A Unifying Completeness Theorem in Quantified Modal Logic

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December 14, 2001

Abstract

A general strategy for proving completeness theorems for quantified modal logics is provided. Starting from free quantified modal logic K, with or without identity, extensions obtained either by adding the principle of universal instantiation or the converse of the Barcan formula or the Barcan formula are considered and proved complete in a uniform way. Completeness theorems are also shown for systems with the extended Barcan rule as well as for some quantified extensions of the propositional modal logic B. The incompleteness of $Q^\circ.B{+}BF$ is proved too.[1]

In this paper we deal with the basic systems of quantified modal logic (q.m.l.) starting from the one presented by S.Kripke in his pioneer paper [6], 1963. The lack of "... a common completeness proof that can cover constant domains, varying domains, and models meeting other conditions..." has often been felt, see [2], p.132. In [3], p.273 or [4] we read " Ideally, we would like to find a completely general completeness proof." We intend to answer in a positive way to the worries concerning the proof theory of q.m.l. by showing that the system $Q^\circ.K$, Kripke's original one with the addition of individual constants, is

*This paper was written in the Fall 2001 during my visit to the Institute of Logic, Language and Computation in Amsterdam. I am very grateful to Prof. Dick de Jongh who made my stay at ILLC profitable and enjoyable. His comments to the present paper have been most valuable. My thanks also to the Netherlands Institute for Advanced Study, NIAS, who provided me, even if indirectly, with a very pleasant place to live near Wassenaarseslag.

[1]The first part of this paper was presented at the workshop "Methods for Modalities 2", Amsterdam, 29-30 November 2001.

complete with respect to Kripke frames with varying domains, and that that very proof yields completeness results for well known extensions of $Q^\circ.K$ such as those characterized either by increasing or constant domains, both with or without non-existing objects, with or without identity. In the first part of the paper we will deal with the systems mentioned in the diagram below:

$$Q^\circ.K$$

$$Q^\circ.K + CBF \qquad\qquad\qquad\qquad Q^\circ.K + CBF + BF$$

$$Q.K$$

$$Q.K + BF$$

In the second part we will consider systems containing the identity relation. The main goal we are aiming at is offering a clear framework in which each completeness result considered, old or new, will find its natural place. Sometimes we will follow through the proof of a known result just to show how it fits into our framework. Alternatively we may make use of strategies present, one way or the other, in the literature, and we will give appropriate acknowledgement in each case.

As we will see, $Q^\circ.K$ ($Q.K$) is obtained by adding to the propositional normal modal logic K the quantificational axioms and rules of free (classical) logic. The main feature of $Q^\circ.K$ is that the principle of Universal Instantiation is not a theorem, but only its universal closure is. Semantically this fact has the important consequence that each world w of a model for $Q^\circ.K$ is endowed both with an *inner* domain, D_w, that represents the set of objects existing at w and coincides with the domain of variation of the quantifiers and with an *outer* domain U_w, $U_w \supseteq D_w$, that contains also possible non-existing objects and that coincides with the domain of interpretation of the variables, the predicate and constant symbols at w. Once the full axiom of Universal Instantiation is present no distinction is made between existing and non-existing individuals and only one domain is associated with each world. Here are the four formulas we will be most concerned with:

UI°	$\forall y(\forall x A(x) \rightarrow A(y/x))$
UI	$\forall x A(x) \rightarrow A(t/x)$
CBF	$\Box\forall x A \rightarrow \forall x\Box A$
BF	$\forall x\Box A \rightarrow \Box\forall x A$

1 Modal systems without identity

First Order Modal Languages and Kripke Semantics

The *alphabet* of first-order modal language $\mathcal{L}$ (without identity) contains the unary connective $\Box$ (box) in addiction to the boolean connectives $\neg$ (not) and $\vee$ (or) and the quantifier $\exists$

(there is). Moreover $\mathcal{L}$ contains a countable set, *Var*, of variables, $x_1, x_2, x_3, \ldots$, the symbol of falsehood, $\bot$, and the following two sets, at most countable, J and P, of, respectively, individual constants $b, c, d, b_1, c_1, d_1, b_2, c_2, d_2, \ldots$, and predicate symbols, $P^n, Q^n, R^n, \ldots$ of arity n, $0 \leq n \leq \omega$.

A *term* is either a variable or an individual constant.[2]

Well formed formulas (wffs)

1. $\bot$ is an *atomic formula*,
2. If P^n is an n-ary predicate symbol and $t_1, \ldots, t_n$ are n terms, then $P^n(t_1, \ldots, t_n)$ is an *atomic formula*,
3. If A and B are wffs and x is a variable, then $\neg A$, $\Box A$, $A \vee B, \exists x A$ are wffs,
4. Nothing else is a well formed formula.

The formulas $A \wedge B, A \rightarrow B, \Diamond A, \forall x A$ are defined in the usual way. By $A(t/s)$ we denote the formula obtained from the wff $A(s)$ by replacing all free occurrences of s by t, changing the name of bound variables, if necessary, to avoid rendering the new occurrences of t bound in $A(t/s)$. $A(t//s)$ denotes that *some* (*all, none*) occurrences of s are replaced by t. $A(s_1, ..., t/s_i, ...s_n)$ stands for $A(s_1, ..., s_i, ...s_n)(t/s_i)$.

A *Kripke-frame*, K-frame, is a quadruple $\mathcal{F} = \langle W, R, D, U \rangle$ where

W is a non-empty set,

R is a binary relation on W, the *accessibility relation*,

D is a function which associates to each $w \in W$ a set D_w. D_w is the *inner* domain of w, and it can be empty.

U is a function which associates to each $w \in W$ a set U_w such that:

$U_w \neq \emptyset$ and if wRv than $U_w \subseteq U_v$. U_w is the *outer* domain of w.

The fact that $U_w \subseteq U_v$, if wRv, doesn't prevent D_w from being disjoint from D_v. In [6], Kripke stipulates that for all $v \in W$, $U_v = \bigcup_{w \in W} D_w$. We generalize Kripke's original semantics by allowing $U_w \subseteq U_v$ if wRv, and $\bigcup_{w \in W} U_w \supseteq \bigcup_{w \in W} D_w$.[3] $\bigcup_{w \in W} U_w$ may contain individuals that never happen to come into existence.

When no condition is imposed on the domain function D, $\mathcal{F}$ is said to have *varying* domains, when wRv implies $D_w \subseteq D_v$ ($D_w \supseteq D_v$, $D_w = D_v$), $\mathcal{F}$ is said to have *increasing* (*decreasing, constant*) domains. (The outer domains are always increasing.)

A *K-model* is given by a K-frame plus a function I that together with every $w \in W$ determines an *interpretation* I_w of the descriptive symbols of the language. In particular,

$$I_w(P^n) \subseteq (U_w)^n \qquad \text{and} \qquad I_w(c) \in U_w.$$

[2] In [6] Kripke considers languages containing no individual constants.

[3] Imposing the condition that for all v, $U_v = \bigcup_{w \in W} D_w$ rules out models for a set of formulas like $\{\Box\bot, \forall x P(x), \neg P(x)\}$.

For each $w \in W$, a *w-assignment* is a function $\sigma : Var \to U_w$. Let σ and τ be two w-assignments. τ is said to be an *x-variant* of σ if σ and τ agree on all variables except possibly on the variable x. If σ is a w-assignment, it is also a v-assignment for any v such that wRv, because $U_w \subseteq U_v$. Given a w-assignment σ, we can interpret all terms of the language, by letting $I^\sigma_w(c) = I_w(c)$ and $I^\sigma_w(x) = \sigma(x)$.

The notion of a formula being *satisfied* by a w-assignment σ at w in a K-model $\mathcal{M}$ is defined so:

$\mathcal{M} \models^\sigma_w P^n(t_1, \ldots, t_n)$	iff	$\langle I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n)\rangle \in I_w(P^n)$
$\mathcal{M} \not\models^\sigma_w \perp$		
$\mathcal{M} \models^\sigma_w \neg B$	iff	$\mathcal{M} \not\models^\sigma_w B$
$\mathcal{M} \models^\sigma_w B \vee C$	iff	$\mathcal{M} \models^\sigma_w B$ or $\mathcal{M} \models^\sigma_w C$
$\mathcal{M} \models^\sigma_w \exists x B$	iff	for some x-variant τ of σ, such that $\tau(x) \in D_w$, $\mathcal{M} \models^\tau_w B$
$\mathcal{M} \models^\sigma_w \Box B$	iff	for all v such that wRv, $\mathcal{M} \models^\sigma_v B$.

A formula B is *true in a K-model $\mathcal{M}$ at w*, $\mathcal{M} \models_w B$, iff for all σ, $\mathcal{M} \models^\sigma_w B$.
A formula B is *valid on a K-model* $\mathcal{M}$, $\mathcal{M} \models B$, iff for all $w \in W$, $\mathcal{M} \models_w B$.
A formula B is *valid on a K-frame* $\mathcal{F}$, $\mathcal{F} \models B$, iff for all K-models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models B$.
A formula B is *K-valid* iff for all K-frames $\mathcal{F}$, $\mathcal{F} \models B$.

As is well-known, the following formulas are not K-valid[4]:

$\forall x \Box A \to \Box \forall x A$ (BF)	$\exists x \Box A \to \Box \exists x A$ (GF)	$\forall x A(x) \to A(t/x)$ (UI)
$\Box \forall x A \to \forall x \Box A$ (CBF)	$\Box \exists x A \to \exists x \Box A$ (CGF)	$\forall x A(x) \to \exists x A(x)$

Denotation, existence and rigidity. In the K-semantics just introduced, every constant is *denoting*, in fact for all $w \in W$ and for all constants c, $I_w(c)$ is defined, but nothing is said whether it denotes an *existing* or a *non-existing* individual, $I_w(c)$ can be in D_w as well as in $(U_w$ - $D_w)$. Moreover it is not assumed that constants are rigid designators, where an individual constant c is said to be a *rigid* designator iff

$$wRv \quad \text{implies} \quad I_w(c) = I_v(c).$$

In a language without identity no formula expresses that a constant is a rigid designator. At the semantical level, rigidity corresponds to the *standard* correlation between satisfaction and substitution as stated by the following lemma.

Lemma 1 *Let $\mathcal{M}$ be a K-model and σ a w-assignment. An individual constant c is a rigid designator iff ($\mathcal{M} \models^\sigma_w A(c/x)$ iff $\mathcal{M} \models^\tau_w A(x)$), where τ is an x-variant of σ such that $\tau(x) = I_w(c)$.*[5]

[4] As to the role of the Ghilardi formula (GF) in counterpart semantics, see [1].

[5] The equivalence stated in the lemma, ($\mathcal{M} \models^\sigma_w A(c/x)$ iff $\mathcal{M} \models^\tau_w A(x)$), corresponds to the equivalence between the *de dicto* and *de re* readings of substituted formulas, e.g., $\Box P(c/x)$ is equivalent to $\langle c\rangle \Box P(x)$.

PROOF. Suppose c is a rigid designator. The proof is by induction on A, we consider just one case. $\mathcal{M} \models^\sigma_w \Box B(c/x)$ iff for all $v.wRv$. $\mathcal{M} \models^\sigma_v B(c/x)$ iff, by induction hypothesis, $\mathcal{M} \models^\tau_v B(x)$, where τ is an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator, then $\tau(x) = I_w(c)$, and so $\mathcal{M} \models^\tau_w \Box B(x)$.
Suppose c is not a rigid designator. Take a model $\mathcal{M}$ based on two worlds, w and v such that wRv, moreover let $D_w = \{u_1\}$, $D_v = \{u_1, u_2\}$, $I_w(c) = u_1$, $I_v(c) = u_2$, $I_w(P) = I_v(P) = \{u_1\}$, where P is a unary predicate letter. Then $\mathcal{M} \not\models^\sigma_w \Box P(c/x)$ and $\mathcal{M} \models^\tau_w \Box P(x)$. $\Box$

A particular case of Kripke semantics which has been widely studied in the literature is the one we will call Tarski-Kripke semantics, TK-semantics, in order to stress the fact that a Tarski-Kripke model is just a family of classical models interconnected by the accessibility relation. A TK-frame is a Kripke frame in which for all $w \in W$, $U_w = D_w$, so each world w is endowed with just *one* domain, D_w, which is both the domain of variation of the quantifiers and the domain of interpretation of the constant and predicate symbols. Of course $D_w \neq \emptyset$ and wRv implies that $D_w \subseteq D_v$. TK-models are defined exactly as K-models.

Each world of a TK-model is a Tarskian model, and so classically valid formulas such as $\forall x A(x) \to A(x)$ or $\forall x A(x) \to A(c/x)$ or $\forall x A(x) \to \exists x A(x)$ turn out to be valid. Moreover CBF and GF are TK-valid too. On the contrary BF and CGF are not TK-valid.

A comparison with the semantics as presented in Kripke, 1963.

We will use the expression *original Kripke semantics*, OK-semantics, to refer to the semantics of Kripke, [6], 1963. An OK-model is a quadruple $\langle W, R, D, I\rangle$ where W, R and D are defined as in K-semantics. The interpretation function I, instead, differs for now I is such that $I_w(P^n) \subseteq V$ and $I_w(c) \in V$, where $V = \bigcup_{w \in W} D_w$. Analogously, the codomain of any assignment function is V. At first sight, OK-models look more general than K-models because both the assignment and the interpretation functions are not *world-bound*, in the sense that the interpretation at w of a unary predicate P, $I_w(P)$, doesn't need to be a subset of D_w, and the interpretation at w of a constant c, $I_w(c)$, doesn't need to be an element of D_w. A way of looking at this semantics is that each world has an 'inner' domain, D_w, the domain of variation of the quantifiers, that varies from world to world and can be empty, and an 'outer' domain, V, which remains fixed and is the domain of interpretation of the free variables, the predicate and constant symbols. V is the global domain of discourse, the set of all things of which we are intitled to say at each world if a predicate is true or false of them at that world. Keeping the outer domain V fixed is a heavy limitation in building models, for suppose we want to define a model based on a frame with two worlds, w and v, that first we want to define D_w and I_w and then, D_v and I_v. In defining the function I_w, we are bound to establish once and for all what the set V is like, so, no way to add new individuals when we come to define either D_v or I_v. This has induced us to generalize OK-semantics by allowing the outer domains to increase and at the same time to have *world-bound* interpretations and assignment functions. But this is no limitation because in K-semantics, as we have defined it, the codomain of I_w as well as of any w-assignment is U_w, the outer domain, and nothing prevents from each U_w to include $\bigcup_{w \in W} D_w$.

A further advantage of K-semantics is that TK-models are particular cases of K-models, just let $U_w = D_w$. Of course we can define a subclass $\mathcal{H}$ of OK-models such that a formula is $\mathcal{H}$-valid iff it is TK-valid, but then we run into problems. "In a formula like ϕx in which x is free what happens in a world w in which x is assigned as a value an individual which is not in D_w?". If we want to stick to the feature of OK-semantics that V is the codomain of any assignment function, then the only way out "is to say ϕx lacks a truth value in such worlds", [5], p.275. By doing so, TK-valid formulas such as $\forall x P(x) \to P(x)$ turn out to be $\mathcal{H}$-valid, because worlds where x is assigned an individual which is not in their inner domain are not taken into consideration since $P(x)$ lacks a truth-value there.[6] But we can hardly say that a semantics with truth-value gaps is in tune with the original Kripke semantics.

The system $Q^{\circ}.K$ and some of its extensions.

The system $Q^{\circ}.K$ contains the following axioms and inference rules.[7]

Axiom schemata:

truth-functional tautologies	$\Box(A \to B) \to (\Box A \to \Box B)$	$A \to \forall x A$, x not free in A
$\forall y(\forall x A(x) \to A(y/x))$	$\forall x(A \to B) \to (\forall x A \to \forall x B)$	$\forall x \forall y A \leftrightarrow \forall y \forall x A$

Inference rules : Modus Ponens (from A and $A \to B$ infer B), Necessitation (from A infer $\Box A$), and Universal Generalization (from A infer $\forall x A$).

The system $Q.K$ is just the system $Q^{\circ}.K$ with UI° replaced by UI.[8]

Definition 2 *Let L be any quantified modal logic which extends $Q^{\circ}.K$, $L \supseteq Q^{\circ}.K$. A* proof *in L is a sequence of formulas such that each of them is either an axiom of L or it is obtained from preceding formulas in the sequence by application of an inference rule.*

A wff A is a theorem *of L, $\vdash_L A$, iff there is proof in L whose last formula is A.*

A wff A is derivable *in L from a set Δ of formulas, $\Delta \vdash_L A$, iff for some finite number of formulas $A_1, ..., A_n$ in Δ, $\vdash_L A_1 \wedge ... \wedge A_n \to A$.*

Lemma 3 *Theorems of $Q^{\circ}.K$ that we will use in the sequel (often without mentioning them).*

[6] See [5], pp.277-280.
[7] We follow Fitting and Mendelsohn, [2], for the choice of this axiom system.
[8] To rule out empty domains add to $Q^{\circ}.K$ the axiom $\forall x A \to A$, where x is not free in A.

(i)	$\forall y(A(y) \to \exists x A(x//y))$, *Existential Instantiation.*
(ii)	$\forall z_1 \ldots \forall z_n \forall z[\forall x A(z_1, \ldots z_n, x) \to A(z_1, \ldots z_n, z/x)]$, *($z$ may or may not occur in $A(z_1, \ldots z_n, x)$.*
(iii)	*If* $\vdash_{Q^\circ.K} A_1 \wedge \cdots \wedge A_n \to B$, *then* $\vdash_{Q^\circ.K} \forall\vec{x}A_1 \wedge \cdots \wedge \forall\vec{x}A_n \to \forall\vec{x}B$, *where* $\forall\vec{x} = \forall x_1 \ldots \forall x_k$, *for some* $k \geq 0$.
(iv)	$(A \vee \forall y B(y)) \leftrightarrow \forall y(A \vee B(y))$, *where y is not free in A.*
(v)	$(A \to \forall y B(y)) \leftrightarrow \forall y(A \to B(y))$, *where y is not free in A.*
(vi)	$\forall x(A(x) \to B) \leftrightarrow (\exists x A(x) \to B)$, *where y is not free in B.*
(vii)	$\forall x A \leftrightarrow \forall y A(y/x)$.
(viii)	$\forall x A(x) \wedge \exists y B(y) \to \exists y(A(y/x) \wedge B(y))$.

Lemma 4 *Here is a list of well-known soundness results.*

q.m.l.	*is sound w.r.t. the class of K-frames with domains*	
	inner	outer
$Q^\circ.K$	*varying*	*increasing*
$Q^\circ.K$+CBF	*increasing*	*increasing*
$Q^\circ.K$+BF	*decreasing*	*constant*
$Q^\circ.K$+CBF+BF	*constant*	*constant*
$Q.K$	*increasing*	$=$ *inner*
$Q.K$+BF	*constant*	$=$ *inner*

Completeness results

The main idea behind the completeness proof we are going to present stems from a simple observation: the affinity of meaning between CBF and UI. Take an instance of UI, $\forall x P(x) \to P(x)$. The falsity at a world w of $\forall x P(x) \to P(x)$ under a w-assignment σ, implies that the individual $\sigma(x)$ does not belong to the domain of variation of the quantifiers, so $\sigma(x)$ does not exist at w. The falsity of an instance of CBF, $\Box\forall x P(x) \to \forall x \Box P(x)$, at a world w implies that $\forall x P(x)$ is true at some future moment v, whereas $P(x)$ is false at v under some w-assignment σ such that $\sigma(x) \in D_w$, and so at v under σ it is false that $\forall x P(x) \to P(x)$. UI discriminates between existing and non-existing individuals in the current world, whereas CBF discriminates between existing and non-existing individuals in *future* worlds. Falsifying CBF has fatal consequences for some individual: any u which is a witness for $\exists x \Diamond \neg P(x)$ at a world w where $\Box\forall x P(x)$ is true, is bound to die in some subsequent world. This induces us to stipulate that

an individual constant c denotes an *existing* individual at w iff
for all sentences $\forall x A(x)$, $\forall x A(x) \to A(c/x)$ is true at w.

We would like to stress that in this way we are able to distinguish between existing and non-existing individuals without having to make recourse to the identity relation (or the existence predicate), as it is usually the case,

$$c \text{ denotes an } \textit{existing} \text{ individual at } w \text{ iff } \exists x(x=c) \text{ is true at } w,$$

and so without becoming entangled with problems linked to identity, modalities and rigid designators.[9]

Notational convention. By L we shall denote any q.m.l. which extends $Q^\circ.K$. If L is a q.m.l. with language $\mathcal{L}$ and C is a denumerable set of individual constants not occurring in $\mathcal{L}$, then $\mathcal{L}^C$ denotes the language obtained by adding all the constants of C to $\mathcal{L}$, and L^C denotes the logic L in the language $\mathcal{L}^C$. From now on we agree that $\mathcal{L}$ is the language of L and $\mathcal{L}^C$ is the language of L^C.

Definition 5 *A set Δ of formulas is L-*consistent *iff* $\Delta \nvdash_L \bot$.

Note. It might well be that a set of sentences Δ is L-consistent and at the same time $\Delta \vdash_L \forall u_1 \ldots \forall u_k \bot$, for some $k \geq 1$. Take $\Delta = \{\forall x A \wedge \neg A\}$, where x doesn't occcur in A or $\Delta = \{\forall x \bot\}$.

Lemma 6 *on constants.*

- *(i)* *If* $\vdash_{L^C} A(c_1, \ldots, c_n)$ *then* $\vdash_{L^C} A(w_1/c_1, \ldots, w_n/c_n)$, *where* $w_1, \ldots w_n$ *are variables not occurring in* $A(c_1, \ldots, c_n)$.
- *(ii)* *If* $\vdash_{L^C} A$ *and no constant of C occurs in A, then* $\vdash_L A$.
- *(iii)* *If Δ is an L-consistent set of sentences and no constant of C occurs in Δ, then Δ is L^C-consistent.*

Proof. (i) As for classical logic, by choosing variables $w_1, \ldots, w_n$ not occurring in the proof $\mathcal{D}$ of $A(c_1, \ldots, c_n)$ and by replacing uniformly in $\mathcal{D}$, c_i by w_i, $1 \leq i \leq n$. (ii) follows from (i), and (iii) from (ii). □

Let C be a set of individual constants. Now we define a set of sentences, which if true, do guarantee that C is a set of constants denoting 'existing' individuals.

Definition 7 $\mathcal{E}(C) =_{df} \{\forall x A(x) \rightarrow A(c/x) : c \in C$ *and* $\forall x A(x)$ *is a* sentence *of* $\mathcal{L}^C\}$.

Lemma 8 *Let Δ be a set of sentences of $\mathcal{L}^C$ not containing the individual constant $c \in C$. Then*

$$\mathcal{E}(C) \cup \Delta \vdash_{L^C} A(c) \quad \textit{implies} \quad \mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z A(z/c),$$

where z doesn't occur in $A(c)$.

[9]Hughes and Cresswell in [5], pp.306-309, present, to our knowledge, the first completeness proof for the system $Q^\circ.K$ without individual constants in which they exploit the principle of UI. The present approach is more general and leads to new, as far as we can tell, completeness results such as those for $Q^\circ.K{+}CBF$, $Q^\circ.K{+}CBF{+}BF$, $Q^\circ_{=}.K{+}BF$.

PROOF. Let $\mathcal{E}(C) \cup \Delta \vdash_{L^C} A(c)$. Then
(*) $\vdash_{L^C} E_1 \wedge \cdots \wedge E_j \rightarrow (D_1 \wedge \cdots \wedge D_k \rightarrow A(c))$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(C)$, $\{D_1, \ldots, D_k\} \subseteq \Delta$. Let $\vec{c} = c_1, \ldots, c_n$ and $c_1, \ldots, c_n, c$ be all the constants of C occurring in (*), then
$\vdash_{L^C} E_1(\vec{c}, c) \wedge \cdots \wedge E_j(\vec{c}, c) \rightarrow (D_1(\vec{c}) \wedge \cdots \wedge D_k(\vec{c}) \rightarrow A(\vec{c}, c))$. (By $E_i(\vec{c}, c)$ we mean that the constants of C occuring in E_i are among $c_1, \ldots, c_n, c$.) Let $z_1, \ldots, z_n, z$ be variables not occurring in (*), so by lemma 6(i),
$\vdash_{L^C} E_1(\vec{z}/\vec{c}, z/c) \wedge \cdots \wedge E_j(\vec{z}/\vec{c}, z/c) \rightarrow (D_1(\vec{z}/\vec{c}) \wedge \cdots \wedge D_k(\vec{z}/\vec{c}) \rightarrow A(\vec{z}/\vec{c}, z/c))$. Then
$\vdash_{L^C} \forall\vec{z}\forall z E_1(\vec{z}, z) \wedge \cdots \wedge \forall\vec{z}\forall z E_j(\vec{z}, z) \rightarrow \forall\vec{z}\forall z[D_1(\vec{z}) \wedge \cdots \wedge D_k(\vec{z}) \rightarrow A(\vec{z}, z)]$.

Now, in each $E_i(\vec{c}, c)$, $1 \leq i \leq j$, c may or may not occur in $\vec{c}$, so $\forall\vec{z}\forall z E_i(\vec{z}, z)$, is either of the form $\forall\vec{z}\forall z[\forall x A(\vec{z}, z, x) \rightarrow A(\vec{z}, z, z/x)]$ or of the form $\forall\vec{z}\forall z[\forall x A(\vec{z}, z, x) \rightarrow A(\vec{z}, z, z_i/x)]$, for some wff $A(\vec{z}, z, x)$, where z_i is one of the variables of $\vec{z}$. In both cases, by lemma 3 (ii), it is a theorem of L^C, consequently
$\vdash_{L^C} \forall\vec{z}\forall z[D_1(\vec{z}) \wedge \cdots \wedge D_k(\vec{z}) \rightarrow A(\vec{z}, z)]$. Hence
$\vdash_{L^C} \forall z_1 \ldots \forall z_n[D_1(z_1, \ldots z_n) \wedge \cdots \wedge D_k(z_1, \ldots z_n) \rightarrow \forall z A(z_1, \ldots z_n, z)]$, since z doesn't occur in $D_1 \wedge \cdots \wedge D_k$. Then, by Modus Ponens with sentences of $\mathcal{E}(C)$, it obtains
$\mathcal{E}(C) \vdash_{L^C} \forall z_2 \ldots \forall z_n[D_1(c_1/z_1, z_2 \ldots z_n) \wedge \cdots \wedge D_k(c_1/z_1, z_2 \ldots z_n) \rightarrow \forall z A(c_1/z_1, z_2 \ldots z_n, z)]$,
$\mathcal{E}(C) \vdash_{L^C} \forall z_3 \ldots \forall z_n[D_1(c_1/z_1, c_2/z_2, z_3 \ldots z_n) \wedge \cdots \wedge D_k(c_1/z_1, c_2/z_2, z_3 \ldots z_n) \rightarrow$
$\forall z A(c_1/z_1, c_2/z_2, z_3 \ldots z_n, z)]$,
...
$\mathcal{E}(C) \vdash_{L^C} D_1(\vec{c}/\vec{z}) \wedge \cdots \wedge D_k(\vec{c}/\vec{z}) \rightarrow \forall z A(\vec{c}/\vec{z}, z)$, therefore
$\mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z A(z)$. □

In the presence of the axiom of Universal Instantiation, lemma 8 is trivial.

Lemma 9 *Let $\Delta \cup \{\exists y A\}$ be a set of sentences of $\mathcal{L}^C$ not containing the constant $c \in C$. If $\mathcal{E}(C) \cup \Delta \cup \{\exists y A\}$ is L^C-consistent, then $\mathcal{E}(C) \cup \Delta \cup \{A(c/y)\}$ is L^C-consistent.*

PROOF. Suppose by *reductio* that $\mathcal{E}(C) \cup \Delta \cup \{A(c/y)\} \vdash_{L^C} \bot$, then $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \neg A(c/y)$. Hence by lemma 8, $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z \neg A(z/y)$, so $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \neg\exists z A(z/c)$, contrary to the L^C-consistency of $\mathcal{E}(C) \cup \Delta \cup \{\exists y A\}$. □

Definition 10 *Let Δ be a set of sentences of $\mathcal{L}$ and $Q \subseteq Const(\mathcal{L})$.*

Δ *is* L-deductively closed	*iff*	*for any sentence A of $\mathcal{L}$, if $\Delta \vdash_L A$, then $A \in \Delta$.*
Δ *is* L-complete	*iff*	*for any sentence A of $\mathcal{L}$, either $A \in \Delta$, or $\neg A \in \Delta$.*
Δ *is* L-maximal	*iff*	*Δ is L-consistent and L-complete.*
Δ *is* Q-universal	*iff*	*for any wff $A(x)$ of $\mathcal{L}$, if $\forall x A(x) \in \Delta$, then $A(c/x) \in \Delta$, for all individual constants $c \in Q$.*
Δ *is* L-deductively closed	*iff*	*for any sentence A of $\mathcal{L}$, if $\Delta \vdash_L A$, then $A \in \Delta$.*
Δ *is* L-complete	*iff*	*for any sentence A of $\mathcal{L}$, either $A \in \Delta$, or $\neg A \in \Delta$.*

Δ *is* L-maximal	*iff*	Δ *is* L*-consistent and* L*-complete.*
Δ *is* Q-universal	*iff*	*for any wff* $A(x)$ *of* $\mathcal{L}$, *if* $\forall x A(x) \in \Delta$, *then* $A(c/x) \in \Delta$, *for all individual constants* $c \in Q$.
Δ *is* Q-existential	*iff*	*for any wff* $A(x)$ *of* $\mathcal{L}$, *if* $A(c/x) \in \Delta$ *for some constant* $c \in Q$, *then* $\exists x A(x) \in \Delta$.
Δ *is* Q-inductive	*iff*	*for any wff* $A(x)$ *of* $\mathcal{L}$, *if* $A(c/x) \in \Delta$ *for all constants* $c \in Q$, *then* $\forall x A(x) \in \Delta$.
Δ *is* Q-rich	*iff*	*for any wff* $A(x)$ *of* $\mathcal{L}$, *if* $\exists x A(x) \in \Delta$, *then* $A(c/x) \in \Delta$, *for some individual constant* $c \in Q$.
Δ *is* L-saturated	*iff*	Δ *is* L*-maximal and for some subset* Q *of individual constants of* $\mathcal{L}$, Δ *is* Q*-universal and* Q*-rich.*

Lemma 11 *Let* Δ *be a set of sentences and* $Q \subseteq Const(\Delta)$, *where* $Const(\Delta)$ *is the set of constants occurring in sentences of* Δ.

(i) *If* Δ *is* Q*-universal and* $Q^* \subseteq Q$, *then* Δ *is* Q^**-universal.*
(ii) *If* Δ *is* Q*-rich and* $Q^* \supseteq Q$, *then* Δ *is* Q^**-rich.*
If Δ *is* L*-maximal, then*
(iii) Δ *is* Q*-universal iff* Δ *is* Q*-existential,*
(iv) Δ *is* Q*-inductive iff* Δ *is* Q*-rich.*

Definition 12 *Let* Δ *be a set of wffs.*

$Cl_L(\Delta) = \{A : \Delta \vdash_L A\}$, $Cl_L(\Delta)$ *is said to be* the L-deductive closure *of* Δ. *When no confusion can possibly arise, we write* $Cl(\Delta)$ *instead of* $Cl_L(\Delta)$.

$\Box^-(\Delta) = \{A : \Box A \in \Delta\}$.

Lemma 13 *Let* Δ *be a set of wffs.*

(i) Δ *is* L*-consistent iff* $Cl_L(\Delta)$ *is* L*-consistent.*
(ii) *If* Δ *is* L*-consistent and* $\Diamond B \in \Delta$, *then* $\Box^-(\Delta) \cup \{B\}$ *is* L*-consistent.*
(iii) *If* Δ *is* L*-deductively closed, then* $\Box^-(\Delta)$ *is* L*-deductively closed.*
(iv) *If* Δ *is* L*-deductively closed and* $\Box^-(\Delta) \vdash_L A$, *then* $\Delta \vdash_L \Box A$.

Lemma 14 *Let* Δ *be an* L*-consistent set of formulas. Then there is an* L*-consistent set* $\Gamma \supseteq \Delta$ *such that for any sentence* A *of* $\mathcal{L}$ *either* $A \in \Gamma$ *or* $\neg A \in \Gamma$.

Lemma 15 *If* Δ *is an* L*-consistent set of sentences and no constants of* C *occur in* Δ, *then either* $\Delta \vdash_L \forall z_1 ... \forall z_h \bot$, *for some* $h \geq 1$ *or* $\mathcal{E}(C) \cup \Delta$ *is* L^C*-consistent.*

Proof. Assume that $\Delta \nvdash_L \forall z_1 ... \forall z_h \bot$, for any $h \geq 1$ and suppose by *reductio* that $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \bot$. Then (*) $\vdash_{L^C} E_1 \wedge \cdots \wedge E_j \to [D_1 \wedge \cdots \wedge D_k \to \bot]$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(C)$ and $\{D_1, \ldots, D_k\} \subseteq \Delta$. Let $\vec{c} = c_1, \ldots, c_n$ be all the individual constants of C occurring

in (*). Then $\vdash_{L^C} E_1(\vec{c}) \wedge \cdots \wedge E_j(\vec{c}) \to [D_1 \wedge \cdots \wedge D_k \to \bot]$. Let $z_1, \ldots, z_n$ be variables not occurring in (*) and $\vec{z} = z_1, \ldots, z_n$, so by the lemma 6$(i)$,
$\vdash_{L^C} E_1(\vec{z}/\vec{c}) \wedge \cdots \wedge E_j(\vec{z}/\vec{c}) \to [D_1 \wedge \cdots \wedge D_k \to \bot]$. Then, by lemma 3$(iii)$,
$\vdash_{L^C} \forall\vec{z}E_1(\vec{z}) \wedge \cdots \wedge \forall\vec{z}E_j(\vec{z}) \to [\forall\vec{z}D_1 \wedge \cdots \wedge \forall\vec{z}D_k \to \forall\vec{z}\bot]$.

Now, each $\forall\vec{z}E_i(\vec{z})$, $1 \le i \le j$, is of the form $\forall\vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, z_i/x))$, for some wff $A(\vec{z}, x)$ (z_i may or may not occur in $A(\vec{z}, x)$).
By lemma 3(ii), $\vdash_{L^C} \forall\vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, z_i/x))$, consequently
$\vdash_{L^C} \forall\vec{z}D_1 \wedge \cdots \wedge \forall\vec{z}D_k \to \forall\vec{z}\bot$. But $\Delta \vdash_{L^C} D_1 \wedge \cdots \wedge D_k$, hence $\Delta \vdash_{L^C} \forall\vec{z}D_1 \wedge \cdots \wedge \forall\vec{z}D_k$, so $\Delta \vdash_{L^C} \forall\vec{z}\bot$, contrary to the assumption. $\square$

In presence of the axiom of Universal Instantiation, lemma 15 is trivial.

Lemma 16 *Let Δ be an L-consistent set of sentences of $\mathcal{L}$ and C be a countable set of new constants. Then there is a set Π of sentences of $\mathcal{L}^C$ such that $\Delta \subseteq \Pi$, Π is L^C-maximal, Π is Q-universal and Q-rich for some set $Q \subseteq Const(\mathcal{L}^C)$.*

Proof. If $\Delta \vdash_L \forall z_1 ... z_h \bot$, for some $h \ge 1$, let Π be an L^C-maximal extension of Δ. No existential sentence $\exists x(A(x)$ is in Π. For otherwise, by lemma 3$(viii)$, $\exists x A(x) \wedge \forall z_2 ... z_h \bot) \in \Pi$, $\exists x \forall z_2 ... z_h \bot \in \Pi$, $\forall z_2 ... z_h \bot \in \Pi$, and so by induction, $\bot \in w$ contrary to the L^C-consistency of Π. Therefore Π is trivially Q-universal and Q-rich.
If $\Delta \not\vdash_L \forall z_1 ... z_h \bot$, let $H_1, H_2, H_3, ...$ be an enumeration of all the existential sentences of $\mathcal{L}^C$. Define the following chain of sets of sentences of $\mathcal{L}^C$.

$\Gamma_0 = \Delta \cup \mathcal{E}(C)$.

Suppose the set Γ_n has already been defined and the constants of C occurring in Γ_n are $c_1, ..., c_k$. Choose the first sentence in the given enumeration (and cancel it) which contains at most the constants $c_1, ..., c_k$. Let it be $\exists x F(x)$. If $\Gamma_n \cup \{\exists x F(x)\}$ is L^C-consistent, define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$, where $c \in C$ and c is different from $c_1, ..., c_k$; if $\Gamma_n \cup \{\exists x F(x)\}$ is not L^C-consistent, define $\Gamma_{n+1} = \Gamma_n$. Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$.
Γ_0 is L^C-consistent in virtue of lemma 15 and so is $Cl(\Gamma_0)$. Each Γ_{n+1} is L^C-consistent in virtue of lemma 9, and so is $Cl(\Gamma_{n+1})$, consequently Γ is L^C-consistent. By construction Γ is also C-rich, and trivially C-universal because it includes Γ_0. In virtue of lemma 14, Γ can be extended to a set Π which is L^C-maximal. Therefore Π is Q-universal and Q-rich for some $Q \subseteq Const(\mathcal{L}^C)$. $\square$

Definition 17 *Let a q.m.l. $L \supseteq Q^\circ.K$ be given with language $\mathcal{L}$. Let V be a set of constants of cardinality $\aleph_0$ such that $V \supseteq Const(\mathcal{L})$ and $|V - Const(\mathcal{L})| = \aleph_0$. A* canonical model $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ for L *is defined as follows:*

- *W is the class of all L^C-saturated set of sentences for some set C of constants such that $Const(\mathcal{L}^C) \neq \emptyset$, $C \subset V$ and $|V - Const(\mathcal{L}^C)| = \aleph_0$,*
- *wRv iff $\Box^-(w) \subseteq v$, for any $w, v \in W$,*

- $D_w = \{c : \forall x A(x) \to A(x/c) \in w$, *for all sentences* $\forall x A(x)$ *of* $\mathcal{L} \cup Const(w)\}$,
- $U_w = Const(w)$,
- $I_w(c) = c$,
- $I_w(P^n) = \{\langle c_1, ..., c_n\rangle : P^n(c_1, ..., c_n) \in w\}$.

Let us check that every canonical model is based on a K-frame. If a logic L is consistent, then the empty set of sentences is L-consistent, so by lemma 16 there is an L^C-saturated set of sentences for some set C of constants, therefore $W \neq \emptyset$. $Const(w) \neq \emptyset$ by definition of W, so $U_w \neq \emptyset$. If wRv then $U_w \subseteq U_v$, for if A is a tautology, $\Box A \in w$ and so $A \in v$; therefore $Const(w) \subseteq Const(v)$. $D_w \subseteq U_w$ by definition. An important fact to notice is that every w is D_w-universal by definition and since every w is Q-rich for some $Q \subseteq D_w$, w is also D_w-rich in virtue of lemma 11(ii). If $\forall z_1...z_h\bot \in w$, for some $h \geq 1$, then $D_w = \emptyset$. In fact if $D_w \neq \emptyset$, then for some constant c, $\forall x\neg(P(x) \to P(x)) \to \neg(P(c/x) \to P(c/x)) \in w$ and so $\exists x(P(c/x) \to P(c/x)) \in w$, contrary to the consistency of w, as we saw in lemma 16.

Where w is a world of a canonical model, $\mathcal{L}_w =_{df} \mathcal{L} \cup Const(w)$ and L_w is the logic L in the language $\mathcal{L}_w$.

Lemma 18 *Let* $\mathcal{M}^L = \langle W, R, D, U, I\rangle$ *be a canonical model. If* $w \in W$ *and* $\Diamond A \in w$, *then there is a* $v \in W$ *such that* $\Box^-(w) \subseteq v$, $A \in v$ *and* $Const(w) \subseteq Const(v)$.

Proof. By lemma 13(ii), $\Box^-(w) \cup \{A\}$ is L_w-consistent. If $\Box^-(w) \cup \{A\} \vdash \forall z_1 \ldots z_h\bot$, for some $h \geq 1$, let v be an L_w-maximal extension of w. Otherwise, let C be a countable set of constants such that $Const(\mathcal{L}_w) \cap C = \emptyset$, $C \subset V$ and $|V - (Const(\mathcal{L}_w \cup C))| = \aleph_0$. Since, by definition of canonical model, $|V - Const(\mathcal{L}_w)| = \aleph_0$, C exists. Let $\mathcal{L}_v = \mathcal{L}_w \cup C$. By lemma 16, there is an L_v-saturated set of sentences v such that $v \supseteq (\Box^-(w) \cup \{A\})$. □

Lemma 19 *Let* $\mathcal{M}^L$ *be a canonical model for* $L \supseteq Q^\circ.K$. *For all formulas* $A(x_1, \ldots, x_n)$ *of* $\mathcal{L}$ *and for any* w*-assignment* σ,

$$\mathcal{M}^L \models^\sigma_w A(x_1, \ldots, x_n) \quad \textit{iff} \quad A(\sigma(x_1)/x_1, \ldots, \sigma(x_n)/x_n) \in w,$$

where $x_1, \ldots, x_n$ *are all the variables occurring free in* A.

Proof. For simplicity's sake we will write in the following $A(\sigma(x_1), \ldots, \sigma(x_n))$ instead of $A(\sigma(x_1)/x_1, \ldots, \sigma(x_n)/x_n)$.

$\mathcal{M}^L \models^\sigma_w P^k(x_{i_1}, \ldots, x_{i_n}, c_{i_{n+1}}, \ldots, c_{i_k})$ iff $\langle\sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), I_w(c_{i_{n+1}}), \ldots, I_w(c_{i_k})\rangle \in I_w(P^n)$ iff by the definition of I_w in $\mathcal{M}^L$, $\langle\sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), c_{i_{n+1}}, \ldots, c_{i_k}\rangle \in I_w(P^n)$ iff, again by definiton of I_w in $\mathcal{M}^L$, $P^n(\sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), c_{i_{n+1}}, \ldots, c_{i_k}) \in w$.

$\bot \notin w$, since w is L_w-consistent.

If $\mathcal{M}^L \not\models^\sigma_w \Box B(x_1, \ldots, x_n)$, then there is a v, such that $\Box^-(w) \subseteq v$ and $\mathcal{M}^L \not\models^\sigma_v B(x_1, \ldots, x_n)$. Hence by induction hypothesis, $B(\sigma(x_1), \ldots, \sigma(x_n)) \notin v$, and so $\Box B(\sigma(x_1), \ldots, \sigma(x_n)) \notin w$.

If $\Box B(\sigma(x_1), \ldots, \sigma(x_n)) \notin w$, then, by the L_w-maximality of w, $\Diamond\neg B(\sigma(x_1), \ldots, \sigma(x_n)) \in w$. By lemma 18, there is a $v \in W$ such that $\Box^-(w) \subseteq v$ and $\neg B(\sigma(x_1), \ldots, \sigma(x_n)) \in v$. By induction hypothesis, $\mathcal{M}^L \models^\sigma_v \neg B(x_1, \ldots, x_n)$, so $\mathcal{M}^L \not\models^\sigma_v B(x_1, \ldots, x_n)$. Moreover, by definition of R, wRv holds, so $\mathcal{M}^L \not\models^\sigma_w \Box B(x_1, \ldots, x_n)$.

Before examining the cases of the quantifiers, let us recall that in canonical models individual constants are rigid designators, for $I_w(c) = c$, for all $w \in W$.

If $\mathcal{M}^L \models^\sigma_w \forall x B(x, x_1, \ldots, x_n)$ and $D_w = \emptyset$, then $\forall x B(x, x_1, \ldots, x_n) \in w$. If $D_w \neq \emptyset$, then $\mathcal{M}^L \models^\tau_w B(x, x_1, \ldots, x_n)$, where τ is any w-assignment which is an x-variant of σ such that $\tau(x) = d$ and $d \in D_w$. By lemma 1, $\mathcal{M}^L \models^\sigma_w B(d, x_1, \ldots, x_n)$, for all $d \in D_w$, therefore by induction hypothesis, $B(d, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, for all $d \in D_w$, so $\forall x B(x, \sigma(x_1), \ldots, \sigma(x_n)) \in w$ since w is D_w-inductive.

If $\forall x B(x, \sigma(x_1), \ldots, \sigma(x_n)) \in w$ and $D_w = \emptyset$, then $\mathcal{M}^L \models^\sigma_w \forall x B(x, x_1, \ldots, x_n)$. If $D_w \neq \emptyset$, then, since w is D_w-universal, $B(d, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, for all constant $d \in D_w$. By induction hypothesis, $\mathcal{M}^L \models^\sigma_w B(d, x_1, \ldots, x_n)$ and by lemma 1, $\mathcal{M}^L \models^\tau_w B(x, x_1, \ldots, x_n)$ where τ is any x-variant of σ such that $\tau(x) = d$ and $d \in D_w$, therefore $\mathcal{M}^L \models^\sigma_w \forall x B(x, x_1, \ldots, x_n)$.$\Box$

Lemma 20 *Let $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ be a canonical model for $L \supseteq Q^\circ.K$.*

- *(i)* *If Δ is an L-consistent set of formulas, then for some $w \in W$ and some w-assignment σ, $\mathcal{M}^L \models^\sigma_w D$, for all $D \in \Delta$.*
- *(ii)* *If Δ is an L-consistent set of sentences, then for some $w \in W$, $\mathcal{M}^L \models_w D$, for all $D \in \Delta$.*
- *(iii)* *$\mathcal{M}^L$ is a model for L.*
- *(iv)* *If $\not\vdash_L A$, then $\mathcal{M}^L \not\models A$.*

Proof. (i) Let $C = \{c_1, c_2, c_3, ...\}$ be a set of constants not occurring in $\mathcal{L}$ and $z_1, z_2, z_3, ...$ be all the variables occurring free in formulas of Δ. Then $\Delta^C = \{D(c_{i_1}/z_{i_1}, \ldots, c_{i_n}/z_{i_n}) : D(z_{i_1}, \ldots, z_{i_n}) \in \Delta$ and $c_{i_1} \ldots c_{i_n} \in C\}$ is L^C-consistent by lemma 6(i). Then by lemma 16 there is a set $\Pi \supseteq \Delta$ which is $L^{C \cup C^*}$-saturated, for some set C^* of new constants. Consider a canonical model $\mathcal{M}^L$ for L such that $V \supseteq Const(L^{C \cup C^*})$ and $|V - Const(L^{C \cup C^*})| = \aleph_0$. Then Π is a world, say w, of $\mathcal{M}^L$ and so $\Delta^C \subseteq w$. Given a w-assignment σ such that $\sigma(z_{i_j}) = c_{i_j}$, $\mathcal{M}^L \models^\sigma_w D(z_{i_1}, \ldots, z_{i_n})$ for any $D(z_{i_1}, \ldots, z_{i_n}) \in \Delta$, in virtue of lemma 19.

The standard pattern to prove that a logic $L \supseteq Q^\circ.K$ is complete with respect to a class of frames $\mathcal{H}$ goes as follows. Take any wff A which is not a teorem of L, so, by (iv), $\{\neg A\}$ is L-consistent. By lemma 16, there is a w in the canonical model for L such that $\neg A \in w$. Because of lemmas 18 and 19, $\mathcal{M}^L \models^\sigma_w \neg A$, therefore $\mathcal{M}^L \not\models A$. If $\mathcal{M}^L$ is based on a frame for L, L is complete. Now, lemma 18 allows us to build canonical models of the most general kind: nothing is said of the inner domains and the outer domains are increasing. In order to prove that $\mathcal{M}^L$ is based on a frame for L, we need to prove *variations* of lemma 18 to the effect that the inner and outer domains fulfill the specific conditions of the frames for L.

$\boxed{Q^\circ.K}$

Since no condition is required on frames for $Q^\circ.K$, lemma 18 yields

Theorem 21 *$Q^\circ.K$ is complete with respect to the class of all K-frames with varying inner domains and increasing outer domains.*

$\boxed{Q^\circ.K{+}CBF}$

Let us move to the system $Q^\circ.K + CBF$. The core fact to notice is that for any world w of a canonical model for $Q^\circ.K + CBF$, $\Box^-(w)$ is D_w-universal. For, if $\forall x A(x) \in \Box^-(w)$, then $\Box\forall x A(x) \in w$, so by CBF, $\forall x \Box A(x) \in w$. Since w is D_w-universal, $\Box A(c/x) \in w$, for all $c \in D_w$, therefore $A(c/x) \in \Box^-(w)$, for all $c \in D_w$. So individuals 'existing' at w, are bound to exist in all accessible worlds. The following two lemmas elaborate this fact.

Lemma 22 *Let w be a world of a canonical model for $L \supseteq Q^\circ.K{+}CBF$. Then $\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},d/x)) \in \Box^-(w)$, for all $d \in D_w$.*

PROOF. $\vdash_{Q^\circ.K+CBF} \quad \forall y(\forall x A(\vec{z},x) \to A(\vec{z},y/x))$,
$\vdash_{Q^\circ.K+CBF} \quad \forall\vec{z}\forall y(\forall x A(\vec{z},x) \to A(\vec{z},y/x))$,
$\vdash_{Q^\circ.K+CBF} \quad \forall y\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},y/x))$,
$\vdash_{Q^\circ.K+CBF} \quad \Box\forall y\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},y/x))$, and by CBF,
$\vdash_{Q^\circ.K+CBF} \quad \forall y\Box\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},y/x))$, consequently
$\forall y\Box\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},y/x)) \in w$. Since w is D_w-universal, for all $d \in D_w$, $\Box\forall\vec{z}(\forall x A(\vec{z}) \to A(\vec{z},d/x)) \in w$ and so $\forall\vec{z}(\forall x A(\vec{z},x) \to A(\vec{z},d/x)) \in \Box^-(w)$, for all $d \in D_w$. $\Box$

Lemma 23 *Let w be a world of a canonical model for $L \supseteq Q^\circ.K + CBF$ and C be a set of constants disjoint from $Const(w)$. If $\forall z_1 ... z_h \bot \notin w$, for any $h \geq 1$, and $\Diamond B \in w$, then $\mathcal{E}(D_w \cup C) \cup \Box^-(w) \cup \{B\}$ is L^C_w-consistent.*

PROOF. We recall that $\mathcal{E}(D_w \cup C) = \{\forall x A(x) \to A(c/x) : A(x) \in \mathcal{L}^C_w$ and $c \in (D_w \cup C)\}$. Suppose by *reductio* that $\mathcal{E}(D_w \cup C) \cup \Box^-(w) \cup \{B\}$ is not L^C_w-consistent, then
(*) $\vdash_{L^C_w} E_1 \wedge \cdots \wedge E_j \to [D_1 \wedge \cdots \wedge D_k \to \neg B]$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(D_w \cup C)$ and $\{D_1, \ldots, D_k\} \subseteq \Box^-(w)$. Let $\vec{d} = d_1 \ldots d_m$ be all the constants of $\mathcal{L}_w$ occurring in (*) and $\vec{c} = c_1 \ldots c_n$ be all the individual constants of C occurring in (*). Then
$\vdash_{L^C_w} E_1(\vec{d},\vec{c}) \wedge \cdots \wedge E_j(\vec{d},\vec{c}) \to [D_1(\vec{d}) \wedge \cdots \wedge D_k(\vec{d}) \to \neg B(\vec{d})]$. Let $z_1 \ldots z_n$ be variables not occurring in (*), so by the lemma 6(i),
$\vdash_{L^C_w} E_1(\vec{d},\vec{z}/\vec{c}) \wedge \cdots \wedge E_j(\vec{d},\vec{z}/\vec{c}) \to [D_1(\vec{d}) \wedge \cdots \wedge D_k(\vec{d}) \to \neg B(\vec{d})]$,
$\vdash_{L^C_w} \forall\vec{z}E_1(\vec{d},\vec{z}) \wedge \cdots \wedge \forall\vec{z}E_j(\vec{d},\vec{z}) \to [\forall\vec{z}D_1(\vec{d}) \wedge \cdots \wedge \forall\vec{z}D_k(\vec{d}) \to \forall\vec{z}\neg B(\vec{d})]$.
Now, each $\forall\vec{z}E_i(\vec{d},\vec{z})$, $1 \leq i \leq j$, is either of the form $\forall\vec{z}[\forall x A(\vec{d},\vec{z},x) \to A(\vec{d},\vec{z},z_k/x)]$ for some $k, 1 \leq k \leq n$, and so it is a theorem of $Q^\circ.K$, by lemma 3(ii), or of the form $\forall\vec{z}(\forall x A(\vec{d},\vec{z},x) \to A(\vec{d},\vec{z},d_h/x))$ for some $h, 1 \leq h \leq m$, and so it is in $\Box^-(w)$, by lemma 22. Hence

$\Box^-(w) \vdash_{L^C_w} \forall\vec{z}D_1(\vec{d}) \wedge \cdots \wedge \forall\vec{z}D_k(\vec{d}) \to \forall\vec{z}\neg B(\vec{d})$. But
$\Box^-(w) \vdash_{L^C_w} D_1(\vec{d}) \wedge \cdots \wedge D_k(\vec{d})$, hence
$\Box^-(w) \vdash_{L^C_w} \forall\vec{z}D_1(\vec{d}) \wedge \cdots \wedge \forall\vec{z}D_k(\vec{d})$ since $\vec{z}$ do not occur in $D_1, \ldots D_k$, therefore
$\Box^-(w) \vdash_{L^C_w} \forall\vec{z}\neg B(\vec{d})$ since z doesn't occur in B,
$w \vdash_{L^C_w} \Box\forall\vec{z}\neg B(\vec{d})$ by lemma 13,
$w \vdash_{L^C_w} \forall\vec{z}\Box\neg B(\vec{d})$ by CBF. But
$w \vdash_{L^C_w} \Diamond B(\vec{d})$, hence
$w \vdash_{L^C_w} \forall\vec{z}\Diamond B(\vec{d})$, so
$w \vdash_{L^C_w} \forall\vec{z}\bot$, contrary to the hypothesis of the lemma. □

Lemma 24 (CBF-variation of lemma 18) *Let $\mathcal{M}^L = \langle W, R, D, U, I\rangle$ be a canonical model for $L \supseteq Q^\circ.K + CBF$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$, $Const(w) \subseteq Const(v)$ and $D_w \subseteq D_v$.*

PROOF. As for lemma 16. Lemma 23 assures us that $D_w \subseteq D_v$. □

Theorem 25 *$Q^\circ.K+CBF$ is complete with respect to the class of K-frames with increasing inner and outer domains.*

$\boxed{Q.K}$

Consider the system $Q.K$ obtained from $Q^\circ.K$ by adding the axiom of Universal Instantiation. As well known, CBF is a theorem of $Q.K$,[10] so by lemma 24, if wRv, $D_w \subseteq D_v$. Moreover, because of axiom UI, each w is U_w-universal, consequently $U_w = D_w$, therefore

Theorem 26 *$Q.K$ is complete with respect to the class of TK-frames with increasing domains.*

$\boxed{Q^\circ.K + CBF + BF}$

Let us now turn our attention to the Barcan Formula and consider canonical models for systems $L \supseteq Q^\circ.K$+CBF+BF. The core fact to notice is that for any world w of a canonical model for $Q^\circ.K + BF$ (CBF is not needed), $\Box^-(w)$ is D_w-inductive.

Lemma 27 *Let w be a world of the canonical model for $L \supseteq Q^\circ.K$+BF and $\{B_1, \ldots, B_n\}$ be a finite set of sentences of $\mathcal{L}$. If $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_L A(c)$, for all $c \in D_w$, then $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_L \forall x A(x)$, consequently, $Cl_L(\Box^-(w) \cup \{B_1, \ldots, B_n\})$ is D_w-inductive.*

PROOF Suppose that[11]
$\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_L A(c)$, for all $c \in D_w$, then where $B = B_1 \wedge \cdots \wedge B_n$,

[10] $Q.K \vdash \forall x A(x) \to A(x)$, $Q.K \vdash \Box\forall x A(x) \to \Box A(x)$, $Q.K \vdash \Box\forall x A(x) \to \forall x \Box A(x)$.
[11] This proof is standard and it is due to Thomason [7].

$\Box^-(w) \vdash_L B \to A(c)$, for all $c \in D_w$, hence

$w \vdash_L \Box(B \to A(c))$, for all $c \in D_w$. Hence

$\Box(B \to A(c)) \in w$, for all $c \in D_w$. (Notice that the constant c could occur also in B.) Take a variable y not occurring either in B or in $A(c)$ and consider the wff $\forall y \Box(B \to A(y/c))$. Since, $\Box(B \to A(c)) \in w$, for all $c \in D_w$ and w is D_w-inductive, then

$\forall y \Box(B \to A(y/c) \in w$, and by BF,

$\Box \forall y(B \to A(y/c)) \in w$. Therefore,

$\Box(B \to \forall y A(y/c)) \in w$,

$(B \to \forall y A(y/c)) \in \Box^-(w)$,

$\Box^-(w) \cup \{B_1, \dots, B_n\} \vdash_L \forall y A(y)$, and so

$\Box^-(w) \cup \{B_1, \dots, B_n\} \vdash_L \forall x A(x)$. □

Now, if w is a world of the canonical model for $L = Q^\circ.K{+}CB\mathrm{F}{+}BF$ and $\Diamond B \in w$, then because of CBF (lemma 22), $(\Box^-(w) \cup \{B\})$ is Q-universal and because of BF (lemma 27), $Cl_L(\Box^-(w) \cup \{B\})$ is Q-inductive. This leads to the following lemma.

Lemma 28 ($CBF{+}BF$-variation of lemma 18) *Let $\mathcal{M}^L = \langle W, R, D, U, I\rangle$ be a canonical model for $L \supseteq Q^\circ.K + CBF + BF$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$, $Const(w) = Const(v)$ and moreover v is D_w-universal and D_w-rich.*

Proof. Let $H_1, H_2, H_3, ...$ be an enumeration of all the existential sentences of $\mathcal{L}_w$. Define the following chain of sets of sentences of $\mathcal{L}_w$.

$\Gamma_0 = \Box^-(w) \cup \{A\}$.

Suppose the set Γ_n has already been defined. Consider the sentence H_{n+1}. Let it be $\exists x F(x)$. If $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent, define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$, where c is a constant of D_w such that $\Gamma_n \cup \{F(c/x)\}$ is L_w-consistent. If $\Gamma_n \cup \{\exists x F(x)\}$ is not L_w-consistent, define $\Gamma_{n+1} = \Gamma_n$. Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$. Γ is D_w-rich. For, the existence of a $c \in D_w$ such that $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$ is L_w-consistent is guaranteed by the fact that otherwise $\Gamma_n \vdash \neg F(c/x)$ for all $c \in D_w$. But Γ_n is $\Box^-(w)$ united with a finite set of sentences $\{A, B_1, \dots B_k\}$, so by lemma 27, $\Gamma_n \vdash \forall x \neg F(x/c)$, contrary to the fact that $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. Therefore $F(c/x) \in Cl_L(\Gamma_n)$, for some $c \in D_w$ and so Γ is D_w-rich. Now extend Γ to a set v which is L_w-maximal. Trivially $Const(w) = Const(v)$. Because of CBF, $\Box^-(w)$ is D_w-universal, therefore v is D_w-universal. □

Theorem 29 *$Q^\circ.K + CBF + BF$ is complete with respect to the class of K-frames with constant inner and outer domains.*

$\boxed{Q.K + BF}$

Let $L \supseteq Q.K + BF$. As well known, $Q.K \vdash CBF$, moreover, thanks to UI, each w is U_w-universal, so lemma 28 yields

Theorem 30 *$Q.K + BF$ is complete with respect to the class of TK-frames with constant domains.*

The following table summarizes the completeness results obtained so far.

q.m.l.	is complete w.r.t. the class of K-frames with domains	
	inner	*outer*
$Q^{\circ}.K$	varying	increasing
$Q^{\circ}.K + CBF$	increasing	increasing
$Q^{\circ}.K + CBF + BF$	constant	constant
$Q.K$	increasing	= inner
$Q.K{+}BF$	constant	= inner

2 Modal systems with identity

We will start by examining the systems of the diagram below.

$Q^{\circ}_{=}.K$

$Q^{\circ}_{=}.K + CBF$ $\qquad\qquad$ $Q^{\circ}_{=}.K + BF$

$Q_{=}.K$ $\qquad\qquad$ $Q^{\circ}_{=}.K + CBF + BF$

$Q.K_{=} + BF$

Let us add to $Q^{\circ}.K$ the identity predicate ' = ' together with the following three axioms and let $Q^{\circ}_{=}.K$ be the resulting system.[12]

REF	$t = t$
SUBS	$s = t \rightarrow (A(s//x) \rightarrow A(t//x))$.
ND	$s \neq t \rightarrow \Box(s \neq t)$.

Lemma 31 *Some theorems about identity.*

i	$\vdash_{Q^{\circ}_{=}.K}$	$x = c \rightarrow \Box(x = c)$
ii	$\vdash_{Q^{\circ}_{=}.K}$	$\forall x \exists y(x = y)$,
iii	$\vdash_{Q^{\circ}_{=}.K}$	$\exists z(z = y) \wedge A(y) \rightarrow \exists x A(x//y)$,
iv	$\vdash_{Q^{\circ}_{=}.K}$	$\exists y(y = c) \rightarrow (\forall x A(x) \rightarrow A(c/x))$,
v	$\vdash_{Q^{\circ}_{=}.K}$	$\exists y(y = c)$ *if* $\vdash_{Q^{\circ}_{=}.K} \forall x B(x) \rightarrow B(c/x)$, *for all wffs* $B(x)$.

[12] The system $Q^{\circ}_{=}.K$ is often called in the literature FK, free quantified K.

vi	$\vdash_{Q^\circ_=.K+CBF}$	$\forall x \Box \exists y(x = y)$,
vii	$\vdash_{Q^\circ_=.K+BF}$	$\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$.
viii	$\vdash_{Q^\circ_=.K+BF}$	$\Diamond \exists y(x = y) \to \exists y(x = y)$.

PROOF

(*i*) $\vdash_{Q^\circ_=.K}$ $x = c \to (\Box(x = x)(x//x) \to \Box(x = x)(c//x))$ by $SUBS$,
$\vdash_{Q^\circ_=.K}$ $x = c \to (\Box(x = x) \to \Box(x = c))$,
$\vdash_{Q^\circ_=.K}$ $\Box(x = x)$,
$\vdash_{Q^\circ_=.K}$ $x = c \to \Box(x = c)$.

(*ii*) $\vdash_{Q^\circ_=.K}$ $\forall y(y = y \to \exists x(y = y)(x//y))$ by lemma 3(*i*),
$\vdash_{Q^\circ_=.K}$ $\forall y(y = y \to \exists x(x = y))$,
$\vdash_{Q^\circ_=.K}$ $\forall y(y = y) \to \forall y \exists x(x = y)$,
$\vdash_{Q^\circ_=.K}$ $\forall y \exists x(x = y)$.

(*iv*) $\vdash_{Q^\circ_=.K}$ $\exists y(y = c) \to [\exists y(y = c) \wedge \forall y(\forall x A(x) \to A(y/x))]$,
$\vdash_{Q^\circ_=.K}$ $\exists y(y = c) \to \exists y[(y = c) \wedge (\forall x A(x) \to A(y/x))]$,
$\vdash_{Q^\circ_=.K}$ $y = c \wedge (\forall x A(x) \to A(y/x)) \to (\forall x A(x) \to A(c/x))$,
$\vdash_{Q^\circ_=.K}$ $\exists y[y = c \wedge (\forall x A(x) \to A(y/x))] \to (\forall x A(x) \to A(c/x))$,
$\vdash_{Q^\circ_=.K}$ $\exists y(y = c) \to (\forall x A(x) \to A(c/x))$.

(*v*) $\vdash_{Q^\circ_=.K}$ $\forall x \exists y(x = y) \to \exists y(c = y)$, let $B(x)$ be $\exists y(x = y)$,
$\vdash_{Q^\circ_=.K}$ $\forall x \exists y(x = y)$,
$\vdash_{Q^\circ_=.K}$ $\exists y(c = y)$.

(*vi*) $\vdash_{Q^\circ_=.K+CBF}$ $\forall x \exists y(x = y)$,
$\vdash_{Q^\circ_=.K+CBF}$ $\Box \forall x \exists y(x = y)$,
$\vdash_{Q^\circ_=.K+CBF}$ $\forall x \Box \exists y(x = y)$ by CBF.

(*vii*) $\vdash_{Q^\circ_=.K+BF}$ $x = y \to x = y$,
$\vdash_{Q^\circ_=.K+BF}$ $\Diamond(x = y) \to \Diamond(x = y)$,
$\vdash_{Q^\circ_=.K+BF}$ $\exists y \Diamond(x = y) \to \exists y \Diamond(x = y)$,
$\vdash_{Q^\circ_=.K+BF}$ $\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$ by BF.

(*viii*) $\vdash_{Q^\circ_=.K+BF}$ $\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$ (*vi*),
$\vdash_{Q^\circ_=.K+BF}$ $\Diamond(x = y) \to (x = y)$ ND,
$\vdash_{Q^\circ_=.K+BF}$ $\exists y \Diamond(x = y) \to \exists y(x = y)$ by ND,
$\vdash_{Q^\circ_=.K+BF}$ $\Diamond \exists y(x = y) \to \exists y(x = y)$ by BF.

$\Box$

Again on rigidity In a language with identity the fact that a constant c is a rigid designator is expressed by the formula $x = c \to \Box(x = c)$. Therefore, thanks to lemma 31(*i*), all the systems of q.m.l. with identity we are going to discuss dare bound to be systems with rigid terms. Notice however that this is the case *given* general features of the K-semantics. The main feature is that universes of accessible worlds are related by the *inclusion function*: $U_w \subseteq U_v$. If we think of individuals of U_v as *counterparts* are individuals of U_w, each individual has one and only one counterpart in each related world (in fact it is the very same individual). It is because of this correlation that rigidity corresponds to NI or to the equivalence between *de dicto* and *de re* readings of substituted formulas, as pointed out in the footnote of lemma 1. For a more general semantics in which these notions are shown to be distinct from one another, see [1].

Definition 32 *Let* $\mathcal{M} = \langle W, R, D, U, I\rangle$ *be a* K*-model.* $\mathcal{M}$ *is said to be* normal *iff*
(a) *for all* $w \in W$, $I_w(=) = \{\langle d, d\rangle : d \in U_w\}$, *and*
(b) *for all individual constants* c, wRv *implies* $I_w(c) = I_v(c)$.

Lemma 33 *Each of the logics mentioned in the diagram at the beginning of this section is sound with respect to the class of normal* K*-models based on frames with respect to which the corresponding systems without identity are sound.*

As to the completeness of systems $L_= \supseteq Q^\circ_=.K$, if w is an $L_=$-saturated set of sentences, the relation

$$a \sim b \quad \text{iff} \quad (a = b) \in w$$

is an equivalence relation and hence divides $Const(w)$ into disjoint partitions. A problem presents itself immediately: the standard canonical model tecnique does not, in general, satisfy both the conditions (a) and (b). In fact, one way of matching condition (a) is to interpret each constant c in w on its equivalence classs $[c]_w$, and to define U_w as the set of equivalence classes of the constants mentioned in w. But by so doing we, in general, violate condition (b), because it might well happen that wRv, $[c]_w \in U_w$ and a new constant c^* belongs to $[c]_v$, i.e. $(c = c^*) \in v$, with the consequence that $[c]_w \neq [c]_v$. We show how to overcome this difficulty by constructing canonical models where the set W is a class of sets satisfying the following condition:

$$\text{if } wRv \text{ then } [c]_w = [c]_v, \text{ for all constants } c \in Const(w).$$

This can be achieved because each time we need a witness for an existential quantifier, we can introduce a constant c together with the assuption that 'c exists' and that c is different from all the constants present in the set built so far. The justification that we *can* do this varies from case to case, either c is a new constant or the set built so far has special properties (D_w-inductive or $\Box$-inductive, as in lemmas 39 or 45).

Definition 34 *Let $L_= \supseteq Q^\circ_=.K$ be given with language $\mathcal{L}_=$. Let V be a set of constants of cardinality $\aleph_0$ such that $V \supseteq Const(\mathcal{L}_=)$ and $|V - Const(\mathcal{L}_=)| = \aleph_0$.*
A canonical model $\mathcal{N}^{L_=}$ for $L_=$ is a quintuple $\langle W, R, D, U, I\rangle$ *such that*

- *W is the class of all $L^C_=$-saturated set of sentences for some set C of constants such that $Const(\mathcal{L}^C_=) \neq \emptyset$, $C \subset V$ and $|V - Const(\mathcal{L}^C_=)| = \aleph_0$,*
- *wRv iff $\Box^-(w) \subseteq v$ and*
  (#) for all $c \in Const(w)$, $[c]_w = [c]_v$, where $[c]_w = \{b \in Const(w) : (c = b) \in w\}$,
- *$D_w = \{[c]_w : \exists y(y = c) \in w\}$,*
- *$U_w = \{[c]_w : c \in Const(w)\}$,*
- *$I_w(c) = [c]_w$,*
- *$I_w(P^n) = \{\langle [c_1]_w, ..., [c_n]_w\rangle : P^n(c_1, ..., c_n) \in w\}$.*

It is easy to see that canonical models so defined are normal K-models. That $W \neq \emptyset$ is due, as before, to lemma 16. Each w is D_w-universal thanks to lemma 31(iv) and (v). Thanks to condition (#), wRv implies that $U_w \subseteq U_v$ and $I_w(c) = I_v(c)$ for all constants $c \in Const(w)$. Moreover $I_w(=) = \{\langle u, u\rangle : u \in U_w\rangle\}$.

If $Const(w) = Const(v)$, then condition (#) always holds. For, if $b \in [c]_w$, then $(b = c) \in w$, so $\Box(b = c) \in w$ by NI, hence $(b = c) \in v$, therefore $b \in [c]_v$. If $b \in [c]_v$, then $(b = c) \in v$, so $\Diamond(b = c) \in w$ since $b \in Const(w)$, so by ND, $(b = c) \in w$, hence $b \in [c]_w$.

To make sure that (#) holds also when $Const(w) \subseteq Const(v)$, we prove the following lemma:

Lemma 35 *(Variation of lemma 18 for systems with identity) Let $\mathcal{N}^{L_=} = \langle W, R, D, U, I\rangle$ be a canonical model for $L_= \supseteq Q^\circ_=.K$. Let C be a set of new constants. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $A \in v$, $\Box^-(w) \subseteq v$, $Const(w) \subseteq Const(v)$ and for all $c \in Const(w)$, $[c]_w = [c]_v$.*

PROOF. Let $H_1, H_2, H_3, ...$ be an enumeration of all the existential sentences of $\mathcal{L}^C_{=,w} = \mathcal{L}_{=,w} \cup C$. Define the following chain of sets of sentences of $\mathcal{L}^C_{=,w}$.

$\Gamma_0 = \Box^-(w) \cup \{A\}$.

Suppose the set Γ_n has already been defined and the constants of C occurring in Γ_n are $c_1, ..., c_k$. Choose the $n+1$-th sentence in the given enumeration. Let it be $\exists x F(x)$.

$Case(1)$. $\Gamma_n \cup \{\exists x F(x)\}$ is $L^C_{=,w}$-consistent.

$Case(1.1)$ For some constant b of $\mathcal{L}_{=,w} \cup \{c_1, ..., c_k\}$, $\Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$ is $L^C_{=,w}$-consistent. Define $\Gamma_{n+1} = \Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$.

$Case(1.2)$ For all constants b of $\mathcal{L}_{=,w} \cup \{c_1, ..., c_k\}$, $\Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$ is not $L^C_{=,w}$-consistent. Take a constant $c \in C$ not occurring in $\mathcal{L}_{=,w} \cup \{c_1, ..., c_k\}$ and define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\} \cup \{\exists y(c = y)\} \cup \{c \neq b$: for all constants b of $\mathcal{L}_w \cup \{c_1, ..., c_k\}\}$.

$Case(2)$. $\Gamma_n \cup \{\exists x F(x)\}$ is not $L^C_{=,w}$-consistent. Define $\Gamma_{n+1} = \Gamma_n$.

Then let $\Gamma = \bigcup_{n\in N} Cl(\Gamma_n)$ and $Q = \{c : \exists y(y = c) \in \Gamma\}$. Γ is Q-rich by construction and Q-universal because of lemma 31(iv) and (v). Extend Γ to a set v which is $L^C_{=,w}$-saturated. What remains to be proved is that Γ_{n+1} as defined in case (1.2) is $L^C_{=,w}$-consistent. First we prove that $\Gamma_n \cup \{F(c/x)\} \cup \{\exists y(c = y)\}$ is $L^C_{=,w}$-consistent. Suppose not, then
$\Gamma_n \vdash_{L^C_{=,w}} \exists y(y = c) \rightarrow \neg F(c)$, since c doesn't occur in Γ_n, then for some z,
$\Gamma_n \vdash_{L^C_{=,w}} \exists y(y = z) \rightarrow \neg F(z/c)$, therefore
$\Gamma_n \vdash_{L^C_{=,w}} \forall z \exists y(y = z) \rightarrow \forall z \neg F(z)$, so
$\Gamma_n \vdash_{L^C_{=,w}} \forall z \neg F(z)$, contrary to the $L^C_{=,w}$-consistency of $\Gamma_n \cup \{\exists x F(x)\}$.
Suppose now that Γ_{n+1} is not $L^C_{=,w}$-consistent, then
$\Gamma_n \vdash_{L^C_{=,w}} \exists y(y = c) \wedge F(c) \rightarrow [(c \neq b_1 \wedge ... \wedge c \neq b_h) \rightarrow \bot]$, for some constants $b_1 \dots b_h \in \mathcal{L}_= \cup \{c_1, ..., c_k\}$, so
$\Gamma_n \vdash_{L^C_{=,w}} \exists y(y = c) \wedge F(c) \rightarrow (c = b_1 \vee ... \vee c = b_h)$.

Therefore $\Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c)\} \cup \{c = b_j\}$ is $L^C_{=,w}$-consistent for some j, $1 \leq j \leq h$, consequently $\Gamma_n \cup \{\exists y(y = b_j)\} \cup \{F(b_j)\}$ is $L^C_{=,w}$-consistent, contrary to the fact that for no $b \in \mathcal{L}_{=,w} \cup \{c_1, \dots, c_k\}$, $\Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$ is $L^C_{=,w}$-consistent. □

Definition 36 *For each equivalence class $[c]_w \in U_w$, $f[c]_w$ is the canonical representative of $[c]_w$. When no confusion can possibly occur, we write $f[c]$.*

Lemma 37 *Let $\mathcal{N}^{L_=}$ be a normal canonical model for $L_= \supseteq Q^\circ_=.K$. For all formulas $A(x_1, \dots, x_n)$ of $\mathcal{L}_=$ and for any w-assignment σ,*

$$\mathcal{N}^{L_=} \models^\sigma_w A(x_1, \dots, x_n) \qquad \text{iff} \qquad A(f\sigma(x_1)/x_1, \dots, f\sigma(x_n)/x_n) \in w,$$

where $x_1, \dots, x_n$ are all the variables occurring free in A.

Proof $\mathcal{N}^{L_=} \models^\sigma_w P^k(x_{i_1}, ..., x_{i_n}, c_{i_{n+1}}, ..., c_{i_k})$ iff $\langle \sigma(x_{i_1}), ..., \sigma(x_{i_n}), I_w(c_{i_{n+1}}), ..., I_w(c_{i_k})\rangle \in I_w(P^n)$ iff $\langle \sigma(x_{i_1}), ..., \sigma(x_{i_n}), [c_{i_{n+1}}], ..., [c_{i_k}]\rangle \in I_w(P^n)$ iff $\langle f\sigma(x_{i_1}), ..., f\sigma(x_{i_n}), c_{i_{n+1}}, ..., c_{i_k}\rangle \in I_w(P^n)$ since $\sim$ is a congruence relation. The other cases are as in lemma 37.

Theorem 38 *(i) $Q^\circ_=.K$ is complete with respect to the class of normal models based on all K-frames.*
(ii) $Q^\circ_=.K$+CBF is complete with respect to the class of normal models based on all K-frames with increasing inner and outer domains.
(iii) $Q_=.K$+CBF is complete with respect to the class of normal models based on TK-frames with increasing domains.

Proof (i) follows from lemmas 35 and 37 as for the systems without identity. As to (ii), the construction of lemma 35 yields trivially that $D_w \subseteq D_v$, for if $c \in D_w$, then $\exists y(y = c) \in w$, so by lemma 31(vi), $\Box\exists y(y = c) \in w$, therefore $\exists y(y = c) \in v$, so $c \in D_v$. (iii) follows from (ii) and the fact that $Q_=.K + CBF \vdash \exists y(y = c)$, for all constants c.

Let us consider now systems with BF.

Lemma 39 *(BF-variation of lemma 18) Let $\mathcal{N}^{L_=} = \langle W, R, D, U, I\rangle$ be a canonical model for $L_= \supseteq Q^\circ_=.K+BF$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$, $Const(w) = Const(v)$ and moreover v is Q-universal and Q-rich for some $Q \subseteq D_w$.*

PROOF The proof is analogous to that of lemma 35 when $C = \emptyset$. All we have to show is that case (1.1) always holds, in particular we show that if $\Gamma_n \cup \{\exists x F(x)\}$ is $L_{=,w}$-consistent, then $\Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c/x)\}$ is $L_{=,w}$-consistent for some $c \in D_w$. Suppose not, then $\Gamma_n \vdash_{L_{=,w}} \exists y(y = c) \to \neg F(c)$, for all $c \in D_w$. But $Cl(\Gamma_n)$ is D_w-inductive by lemma 27, so $\forall z(\exists y(y = z) \to \neg F(z/c)) \in Cl(\Gamma_n)$, where z does not occur in any of the wffs of $(\Gamma_{n+1} - \Box^-(w))$, then as in lemma 35, this is impossible because of the $L_{=,w}$-consistency of $\Gamma_n \cup \{\exists x F(x)\}$. □

Let us see that, in general, $D_v \subseteq D_w$ if wRv. Since CBF is not a theorem of $Q^\circ_=.K + BF$, the set $\{\Box\forall x P(x), \exists x\Diamond\neg P(x)\}$ is $Q^\circ_=.K + BF$-consistent, therefore there is a world w of the canonical model for $Q^\circ_=.K + BF$ such that $\{\Box\forall x P(x), \exists x\Diamond\neg P(x)\} \subseteq w$. Then for some $[d] \in D_w$, $\Diamond\neg P(d) \in w$. But then there is a v, related to w, such that $\{\forall x P(x), \neg P(d)\} \subseteq v$, so $[d] \notin D_v$ consequently v is bound to be D_v-universal and D_v-rich for some $D_v \subset D_w$.

Theorem 40 *(i) $Q^\circ_=.K + BF$ is complete with respect to the class of normal models based on K-frames with decreasing inner domains and constant outer domains.*
(ii) $Q^\circ_=.K+CBF+BF$ is complete with respect to the class of normal models based on K-frames with constant inner domains and constant outer domains.
(iii) $Q_=.K+CBF+BF$ is complete with respect to the class of normal models based on TK-frames with constant domains.

SYSTEMS WITH THE EXTENDED BARCAN RULE

$$BR(n{+}1)\quad \frac{A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(x))\ldots)}{\forall A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1}(x))\ldots)}\quad \text{where } x \text{ is not free in } A_1, \ldots, A_n$$

By EBR, Extended Barcan Rule, we denote the set of all rules $BR(n+1), n \geq 0$. The rule EBR was first introduced by R.Thomason in [7] and since then discussed at various points in the literature.[13] EBR is valid on K-frames in which the outer domains are constant, i.e.

$$\text{for all } w, v \in W, \quad U_w = U_v.$$

Suppose that for some $\mathcal{M}$, w and w-assignment σ, $\mathcal{M} \not\models^\sigma_w A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1}(x)$. Therefore for some v, $wR^{n+1}v$, $\mathcal{M} \not\models^\sigma_v \forall x A_{n+1}(x)$, hence for some x-variant τ of σ such that $\tau(x) \in U_v$, $\mathcal{M} \not\models^\tau_v A_{n+1}(x)$. Since $U_w = U_v$, this is impossible because τ is also a w-assignment and, by hypothesis, $\mathcal{M} \models^\tau_w A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(x))\ldots)$, consequently, $\mathcal{M} \models^\tau_v A_{n+1}(x)$. □

[13] Our treatment of $Q^\circ_=.K + EBR$ has many points in common with [5], pp.296-302.

If the outer domains of a given K-frame $\mathcal{F}$ are not constant, then it might well be that, e.g., $BR(1)$ is valid on $\mathcal{F}$ (just take K-frames in which wRv implies $D_v \subseteq U_w$) and $BR(2)$ is not. Here is an instance in case. Let $\mathcal{F} = \langle W, R, D, U\rangle$, where $W = \{w, v, z\}$, $R = \{\langle w, v\rangle, \langle v, z\rangle\}$, $D_w = \{u\}$, $D_v = \{u\}$, $D_z = \{u, u^*\}$, $U_w = \{u\}$, $U_v = \{u, u^*\}$, $D_z = \{u, u^*\}$. We can easily see that $BR(1)$, i.e.

$$\frac{A_0 \to \Box A_1(x)}{A_0 \to \Box \forall x A_1(x)}$$

is valid on $\mathcal{F}$. On the contrary, if $\mathcal{M} = \langle \mathcal{F}, I\rangle$ is such that $I_w(P) = I_v(Q) = I_z(R) = \{u\}$, then

$\mathcal{M} \models_w P(a)$,
$\mathcal{M} \not\models_v P(a)$, $\mathcal{M} \models_v Q(a)$,
$\mathcal{M} \not\models_z P(a)$, $\mathcal{M} \models_z R(a)$, $\mathcal{M} \not\models_z R(b)$,

therefore the following instance of $BR(2)$ is not true on $\mathcal{M}$:

$$\frac{P(a) \to \Box(Q(a) \to \Box R(x))}{P(a) \to \Box(Q(a) \to \Box \forall x R(x))}$$

In the remaining of this section we will be concerned with the systems obtained by adding the rule EBR to $Q^\circ_=.K$, and this for the simple reason that on one side BF is a theorem of $Q.K + BR(1)$,[14] on the other we do not know of any completeness proof of systems obtained by adding EBR to $Q^\circ.K$. Proving completeness of systems with EBR imposes us to go through lemmas analogous to 16, 35, 27 and 39.

Lemma 41 *(EBR-counterpart of lemma 16) Let $L_= \supseteq Q^\circ_=.K + EBR$, and let Δ be an $L_=$-consistent set of sentences. Then for some set C of new constants there is an extension Π of Δ which is $L^C_=$-maximal, Q-rich and Q-universal, for some $Q \subseteq Const(\mathcal{L}^C_=)$ and moreover Π is $\Diamond$-rich , where Π is $\Diamond$-rich iff if*

$A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond \exists x A_{n+1}(x)) \ldots) \in \Pi$, where x is not free in $A_1, \ldots, A_n$, then for some $c \in \mathcal{L}^C_=$, $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x)) \ldots) \in \Pi$.

Proof Let $H_1, H_2, H_3, \ldots$ be an enumeration of all the sentences of $\mathcal{L}^C_=$ which are either of the form $\exists x F(x)$, for some wff $F(x)$ or of the form $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond \exists x A_{n+1}(x)) \ldots)$ for some wffs $A_1 \ldots A_{n+1}$ such that x is not free in $A_1 \ldots A_n$. Define the following chain of sets of sentences of $\mathcal{L}^C_=$.

$\Gamma_0 = \Delta$.

[14]Since $Q.K \vdash \forall x \Box A(x) \to \Box A(x)$, $Q.K + BR(1) \vdash \forall x \Box A(x) \to \Box \forall x A(x)$.

Suppose the set $\Gamma_n = \Delta \cup \{B_1, \dots, B_k\}$ has already been defined. Consider the sentence H_{n+1} in the given enumeration.
$Case(1)$ $\Gamma_n \cup \{H_{n+1}\}$ is $L^C_=$-consistent.
$Case(1.1)$ H_{n+1} is $\exists x F(x)$. Define $\Gamma_{n+1} = \Gamma_n \cup \{\exists y(y = c), F(c/x)\}$, for some $c \in C$ such that Γ_{n+1} is $L^C_=$-consistent.
$Case(1.2)$ H_{n+1} is $A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x)) \dots)$. Define $\Gamma_{n+1} = \Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x)) \dots)\}$, for some $c \in \mathcal{L}^C_=$ such that Γ_{n+1} is $L^C_=$-consistent.
$Case(2)$ $\Gamma_n \cup \{H_{n+1}\}$ is not $L^C_=$-consistent. Define $\Gamma_{n+1} = \Gamma_n$. Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$ and Π be an $L^C_=$-maximal extension of Γ.
Now we show that in both cases the appropriate constant c is always available.
$Case(1.1)$ As for case (1.2) of lemma 35.
$Case(1.2)$ Suppose that Γ_{n+1} is not $L^C_=$-consistent. Then
$\Gamma_n \vdash A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x)) \dots) \to \perp$, for all $c \in \mathcal{L}^C_=$, so in particular for some c which doesn't occur in $B_1, \dots B_k$. It follows that for some z which doesn't occur in $B_1, \dots B_k$,
$\Gamma_n \vdash A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(z/x)) \dots) \to \perp$, then
$\Gamma_n \vdash A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(z)) \dots) \to \perp$,
$\Gamma_n \vdash A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\neg A_{n+1}(z)) \dots)$,
$\Gamma_n \vdash A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\forall x \neg A_{n+1}(z)) \dots)$, via EBR.
But then $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x)) \dots)\} \vdash \Diamond \dots \Diamond\Diamond\perp$, therefore
$\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x)) \dots)\} \vdash \perp$, contrary to the $L^C_=$-consistency of $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x)) \dots)\}$. $\Box$

Lemma 42 *Let $L_= \supseteq Q^\circ_=.K + EBR$. If Δ is a set of sentences which is $L_=$-maximal and $\Diamond$-rich, then it is $\Box$-inductive, where Δ is said to be $\Box$-inductive iff if*

$A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box A_{n+1}(c)) \dots) \in \Delta$, *for all* $c \in \mathcal{L}_=$, *then*

$A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\forall x A_{n+1}(x)) \dots) \in \Delta$.

Definition 43 *Let $L_= \supseteq Q^\circ_=.K+ EBR$. Let C be a set of constants such that $C \supseteq Const(\mathcal{L}_=)$ and $|C - Const(\mathcal{L}_=)| = \aleph_0$. A* canonical model $\mathcal{N}^{L_=} = \langle W, R, D, U, I\rangle$ for *$L_=$ is defined as follows:*

- *W is the class of all $L^C_=$-saturated and $\Box$-inductive set of sentences,*
- *wRv iff $\Box^-(w) \subseteq v$, for any $w, v \in W$,*
- $D_w = \{[c] : \exists y(y = c) \in w\}$,
- $U_w = \{[c] : c \in Const(\mathcal{L}^C_=)\}$,
- $I_w(c) = [c]$,
- $I_w(P^n) = \{\langle [c_1], ..., [c_n]\rangle : P^n(c_1, ..., c_n) \in w\}$.

Lemmas 41 and 42 guarantee that the set W is not-empty.

Lemma 44 *Let $\mathcal{M}^{L_=} = \langle W, R, D, U, I\rangle$ be a canonical model for $L_= \supseteq Q^{\circ}_{=}.K{+}EBR$. For all $w \in W$, $\Box^-(w) \cup \{B_1, \dots, B_k\}$ is $\Box$-inductive.*

Proof Let $B = B_1 \wedge \dots \wedge B_k$. Suppose that for all $c \in Const(w)$,
$\Box^-(w) \cup \{B\} \vdash A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box A_{n+1}(c)) \dots)$, then
$\Box^-(w) \vdash (B \wedge A_0) \to \Box(A_1 \to \dots \to \Box(A_n \to \Box A_{n+1}(c)) \dots)$, for all $c \in Const(w)$,
$w \vdash \Box[(B \wedge A_0) \to \Box(A_1 \to \dots \to \Box(A_n \to \Box A_{n+1}(c)) \dots)]$, but w is $\Box$-inductive,[15] so
$w \vdash \Box[(B \wedge A_0) \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\forall A_{n+1}(x)) \dots)]$,
$[(B \wedge A_0) \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\forall A_{n+1}(x)) \dots)] \in \Box^-(w)$,
$\Box^-(w) \cup \{B_1, \dots, B_k\} \vdash A_0 \to \Box(A_1 \to \dots \to \Box(A_n \to \Box\forall x A_{n+1}(x)) \dots)$. □

Lemma 45 *(EBR-variation of lemma 18) Let $\mathcal{M}^{L_=} = \langle W, R, D, U, I\rangle$ be a canonical model for $L_= \supseteq Q^{\circ}_{=}.K{+}EBR$. If $w \in W$ and $\Diamond A \in w$ then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$ and $Const(w) = Const(v)$.*

Proof The proof is as for lemma 35. All we have to show is that case (1.1) always holds, in particular we show that if $\Gamma_n \cup \{\exists x F(x)\}$ is $L^C_{=,w}$-consistent, then $\Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c/x)\}$ is $L_{=,w}$-consistent for some $c \in \mathcal{L}^C_{=,w}$. Suppose not, then for all $c \in \mathcal{L}^C_{=,w}$,
$\Box^-(w) \cup \{A, B_1, \dots, B_k\} \cup \{F(c/x)\} \cup \{\exists y(y = c)\}$ is not $L^C_{=,w}$-consistent, where $\Gamma_n = \Box^-(w) \cup \{A, B_1, \dots, B_k\}$. If $G = (A \wedge B_1 \wedge \dots \wedge B_k)$, then
$\Box^-(w) \vdash G \wedge \exists y(y = c) \to \neg F(c)$,
$w \vdash \Box(G \wedge \exists y(y = c) \to \neg F(c))$, since w is $\Box$-inductive,
$w \vdash \Box\forall z(G \wedge \exists y(y = z) \to \neg F(z/c))$,
$w \vdash \Box(G \wedge \forall z \exists y(y = z) \to \forall z \neg F(z/c))$,
$w \vdash \Box(G \to \forall z \neg F(z/c))$,
$(G \to \forall z \neg F(z/c) \in \Box^-(w)$, contrary to the fact that $\Box^-(w) \cup \{A, B_1, \dots, B_k\} \cup \{\exists x F(x)\}$ is $L^C_{=,w}$-consistent. □

Theorem 46 *(i) $Q^{\circ}_{=}.K{+}EBR$ is complete with respect to the class of K-frames with varying inner domains and constant outer domains.*
(ii) $Q^{\circ}_{=}.K{+}EBR{+}CBF$ is complete with respect to the class of K-frames with increasing inner domains and constant outer domains.

The following table summarizes the results obtained so for for systems with identity.

[15] Being $\Box$-inductive can be paraphrased as being EBR-closed. Now, in order to show that $\Box^-(w) \cup \{B_1, \dots, B_k\}$ is $BR(n+1)$ closed, we need to make use of the fact that w is $BR(n+2)$ closed. Therefore we cannot limit ourselves to any finite set $BR(1)$ .... $BR(n+1)$ of rules.

q.m.l.	is complete w.r.t. the class of normal models based on K-frames with domains	
	inner	*outer*
$Q^\circ_=.K$	varying	increasing
$Q^\circ_=.K{+}EBR$	varying	constant
$Q^\circ_=.K{+}CBF$	increasing	increasing
$Q^\circ_=.K{+}CBF{+}EBR$	increasing	constant
$Q^\circ_=.K{+}CBF{+}BF$	constant	constant
$Q_=.K$	increasing	= inner
$Q_=.K{+}BF$	constant	= inner

Quantified extensions of the propositional modal logic B.

By $Q^\circ.B$, $(Q.B,\ Q^\circ_=.B)$ we denote the logic $Q^\circ.K$, $(Q.K,\ Q^\circ_=.K)$ plus the propositional axiom B: $A \to \Box\Diamond A$, i.e. the axiom characteristic of frames whose accessibility relation is symmetric.

Lemma 47 *$(Q^\circ_=.B$ - $ND) \vdash ND$, and $Q^\circ_=.B{+}BF \vdash \exists y(x = y) \to \Box\exists y(x = y)$.*

Proof

$Q^\circ_=.B \vdash x = y \to \Box(x = y)$ $\quad NI$,
$Q^\circ_=.B \vdash \Diamond(x \neq y) \to (x \neq y)$,
$Q^\circ_=.B \vdash \Box\Diamond(x \neq y) \to \Box(x \neq y)$,
$Q^\circ_=.B \vdash (x \neq y) \to \Box(x \neq y)$ $\quad$ by B.

$Q^\circ_=.B{+}BF \vdash x = y \to \Box(x = y)$ $\quad NI$,
$Q^\circ_=.B{+}BF \vdash \Diamond(x = y) \to (x = y)$ $\quad$ via B,
$Q^\circ_=.B{+}BF \vdash \exists y\Diamond(x = y) \to \exists y(x = y)$,
$Q^\circ_=.B{+}BF \vdash \Diamond\exists y(x = y) \to \exists y(x = y)$ $\quad$ via BF,
$Q^\circ_=.B{+}BF \vdash \Box\Diamond\exists y(x = y) \to \Box\exists y(x = y)$,
$Q^\circ_=.B{+}BF \vdash \exists y(x = y) \to \Box\exists y(x = y)$, $\quad$ via B. $\Box$

Lemma 48 .

	q.m.l.	*is complete w.r.t. the class of K-frames where R is symmetric and the domains are*	
		inner	outer
(a)	$Q^\circ_=.B$	*varying*	*constant*
(b)	$Q^\circ.B\ +CBF$	*constant*	*constant*
(c)	$Q^\circ_=.B\ +CBF$	*constant*	*constant*
(d)	$Q^\circ_=.B\ +BF$	*constant*	*constant*
(e)	$Q.B$	*constant*	*= inner*

PROOF That the accessibility relation R is symmetric is easily seen as for the propositional case because the languages of all the worlds are equal.

(a) Since EBR is a derivable rule of $Q^\circ.B$, the completeness proof of $Q^\circ_=.B$ is analogous to that of $Q^\circ_=.K{+}EBR$. For the reader's sake, here is the proof of EBR as given in [5], p.295, for $n = 2$.

$$
\begin{array}{ll}
\vdash_{Q^\circ.B} & A_0 \to \Box(A_1 \to \Box(A_2 \to \Box A_3(x))) \\
\vdash_{Q^\circ.B} & \Diamond A_0 \to (A_1 \to \Box(A_2 \to \Box A_3(x))) \\
\vdash_{Q^\circ.B} & \Diamond A_0 \wedge A_1 \to \Box(A_2 \to \Box A_3(x)) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \to \Box A_3(x)) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond A_0 \wedge A_1) \wedge A_2 \to \Box A_3(x) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to A_3(x) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to \forall x A_3(x) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond A_0 \wedge A_1) \wedge A_2 \to \Box\forall x A_3(x) \\
\vdash_{Q^\circ.B} & \Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \to \Box\forall x A_3(x)) \\
\vdash_{Q^\circ.B} & (\Diamond A_0 \to A_1) \to \Box(A_2 \to \Box\forall x A_3(x)) \\
\vdash_{Q^\circ.B} & \Diamond A_0 \to (A_1 \to \Box(A_2 \to \Box\forall x A_3(x))) \\
\vdash_{Q^\circ.B} & A_0 \to \Box(A_1 \to \Box(A_2 \to \Box\forall x A_3(x)))
\end{array}
$$

(b) Since $Q^\circ.B{+}CBF \vdash BF$, the completeness proof of $Q^\circ.B + CBF$ is analogous to that of $Q^\circ.K{+}CBF{+}BF$. For the reader's sake here is the proof of BF as given in [2], p.138.

$$
\begin{array}{lll}
\vdash_{Q^\circ.B+CBF} & \forall x[\forall x\Box A(x) \to \Box A(x)] & \\
\vdash_{Q^\circ.B+CBF} & \Box\forall x[\forall x\Box A(x) \to \Box A(x)] & \\
\vdash_{Q^\circ.B+CBF} & \forall x\Box[\forall x\Box A(x) \to \Box A(x)] & \text{by } CBF, \\
\vdash_{Q^\circ.B+CBF} & \forall x[\Diamond\forall x\Box A(x) \to \Diamond\Box A(x)] & \\
\vdash_{Q^\circ.B+CBF} & \forall x[\Diamond\forall x\Box A(x) \to A(x)] & \text{via } B \\
\vdash_{Q^\circ.B+CBF} & \forall x\Diamond\forall x\Box A(x) \to \forall x A(x) & \\
\vdash_{Q^\circ.B+CBF} & \Diamond\forall x\Box A(x) \to \forall x A(x) & \\
\vdash_{Q^\circ.B+CBF} & \Box\Diamond\forall x\Box A(x) \to \Box\forall x A(x) & \\
\vdash_{Q^\circ.B+CBF} & \forall x\Box A(x) \to \Box\forall x A(x), & \text{via } B.
\end{array}
$$

(c) from (b).

(d) Since CBF is a theorem of $Q^\circ_=.B\ {+}BF$, the completeness proof of $Q^\circ_=.B + BF$ is as that for $Q^\circ_=.K\ {+}BF{+}CBF$.

$\vdash_{Q^\circ_=.B+BF}$ $\exists y(x=y) \rightarrow \Box\exists y(x=y)$ lemma 47,
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg\Diamond\forall y\neg(x=y)$
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg\Diamond[A(x) \wedge (A(x) \rightarrow \forall y\neg(x=y))]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg[\Diamond A(x) \wedge \Box(A(x) \rightarrow \forall y\neg(x=y))]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg[\Diamond A(x) \wedge \Box\forall y(x=y \rightarrow \neg A(x))]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg[\Diamond A(x) \wedge \Box\forall y\neg A(y)]$ via lemma 31(iii),
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y(x=y) \vee \neg\Diamond A(x) \vee \neg\Box\forall y\neg A(y)$
$\vdash_{Q^\circ_=.B+BF}$ $\neg(\exists y(x=y) \wedge \Diamond A(x)) \vee \Diamond\exists y A(y)$
$\vdash_{Q^\circ_=.B+BF}$ $\neg\exists y\Diamond A(y) \vee \Diamond\exists y A(y)$
$\vdash_{Q^\circ_=.B+BF}$ $\exists y\Diamond A(y) \rightarrow \Diamond\exists y A(y)$.

(e) Because of (b) and the fact that $Q.B \vdash CBF$, BF is a theorem of $Q.B$, consequently $Q.B$ is complete with respect to the class of TK-frames with constant domains. □

Q°.B+BF IS INCOMPLETE

CBF is valid on all frames for $Q^\circ.B{+}BF$ for ach of them is bound to have inner constat domains. Now we present a model for $Q^\circ.B + BF$ in which CBF fails.[16] The model is based on a counterpart Kripke frame. For details about counterpart semantics, see [1]. A counterpart Kripke frame, $\mathfrak{C}K$-frame, is a quintuple $\mathcal{F} = \langle W, R, D, U, \mathfrak{C}\rangle$, where $\langle W, R, D, U\rangle$ is a K-frame and $\mathfrak{C}$, the counterpart relation, is such that

$\mathfrak{C} =_{df} \biguplus_{w,v\in W}\{\mathfrak{C}_{\langle w,v\rangle}\}$, where for any $w, v \in W$ such that wRv, $\mathfrak{C}_{\langle w,v\rangle} \subseteq (U_w \times U_v)$.

It can be easily shown that $Q^\circ.K$ formulated in a language with types is valid with respect to the class of all $\mathfrak{C}K$-models, where the notion of satisfaction is defined thus:

$\langle a_1,...,a_n\rangle \models_w P^n(n : x_1,\ldots,x_n)$	iff	$\langle a_1,...,a_n\rangle \in I_w(P^n)$
$\langle a_1,...,a_n\rangle \models_w \langle n : s_1,...,s_k\rangle B$	iff	$\langle a_1,...,a_n\rangle[n : s_1,...,s_k]_w \models_w B$
$\langle a_1,...,a_n\rangle \models_w \neg C$	iff	$\langle a_1,...,a_n\rangle \not\models_w C$
$\langle a_1,...,a_n\rangle \models_w C \vee D$	iff	$\langle a_1,...,a_n\rangle \models_w C$ or $\langle a_1,...,a_n\rangle \models_w D$
$\langle a_1,...,a_n\rangle \models_w \exists x_{n+1}G$	iff	for some $b \in D_w$, $\langle a_1,...,a_n,b\rangle \models_w G$
$\langle a_1,...,a_n\rangle \models_w \Box C$	iff	for all v such that wRv and for all counterparts $a_1^*,\ldots,a_n^*$ in D_v of $a_1,\ldots,a_n$, respectively, $\langle a_1^*,...,a_n^*\rangle \models_v C$.

A counterpart frame is said to be *symmetric* iff both R and $\mathfrak{C}$ are symmetric. A counterpart relation is said to be *surijective* iff if wRv, then for all $b \in U_v$ there is an $a \in U_w$ such that $a\mathfrak{C}b$ holds. From [1] we know that BF is valid on a counterpart K-frame iff the counterpart relation is surijective.

Consider the following counterpart K-frame $\mathcal{F} = \langle W, R, D, U, \mathfrak{C}\rangle$, where

$W = \{w, v\}$,

[16] This anwer a question raised in [2], p.138, whether CBF is a theorem of $Q^\circ.B{+}BF$.

$R = \{\langle w,v\rangle, \langle v,w\rangle\}$
$D_w = \{a,b\}$, $D_v = \{a^*\}$
$U_w = \{a,b\}$, $U_v = \{a^*, b^*\}$
$\mathfrak{C} = \{\langle a,a^*\rangle, \langle b,a^*\rangle, \langle b,b^*\rangle, \langle a^*,a\rangle, \langle a^*,b\rangle, \langle b^*,b\rangle\}$
Both R and $\mathfrak{C}$ are symmetric relations and $\mathfrak{C}$ is surijective, so $\mathcal{F}$ is a frame for $Q^\circ.B{+}BF$. Consider now a model $\mathcal{M} = \langle \mathcal{F}, I\rangle$ such that $I_w(P) = \{a,b\}$ and $I_v(P) = \{a^*\}$. Then $\mathcal{M} \models_w \Box\forall x P(x)$ because $a^* \in I_v(P)$ and $\mathcal{M} \not\models_w \forall x \Box P(x)$ because $b\mathfrak{C}b^*$ and $b^* \notin I_v(P)$, so $\mathcal{M} \not\models_w \Box\forall x P(x) \to \forall x\Box P(x)$.

Open problems Completeness property of $Q^\circ.K{+}BF$, $Q^\circ.K{+}EBR$ and $Q^\circ.B$.

References

[1] Corsi, Giovanna, "Counterparts and Possible Worlds. A Study on Quantified Modal Logics", *Preprint 21*, Dipartimento di Filosofia, Università di Bologna, Bologna, 2001.

[2] Fitting, Melvin & Mendelsohn, Richard L., *First-Order Modal Logic,* Kluwer A P, 1998.

[3] Garson, James, "Quantification in Modal Logic", in D.Gabbay and F.Guenthner (eds.), *Handbook of Philosophical logic, vol. II*, Kluwer A.P., Dordrecht, 1984, pp. 249-307.

[4] Garson, James, "Quantification in Modal Logic", 2^{nd} edition of [3], *Handbook of Philosophical logic*, Kluwer, Dordrecht, 2001 (to appear).

[5] Hughes, George and Cresswell, Max, *A New Introduction to Modal Logic,* Routledge, London, 1996.

[6] Kripke, Saul, "Semantical Considerations on Modal Logics", *Acta Philoosphica Fennica*, 1963, 16,83-94 [Reprinted in Linsky, *Reference and Modality*, Oxford U.P. 1971, 63-72].

[7] Thomason, R.H. "Some Completeness Results for Modal Predicate Calculi", Karel Lambert (ed.), *Philosophical Problems in Logic: Some Recent Developments*, D.Reidel Pu Co, Dordrecht, 1970, pp.56-76.