Introduction to Turing Nonrigidity

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Abstract

The purpose of this note is to give a short overview of the work on Turing degrees, with particular emphasis on the recent results regarding model-theoretic aspects (definability) and second-order aspects (automorphisms of the structure).

1 Preliminaries

Turing [14] introduced a notion of relative computability, originally as an attempt to better understand the Gödel incompleteness phenomenon. Here is how Turing explains the general idea, in a brief introduction to his paper.

> ... The well known theorem of Gödel shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L'_1, ...$ each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics $L, L_1, L_2, ...$ Proceeding in this way we can associate a system of logic with any constructive ordinal.

In connection with the development such a logic, Turing introduced a new notion, that of computability relative to an *oracle*, i.e. an extrarecursive entity. Formally, suppose that A and B are subsets of the natural numbers ω. We say that A is *Turing reducible to* B or *computable in* B if there is an algorithm to decide whether $x \in A$ when given answers to all questions of the form "Is $y \in B$?". We write $A \leq_T B$ to indicate that A is computable in B. This relation is transitive and reflexive and thus induces an equivalence relation $\equiv_T$, where $A \equiv_T B \leftrightarrow (A \leq_T B)\&(B \leq_T A)$ and a partial order also denoted by $\leq_T$ on the equivalence classes. These equivalence classes are called *Turing degrees* and the equivalence class of a set $A \subseteq \omega$ is called its degree. Degrees will be denoted by lowercase boldface letters. $\mathbf{D}$ will denote the structure of the Turing degrees. Our notation is standard, following that used by Soare [11] and Odifreddi [7].

For instance, Φ_i denotes the i-th partial recursive functional $2^\omega \to 2^\omega$, so that $A \leq_T B$ if and only if $A = \Phi_i^B$, for some $i \in \omega$.

The structure **D** is also called the structure of the *degrees of unsolvability*. They were introduced by Kleene and Post [9] as an attempt to measure the information content of sets of natural numbers. Post was particularly interested in the computably enumerable (c.e.) sets, namely those which can be listed by a computable method, because they appear naturally in most branches of mathematics and there is a noncomputable c.e. set. Post [10] noted that the noncomputable c.e. set K arising from the Gödel incompleteness theorem is *Turing complete*, in the sense that $W \leq_T K$ for every c.e. set W. Post then posed his famous question, called *Post's problem*, of whether there exist noncomputable incomplete c.e. sets. The problem was solved in a positive sense by Friedberg and Muchnik (See theorem 2.1).

Kleene and Post [9] also defined the notion of *jump operator* on sets and degrees. Given a set A, its jump is defined as the set $A' = (x : \Phi_x^A(x) \downarrow)$. In fact, this operation can be iterated: $A^o = A$, and $A^{(n+1)} = (A^n)'$, and we have $A \leq_T A'$, and $A \leq_T B$ implies $A' \leq_T B'$. Thus, we have an infinite hierarchy of degrees, $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0^n} < \cdots$. Degree $\mathbf{0}$ is trivially definable as the least degree and consists precisely of the computable sets. Degree $\mathbf{0}'$ is the degree of the coded theorems of Peano arithmetic and also the degree of the halting problem. Besides, Feferman [1957] and Shoenfield [1958] have shown that the c.e. degrees are exactly those degrees containing (coded) finitely axiomatisable first order theories.

The structure of the c.e. degrees **R** clearly forms a partially ordered set with the least element 0 and the greatest element 1 represented by the degrees $\mathbf{0}$ and $\mathbf{0}'$, respectively. It is also easily seen to be an uppersemilattice (usl) with a join, induced by the disjoint union of representatives, i.e. $\mathbf{a} \cup \mathbf{b} = \mathbf{c}$ where $\mathbf{c}$ is the degree of $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$ for any sets A,B of degree **a**,**b** respectively. However, **D** and **R** are not lattices [8].

2 Characterizing the structure of R

In this section, we describe briefly the result of a long line of research to characterizing the structure of the c.e. degrees. The study of the c.e. degrees begins with Post's problem to construct a c.e. set which is neither computable nor complete. Friedberg and independently Muchnik solved Post's problem, and in doing so introduced the *finite injury priority method.*

Theorem 2.1 *(Friedberg [1957], Muchnik [1956])*
There are two incomparable computably enumerable Turing degrees.

This theorem implies that the computably enumerable degrees are not linearly ordered and consist of more than two degrees, solving Post's problem. Friedberg's and Muchnik's solutions show that the partial ordering with 0,1 and two other incomparable elements can be embedded into **R**. We will use sets as representatives for the degrees, and thus existential properties of degrees can be proved by building appropriate sets. A set will be identified with its characteristic function, which is just a sequence of 0's and 1's. Most of the theorems about Turing degrees are proved by construction. We construct a set or sets of natural numbers of desirable degrees. To do this, in general we need to reduce the statement of the theorem to an equivalent set of *requirements*. We then show how to satisfy all requirements. For example, to prove the Friedberg-Muchnik theorem, we need to build two c.e. sets A_0 and A_1 which satisfy the following requirements, for all e:

$$R_{2e} : A \neq \{e\}^B,$$

$$R_{2e+1} : B \neq \{e\}^A.$$

We say that requirement R_n has *higher* priority than R_m if $n < m$. The finite injury method allows action taken at some stage for a given requirement R_m to be later *injured* (i.e. undone), by the action of some requirement R_n of higher priority, so that R_m must be satisfied once again at a still later stage.

The next question about the structure of **R** is the extension of embedding problems such as density: Given two elements a and b of the partial ordering, can we always find another c, such that $a < c < b$?. Sacks introduced a method stronger than the finite injury method to prove the density theorem. The method of the proof of this theorem is called the *infinite injury priority method* because actions for single requirements can cause infinitely many injuries to ones of lower priority. See [11] for more detailed discussion. The priority method has been viewed as the hallmark of computability theory.

Theorem 2.2 *(Sacks [1964])*
For every pair of c.e. degrees $\mathbf{a} < \mathbf{b}$ *there is a* $\mathbf{c}$ *such that* $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

Initially the prevailing intuitions of the researches about **R** were simplicity. In particular, in 1966 Sacks had conjectured that the theory of **R** is decidable. The above theorem in particular led Shoenfield to his famous conjecture that **R** can be characterized completely in terms of extensions of embeddings.

Conjecture 2.3 *(Shoenfield Conjecture [1965])*
Given any two finite bounded upper semilattices $U \subset V$*, any embedding of* U *into* $\boldsymbol{R}$ *can be extended to an embedding of* V *into* $\boldsymbol{R}$*.*

If this conjecture were true it would have implied a number of nice result about **R**, such as the $\aleph_0$-categoricity and decidability of its first order theory (i.e. Sacks's conjecture). A positive solution to Shoenfield's conjecture would essentially have constituted a characterization of **R**. The density theorem itself motivates the view that the structure **R** should be a homogeneous structure and so a 'simple' one would have allowed back-and-forth constructions. In fact, it is what Shoenfield conjectured. However, this conjecture eventually turned out to be false (Lachlan [1966] and Yates [1966]) by the construction of *minimal pairs*, i.e. nonzero degrees with infimum 0. Furthermore Harrington and Shelah

[1982] succeeded in showing that the theory of **R** is undecidable, thus refuting Sacks's conjecture that the theory of **R** is decidable. These results were the beginning of a trend away from simplicity and towards complexity. But rather than seeing this complexity as an obstacle to characterizing **R**, it suggests that a sufficiently strong proof of complexity would completely characterize **R**. It is this that led Slaman [13] to formulate the view of the structure of **R** as being as complicated as possible.

Conjecture 2.4 *(Biinterpretability Conjecture (Slaman [1991]))*
There is definable coding of standard models of arithmetic in ***R*** *for which the relation between degrees* ***d*** *and the codes of sets of degrees* ***d*** *is definable.*

There is also an appropriate second order version of this conjecture for **D**. The conjecture for **R** is equivalent to the existence of a definable injective map $f : \mathbf{R} \longrightarrow \mathbf{N}$, where **N** is the copy of first order arithmetic coded inside **R**. The definability of f means that the corresponding relation on **R** between an argument $\mathbf{a}$ and a tuple $\mathbf{b}$ representing $f(\mathbf{a})$ is definable. If this conjecture were true it would give complete information about definability in **R**, i.e. every degree in **R** would be definable as would every relation on **R** which is definable in arithmetic. It also would give information about automorphisms for **R**.

3 Definability in the Turing degrees

One of the basic tools of logic is the study of definability of various notions over certain structures. We say that a relation on degrees is *Turing definable* if and only if it is first order definable in **D**. In this section, we will focus our attention on the second order properties of the Turing degrees (Global theory).

We first recall some basic definitions as follows.

Definition 3.1 *Suppose that* Σ *is a model of the finite language* L *and* $\vec{\mathbf{p}}$ *is a finite sequence of parameters from* **D**. Σ *is* coded by $\vec{\mathbf{p}}$ in **D** *if there is an isomorphic image of* Σ *whose universe, relations, functions, constants and quantifiers are all first order definable in the language of* **D** *with additional symbols for the parameters* $\vec{\mathbf{p}}$.

Definition 3.2 *A set of degrees is an* antichain *if its members are mutually incomparable degrees. A set of degrees is an* ideal *if it is closed downward and closed under joins.* A jump ideal *is an ideal closed under application of the Turing jump.*

The origin of the definability approach to understanding the structure of **D** is Spector's result [7] that every countable ideal in **D** is uniformly definable from parameters in **D**. Simpson then uses this result to show that there is an interpretation of the second order theory of arithmetic in the first order theory of **D**. We shall now discuss the Slaman-Woodin coding method which allows us to code sets and relations in **D** directly. We say that a relation R is *countable* if there is a countable subset of the degrees such that all of the solutions to R come from that set.

Theorem 3.3 *(Slaman and Woodin [1986])*
*Every countable relation is definable from finitely many parameters in **D**, in a uniform way.*

By the Slaman-Woodin coding theorem, we can show that second-order arithmetic is interpretable in **D**, as required by the biinterpretability conjecture. The following lemma plays a central role in the proof of the theorem. Here we will sketch the simpler proof, due to Odifreddi [7].

Lemma 3.4 *Every countable antichain is definable from finitely many parameters in **D**, in uniform way.*

Proof (sketch). Given an antichain $\jmath = (\mathbf{c}_n)_{n\in\omega}$. We shall show there are degrees **a** ,**b** and **c** such that the $\mathbf{c}_n$'s are the degrees below **c** that are minimal solutions for **x** for the equation below.

$$\mathbf{x} \in \jmath \longleftrightarrow \mathbf{x} \leq \mathbf{c} \wedge P(\mathbf{x}) \wedge \neg(\exists \mathbf{z} \leq \mathbf{c})(\mathbf{z} \leq \mathbf{x} \wedge P(\mathbf{z})),$$

where

$$P(\mathbf{x}) \longleftrightarrow \mathbf{x} \neq (\mathbf{x} \cup \mathbf{a}) \cup (\mathbf{x} \cup \mathbf{b})$$

We fix $A \subseteq \omega$ and let $\mathbf{a}$ = deg(A), $\mathbf{b}$ = deg(B). We let $C = \oplus_{n\in\omega} C_n$, where $C_n \in \mathbf{c}_n$. The following are two conditions on A and B.

Condition 1 $\mathbf{c_n}$ is a solution of P.

This condition will be achieved by having, for every n, a set D_n such that: $D_n \leq_T C_n \oplus A, C_n \oplus B$, but $D_n \not\leq_T C_n$.

Condition 2 $\mathbf{c_n}$ is a minimal solution of P.

This condition will be achieved by satisfying for any $X \leq_T C$:
$(\exists D)(D \leq_T X \oplus A, X \oplus B \wedge D \not\leq_T X) \longrightarrow C_n \leq_T X$, for some n.

We will construct $A = \bigcup_{s\in\omega} \theta_s$ and $B = \bigcup_{s\in\omega} \vartheta_s$ by coinfinite extensions θ_s and ϑ_s. For this purpose, we let

$$x \in A_n, B_n \longleftrightarrow < n, x > \in A, B,$$

so that A can be thought of as the infinite join of the A_n's. Also each A_n can be thought of as the n-th column of A. It is the same for B. Now we shall code D_n into the n-th columns A_n and B_n. To satisfy the first condition, we define

$$D_n(x) = \begin{cases} 0 & : \quad if x \notin C_n \\ A_n(x) & : \quad otherwise \end{cases}$$

Hence $\mathrm{D}_n \leq_T C_n \oplus$ A and $\mathrm{D}_n \leq_T C_n \oplus B$. It remains to ensure that $D_n \not\leq_T C_n$. To satisfy this condition, we define

$$\tilde{D_n}(x) = \begin{cases} 1 - \{e\}^{C_n}(x) & : \quad if x is C_h \text{ and } \{e\}^{C_n}(x) \downarrow \\ 0 & : \quad otherwise \end{cases}$$

where C_h is the e-th element of C_n.
Indeed, it will be enough to have A_n, and hence D_n, differ only finitely from $\tilde{D_n}$. Recall that the requirements of the second condition are :

$$R_{e,m} : \{e\}^{X_m \oplus A} \simeq \{e\}^{X_m \oplus B} \simeq D \wedge D \not\leq_T X_m \longrightarrow C_n \leq_T X_m,$$

for some n (Here we are using Posner's trick [7]).
We will satisfy them by the following construction. At step (s = 0), we define $\theta_0 = \vartheta_0 = \emptyset$. At each step (s + 1) we are given θ_s dan ϑ_s defined on the first s columns and on finitely many other points, and differing on each of the first s columns only finitely. There are three cases to consider:
Case 1: We can make $\{e\}^{X_m \oplus A}$ not total. Thus, there is a string σ compatible with θ_s, such that: $(\exists y)(\forall \tau \supseteq \sigma)(\tau$ compatible with $\theta_s \rightarrow \{e\}^{\tau}(y) \uparrow)$. We take one such string, and define :

$$\theta_{s+1}(x) = \begin{cases} \theta_s(x) & : \quad if \theta_s(x) \downarrow \\ \sigma(x) & : \quad if \sigma(x) \downarrow \\ \tilde{D_s}(z) & : \quad if x = \langle s, z \rangle, otherwise \end{cases}$$

We can define ϑ_{s+1} similarly.
Case 2: We can make $\{e\}^{X_m \oplus A}$ total but we can force a special kind of disagreement.
Case 3: We can make $\{e\}^{X_m \oplus A}$ total. In this case we can define:

$$\theta_{s+1}(x) = \begin{cases} \theta_s(x) & : \quad if \theta_s(x) \downarrow \\ \tilde{D_s}(z) & : \quad if x = \langle s, z \rangle, otherwise \end{cases}$$

We can define ϑ_{s+1} similarly. This completes the sketch of the construction. $\diamond$

Now we are ready to prove the Slaman-Woodin coding theorem.

Proof of theorem 3.3. The idea is to reduce defining an arbitrary countable set of Turing degrees to defining an antichain, because we know how to define the latter. Given a countable set $\jmath = (\mathbf{c}_n)_{n \in \omega}$. We define a set $(\nabla = \mathbf{a}_n)_{n \in \omega}$ of pairwisely incomparable degrees such that:

$$\mathbf{x} \cup \mathbf{a_m} \leq \mathbf{y} \cup \mathbf{a_n} \longleftrightarrow \mathbf{x} \leq \mathbf{y} \wedge \mathbf{m} = \mathbf{n},$$

for every $\mathbf{x} \leq \mathbf{c}$.

Here we choose $\mathbf{c}$ above every element of $\jmath$. It is standard to construct such a set , that is a set of degrees recursively independent over the degrees below $\mathbf{c}$. Observe that any infinite subset $\hat{\nabla}$ of ∇, for any bijection f, is an antichain, and so is the set $\hat{\jmath} = (\mathbf{c_n} \cup \mathbf{a_{f(n)}})$ by the choice of ∇. Hence both $\hat{\nabla}$ and ∇ are definable with parameters by the previous Lemma. It is easy to see that $\jmath$ is definable, i.e.

$$\mathbf{x} \in \jmath \longleftrightarrow \mathbf{x} \leq \mathbf{c} \wedge (\exists \mathbf{a} \in \hat{\nabla})(\mathbf{x} \cup \mathbf{a} \in \hat{\jmath})$$

Observe that the map $\hat{f}$ from $\jmath$ to $\hat{\jmath}$ induced by f is definable as well, i.e.

$$\hat{f}(\mathbf{x}) = \mathbf{y} \longleftrightarrow \mathbf{x} \in \jmath \wedge \mathbf{x} \cup \mathbf{y} \in \hat{\jmath}$$

Now define the n-projections of Relation R as follows. For each m smaller than n, we have

$$R(m) = (\mathbf{a} : [\exists \mathbf{r} \in \mathbf{R}](\mathbf{r(m)} = \mathbf{a})),$$

Let $\mathbf{c}$ be a uniform upper bound on all of the element R(m). We now pick up n disjoint subset of ∇, to code the projections R(m). We observe that R is definable, because :

$$R(\mathbf{x_1}, \cdots, \mathbf{x_n}) \longleftrightarrow \hat{\mathbf{f_1}}(\mathbf{x_1}) \cup \cdots \cup \hat{\mathbf{f_n}}(\mathbf{x_n}).$$

This completes the proof. $\diamond$

It is worth noting here that the coding theorem allows for a quick proof of the Simpson theorem, which characterizes the complexity of the theory of degrees.

Theorem 3.5 *(S.G. Simpson [1977])*
The first order theory of $\mathbf{D}$ *has the same degree as the theory of second order arithmetic.*

Proof. It is clear that the theory of degrees is interpretable in second-order arithmetic. It remains to show that second-order arithmetic is interpretable in $\mathbf{D}$. Indeed, we can code the standard model of arithmetic and define the collection of codes of standard model, by the Coding theorem. Besides we can interpret second-order quantifiers over a coded countable model by quantifiers over sequences in $\mathbf{D}$ which define unary relations. $\diamond$

Corollary 3.6 *(Lachlan [1968])*
The first order theory of $\mathbf{D}$ *is undecidable and not axiomatizable.*

Definition 3.7 *We say that* $\mathbf{D}$ *is* rigid *if it has no automorphism other than the identity.* $\mathbf{D}$ *is* locally rigid *if each element of* $\mathbf{D}$ *belongs to a rigid countable jump ideal.*

Observe that **D** is rigid if and only if

$$(\mathbf{a}, \mathbf{b} \in D)[< \mathbf{D}, \mathbf{a} > \cong < \mathbf{D}, \mathbf{b} > \longrightarrow \mathbf{a} = \mathbf{b}].$$

We turn now to the following important result.

Theorem 3.8 *(Slaman and Woodin [1986])*
$\boldsymbol{D}$ is locally rigid if and only if the relation "$\vec{\mathbf{c}}$ is code for element of $\mathbf{x}'$" is definable in **D.**

Proof. To prove one direction, it suffices to show the following:
Suppose $\vec{\mathbf{c}}$ is code for a set of integer X. Then X has degree **a** if and only if there is a set of reals I including X whose degrees form a rigid countable jump ideal $\Im$ and an isomorphism ψ mapping the degrees of I to $\Im$ such that ψ maps the degree of X to **a**.

Indeed, this is immediate once it is shown that any two isomorphic jump ideals are identical. We observe that an ideal closed under jump consists exactly of the degrees of sets which are coded by parameters in the ideal. Thus two isomorphic ideals closed under jump code exactly the same sets, and must then be identical. For the converse, we can use a standard Skolem hull construction [6] to show that for each degree **a**, there is a countable jump ideal $\Im$ including **a** that is an elementary substructure of **D**. It must be the case that $\Im$ is rigid. $\diamond$

Corollary 3.9 *(Slaman and Woodin [1991])*
D *is rigid if and only if it is biinterpretable with second-order arithmetic (i.e. the biinterpretability conjecture holds).*

In fact, there are many definable relations on **D**, thus providing circumstantial evidence for the biinterpretability conjecture. We mention here for example: The c.e. degree **R**, every level of the arithmetical hierarchy, the arithmetical degrees and the Turing jump. See [7, 8] for more detailed information.

4 The Nonrigidity of the Turing Universe

Cooper has announced the existence of a nontrivial automorphism of **D** and hence the biinterpretability conjecture fails.

Theorem 4.1 *(Cooper [1997])*
There exists a nontrivial automorphism of the Turing universe.

To prove this theorem, one needs to prove that there is an order-preserving bijective function over the Turing degrees which is not the identity. We need to build deg(f) $\neq$ deg(g) with a bijective induced function $\tilde{*} :< \mathbf{D}, \mathbf{f} > \longrightarrow < \mathbf{D}, \mathbf{g} >$.

Definition 4.2 *A binary relation $* \subset (\omega^{<\omega})^2$, presents $\tilde{*} : \omega^\omega \longrightarrow \omega^\omega$, if and only if for all $f, g \in \omega^\omega$,*

$$f^{\tilde{*}} = g \iff (\exists\{(\sigma_i, \tau_i)\}_{i\in\omega} \subset *)(\forall i)[\sigma_i \subset \sigma_{i+1} \subset f \& \tau_i \subset \tau_{i+1} \subset g].$$

Here **D** is considered as being the degree structure derived from considering Turing reducibility over the tree $\omega^{<\omega}$. The construction is of a partial function $* : \omega^{<\omega} \longrightarrow \omega^{<\omega}$, with a bijective induced function $\tilde{*} : \omega^{\omega} \longrightarrow \omega^{\omega}$, involving stage by stage modification of the identity automorphism to accommodate the required witnesses to nontriviality. We let $\hat{\Phi}$ and $\hat{\Psi}$ denotes partial computable functionals. Then one has requirements of the form:

$$\wp_{\Phi} : (\forall X, Y \in \omega^{\omega})[X = \Phi^{Y} \longrightarrow X^{\tilde{*}} = \hat{\Phi}^{Y^{\tilde{*}}}]$$

$$\Im_{\Psi} : (\forall X, Y \in \omega^{\omega})[Y = \Psi^{X} \longrightarrow Y^{\tilde{*}^{-1}} = \hat{\Psi}^{X^{\tilde{*}^{-1}}}]$$

The actual details of the construction to satisfy the requirements are very complex and difficult. A detailed discussion of this construction lies outside the scope of this paper. See [1, 2] for more detailed discussion.

5 Some philosophical considerations

Cooper argued [3, 4] that the material universe, in a fairly precise sense, is Turing definable. Moreover it is argued that the Turing Universe provides a mathematical model for the heterogeneous mix of computability and incomputability, leading to theoretical explanations of anomalies in the scientific paradigm. In particular, Turing Nonrigidity implies the existence of natural laws which are not fully defined within the material universe, thus providing a theoretical context for the reconciliation of quantum and classical phenomena. The consolidation of quantum entities into classically observable objects has an immediate parallel in the actions of the algebraic operations on the Turing degrees and the invariance of $\mathbf{0}'$ [2, 3].

By Slaman and Woodin, it is known that rigidity is equivalent to the biinterpretability conjecture. Intuitively, one can draw a parallel between Hilbert's program for mathematics and the biinterpretability conjecture for the Turing Universe, insofar as it provides a model for the computationally complex scientific environment. The failure of biinterpretability conjecture then implies the kind of 'theory of everything' predicted by Hawking and others, may be unattainable in practice.

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