Tales From An Old Manuscript

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Abstract

'Possibility semantics' for classical logics is on the rise these days, and it throws interesting new light on the borderline between classical and intuitionistic logic. This note is about a source from 1981 that is attracting some attention, a handwritten Tech Report from the Groningen Institute of Mathematics, never published, but with a facsimile on-line at http://dare.uva.nl/cgi/arno/show.cgi?fid=621016. I will present its content, and discuss what issues this raises that may be of current relevance.

1 Albert Visser in a nutshell

First things first. The main points of this short note are not logical theorems but three empirical observations, based on long experience. Knowing Albert Visser and interacting with him is a pleasure, corresponding with Albert Visser is engaging, and working together with Albert Visser is an educational experience that is one of the best self-improvement methods known to mankind. This concludes the main content of this note, the rest is aftermath.

I have enjoyed all three mentioned dimensions, but my theme for this piece is some thoughts touching on a general issue that has colored my contacts with Albert. My heart beats to the rhythms of *classical logic*, but his (I suspect) to those of *intuitionistic logic*. What follows is a possible new perspective on the interface between these two lifestyles – though I hasten to add that Albert is a master at both, and knows much more about their interfaces than I do.

2 Possible worlds semantics for classical logic

Possibilities Around 1980 possible worlds semantics for modal logics was under attack from various sides. There was a philosophical interest in an ontology of smaller, less elaborate 'possibilities' or 'situations', but there were also related developments in the temporal semantics of natural language where intervals, rather than fully specified points, were emerging as the indices of evaluation (cf. Barwise and Perry 1983, van Benthem 1983). While standard possible worlds semantics has continued to flourish (cf. Blackburn, van Benthem & Wolter, eds. 2006), models with possibilities ordered by inclusion, too, return all the time. For instance, working in an epistemological setting where the options to be considered are often less fine-grained than worlds, Holliday 2014, Holliday 2015 develop a framework of possibilities that allows for a functional view of beliefs, returning to the classic paper Humberstone 1981. [1] Interestingly, going against a prejudice that ontological change must also be logic change, say toward some partial, intuitionistic or hyper-intensional system, the models elaborated in this line support classical logic. [2] And this brings us to the topic of this paper.

An old handwritten tech report As Holliday notes, the possibilities models that he ends up with are close to those in an unpublished report van Benthem 1981, with a short summary included in the published paper van Benthem 1986. However, my motivations in 1981 were different from the modern possibilities program, and this gives me the theme for this paper.

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[1] E.g., coming from mereology, Bochmann 2015, too, develops new possibilities models and languages. Rumfitt 2015 uses possibilities in a sustained philosophical defense of classical logic.

[2] Actually, possibility models are also modal models in the usual style. Many differences that seem vast in the literature are matters of philosophical interpretation, rather than mathematical substance.

Why did I propose these structures (called 'possible worlds models for classical logic' – not a very good name), what results were found, and what relevant logical program might emerge?

The first thing to do was reading the original report from the Mathematical Institute in Groningen. This was not always easy, apart from the fading ink. The author is often terse and apodictic: something which may be all-right to inflict on others, but which seems really objectionable when doing it to *oneself*, even if 35 years later. What follows is my current reconstruction of things written back then, with a selection of some key results, proofs that I have checked, plus some comments on the enterprise as they occur to me now. I hope that this still has some current value: for possibility semantics, and perhaps beyond.

Intuitionistic and classical logic As a student in Amsterdam, I was exposed first-hand to a lot of intuitionistic logic, represented by some of its finest minds: Anne Troelstra, Dick de Jongh, and later on, Albert Visser. Still, probably out of a revolutionary 1960s desire to be different, my heart went out to classical logic. My dissertation on modal logic was squarely in that tradition, and my favorite tool was classical model theory, not constructive proof theory or the like. But of course, for a working logician, things are never that clear-cut. I used proof theory in my later work on categorial grammar and substructural logics, and as for intuitionistic logic, I have always felt a fascination for what it has to offer semantically. In particular, intuitionistic models for coming to know things have always appealed to me, especially their links with information structure and information dynamics (van Benthem 1989, 2009).

More than that, the 1981 report even shows a certain unease with classical logic. It starts with the complaint that classical completeness proofs have the inelegant feature that one needs to pick 'some' maximally consistent extension of a given set of formulas to create a model, and likewise, that basic model-theoretic constructions such as ultraproducts need an ultrafilter going beyond what is often just a filter containing the relevant initial structure. On this basis, the report then proposes what it calls an 'intuitionistic' alternative, though it notes that intuitionistic completeness proofs still have a similar problem, as they use prime filters that split disjunctions in some arbitrary manner, often going beyond the original input.

A modal semantics So, we take models $\boldsymbol{M}$ for the language of first-order logic that are tuples $(W, \leq, D, I)$ where W is a set of worlds (or stages, or information pieces), $\leq$ is a partial order (inclusion between stages), D is a map assigning to each world w a set $D(w)$ of objects (the domain of that world), and I is an interpretation function sending predicates P and worlds w to sets of tuples of the appropriate arity with objects chosen from $D(w)$. In doing so, we impose some further constraints on our structures. The first of these is standard modal:

Cumulation for domains — if $w \leq v$, then $D(w) \subseteq D(v)$.

We also impose, as in models for intuitionistic logic,

Persistence for atomic facts — if $P\boldsymbol{d}$ holds at w and $w \leq v$, then $P\boldsymbol{d}$ holds at v.

However, in line with our classical orientation, we also impose

Cofinality for atomic facts — if for all $v \geq w$, there exists a $u \geq v$ with $P\boldsymbol{d}$ true at u, then $P\boldsymbol{d}$ is already true at w.

The report motivates the latter condition as follows, while pointing out how it embodies the negation law $\neg\neg\varphi \rightarrow \varphi$. If the truth of a formula φ is 'inevitable' cofinally in the model, then we might as well call φ true now. This sounds like the Sure Thing Principle of decision theory: if every available current action of yours will result in believing that φ, believe φ right now. [3]

Remark Later literature on possibility- or stage-based models, in natural language semantics, metaphysics, or epistemology, has added a different intuition motivating cofinality. This time, it appears as a desirable form of 'refinability': if a stage w fails to make φ true, then there must still be some more refined state $v \geq w$ that refutes φ, in the sense of making $\neg\varphi$ true. An early source for this line of motivation is the 'data semantics' of Veltman 1984.

The following truth definition now interprets the first-order language, with a little benign sloppiness in notation. In what follows, in our exposition, we restrict attention to just three logical operations, viz. conjunction, negation, and universal quantification: [4]

$\boldsymbol{M}, w, \boldsymbol{d} \models P\boldsymbol{x}$	iff	the tuple of objects $\boldsymbol{d}$ is in $I(P, w)$
$\boldsymbol{M}, w, \boldsymbol{d} \models \varphi\wedge\psi$	iff	$\boldsymbol{M}, w, \boldsymbol{d} \models \varphi$ and $\boldsymbol{M}, w, \boldsymbol{d} \models \psi$
$\boldsymbol{M}, w, \boldsymbol{d} \models \neg\varphi$	iff	for no $v \geq w$, $\boldsymbol{M}, v, \boldsymbol{d} \models \varphi$
$\boldsymbol{M}, w, \boldsymbol{d} \models \forall x\varphi$	iff	for all $v \geq w$, for all $d \in D(v)$, $\boldsymbol{M}, v, \boldsymbol{d}d \models \varphi$

More formally, instead of tuples $\boldsymbol{d}$, one can use assignment functions s in a format $\boldsymbol{M}, w, s \models \varphi$ where s sends the free variables in φ to objects in the domain $D(w)$. Domain Cumulation makes sure this remains well-defined when we go up to worlds higher in the inclusion order.

Further operations are considered defined, with $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, and $\exists x\varphi$ as $\neg\forall x\neg\varphi$.

With the language interpreted in this way, it can be shown by a straightforward induction that the earlier two semantic properties of persistence and cofinality lift to all formulas.

Completeness by one canonical model With models like this, the completeness proof for a classical first-order axiom system becomes straightforward, referring to a unique canonical model. Consider the set of all *consistent deductively closed* sets Σ of formulas in our language where we assume that each set Σ has an associated language $L(\Sigma)$ which involves only finitely many individual constants. [5] When we refer to inclusion in what follows, we mean inclusion of sets and associated languages. Now we observe that the following standard decomposition principles hold for all formulas and consistent sets in the relevant associated languages:

Fact	$\varphi\wedge\psi \in \Sigma$	iff	$\varphi \in \Sigma$ and $\psi \in \Sigma$
	$\neg\varphi \in \Sigma$	iff	for all $\Delta \supseteq \Sigma$, $\varphi \notin \Delta$
	$\forall x\varphi \in \Sigma$	iff	for $\Delta \supseteq \Sigma$ and all constants c, $[c/x]\varphi \in \Delta$

Proof The proof of these equivalences involves only minimal properties of any proof system. In particular, in the argument for the negation clause, it is not assumed that the underlying proof system is classical. The analogy with the three earlier truth conditions will be clear. ■

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[3] Cofinality is weaker than total `inevitability' of a formula defined as truth on some barrier across the inclusion graph, the second-order clause employed in Beth's original semantics for intuitionistic logic. The difference raises some interesting technical complexities on infinite trees, but we omit them here.

[4] Also, for simplicity, we only consider only variables and constants, a countably infinite set of each (for technical reasons to become clear shortly), but no complex term-forming function symbols.

[5] The restriction ensures that we can extend sets with new witnesses for false universal formulas.

We can exploit these decomposition properties of consistent sets by defining a canonical model ***M*** as consisting of all consistent language-indexed deductively closed sets ordered by inclusion, and deriving its interpretation of atoms as usual, directly from the atomic formulas that are explicitly present in the sets. The key feature of this model is given by the usual equivalence of membership and truth, for all sets Σ in the model and all sentences $[\mathbf{c}/\mathbf{x}]\varphi$ with all free variables in the formula φ replaced by constants in the language $L(\Sigma)$:

Truth Lemma $\mathbf{M}, \Sigma \models [\mathbf{c}/\mathbf{x}]\varphi$ iff $[\mathbf{c}/\mathbf{x}]\varphi \in \Sigma$

Proof This is shown by a straightforward induction on formulas. ■

The further property of Persistence is immediate from the use of set inclusion, but Cofinality does ask for something classical, namely, the derivability of the double negation law $\neg\neg\varphi \to \varphi$.

In this light, completeness for classical logic does not need any arbitrary choices of maximally consistent sets. But what is more, the above proof does not assume any form of the Axiom of Choice, and hence it suggests a more constructive version of standard meta-theory. [6]

Back to classical models: generic branches Now, as pointed out in our source text, this simple and minimal completeness proof comes at a price. For, it does not tell us that a non-derivable model has a counterexample that is 'standard', being just a single-world model of the usual sort. However, a route toward such models is provided by what the report calls the 'usual method' of generic branches, borrowed from set-theoretic forcing.

Consider any countable possibility model as defined above, and look at the branches in it – i.e., the maximal linearly ordered subsets. If a branch ends at some finite stage, we have an obvious classical model for all formulas true there. But such models can also be found for infinite *generic branches B* that satisfy the following two properties:

(a) for each formula φ, either φ or $\neg\varphi$ is true at some stage on B, and
(b) for each formula $\neg\forall x\varphi$ true somewhere on the branch B, some instance $\neg[c/x]\varphi$ is also true somewhere on that branch for some individual constant c.

Each branch B in any model naturally induces a classical model $\mathbf{M}(B)$ whose domain consists of all objects occurring in worlds on B, with the interpretation of predicate atoms copied from what is true on the branch. Persistence makes sure that this stipulation is well-defined. Moreover, for the special case of generic branches that force choices and provide witnesses for false universal formulas, we can prove the following fact by induction on formulas:

Fact $\mathbf{M}(B) \models \varphi$ iff there is some world w on B such that $\mathbf{M}, w \models \varphi$

Proof We use induction on φ. For conjunctions, the key point is that having two formulas true at different points on a branch makes both of them true at the greater of the two points, by persistence. For negation, again persistence is crucial. In one direction, if any point on a branch makes φ true, then no other point on the branch can make $\neg\varphi$ true. The opposite direction uses clause (a) of genericity, plus the inductive hypothesis. Finally, the right to left direction of the universal quantifier case follows immediately by persistence and domain cumulation, while the opposite direction uses clause (b) of genericity plus cofinality. ■

[6] It is of interest to compare this analysis with that of the intuitionistically valid completeness proofs for intuitionistic logic due to Veltman and De Swart in the 1970s: see Troelstra & van Dalen 1988.

What makes these facts useful is the following observation.

Fact Any countable model has generic branches starting from arbitrary points.

Proof Generic branches can be found by a model-internal analogue of the Henkin completeness construction. We enumerate formulas and pick a branch, making sure that all instances of formulas occur with all objects occurring in points on the branch. [7] Using this enumeration as our scheduling, to continue the branch so far, we choose a further point that either makes the scheduled formula or its negation true, and if the latter case is of the form $\neg\forall x\varphi$, we use the fact that $\forall x\varphi$ fails at the point, and continue to a possibly even further point where some instance $\neg[c/x]\varphi$, is true, which must exist by the truth definition plus cofinality. ■

3 Taking the framework further

Possibility models for a classical setting may look like 'beautiful noise' induced by purely philosophical qualms about standard models of first-order logic. However, the remainder, and in fact the bulk of my old report was devoted to taking stock of what a switch to such models would do to working model theory. In particular, new versions of old notions emerge, while we can also use notions and techniques from modal logic. Here is a key example.

From ultraproducts to filter products Consider the fundamental notion of an *ultraproduct*, used in the well-known proof of the compactness theorem for constructing a model for a finitely satisfiable set Σ of formulas out of models for its finite subsets. This proof uses an arbitrary maximal extension of some initial filter carrying all the key information – here, the 'regular filter' of all sets $\{\Delta \subseteq \Sigma \mid \Sigma_0 \subseteq \Delta\}$ with Σ_0 running over all finite subsets of Σ.

The following notion makes the compactness construction unique, while preserving the basic behavior of ultraproducts with respect to first-order formulas according to Los' Theorem.

Consider any family $\{\boldsymbol{M}_i \mid i \in I\}$ of possibility models in our sense, and fix some filter F on I. We define the *filter product* of this family *with respect to F* as the following new possibility model $\boldsymbol{M}$. Let the variable G run over all non-empty filters on I that extend F, and for the worlds, collect all world-like objects in the products $\Pi^G\boldsymbol{M}_i$ of $\{\boldsymbol{M}_i \mid i \in I\}$ in the usual model-theoretic sense: each world is an equivalence class of functions from indices i to worlds in $\boldsymbol{M}_i$, divided out by G. As for objects belonging to such worlds w, these are equivalence classes of functions from indices to objects in $D(w)$, divided out by the relevant filter. [8]

The interpretation of atomic predicates on objects is given locally in the usual way. For the sake of illustrating this product setting and the role of filters, we display one clause:

> P holds of a tuple of equivalence classes $\boldsymbol{d}^G$ in $\Pi^G\boldsymbol{M}_i$ iff
> $\{i \in I \mid \text{in } \boldsymbol{M}_i, P \text{ holds of the tuple of coordinates } (\boldsymbol{d})_i\} \in G$

Analogously, for the stage inclusion order, we put $(w)^G \leq (v)^{G'}$ iff $\{i \in I \mid \text{in } \boldsymbol{M}_i, w_i \leq v_i\} \in G'$. [9]

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[7] I use the countability restriction as a precaution, which need not be necessary for a best result. Also, the precise enumeration pattern requires some care, the details of which are omitted here.

[8] We could make this a bit more stream-lined in terms of two-sorted first-order product models, in which we ban 'mixed' functional objects assigning worlds at some indices and objects at others.

[9] Note that this definition gives the intuitively correct result if the stage does not change, i.e., $G' = G$.

Remark There is a small but interesting issue here with identity across worlds. Worlds $(w)^G$ at stage G need not occur as such at later stages, since an extended filter G' may identify more points in the product of the models $\boldsymbol{M}_i$ than G did. Even so, there is a family of well-defined *functions* $f_{GG'}$ taking worlds $(w)^G$ at earlier stages to worlds $(w)^{G'}$ at later stages (and the same is true for objects at those worlds) – where the well-definedness uses the crucial intersection property of filters. Thus, we have to generalize the earlier models to the case where domain cumulation can run via arbitrary functions, not necessarily the identity. This is a minor but significant modification of our earlier setting, with independent motivation in current semantics of modal predicate logic, for instance, in removing cases of incompleteness. [10] In order to keep notation simple, in what follows, we do not display the maps $f_{GG'}$ explicitly.

Now we can prove the following generalized Los Theorem for filter products $\boldsymbol{M}$ of a family of models $\{\boldsymbol{M}_i \mid I \in I\}$ with respect to F. For any world $(w)^G$, formula φ, and tuple of objects $(\boldsymbol{d})^G$:

Fact $\boldsymbol{M}, (w)^G, (\boldsymbol{d})^G \models \varphi$ iff $\{i \in I \mid \boldsymbol{M}_i, w_i, (\boldsymbol{d})_i \models \varphi\} \in G$

Proof The proof for this equivalence is a simple induction like that for the Los Theorem itself, but now using our definition of the logical constants referring to filter extensions. (a) The atomic clause works by definition. (b) The case of a conjunction uses upward closure plus the intersection property of filters in a straightforward manner. (c1) For a negation $\neg\varphi$, first assume that we do not have the set $\{i \in I \mid \boldsymbol{M}_i, w_i, (\boldsymbol{d})_i \models \neg\varphi\}$ in the filter G. Then, by standard reasoning, there must be an extended filter G' that contains $\{i \in I \mid not\ \boldsymbol{M}_i, w_i, (\boldsymbol{d})_i \models \neg\varphi\}$. Unpacking this by the truth definition plus the upward closure of filters, we have that the set $\{i \in I \mid for\ some\ v_i \geq_i w_i, \boldsymbol{M}_i, v_i, (\boldsymbol{d})_i \models \varphi\}$ is in G'. But then we can collect all these worlds v_i into one functional object v in the domain for the product of the models $\boldsymbol{M}_i$ – where it will not matter what we do on indices outside the preceding set, given the fact that we have divided out by the filter G'. [11] Now, by the inductive hypothesis, $(v)^{G'}$ makes φ true for the relevant objects $\boldsymbol{d}$, while by our definition of filter products, $(v)^{G'}$ is a stage-inclusion successor of $(w)^G$. Unpacking the truth definition for negation, it follows that not $\boldsymbol{M}, (w)^G, (\boldsymbol{d})^G \models \neg\varphi$. (c2) As for the opposite direction, suppose that not $\boldsymbol{M}, (w)^G, (\boldsymbol{d})^G \models \neg\varphi$. So there is a $G' \supseteq G$ and $(v)^{G'} \geq (w)^G$ such that $\boldsymbol{M}, (v)^{G'}, (\boldsymbol{d})^{G'} \models \varphi$. By the inductive hypothesis, then, $\{i \in I \mid \boldsymbol{M}_i, v_i, (\boldsymbol{d})_i \models \varphi\}$ is in G' (#). It follows that the set $\{i \in I \mid \boldsymbol{M}_i, w_i, (\boldsymbol{d})_i \models \neg\varphi\}$ cannot be in G. For, by the truth definition for negation, it is contained in $\{i \in I \mid not\ \boldsymbol{M}_i, v_i, (\boldsymbol{d})_i \models \varphi\}$, which would then also be in the filter G and therefore in G'. But this contradicts *(#)*. (d) The case of the universal quantifier can be proved by similar reasoning. We merely note one direction where a small additional move is needed. If the set $\{i \in I \mid not\ \boldsymbol{M}_i, w_i, (\boldsymbol{d})_i \models \forall x\varphi\}$ is in some filter $G' \supseteq G$, then we can create a further product stage v as we did before for the negation case, but now also with objects d attached to its coordinate worlds such that $\varphi(d)$ fails. And we can do still better: by applying cofinality to the relevant coordinates, and then collecting the worlds obtained into a set in G', we can even make sure that $\neg\varphi(d)$ holds for the matching product world.

Now we are almost done. We still must check that filter products are truly possibility models. Here Persistence is a simple consequence of our product definition and the truth definition,

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[10] Cf. van Benthem 2010 for a discussion of the literature on generalized functional models for modal predicate logic, where objects at one world only need to have 'counterparts' in other worlds.

[11] Here we use the Axiom of Choice, and this seems to be a sin against our earlier aim of reducing our dependence on such principles. It would be of interest to see if we can circumvent the present use.

and it amounts technically to saying that the earlier domain-inclusion maps $f_{GG'}$ are relational homomorphisms. Checking Cofinality for the product model requires a little bit more care (as is in fact done in the original report), using cofinality inside the coordinate models $\boldsymbol{M}_i$ to pass to upward worlds where the negation of the relevant atomic formula holds, and then collecting these worlds into a member of a suitable filter extending F. ■

As a corollary, we get the compactness theorem for possibility semantics by a canonical construction without arbitrary extensions to ultrafilters. Taking a filter product of models for finite subsets with respect to the regular filter will give a model for the whole set.

Further possibility-based model theory The original 1981 report has more material than what we cover here. It defines analogues for all the major modal model constructions: generated submodels, disjoint unions, and in particular also, a notion of 'zigzag invariance' that combines bisimulation for stages with potential isomorphism for objects. In addition, we have the above filter products, plus an inverse operation of taking a 'filter base'. The upshot of the analysis is a complex argument leading up to the following statement:

Claim A class of possibility models is definable by a set of first-order sentences if and only if it is closed under the formation of generated submodels, disjoint unions, zigzag-invariant images, filter products and filter bases.

Given that I have not checked this proof in detail yet, the status of this claim is at present a conjecture. However, in this connection, it is worth noting that Holliday 2015 uses a notion like the above filter product to develop a duality between classical and possibilities models.

Excursion: submodel preservation Instead of pursuing the preceding theme, I will briefly raise a further one, not in the report: namely, what happens to an old interest of Albert and mine (and our friends): preservation theorems for modal and intuitionistic logic, and in particular, characterizing preservation under *submodels*. Which special syntactic forms arise on possibility models, and what is the relation to the classical Los-Tarski Theorem?

We can think in two directions here. One way is to start from the usual syntactic target class of 'universal' sentences of the form *∀-prefix + matrix with only {¬, ∧, atoms),* and ask for its characteristic preservation behavior in the setting of possibility models. This turns out to involve going to 'submodels' of possibility models in the following sense:

> We leave the stage structure the same and only throw away objects
> from domains of worlds, in such a way that the basic constraints
> of cumulation, persistence, and cofinality continue to hold.

I believe that such a preservation theorem can be proved by a variant of the standard compactness-based argument, but now using filter models, rather than standard models.

But there is also an opposite direction. We can investigate what natural notions of submodel arise in the setting of possibility models, and then ask what syntactic preservation classes match these. Then at least three natural options arise for operating on models:

> One option is dropping *objects* from world domains as above,
> a second is dropping *stages*, and a third: stages plus objects.

This time, one has to be even more cautious, since dropping stages can easily endanger the truth of cofinality: concrete counter-examples are easy to find. [12] But assuming that we take care of this desideratum in some way (only taking 'admissible' submodels), it will be clear that we now need strong syntactic restrictions on the matrix part of formulas with only ¬, ∧, and atoms to ensure persistence, since the negation is really a modality over the stage order.

I have no preservation theorem of this sort yet, but now that possibilities semantics is on the table again, I submit that these are interesting questions. Moreover, to get to such a result, I would urge the reader to visit an obvious inspiration point. Preservation theorems for passing to submodels in modal or intuitionistic propositional logic, with appropriate proof techniques for that setting, can be found in van Benthem, de Jongh, Renardel & Visser 1995.

Further directions Finally, here are two directions not studied in any detail in the document discussed here, but which make good sense in the current literature on possibility models.

Language extensions. Any serious generalization of a traditional semantics naturally raises the question whether the old vocabulary of logical operators in the restricted model class is still adequate for capturing essential distinctions in the new broader setting. Indeed, intuitionistic logic itself involves such a language extension, since we now have two natural disjunctions. One is the classical 'slow disjunction' of our semantics, saying that a choice between the disjuncts is made cofinally – where it need not always the same disjunct that is chosen. Another option is the 'fast disjunction' of intuitionistic logic, where we have to choose the disjunct right now at the present stage. In this case, clearly, we must drop the cofinality assumption governing our classical semantics. But the origins of possibility models also suggest other options, with languages that drop even persistence.

In terms of, say, the interval models of the early 1980s, where points may be viewed limits of histories of ever descending subsets, the conjunction of persistence and cofinality amounts to a strong assumption of 'reducibility' of truth at intervals to truth in the points that they contain. [13] Such simply reducible notions clearly fail to exploit the full sui generis structure of intervals, such as the 'collective properties' that hold on them, or the significant *merges* (or lowest upper bounds in the inclusion order) that can be made of them. If we do go this way, we get additional notions, such as a 'sum modality' $\varphi + \psi$ true at those stages that are lowest upper bounds of a stage where φ is true and one where ψ is true. This new notion supports an interesting logic of its own (van Benthem 1989). Thus, possibility semantics may need richer languages to unleash its full potential, whether the resulting logic is classical or not.

Translations. Another natural issue arising in the present setting is syntactic *translation* of languages and proof calculi. It is well-known that merely looking at surface syntax does not tell us the true identity of logical systems: we need to look at their systematic connections. [14] Now the possibility semantics for classical logic presented here is reminiscent of the double-negation based Gödel-Gentzen translation of classical logic into intuitionistic logic.

What this suggests is a dual perspective on what we are doing: as either changing a semantics for a given logic, or as giving a translation into some other logic. Moreover, translating our classical logic into intuitionistic logic is not the end of the road. The Gödel translation of

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[12] One technical way to see the problem is that cofinality is not a statement in syntactic universal form.

[13] In natural language, this would restrict attention to the special class of 'distributive' predicates.

[14] By the way, translations between logical systems are a recurrent theme through Albert's oeuvre.

intuitionistic logic into classical modal *S4* can also be applied here (using ideas going back to McKinsey and Tarski) to rephrase our semantics and logic inside a *bimodal* classical logic, with one modality for the inclusion structure and modalities for the first-order quantifiers. A thorough study of bimodal translational perspectives is made for modal logic itself in van Benthem, Bezhanishvili & Holliday 2015. Thus, we may either use the new semantics, or move up to a standard semantics for an enriched language for possibility models.

4 Conclusion

This admittedly sketchy revival of an old manuscript may have shown that possibility semantics for classical logic is of conceptual interest in its own right – though by now, this point has been made quite persuasively in other literature (Humberstone 1981, Holliday 2015). As an additional benefit, a model theory for classical logic going for bare essentials, such as unique filter-based constructions for completeness and compactness with no appeal to the Axiom of Choice, seems a worthwhile enterprise (cf. Andréka, Bezhanishvili, Németi & van Benthem 2014). Perhaps this will merely restate in a more austere style what is already known by other means, but there may also be more to it. Our discussion of preservation theorems at least suggests how the playing field of classical model theory may get richer in the process. I have more to say on this, but my text already seems a lot for a short Festschrift piece.

Let me end on a personal note, the same way I started. The date on the handwritten report discussed here is December 1981. Thinking about this gave me a shock, and again made me realize what an alien mind wrote this old text. December 1981 is the very month my second son was born, so what was this author thinking? Did he have nothing better to do in that period than write up logic? I am not sure I know all the ins and outs of Albert's mind and its priorities, but one thing I do know. He would never cruelly abandon little babies like that.

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