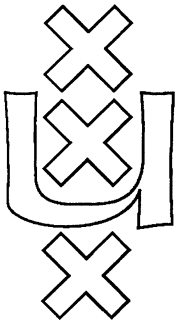
Institute for Language, Logic and Information

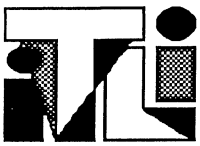
THE MODAL THEORY OF INEQUALITY

Maarten de Rijke

ITLI Prepublication Series
X-89-05



University of Amsterdam



Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

THE MODAL THEORY OF INEQUALITY

Maarten de Rijke
Department of Philosophy
University of Amsterdam

Received September 1989

Master's Thesis
Supervisor J.F.A.K. van Benthem

CONTENTS

PREFACE

The standard semantics of modal and tense logic is based on one binary relation, called the alternative, respectively precedence relation. It is a fairly obvious generalization to allow several binary relations and corresponding operators. One special such relation is the inequality relation. In this thesis we study the formalisms obtained from the modal and tense logical ones by adding an operator corresponding to the inequality relation.

The questions and problems dealt with in this thesis can be divided into three kinds:

(i) old questions/problems, such as: what is the logic of a special structure like $\mathbb{Z}$ or $\mathbb{Q}$; which first order properties are definable, and conversely, which formulas define first order conditions?

(ii) new questions/problems, such as: how does the new operator interact with the modal and tense logical operators; when, i.e. on which class of frames, does each formula in the extended formalism become equivalent to one in the old formalism; and, if any, which sets of formulas in the extended formalism are valid on precisely one frame?

(iii) transfer problems: which techniques and results from the modal and tense logical formalisms generalize to the extensions of these formalisms?

Chapter 1 introduces the basic notions, and examines which of the (anti-) preservation results that are known from ordinary modal logic are still valid in the extended formalism. Next, Chapter 2 studies the expressive powers of the various formalisms. Chapter 3, then, characterizes the translations of formulas into first order formulas, and determines the classes of models that are definable by means of formulas in this new formalism. In Chapter 4 we give complete axiomatizations for several special structures, as well as two incompleteness results and corollaries to these results. Finally, Chapter 5 describes two large classes of first order definable formulas in the new formalism; it ends with a digression on first order definability in other extensions of the modal formalism.

Results and notions belonging to ordinary modal or tense logic that are not credited can be looked up in van Benthem [1985]. Other results or notions not credited are due to the author and/or are trivial.

I am indebted to my teachers at the University of Amsterdam for creating an environment which has been both stimulating for learning and fruitful for doing logic, in particular (in the order in which we met) Frank Veltman, Herman Slangen, Roel de Vrijer, Dick de Jongh and Johan van Benthem.

A number of people have assisted me by reading parts of this thesis and offering their suggestions and comments. I single out for special thanks Johan van Benthem. Evidence of his influence on this thesis can be found throughout it. Harold Schellinx and Frank Veltman generously read and commented on earlier versions. Finally I thank Paul who is responsible for the few paragraphs that are in good English.

Maarten de Rijke
Amstelveen
03-07-1989

CHAPTER 1

INTRODUCTION

In this chapter we first briefly review notation and terminology. Next we examine some of the model-theoretic properties of formulas in the enriched formalisms. Finally we introduce general frames for modal logic extended with an operator for inequality. These structures will be used in Chapter 4.

§1. Basics

We begin with some definitions. The (multi-) modal languages to be considered here have an infinite supply of proposition letters (p, q, r, ...), a propositional constant $\bot$ (falsum), the usual Boolean operators $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (if ... then ...), and $\leftrightarrow$ (if and only if). Furthermore, they contain unary operators. The basic case is the language $L(\Diamond)$ with the operators $\Diamond$ (possibly) and $\Box$ (necessarily) - $\Diamond$ being regarded as primitive, and $\Box$ being defined by $\neg\Diamond\neg$. In general, $L(O_1, ..., O_n)$ denotes the (multi-) modal language with operators $O_1, ..., O_n$. The semantic structures are *frames,* i.e. ordered pairs $\langle W, R\rangle$ consisting of a nonempty set W with a binary relation R on W. To save words, we assume from now on that **F** denotes the frame $\langle W, R\rangle$. In addition to these frames, structures **M** (= $\langle$ **F**, V$\rangle$), called *models,* will be used, consisting of a frame **F** together with a *valuation* V *on* **F** assigning subsets of W to proposition letters.

Now comes the basic truth definition:

1.1. DEFINITION. Let **M** be a model, $w \in W$, and let φ be a modal formula. Then $\mathbf{M} \models \varphi[w]$ (in words: 'φ is true at w in **M**') is defined inductively as follows:

(i) $\mathbf{M} \models p[w]$ iff $w \in V(p)$, for proposition letters p,
(ii) $\mathbf{M} \models \neg\varphi[w]$ iff not $\mathbf{M} \models \varphi[w]$,
(iii) $\mathbf{M} \models (\varphi \wedge \psi)[w]$ iff $\mathbf{M} \models \varphi[w]$ and $\mathbf{M} \models \psi[w]$,
(iv) $\mathbf{M} \models \Diamond\varphi[w]$ iff for some $v \in W$ we have Rwv and $\mathbf{M} \models \varphi[v]$.

The definition of '$\mathbf{M} \models \Box\varphi[w]$' follows from (iv) using the abbreviation $\Box \equiv \neg\Diamond\neg$. For tense logic $\mathbf{M} \models \varphi[w]$ is defined in the same way except that the clause for $\Diamond$ is replaced by two clauses for F and P:

(v) $\mathbf{M} \models F\varphi[w]$ iff for some $v \in W$ we have Rwv and $\mathbf{M} \models \varphi[v]$,
(vi) $\mathbf{M} \models P\varphi[w]$ iff for some $v \in W$ we have Rvw and $\mathbf{M} \models \varphi[v]$.

The definitions of '$\mathbf{M} \models G\varphi[w]$' and '$\mathbf{M} \models H\varphi[w]$' follow from (v) and (vi) using the abbreviations $G \equiv \neg F\neg$ and $H \equiv \neg P\neg$.

Using this definition $\mathbf{F} \models \varphi[w]$ is defined by, for all valuations V on $\mathbf{F}$, $\langle \mathbf{F}, V\rangle \models \varphi[w]$. Next, $\mathbf{F} \models \varphi$ is defined by, for all $w \in W$, $\mathbf{F} \models \varphi[w]$. It is obvious how these notions may be extended to the case of a set of formulas.

The language $L(\Diamond,D)$ will be our main interest in this thesis; here, the operator D is defined by

(vii) $\mathbf{M} \models D\varphi[w] \Leftrightarrow \mathbf{M} \models \varphi[v]$, for some $v \neq w$.

(The proposal to consider this operator is due to several people independently, including Ron Koymans and Gargov, Passy & Tinchev [1987].) D's dual $\neg D \neg$ is denoted by $\overline{D}$. Using the D-operator some useful abbreviations can be defined:

$E\varphi := \varphi \vee D\varphi$ (there exists a point at which φ holds),
$A\varphi := \varphi \wedge \overline{D}\varphi$ (φ holds at all points), and
$U\varphi := E(\varphi \wedge \neg D\varphi)$ (φ holds at a unique point).

Lower case Greek letters φ, ψ, χ, ... will be used to denote (multi-) modal formulas. φ is called a $\Diamond$,D-*formula*, if $\varphi \in L(\Diamond,D)$; φ is called a *modal formula* if $\varphi \in L(\Diamond)$, etcetera.

Clearly, the notion of frame equivalence depends on the language we are using. This fact is reflected in our notation, e.g. $\mathbf{F} \equiv_{\Diamond,D} \mathbf{G}$ will denote the fact that $\mathbf{F}$ and $\mathbf{G}$ validate the same $\varphi \in L(\Diamond,D)$. Likewise, the theory of a frame $\mathbf{F}$ depends on the language we are using, so $Th_{\Diamond,D}(\mathbf{F}) := \{ \varphi \in L(\Diamond,D) \mid \mathbf{F} \models \varphi \}$, etcetera.

Every now and again we want to know whether a (multi-) modal formula corresponds to a first order formula. The next definition fixes the first order languages we will be dealing with:

1.2. DEFINITION. (i) L_0 is the first-order language with one binary predicate constant R as well as identity. L_0-formulas will be denoted by α, β, γ, ...
(ii) L_1 is the first-order language with one binary predicate constant R, identity, and unary predicate constants P_1, P_2, ..., P, Q, ... corresponding to the proposition letters of the (multi-) modal language.

The following notions are useful when dealing with the correspondence theory of $L(\Diamond,D)$.

1.3. DEFINITION. (i) If φ is a $\Diamond$,D-formula and α is an L_0-formula, then
$\mathcal{S}(\varphi,\alpha)$ iff for all $\mathbf{F}$, all $w \in W$, $\mathbf{F} \models \varphi[w] \Leftrightarrow \mathbf{F} \models \alpha[w]$,
$\mathfrak{M}1 = \{ \varphi \mid$ for some $\alpha \in L_0$, $\mathcal{S}(\varphi,\alpha) \}$,
$\mathfrak{P}1 = \{ \alpha \mid$ for some $\varphi \in L(\Diamond,D)$, $\mathcal{S}(\varphi,\alpha) \}$.
(ii) Furthermore, if α is an L_0-sentence, then
$\overline{\mathcal{S}}(\varphi,\alpha)$ iff for all $\mathbf{F}$, $\mathbf{F} \models \varphi \Leftrightarrow \mathbf{F} \models \alpha$,

$\mathfrak{M}_{\mathfrak{E}}1 = \{ \varphi \mid \text{for some } \alpha \in L_0, \mathfrak{E}(\varphi,\alpha) \}$,
$\overline{\mathfrak{M}}1 = \{ \alpha \mid \text{for some } \varphi \in L(\Diamond,D), \overline{\mathfrak{E}}(\varphi,\alpha) \}$.

$\mathfrak{E}$ is the relation of *local equivalence*. $\overline{\mathfrak{E}}$ is the relation of *global equivalence*. If $\mathfrak{E}(\varphi,\alpha)$, where α has the free variable x, then $\overline{\mathfrak{E}}(\varphi,\forall x\alpha)$. From this connection between $\mathfrak{E}$ and $\overline{\mathfrak{E}}$ it follows that $\mathfrak{M}_{\mathfrak{E}}1 \subseteq \overline{\mathfrak{M}}1$. By Theorem 7.4 in van Benthem [1985] the converse inclusion does not hold.

1.4. EXAMPLES. (i) A useful first order condition on R that is not definable in $L(\Diamond)$ is irreflexivity. In $L(\Diamond,D)$ we have $\mathfrak{E}(\Diamond p \to Dp, \neg Rxx)$. To prove this, let $\mathbf{F} \not\models \Diamond p \to Dp[w]$. Then for some valuation V on $\mathbf{F}$ we have $\langle \mathbf{F}, V\rangle \models \Diamond p \wedge \neg Dp[w]$, so we find an R-successor v of w such that p holds at v; by the second conjunct $v = w$, and so Rww holds. Conversely, if Rww holds, putting $V(p) = \{ w \}$ falsifies $\Diamond p \to Dp$ at w.
(ii) By an analogous argument we can prove that $\mathfrak{E}(Dp \to \Diamond p, \forall y\, Rxy)$, and so $(\forall y\, Rxy) \in \mathfrak{M}_{\mathfrak{E}}1$.
(iii) In chapter 2 we show that the L_0-sentence $\exists x\, Rxx$ is outside $\overline{\mathfrak{M}}1$.

We now turn to syntactic matters. A logic is here a set of formulas **L** containing classical tautologies and closed under the rules of Modus Ponens and Substitution

$$\text{SR: } \frac{\varphi(p)}{\varphi(\psi)}$$

We deal only with normal logics, i.e. logics containing the distribution formulas

A0. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and
A1. $\overline{D}(\varphi \to \psi) \to (\overline{D}\varphi \to \overline{D}\psi)$,

as well as the axiom schemes

A2. $\varphi \to \overline{D}D\varphi$ (symmetry),
A3. $DD\varphi \to (\varphi \vee D\varphi)$ (pseudo-transitivity) and
A4. $\Diamond\varphi \to (\varphi \vee D\varphi)$ (relation between $\Diamond$ and D),

and closed under the necessitation rules

$$\frac{\varphi}{\Box\varphi} \qquad \text{and} \qquad \frac{\varphi}{\overline{D}\varphi.}$$

So, our definition of a normal logic extends the 'classical' definition of a normal modal logic. (Of course, this definition applies only to logics in $L(\Diamond,D)$; however, the extension of this definition to e.g. L(F,P,D) is obvious.)

$\mathbf{D}_m$ denotes the basic logic. It was first defined by Ron Koymans in his Koymans [1989]. Finally, we use $\varphi \in \mathbf{L}$ and $\vdash_{\mathbf{L}} \varphi$ or $\mathbf{L} \vdash \varphi$ as synonyms, thus assuming that $\vdash_{\mathbf{L}}$ denotes an axiomatic system in which all formulas of $\mathbf{L}$ are derivable; if φ is derivable in the basic logic we sometimes write $\vdash \varphi$.

We end this section by stating some useful results which are either well-known or easy to prove.

1.5. LEMMA. *Let* $\mathbf{L}$ *be a normal logic. Then*

(i) *If* $\varphi \leftrightarrow \psi \in \mathbf{L}$, *then* $\Box\varphi \leftrightarrow \Box\psi$, $\Diamond\varphi \leftrightarrow \Diamond\psi$, $D\varphi \leftrightarrow D\psi$, $\overline{D}\varphi \leftrightarrow \overline{D}\psi \in \mathbf{L}$.

(ii) *If* $\varphi \rightarrow \psi \in \mathbf{L}$, *then* $\Box\varphi \rightarrow \Box\psi$, $\Diamond\varphi \rightarrow \Diamond\psi$, $D\varphi \rightarrow D\psi$, $\overline{D}\varphi \rightarrow \overline{D}\psi \in \mathbf{L}$.

(iii) $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi) \in \mathbf{L}$.

(iv) $\overline{D}(\varphi \wedge \psi) \leftrightarrow (\overline{D}\varphi \wedge \overline{D}\psi) \in \mathbf{L}$.

(v) $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi) \in \mathbf{L}$.

(vi) $D(\varphi \vee \psi) \leftrightarrow (D\varphi \vee D\psi) \in \mathbf{L}$.

§2. Preservation; anti-preservation; filtrations; cluster theory

To get some idea of the model-theoretic properties of $\Diamond$,D-formulas, we examine their behavior under the four well-known modal operations: p-morphisms, generated subframes, disjoint unions and ultrafilter extensions. Finally, we extend the notion of modal filtration to L($\Diamond$,D), and make a few remarks on the 'cluster theory' of L($\Diamond$,D).

Preservation

It appears that most of the preservation results known from L($\Diamond$) no longer hold for L($\Diamond$,D). To show this we need some definitions.

1.6. DEFINITION. (i) A function f from a frame $\mathbf{F}_1$ to a frame $\mathbf{F}_2$ is called a *p-morphism* if

(1) for all $w, v \in W_1$, if R_1wv then $R_2f(w)f(v)$, and

(2) for all $w \in W_1$ and $v \in W_2$, if $R_2f(w)v$ then there is a $u \in W_1$ such that R_1wu and $f(u) = w$.

(ii) A frame $\mathbf{F}_1$ is called a *generated subframe* of a frame $\mathbf{F}_2$ (notation: $\mathbf{F}_1 \subseteq \mathbf{F}_2$) if

(1) $W_1 \subseteq W_2$,

(2) $R_1 = R_2 \cap (W_1 \times W_2)$, and

(3) for all $w \in W_1$ and $v \in W_2$, if R_2wv then $v \in W_1$.

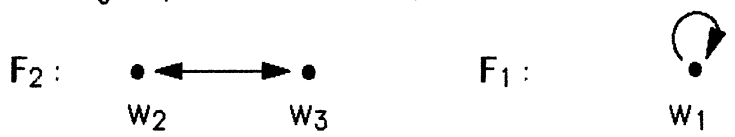
(iii) Let $\{ \mathbf{F}_i \mid i \in I \}$ be a collection of frames. Put $\mathbf{F}_i' := \langle W_i', R_i' \rangle$, where $W_i' = \{ \langle i,w \rangle \mid w \in W_i \}$ and $R_i' = \{ \langle \langle i,w \rangle, \langle i,v \rangle \rangle \mid R_iwv \}$. Then the *disjoint union* $\oplus_i\{ \mathbf{F}_i \mid i \in I \}$ of the collection $\{ \mathbf{F}_i \mid i \in I \}$ is the frame $\langle \cup\{ W_i' \mid i \in I \}, \cup\{ R_i' \mid i \in I \} \rangle$.

A proof of the following fact can be found in van Benthem [1985]:

1.7. FACT. *Modal formulas are preserved under surjective p-morphisms, generated subframes and disjoint unions.*

We now show that, in general, ◇,D-formulas are not preserved under these operations.

1.8. REMARK. (i) ◇,D-formulas are not preserved under p-morphic images. A counterexample is provided by DT. This formula holds on $\mathbf{F}_2$, but fails on its p-morphic image $\mathbf{F}_1$:



(ii) Next, ◇,D-formulas are not preserved under generated subframes: consider the formula DT, as well as the frames $\mathbf{F}_3 \subseteq \mathbf{F}_4$ in the following picture. Obviously, $\mathbf{F}_4 \models$ DT, but $\mathbf{F}_3 \not\models$ DT.

(iii) Finally, ◇,D-formulas are not preserved under disjoint unions either. Let $\mathbf{F}_1$ be as in (i), and define $\mathbf{F}_5 := \oplus_i \{ \mathbf{F}_1 \mid i = 1, 2 \}$. Then $\mathbf{F}_1 \models \neg$DT, although $\mathbf{F}_5 \models$ DT.

Anti-preservation

Another important notion in classical modal logic is that of an ultrafilter extension.

1.9. DEFINITION. (i) Let $\mathbf{F}$ be a frame, and $\mathfrak{X} \subseteq W$. Then

$$\mathbf{l}(\mathfrak{X}) := \{ w \in W \mid \forall v \in W (Rwv \rightarrow v \in \mathfrak{X}) \}.$$

(ii) The *ultrafilter extension* $ue(\mathbf{F})$ of $\mathbf{F}$ is the frame $\langle W_F, R_F \rangle$, where W_F is the set of ultrafilters on W, and $R_F U_1 U_2$ holds if for all $\mathfrak{X} \subseteq W$: $\mathbf{l}(\mathfrak{X}) \in U_1 \Rightarrow \mathfrak{X} \in U_2$.

Now, modal formulas are anti-preserved under ultrafilter extensions, that is, if $ue(\mathbf{F}) \models \varphi$ then $\mathbf{F} \models \varphi$. For L(◇,D) this result still holds good. This fact is readily seen to follow from the next lemma.

1.10. LEMMA. *Let* V *be any valuation on* $\mathbf{F}$. *Define the valuation* V_F *on* $ue(\mathbf{F})$ *by putting* $V_F(p) = \{ U \mid V(p) \in U \}$. *Then, for all ultrafilters* U *on* W, *and all* ◇,D- *formulas* φ, $\langle ue(\mathbf{F}), V_F \rangle \models \varphi[U] \Leftrightarrow V(\varphi) \in U$.

PROOF. This is by induction on φ. The cases $\varphi \equiv p$, $\neg\psi$, $\psi \wedge \chi$, $\Diamond\psi$ are proved in van Benthem [1985], Lemma 2.25. The only new case is $\varphi \equiv D\psi$.

Suppose $V(D\psi) = \{ w \mid \exists v \succ w\ (v \in V(\psi)) \} \in U$. We must find an ultrafilter $U_1 \succ U$ such that $\langle ue(\mathbf{F}), V_F \rangle \models \psi[U_1]$. First assume that U contains a singleton, and hence that it is a principal filter, - say $U = \{ X \subseteq W \mid X \supseteq \{ w_0 \} \}$. Then

$$\{ w_0 \} \subseteq \{ w \mid \exists v \succ w\ (v \in V(\psi)) \}, \text{ i.e. } \exists v \succ w_0\ (v \in V(\psi)).$$

Now, $\{ v \} \notin U$, otherwise we would have $\emptyset = \{ v \} \cap \{ w_0 \} \in U$. Let U_1 be the ultrafilter generated by $\{ v \}$. So, since $\{ v \} \notin U$, we have $U \neq U_1$. Furthermore,

$$\begin{aligned} v \in V(\psi) &\Rightarrow V(\psi) \supseteq \{ v \} \\ &\Rightarrow V(\psi) \in U_1, \\ &\Rightarrow \langle ue(\mathbf{F}), V_F \rangle \models \psi[U_1] \in V_F(\psi), \text{ by the IH}, \\ &\Rightarrow \langle ue(\mathbf{F}), V_F \rangle \models D\psi[U], \text{ since } U_1 \succ U. \end{aligned}$$

Next, suppose that U does not contain a singleton. Since $V(D\psi) \in U$, we find some $w_0 \in V(D\psi)$. Let v be the associated point $\succ w_0$ such that $v \in V(\psi)$. Again, we have $\{ v \} \notin U$. We can now proceed as in the previous case.

Conversely, let $V(D\psi) \notin U$. We have to show that $\langle ue(\mathbf{F}), V_F \rangle \not\models D\psi[U]$. Since $V(D\psi) \notin U$, we have $\mathcal{A} = \{ w \mid \forall v\ (v \succ w \rightarrow v \notin V(\psi)) \} \in U$, whence $\mathcal{A} \neq \emptyset$. Pick some $w_0 \in \mathcal{A}$.
Clearly, if $w_0 \notin V(\psi)$, then $\mathcal{A} = W$ and $V(\psi) = \emptyset$. Consequently,

$$\begin{aligned} &\forall \text{ ultrafilters } U_1 : V(\psi) \notin U_1, \\ &\quad \Rightarrow \forall \text{ ultrafilters } U_1 : \langle ue(\mathbf{F}), V_F \rangle \not\models \psi[U_1], \text{ by the IH}, \\ &\quad \Rightarrow \forall \text{ ultrafilters } U_1 \succ U : \langle ue(\mathbf{F}), V_F \rangle \not\models \psi[U_1], \\ &\quad \Rightarrow \langle ue(\mathbf{F}), V_F \rangle \not\models D\psi[U]. \end{aligned}$$

If $w_0 \in V(\psi)$, then $\mathcal{A} = \{ w_0 \} = V(\psi)$, and U is generated by $\mathcal{A}$. For any ultrafilter $U_1 \succ U$ we have $\mathcal{A} \notin U_1$ - otherwise U_1 would equal U. So,

$$\begin{aligned} &\forall \text{ ultrafilters } U_1 : U_1 \succ U \Rightarrow \mathcal{A} = V(\psi) \notin U_1, \\ &\quad \Rightarrow \forall \text{ ultrafilters } U_1 \succ U : \langle ue(\mathbf{F}), V_F \rangle \not\models \psi[U_1], \text{ by the IH}, \\ &\quad \Rightarrow \langle ue(\mathbf{F}), V_F \rangle \not\models D\psi[U]. \end{aligned}$$

■

1.11. COROLLARY. *For any frame* $\mathbf{F}$, *and all* $\varphi \in L(\Diamond, D)$, $ue(\mathbf{F}) \models \varphi \Rightarrow \mathbf{F} \models \varphi$.

We immediately obtain a non-definability result from this corollary:

1.12. COROLLARY. $\exists x\, Rxx$ *is not* $\Diamond, D$- *definable.*

PROOF. Evidently, $\langle \mathbb{N}, < \rangle \models \neg\exists x\, Rxx$. However, some straightforward cal-

culations show that for any nonprincipal ultrafilter U on $\mathbb{N}$ we have $R_F UU$. Therefore, $ue(\langle \mathbb{N}, < \rangle) \models \exists x Rxx$. ■

Filtrations

Besides the four operations discussed up to now, there is another important modal concept: *filtration*. Modal filtrations - which are defined by the first three clauses in Definition 1.13 - can not be applied to sets of $\Diamond$,D-formulas directly: let $W_1 = \{ v, w \}$, $R_1 = \emptyset$, and let V_1 be any valuation; put $W_2 = \{ u \}$, $R_2 = \{ \langle u,u \rangle \}$, and let V_2 be an arbitrary valuation. Finally, define $\Sigma := \{ T, DT \}$; then $g : W_1 \to W_2$, defined by $g(w) = g(v) = u$, is a modal filtration with respect to Σ from $\langle W_1, R_1, V_1 \rangle$ to $\langle W_2, R_2, V_2 \rangle$. However, $\langle W_1, R_1, V_1 \rangle \models DT$ and $\langle W_2, R_2, V_2 \rangle \not\models DT$.

Things can be mended quite easily: we extend the definition of a modal filtration with only one new clause to obtain the definition of a $\Diamond$,D-filtration.

1.13. DEFINITION. Let $\mathbf{M}_1$ ($= \langle W_1, R_1, V_1 \rangle$) and $\mathbf{M}_2$ ($= \langle W_2, R_2, V_2 \rangle$) be models, and let Σ be a set of $\Diamond$,D-formulas closed under subformulas. A surjective function $g : \mathbf{M}_1 \to \mathbf{M}_2$ is said to be a $\Diamond$,D-*filtration with respect to* Σ, if

(i) for all $w, v \in W_1$, if $R_1 wv$, then $R_2 g(w) g(v)$,

(ii) for all $w \in W_1$, and all proposition letters p in Σ, $w \in V_1(p)$ iff $g(w) \in V_2(p)$,

(iii) for all $w \in W_1$, and all $\Diamond\varphi \in \Sigma$, if $\mathbf{M}_1 \models \Diamond\varphi[w]$, then $\mathbf{M}_2 \models \Diamond\varphi[g(w)]$,

(iv) for all $w \in W_1$, and all $D\varphi \in \Sigma$, if $\mathbf{M}_1 \models D\varphi[w]$, then $\mathbf{M}_2 \models D\varphi[g(w)]$.

1.14. LEMMA. *If* g *is a* $\Diamond$,D-*filtration w.r.t.* Σ *from* $\mathbf{M}_1$ *to* $\mathbf{M}_2$, *then, for all* $w \in W_1$, *and all* $\Diamond$,D-*formulas* $\varphi \in \Sigma$, $\mathbf{M}_1 \models \varphi[w]$ *iff* $\mathbf{M}_2 \models \varphi[g(w)]$.

PROOF. Trivial. ■

Next, we construct a filtration analogous to the standard example of a modal filtration, i.e. analogous to the *modal collapse*.

Let $\mathbf{M}$ ($= \langle W, R, V \rangle$) be a model, and let Σ be a set of $\Diamond$,D-formulas closed under subformulas. Extend Σ as follows. Let $\{ D\psi_i \mid i \in I \}$ be an enumeration of all formulas $D\varphi \in \Sigma$ such that $\mathbf{M} \models \varphi \wedge D\varphi[v]$, for some $v \in W$, and extend the language by adding new proposition letters $\{ q_i \mid i \in I \}$. Expand $\mathbf{M}$ to a model $\mathbf{M}^*$ for this new language, by verifying q_i in one and only one point in which ψ_i holds. Put $\Sigma^* := \Sigma \cup \{ q_i \mid i \in I \}$.

As far as the formulas in Σ are concerned, $\mathbf{M}^*$ behaves just like $\mathbf{M}$, since none of the q_i's occurs in Σ. Now, define the model $\mathbf{M}^*_\Sigma = \langle W_\Sigma, R_\Sigma, V^*_\Sigma \rangle$, using $g(w) = \{ \varphi \in \Sigma^* \mid \mathbf{M}^* \models \varphi[w] \}$, by

$W_\Sigma := g[W]$,
$R_\Sigma g(w)g(v) \equiv \forall\varphi\,[\Box\varphi \in \Sigma \wedge \Box\varphi \in g(w) \rightarrow \varphi \in g(v)]$,
$V^*_\Sigma(p) := \{\, g(w) \mid p \in g(w) \,\}$.

It is obvious that g is a filtration with respect to Σ^* from $\mathbf{M}^*$ to $\mathbf{M}^*_\Sigma$. Restricting V^*_Σ to the original language, yields a filtration g with respect to Σ from $\mathbf{M}$ to $\mathbf{M}_\Sigma = \langle W_\Sigma, R_\Sigma, V_\Sigma\rangle$, where V_Σ is V^*_Σ restricted to the old language. This model is called the $\Diamond$,D-*collapse* of $\mathbf{M}$ with respect to Σ.

REMARK. An alternative way to define the $\Diamond$,D-collapse would be to take the ordinary modal collapse, and to double points that correspond to more than one point in the original model. The inductive proof that corresponding (doubled) points verify the same formulas is similar to e.g. part 3 in the proof of Theorem 2.5.

Using the $\Diamond$,D-collapse it is easily verified that $\Diamond$,D-formulas satisfy the *finite model property*: any formula which is not universally valid is refuted on some finite model. For if φ is not universally valid, then $\mathbf{M} \models \neg\varphi[w]$, for some model $\mathbf{M}$. Taking the $\Diamond$,D-collapse of $\mathbf{M}$ with respect to the set of subformulas of $\neg\varphi$, we see that φ is refuted on a finite model.

Cluster theory

Segerberg [1970] proves the following for L($\Diamond$):

The Bulldozer Theorem. For every transitive, connected (transitive, connected and reflexive) model, there is an equivalent strict linearly (linearly) ordered model.

The increase in expressive power we gain by adding the D-operator to L($\Diamond$) is reflected by the fact that the theorem does not hold for L($\Diamond$,D): consider the model $\mathbf{M} = \langle \{0, 1\}, \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}, V\rangle$, where V is defined by $V(p) = \{0\}$ for all proposition letters p. In each point in $\mathbf{M}$ the formula $(p \wedge \neg Dp) \vee (\neg p \wedge \neg D\neg p)$ holds, for each p. From this it follows that each equivalent $\mathbf{N}$ of $\mathbf{M}$ must have $|\mathbf{N}| = 2$. Furthermore, in each point in $\mathbf{N}$ the formula $(p \rightarrow \Diamond\neg p) \wedge (\neg p \rightarrow \Diamond p)$ must hold; it follows that $\mathbf{N}$'s points must be related to each other. Finally, reflexivity follows since $\Diamond p \wedge \Diamond\neg p$ must hold in each point. So $\mathbf{M} \cong \mathbf{N}$, and $\mathbf{N}$ can not be a linearly ordered model.

Nevertheless, the central notion in Segerberg's proof of the Bulldozer Theorem will appear to be very useful in L($\Diamond$,D). It is the notion of a *cluster*:

1.15. DEFINITION. The *clusters* of a transitive frame $\mathbf{F}$ are the equivalence classes of W under the equivalence relation $x \approx y$ iff $(Rxy \wedge Ryx) \vee x = y$.

Clusters are divided into three kinds: proper, with at least two elements, all reflexive; simple, with one reflexive element; and degenerate with one irreflexive element.

In Chapter 2 we will sometimes want to show two frames equivalent. We will do this as follows: (i) choose a formula φ that is invalid on the one frame, and let V be a valuation refuting φ; (ii) consider the $\Diamond$,D-collapse of the resulting model with respect to the set of subformulas of φ; this model may be viewed as a finite linear sequence of clusters; (iii) turn this model into a model on the second frame, while preserving the invalidity of φ.

A similar application of clusters can be found in Chapter 4, Section 3, where we consider the canonical model of a logic, take a suitable filtration, and turn this filtration into a model on $\mathbb{Z}$, to prove that logic complete for $\mathbb{Z}$.

§3. Algebras; general frames

We define the algebraic semantics for L($\Diamond$,D), and we also introduce general frames for L($\Diamond$,D), as well as functors connecting the algebras and general frames.

The algebraic semantics is a simple modification of the algebraic semantics for modal logics.

1.16. DEFINITION. A *modal algebra with a difference operator* (or *d-algebra* for short) is an ordered tuple $\mathbf{A} = \langle A, \mathbf{m}, \mathbf{d}\rangle$ where A is a Boolean algebra, and $\mathbf{m}$ and $\mathbf{d}$ are unary operations on A such that

$$\begin{aligned}
&\mathbf{m}0 = \mathbf{d}0 = 0,\\
&\mathbf{m}(x \cup y) = \mathbf{m}x \cup \mathbf{m}y,\\
&\mathbf{d}(x \cup y) = \mathbf{d}x \cup \mathbf{d}y,\\
&x \leq \overline{\mathbf{d}}\mathbf{d}x, \text{ where } \overline{\mathbf{d}}z := \overline{\mathbf{d}\bar{z}},\\
&\mathbf{d}\mathbf{d}x \leq (x \cup \mathbf{d}x), \text{ and}\\
&\mathbf{m}x \leq (x \cup \mathbf{d}x)
\end{aligned}$$

hold for all $x, y \in A$.

Any frame $\mathbf{F}$ induces an algebra $A(\mathbf{F}) = \langle \wp(W), 1, -, \cap, \mathbf{m}, \mathbf{d}\rangle$, where $1 = W$, $-$ is complementation with respect to W, $\cap$ is intersection, and $\mathbf{m}$ and $\mathbf{d}$ are defined by

$$\begin{aligned}
&\mathbf{m}(\mathfrak{X}) = \{ v \in W \mid \exists w \in \mathfrak{X}\ (Rvw) \},\\
&\mathbf{d}(\mathfrak{X}) = \{ v \in W \mid \exists w \in \mathfrak{X}\ (v \neq w) \}.
\end{aligned}$$

One easily proves this **d**-operator to be a rather trivial operator, in the sense that it does not create any new sets. If $U \in \wp(W)$ and $|U| \geq 2$, then $\mathbf{d}(U) = W$. For, let $U \supseteq \{ u_1, u_2 \}$ and $w \in W$; if $w \neq u_1$, then $w \in \mathbf{d}(U)$, and if $w = u_1$, then $w \neq u_2$ and $w \in \mathbf{d}(U)$. Similarly, if $|U| = 1$, then $\mathbf{d}(U) = W \backslash U$, and $\mathbf{d}(\emptyset) = \emptyset$. So in all cases, applying the **d**-operator does not yield a new set.

Given a logic **L**, we define an equivalence relation $\approx_L$ on the formulas of **L**, by identifying formulas which can not be distinguished by $\vdash_L$: $\varphi \approx_L \psi$ iff $\mathbf{L} \vdash \varphi \leftrightarrow \psi$. Now, let $[\varphi]_L$ denote $\{ \psi \mid \varphi \approx_L \psi \}$, and consider the Lindenbaum-Tarski algebra $\mathbf{A}_L$ for **L**, where $\mathbf{A}_L = \langle \{ [\varphi]_L \mid \varphi \text{ a formula} \}, 1, -, \cap, \mathbf{m}, \mathbf{d} \rangle$, where

$$\begin{aligned} [\varphi]_L \cap [\psi]_L &= [\varphi \wedge \psi]_L, \\ \overline{[\varphi]}_L &= [\neg\varphi]_L, \\ 1 &= [T]_L, \\ \mathbf{m}[\varphi]_L &= [\Diamond\varphi]_L, \\ \mathbf{d}[\varphi]_L &= [D\varphi]_L. \end{aligned}$$

This definition is justified by the rule of replacement of equivalents for classical logic. The usual properties of classical propositional calculus are then expressed in the fact $\langle \{ [\varphi]_L \mid \varphi \text{ a formula} \}, 1, -, \cap \rangle$ is a Boolean algebra. The last two clauses are justified when **L** is closed under the rules

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \qquad \text{and} \qquad \frac{\varphi \leftrightarrow \psi}{\overline{D}\varphi \leftrightarrow \overline{D}\psi.}$$

By Lemma 1.5, this is indeed the case. And finally, by our definition of a logic $\mathbf{A}_L$ satisfies Definition 1.16.

The following notion will be used in Chapter 4:

1.17. DEFINITION. A *general frame* $\langle \mathbf{F}, \mathfrak{W} \rangle$ consists of a frame **F** and a non-empty set $\mathfrak{W} \subseteq \wp(W)$ such that

(i) $\mathfrak{W}$ is closed under $\cap$ and $-$,

(ii) $U \in \mathfrak{W} \Rightarrow \mathbf{m}(U), \mathbf{d}(U) \in \mathfrak{W}$,

where **m** and **d** are the operators used to define $\mathbf{A}_L$.

A valuation on a general frame is a function taking its values in $\mathfrak{W}$.

It is clear that the (full) general frame $\langle \mathbf{F}, \wp(W) \rangle$ is equivalent to the frame **F**. Furthermore, if $\langle \mathbf{F}, \mathfrak{W} \rangle$ is a general frame, then $A(\langle \mathbf{F}, \mathfrak{W} \rangle) = \langle \mathfrak{W}, 1, -, \cap, \mathbf{m}, \mathbf{d} \rangle$ - with **m** and **d** defined as before - is an algebra in the sense of Definition 1.16.

Finally, a two-way connection between algebras and general frames is established, by defining, for **A** an algebra in the sense of Definition 1.16, $F(\mathbf{A}) = \langle W_A, R_A, \mathfrak{W}_A \rangle$, where

W_A is the set of ultrafilters on **A**,
$\langle w,v\rangle \in R_A$ iff for each $a \in A$, if $a \in v$ then $\mathbf{m}a \in w$,
$\mathfrak{W}_A = \{ \{ w \in W_A \mid a \in w \} \mid a \in A \}$.

Clearly, F(**A**) is a general frame.

We are not going to use d-algebras in this thesis - we only introduced them because we wanted to introduce general frames, and a general frame is nothing but a frame with a d-algebra (on a subset of its powerset) attached to it. Nevertheless, we do want to make a remark about them.

For ordinary modal algebras (i.e. d-algebras without the **d**-operator) we have a representation theorem saying that $A^{\bullet}F(\mathbf{A}) \cong \mathbf{A}$, where $a \in \mathbf{A}$ is mapped onto the set of ultrafilters on **A** containing a. In proving that $A^{\bullet}F$ is an isomorphism one of the things one has to show is that $\mathbf{m}_{AF}(A^{\bullet}F(a)) = A^{\bullet}F(\mathbf{m}(a))$, where **m** is the **m**-operator in **A**, and $\mathbf{m}_{AF}$ is the **m**-operator in $A^{\bullet}F(\mathbf{A})$. Any attempt to prove a corresponding identity for the **d**-operator gets stuck.

The reason for this is that we loose all information about the **d**-operator when we apply the functor F to **A**. Whereas the **m**-operator gives rise to the relation R_A on F(**A**) which in turn gives rise to $\mathbf{m}_{AF}$ in $A^{\bullet}F(\mathbf{A})$, the **d**-operator is lost in passing from **A** to F(**A**). This is because we have a fixed interpretation in mind for the D-operator, notably the inequality relation. If we wanted to treat **d** on a par with **m** we would have to associate a relation S with it in passing from **A** to F(**A**) just as we associated R_A with **m**. But would this relation S be real inequality? Evidently some work needs to be done here.

CHAPTER 2

SOME COMPARISONS

We investigate the expressive powers of L(◇,D) and L(F,P,D), and compare them to those of L(◇) and L(F,P). We point out that in one case at least adding the D-operator to L(F,P) does not enlarge its expressive power. Next we show that, unlike L(◇) and L(F,P), L(◇,D) and L(F,P,D) have enough expressive power to make the notion of categoricity a meaningful one. We end this chapter by making some remarks about L(D).

§1. L(◇) and L(◇,D)

One way to compare the expressive powers of two languages is to examine their ability to discriminate between special (read: well-known) structures. For example, in contrast to L(◇), L(◇,D) is able to distinguish $\mathbb{Z}$ from $\mathbb{N}$:

2.1. PROPOSITION. (i) $\langle \mathbb{N}, < \rangle \equiv_{\Diamond} \langle \mathbb{Z}, < \rangle$.
(ii) $\langle \mathbb{N}, < \rangle \not\equiv_{\Diamond,D} \langle \mathbb{Z}, < \rangle$.

PROOF. (i) If $\langle \mathbb{Z}, < \rangle \not\models \varphi$, for some $\varphi \in L(\Diamond)$, then $\langle \mathbb{Z}, <, V \rangle \models \neg\varphi[w]$ for some $w \in \mathbb{Z}$, and a valuation V. The subframe generated by w is isomorphic to $\langle \mathbb{N}, < \rangle$. So by preservation under generated submodels, we have $\langle \mathbb{N}, <, V' \rangle \models \neg\varphi[w]$, where $V'(p) = V(p) \cap \mathbb{N}$, for all p. Conversely, any valuation V on $\langle \mathbb{N}, < \rangle$ gives rise to a valuation V' on $\langle \mathbb{Z}, < \rangle$ which is equivalent to V on $\langle \mathbb{N}, < \rangle$. Therefore, if $\langle \mathbb{N}, <, V \rangle \models \neg\varphi[n]$, then $\langle \mathbb{Z}, <, V' \rangle \models \neg\varphi[n]$.
(ii) This follows from the fact that the existence of a (different) predecessor is expressible in L(◇,D): we have $\mathfrak{F}(p \to D\Diamond p, \exists y\,(x \neq y \wedge Ryx))$. ■

So $\forall x \exists y\,(x \neq y \wedge Ryx)$ is an L_0-sentence definable in L(◇,D), but not in L(◇). Other well-known L_0-conditions undefinable in L(◇) are irreflexivity and anti-symmetry. By the next result, these conditions do have an L(◇,D)-equivalent:

2.2. PROPOSITION. (Koymans) *Every universal* L_0*-sentence is* ◇,D*-definable.*

PROOF. For $\forall x_1 \ldots x_n\, \mathrm{BOOL}(Rx_ix_j, x_i = x_j)$ take

$$Uq_1 \wedge \ldots \wedge Uq_n \to \mathrm{BOOL}(E(q_i \wedge \Diamond q_j), E(q_i \wedge q_j)).$$ ■

This result can still be improved upon:

2.3. PROPOSITION. *All* Π^1_1*-sentences in* R, = *of the purely universal form* $\forall P_1 \ldots \forall P_m \forall x_1 \ldots \forall x_n \, \mathrm{BOOL}(P_i x_{i_j}, Rx_i x_j, x_i = x_j)$ *are* ◇,D*-definable.*

PROOF. Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be proposition letters such that each of $p_1, \ldots, p_m$ is different from each of $q_1, \ldots, q_n$. Now take

$$Uq_1 \wedge \ldots \wedge Uq_n \rightarrow \mathrm{BOOL}(E(q_{i_j} \wedge p_i), E(q_i \wedge \Diamond q_j), E(q_i \wedge q_j)). \qquad \blacksquare$$

It is a well-known fact that two finite, rooted frames that validate the same L(◇)-formulas, are isomorphic. This result is improved upon in L(◇,D): from Proposition 2.2 it follows that any two finite frames are isomorphic iff they are ◇,D-equivalent. (For, finite frames are isomorphic iff they have the same universal first order theory.) We state this corollary officially, and give an alternative proof.

2.4. PROPOSITION. *If* **F** (= ⟨ W_1, R_1⟩) *and* **G** (= ⟨ W_2, R_2⟩) *are finite frames, then* $\mathbf{F} \cong \mathbf{G}$ *iff* $\mathbf{F} \equiv_{\Diamond,D} \mathbf{G}$.

PROOF. The direction from left to right is obvious. For the converse, let $W_1 = \{ w_1, \ldots, w_n \}$, and suppose that $p_1, \ldots, p_n$ are different proposition letters. Let Φ be defined by

$$\bigwedge_{1 \le i \le n} Ep_i \wedge A[\bigvee_{1 \le i \le n} (p_i \wedge \neg Dp_i)] \wedge A[\bigwedge_{1 \le i \ne j \le n} (p_i \rightarrow \neg p_j)] \wedge A[\bigwedge_{\substack{1 \le i \le n \\ 1 \le j \le n}} (p_i \rightarrow Op_j)],$$

where $O \equiv \Diamond$, if $R_1 w_i w_j$ holds, and $O \equiv \neg\Diamond$, otherwise.
Next, put $V(p_i) = \{ w_i \}$, then ⟨ **F**, V⟩ ⊨ Φ, and so $\neg\Phi \notin \mathrm{Th}_{\Diamond,D}(\mathbf{F})$ and $\neg\Phi \notin \mathrm{Th}_{\Diamond,D}(\mathbf{G})$. Hence, we find some valuation V' on **G** such that ⟨ **G**, V'⟩ ⊨ Φ[v'], for some $v' \in W_2$. By construction there exist $v_1, \ldots, v_n \in W_2$,- which are different and which enumerate W_2 completely -, such that $V(p_1) = \{ v_1 \}, \ldots, V(p_n) = \{ v_n \}$. Finally, defining $f : W_1 \rightarrow W_2$, by $w_i \longmapsto v_i$, gives an isomorphism. $\blacksquare$

You should be convinced by now that adding the D-operator to L(◇) greatly increases the expressive power. On well-orderings, however, both L(◇) and L(◇,D) can only recognize the sort of things they already recognize below ω^2; for L(◇) this result was proved in van Benthem [1989]. For L(◇,D) this result follows from the theorem we are about to prove.

2.5. THEOREM. *If* $\varphi \in$ L(◇,D), *and* **F** *is a well-ordered frame such that* $\mathbf{F} \not\models \varphi$, *then there is a well-ordered frame* $\mathbf{G} < \omega^2$ *having* $\mathbf{G} \not\models \varphi$.

PROOF. Since this proof uses a construction which recurs in the sequel, it will be given in quite some detail. The proof consists of several parts. Suppose that for some valuation V, and a $w_0 \in W$, **M** (= ⟨ **F**, V⟩) ⊨ $\neg\varphi[w_0]$. By

means of filtration we obtain a finite model on which φ fails. Next, this model is made into a well-ordering of order type $< \omega^2$. Finally, we show that φ fails on this well-ordering as well.

(1). Let Σ^- be the set of subformulas of $\neg\varphi$, and define Σ to be $\Sigma^- \cup \{ \Diamond\psi \mid D\psi \in \Sigma^- \}$. Consider the finite model $\mathbf{M}_\Sigma = \langle W_\Sigma, R_\Sigma, V_\Sigma \rangle$, where W_Σ, V_Σ (and the function g) are defined as in the definition of the $\Diamond$,D-collapse, and $R_\Sigma g(w)g(v) \equiv \forall \Diamond\psi \in \Sigma\ (\Diamond\psi \notin g(w) \rightarrow \Diamond\psi, \psi \notin g(v))$. (That g is indeed a filtration with respect to Σ from $\mathbf{M}$ to $\mathbf{M}_\Sigma$ follows from the fact that R is transitive.)

Note that R_Σ inherits some properties of R_0: R_Σ is both transitive and linear. The first property follows from the definition of R_Σ, using transitivity in $\mathbf{M}$. And since g is a R_0-homomorphism, R_Σ is a linear ordering.
Consequently, $\mathbf{M}_\Sigma$ may be viewed as a finite linear sequence of clusters. Each nondegenerated cluster consists of a maximal set of points mutually R_Σ-related. Within each cluster, points verify the same Σ-formulas of the form $\Diamond\psi$.

(2). Next, $\mathbf{M}_\Sigma$ will be blown up into a well-ordered model $\mathbf{N} = \langle W_0, R_0, V_0 \rangle$. Put $W_0 = \emptyset$, $R_0 = \emptyset$ and $V_0(p) = \emptyset$, for all proposition letters p. $\mathbf{N}$ will be defined by examining the consecutive clusters one after another, until all clusters have been taken care of. Each cluster will give rise to extensions of W_0, R_0 and V_0. We start this process with g(0).
Suppose that $\mathfrak{C}$ is the cluster we have to take care of, and that we already have a well-ordering $\langle W_0, R_0 \rangle$ and a valuation V_0,- then the sum of $\langle W_0, R_0 \rangle$ and an ordinal α will again be a well-ordering.
- if $\mathfrak{C}$ is a degenerated, put $W_0 := W_0 + \mathbf{1}$, extend V_0 by verifying a proposition letter $p \in \Sigma$ in the newly added point iff $w \in V_\Sigma(p)$, where $\mathfrak{C} = \{ w \}$;
- if $\mathfrak{C}$ is simple, put $W_0 := W_0 + \omega$, and extend V_0 by verifying a proposition letter $p \in \Sigma$ in all newly added points iff $w \in V_\Sigma(p)$, where $\mathfrak{C} = \{ w \}$;
- if $\mathfrak{C}$ is proper, put $W_0 := W_0 + \omega$. Assume $\mathfrak{C} = \{ w_1 < \ldots < w_k \}$, where $<$ is an arbitrary linear ordering on $\mathfrak{C}$. Informally, V_0 is extended by repeating $w_1 < \ldots < w_k$ ω times on the newly added copy of ω. Formally, V_0 is extended by putting: (all natural numbers mentioned in the next few lines are assumed to be elements of the newly added copy of ω)

$0 \in V_0(p)$ iff $w_1 \in V_\Sigma(p)$;
$1 \in V_0(p)$ iff $w_2 \in V_\Sigma(p)$;
:
$k-2 \in V_0(p)$ iff $w_{k-1} \in V_\Sigma(p)$;
$k-1 \in V_0(p)$ iff $w_k \in V_\Sigma(p)$;
$k \in V_0(p)$ iff $w_1 \in V_\Sigma(p)$;

$k+1 \in V_0(p)$ iff $w_2 \in V_\Sigma(p)$;
:
:

It is evident that this process yields a well-ordered model **N**. As $\mathbf{M}_\Sigma$ is finite, **N** is a well-ordering of order type smaller than ω^2!

Before proceeding to prove that φ fails on **N**, we introduce some notation: if $v \in \mathbf{M}_\Sigma$, $\bar{v}$ will be used to denote (a) point(s) corresponding to v in **N**.

(3). CLAIM. *For all* $\psi \in \Sigma$, *and all* $v \in \mathbf{M}_\Sigma$, $\mathbf{M}_\Sigma \models \psi[v]$ *iff* $\mathbf{N} \models \psi[\bar{v}]$.
PROOF (of the claim). By induction on ψ. The only interesting cases are $\psi \equiv \Diamond\chi$, $\psi \equiv D\chi$.
- $\psi \equiv \Diamond\chi$. $\Rightarrow$: If $\mathbf{M}_\Sigma \models \Diamond\chi[v]$, then $\mathbf{M}_\Sigma \models \chi[u]$, for some u such that $R_\Sigma vu$. g is a R_Σ-homomorphism, so $R_0\bar{v}\bar{u}$. By the IH it follows that $\mathbf{N} \models \chi[\bar{u}]$, for some $\bar{u}$ such that $R_0\bar{v}\bar{u}$. And so $\mathbf{N} \models \Diamond\chi[\bar{v}]$.
$\Leftarrow$: Now suppose $\mathbf{N} \models \Diamond\chi[\bar{v}]$. Then for some $\bar{u} \in W_0$, such that $R_0\bar{v}\bar{u}$, we have $\mathbf{N} \models \chi[\bar{u}]$. By construction $R_\Sigma vu$ holds, and by the IH we have $\mathbf{M}_\Sigma \models \chi[u]$; so $\mathbf{M}_\Sigma \models \Diamond\chi[v]$.
- $\psi \equiv D\chi$. $\Rightarrow$: If $\mathbf{M}_\Sigma \models D\chi[v]$, then $\mathbf{M}_\Sigma \models \chi[u]$, for some $u \neq v$. By construction we have $\bar{v} \neq \bar{u}$, and so the IH yields $\mathbf{N} \models \chi[\bar{u}]$, for some $\bar{u} \neq \bar{v}$. Thus $\mathbf{N} \models D\chi[\bar{v}]$.
$\Leftarrow$: At this point it appears why we began the proof by extending Σ^- to a larger set Σ. Assume $\mathbf{N} \models D\chi[\bar{v}]$, then we can find some $\bar{u} \in W_0$ such that both $\bar{u} \neq \bar{v}$ and $\mathbf{N} \models \chi[\bar{u}]$ hold. Now, suppose that $\bar{u}$ corresponds to v. Then v has to be reflexive in $\mathbf{M}_\Sigma$, i.e. $R_\Sigma vv$. By the IH it follows that $\mathbf{M}_\Sigma \models \Diamond\chi[v]$. Since $D\chi \in \Sigma$, we have $\Diamond\chi \in \Sigma$. By appealing to the original model **M** we find:

$\mathbf{M} \models \Diamond\chi[\text{"}g^{-1}(v)\text{"}]$, by filtration,
$\Rightarrow$ $\mathbf{M} \models D\chi[\text{"}g^{-1}(v)\text{"}]$, **M** is well-ordered, and hence irreflexive,
$\Rightarrow$ $\mathbf{M}_\Sigma \models D\chi[v]$, by filtration.

Next suppose that $\bar{u}$ does not correspond to v, then $u \neq v$, and by the IH we immediately obtain $\mathbf{M}_\Sigma \models D\chi[v]$.
This completes the proof of both the claim and the theorem. ■

§2. L(◇,D) and L(F,P)

Using similar methods we prove the following theorem which compares the expressive power of L(◇,D) to that of L(F,P), by looking at $\mathbb{Q}$ and $\mathbb{R}$.

2.6. THEOREM. (i) $\langle \mathbb{Q}, < \rangle \not\equiv_{F,P} \langle \mathbb{R}, < \rangle$.
(ii) $\langle \mathbb{Q}, < \rangle \equiv_{\Diamond,D} \langle \mathbb{R}, < \rangle$.

PROOF. (i) Consider the formula $\chi := O(G\varphi \to PG\varphi) \to (G\varphi \to H\varphi)$, where $O\varphi$ abbreviates $H\varphi \wedge \varphi \wedge G\varphi$. We have $\langle \mathbb{Q}, < \rangle \not\models \chi$, but $\langle \mathbb{R}, < \rangle \models \chi$.
(ii) First, assume that $\langle \mathbb{R}, < \rangle \not\models \varphi$ for some $\varphi \in L(\Diamond, D)$. So, $\langle \mathbb{R}, <, V \rangle \models \neg\varphi[r]$ for some $r \in \mathbb{R}$, and some valuation V on $\mathbb{R}$. Using the ST-translation as defined in Definition 3.1, we find that $\langle \mathbb{R}, <, V \rangle \models \exists x\, ST(\neg\varphi)$. Now, $\exists x\, ST(\neg\varphi)$ is in L_1, so an application of the Downward Löwenheim-Skolem Theorem yields $\langle \mathbb{Q}, <, V' \rangle \models \exists x\, ST(\neg\varphi)$, where $V'(p) = V(p) \cap \mathbb{Q}$ for all p, so $\langle \mathbb{Q}, <, V' \rangle \models \neg\varphi[q]$ for some $q \in \mathbb{Q}$, and $\langle \mathbb{Q}, < \rangle \not\models \varphi$.
Conversely, assume $\langle \mathbb{Q}, < \rangle \not\models \varphi$. We construct a model $\mathbf{N}$ (= $\langle W, R, V \rangle$) of order type λ such that $\mathbf{N} \not\models \varphi$. The construction is analogous to that of the previous theorem.
(1). Let $\mathbf{M}_\Sigma$ be as in Theorem 2.5. This time R_Σ is not only transitive and linear, but successive as well, both to the right and to the left. It is easily verified that R_Σ has the latter property, by observing that g is a $<$-homomorphism.
Moreover, W_Σ does not contain adjacent degenerated clusters. For, suppose $a, b \in W_\Sigma$ are two adjacent irreflexive points, i.e. we have $R_\Sigma ab$ and b does not succeed any successor of a. Let $q, r \in \mathbb{Q}$ be such that $g(q) = a$, $g(r) = b$. As $\mathbb{Q}$ is linear, we have either $q < r$ or $r < q$. If the latter holds, we also have $R_\Sigma ba$, and by transitivity it follows that a, b are R_Σ-reflexive. So $q < r$. $\mathbb{Q}$'s being dense yields an $s \in \mathbb{Q}$ such that $q < s < r$, and, consequently both $R_\Sigma a g(s)$ and $R_\Sigma g(s) b$ hold. But then a, b can not be adjacent points.
(2). Next, we replace each cluster with an ordering that has either order type λ or $\mathbf{1}+\lambda$. To define a valuation on orderings that replace proper clusters we will use the following trick. Suppose that $\mathfrak{C} = \{ w_1 < \ldots < w_k \}$ is a proper cluster - where $<$ is an arbitrary linear ordering on $\mathfrak{C}$. Consider the interval $[0,1] \subseteq \mathbb{R}$ together with a strictly monotone increasing sequence $(a_n)_n$ such that $a_0 = 0$ and $\mathrm{Lim}\, a_n = 1$. Remove 0 and 1, replace $\mathfrak{C}$ by $(0,1)$ and define V on this copy of λ by repeating $w_1 < \ldots < w_k$ ω times on the sequence $(a_n)_n$, and giving $r \in (0,1) \setminus \{ a_n \}_n$ the same valuation w_1 has.
Now, $\mathbf{M}_\Sigma$ is a sequence of clusters that is successive to the left, so $\mathbf{M}_\Sigma$ 'begins' with a cluster $\mathfrak{C}$ that is either simple or proper. In both cases replace $\mathfrak{C}$ by an ordering of type λ and give all points w's valuation in case $\mathfrak{C} = \{ w \}$ is a simple cluster, or apply the above method in case $\mathfrak{C}$ is proper.
Consider the next cluster $\mathfrak{C}$:
- if $\mathfrak{C} = \{ w \}$ is degenerated, then $\mathfrak{D}$ must be succeeded by a nondegenerated cluster $\mathfrak{D}$, by our previous remarks. Replace $\mathfrak{C}$ and $\mathfrak{D}$ by an ordering of type $\mathbf{1}+\lambda$, give the initial point w's valuation, and treat λ as in the previous case;
- if $\mathfrak{C} = \{ w \}$ is simple, replace it by $\mathbf{1}+\lambda$ and give all new points w's valuation;
- if $\mathfrak{C}$ is a proper cluster, replace it by $\mathbf{1}+\lambda$, give the initial point the valuation of one of $\mathfrak{C}$'s points, and apply our special method to λ.

Notice that the final cluster is either simple or proper, since $\mathbf{M}_\Sigma$ is successive to the right. It follows that the resulting model **N** will have order type $\lambda + \mathbf{m} \cdot (\mathbf{1} + \lambda) = \lambda$, for some $m \in \mathbb{N}$.
(3). Similar to part 3 in Theorem 2.5. ■

REMARK. The proof that W_Σ does not contain adjacent degenerated clusters, is essentially Lemma 1.1 in Segerberg [1970].

Combining Proposition 2.3 and the Theorem we see that $\mathbb{Q}$ and $\mathbb{R}$ can not be distinguished by purely universal Π^1_1-sentences. Of course, we already knew this, since $\mathbb{Q} \equiv \mathbb{R}$ and since all purely universal Π^1_1-sentences are first order definable. However, the only proof of this last fact we know of, requires some heavy machinery, whereas the proofs given here are not too complicated. (Cf. van Benthem [1985], Cor. 3.13.)

Before proceeding to the next section, let us pause to state that when attention is restricted to linearly ordered frames, D is locally definable in L(F,P): on such frames $D\varphi$ is locally equivalent to $P\varphi \vee F\varphi$. However, the D-operator can not be defined globally in L(F,P) - not even on linear orderings. For, the frame $\langle \{ 0 \}, \{ \langle 0,0 \rangle \} \rangle$ is a (tense logical) p-morphic image of $\langle \mathbb{Z}, < \rangle$, and F,P-formulas are preserved under such p-morphic images, but the latter frame validates the formula DT, while the first one refutes it.

By the previous theorem P can not be defined globally in L(◇,D) - not even when we restrict attention to linear orderings.

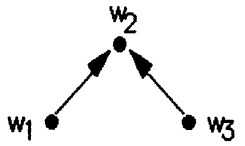
§3. L(F,P) and L(F,P,D)

We describe a class of frames on which every $\varphi \in L(F,P,D)$ is equivalent to a $\psi \in L(F,P)$. As usual, we need some definitions.

Consider the following bounded version of connectedness:

2.7. DEFINITION. (i) Let $i \in \mathbb{N}_{>0}$. A frame **F** is called *i-connected* if for any $w, v \in W$ with $w \neq v$, there exists a sequence of points $w_1, \ldots, w_k$ such that (1) $w_1 = w$, (2) $w_k = v$, (3) for each j ($1 \leq j < k$), either Rw_jw_{j+1} or $Rw_{j+1}w_j$, and (4) $k \leq i$.
(ii) **F** is called *<ω-connected* if **F** is i-connected for some $i \in \mathbb{N}_{>0}$.

Notice that the sequence $w_1, \ldots, w_k$ is not required to be a linearly ordered sequence in the sense that either $Rw_1w_2 \wedge \ldots \wedge Rw_{k-1}w_k$ or $Rw_kw_{k-1} \wedge \ldots Rw_2w_1$ holds. By the definition structures like



are also $<\omega$-connected.

We now define mappings $(\cdot)^*_i$ taking F,P,D-formulas to F,P-formulas.

2.8. DEFINITION. (i) Let $i \in \mathbb{N}_{>0}$. Then $\mathbf{OP}_i$ is the set of sequences of operators of length i, that are built up using only F and P.
(ii) Let $i \in \mathbb{N}_{>0}$. The *i-translation* $(\cdot)^*_i : L(F,P,D) \to L(F,P)$ is defined by:

$$\begin{aligned}
(p)^*_i &\longmapsto p \\
(\varphi \circ \psi)^*_i &\longmapsto (\varphi)^*_i \circ (\psi)^*_i, \text{ where } \circ \equiv \vee, \wedge \text{ or } \to, \\
(\neg\varphi)^*_i &\longmapsto \neg(\varphi)^*_i \\
(P\varphi)^*_i &\longmapsto P(\varphi)^*_i, \\
(F\varphi)^*_i &\longmapsto F(\varphi)^*_i, \\
(D\varphi)^*_i &\longmapsto [\bigvee_{O \in \mathbf{OP}_i} O(\varphi)^*_i] \vee [\bigvee_{O \in \mathbf{OP}_{(i-1)}} O(\varphi)^*_i] \vee \ldots \vee [F(\varphi)^*_i \vee P(\varphi)^*_i].
\end{aligned}$$

These translations are designed to help us remove occurrences of the D-operator in F,P,D-formulas that are evaluated on irreflexive, $<\omega$-connected frames. One might think that the last clause in Definition 2.8 is unnecessarily complicated for that purpose. E.g., why not take

$$(D\varphi)^*_i \longmapsto P^i(\varphi)^*_i \vee \ldots \vee P(\varphi)^*_i \vee F(\varphi)^*_i \vee \ldots \vee F^i(\varphi)^*_i$$

instead? The answer is simple: such a definition can not deal with the frame **F** we pictured above. Notice, for instance, that this frame is 2-connected and that $\mathbf{F} \models \mathrm{DFT}[w_1]$ holds, although $\mathbf{F} \not\models \mathrm{PPFT} \vee \mathrm{PFT} \vee \mathrm{FFT} \vee \mathrm{FFFT}[w_1]$. To be precise, the alternative clause only works when the sequences $w_1, \ldots, w_k$ in Definition 2.7 are required to be linearly ordered.

2.9. PROPOSITION. (i) *Let* $i \in \mathbb{N}_{>0}$, *and let* **F** *be an irreflexive, i-connected frame. If* $\varphi \in L(F,P,D)$, *then* $\mathbf{F} \models \varphi[w]$ *iff* $\mathbf{F} \models (\varphi)^*_i[w]$, *for all* $w \in W$.
(ii) $\mathbf{F} \models \varphi$ *iff* $\mathbf{F} \models (\varphi)^*_i$.

PROOF. (i) An induction on φ. Since i is constant in this proof, we write φ^* instead of $(\varphi)^*_i$. All cases but $\varphi \equiv D\psi$ are trivial. Suppose $\mathbf{F} \not\models D\psi[w]$. Then for some valuation V we have $\langle \mathbf{F}, V\rangle \not\models \varphi[w']$, for all $w' \succ w$. By the IH, $\langle \mathbf{F}, V\rangle \not\models \varphi^*[w']$, for all $w' \succ w$. Since **F** is irreflexive and i-connected, all $w' \succ w$ – and only those – can be reached using sequences of operators in $\mathbf{OP}_i \cup \mathbf{OP}_{(i-1)} \cup \ldots \cup \mathbf{OP}_1$. It follows that

$$\langle \mathbf{F}, V\rangle \not\models [\bigvee_{O \in \mathbf{OP}_i} O(\varphi^*)] \vee [\bigvee_{O \in \mathbf{OP}_{(i-1)}} O(\varphi^*)] \vee \ldots \vee [F(\varphi^*) \vee P(\varphi^*)][w],$$

that is, $\langle \mathbf{F}, V\rangle \not\models (D\varphi)^*[w]$, and so $\mathbf{F} \not\models (D\varphi)^*[w]$, as required. The converse is proved similarly.
(ii) Immediate from (i). ■

2.10. COROLLARY. *Let* $\mathbf{F}_1$, $\mathbf{F}_2$ *be two irreflexive,* $<\omega$*-connected frames. Then* $\mathbf{F}_1 \equiv_{F,P,D} \mathbf{F}_2$ *iff* $\mathbf{F}_1 \equiv_{F,P} \mathbf{F}_2$.

PROOF. One direction is obvious. To prove the other one, assume that $\mathbf{F}_1 \equiv_{F,P} \mathbf{F}_2$ and let $i \in \mathbb{N}_{>0}$ be minimal such that both $\mathbf{F}_1$ and $\mathbf{F}_2$ are i-connected. Choose $\varphi \in L(F,P,D)$ such that $\mathbf{F}_1 \not\models \varphi$. By the Proposition we have $\mathbf{F}_1 \not\models (\varphi)^*_i$. Since $\mathbf{F}_1 \equiv_{F,P} \mathbf{F}_2$, we also have $\mathbf{F}_2 \not\models (\varphi)^*_i$, and another application of the Proposition yields $\mathbf{F}_2 \not\models \varphi$. Similarly, if $\mathbf{F}_2 \not\models \varphi$, for some $\varphi \in L(F,P,D)$, then $\mathbf{F}_1 \not\models \varphi$. So, $\mathbf{F}_1 \equiv_{F,P,D} \mathbf{F}_2$. ■

The following Corollary shows that new results about the ordinary modal and tense logical formalisms can be obtained by studying the extended ones.

2.11. COROLLARY. *Fix* $i \in \mathbb{N}$. *On the class of irreflexive, i-connected frames every purely universal* Π^1_1*-sentence is* F,P*-definable.*

PROOF. Combine Corollary 2.10 and Proposition 2.3. ■

REMARK. It is obvious that Corollary 2.10 can be adapted to obtain a description of a class of frames on which every $\varphi \in L(\Diamond,D)$ is equivalent to some $\psi \in L(\Diamond)$: in the definition of an i-translation we would have to leave out the clause for formulas of the form $P\varphi$, and replace the clause for $D\varphi$ by

$$(D\varphi)^*_i \longmapsto F^i(\varphi)^*_i \vee F^{i-1}(\varphi)^*_i \vee \ldots \vee F(\varphi)^*_i.$$

The resulting analogue of Corollary 2.10 would then be: Let $\mathbf{F}_1$, $\mathbf{F}_2$ be two irreflexive, symmetric and $<\omega$-connected frames. Then $\mathbf{F}_1 \equiv_{\Diamond,D} \mathbf{F}_2$ iff $\mathbf{F}_1 \equiv_{\Diamond} \mathbf{F}_2$.

Corollary 2.10 seems to be a best possible result. As soon as we leave out irreflexivity and replace $<\omega$-connectedness by plain connectedness, the two notions of equivalence no longer coincide. We use the following frames to prove this:

F:

and

G:

Formally: $\mathbf{F} = \langle \mathbb{N}, R\rangle$, where $R = \{ \langle n,m\rangle \mid n = m \text{ or } m = n+1 \}$; and $\mathbf{G} = \langle \mathbb{Z}, S\rangle$, where $S = \{ \langle n,m\rangle \mid n = m \text{ or } m = n+1 \}$. So, both $\mathbf{F}$ and $\mathbf{G}$ are neither

irreflexive nor <ω-connected. We claim that $\mathbf{F} \equiv_{F,P} \mathbf{G}$, but $\mathbf{F} \not\equiv_{F,P,D} \mathbf{G}$. To prove this, we will need a definition and a proposition:

2.12. DEFINITION. (i) Let $\mathbf{H}$ (= $\langle W, T \rangle$) be a frame with $w \in W$. $S_n(\mathbf{H},w)$ is defined by

(1) $S_0(\mathbf{H},w) = \{ w \}$,

(2) $S_{n+1}(\mathbf{H},w) = S_n(\mathbf{H},w) \cup \{ v \in W \mid$ for some $u \in S_n(\mathbf{H},w)$ Tuv or $Tvu \}$.

(ii) The FP-*rank* $R_{FP}(\varphi)$ of an F,P-formula φ is defined by

(1) $R_{FP}(p) = 0$ for proposition letters p,

(2) $R_{FP}(\neg\varphi) = R_{FP}(\varphi)$,

(3) $R_{FP}(\varphi \wedge \psi) = \max(R_{FP}(\varphi), R_{FP}(\psi))$,

(4) $R_{FP}(F\varphi) = R_{FP}(P\varphi) = R_{FP}(\varphi) + 1$.

2.13. PROPOSITION. *Let* V, V' *be two valuations on* $\mathbf{G}$ *resp.* $\mathbf{F}$. *Suppose that for all* $w \in \mathbb{N}$ *and all* $n < w$ *we have* $V'(p) \cap S_n(\mathbf{F},w) = V(p) \cap S_n(\mathbf{G},w)$ *for every proposition letter* p. *If* $\varphi \in L(F,P)$ *and* $R_{FP}(\varphi) \leq n < w \in \mathbb{N}$, *then* $\langle \mathbf{G}, V \rangle \models \varphi[w]$ *iff* $\langle \mathbf{F}, V' \rangle \models \varphi[w]$.

PROOF. An induction on φ. Unravel the relevant definitions, and use the fact that $S_n(\mathbf{F},w) = S_n(\mathbf{G},w)$ for all $w \geq 0$, and $n < w$. ■

2.14. COROLLARY. $\mathbf{F} \equiv_{F,P} \mathbf{G}$.

PROOF. If $\mathbf{G} \models \varphi$, then $\mathbf{F} \models \varphi$, because $\mathbf{F}$ is a (tense logical) p-morphic image of $\mathbf{G}$, via the function $f : \mathbf{G} \to \mathbf{F}$ defined by: $f(a) = 0$, if $a \leq 0$, and $f(a) = a$ if $a > 0$. If $\mathbf{G} \not\models \varphi$, then $\langle \mathbf{G}, V \rangle \models \neg\varphi[w]$, for some $w \in \mathbb{Z}$, and some valuation V. We can assume that $w \geq R_{FP}(\varphi) + 1$, and consequently that $w \in \mathbb{N}$. By putting $V'(p) = \{ v \in \mathbb{N} \mid v \in V(p) \}$ for every proposition letter p, the Proposition yields $\mathbf{F} \not\models \varphi$. ■

On the other hand, one easily sees that $\mathbf{F} \not\equiv_{F,P,D} \mathbf{G}$, by examining the formula $Up \to DFp$.

§4. Categoricity

This section is devoted to showing that the notion of frame categoricity does make sense in L(◇,D) and L(F,P,D).

2.15. DEFINITION. A set $\mathfrak{T}$ of (multi-) modal formulas is called *(frame) categorical* if there is, up to isomorphism, only one frame validating $\mathfrak{T}$. $\mathfrak{T}$ is called *λ-categorical* if, up to isomorphism, $\mathfrak{T}$ has only one frame of power λ validating it.

In $L(\Diamond)$ (and $L(F,P)$) it is quite useless to count the number of (non-isomorphic) frames validating a single formula, or for that matter, a set of formulas. For, any such set Σ having one frame validating it, has arbitrarily many frames validating it:

2.16. PROPOSITION. *Let* $\mathbf{F} \models \Sigma$, *where* $\Sigma \subseteq L(\Diamond)$ *or* $\Sigma \subseteq L(F,P)$, *and let* I *be a set of indices. Then for each* $i \in I$, *there is a frame* $\mathbf{F}_i \models \Sigma$ *such that* $\mathbf{F}_i \not\cong \mathbf{F}_j$ *if* $i \neq j$.

PROOF. Assume that $\Sigma \subseteq L(\Diamond)$. The case that $\Sigma \subseteq L(F,P)$ is proved similarly. Put $\mathbf{F}_0 := \mathbf{F}$. If $i > 0$ and $i \in I$, define $\bar{\lambda}_i$ to be the smallest cardinal λ such that $\lambda > |\mathbf{F}_j|$ holds for all $j < i$. Put $\mathbf{F}_i := \oplus_\kappa \{ \mathbf{F} \mid \kappa < \bar{\lambda}_i \}$. A simple counting argument shows that $\mathbf{F}_i \not\cong \mathbf{F}_j$, if $i \neq j$. Furthermore, using the well-known preservation results for $L(\Diamond)$ (cf. Fact 1.7) it is easily verified that $\mathbf{F}_i \models \Sigma$, for each $i \in I$. ■

However, in $L(\Diamond,D)$ we do have categorical theories. Let $\mathbf{F}$ be a finite frame, having n elements. We claim that $\mathrm{Th}_{\Diamond,D}(\mathbf{F})$ is categorical. Suppose that $\mathbf{G} \models \mathrm{Th}_{\Diamond,D}(\mathbf{F})$, then $|\mathbf{G}| = n$, because by a result in the next section all finite cardinalities are definable in $L(\Diamond,D)$. Now, from Section 1 we know that the notions of isomorphism and $\Diamond,D$-equivalence coincide in the case of finite frames. It follows that $\mathbf{F} \cong \mathbf{G}$.

We now turn to the notion of ω-categoricity. This notion too is rather meaningless in $L(\Diamond)$:

2.17. PROPOSITION. *If* Σ *is a theory in* $L(\Diamond)$ *or* $L(F,P)$ *that is valid on some countably infinite frame, but invalid on every finite frame, then* Σ *is not* ω*-categorical.*

PROOF. Again, assume that $\Sigma \subseteq L(\Diamond)$. Let $\mathbf{F} \models \Sigma$, and $|\mathbf{F}| = \aleph_0$. If $\mathbf{F}$ is connected, then $\mathbf{F} \oplus \mathbf{F}$ validates Σ without being isomorphic to $\mathbf{F}$. If $\mathbf{F}$ is not connected, then let $w \in W$ and consider the subframe $\mathbf{F}_w$ of $\mathbf{F}$ generated by w. Then $\mathbf{F}_w$ validates Σ, so by assumption it has power $\aleph_0$; finally, it can not be isomorphic to $\mathbf{F}$ because it is connected. ■

Notice that the set Σ in the above proof must be invalid on every finite frame. For, if $\Sigma = \{ \Box\bot \}$ and $\mathbf{F}_1 = \langle \{ 0 \}, \emptyset \rangle$, then $\mathbf{F}_1 \models \Box\bot$ and $\mathbf{F}_\omega := \oplus_n \{ \mathbf{F}_1 \mid n \in \mathbb{N} \} \models \Box\bot$. Furthermore, for any frame $\mathbf{G}$, we have $\mathbf{G} \models \Box\bot$ iff $\mathbf{G} \models \forall xy \neg Rxy$. So, if $\mathbf{G}$ is a countably infinite frame such that $\mathbf{G} \models \Box\bot$, then $\mathbf{G} \cong \mathbf{F}_\omega$. Therefore, Σ is ω-categorical.

From the Proposition it follows e.g. that $\mathrm{Th}_\Diamond(\mathbb{Q})$ is not ω-categorical, and more generally, that the complete $\Diamond$-theory of an infinite connected frame is not ω-categorical. However, $\mathrm{Th}_{\Diamond,D}(\mathbb{Q})$ turns out to be ω-categorical, i.e. up to isomorphism $\mathrm{Th}_{\Diamond,D}(\mathbb{Q})$ is valid on exactly one countable frame, notably $\mathbb{Q}$.

2.18. PROPOSITION. *The complete* $\Diamond$,D- *theory of* $\mathbb{Q}$ *is* ω- *categorical.*

PROOF. Obviously it suffices to give $\Diamond$,D-formulas which are equivalent (on frames) to the axioms for the theory of dense linear orderings without endpoints.

(1) $\overline{\mathfrak{S}}(\ \Box\Box p \to \Box p,\ \forall xyz\ (x < y \wedge y < z \to x < z))$,
(2) $\overline{\mathfrak{S}}(\ Up \wedge Uq \to E(p \wedge q),\ \forall xyz\ (x < y \wedge y < x \to x = y))$,
(3) $\overline{\mathfrak{S}}(\ Up \to E(p \wedge \neg\Diamond p),\ \forall x\ (\neg x < x))$,
(4) $\overline{\mathfrak{S}}(\ Up \wedge Uq \to E(p \wedge \Diamond q) \vee E(q \wedge \Diamond p),\ \forall xy\ (x < y \vee y < x))$,
(5) $\overline{\mathfrak{S}}(\ \Box p \to \Box\Box p,\ \forall xy\exists z\ (x < y \to x < z \wedge z < y))$,
(6) $\overline{\mathfrak{S}}(\ DT,\ \exists xy\ (x \neq y))$,
(7) $\overline{\mathfrak{S}}(\ \Diamond T,\ \forall x\exists y\ (x < y))$,
(8) given (3), we have $\overline{\mathfrak{S}}(\ Up \to D\Diamond p,\ \forall x\exists y\ (y < x))$.

The proof that these equivalences do indeed hold is straightforward. ■

The F,P,D-theory of another well-known structure, notably $\mathbb{Z}$, turns out to be categorical. To see this, we repeat the following result from van Benthem [1983] section II.2.2:

2.19. THEOREM. $\mathbb{Z}$ *is* F,P- *definable on the class of connected strict partial orderings.*

From this result it follows that $\mathbb{Z}$ is F,P-definable on the class of all strict linear orderings. Since this class is defined by universal first order sentences, by Proposition 2.2 it is also definable in L(F,P,D). So to end up with $\mathbb{Z}$, first we only consider strict linear orderings, and among these we then single out $\mathbb{Z}$. By Proposition 2.2 and Theorem 2.19 this can be done inside L(F,P,D).

In short, we have proved:

2.20. THEOREM. $\mathbb{Z}$ *is F,P,D-definable.*

We immediately obtain:

2.21. COROLLARY. $Th_{F,P,D}(\mathbb{Z})$ *is categorical.*

§5. L(D)

We finish this chapter by proving some results on L(D). All pure D-formulas turn out to be first-order definable:

2.22. PROPOSITION. $L(D) \subseteq \mathfrak{M}_1 \cap \overline{\mathfrak{M}}_1$.

PROOF. Using the ST-translation as defined in the next chapter, such formulas can be translated into equivalent second-order formulas containing only monadic predicate variables. By a result in Ackermann [1954] these formulas are in turn equivalent to first-order formulas. ■

Although 'infinity' is not D-definable by the previous result, we do have

2.23. PROPOSITION. (Koymans) *All finite cardinalities are definable in* L(D).

PROOF. For all $n \in \mathbb{N}$, $|W| \leq n$ is defined by

$$\bigwedge_{1 \leq i \leq n+1} Up_i \rightarrow \bigvee_{1 \leq i < j \leq n+1} E(p_i \wedge p_j),$$

while $|W| > n-1$ is defined by $A \bigvee_{1 \leq i \leq n} p_i \rightarrow E \bigvee_{1 \leq i \leq n} (p_i \wedge Dp_i)$. ■

All first order formulas over identity can be defined as a Boolean combination of formulas expressing the existence of at least a certain number of elements. Since these formulas are definable in L(D) by the previous proof, it follows that on frames L(D) is equivalent with first order logic over $=$.

CHAPTER 3

MODEL THEORY

We characterize the L_1-formulas that are (locally) equivalent to a $\Diamond$,D-formula on models, and present three conditions on classes of models with a single distinguished point such that such a class is definable by means of a $\Diamond$,D-formula if and only if it satisfies these conditions.

§1. L_1-formulas having a $\Diamond$,D-equivalent on models

Ordinary modal formulas, when interpreted in models, are equivalent to a special kind of first order formulas. Adding the D-operator does not change this. We can simply add a clause in the translation ST for modal formulas. (For the sake of completeness we repeat the entire definition.)

3.1. DEFINITION. Let x be a fixed variable. Then

(i) $ST(p) = Px$,
(ii) $ST(\neg\psi) = \neg ST(\psi)$,
(iii) $ST(\psi \wedge \chi) = ST(\psi) \wedge ST(\chi)$,
(iv) $ST(\Diamond \psi) = \exists y\,(Rxy \wedge ST(\psi)[x:=y])$,
(v) $ST(D\psi) = \exists y\,(x \neq y \wedge ST(\psi)[x:=y])$,

where y is a variable not occurring in $ST(\psi)$.

Since the equivalences

$\mathbf{M} \models \varphi[w]$ iff $\mathbf{M} \models ST(\varphi)[w]$, and
$\mathbf{M} \models \varphi$ iff $\mathbf{M} \models \forall x ST(\varphi)$

hold, some well-known facts about L_1 become applicable for $L(\Diamond,D)$. For example, one has a Löwenheim-Skolem theorem for models, as well as a compactness theorem for $\models_m$, where $\Delta \models_m \varphi$ iff for all models $\mathbf{M}$ we have that $\mathbf{M} \models \Delta$ implies $\mathbf{M} \models \varphi$.

L_1-formulas of the form $ST(\varphi)$ for some $\varphi \in L(\Diamond,D)$ can be described independently in the following way:

3.2. DEFINITION. The set of *md-formulas* is the least set $\mathfrak{X}$ of L_1-formulas such that:

(i) $Px \in \mathfrak{X}$, for unary predicate letter P and all variables x,
(ii) if $\alpha \in \mathfrak{X}$, then $\neg\alpha \in \mathfrak{X}$,
(iii) if $\alpha, \beta \in \mathfrak{X}$, then $(\alpha \wedge \beta) \in \mathfrak{X}$,

(iv) if $\alpha \in \mathfrak{A}$ and x, y are distinct variables, then $\exists y (Rxy \wedge \alpha) \in \mathfrak{A}$,
(v) if $\alpha \in \mathfrak{A}$ and x, y are distinct variables, then $\exists y (x \neq y \wedge \alpha) \in \mathfrak{A}$.

To be precise, the translations $ST(\varphi)$ of $\Diamond$,D-formulas φ are md-formulas having exactly one free variable. To be even more precise, L_1-formulas of the form $ST(\varphi)$, for some φ, belong to the following subset of the set of md-formulas:

3.3. DEFINITION. The set of *Md-formulas* is the least set $\mathfrak{A}$ of L_1-formulas such that:
(i) $Px \in \mathfrak{A}$, for every unary predicate letter P and all variables x,
(ii) if $\alpha \in \mathfrak{A}$, then $\neg\alpha \in \mathfrak{A}$,
(iii) if $\alpha, \beta \in \mathfrak{A}$ have the same free variable, then $(\alpha \wedge \beta) \in \mathfrak{A}$,
(iv) if $\alpha \in \mathfrak{A}$, x, y are distinct variables, and y is α's free variable, then $\exists y (Rxy \wedge \alpha) \in \mathfrak{A}$,
(v) if $\alpha \in \mathfrak{A}$, x, y are distinct variables, and y is α's free variable, then $\exists y (x \neq y \wedge \alpha) \in \mathfrak{A}$.

3.4. LEMMA. *Every md-formula α is equivalent to a Boolean combination of Md-formulas, each with their free variables among those of α.*

PROOF. A simple generalization of van Benthem [1985], Lemma 3.4. ■

3.5. COROLLARY. *Every md-formula having one free variable is equivalent to an Md-formula.*

PROOF. A Boolean combination of Md-formulas having the same free variable is itself an Md-formula. ■

The first result of some importance in this chapter is a semantic characterization of the md-formulas in terms of invariance under p-relations. It generalizes a corresponding result for $L(\Diamond)$ in van Benthem [1985]. However, whereas the proof given there uses an elementary chain construction, the proof we present uses saturated models.

According to Corollary 3.5 the characterization will also be a characterization of the (translations of) $\Diamond$,D-formulas in L_1. As usual we need to state some definitions and facts first:

3.6. DEFINITION. (i) (Koymans) A binary relation Z is said to be a *p-relation* between two models $\mathbf{M}_1$ (= $\langle W_1, R_1, V_1 \rangle$) and $\mathbf{M}_2$ (= $\langle W_2, R_2, V_2 \rangle$), if
(1) if Zwv, then w, v verify the same proposition letters,
(2) if Zwv, and $w' \in W_1$ such that R_1ww', then $Zw'v'$ for some $v' \in W_2$ such that R_2vv',
(3) if Zwv, and $v' \in W_2$ such that R_2vv', then $Zw'v'$ for some $w' \in$

W_2 such that R_1ww',

(4) if $\mathcal{Z}wv$, and $w' \in W_1$ such that $w \neq w'$, then $\mathcal{Z}w'v'$ for some $v' \in W_2$ such that $v \neq v'$,

(5) if $\mathcal{Z}wv$, and $v' \in W_1$ such that $v \neq v'$, then $\mathcal{Z}w'v'$ for some $w' \in W_2$ such that $w \neq w'$,

(6) domain($\mathcal{Z}$) = W_1, range($\mathcal{Z}$) = W_2.

(ii) An L_1-formula $\alpha(x_1, ..., x_n)$ is *invariant for p-relations* if, for all models $\mathbf{M}_1$ and $\mathbf{M}_2$, all p-relations $\mathcal{Z}$ between $\mathbf{M}_1$ and $\mathbf{M}_2$, and all $w_1, ..., w_n \in W_1$, $w_1', ..., w_n' \in W_2$ such that $\mathcal{Z}w_1w_1', ..., \mathcal{Z}w_nw_n'$, we have $\mathbf{M}_1 \models \alpha[w_1, ..., w_n]$ iff $\mathbf{M}_2 \models \alpha[w_1', ..., w_n']$.

REMARK. If $\mathcal{Z}$ is a p-relation and $\mathcal{Z}wv$ holds, then either this is the only $\mathcal{Z}$-connection for w and v, or both w and v are $\mathcal{Z}$-related to at least two other points. So, $\mathcal{Z}$ may be split up in a bijective part where $w \in W_1$ has only one $\mathcal{Z}$-related $v \in W_2$ (and vice versa) and several clusters of $\mathcal{Z}$-related worlds such that each world in such a cluster is $\mathcal{Z}$-related to at least two worlds (of the other model) in that cluster.

3.7. FACTS. (i) *For all* $\diamond$,D- *formulas* φ, $\Pi_U\mathbf{M}_i \models \varphi[f_\approx]$ *iff* $\{ i \in I \mid \mathbf{M}_i \models \varphi[f(i)] \} \in U$.

(ii) *Let* $\mathbf{M}$ *be a structure, let* U *be an ultrafilter. Then* $\mathbf{M}$ *is isomorphic to an elementary submodel of the ultrapower* $\Pi_U\mathbf{M}$.

(iii) *Let* $\mathbf{L}$ *be any countable first-order language, let* U *be a countably incomplete ultrafilter over a set* I, *and let* $\{ \mathbf{M}_i \mid i \in I \}$ *be a collection of* $\mathbf{L}$- *structures. Then the ultraproduct* $\Pi_U\mathbf{M}_i$ *is* ω- *saturated.*

PROOF. (i): $\Pi_U\mathbf{M}_i \models \varphi[f_\approx] \Rightarrow \Pi_U\mathbf{M}_i \models ST(\varphi)[f_\approx]$

$\Rightarrow \{ i \in I \mid \Pi_U\mathbf{M}_i \models ST(\varphi)[f(i)] \} \in U$, the Theorem of Łoś,

$\Rightarrow \{ i \in I \mid \Pi_U\mathbf{M}_i \models \varphi[f(i)] \} \in U$.

(ii) & (iii): Chang & Keisler [1973] Cor. 4.1.13. resp. Thm. 6.1.1. ■

We can now prove the result we announced:

3.8. THEOREM. *An* L_1- *formula* α *containing at least one free variable* x *is equivalent to an md-formula iff* α *is invariant for p-relations.*

PROOF. A simple induction proves that every md-formula is invariant for p-relations.

Conversely: suppose α has this property, and suppose $FV(\alpha) = \{ x_1,...,x_n \}$. Define $md(\alpha) := \{ \beta \mid \beta$ is an md-formula, $\alpha \models \beta$, $FV(\beta) \subseteq FV(\alpha) \}$. We shall prove that $md(\alpha) \models \alpha$. By the compactness theorem it then follows that $\beta \models \alpha$ for some $\beta \in md(\alpha)$ such that $\models \beta \leftrightarrow \alpha$.

Assume $\mathbf{M}_0$ (= $\langle W_0, R_0, V_0 \rangle$) $\models md(\alpha)[w_1, ..., w_n]$. We need to prove that $\mathbf{M}_0 \models \alpha[w_1,...,w_n]$. Introduce new individual constants $\mathbf{w}_1, ..., \mathbf{w}_n$ to stand for the objects $w_1, ..., w_n$, and define $L^* = L_1 \cup \{ \mathbf{w}_1, ..., \mathbf{w}_n \}$. Expand $\mathbf{M}_0$ to an L^*-

model $\mathbf{M}_0^*$ by interpreting $\mathbf{w}_1$ as w_1, ..., $\mathbf{w}_n$ as w_n. In the remainder of the proof we use the following notation: if $\beta \in L_1$, then $\beta^* \equiv \beta[x_1:=\mathbf{w}_1, \ldots, x_n:=\mathbf{w}_n]$; and if $T \subseteq L_1$, then $T^* := \{ \beta^* \mid \beta \in T \}$.
Let $T_0 := \{ \beta \mid \mathbf{M}_0 \models \beta[w_1, \ldots, w_n]$, β is an md-formula, $FV(\beta) \subseteq FV(\alpha)\}$, and suppose $\{ \beta_0, \ldots, \beta_n \}^* = T^* \subseteq T_0^*$ to be finite. Then there exists an L^*-model $\mathbf{N}^*$ such that $\mathbf{N}^* \models \alpha^* \wedge \bigwedge T^*$. For, suppose such a model does not exist, then

$$\begin{aligned}
&\mathbf{N}^* \models \neg(\beta_0^* \wedge \ldots \wedge \beta_n^*), \text{ for every } L^*\text{-model } \mathbf{N}^* \models \alpha^*,\\
&\quad \Rightarrow \models \alpha^* \rightarrow \neg(\beta_0^* \wedge \ldots \wedge \beta_n^*),\\
&\quad \Rightarrow \neg(\beta_0 \wedge \ldots \wedge \beta_n) \in md(\alpha)\\
&\quad \Rightarrow \mathbf{M}_0 \models \neg(\beta_0 \wedge \ldots \wedge \beta_n)[w_1, \ldots, w_n], \text{ since } \mathbf{M}_0 \models md(\alpha)[w_1, \ldots, w_n].
\end{aligned}$$

Contradiction! By the compactness theorem we obtain an L^*-model $\mathbf{M}^* \models \alpha^* \wedge \bigwedge T_0^*$.
Now, let U be a countably incomplete ultrafilter over $\mathbb{N}$, and consider the ω-saturated ultrapowers

$$\Pi_U \mathbf{M}_0^* =: \langle W_1, R_1, w_{11}, \ldots, w_{1n}, V_1 \rangle$$

and

$$\Pi_U \mathbf{M}^* =: \langle W_2, R_2, w_{21}, \ldots, w_{2n}, V_2 \rangle.$$

By the Theorem of Łoś it follows that both $w_{11}, \ldots, w_{1n}$ and $w_{21}, \ldots, w_{2n}$ realize T_0, since in each ultrapower all factors realize T_0. The same argument yields $\Pi_U \mathbf{M}^* \models \alpha^*$.
Define a p-relation $\mathcal{Z} \subseteq W_1 \times W_2$ between the (L_1-reducts of) $\Pi_U \mathbf{M}_0^*$ and $\Pi_U \mathbf{M}^*$ by putting

$$\begin{aligned}
&\mathcal{Z}wv \equiv \text{for all } \Diamond,D\text{-formulas } \varphi:\\
&\qquad \langle W_1, R_1, V_1 \rangle \models \varphi[w] \Leftrightarrow \langle W_2, R_2, V_2 \rangle \models \varphi[v].
\end{aligned}$$

At this point it appears why we defined T_0: to make sure the p-relation $\mathcal{Z}$ can 'start' at the interpretations of $\mathbf{w}_1, \ldots, \mathbf{w}_n$: $\mathcal{Z}w_{11}w_{21}, \ldots, \mathcal{Z}w_{1n}w_{w2n}$. We have, for example, $\mathcal{Z}w_{11}w_{21}$:

$$\begin{aligned}
&\langle W_1, R_1, V_1 \rangle \models \varphi[w_{11}]\\
&\quad \Rightarrow \langle W_1, R_1, V_1 \rangle \models ST(\varphi)[w_{11}]\\
&\quad \Rightarrow \langle W_1, R_1, w_{11}, \ldots, w_{1n}, V_1 \rangle \models ST(\varphi)^*\\
&\quad \Rightarrow ST(\varphi)^* \in T_0^*, \text{ otherwise } \mathbf{M}_0 \models \neg ST(\varphi)[w_1], \text{ and so}\\
&\qquad\qquad \neg ST(\varphi)^* \in T_0^*, \text{ and } \Pi_U \mathbf{M}_0^* \models \neg ST(\varphi)^*,\\
&\qquad\qquad \text{i.e. } \langle W_1, R_1, V_1 \rangle \models \neg ST(\varphi)[w_{11}]\\
&\quad \Rightarrow \langle W_2, R_2, w_{21}, \ldots, w_{2n}, V_2 \rangle \models ST(\varphi)^*\\
&\quad \Rightarrow \langle W_2, R_2, V_2 \rangle \models \varphi[w_{21}];
\end{aligned}$$

the implication from left to right is proved similarly.
Let's verify that $\mathcal{Z}$ is indeed a p-relation by checking the conditions in

Definition 3.6:
(i) By definition.
(ii) Assume R_1ww' and $\mathcal{Z}wv$, with $w, w' \in W_1$ and $v \in W_2$. We have to prove that $\exists v' \in W_2$: $R_2vv' \wedge \mathcal{Z}w'v'$. Define Ψ to be $\{ \Diamond,D\text{-formulas } \varphi \mid \Pi_U\mathbf{M}^*_0 \models \varphi[w'] \}$. We claim that $ST(\Psi) \cup \{R\mathbf{v}y\}$ is finitely satisfiable in $(\Pi_U\mathbf{M}^*,\mathbf{v})$. Suppose for a moment it is not. Then

$$\begin{aligned}(\Pi_U\mathbf{M}^*,\mathbf{v}) &\models \forall y\,(R\mathbf{v}y \rightarrow \neg\Box\wedge ST(\Phi)), \text{ for a finite } \Phi \subseteq \Psi,\\ &\Rightarrow \Pi_U\mathbf{M}^* \models \forall y\,(Rxy \rightarrow \neg\Box\wedge ST(\Phi))[v],\\ &\Rightarrow \Pi_U\mathbf{M}^*_0 \models \forall y\,(Rxy \rightarrow \neg\Box\wedge ST(\Phi))[w], \text{ since } \mathcal{Z}wv,\end{aligned}$$

contradicting the fact that $\Pi_U\mathbf{M}^*_0 \models Rxy \wedge \Box\wedge\Psi[ww']$. Now, $(\Pi_U\mathbf{M}^*,\mathbf{v})$ is ω-saturated, because it is a finite expansion of an ω-saturated model, so for some $v' \in W_2$ we have $(\Pi_U\mathbf{M}^*,\mathbf{v}) \models \Box\wedge ST(\Psi) \wedge R\mathbf{v}x[v']$. But then we have $\mathcal{Z}w'v'$!
(iii) Similar to (ii).
(iv) Assume $w \prec w'$ and $\mathcal{Z}wv$, with $w, w' \in W_1$ and $v \in W_2$. We have to find some $v' \in W_2$, such that $v \prec v'$ and $\mathcal{Z}w'v'$. Define Ψ to be $\{ \Diamond,D\text{-formulas } \varphi \mid \Pi_U\mathbf{M}^*_0 \models \varphi[w'] \}$. Again, we claim that $ST(\Psi) \cup \{\mathbf{v} \prec y\}$ is finitely satisfiable in $(\Pi_U\mathbf{M}^*,\mathbf{v})$. Suppose it is not, then

$$\begin{aligned}(\Pi_U\mathbf{M}^*,\mathbf{v}) &\models \neg\exists y\,(\mathbf{v} \prec y \wedge \neg\Box\wedge ST(\Phi)), \text{ for some finite } \Phi \subseteq \Psi,\\ &\Rightarrow \Pi_U\mathbf{M}^* \models \neg\exists y\,(x \prec y \wedge \neg\Box\wedge ST(\Phi))[v],\\ &\Rightarrow \Pi_U\mathbf{M}^*_0 \models \neg\exists y\,(x \prec y \wedge \neg\Box\wedge ST(\Phi))[w], \text{ since } \mathcal{Z}wv,\end{aligned}$$

contradicting the fact that $\Pi_U\mathbf{M}^*_0 \models x \prec y \wedge \Box\wedge\Psi[ww']$. Finally, $(\Pi_U\mathbf{M}^*,\mathbf{v})$ is ω-saturated, and so we find a $v' \in W_2$ such that $(\Pi_U\mathbf{M}^*,\mathbf{v}) \models \Box\wedge ST(\Psi) \wedge \mathbf{v} \prec x[v']$. But then $\mathcal{Z}w'v'$ holds!
(v) Similar to (iv).
(vi) This is trivial: let $w' \in W_1$, we must find a $v' \in W_2$ such that $\mathcal{Z}w'v'$. If $w' = w_{11}$, we have $\mathcal{Z}w'w_{21}$. Otherwise, $w_{11} \prec w'$, $\mathcal{Z}w_{11}w_{21}$, and condition (iv) gives us the $v' \in W_2$ we are looking for. Hence, $domain(\mathcal{Z}) = W_1$. Of course, using (v) one proves that $range(\mathcal{Z}) = W_2$.
Now, α's invariance for p-relations yields $\Pi_U\mathbf{M}^*_0 \models \alpha^*$. According to Fact 3.7.(ii) we have $\mathbf{M}^*_0 \prec \Pi_U\mathbf{M}^*_0$, and so $\mathbf{M}^*_0 \models \alpha^*$, i.e. $\mathbf{M}_0 \models \alpha[w_1, \ldots, w_n]$. ■

§2. Definability of classes of models

In his Rodenburg [1986] Piet Rodenburg uses a proof similar to the one we gave for Theorem 3.8 to characterize the definable classes of models of intuitionistic propositional logic. A reading of this characterization led to the results in this section.

For the remainder of this section the basic notion of frame is taken to be $\langle \mathbf{F}, w\rangle$, with a distinguished world w (as in Kripke's original publications). Similarly, the basic notion of model is taken to be $\langle \mathbf{F}, w, V\rangle$. Our

definability result will concern classes of such models. In this context, preservation of a formula φ under an operation $\mathbb{O}$ on such models means: if $\mathbb{O}(\langle W, R, w, V\rangle) = \langle W', R', w', V'\rangle$ and $\langle W, R, w, V\rangle \models \varphi[w]$, then also $\langle W', R', w', V'\rangle \models \varphi[w']$.

We need the following definition:

3.9. DEFINITION. A class $\mathfrak{K}$ of first order structures for the first order language L is called an *elementary class* if there is a sentence $\alpha \in L$ such that $\mathfrak{K}$ is the class of all models of α.

The next lemma is the key to our definability result:

3.10. LEMMA. *Let $\mathfrak{K}$ be a class of first order structures* L. *Then $\mathfrak{K}$ is elementary class if and only if both $\mathfrak{K}$ and its complement are closed under ultraproducts and isomorphisms.*

PROOF. Chang & Keisler [1973] Corollary 6.1.16. ■

As a corollary to Theorem 3.8 we give a characterization of the definable classes of models.

3.11. THEOREM. *Let $\mathfrak{M}$ be a class of models. Then $\mathfrak{M} = \{ \mathbf{M} (= \langle W, R, w, V\rangle) \mid \mathbf{M} \models \varphi[w] \}$ for some $\varphi \in L(\Diamond,D)$ iff $\mathfrak{M}$ is closed under p-relations, ultraproducts, while the complement of $\mathfrak{M}$ is closed under ultraproducts.*

PROOF. Introduce an individual constant $\mathbf{w}$ to stand for the object w, and define $L^* = L_1 \cup \{ \mathbf{w} \}$. Obviously, if $\mathbf{M}$ is a model, it is also an L^*-model: one merely has to interpret $\mathbf{w}$ as $\mathbf{M}$'s distinguished point. In the remainder of the proof we use the following notation: if $\beta \in L_1$, then $\beta^* \equiv \beta[x:=\mathbf{w}]$.

If $\mathfrak{M} = \{ \mathbf{M} (= \langle W, R, w, V\rangle) \mid \mathbf{M} \models \varphi[w] \}$ for some $\varphi \in L(\Diamond,D)$, then $\mathfrak{M}$ is closed under p-relations and ultraproducts. The complement of $\mathfrak{M}$ is defined by $\{ \neg ST(\varphi)^* \}$, hence closed under ultraproducts by the theorem of Łoś.

For the other direction, suppose that $\mathfrak{M}$ and its complement satisfy the stated conditions. Since $\mathfrak{M}$ is closed under p-relations, it and its complement are closed under isomorphisms. Both $\mathfrak{M}$ and its complement may be viewed as L^*-models, so by Lemma 3.10 there is a sentence $\alpha^* \in L^*$ such that for all models $\mathbf{M}$ we have $\mathbf{M} \in \mathfrak{M}$ if and only if $\mathbf{M} \models \alpha^*$.
We may safely assume that $\mathbf{w}$ occurs in α^* - for if it doesn't we can consider the equivalent formula $(\alpha^* \wedge \mathbf{w} = \mathbf{w})$. Now $\mathfrak{M}$ is closed under p-relations, so α is invariant under p-relations: let $\mathcal{Z}$ be a p-relation between $\langle W, R, V\rangle$ and $\langle W', R', V'\rangle$ such that $\mathcal{Z}uv$, where $u \in W$ and $v \in W'$. We must prove that $\langle W, R, V\rangle \models \alpha[u]$ if and only if $\langle W', R', V'\rangle \models \alpha[v]$.

Now,

$$\begin{aligned}\langle W, R, V\rangle \models \alpha[u] &\Leftrightarrow \langle W, R, u, V\rangle \models \alpha^*,\\ &\Leftrightarrow \langle W, R, u, V\rangle \in \mathfrak{M},\\ &\Leftrightarrow \langle W', R', v, V'\rangle \in \mathfrak{M}, \text{ by closure under p-relations,}\\ &\Leftrightarrow \langle W', R', v, V'\rangle \models \alpha^*,\\ &\Leftrightarrow \langle W', R', V'\rangle \models \alpha[v].\end{aligned}$$

By Theorem 3.8 α is equivalent to an md-formula with the same free variables. Since α has only got one free variable, the equivalent md-formula must be an Md-formula by Corollary 3.5. Such formulas are translation of $\Diamond$,D-formulas, so α is equivalent to $ST(\varphi)$, for some $\varphi \in L(\Diamond,D)$. ■

CHAPTER 4

AXIOMATIC THEORIES

In this chapter we prove completeness theorems for several logics in $L(\Diamond,D)$. First we define the logic of linear orderings and prove it to be complete. After that, we present a logic that is the logic of both the dense linear orderings without endpoints and the orderings having type η. Next, the logic of $\mathbb{Z}$ is determined; we end this chapter by presenting two incomplete logics, the second of which is used to show that the completeness part of the well-known Sahlqvist theorem has no straightforward extension to $L(\Diamond,D)$.

First we repeat the definition of the logic $\mathbf{D_m}$:

4.1. DEFINITION. (Koymans) The logic $\mathbf{D_m}$ is obtained from the basic modal logic **K** by adding the axiom schemes A1 - A4,

A1. $\overline{D}(\varphi \to \psi) \to (\overline{D}\varphi \to \overline{D}\psi)$,

A2. $\varphi \to \overline{D}D\varphi$,

A3. $DD\varphi \to (\varphi \vee D\varphi)$,

A4. $\Diamond\varphi \to (\varphi \vee D\varphi)$,

as well as a 'necessitation'-rule for $\overline{D}$:

$\vdash_{\mathbf{D_m}} \varphi \Rightarrow \vdash_{\mathbf{D_m}} \overline{D}\varphi$.

$\mathbf{D_m}$ turns out to be the basic logic in $L(\Diamond,D)$:

4.2. THEOREM. (Koymans) *For all* $\Sigma \subseteq L(\Diamond,D)$, $\varphi \in L(\Diamond,D)$ *we have* $\Sigma \vdash_{\mathbf{D_m}} \varphi$ *iff* $\Sigma \models_m \varphi$.

In proving the implication from right to left in Theorem 4.2, one looks for a model verifying $\Sigma + \neg\varphi$. Firstly, one considers the Henkin-model, in which the D-operator is associated with the abstract 'accessibility-relation' R_D. This model turns out to 'verify' $\mathbf{D_m}+\Sigma+\neg\varphi$. Finally, one proves that R_D can be turned into real inequality.

Using similar methods we describe the logic determined by the class of linearly ordered frames.

§1. A logic strongly complete w.r.t. linear orderings

We define $\mathbf{D_{Lin}}$:

4.3. DEFINITION. $\mathbf{D_{Lin}}$ is the logic obtained from $\mathbf{D_m}$ by adding the axiom schemes A5 - A7:

A5. $\Diamond\varphi \to D\varphi$, (irreflexivity)
A6. $\Box\varphi \to \Box\Box\varphi$, (transitivity)
A7. $\varphi \to \Diamond\psi \vee \overline{D}(\psi \to \Diamond\varphi)$. (linearity)

The following Definition and Lemma are used in proving the completeness of $\mathbf{D}_{\mathrm{Lin}}$:

4.4. DEFINITION. Let $\mathbf{L}$ be a normal logic, and suppose that Δ, Γ are maximal $\mathbf{L}$-consistent sets. Then
$R_D\Delta\Gamma \equiv \forall\varphi\,(\varphi \in \Gamma \Rightarrow D\varphi \in \Delta)$, or equivalently $\forall\varphi\,(\overline{D}\varphi \in \Delta \Rightarrow \varphi \in \Gamma)$,
$R_\Diamond\Delta\Gamma \equiv \forall\varphi\,(\varphi \in \Gamma \Rightarrow \Diamond\varphi \in \Delta)$, or equivalently $\forall\varphi\,(\Box\varphi \in \Delta \Rightarrow \varphi \in \Gamma)$.

Some notation: if R is a binary relation then $\check{R}$ is the converse of R, i.e. $\check{R} = \{\langle x,y\rangle \mid \langle y,x\rangle \in R\}$.

4.5. LEMMA. *Let* $\mathbf{L}$ *be a normal logic. Then*
(i) *for all maximal* $\mathbf{L}$*-consistent* Σ, *if* $\Diamond\varphi \in \Sigma$, *then* $\varphi \in \Delta$ *for some maximal* $\mathbf{L}$*-consistent set* $\Delta\check{R}_\Diamond\Sigma$,
(ii) *for all maximal* $\mathbf{L}$*-consistent* Σ, *if* $D\varphi \in \Sigma$, *then* $\varphi \in \Delta$ *for some maximal* $\mathbf{L}$*-consistent set* $\Delta\check{R}_D\Sigma$,
(iii) $\forall\Delta\Gamma\,(R_D\Delta\Gamma \to R_D\Gamma\Delta)$,
(iv) $\forall\Delta\Gamma\Sigma\,(R_D\Delta\Gamma \wedge R_D\Gamma\Sigma \to R_D\Delta\Sigma \vee \Delta = \Sigma)$,
(v) *if* $\mathbf{L} \vdash$ A5, *then* $\forall\Delta\Gamma\,(R_\Diamond\Delta\Gamma \to R_D\Delta\Gamma)$,
(vi) *if* $\mathbf{L} \vdash$ A6, *then* $\forall\Delta\Gamma\Sigma\,(R_\Diamond\Delta\Gamma \wedge R_\Diamond\Gamma\Sigma \to R_\Diamond\Delta\Sigma)$,
(vii) *if* $\mathbf{L} \vdash$ A7, *then* $\forall\Delta\Gamma\,(R_D\Delta\Gamma \to R_\Diamond\Delta\Gamma \vee R_\Diamond\Gamma\Delta)$.

PROOF. (iii) follows from A2, (iv) from A3. (v) - (vii) follow from A5 - A7 respectively. ■

If R_D can indeed be regarded as real inequality in the, if necessary reshaped Henkin-model for $\mathbf{D}_{\mathrm{Lin}}$, then this model has to be a strict linear ordering by parts (v)-(vii) of the previous lemma.

We can now prove the completeness theorem for $\mathbf{D}_{\mathrm{Lin}}$.

4.6. THEOREM. $\mathbf{D}_{\mathrm{Lin}}$ *is strongly complete with respect to linear orderings.*

PROOF. We prove that $\Sigma \vdash \varphi$ iff for any linearly ordered model $\mathbf{M}$, if $\mathbf{M} \models \Sigma$, then $\mathbf{M} \models \varphi$.
'$\Rightarrow$': As always, the proof is by induction on the length of derivations.
'$\Leftarrow$': For future reference we split the proof in two parts.
(1). Assume $\Sigma \nvdash \varphi$, and let Σ_0 be any maximal consistent superset of $\Sigma + \neg\varphi$. For each maximal consistent set Δ_t, choose a unique name t. Next, define a model $\mathbf{M}_0 := \langle W_0, R_\Diamond, R_D, V\rangle$ by putting

$W_0 = \{ t \mid \Delta_t$ is a maximal consistent set $\}$,
$R_D vw$, just in case $R_D \Delta_v \Delta_w$,
$R_\Diamond vw$, just in case $R_\Diamond \Delta_v \Delta_w$,
$V(p) = \{ v \mid p \in \Delta_v \}$, for all proposition letters p.

Then, for all φ, all $v \in \mathbf{M}_0$: $\varphi \in \Delta_v \Leftrightarrow \mathbf{M}_0 \models \varphi[v]$, and so $\mathbf{M}_0 \models \mathbb{A}\Sigma_0 \wedge \neg\varphi[w_0]$, where w_0 is the name for Σ_0.

Now, apply the Generation Theorem to R_D (cf. Fact 1.7 or van Benthem [1985], Lemma 2.11) to obtain a submodel $\mathbf{M}_1$ of $\mathbf{M}_0$ such that $w_0 \in \mathbf{M}_1$ and such that $\mathbf{M}_1$ is closed under R_D. By Lemma 4.5. (v) $\mathbf{M}_1$ is also closed under $R_\Diamond$. Clearly, parts (iii) and (iv) of the same Lemma ensure that R_D holds between any two different points in $\mathbf{M}_1$.
(2). We are not done yet. $\mathbf{M}_1$ might contain R_D-reflexive points. To get a model with real inequality we proceed as follows: let v be an R_D-reflexive point. Since R_D-reflexivity implies $R_\Diamond$-reflexivity by Lemma 4.5.(v), we cannot simply remove the R_D-loop in v. Instead, replace v by a copy $\mathbb{N}(v)$ of $\mathbb{N}$ with its standard ordering, with real inequality, and with v's valuation everywhere. New points $v_n \in \mathbb{N}(v)$ are to be related to the old points u as follows:

$R_D u v_n$, if $R_D uv$ and $u \neq v$;
$R_D v_n u$, if $R_D vu$ and $u \neq v$;
$R_\Diamond u v_n$, if $R_\Diamond uv$ and $u \neq v$;
$R_\Diamond v_n u$, if $R_\Diamond vu$ and $u \neq v$.

Repeat this procedure for all R_D-reflexive, $R_\Diamond$-reflexive points, and let $\mathbf{M}_2$ be the resulting model. (So now, two different points may have the same maximal consistent set associated with them: that's why we started out by taking names for the maximal consistent sets - instead of these sets themselves - as the universe of $\mathbf{M}_0$.)

Now $\mathbf{M}_2$ is a standard model, i.e. a model in which R_D is real inequality. So we are done once we have proved the following:

CLAIM. *For any* φ, *and any* $u \in \mathbf{M}_1$, *we have* $\mathbf{M}_1 \models \varphi[u]$ *iff* $\mathbf{M}_2 \models \varphi[u_n]$, *where* $u_n = u$, *if* u *already was* R_D- *irreflexive in* $\mathbf{M}_1$, *and* $u_n \in \mathbb{N}(u)$ *otherwise*.
PROOF (of the claim). This is an induction on the complexity of φ. The cases $\varphi \equiv p$, $\neg\psi$, $\psi \wedge \chi$ are straightforward.
- $\varphi \equiv \Diamond\psi$. First observe that if $R_\Diamond uz$ holds in $\mathbf{M}_1$, we have $R_\Diamond u_n z_n$ in $\mathbf{M}_2$. Then:

$\mathbf{M}_1 \models \Diamond\psi[u] \Rightarrow \mathbf{M}_1 \models \psi[z]$, for some $z \in \mathbf{M}_1$ such that $R_\Diamond uz$,
$\Rightarrow \mathbf{M}_2 \models \psi[z_n]$, for some $z_n \in \mathbf{M}_2$, by the IH,
$\Rightarrow \mathbf{M}_2 \models \Diamond\psi[u_n]$, since by our remark $R_\Diamond u_n z_n$ holds in $\mathbf{M}_2$.

Conversely,

$$\mathbf{M}_2 \models \Diamond\psi[u_n] \Rightarrow \mathbf{M}_2 \models \psi[z], \text{ for some } z \in \mathbf{M}_2 \text{ such that } R_\Diamond u_n z,$$

Now, take any $z' \in \mathbf{M}_1$ such that $(z')_n = z$; in $\mathbf{M}_1$ $R_\Diamond uz'$ holds, so

$$\Rightarrow \mathbf{M}_1 \models \psi[z'], \text{ by the IH,}$$
$$\Rightarrow \mathbf{M}_1 \models \Diamond\psi[u], \text{ since } R_\Diamond uz' \text{ holds in } \mathbf{M}_1.$$

– $\varphi \equiv D\psi$. Similarly.

This completes the proof of both the claim and the theorem. ■

§2. A strongly complete logic for $\mathbb{Q}$

As we announced in the introduction, the $\Diamond$,D-logic determined by $\mathbb{Q}$ coincides with the $\Diamond$,D-logic determined by the class of all dense linear orderings without endpoints. To fix the logic of this class, we employ the method used in the proof of Theorem 4.6. To determine the logic of $\mathbb{Q}$, we adapt the method used in de Jongh, Veltman and Verbrugge [1988] to L($\Diamond$,D).

Before moving on to the completeness proofs, we have to define the logic we want to prove complete:

4.7. DEFINITION. $\mathbf{D}_\eta$ is the logic obtained from $\mathbf{D}_{Lin}$ by adding the axiom schemes A8 – A10:

A8. $\Diamond\top$, (successiveness to the right)
A9. $\varphi \to D\Diamond\varphi$, (successiveness to the left)
A10. $\Box\Box\varphi \to \Box\varphi$. (denseness)

Of course, the F,P,D-logic for $\mathbb{Q}$ would contain PT instead of A9. However, as the following Lemma proves, we can simulate the P-operator just enough in L($\Diamond$,D).

4.8. LEMMA. *Let* $\mathbf{L}$ *be a normal logic, and suppose that* Δ, Γ, Σ *range over maximal* $\mathbf{L}$*-consistent sets. Then*

(i) *if* $\mathbf{L} \vdash$ A8, *then* $\forall\Delta\exists\Gamma\, R_\Diamond\Delta\Gamma$,
(ii) *if* $\mathbf{L} \vdash$ A9, *then* $\forall\Delta\exists\Gamma\,(R_D\Delta\Gamma \wedge R_\Diamond\Gamma\Delta)$,
(iii) *if* $\mathbf{L} \vdash$ A10, *then* $\forall\Delta\Gamma\exists\Sigma\,(R_\Diamond\Delta\Gamma \to R_\Diamond\Delta\Sigma \wedge R_\Diamond\Sigma\Gamma)$.

PROOF. Since parts (ii) and (iii) are more or less non-trivial, we prove both of them.

(ii) Obviously, it suffices to prove $\{\psi \mid \bar{D}\psi \in \Delta\} \cup \{\Diamond\chi \mid \chi \in \Delta\}$ consistent. Assume the contrary, then we have (omitting the subscript $\mathbf{L}$ in $\vdash_{\mathbf{L}}$)

$\psi_1, \ldots, \psi_n, \Diamond\chi_1, \ldots, \Diamond\chi_m \vdash \bot$, for some $\overline{D}\psi_1, \ldots, \overline{D}\psi_n, \chi_1, \ldots, \chi_m \in \Delta$
$\Rightarrow \vdash \bigwedge\psi_i \rightarrow \neg(\bigwedge\Diamond\chi_j)$,
$\Rightarrow \vdash \overline{D}(\bigwedge\psi_i \rightarrow \neg(\bigwedge\Diamond\chi_j))$, by $\overline{D}$-necessitation,
$\Rightarrow \vdash \bigwedge\overline{D}\psi_i \rightarrow \overline{D}\neg(\bigwedge\Diamond\chi_j)$, by A1 and $\vdash \overline{D}\bigwedge\psi_i \rightarrow \bigwedge\overline{D}\psi_i$. (*)

But, for any γ, δ,

$\vdash \neg(\Diamond\gamma \wedge \Diamond\delta) \rightarrow \neg\Diamond(\gamma \wedge \delta)$,
$\Rightarrow \vdash \overline{D}\neg(\Diamond\gamma \wedge \Diamond\delta) \rightarrow \overline{D}\neg\Diamond(\gamma \wedge \delta)$, by $\overline{D}$-necessitation.

Applying this result to (*), we get

$\vdash \bigwedge\overline{D}\psi_i \rightarrow \overline{D}\neg\Diamond(\bigwedge\chi_j)$,
$\Rightarrow \overline{D}\neg\Diamond(\bigwedge\chi_j) \in \Delta$, since $\bigwedge\overline{D}\psi_i \in \Delta$,
$\Rightarrow \neg D\Diamond(\bigwedge\chi_j) \in \Delta$, by definition of $\overline{D}$,
$\Rightarrow \neg(\bigwedge\chi_j) \in \Delta$, by A9,
$\Rightarrow \bigwedge\chi_j \notin \Delta$. Contradiction.

(iii) Suppose $R_\Diamond vw$. Again, we only have to show that $\{ \psi \mid \Box\psi \in \Delta \} \cup \{ \Diamond\chi \mid \chi \in \Gamma \} \nvdash \bot$. Note:

$\psi_1, \ldots, \psi_n, \Diamond\chi_1, \ldots, \Diamond\chi_m \vdash \bot$, for some $\Box\psi_1, \ldots, \Box\psi_n \in \Delta$,
and $\chi_1, \ldots, \chi_m \in \Gamma$,
$\Rightarrow \bigwedge\psi_i \vdash \neg(\bigwedge\Diamond\chi_j)$,
$\Rightarrow \bigwedge\psi_i \vdash \bigvee\Box\neg\chi_j$,
$\Rightarrow \bigwedge\psi_i \vdash \Box\bigvee\neg\chi_j$,
$\Rightarrow \bigwedge\Box\psi_i \vdash \Box\Box\bigvee\neg\chi_j$,
$\Rightarrow \bigwedge\Box\psi_i \vdash \Box\bigvee\neg\chi_j$, by A10.

Now, $\bigwedge\Box\psi_i \in \Delta$, so $\Box\bigvee\neg\chi_j \in \Delta$, and $\bigvee\neg\chi_j \in \Gamma$. Contradiction. ■

REMARK. Another proof of parts (i) and (iii) of the Lemma would run as follows: axioms A8 and A10 satisfy the conditions of the well-known Sahlqvist Theorem for $L(\Diamond)$ (cf. Section 5, and Sahlqvist [1975], Sambin [1980]). Among other things this result tells us that A8 and A10 are first order definable, and that their corresponding first order properties hold in the canonical model.

Here come the completeness theorems for $\mathbf{D}_\eta$:

4.9. THEOREM.
(i) $\mathbf{D}_\eta$ *is strongly complete with respect to dense linear orderings without endpoints.*
(ii) $\mathbf{D}_\eta$ *is strongly complete with respect to orderings of type* η.

PROOF. (i) Copy part (1) of the proof of Theorem 4.6, and modify its part (2) by replacing R_D-reflexive points u by a copy $\mathbb{Q}(u)$ of $\mathbb{Q}$ with its standard ordering, with real inequality, and with u's valuation everywhere, etcetera.
(ii) There's an uninteresting proof which runs as follows: if $\Delta \nvdash \varphi$ in $\mathbf{D}_\eta$ then (i) yields a dense linearly ordered model without endpoints in which $\Delta+\neg\varphi$ holds at some point. Using the Downward Löwenheim-Skolem Theorem we can take a suitable countable elementary submodel in which $\Delta + \neg\varphi$ still holds at some point. By Cantor's Theorem this model has to be isomorphic to $\mathbb{Q}$.

Here is a more interesting proof which uses the methods of de Jongh et al. [1988]:
'$\Rightarrow$': As usual, proving soundness is left to the reader.
'$\Leftarrow$': Let Δ be a maximal $\mathbf{D}_\eta$-consistent set. Of course, it suffices to define a countable dense linear ordering $\langle \mathfrak{T}, < \rangle$ without endpoints, and a valuation V on $\langle \mathfrak{T}, < \rangle$, such that for some $t \in \mathfrak{T}$, $t \in V(\varphi)$ iff $\varphi \in \Delta$. More precise, we construct such an ordering and associate a maximal $\mathbf{D}_\eta$-consistent set Γ_t with every $t \in \mathfrak{T}$, where

(a) there is a $t \in \mathfrak{T}$ with $\Gamma_t = \Delta$,
(b) if $t < t'$, then $R_\Diamond \Gamma_t \Gamma_{t'}$,
(c) if $t \not< t'$, then $R_D \Gamma_t \Gamma_{t'}$,
(d) if $\Diamond\varphi \in \Gamma_t$, then $\varphi \in \Gamma_{t'}$ for some $t' > t$,
(e) if $D\varphi \in \Gamma_t$, then $\varphi \in \Gamma_{t'}$ for some $t' \not< t$.

Next, putting $V(p) = \{ t \mid p \in \Gamma_t \}$, one easily verifies that for all φ, $t \in V(\varphi)$ iff $\varphi \in \Gamma_t$ - which completes the proof.

Let $\{ \psi_i \mid i \in \mathbb{N} \}$ enumerate all formulas of the forms $\Diamond\psi$ and $D\psi$ in such a way that each such formula occurs infinitely many times. For each $n \in \mathbb{N}$ we construct a finite structure $\langle T_n, < \rangle$ such that (b), (c) hold for the Γ_t's associated with the t's $\in T_n$. At even stages we will select Γ_t's in such a way that all of T_n's elements satisfy (d) for some specific $\Diamond\varphi$, or (e) for some specific $D\varphi$. By adding 'enough' points at the odd stages we make sure that the resulting ordering will be dense and without endpoints.

Stage -1.
$T_{-1} := \{ t_{-1} \}$, $\Gamma_{t_{-1}} := \Delta$.

Stage 2n.
Let $\Diamond\varphi$ be the n-th formula. We can distinguish several possibilities:
I. If $\Diamond\varphi \notin \Gamma_t$, for all $t \in T_{2n}$, put $T_{2n+1} := T_{2n}$.
II. If $\Diamond\varphi \in \Gamma_t$, for some $t \in T_{2n}$, and if for all such t there is a $t' \in T_{2n}$ such that $t < t'$ and $\varphi \in \Gamma_{t'}$, put $T_{2n+1} := T_{2n}$.
Let $T_{2n} = \{ t_0, \ldots, t_k \}$, where $t_0 < \ldots < t_k$. Assume that $\Diamond\varphi \in \Gamma_t$, for some $t \in T_{2n}$, while for no $t' > t$, $\varphi \in \Gamma_{t'}$. There are two possibilities:

$\Diamond\varphi \in \Gamma_{t_k}$ and $\Diamond\varphi \notin \Gamma_{t_{k'}}$.

III. $\Diamond\varphi \in \Gamma_{t_k}$. Let t be a new point, and put $t > t'$, for all $t' \in T_{2n}$. Since $\Diamond\varphi \in \Gamma_{t_k}$, Lemma 4.5 (i) yields a $\Gamma_t \check{R}_\Diamond \Gamma_{t_k}$ such that $\varphi \in \Gamma_t$. By R_D-transitivity and by parts (v) and (iii) of the same Lemma, we have that for all Γ_{t_i} such that $t_i \in T_{2n}$, both $R_D\Gamma_{t_i}\Gamma_t$ and $R_D\Gamma_t\Gamma_{t_i}$ hold. Extend T_{2n} in the obvious way to obtain T_{2n+1}.

IV. $\Diamond\varphi \notin \Gamma_{t_k}$. Let i be the largest index such that $\Diamond\varphi \in \Gamma_{t_i} \backslash \Gamma_{t_{i+1}}$. We may assume that for all $t_j > t_i$ we have $\neg\varphi \in \Gamma_{t_j}$. Another application of Lemma 4.5.(i) yields a $\Gamma\check{R}_\Diamond\Gamma_{t_i}$ such that $\varphi \in \Gamma$. We have

1. $R_\Diamond\Gamma_{t_i}\Gamma \Rightarrow R_D\Gamma_{t_i}\Gamma$ and $R_D\Gamma\Gamma_{t_i}$, by Lemma 4.5.(v) and (iii),
2. $R_\Diamond\Gamma_{t_i}\Gamma_{t_{i+1}} \Rightarrow R_D\Gamma_{t_i}\Gamma_{t_{i+1}}$ and $R_D\Gamma_{t_{i+1}}\Gamma_{t_i}$,
3. $\Diamond\varphi \notin \Gamma_{t_{i+1}}$ and $\varphi \in \Gamma \Rightarrow$ *not* $R_\Diamond\Gamma_{t_{i+1}}\Gamma$,
4. $\varphi \notin \Gamma_{t_{i+1}} \Rightarrow \Gamma \not\asymp \Gamma_{t_{i+1}}$,
   $\Rightarrow R_D\Gamma\Gamma_{t_{i+1}}$, by Lemma 4.5.(iv), 1. and 2.,
   $\Rightarrow R_\Diamond\Gamma_{t_{i+1}}\Gamma$ or $R_\Diamond\Gamma\Gamma_{t_{i+1}}$, by Lemma 4.5.(vii),
   $\Rightarrow R_\Diamond\Gamma\Gamma_{t_{i+1}}$, by 3.

Let t' be a new point in between t_i and t_{i+1}, and put $\Gamma_{t'} = \Gamma$. Then, if $s < t'$ we have $R_\Diamond\Gamma_s\Gamma_{t'}$, and if $t' < s$ we have $R_\Diamond\Gamma_{t'}\Gamma_s$. Finally, using $R_\Diamond$-transitivity (Lemma 4.5.(vi)) and parts (v) and (iii) of the same Lemma, we see that if $s \not\asymp t'$ then both $R_D\Gamma_{t'}\Gamma_s$ and its converse hold.

Next, suppose that $D\varphi$ is the n-th formula. Once again, we can distinguish several possibilities:

I. If $D\varphi \notin \Gamma_t$, for all $t \in T_{2n}$, put $T_{2n+1} := T_{2n}$.

II. If $D\varphi \in \Gamma_t$, for some $t \in T_{2n}$, and if for all such t there is a $t' \in T_{2n}$ such that $t \not\asymp t'$ and $\varphi \in \Gamma_{t'}$, put $T_{2n+1} := T_{2n}$.

III. Let $T_{2n} = \{ t_0, \ldots, t_k \}$, where $t_0 < \ldots < t_k$. Assume that $D\varphi \in \Gamma_{t_i}$, for some $t_i \in T_{2n}$, while for no $t \in T_{2n}$ we have both $t \not\asymp t_i$ and $\varphi \in \Gamma_t$.
Lemma 4.5.(ii) yields a Γ such that $R_D\Gamma_{t_i}\Gamma$ and $\varphi \in \Gamma$. By part (vii) of the same Lemma $R_D\Gamma_{t_i}\Gamma$ implies $R_\Diamond\Gamma_{t_i}\Gamma$ or $R_\Diamond\Gamma\Gamma_{t_i}$. Assume that $R_\Diamond\Gamma_{t_i}\Gamma$ holds. (The other case is similar.)
(*) If $i = k$, then t_i is maximal in T_{2n}. By $R_\Diamond$-transitivity it follows that $R_\Diamond\Gamma_s\Gamma$ holds for all $s \in T_{2n}$. So $R_D\Gamma_s\Gamma$ and $R_D\Gamma\Gamma_s$ hold for all such s, by Lemma 4.5.(v) and (iii). Now, let t be a new point, put $t > s$ for all s in T_{2n}, let $\Gamma_t = \Gamma$, and add t to T_{2n} to obtain T_{2n+1}. We are done.
If $i \not\asymp k$, then

$R_\Diamond\Gamma_{t_i}\Gamma_{t_{i+1}}$
$\Rightarrow R_D\Gamma_{t_i}\Gamma_{t_{i+1}}$, by Lemma 4.5.(v),
$\Rightarrow R_D\Gamma_{t_{i+1}}\Gamma_{t_i}$, by Lemma 4.5.(iii),
$\Rightarrow R_D\Gamma_{t_{i+1}}\Gamma$ or $\Gamma_{t_{i+1}} = \Gamma$, by Lemma 4.5.(iv) and the fact that $R_D\Gamma_{t_i}\Gamma$,
$\Rightarrow R_D\Gamma_{t_{i+1}}\Gamma$, since $\varphi \in \Gamma \backslash \Gamma_{t_{i+1}}$,
$\Rightarrow R_\Diamond\Gamma_{t_{i+1}}\Gamma$ or $R_\Diamond\Gamma\Gamma_{t_{i+1}}$, by Lemma 4.5.(vii).

If $R_\Diamond\Gamma_{t_{i+1}}\Gamma$ holds, go back to (*) and repeat the procedure with i+1 instead of i. Otherwise, $R_\Diamond\Gamma\Gamma_{t_{i+1}}$ holds and we are done: again, by $R_\Diamond$-transitivity it follows that Γ is R_D-related to Γ_s for all $s \in T_{2n}$. Now, let t be a new point, put $\Gamma_t = \Gamma$ and $s < t$ if $s \leq t_i$, and $t < s$ if $s \geq t_{i+1}$. Adding t to T_{2n} defines T_{2n+1}. (Since T_{2n} is finite this procedure will eventually decide where we have to put Γ among the Γ_t's.)

Stage 2n + 1.
This is where we make sure that < will be a dense linear ordering which has no first or last element. Let $T_{2n+1} = \{ t_0, \ldots, t_k \}$, where $t_0 < \ldots < t_k$. Lemma 4.8 parts (i) and (ii) yield an $R_\Diamond$-predecessor $\Gamma_{t'}$ for each Γ_t, $0 \leq t \leq k$, as well as an $R_\Diamond$-successor for Γ_{t_k}. The third part of that Lemma gives new points in between each pair of points. It is obvious how to obtain T_{2n+2}.

Finally, let $\mathfrak{T} := T_{-1} \cup (\bigcup_{n \in \mathbb{N}} T_n)$. Then $\mathfrak{T}$ is a countable dense linear ordering without endpoints satisfying (a)-(e). By Cantor's Theorem $\langle \mathfrak{T}, < \rangle$ has to have order type η. ■

REMARK. Notice that leaving out the odd stages in the preceding proof yields an alternative proof for the completeness of $\mathbf{D}_{\mathrm{Lin}}$.

§3. A complete logic for $\mathbb{Z}$

We can not hope to prove any logic strongly complete with respect to $\langle \mathbb{Z}, < \rangle$. This is easily concluded from the fact that compactness fails. A well-known example is provided by $\{ \Diamond\Box\neg p, \Diamond p, \Diamond^2 p, \Diamond^3 p, \ldots \}$.

This failure implies that completeness of the D-logic for $\mathbb{Z}$ will have to be proved differently than in §1 and §2. The method we will use is inspired by the method H.C. Doets used in his Doets [1987] to prove the standard tense logic for $\mathbb{Z}$ complete. Now, let us begin by defining $\mathbf{D}_\zeta$:

4.10. DEFINITION. The logic $\mathbf{D}_\zeta$ is obtained from $\mathbf{D}_{\mathrm{Lin}}$ by adding the axiom schemes A8, A9 as well as A11:
A11. $\Box(\Box\varphi \rightarrow \varphi) \rightarrow (\Diamond\Box\varphi \rightarrow \Box\varphi)$.

What is essentially D-logical about $\mathbf{D}_\zeta$? Compare $\mathbf{D}_\zeta$ with the well-known modal and tense logical logics for $\mathbb{Z}$:

Modal logic for $\mathbb{Z}$	Tense logic for $\mathbb{Z}$	D-logic for $\mathbb{Z}$
$\Box\varphi\rightarrow\Box\Box\varphi$	$G\varphi\rightarrow GG\varphi$	$\Box\varphi\rightarrow\Box\Box\varphi$
$\Box(\varphi\wedge\Box\varphi\rightarrow\psi) \vee \Box(\psi\wedge\Box\psi\rightarrow\varphi)$	$F\varphi \rightarrow G(\varphi \vee P\varphi \vee F\varphi)$	$\psi\wedge D\varphi \rightarrow \Diamond\psi \vee D(\varphi\rightarrow\Diamond\psi)$
--	$P\varphi \rightarrow H(\varphi \vee P\varphi \vee F\varphi)$	--

$\Diamond T$	FT	$\Diamond T$
--	PT	$\varphi \to D\Diamond\varphi$
$\Box(\Box\varphi \to \varphi) \to (\Diamond\Box\varphi \to \Box\varphi)$	$G(G\varphi \to \varphi) \to (FG\varphi \to G\varphi)$	$\Box(\Box\varphi \to \varphi) \to (\Diamond\Box\varphi \to \Box\varphi)$
--	$H(H\varphi \to \varphi) \to (PH\varphi \to H\varphi)$	--
--	--	$\varphi \to D\Diamond\varphi$

So $\mathbf{D}_\zeta$ does not differ too much from either the modal or tense logic for $\mathbb{Z}$ - although $\mathbf{D}_\zeta$ contains real irreflexivity and linearity, whereas the other two only contain right-linearity or left- and right-linearity.

Moreover, our proof that $\mathbf{D}_\zeta$ is indeed the D-logic of $\mathbb{Z}$ is not really different from the completeness proofs of the modal and tense logics for $\mathbb{Z}$. (Cf. Segerberg [1970] and Doets [1987].) We need the following simple, but very useful lemma:

4.11. LEMMA. *If* $\mathbf{M} \models \mathbf{D}_\zeta$, *and* $\varphi^M = \{ w \mid \mathbf{M} \models \varphi[w] \}$ *is non-empty and upward-bounded, then* φ^M *has a maximum.*

PROOF. Doets [1987]. ■

4.12. THEOREM. $\mathbf{D}_\zeta$ *is complete with respect to* $\mathbb{Z}$.

PROOF. Suppose $\Phi \nvdash_{\mathbf{D}_\zeta} \varphi$, where Φ is finite. Consider the normalized Henkin-model as defined in e.g. Theorem 4.6, where R_D-reflexive points are now replaced by a copy of $\mathbb{Z}$. By Lemmas 4.5 and 4.8 the resulting model $\mathbf{M}$ (= $\langle W, R, V \rangle$) is a strict linear ordering without endpoints.
Next, consider the $\Diamond$,D-collapse $\mathbf{M}_\Sigma$ - as defined in Ch.1, §2 - of $\mathbf{M}$ w.r.t. the set Σ of subformulas of $\Phi \cup \{ \neg\varphi \} \cup \{ \Diamond\psi \mid D\psi \in \Phi \cup \{ \neg\varphi \} \}$. As in the proof of e.g. Theorem 2.6 we find that R_Σ is transitive, linear and successive, both the right and to the left. Moreover, W_Σ consists of a finite linear sequence of clusters. We describe a method for getting rid of nondegenerated clusters that are neither initial nor final.

Let $\mathfrak{C}$ be such a cluster, and let $a \in W_\Sigma \backslash \mathfrak{C}$ be an immediate successor of $\mathfrak{C}$. Then for some $\Box\psi \in \Sigma$ we have $\mathbf{M}_\Sigma \models \Box\psi[a]$ and $\mathbf{M}_\Sigma \nvDash \Box\psi[c]$ for all $c \in \mathfrak{C}$. So, by filtration $(\neg\Box\psi)^M$ is upward bounded and non-empty, and according to Lemma 4.11 it has an R-maximum m. Notice that $g(m) \in \mathfrak{C}$: by the maximality of m we have Rxm for each x such that $g(x) \in \mathfrak{C}$, so $R_\Sigma cg(m)$ for all $c \in \mathfrak{C}$; furthermore, we can not have $R_\Sigma ag(m)$, since $\mathbf{M}_\Sigma \nvDash \Box\psi[m]$ and $\mathbf{M} \models \Box\psi[a]$, so $R_\Sigma g(m)a$ - but then g(m) must be an element of $\mathfrak{C}$, since g(m) is a successor of $\mathfrak{C}$, while a is an immediate successor of $\mathfrak{C}$.

Now choose a strict linear ordering $<'$ on $\mathfrak{C}$ such that g(m) is the $<'$-last element. Letting $\mathfrak{C}^- := \{ a \in W_\Sigma \backslash \mathfrak{C} \mid \forall b \in \mathfrak{C}\ R_\Sigma ab \}$ and $\mathfrak{C}^+ := \{ a \in W_\Sigma \backslash \mathfrak{C} \mid \forall b \in \mathfrak{C}\ R_\Sigma ba \}$, define $\mathbf{F}$ to be the frame

$$\langle \mathcal{C}^-, R_\Sigma\restriction(\mathcal{C}^-\times\mathcal{C}^-)\rangle + \langle \mathcal{C}, <'\rangle + \langle \mathcal{C}^+, R_\Sigma\restriction(\mathcal{C}^+\times\mathcal{C}^+)\rangle,$$

and $\mathbf{M}^*_\Sigma$ to be the model $\langle \mathbf{F}, V_\Sigma\rangle$. One can prove by induction that for all $\psi \in \Sigma$, and all $v \in W_\Sigma$, we have $\mathbf{M}_\Sigma \models \psi[v]$ iff $\mathbf{M}^*_\Sigma \models \psi[v]$. The only non-trivial case is when $\mathbf{M}_\Sigma \not\models \Box\psi[v]$, for some $v \in \mathcal{C}$. Then $\mathbf{M}_\Sigma \not\models \Box\psi[g(m)]$, and hence by filtration we find that $\mathbf{M} \not\models \Box\psi[m]$. So there exists an element k in $\mathbf{M}$ such that Rmk and $\mathbf{M} \not\models \psi[k]$. Obviously $g(k)$ succeeds $\mathcal{C}$. By filtration again we find that $\mathbf{M}_\Sigma \not\models \psi[g(k)]$, so by the IH also $\mathbf{M}^*_\Sigma \not\models \psi[g(k)]$. By the definition of $\mathbf{M}^*_\Sigma$ it follows that $\mathbf{M}^*_\Sigma \not\models \Box\psi[v]$.

Repeating this trick a finite number of times, we end up with a finite model $\mathbf{M}^*$ having the form:

where the first ellipse represents the initial cluster, and the second one represents the final cluster. By our previous remarks we have for all $\psi \in \Sigma$ and all $v \in W_\Sigma$, $\mathbf{M}_\Sigma \models \psi[v]$ iff $\mathbf{M}^* \models \psi[v]$. Now the initial cluster $\mathcal{C}_{in}$ gives rise to a linear ordering $w_1 <' \ldots <' w_n$ just like the other nondegenerated clusters did. ($\mathcal{C}_{in}$ is nondegenerated because $\mathbf{M}_\Sigma$ is successive to the left.) $\mathcal{C}_{in}$ is to be replaced by ω^*, and the valuation is to be expanded by repeating $w_1 <' \ldots <' w_n$ ω times on ω^*:

$$\ldots <' w_1 <' \ldots <' w_n <' w_1 <' \ldots <' w_n.$$

Let $\mathbf{N}$ be the resulting model. One proves by induction that for each $\psi \in \Sigma$ and each $w \in \mathbf{M}^* \models \psi[w]$ iff $\mathbf{N} \models \psi[\overline{w}]$, where $\overline{w}$ is a copy of w, if w has been multiplied, and w otherwise. The only non-trivial case is when $\mathbf{N} \models D\psi[\overline{w}]$, for some w in the initial cluster $\mathcal{C}_{in}$. So for some $v \neq w$ we find $\mathbf{N} \models \psi[v]$. The case that $v \neq \overline{x}$ for all $x \in \mathcal{C}_{in}$ is trivial, so assume that $v = \overline{x}$, for some $x \in \mathcal{C}_{in}$. Then

$\mathbf{M}^* \models \psi[x]$, by the IH,
$\Rightarrow \mathbf{M}_\Sigma \models \psi[x]$,
$\Rightarrow \mathbf{M}_\Sigma \models \Diamond\psi[x]$, since R_Σ is reflexive,
$\Rightarrow \mathbf{M} \models \Diamond\psi[k]$, for some k such that $g(k) = x$, by filtration, (notice that by definition $\Diamond\psi \in \Sigma$!),
$\Rightarrow \mathbf{M} \models D\psi[k]$, because $\mathbf{M} \models \Diamond\psi \rightarrow D\psi$,
$\Rightarrow \mathbf{M}_\Sigma \models D\psi[x]$, by filtration,
$\Rightarrow \mathbf{M}^* \models D\psi[x]$,
$\Rightarrow \mathbf{M}^* \models \psi[y]$, for some $y \neq x$.

So we find two different points (x and y) at which ψ holds - at least one of these must be different from w. Consequently, $\mathbf{M}^* \models D\psi[w]$.

Similarly, the final cluster $\mathcal{C} = \{ w_1 <' \ldots <' w_n \}$ - where $<'$ is an

arbitrary linear ordering on $\mathfrak{C}$ - is to be replaced by a copy of ω, on which $w_1 <' \ldots <' w_n$ is repeated ω times.

It is easily verified that the resulting model $\mathbf{N}^*$ is isomorphic to $\langle \mathbb{Z}, < \rangle$, and that $\Phi \cup \{ \neg\varphi \}$ holds at some point in $\mathbf{N}^*$. ■

§4. A simple incomplete logic

Thomason [1972] gives an example of an incomplete logic, a simplified version of which is described in the sequel. We adapt this example to obtain an easy incompleteness result in L(◇,D).

4.13. DEFINITION. The logic **ID** is obtained from $\mathbf{D}_m$ by adding the axioms A5, A6 and A12, where

A5. $\Diamond\varphi \rightarrow D\varphi$,

A6. $\Box\varphi \rightarrow \Box\Box\varphi$,

A12. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$. (The so-called McKinsey Axiom.)

We need the following result:

4.14. LEMMA. *Let* $\mathbf{F} \models \mathbf{ID}$, *then*

(i) R *is irreflexive,*

(ii) R *is transitive,*

(iii) $\mathbf{F} \models \forall x \exists y\, (Rxy \wedge \forall z\, (Ryz \rightarrow z = y))$.

PROOF. (i) and (ii) are straightforward. (iii) is Lemma 7.2 in van Benthem [1985]. ■

In fact A5 defines irreflexivity, A6 defines transitivity, and given A6, A12 is equivalent to the condition mentioned in Lemma 4.14.(iii). So all **ID**-axioms are first order definable. This means that **ID** is a much simpler example of an incomplete logic then Thomason's incomplete logic, which, in its simplified version, consists of the above McKinsey and the Löb Axiom $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$. It is well-known that the Löb axiom is outside of $\mathfrak{M}1$: it defines transitivity plus well-foundedness of the converse relation. (Cf. van Benthem [1985].)

One proves that the logic consisting of these axioms is incomplete by proving (a) that the McKinsey axiom forces the existence of an R-irreflexive point in frames validating it, while the Löb axiom forbids the existence of such points, and (b) that nonetheless, this logic is consistent. Using a similar method we will show that **ID** is incomplete.

So, first we prove that there are no frames validating **ID**:

4.15. LEMMA. $\{ \mathbf{F} \mid \mathbf{F} \models \mathbf{ID} \} = \emptyset$.

PROOF. Suppose that $\mathbf{F} \models \mathbf{ID}$, and let $w \in W$. By part (iii) in Lemma 4.14, there exist w_1, w_2 such that Rww_1, Rw_1w_2 and Rw_2w_2. However, by part (i) in the Lemma w_2 must be an irreflexive point. ■

4.16. LEMMA. **ID** *is consistent.*

PROOF. Let $\mathcal{W}$ be the set of finite and cofinite subsets of $\mathbb{N}$. We claim that the general frame $\langle \mathbb{N}, <, \mathcal{W} \rangle$ validates **ID**. It is easily verified that all the closure conditions of Definition 1.16 are satisfied. Both the transitivity and irreflexivity axiom are valid already on $\langle \mathbb{N}, < \rangle$, and so on $\langle \mathbb{N}, <, \mathcal{W} \rangle$. Since all valuations have to take their values in $\mathcal{W}$, it follows that for any formula φ and any valuation V, we have that either $V(\varphi)$ or $V(\neg\varphi)$ contains an interval $[m,\rightarrow)$ for some m. From this it follows that for all $n \in \mathbb{N}$ we have $\langle \mathbb{N}, <, \mathcal{W} \rangle \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi[n]$. ■

4.17. THEOREM. **ID** *is incomplete.*

PROOF. If **ID** were complete, it would be inconsistent by Lemma 4.15 - contradicting the previous Lemma. ■

What this incompleteness result shows is that the minimal $\Diamond$,D-logic $\mathbf{D}_m$ is too weak to produce all valid inferences in $L(\Diamond,D)$. Of course, there may be stronger 'base logics': in the context of incompleteness phenomena in $L(\Diamond)$ van Benthem [1979] considers *weak second order logic* as a particular example. This deductive system contains some first order base complete with respect to modus ponens, similar axioms for the second order quantifiers, and the following form of 'first order instantiation' for first order formulas ψ:

$$\forall P\varphi \rightarrow \varphi[P:=\psi].$$

Deducibility in this system will be denoted by $\vdash_2$. Let $\varphi \in L(\Diamond,D)$ contain the proposition letters $p_1, \ldots, p_n$. Then the universal closure of the standard translation $\forall ST(\varphi)$ is $\forall P_1 \ldots \forall P_n\, ST(\varphi)$. Through the second order transcription $\varphi \longmapsto \forall ST(\varphi)$, weak second order logic may be used as a modal base logic. We claim that $\bot$ is still not derivable from **ID** in weak second order logic:

4.18. PROPOSITION. $\mathbf{ID} \nvdash_2 \bot$.

PROOF. Consider the general frame $\langle \mathbb{N}, <, \mathcal{W} \rangle$ as defined in the proof of Lemma 4.16. By Lemma 9.16 in van Benthem [1985] the only L_0-definable subsets of $\mathbb{N}$ are the finite and cofinite ones. So by definition $\mathcal{W}$ is closed

under L_0-definability.

Following van Benthem [1979] we define the following notion of 'weak second order consequence': let $\Sigma \cup \{ \varphi \}$ be a set of formulas in the second order language with one binary first order predicate constant R and unary predicate variables; then $\Sigma \models_2 \varphi$ iff

for all general frames $\langle \mathbf{F}, \mathcal{W} \rangle$ satisfying
(i) $\mathcal{W}$ is closed under L_0-definability, and
(ii) $\langle \mathbf{F}, \mathcal{W} \rangle \models \Sigma[f]$, where f is an assignment of points in W to individual variables, and of sets of points in $\mathcal{W}$ to (unary) predicate variables,
we have $\langle \mathbf{F}, \mathcal{W} \rangle \models \varphi[f]$.

The first condition ensures that $\forall P\varphi \rightarrow \varphi[P:=\psi]$ will be true in $\langle \mathbb{N}, <, \mathcal{W} \rangle$ under any assignment. An easy induction on the length of derivations shows that $\Sigma \vdash_2 \varphi$ implies $\Sigma \models_2 \varphi$. Finally, an application of this result to the (second order translations) of the **ID**-axioms proves the Proposition. ■

So, having weak second order logic as our base logic does not safeguard us from incompleteness phenomena.

§5. An even simpler incomplete logic

We prove (i) that the obvious extension to $L(\Diamond,D)$ of the completeness part of the Sahlqvist Theorem does not hold, and (ii) that not every logic $\mathbf{L} \subseteq L(\Diamond,D)$ that has the finite model property is complete. We will prove claims (i) and (ii) using an extremely simple incomplete logic.

Here it is:

4.19. DEFINITION. The logic $\mathbf{D_m}$A13 is obtained from $\mathbf{D_m}$ by adding axiom scheme A13:
A13: $\varphi \rightarrow D\varphi$.

4.20. PROPOSITION. (i) *Let* Δ *range over maximal* $\mathbf{D_m}$A13*-consistent sets. Then* $\forall \Delta\, (R_D \Delta\Delta)$.
(ii) $\mathfrak{C}(\varphi \rightarrow D\varphi, \bot)$.
(iii) $\{ \mathbf{F} \mid \mathbf{F} \models \mathbf{D_m}A13 \} = \emptyset$.

PROOF. To prove (i) use the axiom A13. (ii) is easy, and (iii) follows from (ii). ■

4.21. PROPOSITION. $\mathbf{D_m}$A13 *is consistent.*

PROOF. Consider the general frame $\langle \mathbf{F}, \mathfrak{W} \rangle$, where $\mathbf{F} = \langle \{0, 1\}, \emptyset, \preceq \rangle$ and $\mathfrak{W} = \{\emptyset, \{0, 1\}\}$. Obviously, $\langle \mathbf{F}, \mathfrak{W} \rangle \models \mathbf{D}_m$. Moreover, $\langle \mathbf{F}, \mathfrak{W} \rangle \models \varphi \rightarrow D\varphi$, for if $\langle \mathbf{F}, \mathfrak{W}, V \rangle \models \varphi[0]$ or $\langle \mathbf{F}, \mathfrak{W}, V \rangle \models \varphi[1]$, for some valuation V on $\langle \mathbf{F}, \mathfrak{W} \rangle$, then $V(\varphi) = \{0, 1\}$, and so $\langle \mathbf{F}, \mathfrak{W}, V \rangle \models D\varphi[0]$ and $\langle \mathbf{F}, \mathfrak{W}, V \rangle \models D\varphi[1]$. ■

4.22. THEOREM. $\mathbf{D}_m$A13 *is incomplete.*

PROOF. Combine Proposition 4.20 (iii) and Proposition 4.21. ■

Before stating our next result we repeat an important theorem about L(◇):

THE SAHLQVIST THEOREM. *Let* φ *be a modal formula which is equivalent to a conjunction of formulas of the form* $\Box^m(\psi \rightarrow \chi)$ *where*

(i) χ *is positive,*

(ii) *after eliminating* $\rightarrow$ *from* ψ *and rewriting* ψ *with* $\neg$ *occurring only in front of proposition letters, no positive occurrence of a proposition letter is in a subformula of* ψ *of the form* $\psi_1 \vee \psi_2$ *or* $\Diamond\psi_1$ *within the scope of some* $\Box$.

Then $\mathbf{K}\varphi$ (= *the logic obtained from* **K** *by adding* φ *as an axiom scheme*) *is complete and* φ *corresponds to a first order formula effectively obtainable from* φ.

A Sahlqvist Theorem for L(◇,D) would describe a class $\mathfrak{S}$ of ◇,D-formulas such that for any $\varphi \in \mathfrak{S}$ the logic $\mathbf{D}_m\varphi$ (= the logic obtained from the basic logic in L(◇,D) by adding φ as an axiom scheme) is complete, and such that any such φ corresponds to a first order condition.

Any such class $\mathfrak{S}$ should extend the original class defined above. The obvious candidate would be the class of all ◇,D-formulas $\varphi \rightarrow \psi$, where ψ is positive and φ satisfies some condition similar to the above condition (ii). However, if we want the completeness-part of the Sahlqvist Theorem to hold for formulas in this class, the condition on the antecedent formulas can not be as 'simple' as condition (ii). Otherwise the formula $p \rightarrow Dp$ would be an admissible formula - but it is incomplete by Theorem 4.22! In short, Theorem 4.22 has the following

4.23. COROLLARY. *The completeness part of the Sahlqvist Theorem has no obvious extension to* L(◇,D).

On the other hand, in Chapter 5 we will show that the second (correspondence-) part of the Sahlqvist Theorem does have an obvious extension to L(◇,D).

In fact the completeness part of the Sahlqvist Theorem does not even have an obvious extension to L(D)! Simply take the basic logic in L(D) - which is $\mathbf{D}_m$ with all the axioms in which a ◇ or □ occurs left out - and

extend it by adding axiom schema A13. (Cf. Koymans [1989] for a precise definition of the basic logic in L(D).)

What's next? A useful result about $L(\Diamond)$ says that any logic $\mathbf{L} \subseteq L(\Diamond)$ that has the finite model property (f.m.p.) is (weakly) complete. We will show that this result does not hold for logics $\mathbf{L} \subseteq L(\Diamond,D)$, by proving that $\mathbf{D}_m$A13 has the f.m.p.

We need the following Proposition:

4.24. PROPOSITION. $\mathbf{D}_m$A13, *considered as a bimodal logic whose semantics is based on two abstract relations* $R_\Diamond$ *and* R_D, *is complete w.r.t. the class of all frames* $\langle W, R_\Diamond, R_D \rangle$ *that satisfy*:

(i) $\forall xy\,(R_Dxy \to R_Dyx)$,
(ii) $\forall xyz\,(R_Dxy \wedge R_Dyz \to R_Dxz \vee x = z)$,
(iii) $\forall xy\,(R_\Diamond xy \to x = y \vee R_Dxy)$,
(iv) $\forall x\, R_Dxx$.

PROOF. The Proposition may be proved using a Henkin-type completeness proof. ■

4.25. COROLLARY. $\mathbf{D}_m$A13 *has the f.m.p.*

PROOF. Assume that $\mathbf{D}_m\text{A13} \nvdash \varphi$. By Proposition 4.24 we find a model $\mathbf{M}_1$ $(= \langle W_1, R^1_\Diamond, R^1_D, V_1 \rangle)$ satisfying conditions (i)-(iv) in Proposition 4.24 such that $\mathbf{M}_1 \models \neg\varphi[w_0]$ for some $w_0 \in W$. By the Generation Theorem we may assume that w_0 generates $\mathbf{M}_1$. Using this fact and condition (ii) it is easily verified that any two different points are R^1_D-related. Moreover, by condition (iv) R^1_D is reflexive, so R^1_D is total.

Let Σ be the set of subformulas of $\neg\varphi$, and for $w \in W$ put $\Sigma(w) = \{ \sigma \in \Sigma \mid \mathbf{M}_1 \models \sigma[w] \}$. We define a filtrated model $\mathbf{M}_2 = \langle W_2, R^2_\Diamond, R^2_D, V_2 \rangle$ by putting

$W_2 = \{ \Sigma(w) \mid w \in W_1 \}$,
$R^2_\Diamond ab \equiv \forall \psi \in \Sigma\ (\Box\psi \in a \Rightarrow \psi \in b)$,
$R^2_D ab \equiv \forall \psi \in \Sigma\ (\overline{D}\psi \in a \Rightarrow \psi \in b)$,
$V_2(p) = \{ \Sigma(w) \mid p \in \Sigma(w) \}$.

A simple induction shows that for all $w \in W_1$ and all $\psi \in \Sigma$ we have

$\mathbf{M}_1 \models \psi[w]$ iff $\mathbf{M}_2 \models \psi[\Sigma(w)]$.

So $\mathbf{M}_2 \models \psi[\Sigma(w_0)]$. Moreover, $\mathbf{M}_2$ is finite and $\mathbf{M}_2 \models \mathbf{D}_m$A13. The first of these claims is obvious, and to prove the latter we only have to show that $\mathbf{M}_2$ satisfies conditions (i)-(iv). Let's do so.

(i) Assume $R_D^2\Sigma(u)\Sigma(v)$, and let $\overline{D}\psi \in \Sigma(v)$; we have to show that $\psi \in \Sigma(u)$. Since $\overline{D}\psi \in \Sigma(v)$ it follows that $\mathbf{M}_1 \models \overline{D}\psi[v]$, and since R_D^1 is total on W_1 we have R_D^1vu, so $\mathbf{M}_1 \models \psi[u]$ and $\psi \in \Sigma(u)$.

(ii) Assume $R_D^2\Sigma(u)\Sigma(v)$, $R_D^2\Sigma(v)\Sigma(w)$ and $\Sigma(u) \neq \Sigma(w)$. Let $\overline{D}\psi \in \Sigma(u)$; we have to show that $\psi \in \Sigma(w)$. Again, we have $\mathbf{M}_1 \models \overline{D}\psi[u]$, because $\overline{D}\psi \in \Sigma(u)$. Since R_D^1 is total on W_1 we have R_D^1uw, so $\mathbf{M}_1 \models \psi[w]$ and $\psi \in \Sigma(w)$.

(iii) Assume $R_D^2\Sigma(u)\Sigma(v)$ and $\Sigma(u) \neq \Sigma(v)$. Let $\overline{D}\psi \in \Sigma(u)$; we have to show that $\psi \in \Sigma(w)$: but this is similar to the previous case.

(iv) Let $\Sigma(u) \in W_2$ and $\overline{D}\psi \in \Sigma(u)$. We have to show that $\psi \in \Sigma(u)$. Yet again, we have $\mathbf{M}_1 \models \overline{D}\psi[u]$, because $\overline{D}\psi \in \Sigma(u)$. Since R_D^1 is reflexive it follows that $\mathbf{M}_1 \models \psi[u]$. Hence $\psi \in \Sigma(u)$.

To complete the proof we only have to show that R_D^2 can be turned into real inequality. To this end we apply the method of doubling R_D-reflexive points to $\mathbf{M}_2$: let $n \in \mathbf{M}_2$, then n is R_D^2-reflexive; replace n by two points n_1, n_2 and put $R_Dn_1n_2$ and $R_Dn_2n_1$, and for all $m \neq n$, R_Dn_1m, R_Dn_2m, R_Dmn_1 and R_Dmn_2. Finally, n_1 and n_2 are to have the same valuations as n, i.e. $n_1, n_2 \in V(p)$ if $n \in V_2(p)$, for all proposition letters p. Repeat this procedure for all R_D^2-reflexive points, and let $\mathbf{M}_3$ be the resulting model. A straightforward inductive proof similar to that of the Claim in the proof of Theorem 4.6 establishes that

$$\mathbf{M}_2 \models \psi[n] \text{ iff } \mathbf{M}_3 \models \psi[n_1] \text{ and } \mathbf{M}_3 \models \psi[n_2]$$

holds for all formulas ψ. So $\mathbf{M}_3 \models \mathbf{D}_m\mathrm{A13}$ and $\mathbf{M}_3 \models \neg\psi[w]$ for some w in $\mathbf{M}_3$.

Now, R_D is real inequality in $\mathbf{M}_3$: it holds between any two different points, and it is irreflexive, so $\mathbf{M}_3$ is a standard model. Finally, since $\mathbf{M}_2$ is finite, $\mathbf{M}_3$ is finite as well. ■

4.26. THEOREM. *Not every logic* $\mathbf{L} \subseteq \mathrm{L}(\Diamond,D)$ *that has the f.m.p. is complete.*

PROOF. $\mathbf{D}_m$A13 has the f.m.p. by Lemma 4.25, but by Theorem 4.22 it is incomplete. ■

Notice that the Theorem also holds for L(D): take the basic logic in L(D) and extend this logic by adding axiom schema A13. It is easily verified that our entire argument can be adapted to this logic. (Cf. Koymans [1989] for a precise definition of the basic logic in L(D).)

We end this Chapter by stating some speculations. Recall that according to *Bull's Theorem* all modal extensions of S4.3 are complete. (Here S4.3 is the modal logic of the reflexive linear orderings.) We

conjecture that it no longer holds for ◇,D-logic. Another conjecture of ours is that there is some general theorem saying that most of the well-known modal logics like T, S4, S5 have a straightforward extension to complete logics in L(◇,D). Finally, we think that simple examples can be found in L(◇,D) for most of the well-known 'pathologies'.

CHAPTER 5

FIRST ORDER DEFINABILITY

We describe two large classes of first order definable $\Diamond$,D-formulas. We show that these results have no straightforward generalization to languages with n-ary modal operators, where $n \geq 2$.

§1. Two theorems on first order definability in L($\Diamond$,D)

5.1. DEFINITION. (i) A formula φ is said to be *monotone in the proposition letter p,* if, for all models $\mathbf{M} = \langle W, R, V\rangle$, for all $w \in W$ and all valuations V' satisfying $V(p) \subseteq V'(p)$, if $\mathbf{M} \models \varphi[w]$, then $\langle W, R, V'\rangle \models \varphi[w]$.
(ii) φ is said to be *positive* if φ is built up using $\top$, $\bot$, proposition letters, $\wedge$, $\vee$, $\Diamond$, $\Box$, D and $\overline{D}$. Notice that each positive formula is monotone in all its proposition letters.

5.2. PROPOSITION. *For all* $\Diamond$,D-*formulas* φ, *all* $\alpha \in L_0$, *and all proposition letters* p *we have*:

(i) $\mathcal{S}(\varphi,\alpha) \Leftrightarrow \mathcal{S}(\varphi[p:=\neg p],\alpha)$, *and*

(ii) *if* φ *is monotone in* p, *then* $\varphi \in \mathfrak{ML}1$ *iff* $\varphi[p:=\bot] \in \mathfrak{ML}1$.

PROOF. (i) Straightforward.
(ii) Let φ be monotone in p. Then, for every frame **F**, and all $w \in W$, $\mathbf{F} \models \varphi[w]$ iff $\mathbf{F} \models \varphi[p:=\bot][w]$. From left to right this is obvious. The other direction follows from the fact that $\{ w \in W \mid \mathbf{F} \models \bot[w] \} = \emptyset$ and the assumption that φ is monotone in p. ■

The first theorem in this section extends Theorem 9.8 of van Benthem [1985] - which applies to L($\Diamond$) - to L($\Diamond$,D). Before proving it, we introduce some useful abbreviations:

$\Box^i\varphi$ abbreviates $\Box$...(i times)...$\Box\varphi$, $\Diamond^i\varphi$, $D^i\varphi$, and $\overline{D}^i\varphi$: similarly;
$R^{i+1}xy$ $(i > 0)$ denotes $\exists z_i\,(R^ixz_i \wedge Rz_iy)$,
$x \neq^{i+1} y$ $(i > 0)$ denotes $\exists z_i\,(x \neq^i z_i \wedge z_i \neq y)$.

One definition is needed:

5.3. DEFINITION. (i) We write $[\chi_1, \chi_2, \ldots \mid O_1, O_2, \ldots]$ to denote the set of objects generated by $\chi_1, \chi_2, \ldots$, using the operators $O_1, O_2, \ldots$. (In the sequel these objects will either be operators or formulas.)
(ii) $\mathbf{OP} := [\langle\rangle \mid \Box, \overline{D}]$, where $\langle\rangle$ denotes the empty sequence.
(iii) If $\vec{O} \in \mathbf{OP}$, and x, y are variables then the L_0-formula $RT(\vec{O},x,y)$, called

the route from x to y described by $\vec{\sigma}$, is defined as follows:

- if $\vec{\sigma} \equiv \langle\,\rangle$, then $RT(\vec{\sigma},x,y) \equiv x = y$;
- if $\vec{\sigma} \equiv \Box^i\overline{D}\vec{\sigma}'$, $i > 0$, then
  $RT(\vec{\sigma},x,y) \equiv \exists z_{i+1}\,(R^i x z_{i+1} \wedge RT(\overline{D}\vec{\sigma}', z_{i+1}, y))$,
- if $\vec{\sigma} \equiv \overline{D}^i\Box\vec{\sigma}'$, $i > 0$, then
  $RT(\vec{\sigma},x,y) \equiv \exists z_{i+1}\,(x \not\approx^i z_{i+1} \wedge RT(\Box\vec{\sigma}', z_{i+1}, y))$,
- if $\vec{\sigma} \equiv \Box^i$, $i > 0$, then $RT(\vec{\sigma},x,y) \equiv R^i xy$,
- if $\vec{\sigma} \equiv \overline{D}^i$, $i > 0$, then $RT(\vec{\sigma},x,y) \equiv x \not\approx^i y$.

5.4. THEOREM. *Let*

(i) $\varphi \in [\,[p, q, r, \ldots \mid \{\vec{\sigma} \mid \vec{\sigma} \in \mathbf{OP}\}]\mid \vee, \wedge, \Diamond, D\,]$, *and*

(ii) ψ *be a positive formula.*

Then $\varphi \to \psi \in \mathfrak{M}1$.

PROOF. First we reduce the theorem to the case without occurrences of '$\vee$' in φ. To this end the obvious propositional and ◇,D-equivalences can be employed to rewrite φ as a disjunction of formulas built up using $\vec{\sigma}p$, $\bot$, T, $\wedge$, ◇ and D:

$$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi),$$
$$D(\varphi \vee \psi) \to (D\varphi \vee D\psi),$$
$$((\varphi \vee \psi) \to \chi) \to ((\varphi \to \chi) \wedge (\psi \to \chi)),$$
$$\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi).$$

Next, write $\varphi \to \psi$ as a conjunction of implications, each of which has one of these disjuncts as its antecedent.

Then remove all proposition letters occurring in $\varphi \to \psi$ but not in both φ and ψ. Let p be such a proposition letter. If p occurs in ψ, then $\varphi \to \psi$ is monotone in p, and we are allowed to substitute $\bot$ for p by Proposition 5.2.(ii). Otherwise, use Proposition 5.2.(i) and consider $(\varphi \to \psi)[\neg p := p]$ in stead of $\varphi \to \psi$. Then $\bot$ can be substituted for p in this formula, since it is monotone in p.

Let $\varphi \to \psi$ be a formula obtained in this way. $ST(\varphi \to \psi)$ can be written in such a way that no two quantifiers bind the same variable. In this way, we obtain a 1-1-correspondence between the occurrences of ◇, □, D and $\overline{D}$ in $(\varphi \to \psi)$ and the bound variables in $ST(\varphi \to \psi)$.

Next, consider the antecedent $ST(\varphi)$ in $ST(\varphi \to \psi)$. Since we only have to pass occurrences of '$\wedge$', all existential quantifiers can be moved to the front. This yields $\exists y_1 \ldots \exists y_k\, \varphi'$, so $ST(\varphi \to \psi)$ may be written as $\forall y_1 \ldots \forall y_k\,(\varphi' \to ST(\psi))$.

Let u be a variable not occurring in $ST(\varphi \to \psi)$, and let $|p|$ be an occurrence of the proposition letter p in φ, and suppose that y_i is the unique bound variable corresponding to the innermost occurrence of a ◇ or D, the scope of which contains $|p|$. Define $v(|p|) := y_i$. If such an occurrence of ◇ or D does not exist, put $v(|p|) = x$.

Now, $|p|$ occurs in the scope of an $\vec{O} \in \mathbf{OP}$. Put $CV(|p|,\varphi) := RT(\vec{O},v(|p|),u)$. $CV(p,\varphi)$ is defined to be the disjunction of all formulas $CV(|p|,\varphi)$, where $|p|$ is an occurrence of p in φ. By taking alphabetic variants we can make sure that the formulas $CV(p,\varphi)$ and $\forall y_1 \ldots \forall y_k\,(\varphi' \rightarrow ST(\psi))$ do not share any bound variables.
By substituting, for each proposition letter p and corresponding predicate constant P, and each variable z, the formula $CV(p,\varphi)[u:=z]$ for Pz in $\forall y_1 \ldots \forall y_k\,(\varphi' \rightarrow ST(\psi))$, we obtain the L_0-equivalent $s(\varphi \rightarrow \psi)$ of $\varphi \rightarrow \psi$. We have proved the theorem, once we have shown that for all frames **F** and all $w \in W$, $\mathbf{F} \models \varphi \rightarrow \psi[w]$ iff $\mathbf{F} \models s(\varphi \rightarrow \psi)[w]$.
The direction from left to right is a universal instantiation, and needs no proof. Conversely, suppose that for some valuation V we have $\langle \mathbf{F}, V\rangle \models \varphi[w]$. We have to show that $\langle \mathbf{F}, V\rangle \models \psi[w]$. Now,

$$\begin{aligned}\langle \mathbf{F}, V\rangle \models \varphi[w] &\Rightarrow \langle \mathbf{F}, V\rangle \models \exists y_1 \ldots \exists y_k\, \varphi'[w] \\ &\Rightarrow \langle \mathbf{F}, V\rangle \models \varphi'[w, w_1, \ldots, w_k], \text{ for some } w_1, \ldots, w_k \in W.\end{aligned}$$

Define a valuation V' by putting, for $v \in W$,

$$V'(p) = \{\, v \mid \mathbf{F} \models CV(p,\varphi)[w, w_1, \ldots, w_k, v] \,\},$$

where v is assigned to u. Obviously, we have, for all proposition letters p, $V'(p) \subseteq V(p)$, as well as $\langle \mathbf{F}, V'\rangle \models \varphi'[w, w_1, \ldots, w_k]$. Let φ'' be the result of substituting the formulas $CV(p,\varphi)$ for the P's in φ'. And let ψ' be obtained from ψ by applying the same substitution to $ST(\psi)$. Then,

$$\begin{aligned}&\langle \mathbf{F}, V'\rangle \models \varphi'[w, w_1, \ldots, w_k] \\ &\quad \Rightarrow \langle \mathbf{F}, V'\rangle \models \varphi''[w, w_1, \ldots, w_k]. \\ &\quad \Rightarrow \langle \mathbf{F}, V'\rangle \models \psi'[w, w_1, \ldots, w_k], \text{ since } \mathbf{F} \models s(\varphi \rightarrow \psi)[w], \\ &\quad \Rightarrow \langle \mathbf{F}, V'\rangle \models ST(\psi)[w].\end{aligned}$$

Applying the fact that ψ is monotone in all its proposition letters, and the fact that for all proposition letters p, and all $v \in W$, we have $V'(p) \subseteq V(p)$, we immediately obtain that $\langle \mathbf{F}, V\rangle \models ST(\psi)[w]$, and so that $\langle \mathbf{F}, V\rangle \models \psi[w]$. ■

Our next theorem generalizes the previous one. Restricted to $L(\Diamond)$ it appears in van Benthem [1985] and Sahlqvist [1975]. Again a definition is needed:

5.5. DEFINITION. We define *positive* and *negative* occurrences of proposition letters p in a formula:

(i) p occurs positively in p,
(ii) p does not occur in $\bot$, T,
(iii) if p occurs positively (negatively) in φ, it occurs positively (negatively) in $\psi \rightarrow \varphi$, and negatively (positively), and

accordingly positively (negatively) in $\varphi \wedge \psi$, $\varphi \vee \psi$, $\psi \wedge \varphi$ and $\psi \vee \varphi$, negatively (positively) in $\neg\varphi$,

(iv) if p occurs positively (negatively) in φ, it occurs positively (negatively) in $\Diamond\varphi$, $\Box\varphi$, $D\varphi$ and $\overline{D}\varphi$.

5.6. THEOREM. *Suppose that* $\varphi \in [\bot, T, p, q, r, \ldots \mid \vee, \wedge, \Diamond, \Box, D, \overline{D}]$ *satisfies for all proposition letters* p *in it, either*

(i) *no positive occurrence of p is in a subformula of the form* $\psi \wedge \chi$, $\Box\psi$ *or* $\overline{D}\psi$ *within the scope of some* $\Diamond$ *or* D, *or*

(ii) *no negative occurrence of* p *is in a subformula of* φ *of the form* $\psi \wedge \chi$ *or* $\Box\psi$ *or* $\overline{D}\psi$ *within the scope of some* $\Diamond$ *or* D.

Then $\varphi \in \mathfrak{M}\mathfrak{b}1$.

PROOF. First we reduce the theorem to a special case. If some proposition letter p occurs only positively in φ, then φ is monotone in p, and by Proposition 5.2 we can consider $\varphi[p:=\bot]$ in stead of φ. If all occurrences of p in φ are negative, then p occurs only positively in $\varphi[p:=\neg p]$, and we can consider this formula in stead of φ, since $\psi \in \mathfrak{M}\mathfrak{b}1$ iff $\psi[p:=\neg p] \in \mathfrak{M}\mathfrak{b}1$ by the same proposition. Then we consider $(\varphi[p:=\neg p])[p:=\bot]$. Applying the proposition again, and removing double negations, we make sure that every remaining proposition letter satisfies condition (2) of the theorem.
Now, consider the negation of formula just obtained, and rewrite it as a formula ψ built up using (negations of) proposition letters, $\bot$, T, $\vee$, $\wedge$, $\Diamond$, $\Box$, D, $\overline{D}$. This can be done by using the equivalences $\neg\Diamond\chi \leftrightarrow \Box\neg\chi$, $\neg\Box\chi \leftrightarrow \Diamond\neg\chi$, $\neg D\chi \leftrightarrow \overline{D}\neg\chi$, $\neg\overline{D}\chi \leftrightarrow D\neg\chi$, $\neg\neg\chi \leftrightarrow \chi$ and the De Morgan laws. The resulting formula has the property that no positive occurrence of a proposition letter in ψ remains in a subformula of ψ of the form $\chi \vee \delta$ or $\Diamond\chi$ or $D\chi$ in the scope of some $\Box$ or $\overline{D}$.

CLAIM 1. *Let* $\Box\chi$ ($\overline{D}\chi$) *be a subformula of* ψ. *Then* $\Box\chi$ ($\overline{D}\chi$) *is equivalent to a conjunction of formulas of the form* $\vec{O}p$ *and n-formulas, - where* $\vec{O} \in$ **OP** *and where an n-formula is a formula in which no proposition letter occurs positively.*

PROOF (of claim 1). An induction on χ. The following cases are trivial: $\chi \equiv p, \neg p, \bot, T, \chi_1 \wedge \chi_2$.
Now, if $\chi \equiv \chi_1 \vee \chi_2$ or $\chi \equiv \Diamond\chi_1$ or $\chi \equiv D\chi_1$, then no proposition letter occurs positively in it, by our remark preceding this claim. That is, in those cases χ already is an n-formula.
If $\chi \equiv \Box\chi_1$, we have - using the IH -:

$$\chi_1 \leftrightarrow \vec{O}_1p_1 \wedge \ldots \wedge \vec{O}_np_n \wedge \chi_2 \wedge \ldots \wedge \chi_m,$$

where $\vec{O}_i \in$ **OP**, and $\chi_2, \ldots, \chi_m$ are n-formulas, and so

$$\Box\chi_1 \leftrightarrow \Box\vec{O}_1p_1 \wedge \ldots \wedge \Box\vec{O}_np_n \wedge \Box\chi_2 \wedge \ldots \wedge \Box\chi_m,$$

and the RHS formula has the required form.
If $\chi \equiv \overline{D}\chi_1$, we can proceed similarly. ■Claim

Now, replace each occurrence of $\Box\chi$ or $\overline{D}\chi$ in ψ which does not lie within the scope of another $\Box$ or $\overline{D}$ by equivalents given in Claim 1. Let ψ' be the resulting formula.

CLAIM 2. *Each subformula χ of ψ' is equivalent to a disjunction of formulas built up using formulas of the form $\vec{\Diamond}p$, n-formulas, $\wedge$, $\Diamond$, D.*
PROOF (of claim 2). Yet another induction. The cases $\chi \equiv p, \neg p, \bot, \top, \chi_1 \vee \chi_2$ are trivial, and if $\chi \equiv \chi_1 \wedge \chi_2$ we can use the propositional distributive laws.
If $\chi \equiv \Diamond\chi_1$, we have - using the IH -:

$$\chi_1 \leftrightarrow \text{"a disjunction of the proper kind"},$$

and

$$\Diamond\chi_1 \leftrightarrow \Diamond(\quad\quad ... \quad\quad),$$

and distributing $\Diamond$ over the disjuncts in the RHS formula again yields a disjunction of the proper kind.
If $\chi \equiv D\chi_1$: similarly.
If $\chi \equiv \Box\chi_1$ or $\overline{D}\chi$, then - by the above - χ is either an n-formula or of the form $\vec{\Diamond}p$. ■Claim

Applying this second claim to ψ', we obtain a disjunction $\psi'' \equiv \psi_1 \vee ... \vee \psi_n$, where $\psi_1, ..., \psi_n$, are built up as indicated. Now, $\varphi \leftrightarrow \neg\psi \leftrightarrow \neg\psi' \leftrightarrow \neg\psi''$, so $\varphi \leftrightarrow \neg\psi_1 \wedge ... \wedge \neg\psi_n$. Since $\varphi \in \mathfrak{ML}1$, if each $\neg\psi_i \in \mathfrak{ML}1$, we only have to consider these formulas $\neg\psi_i$.
For a start, notice that $ST(\psi_i)$ can be written in the form $\exists y_1 ... \exists y_k\, \psi_i'$ as in the proof of Theorem 5.4; this time, however, only with respect to those occurrences of $\Diamond$ and D that have a positive occurrence of a proposition letter in their scope. For each p, define $CV(p,\psi_i)$ as in the proof of the previous theorem, and substitute it in $\forall y_1 ... \forall y_k\, \psi_i'$. This yields the required equivalent $s(\neg\psi_i)$ of $\neg\psi_i$. It's obvious that for all frames $\mathbf{F}$, and all $w \in W$, $\mathbf{F} \models \neg\psi_i[w]$ implies $\mathbf{F} \models s(\neg\psi_i)[w]$.
Conversely, suppose that for some valuation V we have $\langle \mathbf{F}, V\rangle \models \psi_i[w]$, and so $\langle \mathbf{F}, V\rangle \models \psi_i'[w, w_1, ..., w_k]$, for some $w_1, ..., w_k \in W$. Using the formulas $CV(p,\psi_i)$ to define a valuation V' as before, we find that $\langle \mathbf{F}, V'\rangle \models \psi_i'[w, w_1, ..., w_k]$, and $V'(p) \subseteq V(p)$ for all proposition letters p. It is easily verified that $\mathbf{F} \models \psi_i''[w, w_1, ..., w_k]$, where ψ_i'' is obtained from ψ_i' by substituting the formulas $CV(p,\psi_i)$ for the P's. Finally, $s(\psi_i) \equiv \forall y_1 ... \forall y_k\, \psi_i''$, so $\mathbf{F} \models s(\psi_i)[w]$. ■

§2. Excursion: adding other operators to L(◇)

Let I be an index set, and let $L(\Diamond,\Diamond_1,\Diamond_2,\ldots)$ be the language obtained from L(◇) by adding (binary) modal operators $\Diamond_i$, for $i \in I$. A close inspection of the proof of Theorem 5.4 shows that this result can be extended to $L(\Diamond,\Diamond_1,\Diamond_2,\ldots)$. For, as Johan van Benthem pointed out to us, the one feature of the operators occurring in the antecedent formula that is central to that proof, notably their distributivity over ∨, is shared by each $\Diamond_i$: assuming that each $\Diamond_i$ corresponds to a binary relation R_i, one easily verifies that for every $L(\Diamond,\Diamond_1,\Diamond_2,\ldots)$-frame $\langle W, R, R_1, R_2, \ldots\rangle$ we have $\langle W, R, R_1, R_2, \ldots\rangle \models \Diamond_i(\varphi \vee \psi) \leftrightarrow (\Diamond_i\varphi \vee \Diamond_i\psi)$ for all $i \in I$.

So adding more unary modal operators to L(◇) gives rise to a fairly straightforward generalization of Theorem 5.4. As far as possible generalizations of Theorem 5.4 are concerned, the following extension is a less harmless one: let # be a binary modal operator defined by

$$\mathbf{M} = \langle W, S, V\rangle \models \varphi \# \psi[x] \text{ iff } \exists yz\,(Sxyz \text{ and } \mathbf{M} \models \varphi[y] \text{ and } \mathbf{M} \models \psi[z]).^1$$

(Here we assume that the semantics of an n-ary modal operator is to be based on an (n+1)-ary relation.) Its dual $\overline{\#}$ is given by

$$\mathbf{M} = \langle W, S, V\rangle \models \varphi \,\overline{\#}\, \psi[x] \text{ iff } \forall yz \text{ (if } Sxyz \text{ then } \mathbf{M} \models \varphi[y] \text{ or } \mathbf{M} \models \psi[z]).$$

We see that # too distributes over ∨:

$$\begin{aligned}
&\mathbf{M} \models (\varphi_1 \vee \varphi_2) \# \psi[w]\\
&\quad \Leftrightarrow \exists yz\,(\,Swyz \text{ and } \mathbf{M} \models \varphi_1 \vee \varphi_2[y] \text{ and } \mathbf{M} \models \psi[z])\\
&\quad \Leftrightarrow \exists yz\,(\,Swyz \text{ and } \mathbf{M} \models \varphi_1[y] \text{ and } \mathbf{M} \models \psi[z]) \text{ or}\\
&\qquad\qquad \exists yz\,(\,Swyz \text{ and } \mathbf{M} \models \varphi_2[y] \text{ and } \mathbf{M} \models \psi[z])\\
&\quad \Leftrightarrow \mathbf{M} \models (\varphi_1 \# \psi) \vee (\varphi_2 \# \psi)[w].
\end{aligned}$$

Therefore, the following restricted form of Theorem 5.4 holds for L(#): $\varphi \to \psi \in \mathfrak{Ml}1$, if ψ is positive and φ is in $[p, q, r, \ldots \mid \vee, \wedge, \#]$. Notice that the full version of Theorem 5.4 for L(#) introduces universal quantifiers over disjunctions. For, the full version allows antecedent formulas $\varphi \in [\overline{\#}^i p, \overline{\#}^i q, \overline{\#}^i r, \ldots \mid \vee, \wedge, \#]$, where $\overline{\#}^i p$ abbreviates $(\ldots (p \,\overline{\#}\, p) \,\overline{\#}\, \ldots \,\overline{\#}\, p)$ (with i occurrences of p). Now, modal formulas in which universal quantifiers range over a disjunction are known to lead us outside of $\mathfrak{Ml}1$. (Cf. van Benthem [1985] Chapter 10.) So one might expect that things go wrong here - and indeed they do.

To prove this, we present a formula $\varphi \to \psi \in L(\#)$, such that ψ is

---

[1] The recent rise of the so-called 'Interpretability Logics' where a binary modal operator ▷ is added to the provability logic **L**, adds interest to the present considerations.

positive, φ is in $[\overline{\#}^i p, \overline{\#}^i q, \overline{\#}^i r, \ldots \mid \vee, \wedge, \#]$, and $\varphi \to \psi$ is equivalent to a formula χ in $L(\Diamond)$ that is known to be outside of $\mathfrak{Ml}$.

5.7. PROPOSITION. *Let* $\mathbf{M} = \langle W, R, S, V\rangle$ *be an* $L(\Diamond,\#)$*-model such that* $\mathbf{M}$ *satisfies* $\forall xyz\,(Sxyz \leftrightarrow Rxy \wedge y = z)$. *Then for all* $x \in W$:
(i) $\mathbf{M} \models \Diamond\varphi[x]$ *iff* $\mathbf{M} \models \varphi \# \varphi[x]$,
(ii) $\mathbf{M} \models \Box(\varphi \vee \psi)[x]$ *iff* $\mathbf{M} \models \varphi \,\overline{\#}\, \psi[x]$,
(iii) $\mathbf{M} \models \Box\varphi[x]$ *iff* $\mathbf{M} \models \varphi \,\overline{\#}\, \varphi[x]$.

PROOF. (i) $\mathbf{M} \models \Diamond\varphi[x] \Leftrightarrow \exists y\,(Rxy$ and $\mathbf{M} \models \varphi[y])$
$\Leftrightarrow \exists yz\,(Rxy$ and $y = z$ and $\mathbf{M} \models \varphi[y]$ and $\mathbf{M} \models \varphi[z])$
$\Leftrightarrow \exists yz\,(Sxyz$ and $\mathbf{M} \models \varphi[y]$ and $\mathbf{M} \models \varphi[z])$
$\Leftrightarrow \mathbf{M} \models \varphi \,\overline{\#}\, \varphi[x]$
(ii) Similar. (iii) Immediate from (ii). ∎

Let
$$\varphi_0 \equiv \Box(\Box p \vee p) \to \Diamond(\Diamond p \wedge p)$$
and
$$\varphi_1 \equiv \overline{\#}^2 p \to [((p \# p) \wedge p) \# ((p \# p) \wedge p)].$$

5.8. COROLLARY. *Let* $\mathbf{F} = \langle W, R, S\rangle$ *be an* $L(\Diamond,\#)$*-frame such that* $\forall xyz\,(Sxyz \leftrightarrow Rxy \wedge y = z)$. *Then* $\mathbf{F} \models \varphi_0[x]$ *iff* $\mathbf{F} \models \varphi_1[x]$, *for all* $x \in W$.

5.9. LEMMA. (van Benthem [1985]) $\varphi_0 \notin \mathfrak{Ml}$.

PROOF. Consider the sequence of frames $\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \ldots$, where $\mathbf{F}_n = \langle W_n, R_n\rangle$ and $W_n = \{0, 1, \ldots, n\}$ and $R_n = \{\langle 0,i\rangle \mid 1 \le i \le n\} \cup \{\langle 1,2\rangle, \ldots, \langle n-1,n\rangle, \langle n,1\rangle \mid n \ge 2\}$. Notice that $\mathbf{F}_n \models \varphi_0[n]$ for all *odd* n. Now assume that φ_0 is equivalent to $\alpha(x) \in L_0$. Using the compactness theorem for L_0 we find an infinite frame $\mathbf{F}$ containing a point w without predecessors which is succeeded by infinitely many points each having exactly one predecessor other than w, and exactly one successor. Moreover R is irreflexive and there are no loops of finite length.

Now $\alpha(x)$ can be falsified in w by putting a point in $V(p)$ iff both its successor and its predecessor other than w are not in $V(p)$. ∎

5.10. THEOREM. $\varphi_1 \notin \mathfrak{Ml}$.

PROOF. Consider the proof of the previous Lemma. We will modify it in the following way. Extend each frame $\mathbf{F}_n$ in that proof to an $L(\Diamond,\#)$-frame by putting $\forall xyz\,(Sxyz \leftrightarrow Rxy \wedge y = z)$. By Corollary 5.8 we have $\mathbf{F}_n \models \varphi_1[0]$ for *odd* n, since $\mathbf{F}_n \models \varphi_0[n]$ for such n. Since all $\mathbf{F}_n$ have $\mathbf{F}_n \models \forall xyz\,(Sxyz \leftrightarrow Rxy \wedge y = z)$ we may assume that the infinite frame $\mathbf{F}$ we find in the proof of the previous Lemma, also has $\mathbf{F} \models \forall xyz\,(Sxyz \leftrightarrow Rxy \wedge y = z)$. By Corollary 5.8, again, we find that φ_1 is refuted at w. ∎

REFERENCES

ACKERMANN, W.

[1954] *Solvable Cases of the Decision Problem,* North-Holland, Amsterdam.

VAN BENTHEM, J.F.A.K.

[1979] "Syntactical Aspects of Modal Incompleteness Theorems", *Theoria* 45, 1979, 63-77.

[1983] *The Logic of Time,* Reidel, Dordrecht.

[1984] "Correspondence Theory", in: Gabbay & Guenthner [1984], 167-247.

[1985] *Modal Logic and Classical Logic,* Bibliopolis, Naples.

[1989] "A Note on Modal Definability", *Notre Dame J. of Formal Logic* 30, 1989, 20-35.

BULL, R. & SEGERBERG, K.

[1984] "Basic Modal Logic", in: Gabbay & Guenthner [1984], 1-88.

CHANG, C.C. & KEISLER, H.J.

[1973] *Model Theory,* North-Holland, Amsterdam.

DOETS, H.C.

[1987] *Completeness and Definability,* Dissertation, University of Amsterdam.

GABBAY, D.M. & GUENTHNER, F.

[1984] *Handbook of Philosophical Logic, Vol. II: Extensions of Classical Logic,* Reidel, Dordrecht, 1984.

GARGOV, G. & PASSY, S. & TINCHEV, T.

[1987] "Modal Environment for Boolean Speculations", in: Skordev, D. (ed.): *Mathematical Logic and its Applications,* Plenum Press, New York, 253-263.

DE JONGH, D., VELTMAN, F. & VERBRUGGE, R.

[1988] "Tense Logics for Discrete and Dense Time", Institute for Language, Logic and Information, University of Amsterdam.

KOYMANS, R.

[1989] *Specifying Message Passing and Time-Critical Systems with Temporal Logic,* Dissertation, Eindhoven University of Technology.

RODENBURG, P.H.

[1986] *Intuitionistic Correspondence Theory,* Dissertation, University of Amsterdam.

SAHLQVIST, H.

[1975] "Completeness and Correspondence in the First and Second Order Semantics for Modal Logic", in: Kanger, S. (ed.): *Proceedings of the Third Scandinavian Logic Symposium, Uppsala 1973,* North-Holland, Amsterdam, 110-143.

SAMBIN, G.

[1980] "A Simpler Proof of Sahlqvist's Theorem on Completeness of Modal Logics", *Bull. Sect. Logic, Polish Acad. Science* 9, 1980, 50-56.

SEGERBERG, K.

[1970] "Modal Logics with Linear Alternative Relations", *Theoria* 36, 1970, 301-322.

THOMASON, S.K.

[1972] "Semantic Analysis of Tense Logics", JSL 37, 1972, 150-158.

LIST OF SYMBOLS

Abbreviations

IH	induction hypothesis	
LHS	left hand side	
RHS	right hand side	

Axioms and Theories

A0	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	3
A1	$\overline{D}(\varphi \to \psi) \to (\overline{D}\varphi \to \overline{D}\psi)$	3, 31
A2	$\varphi \to \overline{D}D\varphi$	3, 31
A3	$DD\varphi \to (\varphi \vee D\varphi)$	3, 31
A4	$\Diamond\varphi \to (\varphi \vee D\varphi)$	3, 31
A5	$\Diamond\varphi \to D\varphi$	32
A6	$\Box\varphi \to \Box\Box\varphi$	32
A7	$\varphi \to \Diamond\psi \vee \overline{D}(\psi \to \Diamond\varphi)$	32
A8	$\Diamond T$	34
A9	$\varphi \to D\Diamond\varphi$	34
A10	$\Box\Box\varphi \to \Box\varphi$	34
A11	$\Box(\Box\varphi \to \varphi) \to (\Diamond\Box\varphi \to \Box\varphi)$	38
A12	$\Box\Diamond\varphi \to \Diamond\Box\varphi$	41
A13	$\varphi \to D\varphi$	43
$\mathbf{D}_m$	basic logic in $L(\Diamond,D)$	4, 31
$\mathbf{D}_{Lin}$	D-logic of linearly ordered frames	31
$\mathbf{D}\eta$	D-logic of frames of order type η	34
$\mathbf{D}\zeta$	D-logic of frames of order type ζ	38
ID	incomplete D-logic	41
$\mathbf{D}_m$A13	another incomplete D-logic: $\mathbf{D}_m$ + A13	43

Semantics

$\mathbf{F} \equiv_{O_1 \ldots O_n} \mathbf{G}$	$\mathbf{F}, \mathbf{G}$ validate the same φ in $L(O_1, \ldots, O_n)$	2
$\mathbf{F} \subseteq \mathbf{G}$	$\mathbf{F}$ is a generated subframe of $\mathbf{G}$	4
$\oplus_i \{ \mathbf{F}_i \mid i \in I \}$	disjoint union of the frames $\mathbf{F}_i$	4
$\models_m$	$\Delta \models_m \varphi$, if for all models $\mathbf{M}$, if $\mathbf{M} \models \Delta$ then $\mathbf{M} \models \varphi$	24
$\check{R}$	the converse of the relation R	32
$\models_2$	weak second order consequence	43

Syntax

$\varphi[\psi:=\chi]$	simultaneous substitution of χ for ψ in φ	
$\Diamond, \Box, F, P, G, H, D, \overline{D}, E, A, U$	unary modal operators	1, 2
$L(O_1, \ldots, O_n)$	the (multi-) modal language with operators $O_1, \ldots, O_n$	1

$Th_{O_1 \ldots O_n}(\mathbf{F})$	the set of sentences in $L(O_1, \ldots, O_n)$ valid on $\mathbf{F}$	2
$\mathfrak{E}, \bar{\mathfrak{E}}$	relations of local resp. global equivalence	2
$\mathfrak{M}1$, $\bar{\mathfrak{M}}1$	set of $\Diamond$,D-formulas locally resp. globally equivalent to an $\alpha \in L_0$	2,3
$ST(\varphi)$	standard translation of φ	24
$\vdash_2$	weak second order deducibility	42
$[X_1, X_2, \ldots \mid O_1, O_2, \ldots]$	set of objects generated by $X_1, X_2, \ldots$ using the operators $O_1, O_2, \ldots$	48
#	binary modal operator	53

General

$\lvert W \rvert$	cardinality of W	

The ITLI Prepublication Series

1986

86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas — A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem — Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens — A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh — Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem — Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof — Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch — Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer — Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem — Polyadic quantifiers
87-05 Víctor Sánchez Valencia — Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen — Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem — Categorial Grammar and Type Theory
87-08 Renate Bartsch — The Construction of Properties under Perspectives
87-09 Herman Hendriks — Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen — Algorithmic Information Theory
LP-88-02 Yde Venema — Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 — Year Report 1987
LP-88-04 Reinhard Muskens — Going partial in Montague Grammar
LP-88-05 Johan van Benthem — Logical Constants across Varying Types
LP-88-06 Johan van Benthem — Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch — Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof — Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen — A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe — A Blissymbolics Translation Program

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten — Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen — The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman — Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra — On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi — Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid — General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars Leen Torenvliet, Peter van Emde Boas — Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks Gerard R. Renardel de Lavalette — Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas — Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid — A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem — Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars Leen Torenvliet, Peter van Emde Boas — Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen — Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet Peter van Emde Boas — Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel Peter van Emde Boas — Towards implementing RL

Other prepublications:

X-88-01 Marc Jumelet — On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:

LP-89-01 Johan van Benthem — The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema — Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals

Mathematical Logic and Foundations:

ML-89-01 Dick de Jongh, Albert Visser — Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer — Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna — Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna — On the Proof of Solovay's Theorem

Computation and Complexity Theory:

CT-89-01 Michiel H.M. Smid — Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas — Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise Leen Torenvliet, Peter van Emde Boas — On Space efficient Solutions
CT-89-04 Harry Buhrman, Leen Torenvliet — A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid Leen Torenvliet, Willem G. Vree — A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. — Finding isomorphisms between finite fields

Other prepublications:

X-89-01 Marianne Kalsbeek — An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers — New Foundations. a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra — Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke — The Modal Theory of Inequality