A NOTE ON PURE VARIATIONS OF AXIOMS OF BLACKWELL DETERMINACY

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We combine Vervoort's *Pure Variations* and *Axiomatic Variations* to get results on the strength of axioms of pure Blackwell determinacy.

1. Introduction

In this note, we look at weaker versions of the Axiom of Blackwell Determinacy introduced in [**Lö02b**] and [**Lö03**]. We use these results to get a new lower bound for the consistency strength of the Axiom of pure Blackwell Determinacy $\mathsf{pBl\text{-}AD}$.

We shall investigate Blackwell games against opponents with arbitrary mixed strategies $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$, against opponents with usual pure strategies $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$, and (for reasons of expositional completeness) against blindfolded opponents $\mathsf{blBl\text{-}Det}(\mathbf{\Gamma})$. We shall show that under the assumption of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}(\mathbf{\Gamma})$, the axioms $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ and $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$ are equivalent.

We define our axioms of Blackwell determinacy in Section 2. In Section 3, we prove the Purification Theorem 3.3 as an easy application of ideas from Vervoort's *doctoraal* (M.Sc.) thesis [**Ve95**].

This paper will assume knowledge about descriptive set theory as can be found in [**Ke95**] or [**Ka94**]. Since the full axiom of Blackwell determinacy contradicts the full Axiom of Choice AC, we shall work throughout this paper in the theory $\mathsf{ZF} + \mathsf{DC}$.

We will be working on Baire space $\mathbb{N}^\mathbb{N}$ and Cantor space $2^\mathbb{N}$, endowed with the product topology of the discrete topologies on $\mathbb{N}$ and $2 = \{0, 1\}$, $\mathbb{N}^{<\mathbb{N}}$ is the set of finite sequences of natural numbers and $2^{<\mathbb{N}}$ is the set of finite binary sequences. Let us write $\mathbb{N}^{\text{even}}$ and $\mathbb{N}^{\text{odd}}$ for finite sequences of even and odd length, respectively, and $\text{Prob}(\mathbb{N})$ for the set of probability measures on $\mathbb{N}$. Lebesgue measure on $\mathbb{N}^\mathbb{N}$ will be

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denoted by λ, and if Γ is a pointclass, we write $\mathsf{LM}(\Gamma)$ for "all sets in Γ are Lebesgue measurable". A pointclass is called **boldface** if it is closed under continuous preimages (or, equivalently, downward closed under Wadge reducibility $\leq_{\mathrm{W}}$).

We shall be using the standard notation for infinite games: If $x \in \mathbb{N}^{\mathbb{N}}$ is the sequences of moves for player I and $y \in \mathbb{N}^{\mathbb{N}}$ is the sequence of moves for player II, we let $x * y$ be the sequence constructed by playing x against y, *i.e.*,

$$(x * y)(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1. \end{cases}$$

Conversely, if $x \in \mathbb{N}^{\mathbb{N}}$ is a run of a game, then we let x_{I} be the part played by player I and x_{II} be the part played by player II, *i.e.*, $x_{\mathrm{I}}(n) = x(2n)$ and $x_{\mathrm{II}}(n) = x(2n+1)$.

2. Definitions

Blackwell determinacy goes back to imperfect information games of finite length due to von Neumann and was introduced for infinite games by Blackwell [**Bl69**].

We call a function $\sigma : \mathbb{N}^{\mathrm{Even}} \to \mathrm{Prob}(\mathbb{N})$ a **mixed strategy for player I** and a function $\sigma : \mathbb{N}^{\mathrm{Odd}} \to \mathrm{Prob}(\mathbb{N})$ a **mixed strategy for player II**. A mixed strategy σ is called **pure** if for all $s \in \mathrm{dom}(\sigma)$ the measure $\sigma(s)$ is a Dirac measure, *i.e.*, there is a natural number n such that $\sigma(s)(\{n\}) = 1$. This is of course equivalent to being a strategy in the usual (perfect information) sense. A pure strategy σ is called **blindfolded** if for s and t with $\mathrm{lh}(s) = \mathrm{lh}(t)$, we have $\sigma(s) = \sigma(t)$. Playing according to a blindfolded strategy is tantamount to fixing your moves in advance and playing them regardless of what your opponent does. If $x \in \mathbb{N}^{\mathbb{N}}$, we denote the blindfolded strategy that follows x by $\mathbf{bf}_x$:

$$\mathbf{bf}_x(s) = x\left(\left\lfloor \frac{\mathrm{lh}(s)}{2} \right\rfloor\right).$$

We denote the classes of mixed, pure and blindfolded strategies with $\mathcal{S}_{\mathrm{mixed}}$, $\mathcal{S}_{\mathrm{pure}}$, and $\mathcal{S}_{\mathrm{blindfolded}}$, respectively.

Let

$$\nu(\sigma, \tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \mathbb{N}^{<\mathbb{N}}$, we can define

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\upharpoonright}i)(\{s_i\}).$$

This generates a Borel probability measure on $\mathbb{N}^{\mathbb{N}}$. If B is a Borel set, $\mu_{\sigma,\tau}(B)$ is interpreted as the probability that the result of the game ends up in the set B when player I randomizes according to σ and player II according to τ. If σ and τ are both pure, then $\mu_{\sigma,\tau}$ is a Dirac measure concentrated on the unique real that is the outcome of this game, denoted by $\sigma * \tau$. As usual, we call a pure strategy σ for player I (τ for player II) a **winning strategy** if for all pure counterstrategies τ (σ), we have that $\sigma * \tau \in A$ ($\sigma * \tau \notin A$).

Let $\mathcal{S}$ be a class of strategies, σ a mixed strategy for player I, and τ a mixed strategy for player II. We say that σ is **$\mathcal{S}$-optimal** for the payoff set $A \subseteq \mathbb{N}^{\mathbb{N}}$ if for all $\tau_* \in \mathcal{S}$ for player II, $\mu^-_{\sigma,\tau_*}(A) = 1$, and similarly, we say that τ is **$\mathcal{S}$-optimal** for the payoff set $A \subseteq \mathbb{N}^{\mathbb{N}}$ if for all $\sigma_* \in \mathcal{S}$ for player I, $\mu^+_{\sigma_*,\tau}(A) = 0$.[1]

We call a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ **Blackwell determined**, **purely Blackwell determined**, or **blindfoldedly Blackwell determined** if either player I or player II has an $\mathcal{S}_{\text{mixed}}$-, $\mathcal{S}_{\text{pure}}$-, or $\mathcal{S}_{\text{blindfolded}}$-optimal strategy, respectively, and we call a pointclass Γ Blackwell determined (purely Blackwell determined, blindfoldedly Blackwell determined) if all sets $A \in \Gamma$ are Blackwell determined (purely Blackwell determined, blindfoldedly Blackwell determined). We write $\mathsf{Bl\text{-}Det}(\Gamma)$, $\mathsf{pBl\text{-}Det}(\Gamma)$, and $\mathsf{blBl\text{-}Det}(\Gamma)$ for these statements. Furthermore, we write $\mathsf{Bl\text{-}AD}$, $\mathsf{pBl\text{-}AD}$, and $\mathsf{blBl\text{-}AD}$ for the full axioms claiming (pure, blindfolded) Blackwell determinacy for all sets.[2]

There is a peculiar difference between pure and mixed strategies: If a pure strategy is $\mathcal{S}_{\text{blindfolded}}$-optimal, it is already winning. On the other hand, for a mixed strategy, being $\mathcal{S}_{\text{blindfolded}}$-optimal is rather weak.

Proposition 2.1. If σ is a pure strategy and A is an arbitrary payoff, then the following are equivalent for the game on A:

[1]Here, μ^+ denotes outer measure and μ^- denotes inner measure with respect to μ in the usual sense of measure theory. If A is Borel, then $\mu^+(A) = \mu^-(A) = \mu(A)$ for Borel measures μ.

[2]Note that the original definition of Blackwell determinacy was stronger than ours, using imperfect information strategies. By a result of Tony Martin's, the original definition is equivalent to our definition on boldface pointclasses; *cf.* [**Ma98**, p. 1579] and [**MaNeVe03**, p. 618*sq*].

(i) σ is $\mathcal{S}_{\text{blindfolded}}$-optimal, and
(ii) σ is a winning strategy.

Proof. We only have to show **(i)**$\Rightarrow$**(ii)**. Without loss of generality, assume that σ is a strategy for player I. Let τ be a pure counterstrategy such that $\sigma * \tau \notin A$. Then the blindfolded strategy $\tau^* := \mathbf{bf}_{(\sigma*\tau)_{\mathrm{II}}}$ witnesses that σ is not $\mathcal{S}_{\text{blindfolded}}$-optimal, since $\sigma * \tau^* = \sigma * \tau \notin A$. □

Proposition 2.2. Let $A := \{x \in 2^{\mathbb{N}}\,;\, \forall n(x(2n) \neq x(2n+1)\}$. In the game with payoff A, player II has a winning strategy, and player I has an $\mathcal{S}_{\text{blindfolded}}$-optimal strategy. In particular, the strategy of player I is a $\mathcal{S}_{\text{blindfolded}}$-optimal strategy which cannot be $\mathcal{S}_{\text{pure}}$-optimal.

Proof. Obviously "copy the last move of player I" is a winning strategy for player II. But the "randomize" strategy $\sigma(s)(\{0\}) = \sigma(s)(\{1\}) = \frac{1}{2}$ is also $\mathcal{S}_{\text{blindfolded}}$-optimal: Let τ be a blindfolded strategy for player II, then it corresponds to playing a fixed real x digit by digit. The derived measure $\mu_{\sigma,\tau}$ has the following properties: For each set $X \subseteq 2^{\mathbb{N}}$, we have **(a)** $\lambda(X) = \mu_{\sigma,\tau}(\{x\,;\, x_{\mathrm{I}} \in X\})$, and **(b)** $\delta_x(X) = \mu_{\sigma,\tau}(\{x\,;\, x_{\mathrm{II}} \in X\})$ where δ_x is the Dirac measure concentrating on x. Thus $\mu_{\sigma,\tau}(2^{\mathbb{N}}\backslash A) = \mu_{\sigma,\tau}(\{y\,;\, \forall n(y(2n) = y(2n+1) = x(n))\} = \lambda(\{x\}) = 0$. □

The proof of Proposition 2.2 not only displays that for mixed strategies the different notions of optimality are not equivalent, but since player I's winning strategy is also $\mathcal{S}_{\text{blindfolded}}$-optimal, we get an example of a game in which both players have a $\mathcal{S}_{\text{blindfolded}}$-optimal strategy which seems to mess with the usual dichotomy arguments of set-theoretic game theory.

However, while the statement "player I has a $\mathcal{S}_{\text{blindfolded}}$-optimal strategy in the game A" seems to tell us terribly little about the structure of A, the axioms of blindfolded Blackwell determinacy give us logical strength:

Theorem 2.3 (Martin)**.** The statements $\mathsf{blBl\text{-}Det}(\mathbf{\Pi}^1_1)$, $\mathsf{pBl\text{-}Det}(\mathbf{\Pi}^1_1)$, and $\mathsf{Det}(\mathbf{\Pi}^1_1)$ are equivalent.

Proof. *Cf.* [**Lö03**, Corollary 3.9] □

Theorem 2.4. The axioms $\mathsf{blBl\text{-}AD}$ and $\mathsf{pBl\text{-}AD}$ imply the existence of an inner model with a strong cardinal.

Proof. *Cf.* [**Lö03**, Corollary 4.9]. □

3. Purifying mixed opponents

We say that a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is **universally measurable** if for each Borel probability measure μ on $\mathbb{N}^{\mathbb{N}}$, inner and outer μ-measure of A coincide, and write $\mathsf{UM}(\Gamma)$ for "all sets in Γ are universally measurable".

Theorem 3.1. If $\mathbf{\Gamma}$ is a boldface pointclass, then $\mathsf{LM}(\mathbf{\Gamma})$ implies $\mathsf{UM}(\mathbf{\Gamma})$.

Proof. It is enough to show this theorem for subsets of $[0,1]$, so let us assume that our measures live on $[0,1]$.

Let us first assume that μ has a (necessarily countable) nonempty set of atoms A (*i.e.*, $\mu(\{x\}) = 0$ for all $x \in A$). If $\mu(A) = 1$, then every subset of $\mathbb{N}^{\mathbb{N}}$ is μ-measurable. If $\mu(\mathbb{N}^{\mathbb{N}} \backslash A) =: \Phi > 0$, define an atomless measure μ^* by

$$\mu^*(X) := \Phi^{-1} \cdot \mu(X \backslash A).$$

Then μ-measurability and μ^*-measurability coincide. Thus, it is enough to prove the theorem for atomless measures μ.

If μ is atomless, $D_\mu(x) := \mu(\{y\,;\, y \leq x\})$ is a continuous function, and for each set $X \subseteq [0,1]$, μ-measurability of X is equivalent to λ-measurability of $D_\mu^{-1}[X]$. □

The main result of this section is the Purification Theorem 3.3. It is a consequence of a purification theorem for infinite Blackwell games due to Vervoort. The main idea is to understand a mixed strategy as a probability distribution over the set of pure strategies. In this general form, the idea goes back to Harsanyi's famous purification theorem in [**Ha73**].

A pure strategy can be understood as a function $\sigma : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, so we can see the set of pure strategies as a product indexed by $\mathbb{N}^{<\mathbb{N}}$.

Let τ be a mixed strategy. We shall define a probability measure V_τ on $\mathbb{N}^{(\mathbb{N}^{<\mathbb{N}})}$ which we shall call the **Vervoort code of** τ. If p_0, ..., p_n are elements of $\mathbb{N}^{<\mathbb{N}}$ and N_0, ..., N_n are natural numbers, define

$$V_\tau(\{\tau^*\,;\, \tau^*(p_0) = N_0 \,\&\, ... \,\&\, \tau^*(p_n) := N_n\}) = \prod_{i=0}^{n} \tau(p_i)(\{N_i\}),$$

and let V_τ be the unique extension of this function using countable additivity. The measure V_τ is a Borel probability measure on $\mathbb{N}^{(\mathbb{N}^{<\mathbb{N}})}$.

Theorem 3.2 (Vervoort)**.** If σ and τ are mixed strategy and B is a Borel subset of $\mathbb{N}^{\mathbb{N}}$, then

$$\begin{aligned}\mu_{\sigma,\tau}(B) &= \int \mu_{\sigma,x}(B)\,dV_\tau(x)\\ &= \int \mu_{x,\tau}(B)\,dV_\sigma(x)\end{aligned}$$

Proof. This is essentially [**Ve95**, Theorem 6.6]. □

Purification Theorem 3.3. Let $\mathbf{\Gamma}$ be a boldface pointclass and assume that $\mathsf{LM}(\mathbf{\Gamma})$ holds. Let $A \in \mathbf{\Gamma}$ and assume that A is purely Blackwell determined. Then A is Blackwell determined. Moreover, every $\mathcal{S}_{\text{pure}}$-optimal strategy is actually $\mathcal{S}_{\text{mixed}}$-optimal.

Proof. First of all, notice that by Theorem 3.1, we have $\mathsf{UM}(\mathbf{\Gamma})$, so A is universally measurable. Since A is purely Blackwell determined, let σ be (without loss of generality) an $\mathcal{S}_{\text{pure}}$-optimal strategy for player I. Let τ be an arbitrary mixed strategy for player II. We shall show that for every Borel superset $B \supseteq A$, we have $\mu_{\sigma,\tau}(B) = 1$. Because A is universally measurable, this proves the claim.

By our assumption, we know that for every pure strategy τ^* and every Borel $B \supseteq A$, we have $\mu_{\sigma,\tau^*}(B) = 1$. But this means that the function

$$\mathbf{v} : \mathbb{N}^{(\mathbb{N}^{<\mathbb{N}})} \to \mathbb{R} : x \mapsto \mu_{\sigma,x}(B)$$

is the constant function with value 1.

Using Theorem 3.2, we get

$$\mu_{\sigma,\tau}(B) = \int \mu_{\sigma,x}(B)\,dV_\tau(x) = \int \mathbf{v}(x)\,dV_\tau(x) = 1.$$

□

Corollary 3.4. Let $\mathbf{\Gamma}$ be a boldface pointclass such that $\mathsf{LM}(\mathbf{\Gamma})$. Then $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$.

4. Conclusion

In the following, we shall be using the equivalence theorem of Martin, Neeman and Vervoort:

Theorem 4.1 (Martin, Neeman, Vervoort)**.** Let $\mathbf{\Gamma}$ be either $\mathbf{\Delta}^1_{2n}$, $\mathbf{\Sigma}^1_{2n}$, $\mathbf{\Delta}^1_{2n+1}$, $\Game^n(<\omega^2\text{-}\mathbf{\Pi}^1_1)$, or $\wp(\mathbb{N}^{\mathbb{N}})\cap\mathbf{L}(\mathbb{R})$. Then $\mathsf{Bl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Det}(\mathbf{\Gamma})$.

Proof. This is Theorem 5.1, Corollary 5.3, Theorem 5.4, Theorem 5.6, and Theorem 5.7 in [**MaNeVe03**]. □

Corollary 4.2. Let $\mathbf{\Gamma}$ be either $\mathbf{\Delta}^1_{2n}$, $\mathbf{\Sigma}^1_{2n}$, $\mathbf{\Delta}^1_{2n+1}$, $\Game^n(<\omega^2\text{-}\mathbf{\Pi}^1_1)$, or $\wp(\mathbb{N}^\mathbb{N}) \cap \mathbf{L}(\mathbb{R})$, and assume $\mathsf{LM}(\mathbf{\Gamma})$. Then $\mathsf{pBl\text{-}Det}(\mathbf{\Gamma})$ implies $\mathsf{Det}(\mathbf{\Gamma})$. In particular, $\mathsf{pBl\text{-}PD}$ and "all projective sets are Lebesgue measurable" implies PD.

Proof. Clear from Corollary 3.4 and Theorem 4.1. □

Corollary 4.3. The statement $\mathsf{pBl\text{-}Det}(\mathbf{\Delta}^1_2)$ implies the existence of an inner model with a Woodin cardinal.

Proof. Note that by Theorem 2.3, $\mathsf{pBl\text{-}Det}(\mathbf{\Delta}^1_2)$ implies analytic determinacy, and thus –by the usual (Solovay) unfolding argument– the Lebesgue measurability of all $\mathbf{\Sigma}^1_2$ sets. But this is enough to apply Corollary 4.2 and get $\mathsf{Det}(\mathbf{\Delta}^1_2)$ which yields by a famous theorem of Woodin's the existence of an inner model with a Woodin cardinal. □

Building on the results from [**Lö03**], Greg Hjorth (2002, personal communication) found a proof of "$\mathsf{blBl\text{-}AD}$ implies the existence of an inner model with a Woodin cardinal" using Cabal-style descriptive set theory and inner model theory.

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