer level for representing information.

5.1 Evidence models An *evidence model* is a set of possible worlds, viewed as an epistemic range for a K-modality as before, but now with an added family $\mathcal{E}$ of non-empty subsets.[13] We assume for simplicity that evidence sets are uniformly available in the model, not depending on particular worlds. This may be considered a special case of 'neighborhood models' for modal logic. The most straightforward modality in this setting is then interpreted as follows:

$$\boldsymbol{M}, s \models \Box\varphi \text{ iff } \textit{there exists a set } E \in \mathcal{E} \textit{ with } \boldsymbol{M}, t \models \varphi \textit{ for all } t \in E$$

The modality $\Box\varphi$ expresses the existence of evidence available to the agent that supports the proposition φ. As we make no special assumptions on the available evidence, it is easy to see that this modality is only upward monotonic, but it distributes neither over conjunctions nor over disjunctions. For instance, having $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ valid would assume that all available evidence is consistent, and has already been 'processed' to include combinations.

One notion that assumes such processing is belief, viewed as what we can safely conclude by combining our evidence to the utmost. Let a *maximal body of evidence* $\mathcal{X}$ be a family of sets in $\mathcal{E}$ for which all finite intersections are non-empty, and that cannot be extended with further evidence sets to retain this property. In finite models, we can then look at the intersection of the whole family $\mathcal{X}$ as what follows on the basis of this body of evidence. We then define

$$\boldsymbol{M}, s \models B\varphi \text{ iff } \boldsymbol{M}, t \models \varphi \textit{ for all } t \textit{ in all intersections of maximal bodies of evidence.}$$[14]

This can be generalized to conditional belief $B^{\psi}\varphi$ in a fitting manner using maximally consistent sets with respect to including the set $[[\psi]]$. Further attitudes make sense as well, such as *'entertaining* φ*'* in the sense of φ being true throughout the intersection of at least one maximal body of evidence.[15] Yet other evidence-based attitudes will be mentioned below.

Belief as defined here satisfies the axioms of a normal modality, and in particular, we do have the validity of $(B\varphi \wedge B\psi) \rightarrow B(\varphi \wedge \psi)$. It is not trivial to axiomatize the resulting logic that combines both normal and non-normal monotonic attitudes, witness the completeness proof in van Benthem, Fernandez & Pacuit 2012. Such technical details will not concern us here.

5.2 Evidence dynamics As with our earlier two levels for representing information structure, evidence supports a natural dynamics of change. Among its major operations, we can still perform public announcements $!\varphi$ restricting a current model to the subset of all φ–worlds.

Theorem The dynamic logic of public announcement over evidence models is axiomatizable.

Proof The key recursion law for the belief modality is straightforward, though it requires dealing with conditional belief, as we did on plausibility models. Moreover, the law for $[!\varphi]\Box\psi$ needs a new notion of 'conditional evidence' $\Box^{\varphi}\psi$ that we do not spell out here. ■

But then, of greater interest are new operations that are more typical for the evidence setting.

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[13] It would also make sense to model the sources of the evidence, but we ignore this aspect here.

[14] In infinite models with infinite sets E, this stipulation must and can be modified.

[15] Note that this is not the existential dual $\neg B\neg\varphi$ of belief as we have defined it above. Note also that, in models with conflicting evidence, we can 'entertain' contradictory propositions at the same time.

One particularly obvious example is *evidence addition* $+\varphi$. What this model transformation does is add the set $[[\varphi]] = \{t \in W \mid \mathbf{M}, t \models \varphi\}$ as a new set to the current evidence family $\mathcal{E}$.

Theorem The dynamic logic of evidence addition is completely axiomatizable.

Proof This time we give a bit more detail, since more is involved here than in the dynamics of simpler structures such as plausibility models. We start with the evidence modality:

$$[+\varphi]\Box\psi \leftrightarrow (\Box\psi \vee K(\varphi \rightarrow \psi))$$

The rationale for this will be clear, though this particularly simple form only holds for factual propositions. For the belief modality, things get more complex. After we have added $[[\varphi]]$ as a new piece of evidence, there are two sorts of maximal bodies of evidence. The first consists of a family of evidence sets in the old model that is maximally consistent with respect to adding $[[\varphi]]$, which is exactly the basis for conditional belief. The second sort is maximal bodies of evidence in the old model that also satisfy the condition that their intersection is disjoint from $[[\varphi]]$. Now we get this recursion law, again modulo our restriction to factual propositions:

$$[+\varphi]B\psi \leftrightarrow (B^{\varphi}\psi \wedge B(\neg\varphi \rightarrow \psi))$$

It is easy to see that this reduces, much as in our discussion of static suggestion-conditionals $\varphi \rightarrow^{\#}\psi$, to the following equivalence:

$$[+\varphi]B\psi \leftrightarrow (B^{\varphi}\psi \wedge B\psi)$$ [16]

The analogy with suggestion updates is no coincidence, and it will return below. ■

But there are many further natural and appealing operations that affect our current evidence. Indeed, in the present setting, what used to be the basic notion of public announcement $!\varphi$ can be deconstructed into two independent operations: (a) adding evidence that φ is the case, (b) removing all old evidence that supported $\neg\varphi$. The latter notion can be defined by itself.

Deleting evidence $-\varphi$ transforms the current model as follows: all old evidence sets $E \in \mathcal{E}$ that are included in the set $[[\neg\varphi]]$ will be removed. Of course, there can be many reasons for such a removal: one can think of 'retraction' as in belief revision theory (van Benthem & Smets 2015), or of 'forgetting' in some other sense, perhaps just that of cognitive decay.

For a final example of a natural operation on evidence, consider the *K*-axiom for conjunction that failed for the evidence modality. One might say that, given consistent evidence for φ and evidence for ψ, we do have evidence for $\varphi \wedge \psi$, just by combining the two pieces of evidence. Now there are a few technical difficulties in making this work, but clearly, there is a natural operation $\cap$ of *intersecting* all mutually consistent evidence sets to form new evidence sets.

Of greater interest here is that intersection is a sort of *internal* processing or re-arrangement of evidence, unlike addition or deletion. We will give a more precise sense to this notion of internality later on. Intuitively, internal operations are one more illustration of what can be done at a richer level that need not show up at coarser levels of information structure.

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[16] The recursion laws stated in van Benthem & Pacuit 2011 for these and the following operations in this section are more complex syntactically because they are meant to hold also for non-factual propositions. They have to deal with maximal sets of old evidence that are consistent with some given propositions while excluding others. In a full treatment of our themes, we would have to work in this more complex framework – although this is mainly a routine generalization.

5.3 Dynamic-static interactions once more As with plausibility models, the dynamics of evidence models suggests new notions at the level of static attitudes as well. Van Benthem & Pacuit 2011 have several examples of new notions of evidence-based conditional belief, for which axiomatizing the resulting richer doxastic base language is still an open problem. [17]

6 Representation and translation

Again there are connections between our different levels for representing information.

6.1 Projection Evidence structure induces plausibility structure via a natural stipulation that can be found in many areas, from topology to Chu spaces (van Benthem 2000). We require that more plausible worlds satisfy all the evidence satisfied by the less plausible ones

$s \leq t$ iff *for all* $E \in \mathcal{E}$*: if* $s \in E$*, then* $t \in E$ [18]

We will call this the map *ord(**M**)* taking evidence models to plausibility models, where we do not change domains or propositional valuations. There is a natural matching translation here for logical languages. As before, it runs in the opposite direction, from the language of plausibility models to that of evidence models. Actually, the main clauses are simple:

Knowledge *K* goes to knowledge *K,*
Belief *B* goes to belief *B*, and the same is true for conditional belief.

One may find the latter clause unsurprising, as our notion of evidence-based belief mirrored standard maximality notions in relational models. Indeed, any projection map allows us to '*retract'* notions from the plausibility level to the evidence level in a systematic manner.

For another case of such retraction or borrowing of notions, what about the earlier notion of *safe belief*? This now gets related to a natural notion of what may be called *'reliable evidence'*. For any world s, let E_s be the family of sets $\{E \in \mathcal{E} \mid s \in E\}$. This is a consistent family with a non-empty intersection. We now define *evidence-based safe beliefs* φ at s as the propositions φ true at each world in the intersection of $\{E \in \mathcal{E} \mid s \in E\}$. Of course, agents need not know the world they are in, so what is reliable in this objective local sense may be unknown to them.

Of course, we can also look at notions at the evidence level and see if they have plausibility counterparts in this sense. For the most obvious example, this fails.

Fact There is no plausibility counterpart via the map *ord* to the evidence modality.

Proof The reason is as follows. Consider a universe $W = \{1, 2, 3\}$ with two evidence sets {1, 2}, {3, 2} where only the world 2 satisfies the proposition *p*. In this model, the formula $\Box p$ does not hold. Now consider the same model with one evidence set added, viz. {2}. This additional intersection of earlier evidence does not change the induced plausibility order. However, in the new model, $\Box p$ is true, and so cannot have been definable in the plausibility language. ■

6.2 Embedding There is also a natural map *evi(**M**)* running in the opposite direction, sending plausibility models to evidence models. A moment's reflection will show that it should work as follows (for instance, compare the semantics of intuitionistic logic):

the evidence sets are all upward closed sets in the plausibility ordering ≤

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[17] New logics for generalized conditional evidence are developed in van Benthem, Bezhanishvili & Yu 2015.

[18] It is instructive to see this work in concrete cases, and we invite the reader, when following the arguments to come, to draw set diagrams and their corresponding induced plausibility orders.

This embeds plausibility models as special evidence models. In particular, the intersection of any two upward closed sets is upward closed, so the evidence sets are automatically closed under intersections.[19] This also shows in what happens when we repeat the level maps:

$$ord(evi(\boldsymbol{M})) = \boldsymbol{M}, \qquad \text{but not } evi(ord(\boldsymbol{M})) = \boldsymbol{M}$$

Nevertheless, *evi(ord(**M**))* is close to ***M***, being its closure under non-empty intersections.

Again there is a syntactic translation matching the embedding *evi*. It is correct for the identity translation of *K* and *B* from the language of evidence models to that of plausibility models.

Given these translations, there is a natural issue whether they extend to dynamic modalities for changing plausibility order or evidence structure. This kind of correlation will be the topic of our next section, albeit with a slightly different emphasis.

7 Tracking evidence via plausibility

One particular interest we stated at the beginning was how the dynamics of information flow at one level might be tracked at another level. We will look a bit more closely into this now, finding both positive and negative results, and we will discuss what these findings mean.

7.1 Tracking diagrams Let us start with a case of harmony between operations.[20]

Fact For all evidence models ***M*** and factual propositions φ, $ord(\boldsymbol{M}!\varphi) = ord(\boldsymbol{M})!\varphi$.

Proof This follows easily from the definition of public announcement on evidence models: we restrict the domain to the worlds that satisfy φ, and we restrict all evidence sets to this subdomain (dropping the empty set if it arises in this manner). ■

We can picture this situation in the following 'tracking diagram':[21]

$$\begin{tikzcd}
Ord(\boldsymbol{M}) & !\varphi & ord(\boldsymbol{N}) \\
\boldsymbol{M} \arrow[u] & !\varphi & \boldsymbol{N} \arrow[u]
\end{tikzcd}$$

In a diagram like this, we say the upper update map *tracks* the lower one at the richer level.[22]

One might think that this harmony holds here only because public announcement works in much the same way at both levels. But here is a less straightforward example. Evidence addition is tracked by our earlier operation of 'suggestion' on plausibility models.

Fact For all evidence models ***M*** and factual propositions φ, $ord(\boldsymbol{M}+\varphi) = ord(\boldsymbol{M})\#\varphi$.

Proof The essential observation is that, given our definition of the induced order, adding the denotation of φ as an evidence set, the order changes precisely as described in the operation $\#\varphi$. Points where φ holds now satisfy evidence that is not shared by any $\neg\varphi$–point. ■

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[19] In this section we side-step a few technicalities with the empty set being always upward-closed.

[20] The first two harmony results to follow were first stated for preference order in Liu 2011.

[21] Again we assume for simplicity that propositions are subsets of models here, disregarding technical issues of translation between complex formulas at the different levels. In a more detailed treatment, we would have to compare correlated updates of the form $!\varphi$ and $!translation(\varphi)$.

[22] If we think of updates just as generic maps, the way one normally does, tracking really applies to *families* of maps on a whole current level of models. We will leave this matter of uniformity behind diagrams implicit, but it is the way one should really think of our discussion in this section.

To add one more example, sometimes there is harmony, but to see it, we need operations that are new in the literature. For instance, we will now define an evidence counterpart tracked by our earlier plausibility transformation of radical upgrade $\Uparrow\varphi$, again with φ viewed as a subset:

Define the following map $up(\varphi, \boldsymbol{M})$ on evidence models $\boldsymbol{M}$ with a subset φ:

(i) if $E \in \mathcal{E}$ has $E - \varphi$ non-empty, then replace it by $E \cup \varphi$,

(ii) if $E \in \mathcal{E}$ has $E \cap \varphi$ non-empty, then add $E \cap \varphi$.

Fact For all evidence models $\boldsymbol{M}$ and factual propositions φ, $ord(up(\varphi, \boldsymbol{M})) = ord(\boldsymbol{M})\Uparrow\varphi$.

Proof There are four cases for two points x, y: they can satisfy (a) φ φ, (b) φ $-\varphi$, (c) $-\varphi$ φ, *or* (d) $-\varphi$ $-\varphi$. In case (a), if $x \leq y$ initially, then all new evidence satisfied by x holds for y, as the modifications do not affect what happened inside the φ–area. If not $x \leq y$ initially, then new evidence of type (i) no longer distinguishes, but new evidence of type (ii) will. In case (b), the new evidence of type (ii) rules out more plausible $\neg\varphi$–worlds. In case (c), all new evidence true for $\neg\varphi$–worlds also holds for all φ–worlds. Finally, in case (d), the order stays the same for $\neg\varphi$–worlds: only new evidence (i) is relevant which has not changed within this zone. ■

In the background of these three tracking diagrams lie a number of general observations.

Fact For every map f between plausibility models, there exists a map g between evidence models that is tracked by f. One can define g on $\boldsymbol{M}$ as follows: $evi(f(ord(\boldsymbol{M}))$. ■

The choice of g is not unique, any value $g(\boldsymbol{M})$ that induces the same plausibility order will do. If we want uniqueness, we need to sharpen up the definition of our levels and mappings between them. The general background to this and other observations lies in category theory, but we leave these mathematical issues of uniform presentation for another occasion.

There is a general construction behind the preceding argument, our observations are no coincidence. Consider operations on plausibility models defined in the 'flat program format' of van Benthem & Liu 2007, being unions of $?(\neg)\varphi\,;\leq\,;?(\neg)\varphi$ and $?(\neg)\varphi\,;T\,;?(\neg)\varphi$.

Fact There is an algorithm that takes any plausibility transformation in flat program format and returns an evidence transformation tracked by it.

The argument is by a brute-force enumeration that we omit in this version.

One reason why tracking is so appealing is that we can consider the lower-level update as giving a sort of 'best approximate content' for the information provided by the upper-level update. For instance, think of the visible content of some hidden higher-order operation.

Another perspective on tracking diagrams is in terms of *translation*. They allow us to extend the earlier translation from a static evidence language for agent attitudes to the matching plausibility language with dynamic modalities for definable update operations. However, this technical perspective is not our main concern here, and we continue with tracking per se.

7.2 Non-trackable operations But the more interesting question runs in the opposite direction from that of the above algorithm. Given a map g between two evidence models, is it '*trackable*', in the sense that there exists a map f that tracks their plausibility projections? It is not obvious that this always needs to be the case, so let us first give a criterion.

Fact A map g is trackable iff it has the following property: on evidence models with the same *ord* projection, it delivers values with the same *ord* projections.

Proof First consider necessity. If g is tracked by some map f, then, $ord(g(\boldsymbol{M})) = f(ord(\boldsymbol{M}))$. Now let $ord(\boldsymbol{N}) = ord(\boldsymbol{M})$. Then we have that $ord(g(\boldsymbol{N})) = f(ord(\boldsymbol{N})) = f(ord(\boldsymbol{M})) = ord(g(\boldsymbol{M}))$. Conversely, if the stated invariance holds, it is easy to see that it is well-defined to let a tracking function $f(\boldsymbol{M})$ on plausibility models yield the model $ord(g(evi(\boldsymbol{M}))$ as its value. ■

Using this we show that some natural operations on evidence models are not trackable.

Fact The deletion operator $-\varphi$ as defined earlier is not trackable.

Proof Let the domain of $\boldsymbol{M}$ be the set{1, 2, 3, 4}, with evidence sets {1, 2}, {2, 3}, {3, 4}, {4, 1), and proposition φ holding only at 1, 2. The induced plausibility model $ord(\boldsymbol{M})$ consists of four incomparable points. The operation $-\varphi$ turns $\boldsymbol{M}$ into an evidence model with only the evidence sets {2, 3}, {3, 4}, {4, 1). In its induced plausibility model $ord(\boldsymbol{M}-\varphi)$, the points 2, 4 remain incomparable, but now $1 \leq 4$ and $2 \leq 3$.

Now let the evidence model $\boldsymbol{M}^+$ have the same domain and valuation as $\boldsymbol{M}$, but let there also be two additional evidence sets {1, 3}, {2, 4}. This induces the same plausibility model as $\boldsymbol{M}$. However, when we apply the operation $-\varphi$ to $\boldsymbol{M}^+$, deleting {1, 2}, due to the additional evidence sets remaining, the induced plausibility model $ord(\boldsymbol{M}^+)$ still consists of four incomparable points. This observation refutes trackability by the above criterion. ■

Here is another example. Consider the above evidence operation that was tracked by radical upgrade. Now consider only one half, namely the following map:

> $shift(\varphi, \boldsymbol{M})$ sends $E \in \mathcal{E}$ with $E - \varphi$ non-empty to $E \cup \varphi$

Fact $shift(\varphi, \boldsymbol{M})$ is not trackable.

Proof Let $\boldsymbol{M}$ consist of {1, 2, 3}, with evidence sets {1, 2}, {2}, {1, 3}, and φ only true at 1, 2. The plausibility model $ord(\boldsymbol{M})$ has 1, 2 incomparable, while $3 \leq 1$.The map $shift(\varphi, -)$ turns $\boldsymbol{M}$ into an model with evidence sets {1, 2}, {2}, {1, 2, 3}. Its induced plausibility model has $1 \leq 2$, $3 \leq 1$, $3 \leq 2$. Let $\boldsymbol{M}^+$ have the same domain and valuation, but with one more evidence set {1}. This induces the same plausibility model as $\boldsymbol{M}$. However, applying $-\varphi$ to $\boldsymbol{M}^+$ yields evidence sets {1, 2}, {1}, {2}, {1, 2, 3}, and the induced plausibility model has 1 and 2 incomparable. ■

Thus it was the balance between the two earlier clauses that induced the tracking – and what looked like a simpler operation on evidence is in fact more complex at the plausibility level. The preceding results show that operating on evidence is a delicate matter, and it may lack obvious plausibility counterparts, as we have lost the specific sets generating the ordering. [23]

Here is one more illustration, showing the wealth of natural transformations of evidence that emerge on evidence models, even in a very simple set-theoretic format.

Excursion One more case of deletion.
Let's not delete evidence E implying the proposition φ, but do something weaker:

> $\sim\varphi$ 'defuses' such evidence E by replacing it with $E \cup \neg\varphi$.

This operation is not trackable either, as can be seen by our earlier techniques.

On the other hand, somewhat similar-looking operations in this basic set-theoretic – or if one wishes: Boolean – format can in fact be tracked.

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[23] Note also that our non-tracking arguments typically involved evidence models that are not *closed under intersections*. What happens to tracking when we only consider such special evidence models?

One example is this map $/\varphi$ reminiscent of a public announcement update:

> $/\varphi$ replaces each evidence set E with $E \cap \varphi$ if the latter set is non-empty, otherwise it just deletes the set E.

This time, a tracking map in our flat program format is $R := (?\varphi; R\ ;\ ?\varphi) \cup (?\neg\varphi; U\ ;\ ?T)$.[24]

One can view this as a very strong deletion where we no longer care about the $\neg\varphi$–area. ■

One can see non-trackability as a problem, but in our view the opposite is the case. It shows that the richer world of evidence models suggests natural operations that are sui generis.[25]

7.3 Internal operations Next, trackability is not the same as having externally measurable effects. Consider the earlier 'internal' operations on evidence models that did not change the induced plausibility model. Our prime example was intersection of evidence sets.

Fact Evidence combination is tracked by the identity map on plausibility models.

This may be considered a disappointment, but we can also view trackability by the identity map as a defining feature of what we mean by internal operations.

What this still leaves is the issue of what internal operations are good for. We can view them partly as inferences (the combination rule is a sort of conjunction inference), and in that sense, their value may become apparent only at yet higher syntactic levels of representing information (van Benthem & Martinez 2008, van Benthem & Velazquez Quesada 2010).

But we can also think of internal operations as ways of rearranging the current evidence without disturbing the plausibility order. For instance, in finite models, we can prune a given evidence family to smallest families inducing the same order. Or, we can choose new evidence sets by combination that induce the same order in simpler or more perspicuous ways.

7.4 Achieving more generality It is easy to multiply concrete instances of the above tracking arguments and non-tracking counter-examples. Many update functions are in fact definable in simple formats, such as flat PDL programs for plausibility change, or the simple Boolean functions with Boolean case distinctions for evidence change that we considered in the above. It would be of interest to classify all tracking links between operations in these formats, and replace the art of drawing counter-examples by some more algorithmic procedure.

8 Logical aspects I: Languages and invariance relations

In the paper so far, the term 'levels' has been used quite loosely. But really, a level of structure does not just arise by specifying a similarity type of models, but also by giving transformations or invariance relations between what are one takes to be 'the same' structures (van Benthem 1996, 2011). This standard methodology of mathematics also applies to logic, and it has several consequences for any systematic theory incorporating our earlier observations.

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[24] There are a few minor issues here with keeping the whole universe W as an evidence set.

[25] There are also other tests on naturalness of evidence updates that need not favor just the trackable operations. For instance, deletion $-\varphi$ does have a complete dynamic logic axiomatized by recursion laws for evidence and belief, cf. van Benthem & Pacuit 2011. Here is a typical law: $[-\varphi]\Box\psi \leftrightarrow \Box^{\neg\varphi}\psi$, where $\Box^{\alpha}\psi$ says that there is evidence for ψ that is consistent with α. This dynamic logic of evidence change comes with a new matching evidence language for neighborhood models that is closed under the dynamic deletion modality, cf. van Benthem, Bezhanishvili, Engvist & Yu 2015.

8.1 Invariance relations Consider invariances for evidence models. Van Benthem & Pacuit 2011 consider a standard brand of *neighborhood bisimulations*, close to topo-bisimulations (van Benthem & Bezhanishvili 2007). But they also point out that richer languages of evidence models need more discriminating structural bisimulations. One recent example of the latter are the two-way bisimulations of van Benthem, Bezhanishvili & Yu 2015, which have a strengthened symmetric clause where the cross-model relation has to be total between the two neighborhoods compared in the back-and-forth step.

The same variety in defining notions of bisimulation can be found with plausibility models (see van Benthem 2011, Andersen et al. 2013 for some examples), depending on how much structure one wants to preserve at this level of doxastic representation. [26]

8.2 Languages Next, it is well-known that there is another side to this same coin. Invariance relations suggest introducing *languages* that can define the properties appropriate to given invariance level. The model theory of modal logic or first-order logic provides key instances of this harmony (cf. van Benthem 2001, van Benthem & Bonnay 2008 for general discussion).

Thus, there is also an issue of which languages we have in mind when discussing evidence models or plausibility models. In the latter realm, candidates considered included modalities for absolute and conditional belief, but also for safe belief, and so on. On evidence models, we had the basic evidence modality $\Box\varphi$, but also others suggested by the dynamics. One recent stronger candidate is the 'instantial modality'

> $[](\varphi_1, ..., \varphi_n ; \psi)$: there exists an evidence set all of whose points satisfy ψ, while that set also contains φ_i–points for each i with $1 \leq i \leq n$.

8.3 Invariance, dynamics, and definability The choice of an invariance relation also has consequences for the dynamic update actions that are appropriate to models at a given structure level. This issue has been studied extensively in the literature on process algebra (Bergstra, Ponse & Smolka, eds., 2001), dynamic-epistemic logic (van Benthem 2011), or logics of games (van Benthem 2014). Generally speaking, model transformations need to respect a given structural invariance as follows: invariant input models lead to invariant output models.

Given the tight connection between invariances and languages, there is also a syntactic aspect here of definability for update operations in suitable logical languages. [27] Specialized to our tracking setting, here are a few simple remarks on what we did in Section 7. All our examples there involved simply *first-order definable* operations giving the new plausibility order, or the new evidence family, where in the latter case, we can use a two-sorted first-order language over points and evidence sets. Moreover, since the induced plausibility ordering has a first-order definition, there is an obvious translation from plausibility to evidence levels here. [28]

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[26] *Caveat*. One might think that by choosing the right invariance, one can also cross between levels, thereby undermining the whole intuitive picture of different levels that we started with. For instance, one could take 'inducing the same plausibility order' as a strong notion of behavioral equivalence between evidence models. See the formal representation results in van Benthem, Grossi & Liu 2014 for some concrete examples, in the spirit of Andréka, Ryan & Schobbens 2002. While such a rough simulation may indeed blur our level distinctions, we believe that our earlier intuitions will stand up.

[27] This syntactic definability theme has been studied at length for Process Algebra in Hollenberg 1998.

[28] It is easy to give such translations for our earlier concrete update functions discussed in Section 7. In particular, in this setting, we can give a syntactic counterpart for our semantic argument that each type of plausibility update map tracks at least one, suitably defined, type of evidence update map.

This raises new general issues. In particular, our earlier necessary and sufficient semantic criterion for trackability can now be stated as a form of *implicit definability* reminiscent of Beth's Definability Theorem, where the induced order of the updated evidence family only depends on the induced order of the original evidence family. Finding a tracking map then corresponds to giving a syntactic definition making this dependence explicit. [29]

Of course, many update operations live in small fragments of a full first-order language: such as the modal languages mentioned earlier, or the small program formats or Boolean formats used in Section 7. These smaller languages raise trackability questions of their own.

Remark Category theory
Our discussion in this and the following section is extremely sketchy and programmatic. A full logical treatment of the landscape of information levels should probably involve a category-theoretic framework, perhaps of a sort already used for dynamic-epistemic logics (Baltag & Moss 2002), for which we refer to the follow-up study Baltag, van Benthem & Cina 2016.

9 Logical aspects II: Looking across levels

We conclude our logical discussion by highlighting two aspects of logics and languages that occur when we compare different levels of the sort we had so far.

9.1 Tracking and translation Tracking diagrams naturally complement earlier translations between static languages for models at different levels. For instance, what the earlier commuting diagrams said is that the given translation t from the language of plausibility models to that of evidence models can be extended with clauses such as the following:

$[\#\varphi]\psi$ matches $[+t(\varphi)]t(\psi)$

The earlier issue of updates respecting notions of bisimulation then returns in the proof of the invariance of dynamic formulas for bisimulation. The upshot of such an analysis is this:

Fact The earlier translation results for static language of knowledge, belief and evidence extend to the dynamic languages for pairs of operators that track each other.

9.2 Dynamic logic of translation, representation, and change We can also go further than the preceding links. The perspective on information in this paper has two dimensions of dynamics. In a 'horizontal' direction, there are update actions along models of the same level.

However, there is also a natural 'vertical' dynamics in the story of information levels in this paper, namely, of moving back and forth between different levels of representing information. Perhaps a bit outrageously, this suggests a two-level dynamic logic combining update with level change. [30] Its language has the earlier syntax of dynamic logics of attitudes and updates at two levels (that is, we now have two languages in principle, one for each level), but we add explicit *notation for level-connecting operations*, both up and down.

The laws of the logic for the new level-crossing modalities will reflect the recursive clauses of the above translations, just as the recursion laws of definable operations tend to do.

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[29] Baltag, van Benthem & Cina 2016 will contain more details on these model-theoretic matters, including a link from invariance and dependence to interpolation theorems for two-sorted languages. Here, we need to work with Henkin-style generalized models for evidence, something we forego here.

[30] For a static perspective on level shifts in representing time, see the dissertation Montanari 1996.

Special principles in this rich modal language will then express special level connections of the sort we have discussed before. In particular, consider our general tracking diagram

$$\begin{tikzcd} Ord(\mathbf{M}) \arrow[r, phantom, "F"] & ord(\mathbf{N}) \\ \mathbf{M} \arrow[u] \arrow[r, phantom, "G"] & \mathbf{N} \arrow[u] \end{tikzcd}$$

The following commutation axiom will reflect this as a matter of modal frame correspondence in a suitably abstract universe of models: [31]

$$\langle ord\rangle\langle F\rangle\varphi \leftrightarrow \langle G\rangle\langle ord\rangle\varphi$$

The result of these preliminary considerations is a polymodal logic that can be seen as formalizing some elementary meta-theory of what we have discussed so far – just as dynamic-epistemic logics formalize basic properties of model-changing operations.

Coda As one perhaps surprising example of this way of thinking, here is an issue of *complexity* of the meta-theory of levels and tracking. The preceding commutative diagrams with tracking are evidently what logicians like, and in a more general setting, what category theorists love. [32] On the other hand, we also know from modal logic (Blackburn, de Rijke & Venema 2000) that complete logics of frames with commuting relations tend to be of high complexity (at least undecidable), as their grid structure can encode complex geometrical tiling problems.

10 Further information levels

A question that always come up when the preceding ideas are presented to audiences is the nature of the total representation system for information. There are several major directions to go from the above levels of epistemics, plausibility, or flat and ordered evidence.

Ordered evidence A good test on the issues raised so far is how evidence models fare with an obvious next level of structure with a binary ordering that can be stand for entrenchment, trust, or probabilistic weight. There is some concrete logical theory of such models: cf. Girard 2009 and Liu 2011, based on the 'priority graphs' of Andréka, Ryan & Schobbens 2002. One can define new modalities of knowledge and belief based on ordered evidence, depending on how we view the order intuitively. This then leads to an extended theory of representation results and translation between the enriched logics, as well as a richer dynamics on priority graphs, including tracking and non-tracking results extending our earlier analysis.

Probability One view is that our preceding levels represent relative plausibility as a form of qualitative probability, comparable to what was pursued by De Finetti and other pioneers. Finding precise connections here would involve a fresh look at logics for probability (Holliday & Icard 2013), while doing justice to the very different intuitions underlying plausibility models and probability models where many low scorers can team up to form a high-score area (van Benthem 2013). Also relevant in this setting is Kelly & Lin 2012 on the impossibility of tracking quantitative Bayesian update in qualitative plausibility terms.

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[31] We do not spell out the abstract model setting that would give the correspondence its proper bite – and also, we also ignore some small provisos on partiality of update functions.

[32] Tracking diagrams are not exactly categorial commuting diagrams, but we ignore this finesse.

Syntax and proof One can also strike out in a different direction, closer to Mike Dunn's work, and think of evidence sets as *reasons* put forward to ground beliefs, or even knowledge, as in reason-based epistemology. Then the earlier 'internal operations' become crucial, and their corresponding more fine-grained notions of information. In this case, the appropriate setting would seem to be real syntactic proof structure, where detailed formulation can be crucial to information flow consuming coded resources – as in science, where one well-chosen syntactic notation may be much more informative than a semantically equivalent one (van Benthem & Martinez 2008). While there are some dynamic-epistemic logics at this level, the connection between syntactic and semantic approaches to information remains far from clear.

Interpolating levels Once we have the general picture, new questions of their own will arise. E.g., our semantic models and pure syntax seem very far apart. But then, the question is what natural intermediate levels of information structure exist in between the two. One such level is that of algebraic logic, where structures range from rather syntactic ones like free algebras to algebras directly associated with set-theoretic models. In particular, we believe that the representation theory of modal algebras in terms of possible-worlds models (Blackburn, de Rijke & Venema 2000, Andréka, van Benthem, Bezhanishvili & Németi 2014) may have a natural connection with the above uses of plausibility models and evidence models.

Caveat Finally, it goes without saying that we are not claiming that there is a linear hierarchy of information levels. There can well be a more graph-like family with many forks.

11 Further issues and directions

Expansion and compression Many further actions make sense in our perspective of levels and zooming in on richer levels, or zooming out to coarser ones. At the evidence level (or that of reasons and proofs), these include giving reasons to others, say, when answering doubts.

But perhaps the major issue in the grand picture is the interplay of storing a lot of information about the past versus *compressing information*. The latter may be thought of as a necessity due to memory limitations, or intelligent resource constraints in solving tasks. It can also be viewed as *forgetting*, perhaps a human defect, but also a basic feature of civilization.

Cognitive agency The issue of forgetting leads to the issue of cognitive realism for our picture. We can think of all our information levels as mathematical structures with eternal connections, waiting to be used by human agents, but equally perfect if no one ever comes to visit in the course of history. But it is equally tempting to think of it as a world where human cognition takes place, and in that case, many of the above topics might be considered from the viewpoint of agents solving informational tasks. This involves at least two further features:

(a) the computational nature of the agent (that might be modeled in an automata hierarchy),

(b) the nature of the specific issues or tasks that trigger level change.

We may be making local excursions in our landscape of information levels to answer specific questions, rather than engaging in dramatic voyages. [33]

The temporal long-term The dynamics in this paper consisted of local steps, whether horizontal as update at one level of information, or as a vertical step toward a richer or poo-

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[33] The point about issues was made by Fenrong Liu, p.c. A concrete technical illustration is the use of *filtration* in modal logic where models get coarsened using just a small finite set of 'relevant formulas'.

rer level. But informational inquiry also involves global patterns over time, witness the protocols that regulate learning (Hoshi 2009, van Benthem 2011, Kelly 1996, Gierasimczuk 2010). Our style of analysis is not in conflict with this long view: but we just did not develop it here.

Mathematical frameworks What would be a best framework for placing the considerations and observations of this paper? One option are Chu spaces for abstract information, whose category-theoretic treatment can be found in Barwise & Seligman 1995. However, as said earlier, we feel it is probably better to step back and explore a category-theoretic framework for all of this, for which a first attempt may be found in Baltag, van Benthem & Cina 2016.

12 Conclusion

We have argued that information is a multi-faceted notion best studied at a variety of levels, each supporting their own intuitions of invariance and languages bringing out structure.

In doing so, we found many new notions and distinctions, such as the contrast between internal operations that merely rearrange or elucidate and update operations that are visible, and trackable at other levels. We have also emphasized the further dimension of a 'dynamics of zoom', with both information expansion and information reduction.

In all this, we have shown how logical notions and methods apply, making our approach a conscious extension of the logical approach to information that Mike Dunn has advocated for so long. [34] But we are also acutely aware that our current presentation is not the end stage, since we need more general mathematical perspectives to do justice to our landscape.

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[34] Indeed, Wes Holliday, p.c., has pointed out how Mike Dunn's early logical work on information content and relevant logic may provide one more very natural richer level in our hierarchy, connecting models for epistemic-doxastic dynamics with those from the situation-semantic tradition.

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