Justified Belief and the Topology of Evidence

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Abstract. We introduce a new topological semantics for evidence, evidence-based belief, knowledge and learning. This setting generalizes (and in a sense improves on) the evidence models for belief due to van Benthem and Pacuit, as well as our own previous work on (a topological semantics for) Stalnaker's doxastic-epistemic axioms. We prove completeness, decidability and finite model property for the associated logic, and we look at several types of evidential dynamics.

1 Introduction

In this paper we propose a topological semantics for a notion of *evidence-based belief*, as well as for a notion of ("soft", defeasible) *knowledge*, and explore their connections with various notions of *evidence possession*. This work is largely based on looking from a new perspective at the models for evidence, belief and evidence-management proposed by van Benthem and Pacuit [18], and developed further by van Benthem, Fernandez-Duque and Pacuit [17].

The *basic pieces of evidence* possessed by an agent are modeled as non-empty sets of possible worlds. A *combined evidence* (or just "evidence", for short) is any non-empty intersection of finitely many pieces of evidence. This notion of evidence is not necessarily factive[3], since the pieces of evidence are possibly false (and possibly inconsistent with each other). The family of (combined) evidence sets forms a topological basis, that generates what we call *the evidential topology*. This is the smallest topology in which all the basic pieces of evidence are open, and it will play an important role in our setting. We study the operator of "having (a piece of) evidence for a proposition *P*" proposed by van Benthem and Pacuit, but we also investigate other interesting variants of this concept: "having (combined) evidence for *P*", "having a (piece of) *factive* evidence for *P*" and "having (combined) factive evidence for *P* ". We show that the last notion coincides with the *interior operator* in the evidential topology, thus matching Tarski's original topological semantics for modal logic [11]. We also show that the two factive variants of evidence-possession operators are more expressive than the original (non-factive) one, being able (when interacting with the global modality) to define the non-factive variants, as well as many other doxastic/epistemic operators.

[3] Factive evidence is true in the actual world. In Epistemology it is common to reserve the term "evidence" for factive evidence. But we follow here the more liberal usage of this term in [17], which agrees with the common usage in day to day life, e.g. when talking about "uncertain evidence", "fake evidence", "misleading evidence" etc.

A *body of evidence* is a family of evidence such that every finite subfamily is consistent. A body of evidence is *maximal* if it is a "strongest" such body of evidence: it cannot be 'strengthened' (properly extended) to any other body of evidence. Maximal bodies of evidence are essential for the definition of belief proposed by van Benthem and Pacuit, according to which *an agent believes a proposition P if P is entailed by all the maximal bodies of evidence*. This definition works well in the finite case (as well as in the more general case of Alexandroff topologies), but (as already noted in [17]) it has the defect that it can produce *inconsistent beliefs* in the infinite case.[4]

In this paper, we propose an 'improved' semantics for evidence-based belief, obtained by weakening the definition from [18]. According to us, a proposition P is believed if *P is entailed by all finite bodies of evidence that are "sufficiently strong"* (i.e., iff every finite body of evidence can be strengthened to some finite body of evidence that entails P). This definition coincides with the one of van Benthem and Pacuit for finite models, but involves a different generalization of their notion in the infinite case. In fact, our semantics *always ensures consistency of belief*, even when the available pieces of evidence are mutually inconsistent. In this sense, our work can be seen as extending, generalizing and (to an extent) "streamlining" the framework from [18].

We also provide a formalization of a "coherentist" view on justifications. An *argument* essentially consists of one or more evidence sets supporting the same proposition (thus providing multiple evidential paths towards a common conclusion); a *justification* is an argument that is not contradicted by any other evidence. Our definition of belief is equivalent to requiring that *P is believed iff there is some (evidence-based) justification for P*: hence, our notion accurately captures the concept of "justified belief". Our proposal is also very natural from a topological perspective: it is equivalent to saying that *P is believed iff it's true in "almost all" epistemically-possible worlds* (where 'almost all' is interpreted topologically: all except for a nowhere-dense set).

Moving on to 'knowledge', there are a number of different notions one may consider. First, there is "absolutely certain" or "infallible" knowledge, akin to Aumann's concept of 'partitional knowledge' or van Benthem's concept of 'hard information'. In our single-agent setting, this can be simply defined as the global modality (quantifying universally over all epistemically-possible worlds). There are propositions that are 'known' in this infallible way (-e.g. the ones known by introspection or by logical proof), but very few: most facts in science or real-life are unknown in this sense. Hence, it is more interesting to look at notions of knowledge that are less-than-absolutely-certain: so-called 'defeasible knowledge'. The famous Gettier counterexamples [6] show that simply adding "factivity" to belief will not do: true (justified) belief is extremely fragile (i.e. it can be too easily lost), and it is consistent with having only wrong justifications for an (accidentally) true conclusion. One path often discussed by philosophers (e.g. Lehrer and Stalnaker) is to require a *correct justification*.[5] We formalize this notion by saying that *P is known if there is a factive (true) justification for P*.

[4] A more technical defect of that setting is that the corresponding doxastic logic does not have the finite model property [17].

[5] This is sometimes called the "no-false-lemma" approach: a belief is knowledge if it is not based on any false justification.

A stronger requirement (than the no-false-lemma approach) was championed mainly by Lehrer [9, 10], under the name of "Defeasibility Theory of Knowledge". According to this view, P is known (in the in-defeasible sense) only if there is a factive justification for P that cannot be defeated by any further true evidence. This means that the justification is consistent, not only with the currently available evidence, but also with *any potential (new) factive evidence* that the agent might learn in the future. This version of the theory has been criticized as being too strong: some new evidence might be 'misleading' or 'deceiving' despite being true. A weaker version of Defeasibility Theory requires that knowledge is undefeated only by "non-misleading" evidence. In our setting, a proposition P is said to be a *potentially misleading evidence* if it can indirectly generate false evidence (i.e. if by adding P to the family of currently available pieces of evidence we obtain at least one false combined evidence). Misleading propositions include all the false ones, but they may also include some true ones. We show that our notion of knowledge matches this weakened version of Defeasibility Theory (though not the strong version).

Yet another path leading to our setting in this paper goes via our previous work [1, 2] on a topological semantics for the doxastic-epistemic axioms proposed by Stalnaker [14]. These axioms were meant to capture a notion of fallible knowledge, in close interaction with a notion of "strong belief" (defined as "subjective certainty" or the "feeling of knowledge"). The main principle specific to this system was that "believing implies believing that you know" ($Bp \rightarrow BKp$), which goes in direct contradiction to Negative Introspection for Knowledge.[6] The topological semantics that we proposed for these concepts in [1, 2] was overly restrictive (being limited to the rather exotic class of "extremally disconnected" topologies). In this paper, we show that these notions can be interpreted on arbitrary topological spaces, without changing their logic. Indeed, our definitions of belief and knowledge above can be seen as the natural generalizations to arbitrary topologies of the notions in [1, 2].

In the last section, we completely axiomatize the various resulting logics, proving their decidability and finite model property, and we look at various dynamic extensions, encoding several types of evidential dynamics.

2 Evidence, Belief and Knowledge in Topological Spaces

2.1 Topological Models for Evidence

Definition 1 (Evidence Models) *(van Benthem and Pacuit)*[7] Given a countable set of propositional letters Prop, an *evidence model* for Prop is a tuple $\mathcal{M} = (X, E_0, V)$, where: X is a non-empty set of "states"; $E_0 \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is a family of non-empty sets called

[6] Indeed, the logic of Stalnaker's knowledge is not $S5$, but the modal logic $S4.2$.

[7] The notion of evidence model in [18] is more general, covering cases in which evidence depends on the actual world, but we stick with what they call 'uniform' models, since this corresponds to restricting to agents who are "evidence-introspective".

basic evidence sets (or *pieces of evidence*), with $X \in E_0$; and $V : \text{Prop} \to \mathcal{P}(X)$ is a valuation function.

Given an evidence model $\mathcal{M} = (X, E_0, V)$, a *finite body of evidence* is simply a finite set of mutually consistent pieces of evidence (i.e. a finite $F \subseteq E_0$ s.t. $\bigcap F \neq \emptyset$). More generally, a (possibly infinite) *body of evidence* is a family $F \subseteq E_0$ of pieces of evidence s.t. every non-empty finite subfamily is consistent (i.e. $\forall F' \subseteq F\, (F'$ finite $\neq \emptyset \Rightarrow \bigcap F' \neq \emptyset)$). We denote by $\mathcal{F}$ the family of all bodies of evidence over $\mathcal{M}$, and by $\mathcal{F}^{finite}$ the family of all finite bodies of evidence. A body of evidence F *supports a proposition* P iff P is true in every world satisfying all the evidence in F (i.e. if $\bigcap F \subseteq P$).

The *strength order* between bodies of evidence is given by inclusion: $F \subseteq F'$ means that F' is at least as strong as F. Note that stronger bodies of evidence support more propositions: if $F \subseteq F'$ then every proposition supported by F is also supported by F'. A body of evidence is *maximal* ("strongest") if it's not included in any other such body. We denote by $Max_{\subseteq}\mathcal{F} = \{F \in \mathcal{F} : \forall F' \in \mathcal{F}(F \subseteq F' \Rightarrow F = F')\}$ the family of all maximal bodies of evidence. By Zorn's Lemma, *every body of evidence can be strengthened to a maximal body of evidence*: $\forall F \in \mathcal{F}\, \exists F' \in Max_{\subseteq}\mathcal{F}(F \subseteq F')$.

A *combined evidence* (or just "*evidence*", for short) is any non-empty intersection of finitely many pieces of evidence. We denote by $E := \{\bigcap F : F \in \mathcal{F}^{finite}$ s.t. $\bigcap F \neq \emptyset\}$ the family of all (combined) evidence.[8] A (combined) evidence $e \in E$ *supports* a proposition $P \subseteq X$ if $e \subseteq P$. (In this case, we also say that e is *evidence for* P.) Note that the natural strength order between combined evidence sets goes the other way around (reverse inclusion): $e \supseteq e'$ means that e' is at least as strong as e.[9]

The intuition is that $e \in E_0$ represent the basic pieces of "direct" evidence (obtained say by observation or via testimony) that are possessed by the agent, while the combined evidence $e \in E$ represents indirect evidence that is obtained by combining finitely many pieces of direct evidence. Not all of this evidence is necessarily true though.

We say that some (basic or combined) evidence $e \in E$ is *factive evidence* at world $x \in X$ whenever it is true at x (i.e. $x \in e$). A body of evidence F is factive if all the pieces of evidence in F are factive (i.e. $x \in \bigcap F$).

The *plausibility (pre)order* $\sqsubseteq_E$ associated to an evidence model is given by:

$$x \sqsubseteq_E y \;\text{ iff }\; \forall e \in E_0\, (x \in e \Rightarrow y \in e) \;\text{ iff }\; \forall e \in E\, (x \in e \Rightarrow y \in e).$$

Definition 2 (Topological Space) A *topological space* is a pair $\mathcal{X} = (X, \tau)$, where X is a non-empty set and τ is a *topology* on X, i.e. a family $\tau \subseteq \mathcal{P}(X)$ containing X and $\emptyset$, and closed under finite intersections and arbitrary unions. Given a family $E \subseteq \mathcal{P}(X)$ of subsets of X, the *topology generated by* E is the smallest topology τ_E on X such that $E \subseteq \tau_E$. A set $A \subseteq X$ is *closed* iff it is the complement of an open set, i.e. it is of

[8] This is a difference in notation with the setting in [18, 17], where E is used to denote the family of basic evidence sets (denoted here by E_0).

[9] This is both to fit with the strength order on bodies of evidence (since $F \subseteq F'$ implies $\bigcap F \supseteq \bigcap F'$), and to ensure that stronger evidence supports more propositions: since, if $e \supseteq e'$, then every proposition supported by e is supported by e'.

the form $X \setminus U$ with $U \in \tau$. Let $\tau^c = \{X \setminus U \mid U \in \tau\}$ denote the family of all closed sets of $\mathcal{X} = (X, \tau)$. In any topological space $\mathcal{X} = (X, \tau)$, one can define two important operators, namely *interior* $Int : \mathcal{P}(X) \to \mathcal{P}(X)$ and *closure* $Cl : \mathcal{P}(X) \to \mathcal{P}(X)$, given by $IntP := \bigcup\{U \in \tau \mid U \subseteq P\}$, $ClP := \bigcap\{C \in \tau^c \mid P \subseteq C\}$. A set $A \subseteq X$ is called *dense* in $\mathcal{X}$ if $ClA = X$ and it is called *nowhere dense* if $IntClA = \emptyset$. For a topological space $\mathcal{X} = (X, \tau)$, the *specialization preorder* $\sqsubseteq_\tau$ is given by: $x \sqsubseteq_\tau y$ iff $\forall U \in \tau\, (x \in U \Rightarrow y \in U)$.

Special Case: Relational Spaces. A topological space is called *Alexandroff* iff the topology is closed under arbitrary intersections. An Alexandroff topology is fully captured by its specialization preorder: *in this case, the interior operator coincides with the Kripke modality for the specialization relation* (i.e. $IntP = \{x \in X \mid \forall y\, (x \sqsubseteq_\tau y \Rightarrow y \in P)\}$). There is a canonical bijection between Alexandroff topologies $\mathcal{X} = (X, \tau)$ and *preordered spaces*[10] $(X, \leq)$, mapping (X, τ) to $(X, \sqsubseteq_\tau)$; the inverse map takes $(X, \leq)$ into $(X, Up(X))$, where $Up(X)$ is the family of upward-closed sets[11].

An Even More Special Case: (Grove/Lewis) Sphere Spaces. These are topological spaces in which the opens are "nested", i.e. for every $U, U' \in \tau$, we have either $U \subseteq U'$ or $U' \subseteq U$. Sphere spaces are Alexandroff, and moreover they correspond exactly to *totally preordered spaces* (i.e. sets X endowed with a total preorder $\leq$).

Definition 3 (Topological Evidence Models) A *topological evidence model* ("*topo-e-model*", for short) is a structure $\mathcal{M} = (X, E_0, \tau, V)$, where (X, E_0, V) is an evidence model and $\tau = \tau_E$ is the topology generated by the family of combined evidence E (or equivalently, by the family of basic evidence sets E_0)[12], which will be called *the evidential topology*. It is easy to see that *the plausibility order* $\sqsubseteq_E$ *of* $\mathcal{M}$ *coincides with the specialization order* of the associated topology: $\sqsubseteq_E = \sqsubseteq_\tau$.

Since any family $E_0 \subseteq \mathcal{P}(X)$ generates a topology, topo-e-models are just another presentation of (uniform) evidence models. We use this special terminology to stress our focus on the topology, and to avoid ambiguities (since our definition of belief in topo-e-models will be different from the definition of belief in evidence models in [18]).

A topo-e-model is said to be *Alexandroff* iff the underlying topology is Alexandroff. So they can be understood as *relational (plausibility) models*, in terms of a preorder $\leq$ ("plausibility relation"). A special case is the one of *Grove-Lewis (topological) evidence models*: this is the case when the basic pieces of evidence are nested (i.e. for all $e, e' \in E_0$ we have either $e \subseteq e'$ or $e' \subseteq e$). It is easy to see that in this case all the opens of the generated topology are also nested, so the topology is that of a sphere space.

Proposition 1 *Given a topo-e-model* $\mathcal{M} = (X, E_0, \tau, V)$*, the following are equivalent:*

1. $\mathcal{M}$ *is Alexandroff;*
2. *The family E of (combined) evidence is closed under arbitrary non-empty intersections (i.e., if $F \subseteq E$ and $\bigcap F \neq \emptyset$, then $\bigcap F \in E$);*

[10] A *preorder* on X is a reflexive-transitive relation on X.
[11] A subset $A \subseteq X$ is said to be *upward-closed* wrt $\leq$ if $\forall x, y \in X\, (x \in A \wedge x \leq y \Rightarrow y \in A)$.
[12] These families generate the same topology. We denote it by τ_E only because the family E of combined evidence forms a *basis* of this topology.

3. *Every body of evidence is equivalent to a finite body of evidence (i.e.* $\forall F \in \mathcal{F}\ \exists F' \in \mathcal{F}^{finite}$ *s. t.* $\bigcap F = \bigcap F'$*).*

Arguments and Justifications. We can use this setting to formalize a "coherentist" view on justification, in the spirit of Lehrer [9]. An *argument for P* is a disjunction $U = \bigcup_{i\in I} e_i$ of (some non-empty family of) (combined) evidences $e_i \in E$ that all support P (i.e. $e_i \subseteq P$ for all $i \in I$, or equivalently $U \subseteq P$). Thus, an argument may provide *multiple evidential paths* e_i to support a common conclusion P. Topologically, an argument for P is the same as a *non-empty open subset of* P ($U \in \tau_E$ s. t. $U \subseteq P$). Also, the interior $IntP$ is the *weakest (most general) argument for* P.

A *justification for P* is an argument U for P that is consistent with every (combined) evidence (i.e., $U \cap e \neq \emptyset$ for all $e \in E$, which in fact implies that $U \cap U' \neq \emptyset$ for all $U \in \tau_E \setminus \{\emptyset\}$). So justifications are arguments that are not defeated by any available evidence (or any other argument based on this evidence).[13] Topologically, we can see that a justification for P is just an *(everywhere) dense open subset* of P (i.e. $U \in \tau_E$ s. t. $U \subseteq P$ and $Cl_{\tau_E}(U) = X$). As for evidence, an argument or a justification for P is said to be *factive* (or "correct") if it is true in the actual world. The fact that arguments are open in the generated topology encodes the principle that *any argument should be evidence-based*: whenever an argument is correct, then it is supported by some factive evidence. To anticipate further: in our setting, justifications will form the basis of *belief*, while correct justifications will form the basis of *(defeasible) knowledge*. But for now we'll introduce a stronger form of "knowledge": the absolutely-certain and irrevocable kind.

Infallible Knowledge: possessing hard information. We use $\forall$ for the so-called *global* modality, which associates to every proposition $P \subseteq X$, some other proposition $\forall P$, given by putting: $(\forall P) := X$ iff $P = X$, and $(\forall P) := \emptyset$ otherwise. In other words: $(\forall P)$ holds (at any state) iff P holds at all states. In this setting, $\forall$ is interpreted as "absolutely certain, *infallible knowledge*", defined as truth in all the worlds that are consistent with the agent's information.[14] This is not a realistic concept of knowledge, but just a limit notion, encompassing all epistemic possibilities.

Having *Basic* Evidence for a Proposition. van Benthem and Pacuit define, for every proposition $P \subseteq X$, another proposition[15] $E_0 P$ given by putting: $E_0 P := X$ if $\exists e \in E_0\, (e \subseteq P)$, and $E_0 P := \emptyset$ otherwise. Essentially, $E_0 P$ means that "the agent *has basic evidence for P*", i.e. P is supported by some available piece of evidence. One can also introduce a *factive* version of this proposition: $\Box_0 P$, read as "the agent has *factive basic*

[13] This can be connected to Lehrer's Subjective Justification Game [9], in which justified beliefs are based on arguments that cannot be defeated by other arguments (based on the same "justification system", i.e. the same set E_0 of evidence pieces.

[14] In a multi-agent model, some worlds might be consistent with one agent's information, while being ruled out by another agent's information. So, in a multi-agent setting, $\forall_i$ will only quantify over all the states in agent i's current information cell (according to a partition Π_i of the state space reflecting agent i's hard information).

[15] They denote this by EP, but we use $E_0 P$ for this notion, since we reserve the notation EP for having *combined* evidence for P.

evidence for P", is given by putting

$$\Box_0 P := \{x \in X : \exists e \in E_0\, (x \in e \subseteq P)\}.$$

Having (Combined) Evidence for a Proposition. If in the above definitions of E_0P and $\Box_0 P$ we replace basic pieces of evidence by combined evidence, we obtain two other operators EP, meaning that "the agent *has (combined) evidence for P*", and $\Box P$, meaning that "the agent *has factive (combined) evidence for P*". More precisely:

$$EP := X \text{ if } \exists e \in E\, (e \subseteq P), \quad \text{and } EP := \emptyset \text{ otherwise;}$$

$$\Box P := \{x \in X : \exists e \in E\, (x \in e \subseteq P)\}.$$

Observation 1. Note that *the agent has evidence for a proposition P iff she has an argument for P*. So EP can also be interpreted as "having an argument for P". Similarly, $\Box P$ can be interpreted as "having a *correct (i.e. factive) argument* for P".

Observation 2. Note that the agent has factive evidence for P at x iff x is in the interior of P. So our modality $\Box$ *coincides with the interior operator*: $\Box P = IntP$.

2.2 Belief

Belief à la van Benthem-Pacuit [18]. The notion of belief proposed by van Benthem and Pacuit, which we will denote by *Bel*, says *P is believed iff every maximal body of evidence supports P*: $BelP$ holds (at any state of X) iff we have $\bigcap F \subseteq P$ for every $F \in Max_{\subseteq}\mathcal{F}$. This definition can be generalized to *conditional beliefs* $Bel^Q P$, but we skip the details here, referring to [18] instead. As already noticed in [18], this is equivalent to treating evidence models as special cases of plausibility models [3, 4, 15], with the plausibility relation given by $\sqsubseteq_E$ (or equivalently, as Grove-Lewis "sphere models" [8] where the spheres are the sets that are upward closed wrt $\sqsubseteq_E$), and applying the standard definition (due to Grove) of belief as "truth in all the most plausible worlds".[16] Grove's definition works well when the plausibility relation is well-founded (and also in the somewhat more general case given by the Grove-Lewis Limit Assumption), but it yields inconsistent beliefs in the case that there are *no* most plausible worlds. But note that in evidence models $\sqsubseteq_E$ may be non-wellfounded. Indeed, the definition of van Benthem and Pacuit can lead to inconsistent beliefs!

Example 1 *Consider the evidence model* $\mathcal{M} = (\mathbb{N}, E_0, V)$*, where the state space is the set* $\mathbb{N}$ *of natural numbers,* $V(p) = \emptyset$*, and the basic evidence family* $E_0 = \{e \subseteq \mathbb{N} : \mathbb{N} \setminus e \text{ finite}\}$ *consists of all co-finite sets. The only maximal body of evidence in* E_0 *is* E_0 *itself. However,* $\bigcap E_0 = \emptyset$*. So* $Bel\bot$ *holds in* $\mathcal{M}$*.*

[16] Note that all the notions of belief we consider are global: they do not depend on the state of the world, i.e. we have either $BelP = X$ or $BelP = \emptyset$ (similar to the sets $\forall P, E_0P, EP$). This expresses the assumption that belief is a purely internal notion, thus transparent and hence absolutely introspective. This is standard in logic and accepted by most philosophers, though Williamson [20] begs to differ.

This phenomenon only happens in (some cases of) *infinite* models, so it is *not* due to the inherent mutual inconsistency of the available evidence. The "good" examples in [18] are the ones in which (possibly inconsistent) evidence is processed in a way that yields *consistent* beliefs. So it seems to us that the intended goal (only partially fulfilled) in [18] was to ensure that the agents are able to form consistent beliefs based on the available evidence. We think this to be a natural requirement for idealized "rational" agents, and so we consider doxastic inconsistency to be "a bug, not a feature", of the van Benthem-Pacuit framework. Hence, we now propose a notion that agrees with the one in [18] in all the "good" cases, but also produces in a natural way only consistent beliefs.

Our Notion of Belief. The intuition is that something is believed iff *it is entailed by all the "sufficiently strong" (combined) evidence*. Formally, we say that *P is believed*, and write *BP*, iff *every finite body of evidence can be strengthened to some finite body of evidence which supports P*:

$$BP \text{ holds (at any state)} \quad \text{iff} \quad \forall F \in \mathcal{F}^{finite} \exists F' \in \mathcal{F}^{finite} (F \subseteq F' \wedge \bigcap F' \subseteq P).$$

Our notion of belief B coincides with *Bel* in the *finite* case, or, more generally, in *all Alexandroff models*. But, unlike *Bel*, it is always *consistent* i.e. $B\bot = B\emptyset = \emptyset$), and moreover it satisfies the axioms of the standard doxastic logic $KD45$. Another nice feature is that our belief B is a *purely topological notion*, as can be seen from the following:

Proposition 2 *In every evidence model (X, E_0, V), the following are equivalent, for any proposition $P \subseteq X$:*

1. *BP holds (at any state);*
2. *every (combined) evidence can be strengthened to some evidence supporting P ($\forall e \in E \exists e' \in E$ s.t $e' \subseteq e \cap P$);*
3. *every argument (for anything) can be strengthened to an argument for P (i.e. $\forall U \in \tau_E \setminus \{\emptyset\} \exists U' \in \tau_E \setminus \{\emptyset\}$ s.t. $U' \subseteq U \cap P$);*
4. there is a justification for *P: i.e. some argument for P which is consistent with any available evidence ($\exists U \in \tau_E$ s.t. $U \subseteq P$ and $U \cap e \neq \emptyset$ for all $e \in E$) ;*
5. *P includes some dense open set;*
6. *$IntP$ is* dense *in τ_E (i.e. $Cl(IntP) = X$), or equivalently $X \setminus P$* is nowhere dense*;*
7. *$\forall \Diamond \Box P$ holds (at any state: i.e. $\forall \Diamond \Box P \neq \emptyset$, or equivalently $\forall \Diamond \Box P = X$), where $\Diamond P := \neg \Box \neg P$ is the dual of the $\Box$ operator.*

Proof. The equivalence between (1), (2), (3) is easy, and the same goes for the equivalence of (5) and (6). For the equivalence between (3), (4) and (5), recall that arguments are non-empty open sets and justifications are dense open sets. (So, to show (4), just take as justification the largest argument for P, i.e. $IntP$.) Finally, for (6) $\Leftrightarrow$ (7), recall that $\forall$ is the universal quantifier, $\Box$ is interior and $\Diamond$ is closure.

Proposition 2 part (4) can be interpreted as saying that *our notion of belief B is the same as "justified belief"*: a proposition P is believed iff the agent has a justification for P. In this case, there exists a *weakest (most general) justification* for P, namely $IntP$. Moreover, part (6) shows that our proposal is very natural from a topological perspective: it is equivalent to saying that *P is believed iff the complement of P is nowhere dense*. Since nowhere dense sets are one of the topological concepts of "small" or "negligible sets", this amounts to *believing propositions if they are true in "almost all" epistemically-possible worlds* (where 'almost all' is interpreted topologically). Finally, part (7) tells us that *belief is definable in terms of the operators $\forall$ and $\Box$.*

Conditional Belief. We generalize now to *conditional beliefs*. First, for sets $Q, Q' \subseteq X$, we say that Q' is *Q-consistent* iff $Q \cap Q' \neq \emptyset$. We say that a body of evidence F is Q-consistent iff $\bigcap F \cap Q \neq \emptyset$. Finally, we say that *$P$ is believed given Q*, and write $B^Q P$, iff *every finite Q-consistent body of evidence can be strengthened to some finite Q-consistent body of evidence supporting the proposition $Q \to P$* (i.e., $\neg Q \cup P$)). An analogue of Proposition 2 can now be proved for conditional belief:

Similarly to Proposition 2, we can show that $B^Q P$ is equivalent to any of the following: every Q-consistent evidence can be strengthened to some Q-consistent evidence supporting $Q \to P$; every Q-consistent argument can be strengthened to a Q-consistent argument for $Q \to P$; there is a Q-consistent argument for $Q \to P$ with is consistent with any Q-consistent evidence; $Q \to P$ includes some Q-consistent open set which is dense in Q; $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P))) = X$; etc.

Observation 3. In every evidence model (X, E_0, V), the following are equivalent, for any two propositions $P, Q \subseteq X$, with $Q \neq \emptyset$:

1. $B^Q P$ holds (at any state)
2. every Q-consistent evidence can be strengthened to some Q-consistent evidence supporting $Q \to P$ (i.e., $\forall e \in E(e \cap Q \neq \emptyset \Rightarrow \exists e' \in E$ s.t $e' \cap Q \neq \emptyset$ and $e' \subseteq e \cap (Q \to P)))$;
3. every Q-consistent argument can be strengthened to a Q-consistent argument for $Q \to P$;
4. there is some Q-consistent argument for $Q \to P$ which is consistent with any Q-consistent evidence;
5. $Q \to P$ includes some Q-consistent open set which is dense in Q;
6. $Int(Q \to P)$ is *dense* in Q (i.e. $Cl(Int(Q \to P)) \supseteq Q$);
7. $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P)))$ holds (at any state): i.e. $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P))) \neq \emptyset$; or equivalently $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P))) = X$.

Proof. The equivalence between (1), (2), (3) is easy and simply follows from the semantics of $B^Q P$. From (2) to (4): consider the weakest argument $Int(Q \to P)$ for $Q \to P$. Since $X \in E$ and X is Q-consistent, there always exists a stronger $e' \in E$ such that $e' \cap Q \neq \emptyset$ and $e' \subseteq Q \to P$. Since $Int(Q \to P)$ is the largest open with

$Int(Q \to P) \subseteq Q \to P$, we obtain $e' \subseteq Int(Q \to P) \subseteq Q \to P$ for any such e', therefore, $Int(Q \to P)$ is also Q-consistent. Let $e \in E$ such that $e \cap Q \neq \emptyset$. Then, by (2), there exists $e'' \in E$ such that $e'' \cap Q \neq \emptyset$ and $e'' \subseteq e \cap (Q \to P)$. By the previous argument, we know that $e'' \subseteq Int(Q \to P)$, thus, $e'' \subseteq e \cap Int(Q \to P) \neq \emptyset$. From (4) to (2) follows from the fact that any Q-consistent $e \in E$ can be strengthen to the Q-consistent evidence $e \cap Int(Q \to P)$ supporting $Q \to P$. (4)⇔(5) simply follows from the observation that $\forall x \in Q(x \in e \in E \Rightarrow e$ is Q-consistent). Therefore, $Int(Q \to P)$ is the Q-consistent argument included in $Q \to P$, which is consistent with any Q-consistent evidence meaning that it is also dense in Q. (5)⇔(6) also follows from the previous argument. Finally, for (6)⇔(7), recall that $\forall$ is the universal quantifier, $\Box$ is interior and $\Diamond$ is closure.

2.3 Evidential Dynamics

In this section, we consider some of the evidence dynamics introduced in [18]: updates, evidence addition, evidence upgrade and (a feasible version of) evidence combination. Throughout this section, we are given a topo-e-model $\mathcal{M} = (X, E_0, \tau, V)$ and some proposition $P \subseteq X$, with $P \neq \emptyset$.

Updates (also known as public announcements) involve learning a new fact P with absolute certainty: P becomes "hard information". The standard way of modeling this is via model restrictions. For evidence models, this means keeping only the worlds in P and only the P-consistent evidence pieces.[17]

Definition 4 (Updates) *The model* $\mathcal{M}^{!P} = (X^{!P}, E_0^{!P}, \tau^{!P}, V^{!P})$ *is defined as follows:* $X^{!P} = P$*,* $E_0^{!P} = \{e \cap P : e \in E_0 \text{ with } e \cap P \neq \emptyset\}$*,* $\tau^{!P} = \{U \cap P : U \in \tau\}$ *and for each* $p \in \text{Prop}$*,* $V^{!P}(p) = V(p) \cap P$*. It is easy to check that* $\mathcal{M}^{!P}$ *is still a topo-e-model, with combined evidence* $E^{!P} = \{e \cap P : e \in E \text{ with } e \cap P \neq \emptyset\}$*.*

Following [18], we also consider *evidence addition* $+P$, by which P is accepted as a new piece of evidence (without being assumed to be hard information).

Definition 5 (Evidence Addition) *The model* $\mathcal{M}^{+P} = (X^{+P}, E_0^{+P}, \tau^{+P}, V^{+P})$ *is defined as follows:* $X^{+P} = X$*,* $E_0^{+P} = E_0 \cup \{P\}$*,* τ^{+P} *is the topology generated by* E_0^{+P} *and* $V^{+P} = V$*. Again, it is easy to check that* $\mathcal{M}^{+P}$ *is still a topo-e-model, with combined evidence* $E^{+P} = E \cup \{e \cap P :: e \in E \text{ with } e \cap P \neq \emptyset\}$*.*

Observation 4. Our above definition for conditional belief can be justified in terms of dynamics: it is easy to check[18] that *$B^Q P$ holds in $\mathcal{M}$ iff BP holds in $\mathcal{M}^{+Q}$*. So conditional beliefs "pre-encode" the beliefs induced by evidence-addition.

Next, consider the *evidence upgrade* $\Uparrow P$, which incorporates P into all other pieces of evidence, thus making P the most important available evidence:

[17] Topologically, this is a move from the original space (X, τ) to the *subspace* determined by P.

[18] For this to hold, it is essential that P and Q are taken to be sets of states, rather than syntactic formulas in any epistemic language. Otherwise, Moore-type phenomena appear, and the right equivalence is more complicated: see the Reduction Laws in Section 3.

Definition 6 (Evidence Upgrade) *The model* $\mathcal{M}^{\Uparrow P} = (X^{\Uparrow P}, E_0^{\Uparrow P}, \tau^{\Uparrow P}, V^{\Uparrow P})$ *is defined as follows:* $X^{\Uparrow P} = X$, $E_0^{\Uparrow P} = \{e \cup P : e \in E_0\} \cup \{P\}$, $\tau^{\Uparrow P}$ *is the topology generated by* $E_0^{\Uparrow P}$ *and* $V^{\Uparrow P} = V$. *Again,* $\mathcal{M}^{\Uparrow P}$ *is a topo-e-model.*

Another dynamic operation considered in [18] is *evidence combination*. Here, we adapt it to our topological setting, which assumes that agents can combine only finitely many pieces of evidence at a given time. This is what we call *feasible evidence combination*, in contrast to the infinitary combinations allowed in [18].

Definition 7 (Feasible Evidence Combination) *The model* $\mathcal{M}^{\#} = (X^{\#}, E_0^{\#}, \tau^{\#}, V^{\#})$ *is defined as follows:* $X^{\#} = X$, $V^{\#} = V$, $E_0^{\#}$ *is the smallest set closed under non-empty, finite intersections and containing* E_0 *and* $\tau^{\#}$ *is the topology generated by* $E_0^{\#}$. *Again, this gives us a topo-e-model. Note, that as* $E_0^{\#}$ *is obtained by closing* E_0 *under finite and non-empty intersections, the topology stays the same:* $\tau = \tau^{\#}$. *In fact, we have* $E_0^{\#} = E^{\#} = E$.

2.4 Knowledge

We can now define a "softer" notion of knowledge, that approximates better the common usage of the word than the above-defined "infallible" knowledge. Formally, we put $KP := \{x \in X : \exists U \in \tau\, (x \in U \subseteq P \wedge Cl(U) = X)\}$. So KP holds at x iff *P includes a dense open neighborhood of x*; equivalently, iff $x \in IntP$ and $IntP$ is dense. Essentially, this says that *knowledge is "correctly justified belief"*: KP holds at world x iff there exists some justification $U \in \tau$ for P such that $x \in U$. In other words, P is known iff there *exists some correct (i.e. factive) argument for P that is consistent with all the available evidence*.

Note that K satisfies Stalnaker's Strong Belief Principle $BP = BKP$: *from a subjective point of view, belief is indistinguishable from knowledge*.[19]

We illustrate the semantics we proposed for justified belief and knowledge, and the connection between the two notions in the example below:

Example 2 *Consider the model* $\mathcal{X} = ([0, 1], E_0, V)$, *where* $E_0 = \{(a, b) \cap [0, 1] : a, b \in \mathbb{R},\ a < b\}$ *and* $V(p) = \emptyset$. *The generated topology* τ_E *is the standard topology on* $[0, 1]$. *Let* $P = [0, 1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ *be the proposition stating that the actual state is* not *of the form* $\frac{1}{n}$, *for any* $n \in \mathbb{N}$. *Since the complement* $\neg P = \{\frac{1}{n} : n \in \mathbb{N}\}$ *is nowhere dense, the agent* believes *P, and e.g.* $U = \bigcup_{n \geq 1} (\frac{1}{n+1}, \frac{1}{n})$ *is a (dense, open) justification for P. This belief is* true at world $0 \in P$. *But this* true belief is not knowledge *at* 0*: no justification for P is true at* 0, *since P doesn't include any open neighborhood of* 0, *so* $0 \notin IntP$ *and hence* $0 \notin KP$. *(However, P is known at all the* other *worlds* $x \in P \setminus \{0\}$, *since* $\forall x \in P \setminus \{0\} \exists \epsilon > 0$ *s.t.* $x \in (x - \epsilon, x + \epsilon) \subseteq P$, *hence* $x \in IntP$.*)*

[19] As we'll see, K and B satisfy all the Stalnaker axioms for knowledge and belief [1, 2, 12] and further generalizes our previous work on a topological interpretation of Stalnaker's doxastic-epistemic axioms, which was based on extremally disconnected spaces.

This soft type of knowledge is *defeasible*. In contrast, the usual assumption in Logic is that *knowledge acquisition is monotonic*. As a result, logicians typically assume that knowledge is "irrevocable": once acquired, it cannot be defeated by any further evidence. A weaker notion of knowledge is the one proposed by Lehrer and others under the name of Defeasibility Theory: "in-defeasible knowledge" cannot be defeated by any *factive* evidence that might be gathered later (though it may be defeated by false "evidence"). The following example shows that, if we allow as potential defeaters *all* sets of worlds P containing the actual world, then our *KP can* be defeated:

Example 3 *Consider the model* $\mathcal{M} = (X, E_0, V)$*, where* $X = \{x_1, x_2, x_3, x_4, x_5\}$*,* $V(p) = \emptyset$*,* $E_0 = \{X, O_1, O_2\}$*,* $O_1 = \{x_1, x_2, x_3\}$*,* $O_2 = \{x_3, x_4, x_5\}$*. The resulting set of combined evidence is* $E = \{X, O_1, O_2, \{x_3\}\}$*. Assume the actual world is* x_1*. Then* O_1 *is* known*, since* $x_1 \in Int(O_1) = O_1$ *and* $Cl(O_1) = X$*. Now consider the model* $\mathcal{M}^{+O_3} = (X, E_0^{+O_3}, V)$ *obtained by adding the new evidence* $O_3 = \{x_1, x_5\}$*. We have* $E_0^{+O_3} = \{X, O_1, O_2, O_3\}$*, so* $E^{+O_3} = \{X, O_1, O_2, O_3, \{x_1\}, \{x_3\}, \{x_5\}\}$*. Note that the new evidence is* true *(*$x_1 \in O_3$*). But* O_1 *is* not even believed *in* $\mathcal{M}^{+O_3}$ *anymore (since* $O_1 \cap \{x_5\} = \emptyset$*, so* O_1 *is no longer dense in* $\tau_{E^{+O_3}}$*), thus* O_1 *is* no longer known *after the true evidence* O_3 *was added!*

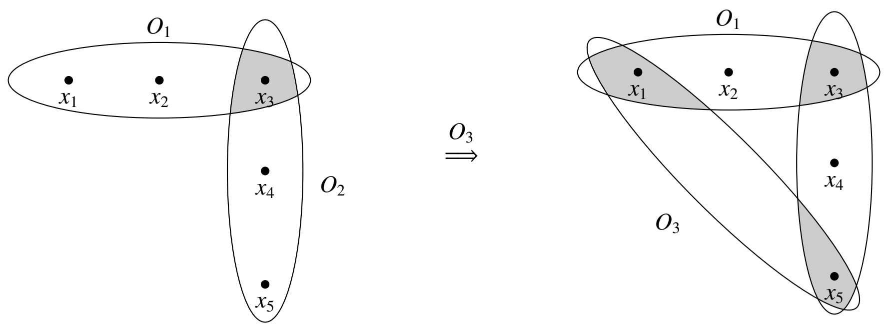

Fig. 1. From $\mathcal{M}$ to $\mathcal{M}^{+O_3}$

In conclusion, our notion of knowledge is incompatible with this strong interpretation of Defeasibility Theory. However, some authors attacked this theory as being *too strong*: one should exclude 'misleading' defeaters, which may unfairly defeat a good justification. Intuitively misleading defeaters are the ones which may lead to false conclusions when combined with the old evidence. A weakened version of Defeasibility Theory would require 'knowledge' (and its justification) only to be *undefeated by non-misleading evidence*.

We proceed now to formalize this distinction. Given a topo-e-model $\mathcal{M}$, a proposition $Q \subseteq X$ is *misleading at* $x \in X$ *wrt* E if evidence-addition with Q produces some false new evidence; i.e. if there is some $e' \in E^{+Q} \setminus E$ s.t. $x \notin e'$; equivalently, there is some $e \in E$ s.t. $x \notin (e \cap Q) \notin E \cup \{\emptyset\}$. It is easy to see that: *old evidence in E is by definition non-misleading wrt E* (i.e. if $e \in E$ then e is non-misleading wrt E), and *new non-misleading evidence must be true* (i.e. if $Q \notin E$ is non-misleading at x then $x \in Q$).

We are now in the position to formulate precisely the "weakened" version of Defeasibility Theory that we are looking for. In fact, there are two possible formulations, which in this setting are non-equivalent. Some authors, e.g. Stalnaker, Rott [13] etc, discussed a version of the Defeasibility Thesis that only requires the believed proposition P to be "undefeated", i.e, to be believed conditional on every true (or, in our formulation, every non-misleading) new evidence": i.e. $B^Q P$ holds for every non-misleading $Q \subseteq X$. This is what Rott calls "the Stability Theory of Knowledge" [13]. As we'll see, although our notion of knowledge is indeed stable in this sense, this definition turns out to be *too weak*. In contrast, Lehrer [9] insists that, in order to know P, not only the belief in P has to be undefeated, but also its justification (i.e. what we call here "an argument for P"). In other words, there must exist an an argument for P that is believed conditional on every non-misleading evidence. Clearly, this implies that P itself is believed conditional on every such non-misleading evidence; but the converse is not at all obvious. Indeed, Lehrer claims that the converse is false. The problem is that, when confronted with various pieces of new non-misleading evidence, the agent might keep switching between different justifications (for believing P); thus, she may keep believing in P conditional on any such non-misleading new evidence, without actually having any "good" justification (i.e., one that remains itself undefeated by all non-misleading evidence). As we'll see, this can indeed happen in our setting. So, to have real knowledge, one has to have at least one "uniformly undefeated" justification: an argument $U \subseteq P$ s.t. $B^Q U$ holds for all non-misleading $Q \subseteq X$.

The next result shows that our notion fits this weakened version of Defeasibility Theory, in its undefeated-justification variant:

Proposition 3 *Let $\mathcal{M}$ be a topo-e-model, and assume $x \in X$ is the actual world. The following are equivalent for all $P \subseteq X$:*

1. *P is known ($x \in KP$);*
2. *there is an argument (justification) for P that cannot be defeated by any non-misleading proposition (i.e. $\exists U \in \tau_E \setminus \{\emptyset\}$ s.t. $U \subseteq P$ and $B^Q U$ for all non-misleading $Q \subseteq X$).*

Proof. To show (1) $\to$ (2): Assume $x \in KP$. Then $x \in IntP$. Since $(IntP) \in \tau_E$ and E is a basis of τ_E, there must exist some $e_0 \in E$ s.t. $x \in e_0 \subseteq IntP \subseteq P$, and $IntP$ is dense. As our argument, we'll take $U := IntP$. Obviously, $U \in \tau_E$ and $U \subseteq P$, so U is an argument for P. Let now $Q \subseteq X$ be non-misleading. We want to show that $B^Q U$ holds. For this, we use the characterization given by clause (2) in Observation 3: let $e \in E$ be any Q-consistent evidence (so $e \cap Q \neq \emptyset$); to show our claim, it is enough to find some Q-consistent $e' \in E$ s.t. $e' \subseteq e \cap U$. We have two cases: (1) $x \in e \cap Q$, and (2) $x \notin e \cap Q$. In case (1), take $e' := e_0 \cap e \in E$. Then e' is Q-consistent (since $x \in e_0 \cap (e \cap Q) = e \cap (e_0 \cap Q) = e' \cap Q$, so $e' \cap Q \neq \emptyset$), and we obviously have $e' = e \cap e_0 \subseteq e \cap IntP = e \cap U$, as desired. In case (2), $x \notin e \cap Q$ implies that $(e \cap Q) \in E$ (since Q is non-misleading and $e \cap Q \neq \emptyset$). Since $IntP$ is dense, this gives us that $IntP \cap (e \cap Q) \neq \emptyset$. Using again the fact that E is a basis of τ_E (and $IntP \in \tau_E$), there must exist some $e'' \in E$ s.t. $e'' \subseteq IntP$ and $e'' \cap (e \cap Q) \neq \emptyset$. Take $e' := e'' \cap (e \cap Q) \in E$

(since $e'', (e \cap Q) \in E$ have non-empty intersection). Again, e' is Q-consistent (since $e' \cap Q = e'' \cap (e \cap Q) \neq \emptyset$) and $e' \subseteq e \cap e'' \subseteq e \cap IntP = e \cap U$, as desired.

To show (2) $\rightarrow$ (1): Assume that $U \in \tau_E$ is s.t. $U \subseteq P$ and $B^Q U$ holds for all non-misleading Q. Clearly, this implies that BU holds (by taking $Q = X$), and hence that BP holds (since B is a normal operator). So, to show that KP holds at x, it is enough to show that $x \in IntP$. For this, it is clearly enough to show that $x \in U$ (since $x \in U \in \tau_E$ and $U \subseteq P$ give us $x \in IntP$). For this, take the proposition $Q = \{x\}$, which obviously is non-misleading at x, hence by (2) we must have $B^Q U$. By using again clause (2) in Observation 3, $B^Q U$ gives us that: for every Q-consistent $e \in E$ there exists some Q-consistent $e' \in E$ with $e' \subseteq e \cap (Q \rightarrow U)$. Take $e := X$, which clearly is Q-consistent (since $X \cap Q = Q = \{x\} \neq \emptyset$); hence, we have some $e' \in E$ with $e' \cap Q \neq \emptyset$ (i.e. $x \in e'$) and $e' \subseteq e \cap (Q \rightarrow U) = (Q \rightarrow U)$, hence $\{x\} = Q = e' \cap Q \subseteq (Q \rightarrow U) \cap Q \subseteq U$, and thus $x \in U$, as desired.

Obviously (2) implies that we have $B^Q P$ for all non-misleading Q, so our knowledge is "stable", in the sense of Rott's Stability Theory [13]. But the next counterexample shows that the "Stability Theory" characterization is too weak.

Example 4 *Consider the model $\mathcal{M} = (X, E_0, V)$, where $X = \{x_0, x_1, x_2\}$, $V(p) = \emptyset$, $E_0 = \{X, O_1, O_2\}$, $O_1 = \{x_1\}$, $O_2 = \{x_1, x_2\}$. The resulting set of combined evidence is $E = E_0$. Assume the actual world is x_0, and let $P = \{x_0, x_1\}$. Then P is* believed *(since its interior $IntP = \{x_1\}$ is dense) but it is* not known *(since $x_0 \notin IntP = \{x_1\}$). However, we can show that P* is believed conditional on any non-misleading proposition. *For this, note that the family of non-misleading propositions (at x_0) is $E \cup \{P, \{x_0\}\} = \{X, O_1, O_2, P, \{x_0\}\}$. It is easy that for each set Q in this family, we have $B^Q P$.*

3 Logics for evidence, belief, knowledge and learning

In this section, we present formal languages for evidence, belief, knowledge and evidence acquisition, and provide sound, complete and decidable proof systems for the resulting logics of evidence, knowledge and belief. We first introduce a language that captures only *topological* properties of our models.

The *topological language* $\mathcal{L}$ is given by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi \mid K\varphi \mid \forall\varphi \mid B^{\varphi}\varphi \mid \Box\varphi \mid E\varphi$$

where $p \in$ Prop. We employ the usual abbreviations for propositional connectives $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$, and for the dual modalities $\langle B\rangle$, $\langle K\rangle$, $\langle E\rangle$ etc, except that some of them have special abbreviations: $\exists\varphi := \langle\forall\rangle\varphi$ and $\Diamond\varphi := \langle\Box\rangle\varphi$.

Several fragments of $\mathcal{L}$ have special importance: $\mathcal{L}_B$ is the fragment having the belief B as the only modality; $\mathcal{L}_K$ has only the knowledge operator K; $\mathcal{L}_{KB}$ has only operators K and B; $\mathcal{L}_{\forall K}$ has only operators $\forall$ and K; $\mathcal{L}_{\forall\Box}$ has only operators $\forall$ and $\Box$.

We also consider an *extension* $\mathcal{L}_{E_0\Box_0}$ of $\mathcal{L}$, called the *evidence language*: this is obtained by extending $\mathcal{L}$ with *two new operators E_0 and $\Box_0$*. The expressivity of $\mathcal{L}_{E_0\Box_0}$ goes beyond purely topological properties: the meaning of E_0 and $\Box_0$ does not depend only on the topology, but also on the basic evidence family E_0. Finally, we will consider one very important fragment of $\mathcal{L}_{E_0\Box_0}$, namely the language $\mathcal{L}_{\forall\Box\Box_0}$ having only the operators $\forall$, $\Box$ and $\Box_0$. Its importance comes from that $\mathcal{L}_{\forall\Box\Box_0}$ is *co-expressive* with $\mathcal{L}_{E_0\Box_0}$.

The *semantics* for these languages is obvious: given a topo-e-model $\mathcal{M} = (X, E_0, \tau, V)$, we recursively extend the valuation map V to an interpretation map $||\varphi||$ for all formulas φ, by interpreting the Boolean connectives and the modalities using the corresponding semantic operators: e.g. $||\forall\varphi|| = \forall||\varphi||$, $||\Box\varphi|| = \Box||\varphi||$ etc.

Moving on to *dynamic extensions*, we consider *PDL*-style languages $\mathcal{L}_{!\forall\Box\Box_0}$, $\mathcal{L}_{+\forall\Box\Box_0}$, $\mathcal{L}_{\Uparrow\forall\Box\Box_0}$ and $\mathcal{L}_{\#\forall\Box\Box_0}$, obtained by adding to $\mathcal{L}_{\forall\Box\Box_0}$ dynamic modalities $[!\varphi]\psi$ for updates, respectively $[+\varphi]\psi$ for evidence addition, $[\Uparrow\varphi]\psi$ for evidence upgrade and $[\#]\psi$ for feasible evidence combination (with the obvious intended interpretations: e.g. $[!\varphi]\psi$ means that "ψ becomes true after an update with φ"). The *semantics* for dynamic operators uses the corresponding model change as standard in Dynamic Epistemic Logic:

$$
\begin{aligned}
x \in ||[!\varphi]\psi|| \quad &\text{iff} \quad x \in ||\varphi|| \text{ implies } x \in ||\psi||_{\mathcal{M}^{!||\varphi||}},\\
x \in ||[+\varphi]\psi|| \quad &\text{iff} \quad x \in ||\exists\varphi|| \text{ implies } x \in ||\psi||_{\mathcal{M}^{+||\varphi||}},\\
x \in ||[\Uparrow\varphi]\psi|| \quad &\text{iff} \quad x \in ||\exists\varphi|| \text{ implies } x \in ||\psi||_{\mathcal{M}^{\Uparrow||\varphi||}},\\
x \in ||[\#]\varphi|| \quad &\text{iff} \quad x \in ||\varphi||_{\mathcal{M}^{\#}},
\end{aligned}
$$

where we denoted by $||\psi||_{\mathcal{M}^{!||\varphi||}}$ the interpretation of ψ in the updated model $\mathcal{M}^{!||\varphi||}$, etc. The precondition $x \in ||\varphi||$ in the above clause for update encodes the fact that updates are factive: so one can only update with *true* sentences φ. The preconditions $x \in ||\exists\varphi||$ in the clauses for evidence addition and upgrade encodes the fact that, in order to qualify as (new) evidence, φ has to be *consistent* (i.e. $||\varphi|| \neq \emptyset$).

Proposition 4 *The following equivalences are valid in all topo-e-models:*

1. $B\varphi \leftrightarrow \langle K\rangle K\varphi \leftrightarrow \exists K\varphi \leftrightarrow \forall\Diamond\Box\varphi$
2. $E\varphi \leftrightarrow \exists\Box\varphi$
3. $E_0\varphi \leftrightarrow \exists\Box_0\varphi$
4. $K\varphi \leftrightarrow \Box\varphi \wedge B\varphi \leftrightarrow \Box\varphi \wedge \forall\Diamond\Box\varphi$
5. $B^{\theta}\varphi \leftrightarrow \forall(\theta \to \Diamond(\theta \wedge \Box(\theta \to \varphi)))$
6. $\forall\varphi \leftrightarrow B^{\neg\varphi}\bot$

So, all the other modalities of $\mathcal{L}_{E_0\Box_0}$ can be defined in $\mathcal{L}_{\forall\Box\Box_0}$. As we'll see, the dynamic modalities are also eliminable, so that $\mathcal{L}_{\forall\Box\Box_0}$ can in fact define all *our* operators.

Theorem 1 *A sound and complete axiomatization for $\mathcal{L}_B$ (on topo-e-models) consists of the axioms and rules of the modal system $KD45$ for the B operator.*

Theorem 2 *A sound and complete axiomatization for $\mathcal{L}_K$ consists of the axioms and rules of the modal system $S4.2$ for the K operator.*

Theorem 3 *A sound and complete axiomatization for $\mathcal{L}_{KB}$ is given by Stalnaker's system* [20] *KB, consisting of the following:*

[20] This shows that the semantics in this paper correctly generalizes the one in [1, 12] for the system KB.

1. *the $S4$ axioms and rules for Knowledge K*
2. *Consistency of Belief: $B\phi \to \neg B \neg \phi$;*
3. *Knowledge implies Belief: $K\phi \to B\phi$;*
4. *Strong Positive and Negative Introspection for Belief: $B\phi \to KB\phi$; $\neg B\phi \to K\neg B\phi$;*
5. *the "Strong Belief" axiom: $B\phi \to BK\phi$.*

Theorem 4 ([7]) *The following system is sound and complete for $\mathcal{L}_{\forall\Box}$:*

1. *the S5 axioms and rules for $\forall$*
2. *the S4 axioms and rules for $\Box$*
3. $\forall\varphi \to \Box\varphi$

Moreover, Proposition 4 shows that $\mathcal{L}_{\forall\Box}$ can define *all* the other operators of $\mathcal{L}$. So a complete system for $\mathcal{L}$ can be obtained from the system for $\mathcal{L}_{\forall\Box}$ by adding the four additional axiom-definitions given in Proposition 4.

Theorem 5 *The following system is sound and complete for $\mathcal{L}_{\forall K}$:*

1. the $S5$ axioms and rules for $\forall$	*3. $\forall\varphi \to K\varphi$*
2. the $S4$ axioms and rules for K	*4. $\exists K\varphi \to \forall\langle K\rangle\varphi$*

Similarly, the language $\mathcal{L}_{\forall K}$ can define belief. So a complete system for the language with this additional belief operator is obtained from the system for $\mathcal{L}_{\forall K}$ by adding the axiom-definition $B\varphi \leftrightarrow \exists K\varphi$ given in Proposition 4.

Theorem 6 (Soundness, Completeness, Finite Model Property and Decidability) *The logic $\mathcal{L}_{\forall\Box\Box_0}$ has* finite model property *and is* completely axiomatized *by the following system $L_{\forall\Box\Box_0}$ (and so it's* decidable*):*

1. *the S5 axioms and rules for $\forall$*
2. *the S4 axioms and rules for $\Box$*
3. $\Box_0\varphi \to \Box_0\Box_0\varphi$
4. *the Monotonicity Rule for $\Box_0$: from $\varphi \to \psi$, infer $\Box_0\varphi \to \Box_0\psi$*
5. $\forall\varphi \to \Box_0\varphi$
6. $\Box_0\varphi \to \Box\varphi$
7. *the Pullout Axiom*[21]*: $(\Box_0\varphi \wedge \forall\psi) \to \Box_0(\varphi \wedge \forall\psi)$*

[21] This axiom originates from [17], where it is stated as an equivalence rather than an implication. But the converse is provable in our system.

Theorem 7 *The sound and complete logic for* $\mathcal{L}_{!\forall\Box\Box_0}$ *is obtained by adding the following recursion axioms to the system* $L_{\forall\Box\Box_0}$*:*

1. $[!\varphi]p \leftrightarrow (\varphi \to p)$
2. $[!\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[!\varphi]\psi)$
3. $[!\varphi](\psi \wedge \chi) \leftrightarrow ([!\varphi]\psi \wedge [!\varphi]\chi)$
4. $[!\varphi]\Box_0\psi \leftrightarrow (\varphi \to \Box_0[!\varphi]\psi)$
5. $[!\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[!\varphi]\psi)$
6. $[!\varphi]\forall\psi \leftrightarrow (\varphi \to \forall[!\varphi]\psi)$
7. $[!\varphi][!\psi]\chi \leftrightarrow [!\langle\varphi\rangle\psi]\chi$

Theorem 8 *The sound and complete logic for* $\mathcal{L}_{+\forall\Box\Box_0}$ *is obtained by adding the axiom K and the Necessitation rule for the evidence addition modalities as well as the following recursion axioms to* $L_{\forall\Box\Box_0}$*:*

1. $[+\varphi]p \leftrightarrow (\exists\varphi \to p)$
2. $[+\varphi]\neg\psi \leftrightarrow (\exists\varphi \to \neg[+\varphi]\psi)$
3. $[+\varphi](\psi \wedge \chi) \leftrightarrow ([+\varphi]\psi \wedge [+\varphi]\chi)$
4. $[+\varphi]\Box_0\psi \leftrightarrow (\exists\varphi \to (\Box_0[+\varphi]\psi \vee (\varphi \wedge \forall(\varphi \to [+\varphi]\psi))))$
5. $[+\varphi]\Box\psi \leftrightarrow (\exists\varphi \to (\Box[+\varphi]\varphi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi)))$
6. $[+\varphi]\forall\psi \leftrightarrow (\exists\varphi \to \forall[+\varphi]\psi)$

Theorem 9 *The sound and complete logic for* $\mathcal{L}_{\Uparrow\forall\Box\Box_0}$ *is obtained by adding the axiom K and the Necessitation rule for the evidence upgrade modalities as well as the following recursion axioms to* $L_{\forall\Box\Box_0}$*:*

1. $[\Uparrow\varphi]p \leftrightarrow (\exists\varphi \to p)$
2. $[\Uparrow\varphi]\neg\psi \leftrightarrow (\exists\varphi \to \neg[\Uparrow\varphi]\psi)$
3. $[\Uparrow\varphi](\psi \wedge \chi) \leftrightarrow ([\Uparrow\varphi]\psi \wedge [\Uparrow\varphi]\chi)$
4. $[\Uparrow\varphi]\Box_0\psi \leftrightarrow (\exists\varphi \to ((\Box_0[\Uparrow\varphi]\psi \vee \varphi) \wedge \forall(\varphi \to [\Uparrow\varphi]\psi)))$
5. $[\Uparrow\varphi]\Box\psi \leftrightarrow (\exists\varphi \to ((\Box[\Uparrow\varphi]\psi \vee \varphi) \wedge \forall(\varphi \to [\Uparrow\varphi]\psi)))$
6. $[\Uparrow\varphi]\forall\psi \leftrightarrow (\exists\varphi \to \forall[\Uparrow\varphi]\psi)$

Theorem 10 *The sound and complete logic for* $\mathcal{L}_{\Uparrow\forall\Box\Box_0}$ *is obtained by adding the axiom K and the Necessitation rule for the evidence upgrade modalities as well as the following recursion axioms to* $L_{\forall\Box\Box_0}$*:*

1. $[\#]p \leftrightarrow p$
2. $[\#]\neg\varphi \leftrightarrow \neg[\#]\varphi$
3. $[\#](\varphi \wedge \psi) \leftrightarrow ([\#]\varphi \wedge [\#]\psi)$
4. $[\#]\Box\varphi \leftrightarrow \Box[\#]\varphi$
5. $[\#]\Box_0\varphi \leftrightarrow \Box[\#]\varphi$
6. $[\#]\forall\varphi \leftrightarrow \forall[\#]\varphi$

The proofs of Theorems 1-3 and 5 are relatively standard, and can be found in Appendices A-D. The proof of Theorem 6 is technically the most difficult result of the paper. The key difficulty consists in guaranteeing that the natural topology for which $\Box$ acts as interior operator is exactly the topology generated by the neighborhood family associated to $\Box_0$. Though the main steps of the proof may look familiar, involving known methods (a canonical quasi-model construction, a filtration argument, and then making multiple copies of the worlds to yield a finite model with the right properties), addressing the above-mentioned difficulty requires a non-standard application of these methods, as well as a number of additional notions and results, and a careful treatment of each of the steps. We give the full proof of Theorem 6 in Appendix E. Finally, the proofs of Theorems 7-10 are also along standard lines, and so we give only a sketch of these proofs in Appendix F.

Appendices

For the proofs of Theorems 1-3, we make use of known completeness results of the corresponding systems with respected to standard *relational* (Kripke) semantics, and of the connection between the Kripke semantics and our proposed topological semantics.

We now recall some frame conditions concerning the relational completeness of the systems $KD45$ and $S4.2$.

Let (X, R) be a transitive Kripke frame. A non-empty subset $C \subseteq X$ is a *cluster* if

(1) for each $x, y \in C$ we have xRy, and

(2) there is no $D \subseteq X$ such that $C \subset D$ and D satisfies (1).

A point $x \in X$ is called a *maximal point* if there is no $y \in X$ such that xRy and $\neg(yRx)$. We call a cluster a *maximal cluster* if all its points are maximal. It is not hard to see that for any maximal cluster C of (X, R) and any $x \in C$, we have $\uparrow x = C$. A transitive Kripke frame (X, R) is called *cofinal* if it has a unique maximal cluster C such that for each $x \in X$ and $y \in C$ we have xRy. We call a cofinal frame a *brush* if $X \setminus C$ is an irreflexive antichain, i.e., for each $x, y \in X \setminus C$ we have $\neg(xRy)$ where C is the maximal cluster. A brush with a singleton $X \setminus C$ is called a *pin*. By definition, every brush and every pin is transitive. Finally, a transitive frame (X, R) is called *rooted*, if there is an $x \in X$, called a *root*, such that for each $y \in X$ with $x \neq y$ we have xRy. Hence, every rooted brush is in fact a pin.

Lemma 1

1. *Each brush is a* $KD45$*-frame. Moreover,* $KD45$ *is sound and complete wrt the class of finite brushes, indeed, wrt the class of finite pins.*

2. *Each reflexive and transitive cofinal frame is an* $S4.2$*-frame. Moreover, S4.2 is sound and complete wrt the class of finite rooted reflexive and transitive cofinal frames.*

Proof. See, e.g., [5, Chapter 5].

It is well-known that given any reflexive and transitive Kripke frame (X, R) (i.e. for any preordered set (X, R)), we can construct a topological space, in fact an Alexandroff space, (X, τ_R) where τ_R is the set of all upward-closed subsets of X with respect to the relation R (see, e.g. [16] for a more detailed discussion on this connection).

We denote the truth set of a formula $\varphi \in \mathcal{L}_{KB}$ in a Kripke model $M = (X, R, V)$ under the standard Kripke semantics by $[\varphi]^M$ and denote by R^+ the reflexive closure of R.

A Proof of Theorem 1

The proof of soundness is a standard validity check. The cases for the axioms D, 4 and 5 are elementary whereas the validity of the K-axiom for B in the class of *all* topological spaces follows from Lemma 2 below.

Lemma 2 *For any topological space (X, τ) and any $U_1, U_2 \subseteq X$, if U_1 is open dense and U_2 is dense, then $U_1 \cap U_2$ is dense too.*

Proof. Let (X, τ) be a topological space and $U_1, U_2 \subseteq X$. Suppose U_1 is an open dense and U_2 is a dense set in (X, τ). Since U_1 is open and dense we have that $W \cap U_1$ is open and non-empty for any non-empty open set W. Thus, since U_2 is dense, we also have that $(W \cap U_1) \cap U_2 \neq \emptyset$. Therefore, $W \cap (U_1 \cap U_2) \neq \emptyset$ for any non-empty $W \in \tau$, i.e., $U_1 \cap U_2$ is dense as well.

For completeness, we use the following connection between the standard Kripke semantics and our proposed semantics for the language $\mathcal{L}_B$:

Proposition 5 *For every Kripke model $M = (X, R, V)$ based on a brush and all $\varphi \in \mathcal{L}_B$,*

$$[\varphi]^M = ||\varphi||^{\mathcal{M}_{\tau_{R^+}}},$$

where $\mathcal{M}_{\tau_{R^+}} = (X, \tau_{R^+}, V)$.

Proof. The proof follows by induction on the complexity of φ. Let $M = (X, R, V)$ be a Kripke model based on the brush (X, R) and $\varphi \in \mathcal{L}_B$. Cases for the propositional variables and Booleans are straighforward.

- Case $\varphi := B\psi$

  Observe that

$$[B\varphi]^M = \begin{cases} X \text{ if } [\varphi]^M \supseteq C \\ \emptyset \ \text{ otherwise} \end{cases} \text{ and, } ||B\varphi||^{\mathcal{M}_{\tau_{R^+}}} = \begin{cases} X \text{ if } ||\varphi||^{\mathcal{M}_{\tau_{R^+}}} \supseteq C \\ \emptyset \ \text{ otherwise} \end{cases}$$

  for any topo-model $\mathcal{M}_{\tau_{R^+}} = (X, \tau_{R^+}, V)$ based on (X, τ_{R^+}) and any formula $\varphi \in \mathcal{L}_B$. By (IH), we have $||\varphi||^{\mathcal{M}_{\tau_{R^+}}} = [\varphi]^M$, therefore, $||B\varphi||^{\mathcal{M}_{\tau_{R^+}}} = [B\varphi]^M$.

Proof of Theorem 1:

Proof. For completeness, let $\varphi \in \mathcal{L}_B$ such that $KD45 \nvdash \varphi$. Then, by Lemma 1(1), there exists a brush $M = (X, R, V)$ with $[\varphi]^M \neq X$. Thus, by Propositition 5, we have that $||\varphi||^{\mathcal{M}_{\tau_{R^+}}} \neq X$, where $\mathcal{M}_{\tau_{R^+}} = (X, \tau_{R^+}, V)$ is the corresponding topological model.

B Proof of Theorem 2

The proof of soundness is again a standard validity check. The relatively harder case of the K-axiom for K follows from Lemma 2 and the fact that the interior operator commutes with finite intersections. For completeness, we follow a similar strategy as in the proof of Theorem 1.

Lemma 3 *For every reflexive and transitive cofinal frame* (X, R) *and non-empty* $U \in \tau_R$*, we have* $Cl(U) = X$ *in* (X, τ_R)*.*

Proof. Let (X, R) be a reflexive and transitive cofinal frame and $C \subseteq X$ be its maximal cluster. By construction, $C \in \tau_R$ and moreover $C \subseteq U$, for all non-empty $U \in \tau_R$. Therefore, for all non-empty $U, V \in \tau_R$, we have $V \cap U \supseteq C \neq \emptyset$. Hence, $Cl(U) = X$ for every non-empty $U \in \tau_R$.

Proposition 6 *For every reflexive, transitive and cofinal Kripke model* $M = (X, R, V)$ *and all* $\varphi \in \mathcal{L}_K$*,*

$$[\varphi]^M = ||\varphi||^{\mathcal{M}_{\tau_R}},$$

where $\mathcal{M}_{\tau_R} = (X, \tau_R, V)$*.*

Proof. The proof follows by induction on the complexity of φ and the cases for propositional variables and Booleans are straightforward. We here only prove the case for K. Let $M = (X, R, V)$ be a reflexive and transitive cofinal Kripke model, $x \in X$ and $\varphi \in \mathcal{L}_K$.

- Case $\varphi := K\psi$

  Left-to-right: Suppose $x \in [K\psi]^M$. This implies that $x \in R(x) \subseteq [\psi]^M$. By (IH), we obtain $R(x) \subseteq ||\psi||^{\mathcal{M}_{\tau_R}}$. Since $x \in R(x) \in \tau_R$, we have $x \in Int(||\psi||^{\mathcal{M}_{\tau_R}})$. Moreover, by Lemma 3, $Cl(Int(||\psi||^{\mathcal{M}_{\tau_R}})) = X$. Therefore, $x \in ||K\psi||^{\mathcal{M}_{\tau_R}}$.

  Right-to-left: Suppose $x \in ||K\psi||^{\mathcal{M}_{\tau_R}}$. This means that $x \in Int(||\psi||^{\mathcal{M}_{\tau_R}})$ and that $Cl(Int(||\psi||^{\mathcal{M}_{\tau_R}})) = X$. Then, by (IH), $x \in Int([\psi]^M)$ and $Cl(Int([\psi]^M)) = X$. The former implies that there is an open set $U \in \tau_R$ such that $x \in U \subseteq [\psi]^M$. In particular, since $R(x)$ is the smallest open neighbourhood of x, we obtain $R(x) \subseteq [\psi]^M$. Therefore, $x \in [K\psi]^M$.

Proof of Theorem 2:

Proof. For completeness, let $\varphi \in \mathcal{L}_B$ such that $S4.2 \nvdash \varphi$. Then, by Lemma 1(2), there exists a reflexive and transitive cofinal $M = (X, R, V)$ with $[\varphi]^M \neq X$. Thus, by Propositition 6, we have that $||\varphi||^{\mathcal{M}_{\tau_R}} \neq X$, where $\mathcal{M}_{\tau_R} = (X, \tau_R, V)$ is the corresponding topological model.

C Proof of Theorem 3

For completeness, we use the following two important features of the system KB (see, e.g. [14, 2, 1]):

1. $KB \vdash B\varphi \leftrightarrow \langle K\rangle K\varphi$, and
2. The $S4.2$ axioms and rules for K can be derived from KB.

Let $\varphi \in \mathcal{L}_{KB}$ such that $KB \nvdash \varphi$. By (1), the formula φ can be reduce to a provably equivalent formula ψ in the language $\mathcal{L}_K$. Moreover, by (2), we have $S4.2 \subseteq KB$, thus, $S4.2 \nvdash \psi$. The proof now follows from the proof of Theorem 2.

D Proof of Theorem 5

It is known that the modal system $S4.2_\forall$ axiomatized by the $S5$ axioms and rules for $\forall$, the $S4.2$ axioms and rules for K and $\forall\varphi \to K\varphi$ is complete with respect to the class of reflexive and transitive cofinal Kriple frames when K is interpreted as the standard Kripke modality and $\forall$ as the global modality [7]. Observe that .2-axiom for K, namely $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$, is derivable in the system given in Theorem 5 (by the axioms 3 and 4 in Theorem 5), hence, it contains $S4.2_\forall$. Let $\varphi \in \mathcal{L}_{\forall K}$ such that φ is not a theorem of the above system. Thus, $S4.2_\forall \nvdash \varphi$. Then, by the relational completeness of $S4.2_\forall$, there exists a reflexive and transitive cofinal Kripke model $\mathcal{M} = (X, R, V)$ such that $[\varphi]^{\mathcal{M}} \neq X$. Then, by Proposition 6 (extended for the language $\mathcal{L}_{\forall K}$), we obtain $||\varphi||^{\mathcal{M}_{\tau_R}} \neq X$, where $\mathcal{M}_{\tau_R} = (X, \tau_R, V)$.

E Proof of Theorem 6

A *quasi-model* is a tuple $\mathcal{M} = (X, E_0, \leq, V)$, where: $E_0 \subseteq \mathcal{P}(X)$ satisfies the same constraints as a topo-e-model, V is a valuation, $\leq$ is a preorder s.t. every $e \in E_0$ is upward-closed wrt $\leq$. The *semantics* is the same as on topo-e-models, except that $\Box$ gets a Kripke semantics: $||\Box\phi|| := \{x \in X \mid \forall y \in X(x \leq y \Rightarrow y \in ||\phi||)\}$.

A quasi-model $\mathcal{M} = (X, E_0, \leq, V)$ is called *Alexandroff* if the topology τ_E is Alexandroff and $\leq = \sqsubseteq_E$ is the specialization preorder. There is a natural *bijection B between Alexandroff quasi-models and Alexandroff topo-e-models*, given by putting, for any Alexandroff quasi-model $\mathcal{M} = (X, E_0, \leq, V)$, $B(\mathcal{M}) := (X, E_0, \tau_E, V)$. Moreover, $\mathcal{M}$ *and* $B(\mathcal{M})$ *satisfy the same formulas of* $\mathcal{L}_{\forall\Box\Box_0}$ *at the same points*. So Alexandroff quasi-models are just another presentation of Alexandroff models.

Proposition 7 *Let $\mathcal{M} = (X, E_0, \leq, V)$ be a quasi-model. The following are equivalent:*

1. *$\mathcal{M}$ is Alexandroff (hence, equivalent to an Alexandroff topo-e-model);*
2. *τ_E coincides with the family of all upward-closed sets (with respect to $\leq$);*
3. *for every $x \in X$, $\uparrow x$ is in τ_E.*

Proof. (1 $\Rightarrow$ 3) Suppose $\mathcal{M}$ is Alexandroff, i.e., τ_E is Alexandroff and $\leq = \sqsubseteq_E$. Let $x \in X$. Then we have: $\uparrow x = \{y \mid x \leq y\} = \{y \mid x \sqsubseteq_E y\} = \{y \mid \forall U \in \tau_E(x \in U \Rightarrow y \in U)\} = \bigcap\{U \in \tau_E \mid x \in U\}$. Since τ_E is an Alexandroff space, we have $\bigcap\{U \in \tau_E \mid x \in U\} \in \tau_E$, and hence $\uparrow x = \bigcap\{U \in \tau_E \mid x \in U\} \in \tau_E$.

(3 $\Rightarrow$ 2) Let $Up(X)$ be the set of all upward-closed subsets of X. It is easy to see that $\tau_E \subseteq Up(X)$ (since τ_E is generated by E_0 and every element of E_0 is upward-closed). Now let $A \in Up(X)$. Since A is upward-closed, we have $A = \bigcup\{\uparrow x \mid x \in A\}$. Then, by (3) (and τ_E being closed under arbitrary unions), we obtain $A \in \tau_E$.

(2 $\Rightarrow$ 1) Suppose (2) and let $\mathcal{A} \subseteq \tau_E$. By (2), every $U \in \mathcal{A}$ is upward-closed; hence, $\bigcap \mathcal{A}$ is upward-closed, so by (2) $\bigcap \mathcal{A} \in \tau_E$. This proves that τ_E is Alexandroff. (2) also implies that $\uparrow x$ is the least open neighbourhood of x in τ_E, i.e., $\uparrow x \subseteq U$, for all U such that $x \in U \in \tau_E$. Therefore, $\leq \subseteq \sqsubseteq_E$. For the other direction, suppose $x \sqsubseteq_E y$. This implies, in particular, $y \in \uparrow x$ (since $x \in \uparrow x \in \tau_E$), i.e., $x \leq y$.

The proof of Theorem 6 goes through *three steps*: (1) strong completeness for quasi-models; (2) finite quasi-model property; (3) every finite quasi-model is modally equivalent to a finite Alexandroff quasi-model (hence, to a topo-e-model).

Proposition 8 *(STEP 1) $L_{\forall\Box\Box_0}$ is sound and strongly complete for quasi-models.*

PROOF Soundness is easy. Completeness goes via a canonical quasi-model:

Lemma 4 *(*Lindenbaum Lemma*) Every consistent set of sentences in $\mathcal{L}_{\forall\Box\Box_0}$ can be extended to a maximally consistent one.*

Proof. Standard.

Let us now fix a consistent set of sentence Φ_0. Our goal is to construct a quasi-model for Φ_0. By Lemma 4, there exists a maximally consistent theory T_0 s. t. $\Phi_0 \subseteq T_0$. For any two maximally consistent theories T and S, we put: $T \sim S$ iff for all $\phi \in \mathcal{L}_{\forall\Box\Box_0}$: $((\forall\phi) \in T \Rightarrow \phi \in S)$; and $T \leq S$ iff for all $\phi \in \mathcal{L}_{\forall\Box\Box_0}$: $((\Box\phi) \in T \Rightarrow \phi \in S)$.

Canonical Quasi-Model for T_0. This is a structure $\mathcal{M} = (X, E_0, \leq, V)$, where: $X := \{T : T$ maximally consistent theory with $T \sim T_0\}$; $E_0 := \{\widehat{\Box_0\phi} : \phi \in \mathcal{L}_{\forall\Box\Box_0}$ with $(\exists\Box_0\phi) \in T_0\}$, where we used notation $\hat{\theta} := \{T \in X : \theta \in T\}$; $\leq$ is the restriction of the above preorder $\leq$ to X; and $V(p) := \hat{p}$. In the following, variables $T, S, \ldots$ range over X.

Lemma 5 *$\mathcal{M}$ is a quasi-model.*

Proof. Easy verification.

Lemma 6 *(*Existence Lemma for $\forall$*) $\widehat{\exists\varphi} \neq \emptyset$ iff $\hat{\varphi} \neq \emptyset$.*

Proof. This is a standard argument:

Left-to-right. Assume $T \in \widehat{\exists\varphi}$, i.e. $(\exists\varphi) \in T \in X$. We prove the following:

Claim: *The set $\Gamma := \{\forall\psi : \forall\psi \in T\} \cup \{\varphi\}$ is consistent*.

Proof of Claim: Suppose that $\Gamma \vdash \bot$. Then there exist finitely many sentences $(\forall\psi_1), \ldots, (\forall\psi_n) \in T$ s.t. $(\forall\psi_1 \wedge \forall\psi_n) \rightarrow \neg\varphi$ is a theorem. But then, by applying the S5 laws to $\forall$, we get that $(\forall\psi_1 \wedge \forall\psi_n) \rightarrow (\forall\neg\varphi)$ is also a theorem; hence $(\forall\neg\varphi) \in T$, which combined with $(\exists\varphi) \in T$, implies that T is inconsistent. Contradiction.

Given the Claim, by Lindenbaum Lemma, there exists some maximally consistent theory S s.t. $\Gamma \subseteq S$. It is easy to see that this implies that $\varphi \in S$ and $S \sim T \sim T_0$ (hence $S \in X$), i.e. $S \in \hat{\varphi}$.

Right-to-left. Assume $T \in \hat{\varphi}$, i.e. $\varphi \in T \in X$. By the theorem $\varphi \to (\exists\varphi)$, we also have $(\exists\varphi) \in T$, i.e. $T \in \widehat{\exists\varphi}$.

Lemma 7 *(Existence Lemma for* $\Box$*)* $T \in \widehat{\Diamond\varphi}$ *iff* $(\exists)\, S \in \hat{\varphi}$ *s. t.* $T \leq S$.

Proof. This is a standard argument:

Left-to-right. Assume $T \in \widehat{\Diamond\varphi}$, i.e. $(\Diamond\varphi) \in T \in X$. We prove the following:

Claim: *The set* $\Gamma := \{\Box\psi : \Box\psi \in T\} \cup \{\varphi\}$ *is consistent*.

Proof of Claim: Suppose that $\Gamma \vdash \bot$. Then there exist finitely many sentences $(\Box\psi_1), \ldots, (\Box\psi_n) \in T$ s.t. $(\Box\psi_1 \wedge \Box\psi_n) \to \neg\varphi$ is a theorem. But then, by applying the S4 laws to $\Box$, we get that $(\Box\psi_1 \wedge \Box\psi_n) \to (\Box\neg\varphi)$ is also a theorem; hence $(\Box\neg\varphi) \in T$, which combined with $(\Box\varphi) \in T$, implies that T is inconsistent. Contradiction.

Given the Claim, by Lindenbaum Lemma, there exists some maximally consistent theory S s.t. $\Gamma \subseteq S$. It is easy to see that this implies that $\varphi \in S$ and $T \leq S$ (hence also $S \in X$, thus $S \in \hat{\varphi}$).

Right-to-left. Assume $T \in \hat{\varphi}$, i.e. $\varphi \in T \in X$. By the theorem $\varphi \to (\Diamond\varphi)$, we also have $(\Diamond\varphi) \in T$, i.e. $T \in \widehat{\Diamond\varphi}$.

Lemma 8 *(Existence Lemma for* $\Box_0$*)* $T \in \widehat{\Box_0\varphi}$ *iff* $(\exists)\, e \in E_0$ *s. t.* $T \in e \subseteq \hat{\varphi}$.

Proof. Left-to-right: Assume $T \in \widehat{\Box_0\varphi}$, i.e. $(\Box_0\varphi) \in T$. From $T \in X$ and $T \sim T_0$ we get $(\exists\Box_0\varphi) \in T_0$. Taking $e := \widehat{\Box_0\varphi}$, we get $e \in E_0$ and $T \in e$. To show that $e \subseteq \hat{\varphi}$, we use the theorem $\Box_0\varphi \to \varphi$, which implies that $\widehat{\Box_0\varphi} \subseteq \hat{\varphi}$, i.e. $e \subseteq \hat{\varphi}$.

Right-to-Left: Let $T \in X$ and $e \in E_0$, s.t. $T \in e \subseteq \hat{\varphi}$. Then $e = \widehat{\Box_0\theta}$ for some θ s.t. $(\exists\Box_0\theta) \in T_0$. So $T \in e = \widehat{\Box_0\theta} \subseteq \hat{\varphi}$. We now prove the following:

Claim: *The set* $\Gamma := \{\Box_0\theta\} \cup \{\forall\psi : \forall\psi \in T\} \cup \{\neg\varphi\}$ *is inconsistent*.

Proof of Claim: Suppose that $\Gamma \nvdash \bot$. By Lemma 4, there exists some $S \in X$ s. t. $\Gamma \subseteq S$. From $(\neg\varphi) \in S$ we get $S \notin \hat{\varphi}$ (by the consistency of S), and from $(\Box_0\theta) \in S$ we get $S \in \widehat{\Box_0\theta}$. So $S \in \widehat{\Box_0\theta} \setminus \hat{\varphi}$, contradicting $\widehat{\Box_0\theta} \subseteq \hat{\varphi}$.

Given the Claim, there exists a *finite* $\Gamma_0 \subseteq \Gamma$ with $\Gamma_0 \vdash \bot$. By the theorem $(\forall\psi_1 \wedge \ldots \forall\psi_n) \leftrightarrow \forall(\psi_1 \wedge \ldots \psi_n)$, we can assume that $\Gamma_0 = \{\Box_0\theta, \forall\psi, \neg\varphi\}$, for some ψ s. t. $(\forall\psi) \in T$. From $\Gamma_0 \vdash \bot$ we get the theorem $(\Box_0\theta \wedge \forall\psi) \to \varphi$. Using the Monotonicity Rule for $\Box_0$, the formula $\Box_0(\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi$ is also a theorem. From the axiom $\Box_0\theta \to \Box_0\Box_0\theta$ and the Pullout Axiom, we get the theorem $(\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi$. Since $(\Box_0\theta) \in T$ and $(\forall\psi) \in T$, it follows that $(\Box_0\varphi) \in T$, i.e. $T \in \widehat{\Box_0\varphi}$, as desired.

Lemma 9 (Truth Lemma) *For every formula* $\phi \in \mathcal{L}_{\forall\Box\Box_0}$, *we have:* $||\phi||_{\mathcal{M}} = \hat{\phi}$.

Proof. Standard proof by induction on the complexity of ϕ, where the inductive step for each modality uses the corresponding Existence Lemma, as usual.

Consequence: $T_0 \models_{\mathcal{M}} \Phi_0$. This proves Step 1 (Proposition 8).

Theorem 11 *(STEP 2) The logic $\mathcal{L}_{\forall\Box\Box_0}$ has Strong Finite Quasi-Model Property.*

PROOF OF THEOREM 11: Let ϕ_0 be a consistent formula. By Step 1, take T_0 a maximal consistent theory s.t. $\phi_0 \in T_0$, and let $\mathcal{M} = (X, E_0, \leq, V)$ be the canonical quasi-model for T_0. We will use two facts about this model:

1. $||\varphi||_{\mathcal{M}} = \hat{\varphi}$, for all $\varphi \in \mathcal{L}_{\forall\Box\Box_0}$,
2. $E_0 = \{\widehat{\Box_0\varphi} : (\exists\Box_0\varphi) \in T_0\} = \{||\Box_0\varphi||_{\mathcal{M}} : (\exists\Box_0\varphi) \in T_0\}$.

Let Σ be a *finite* set such that: (1) $\phi_0 \in \Sigma$; (2) Σ is closed under subformulas; (3) if $(\Box_0\varphi) \in \Sigma$ then $(\Box\Box_0\varphi) \in \Sigma$; (4) Σ is closed under single negations; (5) $(\Box_0\top) \in \Sigma$. For $x, y \in X$, put: $x \equiv_\Sigma y$ iff $\forall\psi \in \Sigma(x \in ||\psi||_{\mathcal{M}} \iff y \in ||\psi||_{\mathcal{M}})$, and denote by $|x| := \{y \in X : x \equiv_\Sigma y\}$ the equivalence class of x modulo $\equiv_\Sigma$. Also, put $X^f := \{|x| : x \in X\}$, and more generally put $e^f := \{|x| : x \in e\}$ for every $e \in E_0$.

We now define a "*filtrated model*" $\mathcal{M}^f = (X^f, E_0^f, \leq^f, V^f)$, by taking: as set of worlds the set X^f (of equivalence classes) defined above; as for the rest, we put: $|x| \leq^f |y|$ iff for all $(\Box\psi) \in \Sigma : (x \in ||\Box\psi||_{\mathcal{M}} \Rightarrow y \in ||\Box\psi||_{\mathcal{M}})$; $E_0^f := \{e^f : e = \widehat{\Box_0\psi} = ||\Box_0\psi||_{\mathcal{M}} \in E_0$ for some ψ s. t. $(\Box_0\psi) \in \Sigma\}$; $V^f(p) := \{|x| : x \in V(p)\}$.

Lemma 10 *$\mathcal{M}^f$ is a finite quasi-model (of size bounded by a computable function of ϕ_0).*

Proof. X^f is finite, since Σ is finite so there are only finitely many equivalence classes modulo $\equiv_\Sigma$. In fact, the size is at most $2^{|\Sigma|}$. It's obvious that $\leq^f$ is a preorder, that $X^f \in E_0^f$ (since $X = ||\Box_0\top||_{\mathcal{M}}$ and $(\Box_0\top) \in \Sigma$, so $X^f \in E_0^f$) and that every $e^f \in E_0^f$ is non-empty (since it comes from some non-empty $e \in E_0$). So we only have to prove that the evidence sets are upward–closed: for this, let $e^f \in E_0^f$, with $e = \widehat{\Box_0\psi} \in E_0$, $(\Box_0\psi) \in \Sigma$ and let $|x| \in e^f$ and $|y| \in X^f$ s.t. $|x| \leq^f |y|$. We need to show that $|y| \in e^f$.

Since $|x| \in e^f$, there exists some $x' \equiv_\Sigma x$ s.t. $x' \in \widehat{\Box_0\psi} = ||\Box_0\psi||_{\mathcal{M}}$. From $(\Box_0\psi) \in \Sigma$ and $x' \equiv_\Sigma x$, we get $x \in ||\Box_0\psi||_{\mathcal{M}}$. By the theorem $\Box_0\psi \rightarrow \Box\Box_0\psi$, we have $x \in ||\Box\Box_0\psi||_{\mathcal{M}}$. But $(\Box\Box_0\psi) \in \Sigma$ (by the closure assumptions on Σ), so $|x| \leq^f |y|$ gives us $y \in ||\Box\Box_0\psi||_{\mathcal{M}}$. By the T-axiom $\Box\phi \rightarrow \phi$, we get $y \in ||\Box_0\psi||_{\mathcal{M}} = \widehat{\Box_0\psi} = e$, hence $|y| \in e^f$.

Lemma 11 (Filtration Lemma) *For every formula $\phi \in \Sigma$: $||\phi||_{\mathcal{M}^f} = \{|x| : x \in ||\phi||_{\mathcal{M}}\}$.*

Proof. Proof by induction on $\phi \in \Sigma$. The atomic case, inductive cases for propositional connectives and modalities $\forall\phi$ and $\Box\phi$ are treated as usual (-in the last case using the filtration property of $\leq^f$). We only prove here the inductive case for the modality $\Box_0\phi$:

Left-to-right inclusion: Let $|x| \in ||\Box_0\phi||_{\mathcal{M}^f}$. This means that there exists some $e^f \in E_0^f$ s.t. $|x| \in e^f \subseteq ||\phi||_{\mathcal{M}^f}$. By the definition of E_0^f, there exists some ψ s.t.: $(\Box_0\psi) \in \Sigma$ and $e = \widehat{\Box_0\psi} = ||\Box_0\psi||_{\mathcal{M}} \in E_0$. From $|x| \in e^f$, it follows that there is some $x' \equiv_\Sigma x$ s.t. $x' \in e = ||\Box_0\psi||_{\mathcal{M}}$, and since $(\Box_0\psi) \in \Sigma$, we have $x \in ||\Box_0\psi||_{\mathcal{M}} = e$. It is easy to see that we also have $e \subseteq ||\phi||_{\mathcal{M}}$. (Indeed, let $y \in e$ be *any* element of e; then $|y| \in e^f \subseteq ||\phi||_{\mathcal{M}^f}$, so

$|y| \in ||\phi||_{\mathcal{M}^f}$, and by the induction hypothesis $y \in ||\phi||_{\mathcal{M}}$.) So we have found an evidence set $e \in E_0$ s.t. $x \in e \subseteq ||\phi||_{\mathcal{M}}$, i.e., shown that $x \in ||\Box_0\phi||_{\mathcal{M}}$.

Right-to-left inclusion: Let $x \in ||\Box_0\phi||_{\mathcal{M}}$, with $(\Box_0\phi) \in \Sigma$. It is easy to see that $(\exists\Box_0\phi) \in x$ (by the theorem $\Box_0\phi \to \exists\Box_0\phi$) and so also $(\exists\Box_0\phi) \in T_0$ (since $x \in X$ so $x \sim T_0$). This means that the set $e := \widehat{\Box_0\phi} = ||\Box_0\phi||_{\mathcal{M}} \in E_0$ is an evidence set in the canonical model, and since $(\Box_0\phi) \in \Sigma$, we conclude that $e^f \in E_0^f$ is an evidence set in the filtrated model. We obviously have $x \in e$, and so $|x| \in e^f$. By the (T) axiom, $e = ||\Box_0\phi||_{\mathcal{M}} \subseteq ||\phi||_{\mathcal{M}}$, and hence $e^f \subseteq \{|y| : y \in ||\phi||_{\mathcal{M}}\} = ||\phi||_{\mathcal{M}^f}$ (by the induction hypothesis). Thus, we have found $e^f \in E_0^f$ s.t. $|x| \in e^f \subseteq ||\phi||_{\mathcal{M}^f}$, i.e., shown that $|x| \in ||\Box_0\phi||_{\mathcal{M}^f}$.

Theorem 12 *(STEP 3) Every finite quasi-model is modally equivalent to a finite Alexandroff quasi-model (and so to a topo-e-model).*

PROOF OF THEOREM 12: Let $\mathcal{M} = (X, E_0, \leq, V)$ be a finite quasi-model. We form a new structure $\tilde{\mathcal{M}} = (\tilde{X}, \tilde{E_0}, \tilde{\leq}, \tilde{V})$, by putting: $\tilde{X} := X \times \{0, 1\}$; $\tilde{V}(p) := V(p) \times \{0, 1\}$; $(x, i)\tilde{\leq}(y, j)$ iff: $x \leq y$ and $i = j$; $\tilde{E_0} := \{e_i : e \in E_0, i \in \{0, 1\}\} \cup \{e_i^y : y \in e \in E_0, i \in \{0, 1\}\} \cup \{\tilde{X}\}$, where we used notations $e_i := e \times \{i\} = \{(x, i) : x \in e\}$ and $e_i^y := \uparrow y \times \{i\} \cup e \times \{1 - i\} = \{(x, i) : y \leq x\} \cup e_{1-i}$.

Lemma 12 *$\tilde{\mathcal{M}}$ is a (finite) quasi-model.*

Proof. Easy verification.

Notation: For any set $\tilde{Y} \subseteq \tilde{X}$, put $\tilde{Y}_X := \{y \in X : (y, i) \in \tilde{Y}$ for some $i \in \{0, 1\}\}$ for the set consisting of first components of all members of $\tilde{Y}$. It is easy to see that we have: $(\tilde{Y} \cup \tilde{Z})_X = \tilde{Y}_X \cup \tilde{Z}_X$, and $\tilde{X}_X = X$.

Lemma 13 *If $y \in e \in E_0$, $i \in \{0, 1\}$ and $\tilde{e} \in \{e_i, e_i^y\}$, then we have:*

1. *$\tilde{e}_X = e$;*
2. *$e_i^y \cap e_i = \uparrow(y, i)$, where $\uparrow(y, i) = \{\tilde{x} \in \tilde{X} : (y, i)\tilde{\leq}\tilde{x}\} = \{(x, i) : y \leq x\}$.*

Proof. 1. If $\tilde{e} = e_i$, then $\tilde{e}_X = (e \times \{i\})_X = e$. If $\tilde{e} = e_i^y$, then $\tilde{e}_X = (\uparrow y \times \{i\})_X \cup (e \times \{1 - i\})_X = \uparrow y \cup e = e$ (since e is upward-closed and $y \in e$, so $\uparrow y \subseteq e$).

2. $e_i^y \cap e_i = (\uparrow y \times \{i\} \cup e \times \{1 - i\}) \cap (e \times \{i\}) = (\uparrow y \cap e) \times \{i\} = \uparrow y \times \{i\} = \uparrow(y, i)$ (since $\uparrow y \subseteq e$).

Lemma 14 *$\tilde{\mathcal{M}}$ is an Alexandroff quasi-model (and thus also a topo-e-model).*

Proof. By Proposition 7, it is enough to show that, for every $(y, i) \in \tilde{X}$, the upward-closed set $\uparrow(y, i)$ is open in the topology τ_E generated by E_0. But this follows directly from part 2 of Lemma 13.

Lemma 15 (Modal-Equivalence Lemma) *For all $\varphi \in \mathcal{L}_{\forall\Box\Box_0}$: $||\varphi||_{\tilde{\mathcal{M}}} = ||\varphi||_{\mathcal{M}} \times \{0, 1\}$.*

Proof. Induction on φ. The base case (for atomic sentences), and the inductive steps for propositional connectives and for the operators $\forall$ and $\Box$, are all straightforward. So we only prove here the inductive step for $\Box_0$:

Left-to-Right Inclusion: Suppose that $(x, i) \in ||\Box_0\varphi||_{\tilde{\mathcal{M}}}$. Then there exists some $\tilde{e} \in \tilde{E}$ such that $(x, i) \in \tilde{e} \subseteq ||\varphi||_{\tilde{\mathcal{M}}} = ||\varphi||_{\mathcal{M}} \times \{0, 1\}$ (where we used the induction hypothesis for φ at the last step). From this, we obtain that $x \in \tilde{e}_X \subseteq (||\varphi||_{\mathcal{M}} \times \{0, 1\})_X = ||\varphi||_{\mathcal{M}}$. But by the construction of $\tilde{E}$, $\tilde{e} \in \tilde{E}$ means that either $\tilde{e} = \tilde{X}$ or there exist $e \in E_0$, $y \in e$ and $j \in \{0, 1\}$ such that $\tilde{e} \in \{e_j, e^y_j\}$. If the former is the case, we have $x \in \tilde{e}_X = X \subseteq ||\varphi||_{\mathcal{M}}$. Since $X \in E_0$, by the semantics of $\Box_0$, we obtain $x \in ||\Box_0\varphi||_{\mathcal{M}}$. If the latter is the case, by part 1 of Lemma 13, we have $\tilde{e}_X = e$, so we conclude that $x \in \tilde{e}_X = e \subseteq ||\varphi||_{\mathcal{M}}$. Therefore, again by the semantics of $\Box_0$, we have $x \in ||\Box_0\varphi||_{\mathcal{M}}$.

Right-to-Left Inclusion: Suppose that $x \in ||\Box_0\varphi||_{\mathcal{M}}$. Then there exists some $e \in E_0$ such that $x \in e \subseteq ||\varphi||_{\mathcal{M}}$. Take now the set $e_i = e \times \{i\} \in \tilde{E}$. Clearly, we have $(x, i) \in e_i \subseteq ||\varphi||_{\mathcal{M}} \times \{i\} \subseteq ||\varphi||_{\mathcal{M}} \times \{0, 1\} = ||\varphi||_{\tilde{\mathcal{M}}}$ (where we used the induction hypothesis for φ at the last step), i.e. we have $(x, i) \in ||\Box_0\varphi||_{\tilde{\mathcal{M}}}$.

Theorem 12 follows immediately from the above Lemma: the same formulas are satisfied at x in $\mathcal{M}$ as at (x, i) in $\tilde{\mathcal{M}}$. In its turn, our Theorem 6 (Completeness and Finite Model property for topo-e-models) is an immediate corollary of Theorem 12.

F Proofs of Theorems 7-10

In this appendix, we provide the soundness proofs for the most complex reduction axioms in Theorems 7-10, and sketch the completeness proof and the expressivity result for Theorem 7. The completeness results in Theorems 8-10 follow from the same argument.

F.1 Proof of Theorem 7

Let us denote the axiomatization given in Thereom 7 by $L_{!\forall\Box\Box_0}$. The soundness proofs of the axioms 1-3 and 6-7 are standard and they do not depend on the topological properties of the model. We here only prove the ones for the modalities $\Box_0$ and $\Box$. Let $\mathcal{M} = (X, E_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}_{!\forall\Box\Box_0}$. In the following, we do not use the subscript $\mathcal{M}$ for the truth sets in the model $\mathcal{M}$.

Axiom-4 of Theorem 7:

$$\begin{aligned} x \in \|[!\varphi]\Box_0\psi\| &\text{ iff } x \in \|\varphi\| \text{ implies } x \in \|\Box_0\psi\|_{\mathcal{M}^{!\|\varphi\|}} \\ &\text{ iff } x \in \|\varphi\| \text{ implies } \exists e^{!\|\varphi\|} \in E_0^{!\|\varphi\|}(x \in e^{!\|\varphi\|} \subseteq \|\psi\|_{\mathcal{M}^{!\|\varphi\|}}) \\ &\text{ iff } x \in \|\varphi\| \text{ implies } \exists e \in E_0(x \in e \cap \|\varphi\| = e^{!\|\varphi\|} \subseteq \|\psi\|_{\mathcal{M}^{!\|\varphi\|}}) \\ &\text{ iff } x \in \|\varphi\| \text{ implies } \exists e \in E_0(x \in e \subseteq \|[!\varphi]\psi\|) \\ &\text{ iff } x \in \|\varphi\| \text{ implies } x \in \|\Box_0[!\varphi]\psi\| \\ &\text{ iff } x \in \|\varphi \to \Box_0[!\varphi]\psi\| \end{aligned}$$

The validity of the axiom $[!\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[!\varphi]\psi)$ follows similarly, where we replace the basic evidence set E_0 by the corresponding combined evidence set E.

Therefore, also by Theorem 6, we obtain that $L_{!\forall\Box\Box_0}$ is sound for topo-e-models. The soundness of the recursion axioms implies that for any formula $\varphi \in \mathcal{L}_{!\forall\Box\Box_0}$, there exists a semantically equivalent formula ψ in the static language $\mathcal{L}_{\forall\Box\Box_0}$. Moreover, the recursion axioms give us an inductive algorithm as to how to reduce a formula in the dynamic language $\mathcal{L}_{!\forall\Box\Box_0}$ to a formula in the pure static language $\mathcal{L}_{\forall\Box\Box_0}$. In other words, by using the recursion axioms, we can also show that any formula $\varphi \in \mathcal{L}_{!\forall\Box\Box_0}$ is provably equivalent to a formula $\psi \in \mathcal{L}_{\forall\Box\Box_0}$ (see, e.g. [19] for a more detailed discussion on the topic). The completeness of $L_{!\forall\Box\Box_0}$ therefore follows from the completeness of $L_{\forall\Box\Box_0}$ and the soundness of the recursion axioms as follows: Let $\varphi \in \mathcal{L}_{!\forall\Box\Box_0}$ such that $\nvdash_{L_{!\forall\Box\Box_0}} \varphi$. Then, by the recursion axioms, there exists $\psi \in \mathcal{L}_{\forall\Box\Box_0}$ with $\vdash_{L_{!\forall\Box\Box_0}} \varphi \leftrightarrow \psi$. As $L_{\forall\Box\Box_0} \subset L_{!\forall\Box\Box_0}$ and $\psi \in \mathcal{L}_{\forall\Box\Box_0}$, we then have $\nvdash_{L_{\forall\Box\Box_0}} \psi$. Then, by completeness of $L_{\forall\Box\Box_0}$, there exists a topo-e-model such that $\|\psi\| \neq X$. Then, by soundness of $L_{!\forall\Box\Box_0}$, we conclude $\|\varphi\| \neq X$.

Moreover, since $L_{!\forall\Box\Box_0}$ extends $L_{\forall\Box\Box_0}$ and any dynamic formula is provably equivalent to a formula in the static language, the logics $\mathcal{L}_{!\forall\Box\Box_0}$ and $\mathcal{L}_{\forall\Box\Box_0}$ are equally expressive.

F.2 Proof of Theorem 8

Let $\mathcal{M} = (X, E_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}_{!\forall\Box\Box_0}$ and observe that,

$$x \in \|\exists\varphi\| \text{ implies } \|\psi\|_{\mathcal{M}^{+\|\varphi\|}} = \|[+\varphi]\psi\| \tag{1}$$

Axiom-4 of Theorem 8:

$x \in ||[+\varphi]\Box_0\psi||$ iff $x \in ||\exists\varphi||$ implies $x \in ||\Box_0\psi||_{\mathcal{M}^{+||\varphi||}}$

iff $x \in ||\exists\varphi||$ implies $\exists e^{+||\varphi||} \in E_0^{+||\varphi||}(x \in e^{+||\varphi||} \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}})$

iff $x \in ||\exists\varphi||$ implies $(\exists e \in E_0(x \in e \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}})$ or $(x \in ||\varphi|| \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}}))$

(by defn. of $E_0^{+||\varphi||}$)

iff $x \in ||\exists\varphi||$ implies $(\exists e \in E_0(x \in e \subseteq ||[+\varphi]\psi||)$ or $x \in ||\varphi|| \subseteq ||[+\varphi]\psi||)$

(by (1))

iff $x \in ||\exists\varphi||$ implies $((x \in ||\Box_0[+\varphi]\psi||)$ or $(x \in ||\varphi||$ and $x \in ||\forall(\varphi \to [+\varphi]\psi||))$

iff $x \in ||\exists\varphi||$ implies $(x \in ||\Box_0[+\varphi]\psi||$ or $x \in ||\varphi \wedge \forall(\varphi \to [+\varphi]\psi||)$

iff $x \in ||\exists\varphi \to (\Box_0[+\varphi]\psi \vee (\varphi \wedge \forall(\varphi \to [+\varphi]\psi)))||$

The proof for the axiom 5 follows in a similar way with minor differences because of the fact that for every $e^{+||\varphi||} \in E^{+||\varphi||}$ there is some combined evidence $e \in E$ such that either $e^{+||\varphi||} = e$ or $e^{+||\varphi||} = e \cap ||\varphi||$. Therefore, we have

Axiom-5 of Theorem 8:

$x \in ||[+\varphi]\Box\psi||$

iff $x \in ||\exists\varphi||$ implies $\exists e^{+||\varphi||} \in E^{+||\varphi||}(x \in e^{+||\varphi||} \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}})$

iff $x \in ||\exists\varphi||$ implies $\exists e \in E(x \in e \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}}$ or $x \in e \cap ||\varphi|| \subseteq ||\psi||_{\mathcal{M}^{+||\varphi||}})$

iff $x \in ||\exists\varphi||$ implies $\exists e \in E((x \in e \subseteq ||[+\varphi]\psi||)$ or $(x \in ||\varphi||$ and $x \in e \subseteq ||\varphi \to [+\varphi]\psi||))$

iff $x \in ||\exists\varphi||$ implies $(x \in ||\Box[+\varphi]\psi||$ or $(x \in ||\varphi||$ and $x \in ||\Box(\varphi \to [+\varphi]\psi||))$

iff $x \in ||\exists\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi)))||$

F.3 Proof of Theorem 9

Let $\mathcal{M} = (X, E_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}_{!\forall\Box\Box_0}$. Similar to the above case, we have

$$x \in ||\exists\varphi|| \text{ implies } ||\psi||_{\mathcal{M}^{\Uparrow||\varphi||}} = ||[\Uparrow\varphi]\psi|| \tag{2}$$

Axiom-4 of Theorem 9:

$x \in ||[\Uparrow\varphi]\Box_0\psi||$

iff $x \in ||\exists\varphi||$ implies $\exists e^{\Uparrow||\varphi||} \in E_0^{\Uparrow||\varphi||}(x \in e^{\Uparrow||\varphi||} \subseteq ||\psi||_{\mathcal{M}^{\Uparrow||\varphi||}})$

iff $x \in ||\exists\varphi||$ implies $(\exists e \in E_0(x \in e \cup ||\varphi|| \subseteq ||\psi||_{\mathcal{M}^{\Uparrow||\varphi||}})$ or $(x \in ||\varphi|| \subseteq ||\psi||_{\mathcal{M}^{\Uparrow||\varphi||}}))$

(by defn. of $E_0^{\Uparrow||\varphi||}$)

iff $x \in ||\exists\varphi||$ implies $(\exists e \in E_0(x \in e \cup ||\varphi|| \subseteq ||[\Uparrow\varphi]\psi||)$ or $(x \in ||\varphi|| \subseteq ||[\Uparrow\varphi]\psi||))$

(by (2))

iff $x \in ||\exists\varphi||$ implies $(\exists e \in E_0(x \in e \subseteq ||[\Uparrow\varphi]\psi||$ and $||\varphi|| \subseteq ||[\Uparrow\varphi]\psi||)$ or $(x \in ||\varphi|| \subseteq ||[\Uparrow\varphi]\psi||))$

iff $x \in ||\exists\varphi||$ implies $(x \in ||\Box_0[\Uparrow\varphi]\psi||$ and $x \in ||\forall(\varphi \to [\Uparrow\varphi]\psi)||)$ or $(x \in ||\varphi \wedge \forall(\varphi \to [\Uparrow\varphi]\psi)||)$

iff $x \in ||\exists\varphi \to ((\Box_0[\Uparrow\varphi]\psi \vee \varphi) \wedge \forall(\varphi \to [\Uparrow\varphi]\psi))||$

The validity of the axiom 5 follows similarly where we replace the basic evidence set E_0 by the corresponding combined evidence set E.

F.4 Proof of Theorem 10

Axiom-4 of Theorem 10:

$$\begin{aligned} x \in \|[\#]\Box_0\varphi\| &\text{ iff } x \in \|\Box_0\varphi\|_{\mathcal{M}^\#} \\ &\text{ iff } \exists e^\# \in E_0^\#(x \in e^\# \subseteq \|\varphi\|_{\mathcal{M}^\#}) \\ &\text{ iff } \exists e^\# \in E_0^\#(x \in e^\# \subseteq \|[\#]\varphi\|) \qquad (\text{since } E_0^\# = E^\# = E) \\ &\text{ iff } \exists e \in E(x \in e \subseteq \|[\#]\varphi\|) \\ &\text{ iff } x \in \|\Box[\#]\varphi\| \end{aligned}$$

The validity of the axiom 5 follows similarly since $E = E^\#$.

References

1. Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. The topology of belief, belief revision and defeasible knowledge. Huang, H., Grossi, D., and Roy, O. (eds.), *Logic, Rationality and Interaction, Proceedings of the Fourth International Workshop, LORI 2013*, Springer.
2. Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2015) The topological theory of belief. *Submitted, also avaliable at http://www.illc.uva.nl/Research/Publications/Reports/PP-2015-18.text.pdf*.
3. Baltag, A. and Smets, S. (2006) Conditional doxastic models: A qualitative approach to dynamic belief revision. *In Proceedings of the 13th Workshop on Logic, Language, Information and Computation (WOLLIC 2006)*, pp. 5–21.
4. Baltag, A. and Smets, S. (2008) A qualitative theory of dynamic interactive belief revision. *Logic and the Foundations of Game and Decision Theory (LOFT7)*, pp. 11–58.
5. Chagrov, A. V. and Zakharyaschev, M. (1997) *Modal Logic*, vol. 35 of *Oxford logic guides*. Oxford University Press.
6. Gettier, E. (1963) Is justified true belief knowledge? *Analysis*, **23**, 121–123.
7. Goranko, V. and Passy, S. (1992) Using the universal modality: Gains and questions. *J. Log. Comput.*, **2**, 5–30.
8. Grove, A. Two modellings for theory change. *Journal of Philosophical Logic*, **17**, 157–170.
9. Lehrer, K. (1990) *Theory of Knowledge*. Routledge.
10. Lehrer, K. and Paxson, T. J. (1969) Knowledge: Undefeated justified true belief. *Journal of Philosophy*, **66**, 225–237.
11. McKinsey, J. C. C. and Tarski, A. (1944) The algebra of topology. *Ann. of Math. (2)*, **45**, 141–191.
12. Özgün, A. (2013) *Topological Models for Belief and Belief Revision*. Master's thesis, University of Amsterdam, Amsterdam, The Netherlands.
13. Rott, H. Stability, strength and sensitivity:converting belief into knowledge. *Erkenntnis*, **61**, 469–493.
14. Stalnaker, R. (2006) On logics of knowledge and belief. *Philosophical Studies*, **128**, 169–199.

15. van Benthem, J. (2004) Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, **14**, 2004.
16. van Benthem, J. and Bezhanishvili, G. (2007) Modal logics of space. *Handbook of Spatial Logics*, pp. 217–298.
17. van Benthem, J., Fernández-Duque, D., and Pacuit, E. (2014) Evidence and plausibility in neighborhood structures. *Annals of Pure and Applied Logic*, **165**, 106 – 133.
18. van Benthem, J. and Pacuit, E. (2011) Dynamic logics of evidence-based beliefs. *Studia Logica*, **99(1)**, 61–92.
19. van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007) *Dynamic Epistemic Logic*. Springer Publishing Company, Incorporated, 1st edn.
20. Williamson, T. (2000) *Knowledge and its Limits*. Oxford Univ. Press.