A SOLOVAY FUNCTION FOR THE LEAST 1-INCONSISTENT SUBTHEORY OF PA

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Abstract. We consider the theory $\mathrm{I}\Sigma_\mu$, where μ is the least x such that $\mathrm{I}\Sigma_x$ together with all true Π_1-sentences is inconsistent. We show that the provability logic of $\mathrm{I}\Sigma_\mu$ coincides with that of PA, i.e. the Gödel–Löb provability logic GL. While arithmetical completeness of GL with respect to PA is established by using a single Solovay function, our proof for $\mathrm{I}\Sigma_\mu$ relies on a uniformly defined infinite sequence of such functions.

1. Introduction

The provability predicate $\triangle$, introduced and studied in [HV, Section 6], proves handy for obtaining suprema in the lattice of interpretability degrees of finite extensions of Peano Arithmetic (PA). For this reason, $\triangle$ is called a *supremum adapter* in [HV]. We define it as:

$$\triangle\varphi := \exists x \left(\Box_x\varphi \wedge \forall y < x \, \Diamond_y^{\Pi_1}\top\right).$$

where $\Box_x$ is the provability predicate for $\mathrm{I}\Sigma_x + \mathsf{exp}$, and $\Diamond_x^{\Pi_1}\top$ the consistency statement for $\mathrm{I}\Sigma_x + \mathsf{exp}$ together with all true Π_1-sentences. Modulo an index shift, this almost coincides with the definition in [HV]. The equivalence of the two versions holds in $\mathrm{I}\Sigma_1$, but not in $\mathrm{I}\Delta_0+\mathsf{exp}$. Our present amended variant allows us to keep the induction footprint of the arguments in subsection 4.2 reasonably minimal.

We say that a theory T is 1-*inconsistent* if there is a proof of contradiction in T together with all true Π_1-sentences. If PA is 1-inconsistent, we denote by μ the smallest x such that $\mathrm{I}\Sigma_x$ is 1-inconsistent. In a sufficiently strong metatheory, $\triangle$-provability is easily seen to coincide with provability in $\mathrm{I}\Sigma_\mu$, where $\mathrm{I}\Sigma_\mu :=$ PA in case PA is 1-consistent. In the standard model, $\triangle$-provability must therefore coincide with ordinary PA-provability. On the other hand, this equivalence is not verifiable in PA since, by the Second Incompleteness Theorem, PA does not prove its own 1-consistency.

In this paper we are interested in the PA-provable propositional schemata involving $\triangle$, i.e. the provability logic of $\triangle$. We show that the latter coincides with the Gödel–Löb provability logic GL of the ordinary PA-provability predicate $\Box$.

Our $\triangle$ is a cross between two well-studied notions of provability, namely Feferman-provability and 1-provability. The Feferman provability predicate $\triangle_{\mathsf{f}}$ is defined as:

$$\triangle_{\mathsf{f}}\varphi := \exists x \left(\Box_x\varphi \wedge \Diamond_x\top\right),$$

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where $\Diamond_x\top$ is the consistency statement for $\mathrm{I}\Sigma_x + \mathsf{exp}$. The joint provability logic of $\triangle_{\mathsf{f}}$ and $\Box$ was worked out in [Sha94]. We note here that since $\mathsf{PA} \vdash \neg\triangle_{\mathsf{f}}\bot$, the provability logic of $\triangle_{\mathsf{f}}$ is different from GL.

Given a theory T containing $\mathrm{I}\Delta_0+\mathsf{exp}$, *n-provability* refers to provability in T together with all true Π_n-sentences. We write $\Box_T^{\Pi_n}$ for the provability predicate of this theory. Smoryński ([Smo85, 3.3.9]) showed that the provability logic of each $\Box_T^{\Pi_n}$ is GL. The polymodal logic GLP, introduced by Japaridze ([Dzh88]), contains a modality $[n]$ for each n. Generalizing Japaridze's result, Ignatiev ([Ign93]) showed that GLP is the joint provability logic of $\Box_T^{\Pi_n}$ for all n. A simplified proof of this result was given by Beklemishev ([Bek11]).

Prerequisites. We assume the reader to be familiar with basic notions and facts concerning arithmetical theories as presented, for example, in [HP93, Chapter I]. Knowledge of Solovay's proof of arithmetical completeness of GL ([Sol76]) is useful.

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2. Preliminaries

We introduce the basic notions and results used in the paper. The reader is advised to go through this section lightly in order to return when some fact or definition is used.

2.1. **Modal logic.** Denote by $\mathcal{L}_\triangle$ the language of propositional modal logic containing a unary modality $\triangle$. The axiom schemata of the Gödel–Löb logic GL include all propositional tautologies in the language $\mathcal{L}_\triangle$, together with (K) and (L):

(K) $\triangle(A \to B) \to (\triangle A \to \triangle B)$

(L) $\triangle(\triangle A \to A) \to \triangle A$

The inference rules of GL are modus ponens and necessitation:

$$\text{if } \mathsf{GL} \vdash A, \text{ then } \mathsf{GL} \vdash \triangle A.$$

We recall that $\mathsf{GL} \vdash \triangle A \to \triangle\triangle A$ (see e.g. [Boo93, p.11]).

A GL-*frame* is a finite non-empty set W together with a transitive irreflexive binary relation $\prec$ on W. For $a, b \in W$, we write $a \preceq b$ if either $a = b$ or $a \prec b$. A GL-frame $\langle W, \prec\rangle$ is said to be *rooted* in case there is some $w \in W$ such that $w \preceq a$ for all $a \in W$. A GL-*model* is a triple $\langle W, \prec, \Vdash\rangle$, where $\langle W, \prec\rangle$ is a GL-frame, and $\Vdash$ a valuation assigning to every propositional letter a subset of W. We extend $\Vdash$ to all $\mathcal{L}_\triangle$-formulas by requiring that it commutes with propositional connectives, and interpreting $\prec$ as the accessibility relation for $\triangle$:

$$\mathcal{M}, a \Vdash \triangle A \text{ if for all } b \text{ with } a \prec b,\ \mathcal{M}, b \Vdash A.$$

Given $\mathcal{M} = \langle W, \prec, \Vdash\rangle$, we write $\mathcal{M} \Vdash A$ if $\mathcal{M}, a \Vdash A$ for every $a \in W$. We write $\mathcal{F} \Vdash A$ if $\mathcal{M} \Vdash A$ for any model $\mathcal{M}$ whose underlying frame is $\mathcal{F}$.

It is well-known (see for example [Boo93, Chapter 5]) that GL is modally complete with respect to the class of GL-frames:

Theorem 1. *$\mathsf{GL} \vdash A$ iff for every GL-frame $\mathcal{F}$, $\mathcal{F} \Vdash A$.* □

2.2. **Arithmetical theories.** We work with first-order theories formulated in the language $\mathcal{L}$ of arithmetic containing 0, S, $+$, $\times$, and $\leq$. A formula is *bounded* or Δ_0 (equivalently, Σ_0 or Π_0) if all quantifiers occurring in it are of the form $\exists x \leq y$ or $\forall x \leq y$. A formula is Σ_{n+1} (Π_{n+1}) if it is of the form $\exists x_1 \ldots \exists x_k\, \varphi$ ($\forall x_1 \ldots \forall x_k\, \varphi$), with φ a Π_n (Σ_n)-formula, and $\Delta_0(\Sigma_1)$ if it is obtained from Σ_1-formulas by using propositional connectives and bounded quantification.

A formula φ is Δ_n *in* a theory T if $T \vdash \varphi \leftrightarrow \sigma$ and $T \vdash \varphi \leftrightarrow \pi$ for some Σ_n-formula σ and Π_n-formula π.

Given a class Γ of formulas, $\mathrm{I}\Gamma$ is the theory obtained by adding to Q the induction schema for Γ-formulas. The theory of Peano Arithmetic (PA) is given as $\bigcup_{n\in\omega} \mathrm{I}\Sigma_n$. For $n > 0$, $\mathrm{I}\Sigma_n$ is finitely axiomatizable ([HP93, Theorem I.2.52]).

The graph of exponentiation is definable in $\mathrm{I}\Delta_0$ by a Δ_0-formula exp (see [HP93, Theorem V.3.15]). We denote by exp the axiom stating the totality of exponentiation. We recall that $\mathrm{I}\Delta_0+\mathsf{exp}$ is finitely axiomatizable ([HP93, Theorem V.5.6]), and that $\mathrm{I}\Sigma_1 \vdash \mathsf{exp}$ ([HP93, Theorem I.1.50]).

A formula is $\Delta_0^{\exp}$ if all quantifiers occurring in it have the form $\exists x \leq \exp y$ or $\forall x \leq \exp y$. It is well-known that any $\Delta_0^{\exp}$-formula is Δ_1 in $\mathrm{I}\Delta_0+\mathsf{exp}$.

Theorem 2 ([HP93, Remark I.1.59(3)])**.** *Let $\mathtt{f}$, $\mathtt{g}$ and $\mathtt{k}$ be $\Delta_0^{\exp}$-defined provably total functions of $\mathrm{I}\Delta_0+\mathsf{exp}$. Suppose that $\mathtt{h}$ is defined from $\mathtt{f}$ and $\mathtt{g}$ by primitive recursion, and majorized by $\mathtt{k}$. Then $\mathtt{h}$ is $\Delta_0^{\exp}$-definable and provably total in $\mathrm{I}\Delta_0+\mathsf{exp}$, and moreover the defining equations of $\mathtt{h}$ are provable in $\mathrm{I}\Delta_0+\mathsf{exp}$.* □

We assume some standard formalization of syntax and provability in $\mathrm{I}\Delta_0+\mathsf{exp}$. In particular, $\Box_x$ is the conventional provability predicate for $\mathrm{I}\Sigma_x + \mathsf{exp}$, with x a free variable. We write $\Box$ for the provability predicate of PA, where we assume that for all φ, $\Box\varphi$ is provably equivalent in $\mathrm{I}\Delta_0+\mathsf{exp}$ to $\exists x\, \Box_x\varphi$. The symbol $\Diamond_x\varphi$ is used as shorthand for $\neg\Box_x\neg\varphi$.

The sequence $(\Box_x)_{x<\omega}$ of provability predicates is *monotone* in the sense that:

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \Box_x\varphi \wedge x \leq y \to \Box_y\varphi.$$

As usual, $\Box_x\varphi(\dot{y})$ means that the numeral for the value of y has been substituted for the free variable of the formula φ inside $\Box_x$. If the intended meaning is clear from the context, we will often write $\Box_x\varphi(y)$ instead of $\Box_x\varphi(\dot{y})$. Recall that $\mathrm{I}\Delta_0+\mathsf{exp}$ is provably Σ_1-complete, i.e. that for any Σ_1-formula σ:

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \sigma(y) \to \Box_0\sigma(\dot{y});$$

and that $\mathrm{I}\Delta_0+\mathsf{exp}$ verifies the Hilbert–Bernays–Löb (HBL) derivability conditions for $\Box_x$:

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$$
$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \Box_x\varphi \to \Box_x\Box_{\dot{x}}\varphi$$

The validity of Löb's principle for $\Box_x$ follows from the above together with the Fixed Point Lemma (see [Boo93, Theorem 3.2]). Thus

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \Box_x(\Box_{\dot{x}}\varphi \to \varphi) \to \Box_x\varphi,$$

and so principles valid in GL can be used when reasoning about $\Box_x$ in $\mathrm{I}\Delta_0{+}\mathsf{exp}$.

Löb's Theorem for $\mathrm{I}\Delta_0{+}\mathsf{exp}$ facilitates a short proof (see for example [Bek03, proof of Lemma 2.4]) of the closure of $\mathrm{I}\Delta_0{+}\mathsf{exp}$ under the *reflexive induction* rule:

$$\frac{\forall x\ (\Box_0 \forall y < \dot{x}\, \varphi(y) \to \varphi(x))}{\forall x\, \varphi(x)}.$$

Reflexive induction, as a general method for reasoning about linear orderings in arithmetical theories, first appears in the work of Schmerl ([Sch79, p. 337]).

It is well-known that in $\mathrm{I}\Delta_0{+}\mathsf{exp}$ there is a partial satisfaction predicate $\mathrm{Sat}_{\Pi_1}(\varphi, y)$ for Π_1-formulas, where y and φ are internal variables ranging, respectively, over assignments and $\mathcal{L}$-formulas. The formula Sat_{Π_1} is Π_1 and satisfies Tarski's conditions ([HP93, Theorem I.2.55]). Defining $\mathrm{Tr}_{\Pi_1}(\varphi)$ to be the formula saying that φ is a sentence and $\forall y\, \mathrm{Sat}_{\Pi_1}(\varphi, y)$, it is clear that Tr_{Π_1} is Π_1, and that for any Π_1-formula $\pi(x)$,

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \pi(x) \leftrightarrow \mathrm{Tr}_{\Pi_1}(\pi(\dot{x})).$$

(By our conventions for the dot notation, $\pi(\dot{x})$ is a sentence from the point of view of Tr_{Π_1}.) With Tr_{Π_1}, we can define the provability predicate $\Box_x^{\Pi_1}$ for 1-provability in $\mathrm{I}\Sigma_x + \mathsf{exp}$:

$$\Box_x^{\Pi_1}\varphi := \exists\pi\, (\mathrm{Tr}_{\Pi_1}(\pi) \wedge \Box_x(\pi \to \varphi)).$$

Similarly, the provability predicate $\Box^{\Pi_1}$ for 1-provability in PA is defined as:

$$\Box^{\Pi_1}\varphi := \exists\pi\, (\mathrm{Tr}_{\Pi_1}(\pi) \wedge \Box(\pi \to \varphi)).$$

It is then clear that for all φ, $\Box^{\Pi_1}\varphi$ is provably equivalent in $\mathrm{I}\Delta_0{+}\mathsf{exp}$ to $\exists x\, \Box_x^{\Pi_1}\varphi$. We note that $\Box_x^{\Pi_1}$ is Σ_2. It follows from [Smo85, 3.3.9] that the principles of GL for $\Box_x^{\Pi_n}$ are valid in $\mathrm{I}\Delta_0{+}\mathsf{exp}$. It is well-known that $\Box_x^{\Pi_1}$ is Σ_2-complete, i.e. that for any Σ_2-formula ς,

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \varsigma(y) \to \Box_x^{\Pi_1}\varsigma(\dot{y}).$$

[HP93, Theorem I.4.33] shows that $\mathrm{I}\Sigma_{k+1}$ proves the consistency of the set of all true Π_{k+2}-sentences. Since $\mathrm{I}\Sigma_k + \mathsf{exp}$ is axiomatized by a single Π_{k+2}-sentence, $\mathrm{I}\Sigma_{k+1}$ proves the consistency of $\mathrm{I}\Sigma_k + \mathsf{exp} + \Pi_1$-truth. An inspection of the proof reveals that it can be formalized in $\mathrm{I}\Delta_0{+}\mathsf{exp}$; thus we have

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \Box_x \forall y < \dot{x}\ \Diamond_y^{\Pi_1}\top.$$

We shall refer to the above property as *reflection*.

3. $\triangle$ as provability in $\mathrm{I}\Sigma_\mu$

Recall that the provability predicate $\triangle$ is defined as $\triangle\varphi := \exists x\, (\Box_x\varphi \wedge \forall y < x\, \Diamond_y^{\Pi_1}\top)$. As mentioned in Section 1, we can think of $\triangle$ as provability in $\mathrm{I}\Sigma_\mu$, where μ is the least x such that the theory $\mathrm{I}\Sigma_x$ is 1-inconsistent. In order to make this precise, let $\mu = x$ be the formula

$$\left(\neg\Box^{\Pi_1}\bot \wedge x = \infty\right) \vee \left(\Box_x^{\Pi_1}\bot \wedge \forall y < x\, \Diamond_y^{\Pi_1}\top\right).$$

The uniqueness of μ is provable in $\mathrm{I}\Delta_0+\mathsf{exp}$:

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \mu = x \wedge \mu = y \to x = y.$$

As for existence, note that since $\exists x\, \Box_x^{\Pi_1}\bot$ is Σ_2 and the least number principle for Σ_2-formulas is equivalent to induction for Σ_2-formulas ([HP93, Theorem I.2.4]), it is clear that $\mathrm{I}\Sigma_2 \vdash \exists x\, \mu = x$. Using this, it is easy to see that according to $\mathrm{I}\Sigma_2$, provability in $\mathrm{I}\Sigma_\mu$ coincides with $\triangle$-provability:

$$\mathrm{I}\Sigma_2 \vdash \forall \varphi\, (\triangle\varphi \leftrightarrow \Box_\mu \varphi),$$

where $\Box_\infty$ is defined to be $\Box$.

The following theorem is due to F. Pakhomov.

Theorem 3. $\mathrm{I}\Sigma_1 \nvdash \exists x\ \mu = x$

Proof. We show that there is a model $\mathfrak{M}$ of $\mathrm{I}\Sigma_1$ where PA is 1-inconsistent, but there is no smallest a such that $\mathrm{I}\Sigma_a$ is inconsistent. $\mathfrak{M}$ is constructed as the union of an ascending chain $(\mathfrak{M}_i)_{i<\omega}$ of models, where $\mathfrak{M}_i \vDash \Box^{\Pi_1}\bot$ and $\mathfrak{M}_i \vDash \mathsf{PA}$ for all i. Denote by m_i the least element such that $\mathfrak{M}_i \vDash \Box^{\Pi_1}_{m_i}\bot$. We shall ensure that for all i,

i. $m_i > m_{i+1}$

ii. $\mathfrak{M}_i \prec_{\Sigma_1} \mathfrak{M}_{i+1}$ ($\mathfrak{M}_{i+1}$ is a Σ_1-elementary extension of $\mathfrak{M}_i$)

We note that $\mathfrak{M} := \bigcup_{i<\omega} \mathfrak{M}_i$ is a model with the desired properties: from (ii) it follows that for all i,

$$\mathfrak{M}_i \prec_{\Sigma_1} \mathfrak{M}.$$

Using this, it is easy to show that $\mathfrak{M} \vDash \mathrm{I}\Sigma_1$ and furthermore for all $a \in \mathfrak{M}$, we have that $\mathfrak{M} \vDash \Box^{\Pi_1}_a\bot$ if and only if $\mathfrak{M}_i \vDash \Box^{\Pi_1}_a\bot$ for some i.

It remains to show that a chain $(\mathfrak{M}_i)_{i<\omega}$ with the required properties exists. We proceed by induction on i. Let $\mathfrak{M}_0$ be any model of PA with $\mathfrak{M}_0 \vDash \Box^{\Pi_1}\bot$. Now suppose that we have constructed a model $\mathfrak{M}_i$ of PA with $\mathfrak{M}_i \vDash \Box^{\Pi_1}_{m_i}\bot \wedge \Diamond^{\Pi_1}_{m_i-1}\top$. Since $\mathfrak{M}_i \vDash \Diamond^{\Pi_1}_{m_i-1}\top$, we have $\mathfrak{M}_i \vDash \Diamond^{\Pi_1}_{m_i-1}\Box^{\Pi_1}_{\dot{m}_i-1}\bot$ by Löb's principle for $\Box^{\Pi_1}_{m_i-1}$. In fact we have $\mathfrak{M}_i \vDash \Diamond^{\Pi_1}_{m_i-1}(\Box^{\Pi_1}_{\dot{m}_i-1}\bot \wedge \Diamond^{\Pi_1}_{\dot{m}_i-2}\top)$, for $\mathfrak{M}_i \vDash \Box_{m_i-1}\Diamond^{\Pi_1}_{m_i-2}\top$ by reflection. In other words, $\mathfrak{M}_i$ thinks that the theory

$$T := \mathrm{I}\Sigma_{m_i-1} + \Pi_1\text{-truth} + \Box^{\Pi_1}_{\dot{m}_i-1}\bot + \Diamond^{\Pi_1}_{\dot{m}_i-2}\top$$

is consistent (where Π_1-truth is to be understood in the sense of $\mathfrak{M}_i$). By the Arithmetized Completeness Theorem (see e.g. [McA78, Theorems 1.7 and 2.2]), there is an end-extension $\mathfrak{M}_{i+1}$ of $\mathfrak{M}_i$ with $\mathfrak{M}_{i+1} \vDash T$. Since T contains $\pi(\dot{a})$ whenever $\mathfrak{M}_i \vDash \pi(a)$ and $\pi(x)$ is a Π_1-formula, we see that $\mathfrak{M}_{i+1}$ is in fact a Σ_1-elementary end-extension of $\mathfrak{M}_i$. Since m_i-1 is nonstandard, we have $\mathfrak{M}_i \vDash \Box_{m_i-1}\varphi$ for each axiom φ of PA whence, from the external point of view, $\mathfrak{M}_{i+1}$ is a model of PA. We now have $\mathfrak{M}_i \prec_{\Sigma_1} \mathfrak{M}_{i+1} \vDash \mathsf{PA}$ and $\mathfrak{M}_{i+1} \vDash \Box^{\Pi_1}_{m_i-1}\bot \wedge \Diamond^{\Pi_1}_{m_i-2}\top$. Thus it suffices to put $m_{i+1} = m_i - 1$. $\Box$

4. The provability logic of △

We are interested in interpreting the modality of $\mathcal{L}_\vartriangle$ as the provability predicate $\vartriangle$.

Definition 4. A $\vartriangle$-*realization* is a function * from the propositional letters of $\mathcal{L}_\vartriangle$ to $\mathcal{L}$-sentences. The domain of * is extended to all $\mathcal{L}_\vartriangle$-formulas by requiring that it commutes with propositional connectives, and

$$(\vartriangle A)^* := \exists x\, (\Box_x A^* \wedge \forall y < x \, \Diamond_y^{\Pi_1} \top).$$

We prove that GL is the provability logic of $\vartriangle$, i.e. the following:

Theorem 5. $\mathsf{GL} \vdash A$ *iff for all* $\vartriangle$*-realizations* *, $\mathsf{PA} \vdash A^*$.

We first prove arithmetical soundness, i.e. the left to right direction of Theorem 5.

Lemma 6. *i. If* $\mathsf{PA} \vdash \varphi$*, then* $\mathsf{PA} \vdash \vartriangle\varphi$*.*

ii. $\mathrm{I}\Sigma_1 \vdash \vartriangle(\varphi \to \psi) \to (\vartriangle\varphi \to \vartriangle\psi)$

iii. $\mathrm{I}\Sigma_1 \vdash \vartriangle\varphi \to \vartriangle\vartriangle\varphi$

Proof. (i) Assume $\mathsf{PA} \vdash \varphi$. Then $\mathrm{I}\Sigma_n \vdash \varphi$ for some n, whence $\mathsf{PA} \vdash \Box_n\varphi$. Since $\mathsf{PA} \vdash \forall y < n \, \Diamond_y^{\Pi_1} \top$ by reflection, we thus have $\mathsf{PA} \vdash \vartriangle\varphi$.

(ii) follows from monotonicity together with the HBL-conditions for $\Box_x$.

(iii) Assume $\vartriangle\varphi$, and let x be such that $\Box_x\varphi$ and $\forall y < x \, \Diamond_y^{\Pi_1}\top$. By Σ_1-completeness and reflection we obtain $\Box_x(\Box_x\varphi \wedge \forall y < x \, \Diamond_y^{\Pi_1}\top)$, thus $\vartriangle\vartriangle\varphi$ as required. □

Verifiability of Löb's principle for $\vartriangle$ follows from Lemma 6 by the usual argument (see [Boo93, Theorem 3.2]). This concludes the proof of arithmetical soundness. □

4.1. **Arithmetical Completeness.** Our proof of the remaining direction of Theorem 5 proceeds, as usual (see [Sol76]), by showing that any finite Kripke frame for GL can be suitably embedded into PA, and is closely related to Beklemishev's arithmetical completeness proof for GLP ([Bek11]). The latter uses a sequence $(h_i)_{i<m}$ of Solovay functions, where m is some standard number. In contrast, we need an infinite sequence $(h_y)_{y<\omega}$ of such functions, uniformly defined by a single formula.

We start with an informal description of the construction, based on the view of $\vartriangle$ as provability in $\mathrm{I}\Sigma_\mu$. Given a finite GL-frame $\mathcal{F} = \langle W, \prec \rangle$ with root 0, we consider a family of Solovay functions $(h_y)_{y<\omega}$ climbing up the accessibility relation $\prec$ of $\mathcal{F}$. The function h_0 is the usual Solovay function for $\mathrm{I}\Delta_0{+}\mathsf{exp}$: it starts at 0 and moves upon the emergence of $\mathrm{I}\Delta_0{+}\mathsf{exp}$-proofs concerning its own limit. Similarly, each h_{y+1} is like the usual Solovay function for $\mathrm{I}\Sigma_{y+1}$, except that it starts where the previous function h_y came to rest.

We write $x : \Box_y\varphi$ to mean that x is (the code of) a $\Box_y$-proof of φ. Given a reasonable coding of proofs, we have that the formula $x : \Box_y\varphi$ is Δ_0^{exp}, and furthermore that every x is the proof of at most one sentence.

Letting ℓ_y denote the limit of h_y, we would like the functions $(h_y)_{y<\omega}$, with $h_y : \omega \to W$, to satisfy:

$$h_0(0) = 0, \qquad h_{y+1}(0) = \ell_y, \qquad \text{and}$$

$$h_y(x+1) = \begin{cases} a & \text{if } h_y(x) \prec a \text{ and } x : \Box_y \ell_y \neq a, \\ h_y(x) & \text{otherwise.} \end{cases}$$

We are interested in the value ℓ_μ, i.e. the limit of the function h_μ, where ℓ_μ is defined to be $\lim_{y\to\infty} \ell_y$ in case $\mu = \infty$. We would like to to show that the sentence $\ell_\mu = a$ is a natural arithmetical representative for the node a, in the sense that for some theory $T \subseteq \mathsf{PA}$,

i. if $a \neq 0$, then $T \vdash \ell_\mu = a \to \Box_\mu a \prec \ell_\mu$,

ii. if $a \prec b$, then $T \vdash \ell_\mu = a \to \Diamond_\mu \ell_\mu = b$.

It might seem, at first sight, that T has to be at least as strong as $\mathrm{I}\Sigma_2$: as seen above (Theorem 3), the existence of μ is not known to $\mathrm{I}\Sigma_1$. Moreover, each function h_{y+1}, as presented above, is genuinely more complex than the usual Solovay function: it is defined by using the limit of h_y, the natural representation of which is at least Σ_2. It is therefore not obvious that the basic basic properties of h_y and ℓ_y can be verified in $\mathrm{I}\Delta_0+\mathsf{exp}$ or even $\mathrm{I}\Sigma_1$. By tweaking the construction we shall nevertheless succeed in making everything work smoothly in $\mathrm{I}\Delta_0+\mathsf{exp}$.

4.2. **A multi-stage Solovay function.** We start by defining an auxiliary function $h_{y,a}$ — the Solovay function for $\Box_y$, starting off at node a (where both y and a are parameters represented by free variables).

Definition 7 ($\mathrm{I}\Delta_0+\mathsf{exp}$)**.** For $y < \omega$, $a \in W$, the function $h_{y,a} : \omega \to W$ is defined by:

$$h_{y,a}(0) = a$$

$$h_{y,a}(x+1) = \begin{cases} b & \text{if } h_{y,a}(x) \prec b \text{ and } x : \Box_y \ell_{\dot{y}} \neq b, \\ h_{y,a}(x) & \text{otherwise.} \end{cases}$$

The formula $\ell_y \neq b$ (Definition 9 below) depends on the formula χ representing the family of functions $(h_{y,a})_{y<\omega, a\in W}$. The self-reference in the definition of $h_{y,a}$ is handled, as usual, by the Fixed Point Lemma. We note here that the definition of $h_{y,a}$ only relies on the gödelnumber of $\ell_{\dot{y}} \neq b$, and the latter can be obtained from y, b and $\ulcorner \chi \urcorner$ by a function whose totality is known to $\mathrm{I}\Delta_0+\mathsf{exp}$.

It follows from Theorem 2 — for example, by using that W is finite — that the function $\mathtt{h}(y, a, x) = h_{y,a}(x)$ is Δ_0^{exp}-defined and provably total in $\mathrm{I}\Delta_0+\mathsf{exp}$, with its defining equations also provable in $\mathrm{I}\Delta_0+\mathsf{exp}$.

We write $\lim h_{y,a} = b$ for the formula

$$\exists x\; h_{y,a}(x) = b \wedge \forall x\; h_{y,a}(x) \preceq b.$$

Since $h_{y,a}(x)$ is Δ_0^{exp}, we have that $\lim h_{y,a} = b$ is provably equivalent in $\mathrm{I}\Delta_0+\mathsf{exp}$ to a $\Delta_0(\Sigma_1)$-formula. The formula $\lim h_{y,a} = b$ states that b is the $\preceq$-largest element in the range of $h_{y,a}$. By using the following lemma, we can think of $\lim h_{y,a} = b$ as saying that b is the limit of $h_{y,a}$.

Lemma 8. *i.* $\mathrm{I}\Delta_0+\mathsf{exp} \vdash x' \leq x \to h_{y,a}(x') \preceq h_{y,a}(x)$

ii. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \exists! b \lim h_{y,a} = b$

Proof. (i) is proven by internal induction on x. Since $h_{y,a}(x) \preceq h_{y,a}(x+1)$ by definition, the inductive step follows by using the transitivity of $\preceq$.

(ii) Since $\preceq$ is antisymmetric, uniqueness is immediate from the definition of $\lim h_{y,a}$. For existence, we show by external induction on the converse of $\prec$ that for all $c \in W$,

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash h_{y,a}(x) = c \to \exists b \lim h_{y,a} = b.$$

This is sufficient, since $\mathrm{I}\Delta_0+\mathsf{exp}$ proves that $h_{y,a}(0) = a$. From (i) we have that

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash h_{y,a}(x) = c \to (\forall x' \geq x\ h_{y,a}(x') = c \vee \exists x' \geq x\ c \prec h_{y,a}(x'))\,. \tag{1}$$

Argue in $\mathrm{I}\Delta_0+\mathsf{exp}$, assuming $h_{y,a}(x) = c$. If the first disjunct in (1) holds then, using (i), we have $\lim h_{y,a} = c$, while if the second disjunct holds, then $\exists b \lim h_{y,a} = b$ by the induction assumption. Thus in either case $\exists b \lim h_{y,a} = b$ as required. □

Definition 9 ($\mathrm{I}\Delta_0+\mathsf{exp}$)**.** The formula $\ell_y = a$, with free variable y, is defined as:

$$\exists s\ (s = (l_0, l_1, \ldots, l_y)\ \wedge\ l_0 = \lim h_{0,0}\ \wedge\ \forall z < y\ l_{z+1} = \lim h_{z+1,l_z}\ \wedge\ l_y = a)\,.$$

A sequence s is a *y-witness* if it satisfies the first three conjuncts in the formula above. If s also satisfies the fourth conjunct, then s is a *witness for* $\ell_y = a$. We write $\ell_y \neq a$ for the negation of $\ell_y = a$.

Given Definition 7, it is clear that any y-witness is a sequence of elements of W. Since the latter is finite, the leading existential quantifier in $\ell_y = a$ can be bounded by a term of the form $\exp(k \cdot y)$, where k is a sufficiently large standard number. Recalling that $\lim h_{y,b} = c$ is a $\Delta_0(\Sigma_1)$-formula, we thus see that $\ell_y = a$ is a $\Delta_0^{\exp}(\Sigma_1)$-formula. Reasoning about the formula $\ell_y = a$ by induction on y can therefore be problematic in $\mathrm{I}\Delta_0+\mathsf{exp}$. The following lemmas state that several properties of $\ell_y = a$ are nevertheless verifiable in $\mathrm{I}\Delta_0+\mathsf{exp}$.

Lemma 10 ($\mathrm{I}\Delta_0+\mathsf{exp}$)**.** *For each y, there is at most one y-witness. In particular, $\ell_y = a$ and $\ell_y = b$ imply $a = b$.*

Proof. We reason in $\mathrm{I}\Delta_0+\mathsf{exp}$. Suppose that $(l_0, \ldots, l_y)$ and $(l'_0, \ldots, l'_y)$ are both y-witnesses. We prove by $\Delta_0^{\exp}$-induction that for all $i \leq y$, $l_i = l'_i$. By Lemma 8(ii) it is clear that $l_0 = \lim h_{0,0} = l'_0$. Supposing that $l_i = l'_i$, we have again by Lemma 8(ii) that $l_{i+1} = \lim h_{i+1,l_i} = \lim h_{i+1,l'_i} = l'_{i+1}$. □

If follows from Lemma 10 that we can treat ℓ_y as a partial function in $\mathrm{I}\Delta_0+\mathsf{exp}$.

Lemma 11. *i.* $\mathrm{I}\Delta_0+\mathsf{exp} \vdash x < y \wedge \ell_y = b \to \exists a\, \ell_x = a$

ii. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash x < y \wedge \ell_y = b \to \ell_x \preceq b$

iii. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \ell_y = 0 \leftrightarrow \forall z \leq y\ \forall x\ h_{z,0}(x) = 0.$

iv. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \ell_y = a \prec b \to \Diamond_y \ell_{\dot{y}} = b$

v. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \ell_y = a \neq 0 \to \Box_y \ell_{\dot{y}} \neq a.$

Proof. We argue in $\mathrm{I}\Delta_0+\mathsf{exp}$. (i) Suppose that $(l_0,\ldots,l_y)$ is a witness for $\ell_y = b$. If $x < y$, then clearly $(l_0,\ldots,l_x)$ is a witness for $\ell_x = l_x$, so we can put $a = l_x$.

(ii) Suppose that $(l_0,\ldots,l_y)$ is a witness for $\ell_y = b$. If $x < y$, then an x-witness exists by clause (i), and by Lemma 10 it is an initial segment of $(l_0,\ldots,l_y)$. We prove by Δ_0^{exp}-induction on $z \le y$ that $z' < z$ implies $l_{z'} \preceq l_z$. It follows from the relevant definitions, together with Lemma 8, that $l_z \preceq \lim h_{z+1,l_z} = l_{z+1}$. Assuming $l_{z'} \preceq l_z$, we thus obtain $l_{z'} \preceq l_{z+1}$ by transitivity of $\preceq$.

(iii) We have $\forall z \le y\ \forall x\ h_{z,0}(x) = 0$ iff $\forall z \le y\ \lim h_{z,0} = 0$ iff $(l_0,\ldots,l_z)$, with all $l_i = 0$, is a witness for $\ell_z = 0$.

(iv) By Definitions 7 and 9, together with the transitivity and antisymmetry of $\preceq$.

(v) Assume that $\ell_y = a \neq 0$, and let $(l_0,\ldots,l_y)$ be the witness for $\ell_y = a$. By Δ_0^{exp}-induction, let $y' \le y$ be minimal such that $l_{y'} = a$. Since $a \neq 0$, by the definition of ℓ_y and $h_{y,c}$, we have that $\Box_{y'}\ell_{y'} \neq a$. We show $\Box_{y'}\ell_y \neq a$, from which $\Box_y\ell_y \neq a$ clearly follows. Argue in $\Box_{y'}$:

> Suppose that $\ell_y = a$. Since $\ell_{y'} \neq a$, we have $y' < y$, and so $\ell_{y'} \prec \ell_y$ by clause (ii). Let b be such that $\ell_{y'} = \lim h_{y',b}$. We thus have $\forall x\ h_{y',b}(x) \prec a$. By Σ_1-completeness, we also have $\Box_{y'}\ell_{y'} \neq a$. By definition of $h_{y',b}$, this implies $\exists x\, h_{y',b}(x) = a$, hence $a \preceq \lim h_{y',b}$ i.e. $\ell_y \preceq \ell_{y'}$, a contradiction. □

Lemma 12. *i.* $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \ell_y = a \neq 0 \to \Box_y^{\Pi_1}\bot$

ii. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \Diamond_x^{\Pi_1}\top \to \ell_x = 0$

iii. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \forall x < y\, \Diamond_x^{\Pi_1}\top \to \ell_y = \lim h_{y,0}$

iv. $\mathrm{I}\Delta_0+\mathsf{exp} \vdash \lim h_{y,0} = a \to \Box_y a \preceq \ell_{\dot{y}}$.

Proof. (i) Argue in $\mathrm{I}\Delta_0+\mathsf{exp}$, letting $(l_0,\ldots,l_y)$ be the witness for $\ell_y = a \neq 0$. Using Δ_0^{exp}-induction, we can assume that y is minimal such that $l_y \neq 0$, thus either $y = 0$ or $l_{y-1} = 0$. It follows that $\ell_y = \lim h_{y,0}$, and so

(2) $$\lim h_{y,0} = a.$$

Since $a \neq 0$, we have by Lemma 11(v) that

(3) $$\Box_y\ell_y \neq a.$$

Reason in $\Box_y^{\Pi_1}$:

> We claim first that $\ell_y = \lim h_{y,0}$. If $y = 0$, this is clear from the definition. And if $y > 0$, then we have $\ell_{y-1} = 0$ since, using Lemma 11(iii), the latter is equivalent to a true Π_1-formula. Since (2) is a true conjunction of a Σ_1- and a Π_1-formula, it is also true here, whence it follows that $\ell_y = a$, contradicting (3).

(ii) By reflexive induction, it suffices to show:

$$\mathrm{I}\Delta_0+\mathsf{exp} \vdash \Box_0\forall z < x\,(\Diamond_z^{\Pi_1}\top \to \ell_z = 0) \to (\Diamond_x^{\Pi_1}\top \to \ell_x = 0).$$

Argue in $\mathrm{I}\Delta_0+\mathsf{exp}$. If $x = 0$, then, since $\ell_0 = \lim h_{0,0}$ and the latter exists by Lemma 8(ii), we have the claim from clause (i) by contraposition. So let $x > 0$, and suppose that $\Box_0(\Diamond_{x-1}^{\Pi_1}\top \to \ell_{x-1} = 0)$ and $\Diamond_x^{\Pi_1}\top$. Since $\Box_x\Diamond_{x-1}^{\Pi_1}\top$ by reflection, it follows that $\Box_x\ell_{x-1} = 0$. Since $\Diamond_x^{\Pi_1}\top$ is equivalent to Σ_1-reflection for $\Box_x$ and $\ell_{x-1} = 0$ is equivalent to a Π_1-formula by Lemma 11(iii), we now have $\ell_{x-1} = 0$.

But this means that ℓ_x is equal to $\lim h_{x,0}$, and thus it exists by Lemma 8(ii). Finally, $\ell_x = 0$ follows by contraposition from clause (i).

(iii) Argue in $\mathrm{I}\Delta_0{+}\mathsf{exp}$, assuming $\forall x < y\, \Diamond_x^{\Pi_1}\top$. For $y = 0$ we have $\ell_y = \lim h_{y,0}$ by definition. For $y > 0$ we have $\ell_{y-1} = 0$ from clause (ii) together with the assumption, and so again by definition $\ell_y = \lim h_{y,\ell_{y-1}} = \lim h_{y,0}$.

(iv) Suppose that $\lim h_{y,0} = a$, whence in particular $\exists x\, h_{y,0}(x) = a$. Reason in $\Box_y$:

> Using reflection, we obtain $\forall x < y\, \Diamond_x^{\Pi_1}\top$, and so $\ell_y = \lim h_{y,0}$ by clause (iii). By Σ_1-completeness we have $\exists x\, h_{y,0}(x) = a$ from outside, and so $a \preceq \ell_y$ by the definition of $\lim h_{y,0}$. □

Remark 13. While, as shown above, ℓ_y is a partial function in $\mathrm{I}\Delta_0{+}\mathsf{exp}$, its totality is, in general, not provable in $\mathrm{I}\Delta_0{+}\mathsf{exp}$. Consider the frame with $W = \{0,1\}$ and $0 \prec 1$. From the definition of $h_{y,a}$ it is clear that

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \Box_y\bot \leftrightarrow \lim h_{y,0} = 1. \tag{4}$$

As in the proof of Theorem 3, we see that there is a model $\mathfrak{M}$ of $\mathrm{I}\Delta_0{+}\mathsf{exp}$ and a sequence $(m_i)_{i\in\omega}$ of elements of $\mathfrak{M}$, such that

i. $\mathfrak{M} \vDash \Box_{m_i}\bot$ and $\mathfrak{M} \vDash m_i > m_{i+1}$ for all i

ii. For all $k \in \mathfrak{M}$, $\mathfrak{M} \vDash \Box_k\bot$ if and only if for some $i \in \omega$, $m_i < k$

It follows from the above that $\mathfrak{M} \vDash \Box\bot$, and $\mathfrak{M} \vDash \Diamond_n\top$ for all standard n.

Let m be any element from $(m_i)_{i\in\omega}$, and suppose for a contradiction that $\ell_m = a$ is witnessed by $s = (l_0, \ldots, l_m)$. If $a = 0$, then by Lemma 11(ii) also $\ell_{m-1} = 0$; thus $\ell_m = \lim h_{m,0} = 0$. However since $\mathfrak{M} \vDash \Box_m\bot$, from (4) we have that $\lim h_{m,0} = 1$. Thus it must be that $\ell_m = 1$. Let $i \le m$ be the minimal coordinate of s with $l_i = 1 = \ell_i$. Since $\mathfrak{M} \vDash \Diamond_0\top$, it follows from (4) that $i > 0$. Thus $\ell_{i-1} = \lim h_{i-1,0} = 0$, and so $\neg\Box_{i-1}\bot$ by (4). On the other hand, $\ell_i = \lim h_{i,0} = 1$, and so $\Box_i\bot$, contradicting the properties of $\mathfrak{M}$. □

Write $L = a$ for the formula

$$\exists y\,\left(\ell_y = a \wedge \forall x < y\, \Diamond_x^{\Pi_1}\top\right) \wedge \forall z\,\left(\forall x < z\, \Diamond_x^{\Pi_1}\top \to \ell_z \preceq a\right)$$

stating, intuitively, that $\ell_\mu = a$.

Lemma 14. *i.* $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \exists! a\, L = a$

ii. $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \forall x < y\, \Diamond_x^{\Pi_1}\top \to \ell_y \preceq L$

iii. $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \Box_z \ell_{\dot z} \preceq L$

iv. $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \Box_z(\Box_{\dot z}^{\Pi_1}\bot \to L = \ell_{\dot z})$

Proof. (i). Since $\preceq$ is antisymmetric, uniqueness is immediate from the definition. For existence, we show by external induction on the converse of $\prec$ that for all $a \in W$,

$$\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \ell_y = a \wedge \forall x < y\, \Diamond_x^{\Pi_1}\top \to \exists b\; L = b.$$

We note that this is sufficient, for $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash \ell_0 = \lim h_{0,0} \wedge \forall x < 0\, \Diamond_x^{\Pi_1}\top$. Argue in $\mathrm{I}\Delta_0{+}\mathsf{exp}$. From $\ell_y = a \wedge \forall x < y\, \Diamond_x^{\Pi_1}\top$ we have by Lemmas 11(ii) and 12(iii):

$$\forall z > y\,\left(\forall x < z\, \Diamond_x^{\Pi_1}\top \to \ell_z = a\right) \vee \exists z > y\,\left(\forall x < z\, \Diamond_x^{\Pi_1}\top \wedge a \prec \ell_z\right).$$

If the first disjunct holds, then, using Lemma 11(ii), we have that $L = a$. And if the second disjunct holds, then $\exists b\, L = b$ follows by the induction assumption.

(ii) Immediate from the definition of L, by using Lemma 12(iii) to see that ℓ_y exists.

(iii) Within $\Box_z$ we have $\forall x < z\, \Diamond_x^{\Pi_1} \top$ by reflection, and thus $\ell_z \preceq L$ by clause (ii).

(iv) Argue in $\Box_z$, assuming $\Box_z^{\Pi_1} \bot$. By clause (iii) we have that $\ell_z \preceq L$. Suppose for a contradiction that $\ell_z \prec L$. In particular, there is some x with

$$L = \ell_x \wedge \forall y < x\, \Diamond_y^{\Pi_1} \top.$$

Since $\Box_z^{\Pi_1} \bot$, the second conjunct implies that $x \leq z$. On the other hand the assumption $\ell_z \prec \ell_x$ implies, using Lemma 11(ii), that $z < x$, a contradiction. □

Lemma 15. *If $a \prec b$, then* $\mathrm{I}\Delta_0 + \mathsf{exp} \vdash L = a \to \triangledown L = b$.

Proof. Argue in $\mathrm{I}\Delta_0+\mathsf{exp}$. Assuming $L = a$, we have

$$\forall z\, \big(\forall x < z\, \Diamond_x^{\Pi_1} \top \to \ell_z \preceq a\big) \tag{5}$$

If $\triangle L \neq b$ for some $b \succ a$, then there is some y with $\Box_y L \neq b \wedge \forall x < y\, \Diamond_x^{\Pi_1} \top$. Using (5) we have that $\ell_y \preceq a \prec b$. Now $\Box_y$ thinks:

> Suppose that $\ell_y = b$. Since $a \prec b$, we have $b \neq 0$ whence $\Box_y^{\Pi_1} \bot$ by Lemma 12(i). By Lemma 14(iv), the latter implies $L = \ell_y$ i.e. $L = b$, a contradiction.

Back in $\mathrm{I}\Delta_0+\mathsf{exp}$, we conclude $\Box_y \ell_y \neq b$, contradicting Lemma 11(iv). □

Lemma 16. *If $a \neq 0$, then* $\mathrm{I}\Delta_0+\mathsf{exp} \vdash L = a \to \triangle\, a \prec L$.

Proof. Argue in $\mathrm{I}\Delta_0+\mathsf{exp}$. Assume $L = a$, and let y be such that

$$\ell_y = a \wedge \forall x < y\, \Diamond_x^{\Pi_1} \top.$$

It follows from Lemma 12(iii)–(iv) that $\Box_y a \preceq \ell_y$. Given that $a \neq 0$, we also have $\Box_y a \neq \ell_y$ from Lemma 11(v). Combining the above yields $\Box_y a \prec \ell_y$. Since $\Box_y \ell_y \preceq L$ by Lemma 14(iii), we obtain $\Box_y a \prec L$, whence clearly also $\triangle\, a \prec L$. □

Definition 17. Let $\mathcal{M} = \langle \mathcal{F}, \Vdash \rangle$ be a finite GL-model. The model $\mathcal{M}_0$ is obtained by appending a new root 0 to $\mathcal{M}$; the truth values of propositional formulas at 0 are set arbitrarily. Apply Definition 7 to $\mathcal{M}_0$, and define the $\triangle$-realization * by letting

$$p^* := \bigvee_{\mathcal{M}_0, a \Vdash p} L = a.$$

Lemma 18. *Let $\mathcal{M}$ and * be as in Definition 17. For all $B \in \mathcal{L}_\triangle$, $a \neq 0$,*

$$\textit{if } \mathcal{M}, a \Vdash B, \textit{ then } \mathrm{I}\Delta_0+\mathsf{exp} \vdash L = a \to B^*.$$

Proof. Using Lemmas 15 and 16, we prove the claim simultaneously with

$$\text{if } \mathcal{M}, a \Vdash \neg B, \text{ then } \mathrm{I}\Delta_0+\mathsf{exp} \vdash L = a \to \neg B^*$$

by induction on the structure of B. □

Lemma 19. *i.* $\mathbb{N} \vDash L = 0$, *where $\mathbb{N}$ is the standard model.*

ii. For all $a \neq 0$, $L = a$ is consistent with PA.

Proof. (i) follows from Lemma 12(i).

(ii) Note that by (i) and Lemma 15, we have $\mathbb{N} \vDash \triangledown L = a$ for all a, whence also $\mathbb{N} \vDash \Diamond L = a$ (recall that since $\mathbb{N} \vDash \Diamond^{\Pi_1} \top$, we have $\mathbb{N} \vDash \Box\varphi \leftrightarrow \triangle\varphi$ for all φ). □

We prove the remaining direction of Theorem 5.

Proof. If $\mathsf{GL} \nvdash A$, then by Theorem 1 there is a finite rooted GL-model $\mathcal{M}$ with $w \nVdash A$ for some w in $\mathcal{M}$. Let * be the $\triangle$-realization as in Definition 17. By Lemma 18, $\mathrm{I}\Delta_0{+}\mathsf{exp} \vdash L = w \to \neg A^*$. Since PA does not prove $L \neq w$ by Lemma 19(ii), it therefore cannot prove A^* either. □

Open Question 20. What is the joint provability logic of $\triangle$ and $\Box$? For a candidate, see [HV, Definition 6]. □

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