Amalgamation in finite dimensional cylindric algebras

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February 22, 1999

Abstract

For every finite $n > 1$, the embedding property fails in the class of all n-dimensional cylindric type algebras which satisfy the following. Their boolean reducts are boolean algebras and two of the cylindrifications are normal, additive and commute. This result also holds for all subclasses containing the representable n-dimensional cylindric algebras. This considerably strengthens a result of S. Comer on CA_n and provides a strong counterexample for interpolation in finite variable fragments of first order logic. We provide a new modern proof, using an argument inspired by modal logic.

S. Comer showed that for finite n larger than 1, the embedding property fails in any class of cylindric type algebras of dimension n lying between RCA_n and CA_n (Comer, 1969). On the other hand, the proof method employed by I. Németi in (Németi, 1985) can be used to show that for any α, the class NCA_α (defined by all CA_α-axioms except commutativity of the cylindrifications) and its subclass D_α of cylindric-relativised set algebras have the strong amalgamation property. In this last class, commutativity only holds in the restricted sense of $\mathsf{c}_i(\mathsf{c}_j(x \wedge \mathsf{d}_{jk}) \wedge \mathsf{d}_{ik}) = \mathsf{c}_j(\mathsf{c}_i(x \wedge \mathsf{d}_{ik}) \wedge \mathsf{d}_{jk})$. So some combination of the NCA axioms together with some commutativity of the cylindrifications can be seen as the cause of the failure of the embedding property. It is a natural question then to ask how much commutativity and how much of the other axioms is really needed to obtain Comer's result. It turns out that for every finite n, we only need to ask that the first two cylindrifications are normal, additive and commute in only one direction. So we find a unique reason for the failure of the embedding property for every n. This considerably strengthens Comer's result. The difference lies not only in the required properties of the cylindrifications (e.g., in CA_n, they are complemented closure operators, in our case they are just operations), but in particular that for any finite n, the failure of the embedding property is caused by the behaviour of the same two cylindrifications. We provide a new modern proof, using an argument inspired by modal logic. These algebraic results have –by the correspondence theorems developed in (Pigozzi, 1972)– a direct logical meaning: they provide a strong counterexample for interpolation in first order logic with only finitely many variables.

The author is supported by UK EPSRC grant No. GR/K54946.

Mathematics Subject Classification: 03G15, 03C40. Keywords: cylindric algebras, amalgamation, interpolation.

We recall the definitions of the above mentioned classes of cylindric type algebras and their operations from (Henkin et al., 1971 1985). An algebra $\mathfrak{A} = (A, \wedge, \vee, -, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ is a cylindric type algebra of dimension α if $0, 1, \mathsf{d}_{\kappa\lambda}$ are constants, $-, \mathsf{c}_\kappa$ are unary, and $\wedge, \vee$ are binary operations. The c_κ are called cylindrifications, and the $\mathsf{d}_{\kappa\lambda}$, diagonal elements.

The class CA_α of cylindric algebras of dimension α, consists of all cylindric type algebras $\mathfrak{A} = (A, \wedge, \vee, -, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ satisfying the following conditions, for all $x, y \in A$, $\kappa, \lambda, \mu < \alpha$,

C_0 $(A, \wedge, \vee, -, 0, 1)$ is a boolean algebra
C_1 $\mathsf{c}_\kappa 0 = 0$
C_2 $x \leq \mathsf{c}_\kappa x$
C_3 $\mathsf{c}_\kappa(x \wedge \mathsf{c}_\kappa y) = \mathsf{c}_\kappa x \wedge \mathsf{c}_\kappa y$
C_4 $\mathsf{c}_\kappa \mathsf{c}_\lambda x = \mathsf{c}_\lambda \mathsf{c}_\kappa x$
C_5 $\mathsf{d}_{\kappa\kappa} = 1$
C_6 if $\kappa \neq \lambda$, then $\mathsf{d}_{\lambda\mu} = \mathsf{c}_\kappa(\mathsf{d}_{\lambda\kappa} \wedge \mathsf{d}_{\kappa\mu})$
C_7 if $\kappa \neq \lambda$, then $\mathsf{d}_{\kappa\lambda} \wedge \mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \wedge x) \leq x$.

A unary extra boolean operation f is called an operator if it is additive as in $f(x \vee y) = fx \vee fy$. It is called normal if $f0 = 0$. The class Bo_α of α–dimensional boolean algebras with operators consists of all α–dimensional cylindric type algebras which satisfy C_0 and in which all the c_κ are additive. One can prove that all the c_κ are normal operators in CA_α. The class CA_α is an axiomatic approximation of the class RCA_α of representable cylindric algebras. The latter class is defined as follows. For a nonempty set U and an ordinal α, consider the system

$$(\mathcal{P}({}^\alpha U), \cap, \cup, \sim, \emptyset, {}^\alpha U, C_\kappa, D_{\kappa\lambda})_{\kappa,\lambda<\alpha},$$

where $\mathcal{P}({}^\alpha U)$ is the powerset of ${}^\alpha U$, $\cap, \cup, \sim$ are the usual set-theoretic operations, and for $\kappa, \lambda < \alpha$ and $X \subseteq {}^\alpha U$, $C_\kappa X$ and $D_{\kappa\lambda}$ are defined by

$$D_{\kappa\lambda} = \{s \in {}^\alpha U \mid s(\kappa) = s(\lambda)\}, \text{ and}$$
$$C_\kappa X = \{s \in {}^\alpha U \mid \text{ there exists a } t \in X \text{ such that } s(\lambda) = t(\lambda) \text{ for all } \lambda \neq \kappa\}.$$

The above system is called the full cylindric set algebra of dimension α and base U, notation $\mathfrak{A}(\alpha, U)$. The class RCA_α of representable cylindric algebras of dimension α consists of all algebras isomorphic to a subdirect product of subalgebras of algebras $\mathfrak{A}(\alpha, U)$. RCA_α is a strict subclass of CA_α.

If we delete the $\mathsf{d}_{\kappa\lambda}$ constants from these definitions, we obtain algebras of the diagonal free cylindric type. More precisely, Df_α denotes all algebras $\mathfrak{A} = (A, \wedge, \vee, -, 0, 1, \mathsf{c}_\kappa)_{\kappa<\alpha}$ satisfying $C_0 - C_4$. The notation and definitions given above extend in the obvious way to diagonal free cylindric set algebras. In particular, we denote the full diagonal free cylindric set algebra of dimension α and base U by $\mathfrak{A}_{df}(\alpha, U)$ and the class of all representable diagonal free cylindric algebras by RDf_α.

For finite dimensional cylindric type algebras we use n instead of α. Axiom C_4 is called the commutativity axiom for the cylindrifications. It corresponds to the law $\exists x \exists y \varphi \leftrightarrow \exists y \exists x \varphi$ in first order logic.

Remark 0.1 In (Comer, 1969), the similarity type of diagonal free cylindric algebras is called *cylindrification type*, a full diagonal free cylindric set algebra is called a *full cylindrification set algebra*, etc. He denotes the class Df_α by Cy_α, and RDf_α by RCy_α. Here we use the, nowadays standard, terminology from (Henkin et al., 1971 1985).

Since we will give a proof inspired by modal logic, we focus attention on atom structures and use a bit of duality theory from (Goldblatt, 1989). We recall some definitions. A cylindric type atom structure of dimension α is a structure $(B, T_\kappa, E_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ with B a set, $T_\kappa \subseteq B \times B$ and $E_{\kappa\lambda} \subseteq B$. The complex algebra of an atom structure $(B, T_\kappa, E_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ is the algebra $(\mathcal{P}({}^\alpha U), \cap, \cup, \sim, \emptyset, {}^\alpha U, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ in which the non–boolean operations are defined as follows. For $X \subseteq B$,

$$\begin{aligned} \mathsf{c}_\kappa X &= \{w \in B \mid (\exists x \in B) : wT_\kappa x \ \&\ x \in X\} \\ \mathsf{d}_{\kappa\lambda} &= \{w \in B \mid E_{\kappa\lambda} w\}. \end{aligned}$$

As an example, consider for some set U, the structure $({}^\alpha U, T_\kappa, E_{\kappa\lambda})_{\kappa,\lambda<\alpha}$, defined as follows

$$\begin{aligned} sT_\kappa t &\iff s(\lambda) = t(\lambda) \text{ for all } \lambda \neq \kappa \\ E_{\kappa\lambda} &\iff s(\kappa) = s(\lambda). \end{aligned}$$

Then $\mathfrak{A}(U, \alpha)$ is nothing but the complex algebra of this atom structure.

Duality theory (cf. (Goldblatt, 1989)) studies the relations between operations on boolean algebras with operators and operations on atom structures. The dual of an embedding is a surjective bounded morphism. Let $(A, T^A_\kappa, E^A_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ and $(B, T^B_\kappa, E^B_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ be two α–dimensional cylindric type atom structures. A function $f : A \longrightarrow B$ is a bounded morphism if it satisfies for all x, y,

diag $E^A_{\kappa\lambda} x$ if and only iff $E^B_{\kappa\lambda} f(x)$

zig $xT^A_\kappa y$ implies $f(x) T^B_\kappa f(y)$, and

zag $f(x) T^B_\kappa y$ only if there exists a $y' \in A$ such that $f(y') = y$ and $xT^A_\kappa y'$.

All these notions are extended in the obvious way to the diagonal free case.

An extensive study of amalgamation in cylindric type algebras is (Pigozzi, 1972). We recall the definition of the embedding and the amalgamation property from it:

We say that a class K of algebras has the *Embedding Property (EP)* if, for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$, and embeddings f, h such that $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$, there exists $\mathfrak{D} \in \mathsf{K}$, and embeddings m, n such that $\mathfrak{B} \overset{m}{\rightarrowtail} \mathfrak{D} \overset{n}{\leftarrowtail} \mathfrak{C}$. We say that K has the *Amalgamation Property (AP)* if in addition to the above, we also require that $m \circ f = n \circ h$. We say that K has the *Strong Amalgamation Property (SAP)* if in addition to the requirements for AP, we also require that $m(B) \cap n(C) = m \circ f(A)$.

We first show how the positive results mentioned in the introduction can be obtained using the proof methods and results in (Németi, 1985). For a definition of the class Crs_α

of cylindric relativised set algebras we refer to (Henkin et al., 1971 1985) or (Andréka and Thompson, 1988). The class D_α is defined as the class of all $\mathfrak{A} \in \mathsf{Crs}_\alpha$ which satisfy $(\forall i, j \in \alpha)(\forall s \in 1^{\mathfrak{A}}) s^i_j \in 1^{\mathfrak{A}}$, where s^i_j denotes the sequence which agrees with s on $\alpha \setminus \{i\}$ and which is $s(j)$ on its ith place. The class NCA_α of non–commutative cylindric algebras of dimension α consists of all α–dimensional cylindric type algebras satisfying all CA_α equations, except for the commutative law C_4.

Theorem 0.2 *Let α be any ordinal.*
(i) The class NCA_α has SAP.
(ii) The class D_α has SAP.

PROOF. *(i)* We use Lemma 1.2 in (Sain, 1990). It implies that any variety $\mathsf{V} \subseteq \mathsf{Bo}_\alpha$ has SAP if it can be defined by a set of positive equations which correspond (in the sense of (de Rijke and Venema, 1995)) to universal Horn sentences. By (Henkin et al., 1971 1985):Theorem 1.2.6, every variety of cylindric type which satisfies C_0, C_1, C_2, C_3 is a subclass of Bo_α. The equations C_2 and C_3 correspond to every T_i being an equivalence relation, and C_7 to $\forall xy((E_{\kappa\lambda}x \wedge xT_\kappa y \wedge E_{\kappa\lambda}y) \rightarrow x = y)$. (See (Henkin et al., 1971 1985) or (Venema, 1995).) Thus the variety defined by the equations C_0, C_1, C_2, C_3, C_7 has SAP. But every subclass of a class with SAP obtained by adding equations in which no variables occur has SAP as well, as is easily verified. The equations C_5 and C_6 contain no variables, so NCA_α has SAP.
(ii) The variety D_α can be equationally defined by adding the set of equations $\{\mathsf{c}_\kappa \mathsf{d}_{\kappa\lambda} = 1 \mid \kappa\lambda < \alpha\}$ to the equations defining Crs_α. This is immediate from the above definition. Crs_α has SAP by (Németi, 1985), and we added equations without variables, whence D_α has SAP. QED

Now we turn to the central results of the paper.

Theorem 0.3 *Let $1 < n < \omega$. Let K_n be a class of n-dimensional cylindric type algebras closed under subalgebras such that*
(0) the boolean reduct of each member in K_n is a boolean algebra,
(i) c_0 and c_1 are normal operators, i.e., $\mathsf{K}_n \models \mathsf{c}_i 0 = 0$ and $\mathsf{K}_n \models \mathsf{c}_i(x \vee y) = \mathsf{c}_i x \vee \mathsf{c}_i y$ for $i \in \{0, 1\}$,
(ii) every finite RCA_n belongs to K_n, and
(iii) $\mathsf{K}_n \models \mathsf{c}_0\mathsf{c}_1 x \leq \mathsf{c}_1\mathsf{c}_0 x$.
Then the embedding property fails in K_n.

Theorem 0.4 *Let $1 < n < \omega$. Let K_n be a class of n-dimensional diagonal free algebras closed under subalgebras such that*
(0) the boolean reduct of each member in K_n is a boolean algebra,
(i) c_0 and c_1 are normal operators,
(ii) every finite RDf_n belongs to K_n, and
(iii) $\mathsf{K}_n \models \mathsf{c}_0\mathsf{c}_1 x \leq \mathsf{c}_1\mathsf{c}_0 x$.
Then the amalgamation property fails in K_n.

Comer's results on CA_n and Df_n were obtained as corollaries from two much stronger theorems (Theorems 1 and 2 in (Comer, 1969)). The main lemma on which these theorems depend is (Comer, 1969):Lemma 3. Lemma 0.5 below is our stronger version of Comer's Lemma. The only difference between the lemmas is the class of algebras used. In Comer's version, all cylindrifications in the algebra $\mathfrak{B}$ are commuting complemented closure operators; in our version only the first two cylindrifications are operators which satisfy $\mathsf{c}_0\mathsf{c}_1 x \leq \mathsf{c}_1\mathsf{c}_0 x$.

After proving the lemma we will show how the theorems follow.

Lemma 0.5 *Let K_n be as in Corollary 0.4. Suppose $1 < n \leq |U_0| < |U_1|$ and $|U_0| < \omega$. Then there does not exist a $\mathfrak{B} \in \mathsf{K}_n$ and isomorphisms f_0, f_1 such that for $i = 0, 1$, f_i embeds the full n–dimensional diagonal free cylindric set algebra $\mathfrak{A}_i$ with base U_i into $\mathfrak{B}$, and $f_0(D_0) = f_1(D_1)$ where D_i is the set of all sequences s in ${}^nU_i$ satisfying $s(0) = s(1)$.*

Proof. Fix a natural number n bigger than 1. Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be full diagonal free cylindric set algebras with bases $n = \{0, 1, \ldots, n-1\}$ and $n+1 = \{0, \ldots, n\}$ respectively. We only prove the lemma for this case since it is most instructive and avoids cumbersome notation. It will be obvious how to extend the proof to the general case.

In order to derive a contradiction, we assume that there exists a $\mathfrak{B} \in \mathsf{K}_n$ and embeddings f_i from $\mathfrak{A}_i$ into $\mathfrak{B}$ and $f_0(D_0) = f_1(D_1)$ where D_i is the set of all sequences s in ${}^nU_i$ satisfying $s(0) = s(1)$.

Let $\mathfrak{B}'$ denote the reduct $(B, \wedge, \vee, -, 0, 1, \mathsf{c}_0, \mathsf{c}_1)$ of $\mathfrak{B}$, and similarly for the $\mathfrak{A}'_i$. Then clearly the f_i embed the $\mathfrak{A}'_i$ into $\mathfrak{B}'$. We consider these reducts because our proof only depends on the first two cylindrifications.

Without loss of generality we may assume that $\mathfrak{B}'$ is the complex algebra of some atom structure (B, T_0, T_1) because of the following. Since $\mathfrak{B}' \in \mathsf{Bo}_2$ (by assumptions (0) and (i)), the canonical embedding Theorem ((Henkin et al., 1971 1985): Theorem 2.7.5) applies. By (Henkin et al., 1971 1985):Remark 2.7.14, all positive equations are preserved under canonical embeddings, whence $\mathsf{c}_0\mathsf{c}_1 x \leq \mathsf{c}_1\mathsf{c}_0 x$ remains true in the complex algebra $\mathfrak{B}'$. We denote the atom structure (B, T_0, T_1) of $\mathfrak{B}'$ by $\mathfrak{B}'_+$. By (Henkin et al., 1971 1985):Theorem 2.7.40, we have that

$$\mathfrak{B}'_+ \models \forall xyz((xT_0y \wedge yT_1z) \to \exists w(xT_1w \wedge wT_0z)). \tag{1}$$

In a suggective picture:

$$\begin{tikzcd} & z \\ x \arrow[r, "T_0"'] & y \arrow[u, "T_1"'] \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} w \arrow[r, "T_0"] & z \\ x \arrow[u, "T_1"] \arrow[r, "T_0"'] & y \arrow[u, "T_1"'] \end{tikzcd}$$

By (Goldblatt, 1989): Corollary 3.2.5, the embeddings f_i induce surjective bounded morphisms f_{i+} from $\mathfrak{B}'_+$ onto $\mathfrak{A}'_{i+}$ which satisfy for all atoms x,

$$f_{0+}(x) \in D_0 \text{ if and only if } f_{1+}(x) \in D_1.$$

Since the $\mathfrak{A}'_i$ are finite, the canonical embedding algebra of $\mathfrak{A}'_i$ is isomorphic to $\mathfrak{A}'_i$ and the atoms in $\mathfrak{A}'_{i+}$ are precisely the n-sequences.

By using zig, zag and condition (1) in a specific order we obtain a picture of $\mathfrak{B}'_+$ which will lead to our desired contradiction. We use $\vec{0}$ to denote a $n-2$ long sequence of zero's. By surjectiveness and zag, we have atoms $x_0^0, x_0^1, \ldots, x_0^n$ and x_1^n in $\mathfrak{B}'_+$ such that $f_{1+}(x_j^i) = \langle i, j, \vec{0}\rangle$, and $x_0^i T_0 x_0^{i+1}$ and $x_0^n T_1 x_1^n$ holds, see Figure 1.a. Then by (1), we find atoms $x_1^0, \ldots, x_1^{n-1}$ such that $x_0^i T_1 x_1^i$ and $x_1^i T_0 x_1^{i+1}$ holds in $\mathfrak{B}'_+$. See Figure 1.b. By zig then it must be that also for these atoms $f_{1+}(x_j^i) = \langle i, j, \vec{0}\rangle$. We call the set $\{s \in {}^nU_i \mid (\exists m < n)(\forall j \geq 2)s(j) = m\}$ a slice of $\mathfrak{A}_i$. Repeating this argument in the same order, we find $n+1$ rows and columns, all linked in this grid style manner, corresponding to the first slice of $\mathfrak{A}_1$. See Figure 1.c. Formally $\mathfrak{B}'_+$ contains atoms x_j^i for $0 \leq i, j \leq n$ such that

$$x_j^i T_0 x_j^{i+1} \text{ and } x_j^i T_1 x_{j+1}^i \text{ and } f_{1+}(x_j^i) = \langle i, j, \vec{0}\rangle. \tag{2}$$

Note that $f_{1+}(x_j^i)$ is below D_1 if and only if $i = j$. The function f_{0+} from $\mathfrak{B}'_+$ to $\mathfrak{A}'_{0+}$ is also a surjective bounded morphism. The zig-condition on T_0 and T_1 restricts the values of f_{0+} as follows:

$$\text{there exists an } m < n \text{ such that for all } x, \text{ for all } i \text{ and for all } k \geq 2\text{: } f_{0+}(x_i^i)(k) = m, \tag{3}$$

i.e., all the f_{0+}- images of the x_i^i come from one slice in $\mathfrak{A}_0$ as well. This is because of the following. The atoms x_i^i are sent by f_{1+} to atoms below D_1, so f_{0+} must send them below D_0 in $\mathfrak{A}_0$. Let x_i^i and x_j^j be two atoms in $\mathfrak{B}'_+$. Suppose $i < j$. Then $\mathfrak{B}'_+ \models x_i^i T_0^* x_i^j T_1^* x_j^j$ by (2), where T_i^* abbreviates a finite T_i path. But then by zig of f_{0+} $\mathfrak{A}_{0+} \models f_{0+}(x_i^i) T_0^* f_{0+}(x_i^j) T_1^* f_{0+}(x_j^j)$, and this means that the other coordinates cannot have been changed.

Since there are $n+1$ of these x_i^i atoms in $\mathfrak{B}'_+$, and only n D_0-atoms in one "slice" in $\mathfrak{A}_0$, we find $i \neq i'$ such that $f_{0+}(x_i^i) = f_{0+}(x_{i'}^{i'})$. Assume $i < i'$. Then in $\mathfrak{B}'_+$, $x_i^i T_0^* x_i^{i'}$ and $x_i^{i'} T_1^* x_{i'}^{i'}$. Since $i \neq i'$, $f_{1+}(x_i^{i'})$ is not below D_1, whence also $f_{0+}(x_i^{i'})$ cannot be below D_0.

Now zig implies that $f_{0+}(x_i^i) T_0^* f_{0+}(x_i^{i'})$ and $f_{0+}(x_i^{i'}) T_1^* f_{0+}(x_{i'}^{i'})$ holds in $\mathfrak{A}'_{0+}$. Since T_0 and T_1 are equivalence relations in $\mathfrak{A}'_{0+}$ and $f_{0+}(x_i^i) = f_{0+}(x_{i'}^{i'})$, this means that $\mathfrak{A}_0 \models \{f_{0+}(x_i^i)\} \leq \mathsf{c}_0\{f_{0+}(x_i^{i'})\}$ and $\mathfrak{A}_0 \models \{f_{0+}(x_i^{i'})\} \leq \mathsf{c}_1\{f_{0+}(x_i^i)\}$. Whence, since $f_{0+}(x_i^i)$ is an atom below D_0 and $f_{0+}(x_i^{i'})$ an atom not below D_0 we have for some singleton set $x \in A$, $\mathfrak{A} \models D_0 \wedge x \wedge \mathsf{c}_0(-D_0 \wedge \mathsf{c}_1 x) \neq 0$. Since D_0 is interpreted as the diagonal D_{01} in $\mathfrak{A}_0$, this is clearly impossible.

QED

PROOF OF THEOREM 0.3. Let K_n be as stated in the theorem. The algebras $\mathfrak{A}_i$ from Lemma 0.5 can be embedded into full cylindric set algebras $\mathfrak{A}(n, U_i)$ mapping the D_i–elements onto d_{01}. By condition (ii) they are in K_n. By (Henkin et al., 1971 1985):Theorem 2.1.17ff they have a common isomorphic subalgebra generated by the d_{ij} elements. (The intuitive argument is that a first-order sentence using n variables and only identity cannot express that the domain has size larger than n.) Applying the lemma we obtain that there is no $\mathfrak{B}$ in K_n in which the $\mathfrak{A}_i$ can be jointly embedded. QED

$$\begin{tikzcd}
 & & & & x_1^n \\
x_0^0 \arrow[r, "T_0"'] & x_0^1 \arrow[r] & \cdots & x_0^{n-1} \arrow[r, "T_0"'] & x_0^n \arrow[u, "T_1"']
\end{tikzcd}$$

Figure 1.a

$$\begin{tikzcd}
x_1^0 \arrow[r] & x_1^1 \arrow[r] & \cdots & x_1^{n-1} \arrow[r] & x_1^n \\
x_0^0 \arrow[u] \arrow[r, "T_0"'] & x_0^1 \arrow[u] \arrow[r] & \cdots & x_0^{n-1} \arrow[u] \arrow[r, "T_0"'] & x_0^n \arrow[u, "T_1"']
\end{tikzcd}$$

Figure 1.b

$$\begin{tikzcd}
x_n^0 \arrow[r] & x_n^1 \arrow[r] & \cdots & {} \arrow[r] & x_n^n \\
x_{n-1}^0 \arrow[u] & & \cdots & & x_{n-1}^n \arrow[u, "T_1"'] \\
\vdots & & \vdots & & \vdots \\
x_1^0 \arrow[r] & x_1^1 \arrow[r] & \cdots & x_1^{n-1} \arrow[r] & x_1^n \\
x_0^0 \arrow[u] \arrow[r, "T_0"'] & x_0^1 \arrow[u] \arrow[r] & \cdots & x_0^{n-1} \arrow[u] \arrow[r, "T_0"'] & x_0^n \arrow[u, "T_1"']
\end{tikzcd}$$

Figure 1.c

Figure 1: The atom structure $\mathfrak{B}'_+$.

Proof of Theorem 0.4. Apply a similar argument. QED

Relation Algebras. (Comer, 1969):Theorem 1 implies that for any class of relation algebras between RA and RRA, the embedding property fails. Using our stronger lemma we can extend the range of relational type algebras for which this holds. We outline the differences in the proof, for the full argument we refer to (Comer, 1969). In order to obtain an equational definitional embedding into the class K_2 defined in Theorem 0.3, it is only required that the boolean laws hold and that (the composition operator) ; is normal, additive, and satisfies $1;(x;1) \leq (1;x);1$. (We use the relationship between RA and CA found on p. 135 of (Jónsson and Tarski, 1951). c_0x is defined as $1;x$ and c_1x as $x;1$.) So, using Comer's argument, the embedding property fails in every class of relational type algebras satisfying at least these equations, and containing the finite representable relation algebras.

(Marx, 1998):Theorem 5 states that also the law $(x;1);1 \leq x;1$ is sufficient for failure of the embedding property. Combining these results we obtain

Theorem 0.6 *Let* V *be a variety of relational type algebras such that*

(0) the boolean reduct of each member in V *is a boolean algebra,*

(i) the composition operator is normal and additive,

(ii) all finite representable relation algebras belong to V*, and*

(iii) $\mathsf{V} \models 1;(x;1) \leq (1;x);1$ ***or*** $\mathsf{V} \models (x;1);1 \leq x;1$.

Then the embedding property fails in V.

Note that the two disjuncts in *(iii)* are not comparable in strength on the relation algebraic level. They differ considerably when we use them to give an equational definitional embedding into the class K_2. The first disjunct causes that $\mathsf{K}_2 \models \mathsf{c}_0\mathsf{c}_1x \leq \mathsf{c}_1\mathsf{c}_0x$, the second only that $\mathsf{K}_2 \models \mathsf{c}_1\mathsf{c}_1x \leq \mathsf{c}_1x$. This last equation holds in NCA_2; it corresponds to a universal Horn sentence on the atom structures, whence is harmless for amalgamation (see the proof of Theorem 0.2).

First Order Logic. The connection between the amalgamation property of cylindric type algebras and the interpolation property in first order logic is established in (Pigozzi, 1972). From the results reported here we can derive the following for every finite $n > 1$. There are first order sentences φ and ψ such that the truth of $\varphi \models \psi$ can be proved using only the axioms corresponding to the equations in $(0), (i), (iii)$ of Theorem 0.3, for which there is no interpolant in only n variables. The only atomic relation symbol that is common in φ and ψ is the equality symbol. Both φ and ψ only contain two variables. Without the restriction on the proof of $\varphi \models \psi$ this was also obtained in (Andréka et al., 1995).

Acknowledgement. The author thanks Yde Venema for a careful reading of the manuscript.

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