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UNDECIDABLE PROBLEMS IN CORRESPONDENCE THEORY

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UNDECIDABLE PROBLEMS IN CORRESPONDENCE THEORY

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Received October 1990

UNDECIDABLE PROBLEMS IN THE CORRESPONDENCE THEORY

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§1. Introduction. In this paper we prove undecidability of first-order definability of propositional formulas. The main result is proved for intuitionistic formulas, but it remains valid for other kinds of propositional formulas by the analogous arguments or with the help of various translations.

For general background on correspondence theory the reader is referred to van Benthem [1], [2] ([3] for survey on recent results).

The method for the proofs of undecidability in this paper will be to simulate calculations of a Minsky machine by intuitionistic formulas. §3 concerns this simulation. In the literature effective procedure to construct these formulas is present (cf.[4]), but only modal formulas were used.

The principal results of the paper are in §4. §5 gives some further undecidability results, that will be proved in another paper by modification of the method of this paper.

§2. First-order definable intuitionistic formulas. Two examples

Intuitionistic formulas are constructed in the usual way from p_0, p_1, ... (sentence letters), $\neg$ (negation), $\supset$ (implication), & (conjunction) and $\vee$ (disjunction). A Kripke frame for the intuitionism (frame) is a pair $\mathfrak{F}=\langle W,\leq\rangle$, where W is a nonempty set (whose elements x,y,w,v,... are called worlds) and $\leq\subseteq W\times W$ is a partial ordering.

$\mathfrak{M}=\langle\mathfrak{F},V\rangle$ is a model (based on the frame $\mathfrak{F}$) if $\mathfrak{F}$ is a frame and V is a function, called valuation, that associates with each propositional variable p a subset V(p) of W such that if $x\in V(p)$ and $x\leq y$ then $y\in V(p)$. Truth ($\models$) relative to a model $\mathfrak{M}$ is defined by

$x\models p$ iff $x\in V(p)$,

$x\models\neg A$ iff $(\forall y\in W)(x\leq y\Rightarrow \text{not}(y\models A))$,

$x\models A\&B$ iff $x\models A$ and $x\models B$,

$x\models A\vee B$ iff $x\models A$ or $x\models B$,

$x\models A\supset B$ iff $(\forall y\in W)(x\leq y,\ y\models A\Rightarrow y\models B)$.

In the case, when not($x\models A$), we write $x\not\models A$.

$\mathfrak{M}\models A$, A is true in $\mathfrak{M}$, if $(\forall x\in W)(x\models A)$. $\mathfrak{F}\models A$, A is valid in $\mathfrak{F}$, if $(\forall\mathfrak{M}$ based on $\mathfrak{F})(\mathfrak{M}\models A)$. Otherwise we write $\mathfrak{F}\not\models A$.

Let an intuitionistic sentence A be an implication $B\supset C$. Then we write $x\not\Vdash A$ for $x\models B$ and $x\not\models C$. Int+A$\vdash$B, B is derivable from A, if from A and the set of theorems of Int one may derive B with the help of modus ponens and substitution.

We say that intuitionistic formulas A and B are deductively equivalent iff Int+A$\vdash$B and Int+B$\vdash$A.

An intuitionistic sentence A is first-order definable iff there is a first-order sentence A^* (A^* is a sentence from the first-order language (with equality) of a single binary predicate) such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \models A$ iff $\mathfrak{F} \models A^*$ in the classical first-order sense. If A^* is $\forall$-formula ($\forall\exists$-formula), A is called $\forall$-definable ($\forall\exists$-definable). For example, an intuitionistic formula $p \vee \neg p$ is $\forall$-definable (by formula $\forall x \forall y (x \leq y \supset x = y)$), and $\neg p \vee \neg\neg p$ is $\forall\exists$-definable (by formula $\forall x \forall y \forall z \exists t (x \leq y \,\&\, x \leq z \supset y \leq t \,\&\, z \leq t)$), but isn't $\forall$-definable.

We need two examples of $\forall$-definable and $\forall\exists$-definable formulas – F_1 and F_2.

The formula F_1 is defined as follows:

$$A^1_{-3} = s_8 \supset t_8, \quad B^1_{-3} = t_8 \supset s_8, \quad A^1_{-2} = s_7 \supset s_8 \vee A^1_{-3},$$

$$B^1_{-2} = t_7 \supset t_8 \vee B^1_{-3}, \quad A^2_{-3} = s_6 \supset s_7 \vee A^1_{-2}, \quad B^2_{-3} = t_6 \supset t_7 \vee B^1_{-2},$$

$$A^2_{-2} = s_5 \supset s_6 \vee A^2_{-3}, \quad B^2_{-2} = t_5 \supset t_6 \vee B^2_{-3}, \quad A^3_{-3} = s_4 \supset s_5 \vee A^2_{-2},$$

$$B^3_{-3} = t_4 \supset t_5 \vee B^2_{-2}, \quad A^3_{-2} = s_3 \supset s_4 \vee A^3_{-3}, \quad B^3_{-2} = t_3 \supset t_4 \vee B^3_{-3},$$

$$C_0 = s_1 \,\&\, t_8 \supset s_2 \vee t_2 \vee t_1, \quad D_0 = t_1 \,\&\, s_8 \supset s_1 \vee s_2 \vee t_2,$$

$$C_1 = s_2 \supset s_3 \vee A^3_{-2}, \quad D_1 = t_2 \supset t_3 \vee B^3_{-2}, \quad C_2 = s_1 \supset s_2 \vee C_1 \vee C_0,$$

$$D_2 = t_1 \supset t_2 \vee D_1 \vee D_0, \quad C_3 = D_0 \,\&\, B^1_{-3} \supset s_1 \vee C_2,$$

$$D_3 = C_0 \,\&\, A^1_{-3} \supset t_1 \vee D_2, \quad F_1 = C_3 \vee D_3.$$

For ease in readability, we may abbreviate a first-order sentence by its English translation with the help of pictures, in quotation marks.

LEMMA 1. The formula F_1 is $\forall$-definable.

PROOF. Clearly, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models F_1$ iff ''there are $x, y, z, \tau_2, \tau_1, \tau_0, \alpha^3_{-2}, \alpha^3_{-3}, \alpha^2_{-2}, \alpha^2_{-3}, \alpha^1_{-2}, \alpha^1_{-3}, \delta_2, \delta_1, \delta_0, \beta^3_{-2}, \beta^3_{-3}, \beta^2_{-2}, \beta^2_{-3}, \beta^1_{-2}, \beta^1_{-3}$ from $\mathfrak{F}$ such that they form the subframe $\mathfrak{F}'$ of a frame $\mathfrak{F}$, where $\mathfrak{F}'$ is depicted graphically in Figure 1''. The sentence in quotation marks is $\exists$-formula, and its negation, being $\forall$-formula, is true precisely in that frames, in which F_1 is valid.

The formula F_2 is obtained from F_1 by replacing the formulas A^1_{-3}, B^1_{-3} by the formulas $\neg(s \,\&\neg t)$, $\neg(t \,\&\neg s)$, respectively.

LEMMA 2. The formula F_2 is $\forall\exists$-definable, but isn't $\forall$-definable.

PROOF. Clearly, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models F_2$ iff ''there are $x, y, z, \tau_2, \tau_1, \tau_0, \alpha^3_{-2}, \alpha^3_{-3}, \alpha^2_{-2}, \alpha^2_{-3}, \alpha^1_{-2}, \alpha^1_{-3}, \delta_2, \delta_1, \delta_0, \beta^3_{-2}, \beta^3_{-3}, \beta^2_{-2}, \beta^2_{-3}, \beta^1_{-2}, \beta^1_{-3}$ from $\mathfrak{F}$ such that they form the subframe $\mathfrak{F}'$ of a frame $\mathfrak{F}$, and, for the worlds $\alpha^1_{-3}, \beta^1_{-3}$; α^1_{-3}, δ_0; β^1_{-3}, τ_0 in given frame, there are no common successors''. The sentence in quotation marks is $\exists\forall$-formula, and its negation, being $\forall\exists$-formula, is true precisely in that frames, in which F_2 is valid.

Show that F_2 isn't $\forall$-definable. Denote by $\mathfrak{F}''$ the frame obtained from $\mathfrak{F}'$ by adding of the world τ accessible from the worlds α^1_{-3}, β^1_{-3}. Clearly $\mathfrak{F}'' \models F_2$ and $\mathfrak{F}' \not\models F_2$. But $\mathfrak{F}'$ is submodel of $\mathfrak{F}''$ in the sense of the classical model theory. Hence F_2 isn't $\forall$-definable by well known criterion (theorem 5.2.4 [5]).

It is clear that if A and B are deductively equivalent then classes of frames, in that they are valid, are equal, and so we obtain

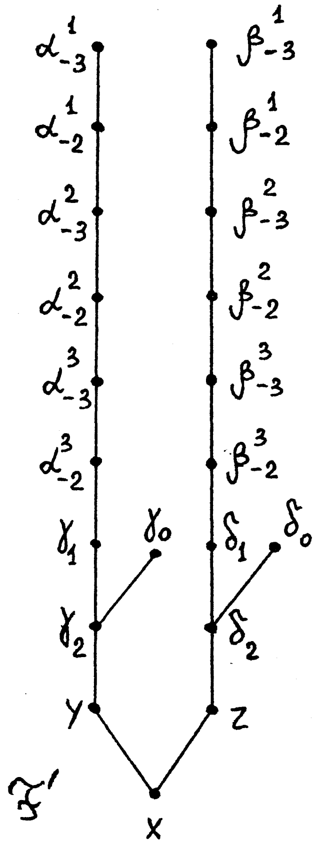

α¹₋₃
α¹₋₂
α²₋₃
α²₋₂
α³₋₃
α³₋₂
γ₁
γ₀
γ₂
y
β¹₋₃
β¹₋₂
β²₋₃
β²₋₂
β³₋₃
β³₋₂
δ₁
δ₀
δ₂
z
𝔉′
x

Figure 1.

lemma 3 by lemmas 1 and 2.

LEMMA 3. a) If an intuitionistic formula is deductively equivalent to F_1, then it is $\forall$-definable.

b) If an intuitionistic formula is deductively equivalent to F_2, then it is $\forall\exists$-definable, but isn't $\forall$-definable.

REMARK. The lemmas 1 and 2 immediately follow from [6] or [7]. In [6] the algorithm is described which, given an intuitionistic formula without negative occurences of disjunction (F_2 is such), constructs a first-order $\forall\exists$-equivalent, and if $\forall$-formula isn't obtained as a result of the algorithm work then given the intuitionistic formula isn't $\forall$-definable. Besides this algorithm, any formula without negative occurences of disjunction and without occurences of negation (F_1 is such), gives its $\forall$-equivalent.

§ 3. Minsky machines and their simulations by intuitionistic formulas

The Minsky machine is the two-tape machine operating on two integers s_1 and s_2. The Minsky machine program is the finite set of instructions I_j of the forms:

(1) $q_\alpha \to q_\beta T_1 T_0$ - being in q_α, add 1 to s_1, go to q_β,

(2) $q_\alpha \to q_\beta T_0 T_1$ - being in q_α, add 1 to s_2, go to q_β,

(3) $q_\alpha \to q_\beta T_{-1} T_0 (q_\gamma T_0 T_0)$ - being in q_α, subtract 1 from s_1, if

$s_1 \neq o$, and go to q_β, otherwise go to q_γ,

(4) $q_\alpha \to q_\beta T_0 T_{-1} (q_\gamma T_0 T_0)$ - being in q_α, subtract 1 from s_2, if $s_2 \neq o$, and go to q_β, otherwise go to q_γ.

A Minsky machine configuration is an ordered triple (i,j,k) of natural numbers, where i is a state number, $j = s_1$, $k = s_2$.

We write $P:(\alpha,m,n) \to (\beta,k,l)$ if the program P starting at configuration (α,m,n) can reach a configuration (β,k,l), otherwise we write $P:(\alpha,m,n) \not\to (\beta,k,l)$. Define

$$P(i,j,k)=\{(l,m,n) \;/\; P:(i,j,k) \to (l,m,n)\}.$$

The basic unsolvable problem supposed to be used is to recognize, given Minsky machine P and two configurations (α,m,n) and (β,k,l), true that $(\beta,k,l) \in P(\alpha,m,n)$ [4].

We introduce the following formulas:

$$A^1_{n+1} = B^1_n \supset A^1_n \vee B^1_{n-1}, \quad B^1_{n+1} = A^1_n \supset B^1_n \vee A^1_{n-1} \quad (n \geq -2);$$

$$Q_{-2} = r, \quad Q'_{-2} = s, \quad Q_{-1} = p, \quad Q'_{-1} = q,$$

$$Q_{n+1} = A^3_{-3} \& B^3_{-3} \& Q'_n \supset Q_n \vee Q'_{n-1} \vee A^2_{-3} \vee B^2_{-3},$$

$$Q'_{n+1} = A^3_{-3} \& B^3_{-3} \& Q_n \supset Q'_n \vee Q_{n-1} \vee A^2_{-3} \vee B^2_{-3} \quad (n \geq -1);$$

$$A^2_{n+1} = A^3_{-3} \& B^3_{-3} \& B^2_n \supset A^2_n \vee B^2_{n-1} \vee A^2_{-3} \vee B^2_{-3},$$

$$B^2_{n+1} = A^3_{-3} \& B^3_{-3} \& A^2_n \supset B^2_n \vee A^2_{n-1} \vee A^2_{-3} \vee B^2_{-3} \quad (n \geq -2);$$

$$R_{-2} = r', \; R'_{-2} = s', \; R_{-1} = p', \; R'_{-1} = q',$$

$$R_{n+1} = C_1 \& D_1 \& R'_n \supset R_n \vee R'_{n-1} \vee A^3_{-3} \vee B^3_{-3},$$

$$R'_{n+1} = C_1 \& D_1 \& R_n \supset R'_n \vee R_{n-1} \vee A^3_{-3} \vee B^3_{-3} \quad (n \geq -1);$$

$$A^3_{n+1} = C_1 \& D_1 \& B^3_n \supset A^3_n \vee B^3_{n-1} \vee A^3_{-3} \vee B^3_{-3},$$

$$B^3_{n+1} = C_1 \& D_1 \& A^3_n \supset B^3_n \vee A^3_{n-1} \vee A^3_{-3} \vee B^3_{-3} \quad (n \geq -2);$$

$$T(n, Q_i, R_j) = A^1_{n+1} \& B^1_{n+1} \& Q_{i+1} \& Q'_{i+1} \& R_{j+1} \& R'_{j+1} \supset A^1_n \vee B^1_n \vee Q_i \vee Q'_i \vee R_j \vee R'_j$$

$(n, i, j \geq o)$.

Because the formulas A^2_i, B^2_i, A^3_i, B^3_i $(i \geq o)$ are obtained from the formulas Q_1, Q'_1, R_1, R'_1 by replacing r, s, p, q by the formulas A^2_{i-3}, B^2_{i-3}, A^2_{i-2}, B^2_{i-2} and r', s', p', q' – by the formulas A^3_{i-3}, B^3_{i-3}, A^3_{i-2}, B^3_{i-2} respectively, we write

$T(n, A^2_i, \Phi_1)$ for $T(n, Q_1, \Phi_1)(A^2_{i-3}/r, \; B^2_{i-3}/s, \; A^2_{i-2}/p, \; B^2_{i-2}/q)$,

$T(n, \Phi_2, A^3_i)$ for $T(n, \Phi_2, R_1)(A^3_{i-3}/r', \; B^3_{i-3}/s', \; A^3_{i-2}/p', \; B^3_{i-2}/q')$, where Φ_1 doesn't contain r, s, p, q, Φ_2 doesn't contain r', s', p', q'.

We are now in a position to define the set of formulas AxI_j which correspond to the instruction set of the program P. For each instruction I_j of P the formula AxI_j is defined as follows:

(1) If I_j is of form (1), then AxI_j will be the formula

$$AxI_j = T(\beta,Q_2,R_1)\supset T(\alpha,Q_1,R_1)\vee F,$$

(2) If I_j is of form (2), then AxI_j will be the formula

$$AxI_j = T(\beta,Q_1,R_2)\supset T(\alpha,Q_1,R_1)\vee F,$$

(3) If I_j is of form (3), then AxI_j will be the formula

$$AxI_j = (T(\beta,Q_1,R_1)\supset T(\alpha, Q_2,R_1)\vee F)\&(T(\tau,A_0^2,R_1)\supset T(\alpha,A_0^2,R_1)\vee F),$$

(4) If I_j is of form (4), then AxI_j will be the formula

$$AxI_j = (T(\beta,Q_1,R_1)\supset T(\alpha, Q_1,R_2)\vee F)\&(T(\tau,Q_1,A_0^3)\supset T(\alpha,Q_1,A_0^3)\vee F).$$

Here F is either F_1 or F_2. Difference between F_1 and F_2 isn't essential almost always, and in those cases, when we need F_1 or F_2, we shall note this fact specially.

Define the axiom AxP as follows:

$$AxP = \mathop{\&}_{I_j \in P} AxI_j .$$

LEMMA 4. If $(x,y,z) \in P(i,j,k)$, then

$$Int+AxP \vdash T(x, A^2_y, A^3_z) \supset T(i, A_j, A_k) \vee F.$$

PROOF. For any (i,j,k) it is proved by induction on the number of steps from (i,j,k) to (x,y,z). For the case where this number is 0, it is obvious that $T(x, A^2_y, A^3_z) \supset T(i, A^2_j, A^3_k) \vee F$ is provable.

Now suppose that the lemma holds for computation r steps long, and let (ξ, ζ, ς) be the configuration after the first r steps an $r+1$ steps computation from (i,j,k). By the induction hypothesis, $T(\xi, A^2_\zeta, A^3_\varsigma) \supset T(i, A^2_j, A^3_k) \vee F$ is provable. Now the next step of the computation is to apply I_ℓ. We shall treat the case, where I_ℓ is of the form:''being in q_ξ, subtract 1 from s_1, if $s_1 \neq 0$, and go to q_χ, otherwise go to q_γ''. The other cases are similar.

First consider possibility that $\zeta = 0$. Then, at the $(r+1)$st step, after the application of instruction, the configuration will be $(\gamma, 0, \varsigma)$, so we must show that $T(\gamma, A^2_0, A^3_\varsigma) \supset T(i, A^2_j, A^3_k) \vee F$ is provable. But the formula AxI_ℓ corresponding to I_ℓ contains a conjunct

$$T(\gamma, A^2_0, R_1) \supset T(\xi, A^2_0, R_1) \vee F,$$

which gives rise to

$$T(\gamma, A_0^2, A_\xi^3) \supset T(\xi, A_0^2, A_\xi^3) \vee F$$

by substitution. Taken with

$$T(\xi, A_\zeta^2, A_\xi^3) \supset T(i, A_j^2, A_k^3) \vee F$$

this leads to the desired result.

If $\zeta \neq 0$, then at the (r+1)st step, after the application of instruction, the configuration will be $(x, \zeta-1, \xi)$, so we must show that $T(x, A_y^2, A_z^3) \supset T(i, A_j^2, A_k^3) \vee F$ is provable. Now we can use the conjunct $T(x, Q_1, R_1) \supset T(\xi, Q_2, R_1) \vee F$ of AxI_ℓ and proceed as above.

REMARK. Lemmas 10 and 11 nave as the consequence the converse of lemma 4. Thus a calculus Int+AxP is undecidable by an appropriate choice of program P.

§4. The proofs of the principal results

Define the following formulas:

$$G = ((C_1 \& D_1 \supset p) \supset (C_2 \& D_2 \supset C_1 \vee D_1),$$

$$E = (p \& D_1 \supset s_2 \vee C_1) \& (p \& C_1 \supset t_2 \vee D_1) \supset (C_2 \& D_2 \supset C_1 \vee D_1),$$

$$H = G \& E \supset F,$$

$$B(P,(\alpha,m,n),(i,j,k)) = A\times P\&((T(\alpha,A^2_m,A^3_n)\supset T(i,A^2_j,A^3_k)\vee F)\supset F)\&(F\vee H).$$

LEMMA 5. If $(\alpha,m,n)\in P(i,j,k)$, then $B(P,(\alpha,m,n),(i,j,k))$ is first--order definable. Besides, if F is F_1, then $B(P,(\alpha,m,n),(i,j,k))$ is $\forall$-definable, and if F is F_2, the $B(P,(\alpha,m,n),(i,j,k))$ is $\forall\exists$-definable, but isn't $\forall$-definable.

PROOF. We use the lemma 3. It's enough for us to show that if $(\alpha,m,n)\in P(i,j,k)$, then $B(P,(\alpha,m,n),(i,j,k))$ is deductively equivalent to F.

By lemma 4 we obtain that

$$Int+B(P,(\alpha,m,n),(i,j,k)) \vdash T(\alpha,A^2_m,A^3_n)\supset T(i,A^2_j,A^3_k)\vee F,$$

and using the second conjunct of $B(P,(\alpha,m,n),(i,j,k)$ and modus ponens we have

$$Int+B(P,(\alpha,m,n),(i,j,k)) \vdash F.$$

Now we note that all conjuncts of $B(P,(\alpha,m,n),(i,j,k))$ have one of the forms: $A\vee F$, $A\supset B\vee F$, and so

$$Int+F \vdash B(P,(\alpha,m,n),(i,j,k)).$$

LEMMA 6. If $(\alpha,m,n)\notin P(i,j,k)$, then $B(P,(\alpha,m,n),(i,j,k))$ isn't first-order definable.

PROOF. Let $(\alpha,m,n)\notin P(i,j,k)$. We shall write B instead of $B(P,(\alpha,m,n),(i,j,k))$. For the proof we construct some uncountable frame $\mathfrak{F}$, in which B is valid, and in some its countable elementary subframe (as submodel [5]) $\mathfrak{F}^*$ the formula B is false (cf.[1]).

Define the frame $\mathfrak{F}$ as follows. The set of elements W of $\mathfrak{F}$ is to contain:

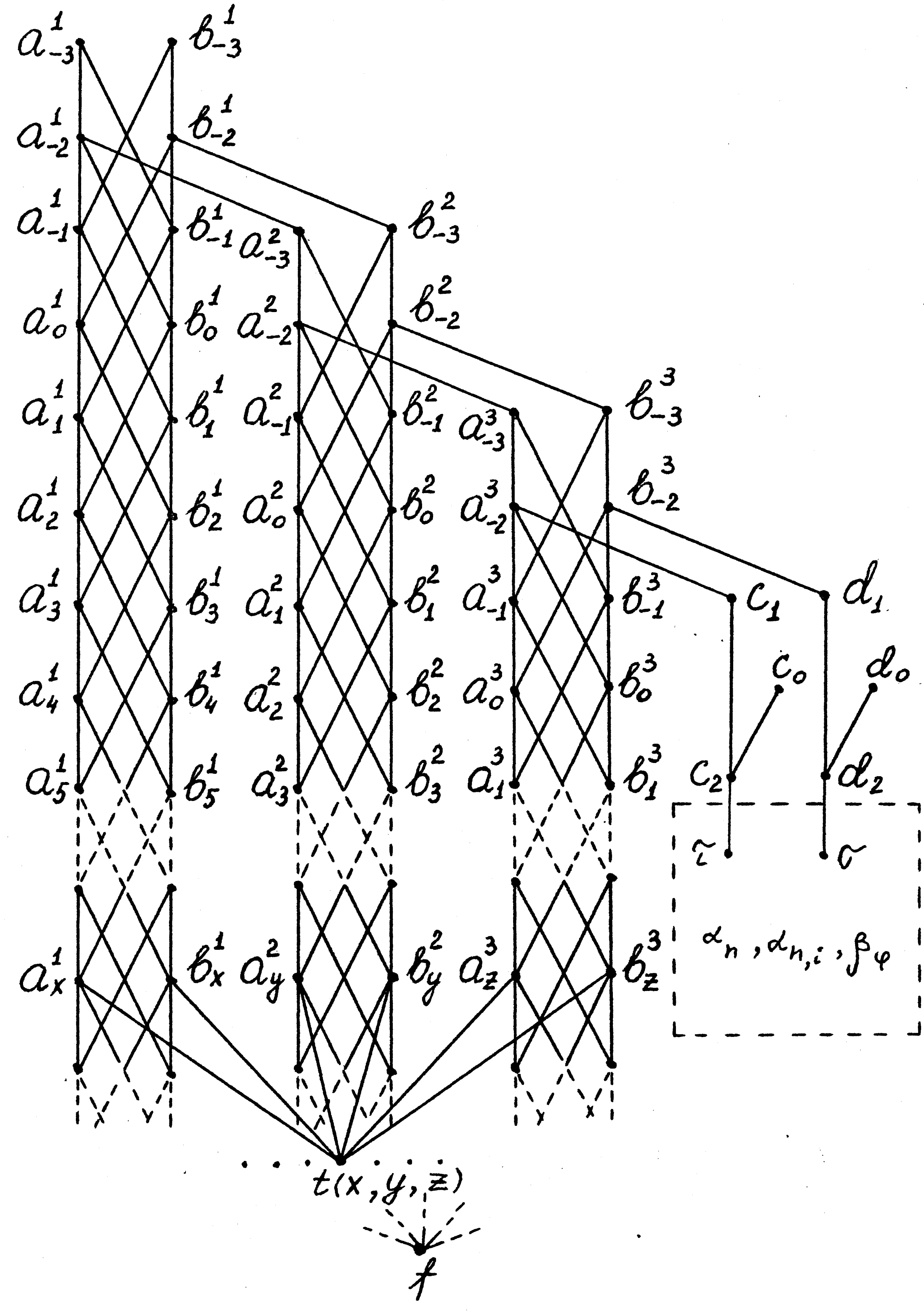
f, c_1, d_1, c_2, d_2, c_0, d_0, τ, σ,

a^n_m, b^n_m for each $n\in\{1,2,3\}$, $-3\le m<\omega$,

$t(p,q,r)$, where $(p,q,r)\in P(i,j,k)$,

$\alpha_n, \alpha_{n,i}$ for each $n\in\omega$, $i\in\{0,1\}$,

β_φ, where $\varphi\in 2^\omega$.



a^1_{-3} b^1_{-3}
a^1_{-2} b^1_{-2}
a^1_{-1} b^1_{-1} a^2_{-3} b^2_{-3}
a^1_0 b^1_0 a^2_{-2} b^2_{-2}
a^1_1 b^1_1 a^2_{-1} b^2_{-1} a^3_{-3} b^3_{-3}
a^1_2 b^1_2 a^2_0 b^2_0 a^3_{-2} b^3_{-2}
a^1_3 b^1_3 a^2_1 b^2_1 a^3_{-1} b^3_{-1} c_1 d_1
a^1_4 b^1_4 a^2_2 b^2_2 a^3_0 b^3_0 c_0 d_0
a^1_5 b^1_5 a^2_3 b^2_3 a^3_1 b^3_1 c_2 d_2
τ σ
a^1_x b^1_x a^2_y b^2_y a^3_z b^3_z
α_n, $\alpha_{n,i}$, β_φ
$t(x, y, z)$
f

Figure 2.

All of the above mentioned elements of W are to be distinct from one another. Now we define the relation $\leqslant$ on W as a closure to a partial ordering of the following binary relation R:

$$
\begin{aligned}
xRy \rightleftharpoons\ & x=f \vee (x=a^s_t \,\&\, y=a^s_p \,\&\, t\geq p) \vee (x=b^s_t \,\&\, y=b^s_p \,\&\, t\geqslant p) \vee (x=a^s_t \,\&\, y=b^s_p \,\&\, p\leq t-2) \vee \\
& \vee (x=b^s_t \,\&\, y=a^s_p \,\&\, p\leq t-2) \vee (x=a^2_{-3} \,\&\, y=a^1_{-2}) \vee (x=a^3_{-3} \,\&\, y=a^2_{-2}) \vee (x=c_1 \,\&\, y=a^3_{-2}) \vee \\
& \vee (x=b^3_{-3} \,\&\, y=b^2_{-2}) \vee (x=b^2_{-3} \,\&\, y=b^1_{-2}) \vee (x=d_1 \,\&\, y=b^3_{-2}) \vee (x=c_2 \,\&\, y=c_1) \vee (x=d_2 \,\&\, y=d_1) \vee \\
& \vee (x=c_2 \,\&\, y=c_0) \vee (x=d_2 \,\&\, y=d_0) \vee (x=t(p,q,r) \,\&\, y\in\{a^1_p,\ b^1_p,\ a^2_q,\ b^2_q,\ a^3_r,\ b^3_r\}) \vee \\
& \vee (x=\tau \,\&\, y=c_2) \vee (x=\sigma \,\&\, y=d_2) \vee (x=\alpha_n \,\&\, y=\alpha_{n,i}) \vee (x=\alpha_{n,0} \,\&\, y=c_1) \vee (x=\alpha_{n,1} \,\&\, y=d_1) \vee \\
& \vee (x=\beta_\varphi \,\&\, ((y=\alpha_{n,0} \,\&\, n\notin\varphi) \vee (y=\alpha_{n,1} \,\&\, n\in\varphi) \vee y\in\{c_1,\ d_1\})),
\end{aligned}
$$

where each disjunctive member is the disjunction on all possible meanings of undefined indexes. Frame $\mathfrak{F}$ is depicted graphically in Figure 2.

LEMMA 7. For any $x\in W$, $x\nvDash F_1$ $(x\nvDash F_2)$ iff $x=f$ and either

$$
\left.
\begin{array}{l}
\{x\ /\ x\not\Vdash A^1_{-3}\}=\{a^1_{-3}\}\quad (\{x\ /\ x\vDash s_8 \,\&\, \neg t_8\}=\{a^1_{-3}\}), \\
\{x\ /\ x\not\Vdash B^1_{-3}\}=\{b^1_{-3}\}\quad (\{x\ /\ x\vDash t_8 \,\&\, \neg s_8\}=\{b^1_{-3}\}), \\
\{x\ /\ x\not\Vdash A^1_{-2}\}=\{a^1_{-2}\},\quad \{x\ /\ x\not\Vdash B^1_{-2}\}=\{b^1_{-2}\}, \\
\{x\ /\ x\not\Vdash A^2_{-3}\}=\{a^2_{-3}\},\quad \{x\ /\ x\not\Vdash B^2_{-3}\}=\{b^2_{-3}\},
\end{array}
\right)
$$

$$\left.\begin{array}{ll}
\{x \ / \ x \nVdash A^2_{-2}\} = \{a^2_{-2}\}, & \{x \ / \ x \nVdash B^2_{-2}\} = \{b^2_{-2}\}, \\
\{x \ / \ x \nVdash A^3_{-3}\} = \{a^3_{-3}\}, & \{x \ / \ x \nVdash B^3_{-3}\} = \{b^3_{-3}\}, \\
\{x \ / \ x \nVdash A^3_{-2}\} = \{a^3_{-2}\}, & \{x \ / \ x \nVdash B^3_{-2}\} = \{b^3_{-2}\}, \\
\{x \ / \ x \nVdash C_0\} = \{c_0\}, & \{x \ / \ x \nVdash D_0\} = \{d_0\}, \\
\{x \ / \ x \nVdash C_1\} = \{c_1\}, & \{x \ / \ x \nVdash D_1\} = \{d_1\}, \\
\{x \ / \ x \nVdash C_2\} = \{c_2\}, & \{x \ / \ x \nVdash D_2\} = \{d_2\}, \\
\{x \ / \ x \nVdash C_3\} = \{\tau\}, & \{x \ / \ x \nVdash D_3\} = \{\sigma\},
\end{array}\right\} \quad (*)$$

or

$$\left.\begin{array}{ll}
\{x \ / \ x \nVdash A^1_{-3}\} = \{b^1_{-3}\} & (\{x \ / \ x \vDash s_8 \,\&\, \neg t_8\} = \{b^1_{-3}\}), \\
\{x \ / \ x \nVdash B^1_{-3}\} = \{a^1_{-3}\} & (\{x \ / \ x \vDash t_8 \,\&\, \neg s_8\} = \{a^1_{-3}\}), \\
\{x \ / \ x \nVdash A^1_{-2}\} = \{b^1_{-2}\}, & \{x \ / \ x \nVdash B^1_{-2}\} = \{a^1_{-2}\}, \\
\{x \ / \ x \nVdash A^2_{-3}\} = \{b^2_{-3}\}, & \{x \ / \ x \nVdash B^2_{-3}\} = \{a^2_{-3}\}, \\
\{x \ / \ x \nVdash A^2_{-2}\} = \{b^2_{-2}\}, & \{x \ / \ x \nVdash B^2_{-2}\} = \{a^2_{-2}\}, \\
\{x \ / \ x \nVdash A^3_{-3}\} = \{b^3_{-3}\}, & \{x \ / \ x \nVdash B^3_{-3}\} = \{a^3_{-3}\}, \\
\{x \ / \ x \nVdash A^3_{-2}\} = \{b^3_{-2}\}, & \{x \ / \ x \nVdash B^3_{-2}\} = \{a^3_{-2}\}, \\
\{x \ / \ x \nVdash C_0\} = \{d_0\}, & \{x \ / \ x \nVdash D_0\} = \{c_0\}, \\
\{x \ / \ x \nVdash C_1\} = \{d_1\}, & \{x \ / \ x \nVdash D_1\} = \{c_1\}, \\
\{x \ / \ x \nVdash C_2\} = \{d_2\}, & \{x \ / \ x \nVdash D_2\} = \{c_2\}, \\
\{x \ / \ x \nVdash C_3\} = \{\sigma\}, & \{x \ / \ x \nVdash D_3\} = \{\tau\}.
\end{array}\right\} \quad (**)$$

PROOF. The statement follows from lemmas 1 and 2, respectively.

In the further we shall suppose that the set of conditions (*) is satisfied. The case, when the conditions (**) hold, is similar.

LEMMA 8. If $\tilde{\mathfrak{F}} \nvDash F$, then, for any $x \in W$ and any number $n \geq -3$ and

$s\in\{1,2,3\}$,

a) $x \Vdash\!\!\not\; A_h^s$ iff $x=a_h^s$, b) $x \Vdash\!\!\not\; B_h^s$ iff $x=b_h^s$.

(In the case, when $F=F_2$ $x\models s_8 \& \neg t_8$ iff $x=a_{-3}^1$, $x\models t_8 \& \neg s_8$ iff $x=b_{-3}^1$).

PROOF. We use induction on n. The cases for n=-2 and n=-3 hold by lemma 7.

Now suppose that $x \Vdash\!\!\not\; A_k^s$ ($k\geq -1$). It means that $x\models B_{k-1}^s$, $x\not\models A_{k-1}^s$, $x\not\models B_{k-2}^s$. By the induction hypothesis, $x\not\leq b_{k-1}^s$, $x\leq a_{k-1}^s$, $x\leq b_{k-2}^s$. Thus $x=a_k^s$.

Similarly $y \Vdash\!\!\not\; B_k^3$ implies $y=b_k^3$.

For the converse we use the fact if $x=a_k^s$ and $y=b_k^s$ then $x \Vdash\!\!\not\; A_k^s$ and

$y \Vdash\!\!\not\; B_k^s$ because $a_k^s\leq a_{k-1}^s$, $a_k^s\leq b_{k-2}^s$, $a_k^s\not\leq b_{k-1}^s$, $b_k^s\leq b_{k-1}^s$, $b_k^s\leq a_{k-2}^s$, $b_k^s\not\leq a_{k-1}^s$ by the induction hypothesis.

LEMMA 9. If $\mathfrak{F}\not\models F$ and, for any $\alpha\in\mathfrak{F}$, natural x; ε , $\delta\in\{1,2\}$, $\alpha \Vdash\!\!\not\; T(x,Q_\varepsilon,R_\delta)$, then $\alpha=t(x,y,z)$ for some triple $(x,y,z)\in P(i,j,k)$ and

a) either $a_y^2 \Vdash\!\!\not\; Q_\varepsilon$, $b_y^2 \Vdash\!\!\not\; Q'_\varepsilon$, $a_{y+1}^2 \Vdash\!\!\not\; Q_{\varepsilon+1}$, $b_{y+1}^2 \Vdash\!\!\not\; Q'_{\varepsilon+1}$,

or $a_y^2 \Vdash\!\!\not\; Q'_\varepsilon$, $b_y^2 \Vdash\!\!\not\; Q_\varepsilon$, $a_{y+1}^2 \Vdash\!\!\not\; Q'_{\varepsilon+1}$, $b_{y+1}^2 \Vdash\!\!\not\; Q_{\varepsilon+1}$,

b) either $a_z^3 \Vdash\!\!\not\; R_\delta$, $b_z^3 \Vdash\!\!\not\; R'_\delta$, $a_{z+1}^3 \Vdash\!\!\not\; R_{\delta+1}$, $b_{z+1}^3 \Vdash\!\!\not\; R'_{\delta+1}$,

or $a_z^3 \Vdash\!\!\not\; R'_\delta$, $b_z^3 \Vdash\!\!\not\; R_\delta$, $a_{z+1}^3 \Vdash\!\!\not\; R'_{\delta+1}$, $b_{z+1}^3 \Vdash\!\!\not\; R_{\delta+1}$.

PROOF. If $\alpha \Vdash\!\!\not\; T(x,Q_\varepsilon,R_\delta)$, by lemma 8 $\alpha\not\leq a_{x+1}^1$, $\alpha\not\leq b_{x+1}^1$, $\alpha\leq a_x^1$, $\alpha\leq b_x^1$, therefore $\alpha=t(x,y,z)$ for some $y\geq 0$, $z\geq 0$. There are β_1, β_2 such that $t(x,y,z)\leq\beta_1$, $t(x,y,z)\leq\beta_2$ and $\beta_1 \Vdash\!\!\not\; Q_\varepsilon$, $\beta_2 \Vdash\!\!\not\; Q'_\varepsilon$, that implies β_1, $\beta_2\in\{a_u^2,\ b_v^2 / u,v\geq -1\}$ and $\beta_1\not\leq\beta_2$, $\beta_2\not\leq\beta_1$ by lemma 7 and constructing of the frame $\mathfrak{F}$.

If $\beta_1=a_u^2$, for some $u\geq -1$, then there are τ_1 , τ_2 such that $a_u^2\leq\tau_1$, $\beta_2\leq\tau_2$, $\tau_1 \Vdash\!\!\not\; Q_{\varepsilon-1}$, $\tau_2 \Vdash\!\!\not\; Q'_{\varepsilon-1}$, that implies τ_1, $\tau_2\in\{a_r^2$, b_s^2 / $r,s\geq -1\}$ and $\tau_1\not\leq\tau_2$, $\tau_2\not\leq\tau_1$, $\beta_1\not\leq\tau_2$, $\beta_2\not\leq\tau_1$, therefore $\tau_1=a_{u-1}^2$, $\beta_2=b_u^2$, $\tau_2=b_{u-1}^2$.

Then we have $a^2_{u+1} \Vdash\!\!\not\;\; Q_{\varepsilon+1}$, $b^2_{u+1} \Vdash\!\!\not\;\; Q'_{\varepsilon+1}$, that implies $t(x,y,z) \not\leq a^2_{u+1}$, $t(x,y,z) \not\leq b^2_{u+1}$, therefore u=y.

If $\beta_1 = b^2_u$ then $\beta_2 = a^2_u$ and u=y.

The clause b) is similar.

LEMMA 10. If V is a valuation on $\mathfrak{F}$ such that $\mathfrak{F} \nvDash F$, then, for any $\alpha \in \mathfrak{F}$ and natural $x, y \geq 0$, $z \geq 0$, $\alpha \Vdash\!\!\not\;\; T(x, A^2_y, A^3_z)$ iff $\alpha = t(x,y,z)$.

PROOF. Clearly $t(x,y,z) \Vdash\!\!\not\;\; T(x, A^2_y, A^3_z)$, by the definition of $\mathfrak{F}$. Converse affirmation is proved similarly to lemma 9.

LEMMA 11. $\mathfrak{F} \vDash AxP$.

PROOF. Show that the formulas AxI_ℓ corresponding to the instructions I_ℓ hold in the $\mathfrak{F}$.

We consider the formulas which arise from rules of form (1). So suppose instruction I_ℓ is ''being in q , add 1 to s ; go to q ''. To show that AxI_ℓ holds we first assume that $w \nvDash T(\alpha, Q_1, R_1) \vee F$. Then $w \nvDash F$, so, by lemma 7, w=f. Then $f \nvDash T(\alpha, Q_1, R_1)$, i.e. there is x such that $f \leq x$ and $x \Vdash\!\!\not\;\; T(\alpha, Q_1, R_1)$. By lemma 9, $x = t(\alpha, y, z)$, and $(\alpha, y, z) \in P(i,j,k)$. Now in such a configuration, the machine will proceed, using I_ℓ, to a configuration $(\beta, y+1, z)$, so the definition of $\leq$ provides that $f \leq t(\beta, y+1, z)$. To show that the formula holds, we must prove that $f \nvDash T(\beta, Q_2, R_1)$. But by the lemmas 8 and 9 we see that $t(\beta, y+1, z) \Vdash\!\!\not\;\; T(\beta, Q_2, R_1)$ and hence $f \nvDash T(\beta, Q_2, R_1)$. Then we have $\mathfrak{F} \vDash T(\beta, Q_2, R_1) \supset T(\alpha, Q_1, R_1) \vee F$.

The other cases are similar.

LEMMA 12. If $(\alpha, m, n) \notin P(i,j,k)$ then $\mathfrak{F} \vDash (T(\alpha, A^2_m, A^3_h) \supset T(i, A^2_j, A^3_k) \vee F) \supset F$.

PROOF. If $\mathfrak{F} \nvDash F$, then, by lemma 7,

(1) $f \nVdash F$.

By lemma 10 we have conditions

(2) $f \models T(\alpha, A^2_m, A^3_h)$,

(3) $f \nVdash T(i, A^2_j, A^3_k)$.

From conditions (1)-(3) $f \nVdash T(\alpha, A^2_m, A^3_h) \supset T(i, A^2_j, A^3_k) \vee F$.

LEMMA 13. If $\mathfrak{F} \nVdash F$, for some valuation V, then by this valuation besides (*) of lemma 7 the following conditions hold

a) $x \nVdash C_2 \& D_2 \supset C_1 \vee D_1$ iff $x = \alpha_n$ or $x = \beta_\varphi$, for some $n \in w$, $\varphi \in 2^\omega$,

b) $x \nVdash D_1 \supset s_2 \vee C_1$ implies $x = \alpha_{n,0}$, for some $n \in w$ or $x = \tau$ or $x = c_2$,

c) $x \nVdash C_1 \supset t_2 \vee D_1$ implies $x = \alpha_{n,1}$, for some $n \in w$ or $x = \sigma$ or $x = d_2$.

PROOF. a) $x \nVdash C_2 \& D_2 \supset C_1 \vee D_1$ iff (by lemma 7) $x \nleq c_2$, $x \nleq d_2$, $x \leq c_1$, $x \leq d_1$ iff $x = \alpha_n$ or $x = \beta_\varphi$,

b) $x \nVdash D_1 \supset s_2 \vee C_1$ implies (by lemma 7) $x \nleq d_1$, $x \nleq c_1$ that implies $x = \alpha_{n,0}$, for some $n \in w$,

c) is proved similarly to b).

LEMMA 14. $\mathfrak{F} \models H$.

PROOF. If $x \nVdash F$, by some valuation, then, by lemma 7, $x = f$. If $x \models E$, then, by lemma 13a), for each $n \in w$, $\alpha_n \nVdash p \& C_1 \supset t_2 \vee D_1$ or $\alpha_n \nVdash p \& D_1 \supset s_2 \vee C_1$, hence, by lemma 13b),c), $\alpha_{n,0} \models p$ or $\alpha_{n,1} \models p$. Choose $\varphi \in 2^\omega$ such that $\alpha_{n,1} \models p$, for each $n \in \varphi$. Then, clearly, $\beta_\varphi \nVdash G$, so $x \nVdash G$.

LEMMA 15. $\mathfrak{F} \models B$.

PROOF. Immediate from lemmas 11, 12 and 14.

LEMMA 16. There is elementary subframe $\mathfrak{F}^*$ of the frame $\mathfrak{F}$ such that $\mathfrak{F}^* \not\models F \vee H$ and so $\mathfrak{F}^* \not\models B$.

PROOF. Let $\mathfrak{F}^*$ be some countable elementary subframe of $\mathfrak{F}$ whose domain contains f, a^h_m, b^h_m, for all $n \in \{1,2,3\}$, $-3 \le m < w$; c_1, d_1, c_2, d_2, τ, σ, t(p,q,r), for all triples $(p,q,r) \in P(i,j,k)$; α_n, $\alpha_{n,1}$, $\alpha_{n,0}$, for all $n \in w$. There must be some $\varphi \in 2^\omega$ such that $\beta_\varphi \in W \setminus W^*$, because W is uncountable. Define V on $\mathfrak{F}^*$ by the following conditions: $V(p) = \{\alpha_{n,1} \,/\, n \in \varphi\} \cup \{\alpha_{n,0} \,/\, n \notin \varphi\} \cup \{x \mid c_1 \le x \vee d_1 \le x\}$, and V(r) for $r \in \{s_0, s_1, s_2, \ldots, s_8, t_0, t_1, t_2, \ldots, t_8\}$ is the same as in lemma 7(*). Then we have 1) $f \not\models F$, $f \models E$, 2) $f \models G$. Here 2) follows from the fact $\alpha_n \models G$ for $n \in w$ and $\beta_g \models G$ for all $\beta_g \in W^*$ (because $g \ne \varphi$). To see that 1) holds, first note that for each $n \in w$ $\alpha_n \not\models p \& C_1 \supset t_2 \vee D_1$ or $\alpha_n \not\models p \& D_1 \supset c_2 \vee C_1$. Moreover, for each $\beta_g \in W$, there is some $n \in w$ such that $n \in g \cap \varphi$. (If $g = W \setminus \varphi$ then β_g would be in W since the existence of ''complementary'' worlds β_φ is elementary expressible.) Because of this fact, $\beta_g \not\models p \& C_1 \supset t_2 \vee D_1$ or $\beta_g \not\models p \& D_1 \supset s_2 \vee C_1$. In other words, $(G \& E \supset F) \vee F$ has been shown to fail (at f) in $\mathfrak{F}^*$.

The lemma 6 is proved.

Thus, since the problem ''$(\alpha,m,n) \in P(i,j,k)$?'' is undecidable, and the formula $B(P,(\alpha,m,n),(i,j,k))$, given P, (α,m,n), (i,j,k), is constructed effectively, from the lemmas 5 and 6 the following theorems are obtained.

THEOREM 1. The problem of first-order definability of intuitionistic formulas is algorithmic undecidable.

THEOREM 2. The problem of $\forall$-definability of intuitionistic formulas is algorithmic undecidable.

THEOREM 3. The problem of $\forall\exists$-definability of intuitionistic formulas is algorithmic undecidable.

THEOREM 4. A set of intuitionistic formulas that are $\forall\exists$-definable, but aren't $\forall$-definable, is undecidable.

REMARK. If we consider the formula F as the formula F_1, then the formula $B(P,(\alpha,m,n),(i,j,k))$ doesn't contain negation, and conjunction, as it is well-known, is eliminated from any formula. Thus, in the theorems 1 and 2 we can consider intuitionistic formulas constructed from sentence letters, implication and disjunction only. Further simplification of formulas in this direction is impossible, because all disjunctionless and all implicationless intuitionistic formulas are first-order definable (cf.[7] or [8]).

4. Further results

In the following paper we suppose to present some other results on undecidability in the correspondence theory. We note some ones.

THEOREM. The problem of first-order definability of intuitionistic formulas in the class of countable frames is undecidable.

THEOREM. A set of intuitionistic formulas that are first-order definable in the class of countable frames but aren't first-order definable is undecidable.

~~THEOREM. The problem of equivalence of any intuitionistic formula and classical first-order formula is undecidable.~~

The proofs of these theorems use variants of the formula AxP that are first-order definable, and the proof of their first-order definability is quite bulky. The variant of the proof of the theorem 1 that we have given here is obtained with the help of one idea of A.V.Chagrov from [9] that is used in the proof of the lemmas 5 and 12.

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