Universal models for the positive fragment of intuitionistic logic

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Abstract. We study the n-universal model of the $[\vee, \wedge, \to]$-fragment of the intuitionistic propositional calculus IPC. We denote it by $\mathcal{U}^\star(n)$ and show that it is isomorphic to a generated submodel of the n-universal model of IPC, which is denoted by $\mathcal{U}(n)$. We show that this close resemblance makes $\mathcal{U}^\star(n)$ mirror many properties of $\mathcal{U}(n)$. Using $\mathcal{U}^\star(n)$, we give an alternative proof of Jankov's theorem stating that the intermediate logic KC, the logic of the weak excluded middle, is the greatest intermediate logic extending IPC that proves exactly the same negation-free formulas as IPC.

1 Introduction

In this paper we use the tools of universal models to study positive formulas in intuitionistic propositional calculus IPC, i.e., formulas containing only propositional variables, $\top, \wedge, \vee$ and $\to$. Fragments of intuitionistic logic have been thoroughly investigated in the literature. For a detailed historic account we refer to [13]. Among these fragments, the locally finite ones, i.e., the fragments where for each $n \in \omega$ there are only finitely many non-equivalent formulas in n variables, attracted more attention. For example, [8] proved a classic result that the $[\wedge, \to]$-fragment of IPC is locally finite. The positive fragment is not locally finite, and as a result it has not received much attention in the literature. The major interest for the study of this fragment comes from minimal logic [12], the fragment of intuitionistic logic obtained by dropping the axiom $\varphi \to \bot$.

Universal models of intuitionistic logic are used to describe finitely generated free Heyting algebras. In some sense universal models are duals to free Heyting algebras. We refer to [9, Section 3.2.1] for an overview of the history of universal models. Universal models of fragments of intuitionistic propositional calculus (IPC) have been investigated in [3, 5, 10, 14].

In this paper, we describe the n-universal model $\mathcal{U}^\star(n)$ of the $[\vee, \wedge, \to]$-fragment of the intuitionistic propositional calculus IPC and show that it is isomorphic to a generated submodel of the n-universal model of IPC. Using $\mathcal{U}^\star(n)$, we give an alternative proof of Jankov's theorem that the intermediate logic KC, the logic of the weak excluded middle, is the greatest intermediate logic extending IPC that proves exactly the same negation-free formulas as IPC.

The paper is organized as follows: In Section 2 we summarise all the basic notions and results used consequently in the paper. In Section 3, we recall the top-model property and its relationship with the positive fragment of IPC. In Section 4, we study the universal models for the positive fragment of IPC. In Section 5, we discuss the relationship between the n-Henkin models and the n-universal models for this fragment, as well as an alternative proof of Jankov's theorem. In Section 6, we study the duality between the $[\vee, \wedge, \rightarrow]$-algebras and the universal models for the positive fragment of IPC. Finally, in Section 7 we use the universal model for the positive fragment of IPC to describe the universal model for minimal logic MPC.

2 Preliminaries

2.1 Basic Notations

In this section we briefly recall the relational semantics for the intuitionistic propositional calculus IPC. For a detailed information about IPC, we refer to [4].

Definition 1 (Kripke Frames and Models) *A* Kripke frame *is a pair* $\mathfrak{F} = (W, R)$ *where* W *is a non-empty set and* R *is a partial order on it. A* Kripke model *is a triple* $\mathfrak{M} = (W, R, V)$ *where* (W, R) *is a Kripke frame and* V *is a partial map* $V : Prop \rightarrow \mathscr{P}(W)$ *(where* $\mathscr{P}(W)$ *is the powerset of* W*) such that for any* $w, w' \in W$*,* $w \in V(p)$ *and* wRw' *imply* $w' \in V(p)$*.*

The valuation can be extended to all formulas as usual. We call the upward closed subsets of W (with respect to R) *upsets*. The set of all upsets of W is denoted by $Up(W)$. As usual $w \in V(\varphi)$ will ask be denoted as $w \models \varphi$.

Definition 2 (General Frames)

1. *A* general frame *is a triple* $\mathfrak{F} = (W, R, \mathcal{P})$*, where* (W, R) *is a Kripke frame and* $\mathcal{P}$ *is a family of upsets containing* $\emptyset$ *and closed under* $\cap, \cup$ *and the following operation* $\supset$*: for every* $X, Y \subseteq W$*,*

$$X \supset Y = \{x \in W : \forall y \in W(xRy \wedge y \in X \rightarrow y \in Y)\}$$

   Elements of the set $\mathcal{P}$ *are called* admissible sets*.*
2. *A general frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *is called* refined *if for any* $x, y \in W$*,*

$$\forall X \in \mathcal{P}(x \in X \rightarrow y \in X) \Rightarrow xRy.$$

3. $\mathfrak{F}$ *is called* compact*, if for any families* $\mathcal{X} \subseteq \mathcal{P}$ *and* $\mathcal{Y} \subseteq \{W \backslash X : X \in \mathcal{P}\}$*, for which* $\mathcal{X} \cup \mathcal{Y}$ *has the finite intersection property,* $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$*.*
4. *A general frame* $\mathfrak{F}$ *is called a* descriptive frame *iff it is refined and compact.*

In this paper our models will usually be n-models with V restricted to n-formulas built up from $p_1, \ldots, p_n$.

Next we introduce some frame and model constructions that will be used consequently.

Definition 3 (Generated Subframe and Generated Submodel)

1. *For any Kripke frame* $\mathfrak{F} = (W, R)$ *and* $X \subseteq W$*, the* subframe of $\mathfrak{F}$ generated by X *is* $\mathfrak{F}_X = (R(X), R')$*, where* $R(X) = \{w' \in W | wRw' \text{ for some } w \in X\}$ *and* R' *is the restriction of* R *to* $R(X)$*. If* $X = \{w\}$*, then we denote* $\mathfrak{F}_X$ *by* $\mathfrak{F}_w$ *and* $R(X)$ *by* $R(w)$*.*
2. *For any Kripke frame* $\mathfrak{F} = (W, R)$*, any valuation* V *on* $\mathfrak{F}$ *and* $X \subseteq W$*, the* submodel of $\mathfrak{M} = (\mathfrak{F}, V)$ generated by X *is* $\mathfrak{M}_X = (\mathfrak{F}_X, V')$*, where* V' *is the restriction of* V *to* $R(X)$*. If* X *is a singleton* $\{w\}$*, then we denote* $\mathfrak{M}_X$ *by* $\mathfrak{M}_w$*.*
3. *For any general frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *and any* $X \subseteq W$*, the* (general) subframe of $\mathfrak{F}$ generated by X *is* $\mathfrak{F}_X = (R(X), R', \mathcal{Q})$*, where* $(R(X), R')$ *is the subframe of* (W, R) *generated by* X*, and* $\mathcal{Q} = \{U \cap R(X) | U \in \mathcal{P}\}$*.*

The next lemma states a basic fact about descriptive frames. For a proof we refer to, e.g., [14].

Lemma 4 *For any descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$*, any* $W' \in \mathcal{P}$*, we have that* $\mathfrak{G} = (W', R', \mathcal{Q})$*, the (general) subframe of* $\mathfrak{F}$ *based on* W'*, where* R' *is* R *restricted to* W' *and* $\mathcal{Q} = \{U \cap W' | U \in \mathcal{P}\}$*, is a descriptive frame.*

Next we introduce p-morphisms:

Definition 5 (p-morphism)

1. *For two Kripke frames* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$*, a* p-morphism *from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ *is a map* $f : W \to W'$ *satisfying:*
   - wRw' *implies* $f(w)R'f(w')$ *for any* $w, w' \in W$*;*
   - $f(w)R'v'$ *implies* $\exists v \in W(wRv \wedge f(v) = v')$*.*
2. *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *and* $\mathfrak{G} = (V, S, \mathcal{Q})$ *be two general frames. We call a Kripke frame p-morphism* f *of* (W, R) *to* (V, S) *a* (general frame) p-morphism *of* $\mathfrak{F}$ *onto* $\mathfrak{G}$*, if it also satisfies the following condition:*
$$\forall X \in \mathcal{Q}, f^{-1}(X) \in \mathcal{P}.$$
3. *A* p-morphism f *from* $\mathfrak{M} = (W, R, V)$ *to* $\mathfrak{M}' = (W', R', V')$ *is a p-morphism from* (W, R) *to* (W', R') *such that* $w \in V(p) \Leftrightarrow f(w) \in V'(p)$ *for every* $p \in Prop$*. For models based on general frames, we also require the condition for p-morphisms between general frames. For* n*-models, the definition is similar.*

Next we give a definition for the n-Henkin model, which is the canonical model used in the completeness proof for the n-variable fragment of IPC.

Definition 6 (n-Henkin Model)

1. *An* n-theory *is a set of* n*-formulas closed under deduction in* IPC*.*
2. *A set of formulas* Γ *has the* disjunction property*, if for all* n*-formulas* φ, ψ*, we have that* $\varphi \vee \psi \in \Gamma$ *implies* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$*.*
3. *The* n-canonical model *or* n-Henkin model $\mathcal{H}_n = (W_n, R_n, V_n)$ *is a model where* W_n *consists of all consistent* n*-theories with the disjunction property,* R_n *is the subset relation, and* $\Gamma \in V_n(p)$ *iff* $p \in \Gamma$*.*

2.2 The n-Universal Model for the Full Language of IPC

In this part we recall the definition of the n-universal model for the full language of IPC, state its properties, define the de Jongh formulas and state the Jankov-de Jongh theorem. All of the content of this part can be found in [2, Section 3], [4, Chapter 8] and [6].

In the following we use the terminology *color* to denote the valuation at a world in an n-model. In general, an n-color (n can be omitted if it is clear from the context) is a sequence $c_1 \dots c_n$ of 0's and 1's. The set of all n-colors is denoted by C^n. We define the order of colors as following: $c_1 \dots c_n \leq c'_1 \dots c'_n$ iff $c_i \leq c'_i$ for $1 \leq i \leq n$. We write $c_1 \dots c_n < c'_1 \dots c'_n$ if $c_1 \dots c_n \leq c'_1 \dots c'_n$ but $c_1 \dots c_n \neq c'_1 \dots c'_n$.

A *coloring* on $\mathfrak{F} = (W, R)$ is a map $col : W \to C^n$ satisfying $uRv \Rightarrow col(u) \leq col(v)$. It is easy to see that colorings and valuations are in 1-1 correspondence. Given $\mathfrak{M} = (W, R, V)$, we can describe the valuation by the coloring $col_V : W \to C^n$ such that $col_V(w) = c_1 \dots c_n$, where for each $1 \leq i \leq n$, $c_i = 1$ if $w \in V(p_i)$, and 0 otherwise. We call $col_V(w)$ the color of w under V.

In any frame $\mathfrak{F} = (W, R)$, we say that $X \subseteq W$ *totally covers* w (notation: $w \prec X$), if X is the set of all immediate successors of w. When $X = \{v\}$, we write $w \prec v$. $X \subseteq W$ is called an *anti-chain* if $|X| > 1$ and for every $w, v \in X$, $w \neq v$ implies that $\neg(wRv)$ and $\neg(vRw)$. If uRv we say u is *under* v.

We can now inductively define the n-universal model $\mathcal{U}(n)$ by its cumulative layers $\mathcal{U}(n)^k$ for $k \in \omega$. (We omit n if it is clear from the context.)

Definition 7 (n-Universal Model)

- *The first layer $\mathcal{U}(n)^1$ consists of 2^n nodes with the 2^n different n-colors under the discrete ordering.*
- *Under each element w in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.*
- *Under any finite anti-chain X with at least one element in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.*

The whole model $\mathcal{U}(n)$ is the union of its layers.

It is easy to see from the construction that every $\mathcal{U}(n)^k$ is finite. As a consequence, the generated submodel $\mathcal{U}(n)_w$ is finite for any node w in $\mathcal{U}(n)$.

We now state some properties of the n-universal model. For a proof of the next lemma we refer to, e.g., [14, Theorem 3.2.3].

Lemma 8 *For any finite rooted Kripke n-model $\mathfrak{M}$, there exists a unique $w \in \mathcal{U}(n)$ and a p-morphism of $\mathfrak{M}$ onto $\mathcal{U}(n)_w$.*

Using this lemma we will show (in the next theorem) that $\mathcal{U}(n)$ is a counter-model to every n-formula not provable in IPC. This justifies the name "universal model" for $\mathcal{U}(n)$. For a proof we refer to, e.g., [14, Theorem 3.2.4].

Theorem 9

1. *For any n-formula φ, $\mathcal{U}(n) \models \varphi$ iff $\vdash_{\mathsf{IPC}} \varphi$.*
2. *For any n-formulas φ, ψ, for all $w \in \mathcal{U}(n)(w \models \varphi \Rightarrow w \models \psi)$ iff $\varphi \vdash_{\mathsf{IPC}} \psi$.*

In the following we define de Jongh formulas for the full language of IPC and prove that they define point generated submodels of universal models.

For any node w in an n-model $\mathfrak{M}$, if $w \prec \{w_1, \dots, w_m\}$, then we let

prop$(w) := \{p_i | w \models p_i, 1 \leq i \leq n\}$,
notprop$(w) := \{q_i | w \not\models q_i, 1 \leq i \leq n\}$,
newprop$(w) := \{r_j | w \not\models r_j$ and $w_i \models r_j$ for each $1 \leq i \leq m$, for $1 \leq j \leq n\}$.

Here newprop(w) denotes the set of atoms which are about to be true in w, i.e., they are true in all of w's proper successors. Next, we define the formulas φ_w and ψ_w, which are called de Jongh formulas. For the notion of a depth a point we refer to [4, ??] or [2, 3.1.9]. Roughly speaking a point w of a universal model has depth n if it belongs to the n-th layer of $\mathcal{U}(n)$. The depth of a point w will be denoted by $d(w)$.

Definition 10 *Let w be a point in $\mathcal{U}(n)$. We inductively define the corresponding* de Jongh formulas *φ_w and ψ_w:*

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge \text{prop}(w) \wedge \bigwedge \{\neg p_k | p_k \in \text{notprop}(w), 1 \leq k \leq n\},$$

and

$$\psi_w := \neg \varphi_w.$$

If $d(w) > 1$, and $\{w_1, \dots, w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w := \bigwedge \text{prop}(w) \wedge (\bigvee \text{newprop}(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$

and

$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

The most important properties of the de Jongh formulas are revealed in the following proposition. For a proof we refer to [2, Theorem 3.3.2.].

Proposition 11 *For every $w \in \mathcal{U}(n)$ we have that*

- $V(\varphi_w) = R(w)$, *where* $R(w) = \{w' \in \mathcal{U}(n) | wRw'\}$*;*
- $V(\psi_w) = \mathcal{U}(n) \backslash R^{-1}(w)$, *where* $R^{-1}(w) = \{w' \in \mathcal{U}(n) | w'Rw\}$*.*

Now we state more properties of the universal model and de Jongh formulas. (For a proof we refer to [6, Corollary 19].) We write $Cn_n(\varphi) = \{\psi | \psi$ is an n-formula such that $\vdash_{\mathsf{IPC}} \varphi \to \psi\}$, $Th_n(\mathfrak{M}, w) = \{\varphi | \varphi$ is an n-formula such that $\mathfrak{M}, w \models \varphi\}$, and we omit n if it is clear from the context.

Lemma 12 *For any point w in $\mathcal{U}(n)$, $Th_n(\mathcal{U}(n), w) = Cn_n(\varphi_w)$.*

The next lemma states that $\mathcal{U}(n)_w$ is isomorphic to the submodel of $\mathcal{H}(n)$ generated by the theory axiomatized by the de Jongh formula of w. For a proof we refer to [6, Lemma 20].

Lemma 13 *For any $w \in \mathcal{U}(n)$, let φ_w be the de Jongh formula of w, then we have that $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$.*

Let $Upper(\mathfrak{M})$ denote the submodel $\mathfrak{M}_{\{w \in W | d(w) < \omega\}}$ generated by all the points with finite depth, where depth is defined as usual. Intuitively, $Upper(\mathfrak{M})$ is the "upper" part of $\mathfrak{M}$. It can be shown that the n-universal model is isomorphic to the upper part of the n-Henkin model, i.e. $Upper(\mathcal{H}(n))$. For a proof we refer to, e.g., [2, Theorem 3.2.9.] and [6, Theorem 39].

Lemma 14 *$Upper(\mathcal{H}(n))$ is isomorphic to $\mathcal{U}(n)$.*

The following is a corollary which follows from the correspondence between $\mathcal{H}(n)$ and $\mathcal{U}(n)$. For a proof we refer to [6, Corollary 21].

Corollary 15 *Let $\mathfrak{M}$ be any model and w be a point in $\mathcal{U}(n) = (W, R, V)$. For any point x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi_w$, then there exists a unique point v satisfying $\mathfrak{M}, x \models \varphi_v, \mathfrak{M}, x \not\models \varphi_{v_1}, \ldots, \mathfrak{M}, x \not\models \varphi_{v_m}$, where $v \prec \{v_1, \ldots, v_m\}$, and wRv.*

In the following we state the Jankov-de Jongh theorem for the full language of IPC. For a proof we refer to [2, Theorem 3.3.3.] and [6, Theorem 26].

Theorem 16 (Jankov-de Jongh Theorem for the Full Language of IPC) *For every finite rooted frame $\mathfrak{F}$, let ψ_w be the de Jongh formula of w in the model $\mathcal{U}(n)_w$, then for every descriptive frame $\mathfrak{G}$, $\mathfrak{G} \not\models \psi_w$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.*

3 The Top-Model Property and its Relationship with $[\vee, \wedge, \to]$-fragment

In this section we will introduce the top-model property and the top-model construction. This is related to the $[\vee, \wedge, \to]$-fragment of IPC, which contains the formulas constructed only by $\vee, \wedge, \to$ (i.e. the negation-free formulas) and we denote this language by $\mathcal{L}_{\vee,\wedge,\to}$. For other fragments, the notations are similar. Here we state the relevant results without proof, except for an algorithm for which we give the procedure.

By replacing every occurrence of $\bot$ by $\neg(p \to p)$, every formula is IPC-equivalent to a $\bot$-free formula. For simplicity of discussion, we restrict our attention to $\bot$-free formulas (i.e. formulas in $\mathcal{L}_{\vee,\wedge,\to,\neg}$) only.

Definition 17 (Top-Model Property) *We say that a formula φ has the* top-model property*, if for all Kripke models $\mathfrak{M} = (W, R, V)$, all $w \in W$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^+, w \models \varphi$, where $\mathfrak{M}^+ = (W^+, R^+, V^+)$ is obtained by adding a top point t (which is a successor of all points) such that all proposition letters are true in t.*

We have the following results about the top-model property. We write $\varphi \sim \psi$ for $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$. For the proof of the next theorem see [7, Theorem 5].

Proposition 18

1. *Every formula in $\mathcal{L}_{\vee,\wedge,\to}$ has the top-model property, and so has $\bot$.*
2. *For any formula φ in $\mathcal{L}_{\vee,\wedge,\to,\neg}$, there exists a formula φ^* in $\mathcal{L}_{\vee,\wedge,\to}$ or $\varphi^* = \bot$ such that for any top-model $(\mathfrak{M}, w)$, we have $(\mathfrak{M}, w) \models \varphi \leftrightarrow \varphi^*$.*

4 The universal models for the $[\vee, \wedge, \to]$-fragment of IPC

We will now proceed by defining the n-universal model, $\mathcal{U}^\star(n)$, for the $[\vee, \wedge, \to]$-fragment of IPC. This model closely resembles the n-universal model for IPC: it is a generated submodel of it. Mirroring the definition of $\mathcal{U}(n)$ we define $\mathcal{U}^\star(n) = (U^\star(n), R^\star, V^\star)$ inductively:

Definition 19

- *The first layer $\mathcal{U}^\star(n)^1$ consists of $2^n - 1$ nodes with all the different n-colors* — excluding the color $1 \dots 1$ — *under the discrete ordering.*
- *Under each element w in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}^\star(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.*
- *Under any finite anti-chain X with at least one element in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}^\star(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.*

The whole model $\mathcal{U}^\star(n)$ is the union of its layers.

We should also note here the sharp contrast between $\mathcal{U}(1)$, also known as the Rieger-Nishimura ladder, and $\mathcal{U}^\star(1)$. The latter consists of a single element that does not satisfy p. The only formulas satisfied at that element are the intuitionistic tautologies.

Given two intuitionistic models (W, R, V) and (W', R', V') and a partial map $f : W \to W'$, if $X \subseteq W'$ we let $f^*(X) = W \setminus R^{-1}(f^{-1}[W' \setminus X])$.

Definition 20 *A* positive morphism *is a partial function* $f : (W, R, V) \to (W', R', V')$ *such that:*

1. *If* $w, v \in \mathrm{dom}(f)$ *and* wRv *then* $f(w)R'f(v)$ *(*forth condition*).*
2. *If* $w \in \mathrm{dom}(f)$ *and* $f(w)R'v$ *then there exists some* $u \in \mathrm{dom}(f)$ *such that* $f(u) = v$ *and* wRu *(*back condition*).*
3. *If* $w \in \mathrm{dom}(f)$ *and* vRw*, then* $v \in \mathrm{dom}(f)$*.*
4. *For every* $p \in \mathrm{Prop}$ *we have* $V(p) = f^*(V'(p))$*.*

If the models are descriptive, then $f : (W, R, \mathcal{P}, V) \to (W', R', \mathcal{Q}, V')$ is a *descriptive positive morphism* if furthermore for every $Q \in \mathcal{Q}$ we have $f^*(Q) \in \mathcal{P}$.

Lemma 21 *Let* $f : W \to W'$ *be a positive morphism. If* $X \subseteq W'$ *such that* X *is an upset of* W'*, then* $f^*(X) = f^{-1}[X] \cup (W \setminus \mathrm{dom}(f))$*.*

Proof. Let X be an upset of W'. Then $W' \setminus X$ is a downset of W'. By (3) from the definition of positive morphisms if $w \in \mathrm{dom}(f)$ and $u \in R^{-1}(w)$ then $u \in \mathrm{dom}(f)$ and by (1) $f(u)Rf(w)$. These, along with the fact that $W' \setminus X$ is a downset imply that if $w \in f^{-1}[W' \setminus X]$ and $u \in \mathrm{dom}(f)$ then $u \in f^{-1}[W' \setminus X]$, hence $R^{-1}(f^{-1}[W' \setminus X]) = f^{-1}[W' \setminus X]$.
Now we have that $f^*(X) = W \setminus R^{-1}(f^{-1}[W' \setminus X]) = W \setminus f^{-1}[W' \setminus X]$. But $W \setminus f^{-1}[W' \setminus X] = f^{-1}[X] \cup (W \setminus \mathrm{dom}(f))$. □

Positive morphisms correspond to *dense subreductions* of [4, Section 9.1] and are related to *strong partial Esakia morphisms* of [1].

Lemma 22 *A a partial function* $f : (W, R, V) \to (W', R', V')$ *is a positive morphism if and only if the following conditions hold:*

1*. *If* $w, v \in \mathrm{dom}(f)$ *and* wRv *then* $f(w)R'f(v)$*.*
2*. *If* $w \in \mathrm{dom}(f)$ *and* $f(w)R'v$ *then there exists some* $u \in \mathrm{dom}(f)$ *such that* $f(u) = v$ *and* wRu *(*back condition*).*
3*. *If* $w \in \mathrm{dom}(f)$ *and* vRw*, then* $v \in \mathrm{dom}(f)$*.*
4*. *For every* $p \in \mathrm{Prop}$ *and* $w \in \mathrm{dom}(f)$ *we have* $w \in V(p) \iff f(w) \in V'(p)$*.*
5*. $\mathrm{dom}(f) \supseteq \{w \in W : \exists p \in \mathrm{Prop}\ w \notin V(p)\}$*.*

Proof. We need to prove that under the assumption of 1 through 3 of the definition of positive morphisms, 4 is equivalent to 4^* and 5^*.

Let us assume 4^* and 5^*. By Lemma 21 we have that $f^*(V'(p)) = f^{-1}[V'(p)] \cup W \setminus \mathrm{dom}(f)$. By 4^* we have that $f^{-1}[V'(p)] = V(p) \cap \mathrm{dom}(f)$. We have that 5^* implies that $W \setminus \mathrm{dom}(f) \subseteq V(p)$ since every element not in the domain of f satisfies all propositional atoms. Therefore $V(p) = (V(p) \cap \mathrm{dom}(f)) \cup W \setminus \mathrm{dom}(f)$ and thus $f^*(V'(p)) = V(p)$.

For the other direction assume 4. Then if $w \in \mathrm{dom}(f)$ and since $f^{-1}[V'(p)] \cup W \setminus \mathrm{dom}(f) = V(p)$ it follows that $w \in V(p)$ if and only if $f(w) \in V'(p)$, i.e. we arrive at 4^*. Also, for any $p \in \mathrm{Prop}$ we have that $W \setminus \mathrm{dom}(f) \subseteq V(p)$ and hence all elements not in the domain of f satisfy all propositional atoms, i.e. we arrive at 5^*. □

From here on we will use this alternative definition of positive morphisms.

We note that if for all $w \in W$, there is some propositional atom p such that p is not satisfied in w, then the positive morphisms are p-morphisms. Also, every p-morphism is a positive morphism. Finally it is easy to check that the composition of two positive morphisms is a positive morphism.

The essential difference between p-morphisms and positive morphisms is that the latter are partial maps - they do not have to contain in their domains worlds that satisfy all propositional atoms. The reason we can ignore these worlds, when dealing with the $[\vee, \wedge, \rightarrow]$-fragment of IPC lies in the simple fact (which can be easily checked by induction) that in such worlds all negation-free formulas hold. Next we show that positive morphisms preserve negation-free formulas:

Proposition 23 *Let $f : (W, R, V) \to (W', R', V')$ be a positive morphism. Then for every negation-free formula φ and $w \in \mathrm{dom}(f)$ we have that*

$$(W, R, V), w \models \varphi \quad \textit{iff} \quad (W', R', V'), f(w) \models \varphi.$$

Proof. We proceed by induction on the complexity of the formulas. The base case, i.e. atoms, follows directly by the definition of positive morphisms. Let us assume that f preserves the negation-free formulas φ and ψ. That this is also the case for $\varphi \vee \psi$ and $\varphi \wedge \psi$ trivially follows from the semantic definitions of the connectives.

Let us now assume that $(W, R, V), w \models \varphi \to \psi$. Let $f(w)R'v$ and assume that $(W', R', V'), v \models \varphi$. Then by the definition of the positive morphisms we have that there is some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu. By the induction hypothesis we have that $(W, R, V), u \models \varphi$, hence $(W, R, V), u \models \psi$, which by the induction hypothesis gives us that $(W', R', V'), f(u) \models \psi$. For the converse, let us assume that $(W', R', V'), f(w) \models \varphi \to \psi$ and for some u such that wRu we have $(W, R, V), u \models \varphi$. If $u \in \mathrm{dom}(f)$, then the induction hypothesis readily implies that $(W, R, V), u \models \psi$. If $u \notin \mathrm{dom}(f)$, then for every propositional atom p we have that $u \in V(p)$, which implies that $(W, R, V), u \models \psi$, since all negation-free formulas are true in such worlds. □

In view of Definition 7, it is clear that $\mathcal{U}^\star(n)$ is isomorphic to $\mathcal{N} = (N, R, V)$ with $N = \{w \in \mathcal{U}(n) : w^\star \notin R(w)\}$, a generated submodel of $\mathcal{U}(n)$, where $w^\star$ is the greatest node of $\mathcal{U}(n)$ such that $\mathrm{col}(w^\star)_i = 1$ for every $i \leq n$. By Proposition 18.1 $(\mathcal{U}^\star(n))^+$ satisfies the same negation-free formulas as $\mathcal{U}^\star(n)$. It is also the case that $(\mathcal{U}^\star(n))^+$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$, whose domain consist of the elements of $U(n)$ whose only successor of depth 1 satisfies all propositional atoms. Let us call this submodel $\mathcal{M}$, and let $G : (\mathcal{U}^\star(n))^+ \to \mathcal{M}$ be this isomorphism.

The models $\mathcal{U}^\star(n)$ and $(\mathcal{U}^\star(n))^+$ can be viewed as two different ways of describing the universal model sof the $[\vee, \wedge, \to]$-fragment of IPC. In the first approach there are IPC-satisfiable positive formulas (for example $p_1 \wedge \ldots \wedge p_n$) that are satisfied nowhere in $\mathcal{U}^\star(n)$ and hence are indistinguishable from $\perp$ in this model, something that is not the case in $\mathcal{U}(n)$, where every IPC-satisfiable formula is satisfied in some world. In $(\mathcal{U}^\star(n))^+$ all positive formulas are satisfied at the topmost point, and hence this model can distinguish positive formulas from $\perp$. As we will see below, every finite rooted model can be embedded onto a generated submodel of $\mathcal{U}^\star(n)$ via a positive morphism, which is not the case for $(\mathcal{U}^\star(n))^+$. On the other hand, for every finite rooted model $\mathfrak{M}$, $\mathfrak{M}^+$ can be embedded onto a generated submodel of $(\mathcal{U}^\star(n))^+$ via a p-morphism. Dually, we can define two distinct types of algebras to describe the $[\vee, \wedge, \to]$-fragment of IPC. This will be discussed more extensively in Section 6.

We can now prove the following:

Lemma 24 *There exists a positive morphism $F : \mathcal{U}(n) \to \mathcal{U}^\star(n)$, which is onto, $\mathrm{dom}(F) = \{w \in \mathcal{U}(n) : \exists p \in \mathrm{Prop}(w \notin V(p))\}$ and for every $w \in \mathrm{dom}(F)$ we have that the restriction of F to $\mathcal{U}(n)_w$ is onto $\mathcal{U}^\star(n)_{F(w)}$.*

Proof. We will define F by induction on the depth of the elements of $\mathcal{U}(n)$ in such a way that the color of $F(w)$ is the same as that of w. If $d(w) = 1$, then $F(w) = w'$, where $d(w') = 1$ and $\mathrm{col}(w) = \mathrm{col}(w')$. Let us now assume that F is defined for the elements of $\mathcal{U}(n)$ of depth m. Let $d(w) = m + 1$ and let us assume that $w \prec \{w_1, \ldots, w_k\}$. Take $A \subseteq F[\{w_1, \ldots, w_k\}]$ as the set that contains the R-minimal elements of $F[\{w_1, \ldots, w_k\}]$. A is finite as a subset of a finite set. If A is empty then let $F(w)$ be the element of $\mathcal{U}^\star(n)$ of depth 1 with the same color as w. If $A = \{u\}$ and u has the same color as w, then let $F(w) = u$. Otherwise by the construction of $\mathcal{U}^\star(n)$, there is a unique $v \prec A$ (by the induction hypothesis about F) with the same color as w and we let $F(w) = v$.

It is left to be shown is that F is a positive morphism. That $w \in V(p) \iff F(w) \in V^\star(p)$ follows directly by the construction of F. Likewise it is easy to see by the above construction that if uRw then $F(u)R^\star F(w)$.

That if $F(w)R^\star v$ implies the existence of some u such that $F(u) = v$ and wRu follows from the fact that the restriction of F to $\mathcal{U}(n)_w$ is onto $\mathcal{U}^\star(n)_{F(w)}$. To prove this we just need to show that for all $u \in \mathrm{dom}(F)$ all the immediate successors of $F(u)$ are images of successors of u, since the depth of all elements of $\mathcal{U}^\star(n)$ have finite depth. That it is the case for the immediate successors follows by the definition of F: $F(w)$'s immediate successors form a subset of $F[\{w_1, \ldots, w_k\}]$ (where $w_1, \ldots, w_k$ are the only immediate successors of w).

Finally, that F is onto can be shown by viewing $\mathcal{U}^\star(n)$ as the generated submodel $\mathcal{N}$ of $\mathcal{U}(n)$ presented above. Then it is routine to check that F is the identity function on $\mathcal{N}$. $\square$

Let us fix the injective positive morphism $i : (\mathcal{U}^\star(n))^+ \to \mathcal{U}^\star(n)$, between the two versions of the universal model described above. This map is the identity where it can be defined. Even though the above proof is constructive we can describe F in a non-constructive way by the following observation: For every $w \in U(n)$ we have that w satisfies the same positive formulas in $\mathcal{U}(n)$ and $(\mathcal{U}(n))^+$. Furthermore, by Lemma 8 there exists a (unique) p-morphism f_w from $((\mathcal{U}(n))^+)_w$ to $\mathcal{U}(n)$, and in particular to $\mathcal{M}$. By the uniqueness we have that $f = \bigcup_{w \in U(n)} f_w$ is a p-morphism from $(\mathcal{U}(n))^+$ onto $\mathcal{M}$. Then we can define $F = i \circ G^{-1} \circ f$.

The above lemma gives us analogues of Lemma 8 and Theorem 9 for positive morphisms.

Theorem 25 *For any finite rooted intuitionistic n-model $\mathfrak{M} = (W, R, V)$ such that for some $x \in W$ and $p \in \mathrm{Prop}$ with $x \notin V(p)$, there exists a unique $w \in U^\star(n)$ and positive morphism of $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_w$.*

Proof. Given any finite rooted intuitionistic n-model $\mathfrak{M}$, Lemma 8 implies that there is a unique $w \in U(n)$ and a p-morphism f from $\mathfrak{M}$ onto $\mathcal{U}(n)_w$. By taking the F from Lemma 24, it follows that $F \circ f$ (with domain $\{x \in M : f(x) \in \mathrm{dom}(F)\}$) is a positive morphism (as a composition of positive morphisms) of $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_{F(w)}$ (assuming of course that $\mathrm{dom}(F \circ f) \neq \varnothing$).
To show the uniqueness we observe that given two positive morphisms g_1, g_2 from $\mathfrak{M}$ to $\mathcal{U}^\star(n)$, $\mathrm{dom}(g_1) = \mathrm{dom}(g_2) = \{x \in M : \exists p \in \mathrm{Prop}(x \notin V(p))\}$, since in no element of $\mathcal{U}^\star(n)$ every propositional atom holds. Thus, if x_0 is the root of $\mathfrak{M}$, $g_1(x_0) \neq g_2(x_0)$ and $g_1[\mathfrak{M}] = \mathcal{U}^\star(n)_{g_1(x_0)}$ and $g_2[\mathfrak{M}] = \mathcal{U}^\star(n)_{g_2(x_0)}$, then there are two different p-morphisms (g_1 and g_2) from $\mathrm{dom}(g_1)$ to $\mathcal{U}(n)$ (since $\mathcal{U}^\star(n)$ is a generated subframe of $\mathcal{U}(n)$), contradicting Lemma 8. $\square$

Theorem 26 *For every n-formula $\varphi \in \mathcal{L}_{\vee,\wedge,\to}$, $\mathcal{U}^\star(n) \models \varphi$ iff $\vdash_{\mathsf{IPC}} \varphi$.*

Proof. One direction is trivial. For the other, let us assume that $\nvdash_{\mathsf{IPC}} \varphi$, i.e. there is a finite rooted model $\mathfrak{M}$ such that $\mathfrak{M}, x \not\models \varphi$, where x is the root of $\mathfrak{M}$. Since φ is negation-free we have that x does not satisfy all propositional atoms. Then by

Theorem 25, there exists a unique $w \in U^\star(n)$ and a positive morphism f from $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_w$. By Proposition 23, it follows that $\mathcal{U}^\star(n), f(x) \not\models \varphi$. □

We will now define the de Jongh formulas for the $[\vee, \wedge, \to]$ fragment of IPC (for the de Jongh formulas for the $[\wedge, \to]$ fragment of IPC, see [3]). We will present two ways of constructing the formulas, one that mirrors the construction of the standard de Jongh formulas, and one that derives the formulas through the algorithm presented in Section 3. For $w \in U^\star(n)$ let $\mathrm{prop}(w), \mathrm{newprop}(w)$ and $\mathrm{notprop}(w)$ be defined as for the elements of $U(n)$.

Definition 27 *Let w be a point of $U^\star(n)$. We will define the formulas $\varphi^\star_w$ and $\psi^\star_w$ by induction on the depth of w:*

- *If $d(w) = 1$ then define*

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{notprop}(w))$$

  and

$$\psi^\star_w = \varphi^\star_w \to \bigwedge_{i \in n} p_i.$$

- *If $d(w) > 1$ then let $w \prec \{w_1, \ldots, w_r\}$ and define*

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i \le r} \psi^\star_{w_i} \to \bigvee_{i \le r} \varphi^\star_{w_i})$$

  and

$$\psi^\star_w = \varphi^\star_w \to \bigvee_{i \le r} \varphi^\star_{w_i}.$$

The construction is motivated by the following observation: As we noted $(\mathcal{U}^\star(n))^+$ is a generated submodel of $\mathcal{U}^\star(n)$. Using the original de Jongh formula φ_w, for w the greatest element of $(\mathcal{U}^\star(n))^+$, we can define the de Jongh formulas from depth 2, using exactly the same construction as for the standard de Jongh formulas. Only now there is no need to take into consideration the ψ_w formula. This is because every negation-free formula is satisfied in a world that satisfies all propositional atoms, and hence all negation-free formulas are true in w.

The above leads us to the second way of constructing the de Jongh formulas for $\mathcal{U}^\star(n)$.

Definition 28 *For every $w \in \mathcal{U}^\star(n)$, we define $\varphi^\star_w$ and $\psi^\star_w$ as $[\varphi_{G(w)}]^*$ and $[\psi_{G(w)}]^*$ respectively, where $[\cdot]^*$ is the operation defined in Proposition 18.2.*

Proposition 29 *The formulas defined in Definition 27 and 28 are equivalent.*

Proof. The proof is by induction on the depth of w. For $d(w) = 1$, we note that $[\varphi_{G(w)}]^*$ is $\bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{prop}(w) \wedge \bigwedge \mathrm{notprop}(w))$, which is clearly equivalent to $\bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{notprop}(w))$ and the ψ_w coincide. For $d(w) = k+1$, since the formulas are inductively constructed in the same manner, the induction hypothesis immediately yields the desired result. □

We can now show that these formulas are indeed analogous to the standard de Jongh formulas:

Proposition 30 *For every $w \in \mathcal{U}^\star(n)$ we have that*

- $V^\star(\varphi_w) = R^\star(w)$
- $V^\star(\psi_w) = \mathcal{U}^\star(n) \setminus (R^\star)^{-1}(w)$

Proof. By Theorem 18.1 we have that any formula σ is equivalent in the top models with $[\sigma]^*$. Hence $\varphi^\star_w$ is satisfied in the same worlds of $\mathcal{M}$ (which is isomorphic to $(\mathcal{U}^\star(n))^+$) as φ_w (and likewise for ψ_w). But since $\varphi^\star_w$ are negation-free formulas, by Proposition 18.1, they will be satisfied in the same worlds in $(\mathcal{U}^\star(n))$. $\square$

The proposition above implies that two distinct points of $\mathcal{U}^\star(n)$ can be distinguished via a positive formula. Indeed, if $w_1 \neq w_2$ are two worlds in $\mathcal{U}^\star(n)$, then either $\neg w_1 R w_2$ or $\neg w_2 R w_1$. In the first case $\mathcal{U}^\star(n), w_2 \not\models \varphi_{w_1}$, while in the second case $\mathcal{U}^\star(n), w_1 \not\models \varphi_{w_2}$. Thus what the positive morphism F defined in Lemma 24 really does is identify the points of $\mathcal{U}(n)$ that satisfy the same positive formulas.

5 n-Henkin models and Jankov's theorem for KC

Let us denote the n-Henkin model for the $[\vee, \wedge, \rightarrow]$-fragment of IPC with $\mathcal{H}^\star(n)$. We write

$$\mathrm{Cn}^\star_n(\varphi) = \{\psi \in [\vee, \wedge \rightarrow] : \psi \text{ is an } n\text{-formula and } \vdash_{\mathsf{IPC}} \varphi \rightarrow \psi\}$$

and we write

$$\mathrm{Th}^\star_n(\mathfrak{M}, w) = \{\varphi \in [\vee, \wedge \rightarrow] : \varphi \text{ is an } n\text{-formula and } \mathfrak{M}, w \models \varphi\}.$$

Proposition 31 *For any point $w \in \mathcal{U}^\star(n)$, $\mathrm{Th}^\star_n(\mathcal{U}^\star(n), w) = \mathrm{Cn}^\star_n(\varphi^\star_w)$.*

Proof. That the right hand side is a subset of the left hand side follows from Proposition 30. For the other direction, assume $\mathcal{U}^\star(n), w \models \sigma$. Then if $\not\vdash_{\mathsf{IPC}} \varphi^\star_w \rightarrow \sigma$, there is a finite model $\mathfrak{M}$ whose root, x, satisfies $\varphi^\star_w$ and does not satisy σ. Then, since x does not satisfy all negation-free formulas, it does not satisfy all propositional atoms, hence there is a nonemtpy positive morphism f from $\mathfrak{M}$ to $\mathcal{U}^\star(n)$. Because x satisfies φ_w Proposition 30 implies that $f(x) \in R^\star(w)$. Since $\mathcal{U}^\star(n), w \models \sigma$ we get $\mathcal{U}^\star(n), f(w) \models \sigma$, a contradiction because f preserves negation-free formulas and specifically σ. $\square$

Lemma 32 *Let Γ be an n-theory of the $[\vee, \wedge, \rightarrow]$-fragment of* IPC*. If $\Gamma \supseteq \mathrm{Cn}^\star(\varphi^\star_w)$, for some $w \in U^\star(n)$, then either there exists some $v \in R^\star(w)$, such that $\Gamma = \mathrm{Cn}^\star(\varphi^\star_v)$, or Γ contains all negation-free formulas.*

Proof. Let $\Gamma \supseteq \mathrm{Cn}^\star(\varphi^\star_w)$ and let v be such that wRv and $\varphi^\star_v \in \Gamma$ while for all immediate successors of v (let $v_1, \dots, v_k$ be all the immediate successors of v) we have that $\Gamma \cap \{\varphi^\star_{v_1}, \dots, \varphi^\star_{v_k}\} = \varnothing$.

If this v is unique we can see that $\Gamma = \mathrm{Cn}^\star_n(\varphi^\star_v)$. One inclusion is trivial; for the other we observe that for every $\sigma \in \Gamma$ we have $\sigma \wedge \varphi^\star_v \nvdash \varphi^\star_{v_1} \vee \dots \vee \varphi^\star_{v_k}$ which implies by Theorem 26 that there is a point of $\mathcal{U}^\star(n)$ that satisfies $\sigma \wedge \varphi^\star_v$ but not $\varphi^\star_{v_1} \vee \dots \vee \varphi^\star_{v_k}$. By Proposition 30, there is only one such element, v. Hence $\sigma \in \mathrm{Th}^\star_n(\mathcal{U}^\star(n), v)$, which by Proposition 31 means that $\sigma \in \mathrm{Cn}^\star_n(\varphi^\star_v)$.

To complete the proof we will show that the aforementioned v is unique or it has depth 1. If $d(v) > 1$ and there is u $(v \neq u)$ with the aforementioned property, then Proposition 30 implies that $\neg(vR^\star w)$ and $\neg(wR^\star v)$ and hence $\psi^\star_v \in \mathrm{Th}^\star_n(\mathcal{U}^\star(n), u)$, thus $\psi^\star_v \in \Gamma$. Therefore, since Γ has the disjunction property, there is some immediate successor v_i of v, such that $\varphi^\star_{v_i} \in \Gamma$. This is a contradiction, hence if $d(v) > 1$ then v is unique.

Finally, if $\varphi^\star_v, \varphi^\star_u \in \Gamma$, where $v \neq u$ and $d(v) = d(u) = 1$, then (without loss of generality we can assume that) there is some propositional atom q true in v but not true in u. By the definition of $\varphi^\star_v$ we have that $q \in \Gamma$. By the definition of $\varphi^\star_u$ we have that $q \to \bigwedge_{i \leq n} p_i \in \Gamma$. Hence all propositional atoms are in Γ, which implies that Γ contains all negation-free formulas. □

Lemma 33 *For any* $w \in \mathcal{U}^\star(n)$ *we have* $\mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi^\star_w)} \cong (\mathcal{U}^\star(n)_w)^+$.

Proof. We will show that the function $g : (\mathcal{U}^\star(n)_w)^+ \to \mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi^\star_w)}$, such that $g(v) = \mathrm{Cn}^\star_n(\varphi^\star_v)$ and the topmost element is mapped to the set of all negation-free formulas, is the isomorphism we are looking for. That it is injective and that the frame relations are preserved follow from Proposition 30. Finally, Lemma 32 implies that g is onto. □

Corollary 34 $\mathit{Upper}(\mathcal{H}^\star(n)) \cong (\mathcal{U}^\star(n))^+$.

Proof. As above, the isomorphism will be given by the function $g : (\mathcal{U}^\star(n))^+ \to \mathit{Upper}(\mathcal{H}^\star(n))$, such that $g(v) = \mathrm{Cn}^\star_n(\varphi^\star_v)$ and the topmost element will be mapped to the set of all negation-free formulas. That this map is injective and preserves the relation is trivial. What is left to show is that it is onto. Let $x \in \mathit{Upper}(\mathcal{H}^\star(n))$, and x does not contain all negation-free formulas. Then, by Theorem 25, there is a positive morphism, f (which is non-empty by the assumptions for x) from $\mathit{Upper}(\mathcal{H}^\star(n))_x$ onto some $\mathcal{U}^\star(n)_w$. Then we observe by Proposition 23 that $\mathrm{Th}^\star_n(\mathcal{U}^\star(n), w) = x$, i.e., by Proposition 31 $x = \mathrm{Cn}^\star_n(\varphi^\star_w)$. □

Corollary 35 *Let* $\mathfrak{M} = (W, R, V)$ *be any* n*-model and let* $X \subseteq V(\varphi^\star_w)$ *for some* $w \in U^\star(n)$ *and* $X \neq \varnothing$. *Then there is a unique positive morphism* f *from* $\mathfrak{M}_X$ *to* $\mathcal{U}^\star(n)_w$. *Furthermore if* $\mathfrak{M}_X$ *is rooted and does not satisfy all negation-free formulas, then there is a unique* $v \in U^\star(n)$ *such that* $wR^\star v$ *and* f *is from* $\mathfrak{M}_X$ *onto* $\mathcal{U}^\star(n)_v$.

Proof. Since $X \subseteq V(\varphi^\star_w)$ for every $y \in W$ such that xRy for some $x \in X$, we have that $\mathrm{Th}^\star_n(\mathfrak{M}, y) \supseteq \mathrm{Cn}^\star_n(\varphi^\star_w)$. By Lemma 32 such a theory is equal to some $\mathrm{Cn}^\star_n(\varphi^\star_v)$ or contains all negation-free formulas. We define the positive morphism f by making $f(y) = u$ such that $\mathrm{Th}^\star_n(\mathfrak{M}, y) = \mathrm{Cn}^\star_n(\varphi^\star_u)$ (if no such u exists then $y \notin \mathrm{dom}(f)$).

If the domain of f is empty then it is vacuously a positive morphism. If the domain is non-empty, by the definition of f the only non-trivial step to show that f is a positive morphism is the back condition. For this we have: If $vR^\star u$ and $f(y) = v$, then by Proposition 30, it is the case that $\mathfrak{M}, y \nvDash \psi^\star_u$; hence there is some $z \in W$ with yRz such that $\mathfrak{M}, z \models \varphi^\star_u$ and $\mathfrak{M}, z \nvDash \bigvee_{i \leq l} \varphi^\star_{u_i}$. This yields that $\mathrm{Th}^\star_n(\mathfrak{M}, z) = \mathrm{Cn}^\star_n(\varphi^\star_u)$, i.e. $f(z) = u$.

Finally, if $\mathfrak{M}_X$ is rooted and it doesn't satisfy all negation-free formulas, then the root, x, is in the domain of f. Then we let $v = f(x)$. $\square$

Note that the underlying Kripke frame of $\mathcal{U}^\star(n)_w = (\mathcal{U}^\star(n)_w, R^\star(n)_w, V^\star(n)_w)$ described in the previous lemma can be viewed as the general frame $(\mathcal{U}^\star(n)_w$, $R^\star(n)_w$, $Up(\mathcal{U}^\star(n)_w))$, which is a descriptive frame since W is finite.
We can prove an analogue of the Jankov's-de Jongh theorem (Theorem 16), which will be used to give an alternative proof of Jankov's theorem for KC.

Theorem 36 (Jankov's theorem for the positive fragment of IPC) *For every descriptive frame $\mathfrak{G}$ and $w \in U^\star(n)$ we have that $\mathfrak{G} \nvDash \psi^\star_w$ if and only if there is an n-valuation V on $\mathfrak{G}$ such that $\mathcal{U}^\star(n)_w$ is the image, through a positive morphism, of a generated submodel of $(\mathfrak{G}, V)$.*

Proof. Let $\mathcal{U}^\star(n)_w$ be the image, through a positive morphism f, of a generated submodel $\mathcal{K}$ of $(\mathfrak{G}, V)$. Proposition 30 implies that $\mathcal{U}^\star(n)_w, w \nvDash \psi^\star_w$. Since f is a positive morphism, Proposition 23 yields that $\mathcal{K}, x \nvDash \psi^\star_w$ for every $x \in f^{-1}[\{w\}]$. Now, because $\mathcal{K}$ is a generated submodel of $(\mathfrak{G}, V)$, we have that $(\mathfrak{G}, V), x \nvDash \psi^\star_w$, i.e. $\mathfrak{G} \nvDash \psi^\star_w$.

For the other direction, let us assume that there is some valuation and some x such that $(\mathfrak{G}, V), x \nvDash \psi^\star_w$. This implies that there is some y_0 such that xRy_0 and $(\mathfrak{G}, V), y_0 \models \varphi^\star_w$, while $(\mathfrak{G}, V), y_0 \nvDash \varphi^\star_{w_i}$, for all immediate successors w_i of w.

We take $(\mathfrak{G}, V)_{V(\varphi^\star_w)}$, the submodel of $(\mathfrak{G}, V)$ generated by $V(\varphi^\star_w)$. We note that by the above observation $V(\varphi^\star_w) \neq \varnothing$. Furthermore, we have that $(\mathfrak{G}, V)_{V(\varphi^\star_w)}$ does not satisfy all negation free formulas since $y_0 \in V(\varphi^\star_w)$ and $(\mathfrak{G}, V), y_0 \nvDash \varphi^\star_{w_i}$, for all immediate successors w_i of w.

Therefore, by Corollary 35, we have that there is a positive morphism f from $(\mathfrak{G}, V)_{V(\varphi^\star_w)}$ to $\mathcal{U}^\star(n)_w$. It is onto because $\mathrm{Th}^\star((\mathfrak{G}, V), y_0) = \mathrm{Cn}^\star(\varphi^\star_w)$ and hence $f(y_0) = w$.

Finally, we have that $(\mathfrak{G}, V)_{V(\varphi_w)}$ is a descriptive model, by Lemma 4, since it is based on $V(\varphi_w)$. To show that the positive morphism is also descriptive, we only need to show that $f^{-1}[R^\star(v)] \cup (\mathfrak{G} \setminus \mathrm{dom}(f)) = V(\varphi_v^\star)$, for $v \in \mathcal{U}^\star(n)_w$. For the left to right inclusion we observe that anything outside the domain of f satisfies all negation-free formulas and f preserves negation-free formulas. For the right to left assume that $x \in V(\varphi_v^\star)$. Then $x \in V(\varphi_w^\star)$ and by Lemma 32 we get that $f(x) \in R^\star(w)$ or x satisfies all propositional atoms and hence it is not in the domain of f. □

We recall that KC is complete with respect to the finite frames with a topmost node. Thus, by reflecting on Proposition 18.1, one can easily see that KC proves exactly the same negation-free formulas as IPC. Jankov, in [11] proved that KC is maximal with that property. In [6] an alternative proof based on the universal model for IPC is given.

Using the universal model for negation-free formulas we can provide a simpler and more insightful proof of that fact:

Lemma 37 *If $\mathfrak{F}$ is a descritpive frame with a topmost element, and $f : (\mathfrak{G}, V) \to (\mathfrak{F}, V')$ is a descriptive positive morphism between models, then f can be extended to a descriptive frame p-morphism.*

Proof. If f is a total then it is a frame p-morphism. If f is not total then, we extend f to f' such that for every $y \in \mathfrak{G} \setminus \mathrm{dom}(f)$ we have $f'(y) = x_0$, where x_0 is the topmost element of $\mathfrak{F}$. We claim that f' is the desired frame p-morphism. That the forth condition holds is trivial, since everything in $\mathfrak{F}$ is below x_0. For the back condition the only possible problem may arise if some $f'(y)Rx_0$. In that case, if $y \in \mathrm{dom}(f)$ then $f(y)Rx_0$ and by the definition of positive morphisms a witness for the back condition exists. If $y \notin \mathrm{dom}(f)$ then the witness is y. To show that it is descriptive we only need to show that $f'^{-1}[Q]$ is admissible, where Q is admissible in $\mathfrak{F}$. But, by the construction of f we have that $f'^{-1}[Q] = f^{-1}[Q] \cup (\mathfrak{G} \setminus \mathrm{dom}(f))$, which is admissible since it is equal to $f^*(Q)$ and f is a descriptive positive morphism. □

Theorem 38 (*Jankov*) *For every logic $\mathcal{L} \nsubseteq$ KC there exists some negation-free formula σ such that $\mathcal{L} \vdash \sigma$ while IPC $\nvdash \sigma$.*

Proof. Let us assume that $\mathcal{L} \nsubseteq$ KC. Then $\mathcal{L} \vdash \chi$ and KC $\nvdash \chi$ for some formula χ. As KC is complete with respect to finite rooted frames with a topmost element (e.g. [4]), there is a a finite rooted frame with a topmost element, $\mathfrak{F} = (W, R)$ with $\mathfrak{F} \nvDash \chi$. We define a valuation, V, on $\mathfrak{F}$ such that each of its elements has a different color and that there is a propositional atom, q, not satisfied at the topmost element. A way to do this is to have a propositional atom p_x for each $x \in F$ such that $V(p_x) = R(x)$ and $V(q) = \varnothing$. By Theorem 25, there is some $w \in U(n)$ and a positive morphism from $(\mathfrak{F}, V)$ onto $\mathcal{U}^\star(n)_w$. Since each element of $(\mathfrak{F}, V)$ has a different color, the positive morphism is 1-1 and since in every element of W at least one propositional atom is not satisfied, the positive morphism has W as its domain, hence $(\mathfrak{F}, V) \cong \mathcal{U}^\star(n)_w$.

We claim that the negation-free formula, σ, that we are looking for is $\psi^\star_w$. Heading towards a contradiction, let us assume that $\mathcal{L} \nvdash \psi^\star_w$. Then, as every logic is complete with respect to descriptive frames (e.g. [2], [4]), there exists a descriptive $\mathcal{L}$-frame, $\mathfrak{G}$ such that $\mathfrak{G} \not\models \psi^\star_w$. By Theorem 36 there is a valuation V' on $\mathfrak{G}$, a generated submodel $\mathcal{K}$ of $(\mathfrak{G}, V')$, and a descriptive positive morphism f, from $\mathcal{K}$ onto $(\mathfrak{F}, V)$. By Lemma 37, f can be extended to a descriptive frame p-morphism, f'. We have reached a contradiction: Since $\mathfrak{G}$ is an $\mathcal{L}$-frame and $\chi \in \mathcal{L}$, we have that $\mathfrak{G} \models \chi$. As f' is a descriptive frame p-morphism, $\mathfrak{G} \models \chi$ implies that $\mathfrak{F} \models \chi$, contrary to the assumption that $\mathfrak{F} \not\models \chi$. □

6 Duality between n-free positive Heyting algebras and the universal model for the $[\vee, \wedge, \to]$-fragment of IPC

In this section, we will present the duality between the n-universal model $\mathcal{U}^\star(n)$ and the n-free positive Heyting algebra. Here we will only consider the case of n-generated free positive Heyting algebras and leave the general case as future work.

Definition 39 (n-free positive Heyting algebra) *Let $\mathbf{A}(X)$ be the free Heyting algebra generated by $X = \{a_1, \ldots, a_n\}$. The* n-free positive Heyting algebra *$\mathbf{A}^\sharp(X)$ is the $\{\vee, \wedge, \to\}$-subalgebra of $\mathbf{A}(X)$ generated by X.*

We can also define the following algebra of positively definable upsets in $\mathcal{U}^\star(n)$:

Definition 40 (n-positive upset algebra) *Let $\mathcal{U}^\star(n)$ denote the n-universal model of positive* IPC*. The* n-positive upset algebra *$Up^+(n)$ is defined as the algebra $(Up^+(\mathcal{U}^\star(n)), \cup, \cap, \supset)$ where $Up^+(\mathcal{U}^\star(n))$ is the collection of all upsets of $\mathcal{U}^\star(n)$ definable by some $\varphi \in \mathcal{L}_{\vee,\wedge,\to}$, and $\cup, \cap, \supset$ are defined as in Definition 2.*

Theorem 41 *The n-positive upset algebra $Up^+(n)$ is isomorphic to the n-free positive Heyting algebra $\mathbf{A}^\sharp(X)$.*

Proof. Let us define the map $i : Up^+(n) \to \mathbf{A}^\sharp(X)$ by $i(V^\star(\varphi(p_1, \ldots, p_n))) = \varphi(a_1, \ldots, a_n)$, for each positive formula φ.

To show that this map is well defined, suppose that given a set Y, there are different formulas φ, $\psi \in \mathcal{L}_{\vee,\wedge,\to}$ such that $V^\star(\varphi) = V^\star(\psi) = Y$. Then $V^\star(\varphi \leftrightarrow \psi) = U^\star(n)$. By Theorem 26, $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$. Therefore, in the n-generated free Heyting algebra, $\varphi(a_1, \ldots, a_n) = \psi(a_1, \ldots, a_n)$, which also holds in the n-free positive Heyting algebra. Therefore i is well defined.

For surjectivity, for every element $a \in \mathbf{A}^\sharp(X)$, consider the formula φ such that $a = \varphi(a_1, \ldots, a_n)$, then $i(V^\star(\varphi(p_1, \ldots, p_n))) = \varphi(a_1, \ldots, a_n)$. For injectivity, consider two sets $V^\star(\varphi(p_1,\dots,p_n))$ and $V^\star(\psi(p_1,\dots,p_n))$ with $V^\star(\varphi(p_1,\dots,p_n)) \neq V^\star(\psi(p_1,\dots,p_n))$, then $V^\star(\varphi \leftrightarrow \psi) \neq U^\star(n)$. By Theorem 26, $\nvdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi$. Therefore, in the n-generated free Heyting algebra, $\varphi(a_1,\dots,a_n) \neq \psi(a_1,\dots,a_n)$, which is also the case in the n-free positive Heyting algebra. Therefore i is injective. Finally, it is easy to check that i is a homomorphism, which finishes the proof. □

Remark 42 As we have already seen in the previous section, there are two different ways to describe the n-universal model for positive IPC, namely $\mathcal{U}^\star(n)$ and $(\mathcal{U}^\star(n))^+$. Theorem 41 states that the positive Heyting algebra of upsets of $\mathcal{U}^\star(n)$ definable by positive formulas is isomorphic to $\mathbf{A}^\sharp(X)$. Similarly one can prove, using Proposition 17, that a positive Heyting algebra of upsets of $(\mathcal{U}^\star(n))^+$ definable by positive formulas is isomorphic to $\mathbf{A}^\sharp(X)$. However, there is more that we can say about definable upsets of $(\mathcal{U}^\star(n))^+$. The algebra $Up^+_\bot(n)$ of all upsets of $(\mathcal{U}^\star(n))^+$ definable in the enriched language $\mathcal{L}_{\vee,\wedge,\to,\bot}$ is isomorphic to the algebra $\mathbf{A}^\sharp_\bot(X)$ obtained by adjoining to $\mathbf{A}^\sharp_\bot(X)$ a new bottom element $\{\bot\}$.

7 Application: Minimal Logic

In this part we give an application of the positive universal model for IPC: the positive $(n+1)$-universal model of IPC is essentially the n-universal model for the minimal logic MPC. In the following we will only give a sketch of the proof and will skip most of the technical details.

Minimal logic is obtained from the $[\vee,\wedge,\to]$-fragment of IPC by adding a weaker negation: $\neg\varphi$ is defined as $\varphi \to f$, where the variable f is a special proposition variable interpreted as the falsum. Therefore, the language of minimal logic is the $[\vee,\wedge,\to]$-fragment of IPC plus f. f has no specific properties, in particular $f \to \varphi$ does not hold, therefore the Hilbert system for the minimal logic is the same as IPC but without $f \to \varphi$. Note that every negation-free formula containing f is equivalent in MPC to a formula not containing f, because $\vdash_{\mathsf{MPC}} f \leftrightarrow \neg p \wedge \neg\neg p$. For the semantics, f is interpreted as an ordinary proposition letter. Therefore, if we regard the f as an ordinary proposition letter in the syntax, we will get the semantics for IPC in the $[\vee,\wedge,\to]$-fragment, with an additional proposition letter f.

Then by defining the n-universal model $\mathcal{U}_{\mathsf{MPC}}(n)$ for minimal logic as $\mathcal{U}^\star_{\mathsf{IPC}}(n+1)$, where the $(n+1)$-th proposition letter is f, we can check that the model is indeed universal for minimal logic. The proof can be easily obtained by using the semantics for IPC and minimal logic.

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