On $HS = SH$ and Duality Theorems of Intuitionistic Descriptive Frames

Fan Yang*
Institute for Logic, Language and Computation
Universiteit van Amsterdam

November 8, 2007

Abstract

The variety of Heyting algebras has a nice property that $\mathbf{HS} = \mathbf{SH}$. Heyting algebras are the algebraic dual of intuitionistic descriptive frames. The goal of this paper is to define proper dual notions so as to formulate this algebraic properties in the frame language, and to give a frame-based proof of this property and some other duality theorems.

1 Introduction

We know that for any algebra $\mathfrak{A}$ in a variety, it holds that $\mathbf{SH}(\mathfrak{A}) \subseteq \mathbf{HS}(\mathfrak{A})$ but in general not the other way around. However, the class of Heyting algebras is a variety for which the other direction does hold. This gives the property that $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$ for every Heyting algebra $\mathfrak{A}$.

Heyting algebras are the algebraic dual of intuitionistic descriptive frames [1]. So it is possible to express the above property in the framework of descriptive frames. The goal of this paper is to define proper dual notions so as to formulate the above algebraic properties in the frame language, and to give a frame-based proof of this property and some other duality theorems.

2 Intuitionistic Kripke Frames

In order to study descriptive frames, we start with intuitionistic Kripke frames, which is the simplest case of intuitionistic frames. In this section, we prove a theorem which corresponds to the algebraic property $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$ for Heyting algebras in the intuitionistic Kripke frame case.

2.1 The Definition of Intuitionistic Kripke Frames

Definition 1. An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of a nonempty set W and a partial order R on W.

*This paper has been written as a MoL project under the advice of Prof. dr. Dick de Jongh.

[1] Hereafter, we only write "descriptive frame" for short.

Definition 2. A set $V \subseteq W$ is called an *upward closed subset*, if for every $w \in V$ and $u \in W$, wRu implies $u \in V$.

Definition 3. An intuitionistic Kripke frame $\mathfrak{G} = \langle V, S\rangle$ is called a *subframe* of an intuitionistic Kripke frame $\mathfrak{F} = \langle W, R\rangle$ if $V \subseteq W$ and S is the restriction of R to V ($S = R \restriction V$, in symbols), i.e., $S = R \cap V^2$. The subframe $\mathfrak{G}$ is a *generated subframe* of $\mathfrak{F}$ if V is an upward closed subset of W.

Definition 4. Suppose $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are two intuitionistic Kripke frames. A map f from W to V is called a *p-morphism* from $\mathfrak{F}$ to $\mathfrak{G}$ if the following conditions hold for every $w, u \in W$:
(P1) wRu implies $f(w)Sf(u)$;
(P2) $f(w)Sf(u)$ implies $\exists v \in W(wRv \wedge f(v) = f(u))$.

2.2 A Frame Instance of $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$

The next theorem is actually the dual theorem of $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$ of Heyting algebra in terms of intuitionistic Kripke frames. Here we give a constructive proof of this algebraic result.

We will generalize the result of Theorem 5 to the case of descriptive frames in Proposition 44 in Section 4.

Theorem 5. *An intuitionistic Kripke frame $\mathfrak{F}'$ is a generated subframe of a p-morphic image of an intuitionistic Kripke frame $\mathfrak{G}$ iff $\mathfrak{F}'$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.*

Proof. For "$\Rightarrow$": Suppose $\mathfrak{F} = \langle V, S\rangle$ is a p-morphic image of $\mathfrak{G} = \langle W, R\rangle$ via f, and $\mathfrak{F}' = \langle V', S'\rangle$ is a generated subframe of $\mathfrak{F}$. Define $\mathfrak{G}' = \langle f^{-1}(V'), R \restriction f^{-1}(V')\rangle$ and $g = f \restriction f^{-1}(V') : f^{-1}(V') \to V'$. We prove the following:

(1) $\mathfrak{G}'$ is a generated subframe of $\mathfrak{G}$.

(2) g is a p-morphism.

For (1): We only need to show that $f^{-1}(V')$ is upward closed in W. Suppose $w \in f^{-1}(V')$ and wRu for some $u \in W$. Since f is a p-morphism, we have $f(w)Sf(u)$. Then, from $f(w) \in V'$ and V' being upward closed, we obtain $f(u) \in V'$, i.e., $u \in f^{-1}(V')$.

For (2): It suffices to show that the map g from $\mathfrak{G}'$ to $\mathfrak{F}'$ satisfies the two conditions of a p-morphism.

(P1) Suppose $w_1 R w_2$ where $w_1, w_2 \in f^{-1}(V')$. Since f is a p-morphism, we have $f(w_1)Sf(w_2)$, i.e., $g(w_1)Sg(w_2)$.

(P2) Suppose $g(w)S'g(u)$. Then $f(w)Sf(u)$, for some $w, u \in f^{-1}(V')$. Since f is a p-morphism, there exists $v \in W$ s.t. $f(v) = f(u)$ and wRv. Thus $v \in f^{-1}(V')$ since $f^{-1}(V')$ is upward closed. Finally we also get $g(v) = f(v) = f(u) = g(u)$ as required.

For "$\Leftarrow$": Suppose g is a p-morphism from $\mathfrak{G}' = \langle W', R'\rangle$ to $\mathfrak{F}' = \langle V', S'\rangle$, where $\mathfrak{G}'$ is a generated subframe of $\mathfrak{G} = \langle W, R\rangle$. Without loss of generality, we may assume $W \cap V' = \emptyset$. We first define a frame $\mathfrak{F} = \langle V, S\rangle$ by putting $V = V' \cup (W - W')$ and $S = S' \cup S_1 \cup S_2$, where $S_1 = R \restriction (W - W')$ and

$$S_2 = \{(w_1, g(w_2)) : w_1 \in W - W', w_2 \in W' \text{ and } w_1 R w_2\}.$$

Note that S', S_1 and S_2 are pairwise disjoint.

Next we verify that the relation S defined above is a partial order.

In fact, for any $v \in V$, if $v \in V'$, then $vS'v$, so vSv. If $v \in W - W'$, then vS_1v, i.e., vSv, hence S is reflexive.

Suppose xSy and ySx. Then by the construction of S, it is impossible that one of x and y is in V' but another in $W - W'$. If both x and y are in V', then it follows that $xS'y$ and $yS'x$, hence $x = y$. If both x and y are in $W - W'$, then xRy and yRx, so $x = y$.

Suppose xSy and ySz. If $xS'y$, then $x, y \in V'$, so it must be the case that $z \in V'$. Thus by the transitivity of S', we have $xS'z$, hence xSz;

If xS_1y, then $x, y \in W - W'$ and xRy. Whenever $z \in W - W'$, from the transitivity of R it follows xS_1z. Whenever $z \in V'$, yRw for some $w \in W'$ such that $z = g(w)$, thus xRw, and so xS_2z by the definition of S_2.

If xS_2y, then $x \in W - W'$, and there exists $w \in W'$ such that xRw and $y = g(w) \in V'$. So it must be the case that $yS'z$. Since $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{G}'$ via g, by (P2), there exists $u \in W'$ such that $wRu, z = f(u)$, thus xRu, and so xS_2z by the definition of S_2.

Next, we define a map $f : W \to V$ by taking

$$f(w) = \begin{cases} g(w), & \text{if } w \in W' \\ w, & \text{if } w \in W - W'. \end{cases}$$

It is easy to see that $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and f is surjective. It remains to show that f is a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}$, i.e., f satisfies (P1) and (P2).

(P1) For each $w_1, w_2 \in W$ s.t., w_1Rw_2, from the construction of S, it is easy to see that $f(w_1)Sf(w_2)$.

(P2) For each $w, u \in W$ s.t., $f(w)Sf(u)$, we find a $v \in W$ s.t., $f(v) = f(u)$ and wRv.

Case 1: $w \in W'$. Then $f(w) = g(w) \in V'$, which implies $f(u) \in V'$ for V' is upward closed. Since g is a p-morphism, there exists $v \in W' \subseteq W$ s.t., $g(v) = g(u)$ and $wR'v$. These are equivalent to $f(v) = f(u)$ and wRv.

Case 2: $w, u \in W - W'$. Then from $f(w)Sf(u)$, we get wSu. It follows that wS_1u. Thus, by the definition of S_1, we have wRu and $v = u$ satisfies (P2).

Case 3: $w \in W - W'$ and $u \in W'$. Then from $f(w)Sf(u)$ and $u \in W'$, we get $wSg(u)$. It follows that $wS_2g(u)$. So by the definition of S_2, there exists $v \in W'$ s.t., $f(v) = f(u)$ and wRv. □

3 Duality Theorems of Intuitionistic General Frames

In this section, we prove some duality theorems of intuitionistic general frames. These results are crucial in proving some nice properties and duality theorems of descriptive frames.

3.1 The Definition of Intuitionistic General Frames and Their Duals

Definition 6. An *intuitionistic general frame* is a triple $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic Kripke frame and $\mathcal{P}$ is a family of some upward closed sets, containing $\emptyset$ and closed under $\cap$, $\cup$ and the following operation $\supset$: for every $X, Y \subseteq W$,

$$X \supset Y = \{x \in W : \forall y \in W(xRy \wedge y \in X \to y \in Y)\}$$

Note 7. *From the definition above, it follows that* $W \in \mathcal{P}$ *since* $W = \emptyset \supset \emptyset$.

The next two theorems show that there is a one-to-one correspondence between intuitionistic general frames and Heyting algebras. For the proof of Theorem 8, see Chapter 8 in [5].

Theorem 8. *Let* $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ *be an intuitionistic general frame. The algebra* $\langle \mathcal{P}, \cap, \cup, \supset, \emptyset\rangle$ *is a Heyting algebra and is called the* dual *of* $\mathfrak{F}$*, denoted by* $\mathfrak{F}^+$.

Theorem 9. *Let* $\mathfrak{A}$ *be a Heyting algebra, define* $\mathfrak{A}_+ = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}}\rangle$ *as follows:*

(i) $W_{\mathfrak{A}} = \{\nabla \subseteq A : \nabla$ *is a prime filter of* $\mathfrak{A}\}$,

(ii) $\nabla_1 R_{\mathfrak{A}} \nabla_2$ *iff* $\nabla_1 \subseteq \nabla_2$,

(iii) $\mathcal{P}_{\mathfrak{A}} = \{\hat{a} : a \in A\}$, *where* $\hat{a} = \{\nabla \in W_{\mathfrak{A}} : a \in \nabla\}$.

Then $\mathfrak{A}_+$ *is an intuitionistic general frame called the* dual *of* $\mathfrak{A}$*. Furthermore,* $\mathfrak{A} \cong (\mathfrak{A}_+)^+ = \langle \mathcal{P}_{\mathfrak{A}}, \cap, \cup, \supset, \emptyset\rangle$.

Proof. We first prove $\mathfrak{A}_+$ is an intuitionistic general frame. Obviously, $R_{\mathfrak{A}}$ is a partial order, and $\mathcal{P}_{\mathfrak{A}}$ is upward closed. We first show that $\mathcal{P}_{\mathfrak{A}}$ is closed under $\cap$. Since the elements in $W_{\mathfrak{A}}$ are filters, we have for any $a, b \in A$,

$$\hat{a} \cap \hat{b} = \{\nabla \in W_{\mathfrak{A}} : a \in \nabla \text{ and } b \in \nabla\} = \{\nabla \in W_{\mathfrak{A}} : a \wedge b \in \nabla\} = \widehat{a \wedge b}.$$

Next we show that $\mathcal{P}_{\mathfrak{A}}$ is closed under the union $\cup$. Indeed, For arbitrary $\hat{a}, \hat{b} \in \mathcal{P}_{\mathfrak{A}}$, since $W_{\mathfrak{A}}$ consists of all prime filters ∇ of A, we have

$$\hat{a} \cup \hat{b} = \{\nabla \in W_{\mathfrak{A}} : a \in \nabla \text{ or } b \in \nabla\} = \{\nabla \in W_{\mathfrak{A}} : a \vee b \in \nabla\} = \widehat{a \vee b}.$$

To show that $\mathcal{P}_{\mathfrak{A}}$ is closed under the operation $\supset$, it suffices to prove that for any $a, b \in A$,

$$\hat{a} \supset \hat{b} = \widehat{a \to b}.$$

First, we show $\widehat{a \to b} \subseteq \hat{a} \supset \hat{b}$. For any $\nabla \in \widehat{a \to b}$ and any $\nabla' \in \hat{a}$ such that $\nabla R_{\mathfrak{A}} \nabla'$, it suffices to show $\nabla' \in \hat{b}$ i.e., $b \in \nabla'$, but this follows immediately from $a \to b \in \nabla \subseteq \nabla'$ and $a \in \nabla'$.

Second, we show $\hat{a} \supset \hat{b} \subseteq \widehat{a \to b}$. Suppose $\nabla \in \hat{a} \supset \hat{b}$. It suffices to show that $\nabla \in \widehat{a \to b}$. By the definition of operation $\supset$, for any $\nabla' \supseteq \nabla$, it holds that

$$\nabla' \in \hat{a} \Rightarrow \nabla' \in \hat{b}.$$

Consider the prime filter

$$\nabla_a = \{x \in A : \exists z \in \nabla(z \wedge a \leq x)\}.$$

Clearly, it holds that $\nabla_a \in \hat{a}$ and $\nabla \subseteq \nabla_a$. Thus we have $\nabla_a \in \hat{b}$, which means $\exists z \in \nabla(z \wedge a \leq b)$, hence $z \leq a \to b$. Therefore $a \to b \in \nabla$ i.e., $\nabla \in \widehat{a \to b}$.

Hence $\mathfrak{A}_+$ is an intuitionistic general frame. And we have also proved that the map f defined by taking $f(a) = \hat{a}$ is a natural homomorphism from $\mathfrak{A}$ onto $(\mathfrak{A}_+)^+ = \langle \mathcal{P}_{\mathfrak{A}}, \cap, \cup, \supset, \emptyset\rangle$.

Lastly we show that f is injective. Suppose $\hat{a} = \hat{b}$, then

$$\hat{\top} = \widehat{a \to a} = \hat{a} \supset \hat{a} = \hat{a} \supset \hat{b} = \widehat{a \to b}.$$

From $\{\top\} \in \hat{\top} = \widehat{a \to b}$, it follows that $a \to b \in \{\top\}$ i.e., $\top = a \to b$. Hence $a \leq b$. By a similar argument, we can prove $b \leq a$, hence $a = b$. □

3.2 Duality Theorems about Operations On General Frames

In the rest of this section, we discuss the relation between two operations on general frames (i.e., the operations of generating generated subframes and forming p-morphic images) and two algebraic operations (i.e., the operations of forming homomorphic images and generating subalgebras) respectively.

Definition 10. An intuitionistic general frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is called a *generated subframe* of an intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ if it satisfies the following conditions:
(S1) $\langle V, S \rangle$ is a generated subframe of $\langle W, R \rangle$,
(S2) $\mathcal{Q} = \{U \cap V : U \in \mathcal{P}\}$.

The notion of generated subframe of intuitionistic general frames corresponds to the notion of homomorphic image of Heyting algebras. To see this clearly, we prove the next two theorems.[2]

Theorem 11. *If h is an isomorphism of $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ onto a generated subframe of $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, then the map h^+ defined by*

$$h^+(X) = h^{-1}(X) = \{x \in V : h(x) \in X\}, \textit{ for every } X \in P,$$

is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.

Proof. W.l.o.g., we may assume $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$. Then h is an identity map and $h^+(X) = X \cap V$.

Clearly, h^+ is a surjection. We show it preserves $\cap, \cup$ and $\supset$. Let $X, Y \in \mathcal{P}$. Then we have

$$h^+(X \cap Y) = (X \cap Y) \cap V = (X \cap V) \cap (Y \cap V) = h^+(X) \cap h^+(Y);$$

$$h^+(X \cup Y) = (X \cup Y) \cap V = (X \cap V) \cup (Y \cap V) = h^+(X) \cup h^+(Y);$$

$$\begin{aligned} h^+(X \supset Y) &= \{x \in W : \forall y \in W(xRy \land y \in X \to y \in Y)\} \cap V \\ &= \{x \in V : \forall y \in V(xSy \land y \in X \cap V \to y \in Y \cap V)\} \\ &\text{(since } V \text{ is upward closed and } S = R \cap V^2) \\ &= (X \cap V) \supset (Y \cap V) \\ &= h^+(X) \supset h^+(Y). \end{aligned}$$

□

Note that in the proof above, we do not need the property that $\mathcal{Q}$ is closed under operations. This gives the following corollary.

[2] The proofs are adapted from those of Theorem 8.57 and Theorem 8.59 in [5].

Corollary 12. *Let* $\langle W', R' \rangle$ *be a Kripke generated subframe of an intuitionistic general frame* $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$*. Then by taking* $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$*, we can define an intuitionistic general frame* $\mathfrak{F}' = \langle W', R', \mathcal{P}' \rangle$ *which is a generated subframe of* $\mathfrak{F}$*.*

Theorem 13. *Suppose* h *is a homomorphism of a Heyting algebra* $\mathfrak{A}$ *onto a Heyting algebra* $\mathfrak{B}$*. Then*

(i) *For any filter* ∇' *in* $\mathfrak{B}$*,* $h^{-1}(\nabla')$ *is a filter in* $\mathfrak{A}$*; in particular,* $h^{-1}(\nabla')$ *is prime whenever* ∇' *is prime;*

(ii) *For any prime filter* ∇ *in* $\mathfrak{A}$ *such that* $h^{-1}(\top) \subseteq \nabla$*,* $\nabla = h^{-1}(h(\nabla))$*, and* $h(\nabla)$ *is a prime filter in* $\mathfrak{B}$*;*

(iii) *The map* h_+ *defined by*

$$h_+(\nabla') = h^{-1}(\nabla'), \text{ for every prime filter } \nabla' \text{ in } \mathfrak{B},$$

is an isomorphisim of $\mathfrak{B}_+$ *onto a generated subframe of* $\mathfrak{A}_+$*.*

Proof. (i) Let ∇' be any filter in $\mathfrak{B}$. Obviously $\top \in h^{-1}(\nabla')$.

$$\begin{aligned} b, b \to a \in h^{-1}(\nabla') &\Rightarrow h(b), h(b \to a) \in \nabla' \\ &\Rightarrow h(b), h(b) \to h(a) \in \nabla' \text{ (Since } h \text{ is a homomorphism)} \\ &\Rightarrow h(a) \in \nabla' \text{(since } \nabla' \text{is a filter)} \\ &\Rightarrow a \in h^{-1}(\nabla'). \end{aligned}$$

Hence $h^{-1}(\nabla')$ is a filter. In particular, if ∇' is prime, then

$$\begin{aligned} a \vee b \in h^{-1}(\nabla') &\Rightarrow h(a \vee b) = (h(a) \vee h(b)) \in \nabla' \\ &\quad \text{(since } h \text{ is a homomorphism)} \\ &\Rightarrow h(a) \in \nabla' \text{ or } h(b) \in \nabla' \text{ (since} \nabla' \text{ is a prime filter)} \\ &\Rightarrow a \in h^{-1}(\nabla') \text{ or } b \in h^{-1}(\nabla'). \end{aligned}$$

Hence $h^{-1}(\nabla')$ is prime.

(ii) It suffices to show $h^{-1}(h(\nabla)) \subseteq \nabla$. Indeed, $a \in h^{-1}(h(\nabla))$ implies $h(a) = h(b)$ for some $b \in \nabla$. Since h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, we have

$$h(b \to a) = h(b) \to h(a) = \top.$$

From the assumption that $h^{-1}(\top) \subseteq \nabla$, it follows $b \to a \in \nabla$, thus $a \in \nabla$.

Observe that $\nabla = h^{-1}(h(\nabla))$ is equivalent to the following:

$$a \in \nabla \text{ iff } h(a) \in h(\nabla). \tag{1}$$

Suppose $h(a), h(a) \to h(b) \in h(\nabla)$. Since h is a homomorphism, $h(a \to b) = h(a) \to h(b) \in h(\nabla)$. From (1) it follows that $a, a \to b \in \nabla$, which implies $b \in \nabla$ since ∇ is a filter. Hence $h(b) \in h(\nabla)$ and $h(\nabla)$ is a filter.

Next we show that $h(\nabla)$ is prime. For any $a, b \in \nabla$, we have

$$\begin{aligned} (h(a) \vee h(b)) \in h(\nabla) &\Leftrightarrow h(a \vee b) = (h(a) \vee h(b)) \in h(\nabla) \\ &\quad \text{(since } h \text{ is a homomorphism)} \\ &\Leftrightarrow a \vee b \in \nabla \text{ (by (1))} \\ &\Leftrightarrow a \in \nabla \text{ or } b \in \nabla \text{ (since } \nabla \text{ is a prime filter)} \\ &\Leftrightarrow h(a) \in h(\nabla) \text{ or } h(b) \in h(\nabla) \text{(by (1))}. \end{aligned}$$

(iii) By (i), h_+ is a well-defined map from $W_{\mathfrak{B}}$ to $W_{\mathfrak{A}}$. (1) indicates that h_+ is surjective. Obviously, h_+ is also injective. So h_+ is a bijection. Let

$$W = \{\nabla \in W_{\mathfrak{A}} : h^{-1}(\top) \subseteq \nabla\}.$$

Clearly W is upward closed in $W_{\mathfrak{A}}$.

Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a generated subframe of $\mathfrak{A}_+$. Note that

$$\begin{aligned} U' \in \mathcal{P} &\Leftrightarrow \exists U \in \mathcal{P}_{\mathfrak{A}} \text{ s.t. } U' = U \cap W \\ &\Leftrightarrow \exists a \in A \text{ s.t. } U' = \{\nabla \in W : a \in \nabla\}. \end{aligned}$$

Then, for any $X \subseteq W_{\mathfrak{B}}$,

$$\begin{aligned} X \in \mathcal{P}_{\mathfrak{B}} &\Leftrightarrow \exists b \in B,\ X = \{\nabla' \in W_{\mathfrak{B}} : b \in \nabla'\} \\ &\Leftrightarrow \exists a \in A,\ h(a) = b \ \text{ and } h^{-1}(X) = \{h^{-1}(\nabla') \in W_{\mathfrak{A}} : a \in h^{-1}(\nabla')\} \\ &\Leftrightarrow \exists a \in A,\ h^{-1}(X) = \{\nabla \in W : a \in \nabla\} \\ &\Leftrightarrow h^{-1}(X) \in \mathcal{P} \\ &\Leftrightarrow h_+(X) \in \mathcal{P} \end{aligned}$$

To show that h_+ is an isomorphism from $\mathfrak{B}_+$ onto $\mathfrak{F}$, it remains to show that for any $\nabla'_1, \nabla'_2 \in W_{\mathfrak{B}}$,

$$\nabla'_1 R_{\mathfrak{B}} \nabla'_2 \text{ iff } h_+(\nabla'_1) R_{\mathfrak{A}} h_+(\nabla'_2).$$

However, this is trivial since $\nabla'_1 \subseteq \nabla'_2$ iff $h^{-1}(\nabla'_1) \subseteq h^{-1}(\nabla'_2)$. □

Definition 14. Given intuitionistic general frames $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$, we say a map f from W to V is a *p-morphism* from $\mathfrak{F}$ to $\mathfrak{G}$, if the following three conditions are satisfied, for all $w, u \in W$ and $X \in \mathcal{Q}$:
(P1) wRu implies $f(w)Sf(u)$;
(P2) $f(w)Sf(u)$ implies $\exists v \in W(wRv \wedge f(v) = f(u))$;
(P3) $f^{-1}(X) \in \mathcal{P}$.

As with generated subframes, the notion of p-morphisms between general frames has its dual notion, i.e., subalgebras of Heyting algebras. We prove the next two theorems.[3]

Theorem 15. *If $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is a p-morphic image of $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ via f, then the map f^+ defined by*

$$f^+(X) = f^{-1}(X), \textit{ for every } X \in \mathcal{Q},$$

is an isomorphism of $\mathfrak{G}^+$ onto a subalgebra of $\mathfrak{F}^+$.

Proof. Since f is a p-morphism, we have for any $X \in \mathcal{Q}$, $f^{-1}(X) \in \mathcal{P}$, which gives $f^+[\mathcal{Q}] \subseteq \mathcal{P}$. We prove $f^+ : \mathcal{Q} \to f^+[\mathcal{Q}]$ is an isomorphism. Clearly, f^+ is a bijection. So it suffices to show that f^+ preserves all the operations in $\mathfrak{G}^+$. Let $X, Y \in \mathcal{Q}$. Then we have

$$f^+(X \cap Y) = f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) = f^+(X) \cap f^+(Y);$$

$$f^+(X \cup Y) = f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) = f^+(X) \cup f^+(Y);$$

[3]The proofs are adapted from those of Theorem 8.65 and Theorem 8.71 in [5].

$$
\begin{aligned}
x \in f^+(X \supset Y) &\Leftrightarrow f(x) \in X \supset Y \\
&\Leftrightarrow \forall f(y) \in V(f(x) S f(y) \wedge f(y) \in X \rightarrow f(y) \in Y) \\
&\Leftrightarrow \forall y \in W(xRy \wedge y \in f^{-1}(X) \rightarrow y \in f^{-1}(Y)) \\
&\text{(since } f \text{ is a p-morphism)} \\
&\Leftrightarrow x \in f^{-1}(X) \supset f^{-1}(Y) \\
&\Leftrightarrow x \in f^+(X) \supset f^+(Y)
\end{aligned}
$$

□

Theorem 16. *If f is an isomorphism of a Heyting algebra $\mathfrak{B}$ onto a subalgebra of $\mathfrak{A}$, then the map f_+ defined by*

$$f_+(\nabla) = f^{-1}(\nabla), \textit{ for every } \nabla \in W_{\mathfrak{A}},$$

is a p-morphism from $\mathfrak{A}_+$ onto $\mathfrak{B}_+$.

Proof. W.l.o.g., we may assume $\mathfrak{B}$ to be a subalgebra of $\mathfrak{A}$ and so f is the identity map and

$$f_+(\nabla) = \nabla \cap B, \text{ for every } \nabla \in W_{\mathfrak{A}}.$$

Clearly, if ∇ is a prime filter in $\mathfrak{A}$ then $f_+(\nabla)$ is a prime filter in $\mathfrak{B}$. So by Lemma 8.70 in [5], f_+ is a map from $W_{\mathfrak{A}}$ onto $W_{\mathfrak{B}}$. Next, we show f_+ satisfies the three conditions of a p-morphism.

(P1) Suppose $\nabla_1 R_{\mathfrak{A}} \nabla_2$, for some $\nabla_1, \nabla_2 \in W_{\mathfrak{A}}$. This means $\nabla_1 \subseteq \nabla_2$, hence $\nabla_1 \cap B \subseteq \nabla_2 \cap B$, i.e., $f_+(\nabla_1) R_{\mathfrak{B}} f_+(\nabla_2)$.

(P2) Suppose $f_+(\nabla_1) R_{\mathfrak{B}} f_+(\nabla_2)$, for some $\nabla_1, \nabla_2 \in W_{\mathfrak{A}}$. Consider the filter ∇_0 in $\mathfrak{A}$ generated by the set

$$\nabla_1 \cup (\nabla_2 \cap B).$$

And consider the ideal $\triangle_0$ in $\mathfrak{A}$ generated by $B - \nabla_2$. By Exercise 7.18 in [5], there is a prime filter ∇' in $\mathfrak{A}$ such that $\nabla_0 \subseteq \nabla'$ and $\triangle_0 \cap \nabla' = \emptyset$. By the definition, $\nabla_1 \subseteq \nabla'$ and $\nabla' \cap B = \nabla_2 \cap B$. These mean $\nabla_1 R_{\mathfrak{A}} \nabla'$ and $f_+(\nabla') = f_+(\nabla_2)$.

(P3) Let $X \in \mathcal{P}_{\mathfrak{B}}$, i.e., there is $b \in B$ such that $X = \{\nabla \in W_{\mathfrak{B}} : b \in \nabla\}$. Then

$$
\begin{aligned}
f_+^{-1}(X) &= \{\nabla' \in W_{\mathfrak{A}} : f_+(\nabla') \in X\} \\
&= \{\nabla' \in W_{\mathfrak{A}} : \nabla' \cap B \in X\} \\
&= \{\nabla' \in W_{\mathfrak{A}} : b \in \nabla'\}
\end{aligned}
$$

and so $f_+^{-1}(X) \in \mathcal{P}_{\mathfrak{A}}$. □

4 Descriptive Frames

In this section, we generalize Theorem 5 to the case of descriptive frames. For this purpose, we first give some definitions.

4.1 The Definition of Descriptive Frames

Definition 17. An intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is called *refined* if for any $x, y \in W$,

$$\forall X \in \mathcal{P}(x \in X \to y \in X) \Rightarrow xRy,$$

or equivalently,

$$\neg xRy \Rightarrow \exists X \in \mathcal{P}(x \in X \wedge y \notin X).$$

Definition 18. A family $\mathcal{X}$ of sets has the *finite intersection property* if every finite subfamily $\mathcal{X}' \subseteq \mathcal{X}$ has a nonempty intersection, i.e., $\bigcap \mathcal{X}' \neq \emptyset$.

Definition 19. An intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is called *compact*, if for any families $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \overline{\mathcal{P}} = \{W - X : X \in \mathcal{P}\}$ for which $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset.$$

Definition 20. An intuitionistic general frame $\mathfrak{F}$ is called *descriptive* iff it is refined and compact.

4.2 Descriptively generated subframes

When trying to define the two operations on descriptive frames, we came to recognize that to restrict all the frames under consideration to be descriptive, we cannot simply use the definitions of the intuitionistic general frame case. To see this clearly, we first give an example concerning generated subframes.

Example 21. *There exists a generated subframe of a descriptive frame which is not a descriptive frame.*

Proof. Define a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, by taking

$$W = \omega \cup \{\omega\},\ R = \geq,\ \mathcal{P} = \{R(n) : n \in W\} \cup \{\varnothing\},$$

where $R(n) = \{m \in W : nRm\} = \{0, 1, ..., n\}$ (note: $R(\omega) = W$).

Obviously $\mathcal{P}$ is closed under the two operations $\cap$ and $\cup$, and $\mathcal{P}$ is refined.

Next we verify that $\mathcal{P}$ is closed under $\supset$.

Indeed, for any $m, n \in W$, whenever $m \geq n$, we have $R(n) \subseteq R(m)$, hence

$$R(m) \supset R(n) = R(n) \in \mathcal{P};$$

whenever $m < n$, we have $R(m) \subsetneq R(n)$, hence

$$R(m) \supset R(n) = W \in \mathcal{P}.$$

Now we show that $\mathcal{P}$ is compact.

Suppose family $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, where

$$\mathcal{X} = \{R(j) : j \in J \subseteq W\} = \{\{0, 1, ..., j\} : j \in J \subseteq W\} \subseteq \mathcal{P},$$

$$\mathcal{Y} = \{W - R(i) : i \in I \subseteq W\} = \{\{i+1, i+2, ..., \omega\} : i \in I \subseteq W\} \subseteq \overline{\mathcal{P}}.$$

By the well-orderedness of W, there exists a least member $k \in J$, thus $\{1, ..., k\}$ is the least member of $\mathcal{X}$ with respect to the relation $\subseteq$, and k is in each member of $\mathcal{X}$. Since $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property,

$\{1, ..., k\}$ intersects every member $\{i+1, i+2, ..., \omega\}$ of $\mathcal{Y}$, hence k is in each member $\{i+1, i+2, ..., \omega\}$ of $\mathcal{Y}$, therefore $k \in \bigcap(\mathcal{X} \cup \mathcal{Y})$.

Now, consider the frame

$$\mathfrak{G} = \langle \omega, \geq, \mathcal{Q} \rangle, \mathcal{Q} = \{R(n) : \ n \in \omega\} \cup \{\varnothing\}.$$

Obviously ω is an upward closed subset of W and $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$. However, $\mathfrak{G}$ is not compact since the family $\{\omega - R(n) : \ n \in \omega\} = \{\{n+1, n+2, ...\} : n \in \omega\}$ has the finite intersection property, but

$$\bigcap\{\omega - R(n) : n \in \omega\} = \varnothing.$$

□

However, we do have the following result.

Proposition 22. *If an intuitionistic general frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is a generated subframe of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, then $\mathfrak{G}$ is refined.*

Proof. Suppose $\neg uSv$ for some $u, v \in V$. Since $\mathcal{P}$ is refined, there exists $U \in \mathcal{P}$ such that $u \in U$ and $v \notin U$. Let $U' = U \cap V$. Then by (S2), $U' \in \mathcal{Q}$. And we have $u \in U'$ and $v \notin U'$. □

In view of Example 21 and Proposition 22, we need to redefine (or even rename for distinguishing use) "generated subfames" of descriptive frames to guarantee the generated subframes to be compact, thus descriptive as well.

Definition 23. An intuitionistic general generated subframe $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ (i.e., $\mathfrak{G}$ satisfies (S1) and (S2)) of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is called a *descriptively generated subframe*, if it also satisfies the following condition:
(S3) $V = \bigcap\{U \in \mathcal{P} : V \subseteq U\}$.

We now show that (S3) is a necessary and sufficient condition for a generated subframe of a descriptive frame to be descriptive as well.

Proposition 24. *Suppose $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is a generated subframe of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$. If $\mathfrak{G}$ is a descriptive frame, then $\mathfrak{G}$ satisfies (S3) in Definition 23, i.e., $\mathfrak{G}$ is a descriptively generated subframe of $\mathfrak{F}$.*

Proof. Assume

$$V \neq \bigcap\{U \in \mathcal{P} : V \subseteq U\}. \tag{2}$$

(2) means $V \not\supseteq \bigcap\{U \in \mathcal{P} : V \subseteq U\}$, i.e., there exists w such that $w \notin V$ and

$$w \in \bigcap\{U \in \mathcal{P} : V \subseteq U\}. \tag{3}$$

Define

$$X = \{V - U : U \in \mathcal{P}, w \notin U\} \subseteq \overline{\mathcal{Q}}.$$

First, we show X has the finite intersection property. For all $V - U_1, ..., V - U_n \in X$, we have

$$(V - U_1) \cap ... \cap (V - U_n) = V - (U_1 \cup ... \cup U_n). \tag{4}$$

Note that $\mathcal{P}$ is closed under finite unions, so we have $(U_1 \cup ... \cup U_n) \in \mathcal{P}$. Thus, since $w \notin U_1 \cup ... \cup U_n$, by (3), we get $V \not\subseteq U_1 \cup ... \cup U_n$. It follows that (4) $\neq \emptyset$, hence $\bigcap X \neq \emptyset$ since $\mathfrak{G}$ is compact. Thus there exists $u \in V$ such that $u \in \bigcap X$. From V being upward closed and $w \notin V$, we derive

$\neg uRw$. Since $\mathcal{P}$ is refined, there exists $U \in \mathcal{P}$ such that $u \in U$ and $w \notin U$. By the definition of X, it follows that $V - U \in X$. Thus, by $u \in \bigcap X$, we conclude $u \in V - U$, which contradicts the fact $u \in U$. $\square$

Proposition 25. *If $\mathfrak{G} = \langle V, R, \mathcal{Q} \rangle$ is a descriptively generated subframe of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, then $\mathfrak{G}$ is also a descriptive frame.*

Proof. By Proposition 22, $\mathfrak{G}$ is refined. It remains to show it is compact.

For any families $\mathcal{X} \subseteq \mathcal{Q}$ and $\mathcal{Y} \subseteq \overline{\mathcal{Q}} = \{V - U' : U' \in \mathcal{Q}\}$, suppose $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

Define

$$\mathcal{X}_1^* = \{U \in \mathcal{P} : U \cap V \in \mathcal{X}\},$$
$$\mathcal{X}_2^* = \{U \in \mathcal{P} : V \subseteq U\},$$
$$\mathcal{X}^* = \mathcal{X}_1^* \cup \mathcal{X}_2^* \subseteq \mathcal{P},$$

and

$$\mathcal{Y}^* = \{W - U' : U' \in \mathcal{P}, V - (U' \cap V) \in \mathcal{Y}\} \subseteq \overline{\mathcal{P}}.$$

Take any

$$X_1^*, ..., X_n^* \in \mathcal{X}_1^*, X_{n+1}^*, ..., X_{n+m}^* \in \mathcal{X}_2^*, \text{ and } Y_1^*, ..., Y_k^* \in \mathcal{Y}^*.$$

We know that there exist $X_1, ..., X_n \in \mathcal{X}$, and $Y_1, ..., Y_k \in \mathcal{Y}$ such that

$$X_i = X_i^* \cap V (1 \leq i \leq n) \text{ and } Y_j = Y_j^* \cap V (1 \leq j \leq k).$$

Note that by (S3) we have $V = \bigcap\{U \in \mathcal{P} : V \subseteq U\} = \bigcap \mathcal{X}_2^*$. Observe that

$$\begin{aligned}
\bigcap_{i=1}^{n} X_i \cap \bigcap_{j=1}^{k} Y_j &= \bigcap_{i=1}^{n} X_i^* \cap V \cap \bigcap_{j=1}^{k} Y_j^* \cap V \\
&= \bigcap_{i=1}^{n} X_i^* \cap \bigcap_{j=1}^{k} Y_j^* \cap \bigcap \mathcal{X}_2^* \\
&\subseteq \bigcap_{i=1}^{n} X_i^* \cap \bigcap_{j=1}^{k} Y_j^* \cap \bigcap_{l=n+1}^{n+m} X_l^*
\end{aligned}$$

So, the fact that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property implies that $\mathcal{X}^* \cup \mathcal{Y}^*$ has the finite intersection property.

Similarly, we also have

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) = \bigcap \mathcal{X} \cap \bigcap \mathcal{Y} = \bigcap \mathcal{X}_1^* \cap \bigcap \mathcal{Y}^* \cap \bigcap \mathcal{X}_2^* = \bigcap(\mathcal{X}^* \cup \mathcal{Y}^*).$$

Thus, by the compactness of $\mathcal{P}$, it holds that $\bigcap(\mathcal{X}^* \cup \mathcal{Y}^*) \neq \emptyset$, from which it follows that $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$. $\square$

4.3 Descriptively p-morphic Images

Next, we generalize the definition of p-morphisms between intuitionistic general frames to the descriptive frame case.

To this end, we first prove some propositions to analyze the properties of p-morphisms between intuitionistic general frames. The next proposition shows that p-morphisms preserve compactness.

Proposition 26. *If $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is a p-morphic image of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ via f, then $\mathfrak{G}$ is compact.*

Proof. For any families $\mathcal{X} \subseteq \mathcal{Q}$ and $\mathcal{Y} \subseteq \overline{\mathcal{Q}} = \{V - Y : Y \in \mathcal{Q}\}$ such that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, define

$$\mathcal{X}^* = \{f^{-1}(X) : X \in \mathcal{X}\} \subseteq \mathcal{P},$$

$$\mathcal{Y}^* = \{W - f^{-1}(Y) : V - Y \in \mathcal{Y}\} \subseteq \overline{\mathcal{P}}.$$

Clearly $\bigcap(\mathcal{X}^* \cup \mathcal{Y}^*) = f^{-1}(\bigcap(\mathcal{X} \cup \mathcal{Y}))$. In order to show $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$, it suffices to show $\bigcap(\mathcal{X}^* \cup \mathcal{Y}^*) \neq \emptyset$. Indeed, for any

$$f^{-1}(X_i) \in \mathcal{X}^*, i = 1, \cdots, n; W - f^{-1}(Y_j) \in \mathcal{Y}^*, j = 1, \cdots, m,$$

since $(\bigcap X_i) \cap (\bigcap(V - Y_j)) \neq \emptyset$ and

$$(\bigcap f^{-1}(X_i)) \cap (\bigcap(W - f^{-1}(Y_j))) = f^{-1}((\bigcap X_i) \cap (\bigcap(V - Y_j))),$$

we have $\bigcap f^{-1}(X_i) \cap (\bigcap W - f^{-1}(Y_j) \neq \emptyset$, which means that $\mathcal{X}^* \cup \mathcal{Y}^*$ has the finite intersection property. From the compactness of $\mathfrak{F}$, it follows that $\bigcap(\mathcal{X}^* \cup \mathcal{Y}^*) \neq \emptyset$. □

We know that the kernel E_f of every surjective p-morphism f, defined by

$$wE_f u \text{ iff } f(w) = f(u),$$

is an equivalence relation. Moreover, the kernel gives rise to a special kind of equivalence relation defined as follows.

Definition 27. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a descriptive frame. An equivalence relation E on W is called a *bisimulation equivalence* on $\mathfrak{F}$ if the following two conditions are satisfied:
(B1) For every $w, u, v \in W$, wEv and vRu imply that there is $z \in W$ such that wRz and zEu;
(B2) If there is no x such that uRx and xEv, then there exists $U \in \mathcal{P}$ such that $E(U) = U$, $u \in U$ and $v \notin U$.
(B3) If wRx, xEu, uRy and yEw, then wEu.

With the bisimulation equivalence, we can define quotient frames of intuitionistic general frames.

Definition 28. Define the *quotient frame* $\mathfrak{F}/E = \langle W_E, R_E, \mathcal{P}_E \rangle$ of an intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ associated with a bisimulation equivalence relation E by taking

$$W_E = \{E(w) : w \in W\}, \text{ where } E(w) = \{u \in W : wEu\},$$

$$E(w)R_E E(u) \text{ iff } w'Ru' \text{ for some } w' \in E(w) \text{ and } u' \in E(u),$$

and

$$\mathcal{P}_E = \{U_E : U \in \mathcal{P}, E(U) = U\},$$

where

$$U_E = \{E(w) : w \in U\}$$

and

$$E(U) = \bigcup\{E(w) : w \in U\} = \{x : wEx, \text{ for some } w \in U\}.$$

Note 29. *In the above definition,*

(i) *R_E is really a partial order;*

(ii) *the set $\mathcal{P}_E$ is really a set of admissible sets.*

Proof. (i) The reflexivity and the antisymmetry of R_E follow immediately from the reflexivity of R and (B3) respectively.

Suppose $E(w)R_EE(u)$ and $E(u)R_EE(v)$. Then by the definition, there exist $w' \in E(w)$, $u', u'' \in E(u)$ and $v' \in E(v)$, such that $w'Ru'$ and $u''Rv'$. From $u'Eu''$ and $u''Rv'$, by (B1), it follows that there exists $z \in W$ such that $u'Rz$ and zEv'. Then since $w'Ru'$, by the transitivity of R, $w'Rz$, which means $E(w)R_EE(v)$.

(ii) First we show $\mathcal{P}_E$ is a family of upward closed sets. For any $E(w) \in U_E \in \mathcal{P}_E$ and $E(w)R_EE(u)$, there exist $w' \in E(w)$, $u' \in E(u)$ such that $w'Ru'$. Since

$$E(w) \subseteq E(U) = U,$$

we have $w' \in U$, hence $u' \in U$ since U is upward closed. It then follows that

$$E(u) = E(u') \in U_E.$$

So $\mathcal{P}_E$ is upward closed.

Since $\emptyset \in \mathcal{P}$ and $E(\emptyset) = \emptyset$, $\emptyset = \emptyset_E \in \mathcal{P}_E$.

In order to show that $\mathcal{P}_E$ is closed under operations, we first show that for any $X_E, Y_E \in \mathcal{P}_E$, the following hold:

$$\begin{aligned} E(X \cup Y) &= E(X) \cup E(Y); \\ E(X \cap Y) &= E(X) \cap E(Y); \\ E(X \supset Y) &= E(X) \supset E(Y). \end{aligned} \tag{5}$$

Let $f : W \to W_E$ be the natural map defined by

$$f(w) = E(w).$$

Observe that for any $U \subseteq W$,

$$E(U) = U \Leftrightarrow U = f^{-1}(f(U)). \tag{6}$$

Hence, we have

$$X \cup Y = f^{-1}(f(X)) \cup f^{-1}(f(Y)) = f^{-1}(f(X) \cup f(Y)) \tag{7}$$

and

$$X \cap Y = f^{-1}(f(X)) \cap f^{-1}(f(Y) = f^{-1}(f(X) \cap f(Y)). \tag{8}$$

It follows that

$$f(X \cup Y) = f(X) \cup f(Y) \text{ and } f(X \cap Y) = f(X) \cap f(Y). \tag{9}$$

From (7)-(9) we obtain

$$X \cup Y = f^{-1}(f(X \cup Y)) \text{ and } X \cup Y = f^{-1}(f(X \cup Y)).$$

So, by (6), the above gives

$$E(X \cup Y) = X \cup Y = E(X) \cup E(Y)$$

and

$$E(X \cap Y) = X \cap Y = E(X) \cap E(Y).$$

Next we show

$$X \supset Y = f^{-1}(f(X) \supset f(Y)), \tag{10}$$

i.e.,

$$x \notin X \supset Y \Leftrightarrow f(x) \notin f(X) \supset f(Y).$$

For any $U_E \in \mathcal{P}_E$, since $E(U) = U$, by (6), we have

$$w \in U \Leftrightarrow f(w) \in f(U). \tag{11}$$

Suppose $x \notin X \supset Y$. Then

$$\exists y \in W(xRy \wedge y \in X \wedge y \notin Y).$$

By the definition of R_E and (11), we obtain

$$\exists f(y) \in f(W)(f(x)R_E f(y) \wedge f(y) \in f(X) \wedge f(y) \notin f(Y)),$$

which means $f(x) \notin f(X) \supset f(Y)$.

Conversely, suppose $f(x) \notin f(X) \supset f(Y)$. Then,

$$\exists f(y) \in f(W)(f(x)Sf(y) \wedge f(y) \in f(X) \wedge f(y) \notin f(Y)).$$

By (B1) and (11), we obtain

$$\exists y' \in W(xRy' \wedge y' \in X \wedge y' \notin Y),$$

which means $x \notin X \supset Y$.

Thus, (10) is obtained. It follows that

$$f(X \supset Y) = f(X) \supset f(Y). \tag{12}$$

By (10) and (12) we obtain

$$X \supset Y = f^{-1}(f(X \supset Y)),$$

which by (6) means

$$E(X \supset Y) = X \supset Y = E(X) \supset E(Y).$$

Now, suppose $U_E, U'_E \in \mathcal{P}_E$. Then $U, U' \in \mathcal{P}$, $E(U) = U$ and $E(U') = U'$. Since $\mathcal{P}$ is closed under $\cap$, $\cup$ and $\supset$, we have $U \cap U', U \cup U', U \supset U' \in \mathcal{P}$. Together with (5), we obtain $U_E \cap U'_E,\ U_E \cup U'_E,\ U_E \supset U'_E \in \mathcal{P}_E$. Thus, $\mathcal{P}_E$ is closed under operations. This completes the proof. □

Now consider the next example.

Example 30. *There exists a p-morphic image of a descripitive frame which is not a descripitive frame.*

Proof. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be any descriptive frame such that $R \neq W \times W$. Define $\mathfrak{G} = \langle W, R, \mathcal{Q} \rangle$ by taking $\mathcal{Q} = \{\emptyset, W\}$. Clearly, $\mathfrak{G}$ is an intuitionistic general frame and a p-morphic image of $\mathfrak{F}$ via the identity map. However, $\mathfrak{G}$ is not a descriptive frame since it is not refined. □

In view of this example, we then define the descriptively p-morphic image which is essentially different from the p-morphic image induced by Definition 14, in the sense that (P4) and (P5) are necessary and sufficient conditions for a p-morphic image of a descriptive frame to be descriptive.

Definition 31. An intuitionistic general frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is called a *descriptively p-morphic image* of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ via a map f if f satisfies (P1)-(P3) and the following conditions:
(P4) the kernel E_f satisfies (B2), i.e., if $\neg f(u)Sf(v)$, then there exists $Y \in \mathcal{P}$ such that $E_f(Y) = Y$, $u \in Y$ and $v \notin Y$;
(P5) if $Y \in \mathcal{P}$ and $E_f(Y) = Y$, then $f(Y) \in \mathcal{Q}$.

Remark 32. Since $E_f(Y) = Y \Leftrightarrow Y = f^{-1}(f(Y))$, the above (P4) and (P5) are equivalent to the following (P4') and (P5') respectively.
(P4') If $\neg f(u)Sf(v)$, then there exists $Y \in \mathcal{P}$ such that $Y = f^{-1}(f(Y))$, $u \in Y$ and $v \notin Y$.
(P5') If $Y \in \mathcal{P}$ and $Y = f^{-1}(f(Y))$, then $f(Y) \in \mathcal{Q}$

Next, we justify that (P4) and (P5) are sufficient conditions.

Proposition 33. *If $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is a descriptively p-morphic image of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ via f, then $\mathfrak{G}$ is also a descriptive frame.*

Proof. By Proposition 26, $\mathfrak{G}$ is compact. It remains to show that $\mathfrak{G}$ is refined. Suppose $\neg w'Su'$ for some $w', u' \in V$. Since f is surjective, there exist $w, u \in W$ such that $f(w) = w'$ and $f(u) = u'$. So we have $\neg f(w)Sf(u)$. Thus, by (P4), there exists $U \in \mathcal{P}$ such that $E_f(U) = U$, $w \in U$ and $u \notin U$. By (P5), $f(U) \in \mathcal{Q}$, and so $w' = f(w) \in f(U)$ and $u' = f(u) \notin f(U)$. ☐

To show that (P4) and (P5) are necessary conditions, we first prove the following lemma.

Lemma 34. *Suppose $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{F}' = \langle W, R, \mathcal{Q}\rangle$ are general frames. If $\mathfrak{F}$ is descriptive and $\mathcal{Q} \subsetneq \mathcal{P}$, then $\mathfrak{F}'$ is not refined, hence is not descriptive.*

Proof. Suppose $V \in \mathcal{P}$ and $V \notin \mathcal{Q}$. Consider the sets

$$\mathcal{X} = \{U \in \mathcal{Q} : V \subsetneq U\},$$

$$\mathcal{Y} = \{\overline{U'} \in \overline{\mathcal{Q}} : \overline{V} \subsetneq \overline{U'}\},$$

where

$$\overline{\mathcal{Q}} = \{W - U : U \in \mathcal{Q}\}.$$

Since $\overline{V} \subseteq \bigcap \mathcal{Y}$ and $\mathcal{X}$ is closed under finite intersection, $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, hence $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$ by the compactness of $\mathfrak{F}$.

Case 1. $V \cap (\bigcap(\mathcal{X} \cup \mathcal{Y})) \neq \emptyset$.

Let $w \in V \cap (\bigcap(\mathcal{X} \cup \mathcal{Y}))$. To show $\mathfrak{F}'$ is not refined, we show that there exists a $u \in \overline{V}$, hence $\neg wRu$ and

$$\forall U \in \mathcal{Q}(w \in U \Rightarrow u \in U).$$

That is to show that

$$(u \in) \bigcap(\mathcal{Z} \cup \{\overline{V}\}) \neq \emptyset, \tag{13}$$

where $\mathcal{Z} = \{U \in \mathcal{Q} : w \in U\}$.

Claim 1: each $U \in \mathcal{Z}$ intersects $\overline{V}$, and $\mathcal{Z}$ is closed under finite intersection.

Suppose there exists $U \in \mathcal{Z}$ such that $U \cap \overline{V} = \emptyset$. Then $\overline{V} \subsetneq \overline{U}$, so $\overline{U} \in \mathcal{Y}$. From $w \in \bigcap(\mathcal{X} \cup \mathcal{Y})$ it follows that $w \in \overline{U}$, which contradicts $w \in U$. Since $\mathcal{Q}$ is closed under finite intersection, $\mathcal{Z}$ is closed under finite intersection.

From claim 1, it follows immediately that $\mathcal{Z} \cup \{\overline{V}\}$ has the finite intersection property. Since $\mathcal{Z} \cup \{\overline{V}\} \subseteq \mathcal{P} \cup \overline{\mathcal{P}}$, by the compactness of $\mathfrak{F}$, (13) is obtained.

Case 2. $V \cap (\bigcap(\mathcal{X} \cup \mathcal{Y})) = \emptyset$.

Let $w' \in \bigcap(\mathcal{X} \cup \mathcal{Y})$. Then $w' \notin V$. To show that $\mathfrak{F}'$ is not refined, we show that there exists a $u' \in V$, hence $\neg u'Rw'$, and

$$\forall U' \in \mathcal{Q}(u' \in U' \Rightarrow w' \in U'),$$

which is equivalent to

$$\forall \overline{U'} \in \overline{\mathcal{Q}}(w' \in \overline{U'} \Rightarrow u' \in \overline{U'}).$$

That is to show that

$$(u' \in) \bigcap(\overline{\mathcal{Z}} \cup \{V\}) \neq \emptyset, \tag{14}$$

where $\overline{\mathcal{Z}} = \{\overline{U'} \in \overline{\mathcal{Q}} : w' \in \overline{U'}\}$.

Claim 2: each $\overline{U'} \in \overline{\mathcal{Z}}$ intersects V, and $\overline{\mathcal{Z}}$ is closed under finite intersection.

Suppose there exists some $\overline{U'} \in \overline{\mathcal{Z}}$ such that $\overline{U'} \cap V = \emptyset$. Then $V \subsetneq U'$, so $U' \in \mathcal{X}$. From $w' \in \bigcap(\mathcal{X} \cup \mathcal{Y})$ it follows that $w' \in U'$, which contradicts $w' \in \overline{U'}$. Since $\mathcal{Q}$ is closed under finite union, $\overline{\mathcal{Z}}$ is closed under finite intersection.

From claim 2, it follows immediately that $\overline{\mathcal{Z}} \cup \{V\}$ has the finite intersection property. Since $\overline{\mathcal{Z}} \cup \{V\} \subseteq \overline{\mathcal{P}} \cup \mathcal{P}$, by the compactness of $\mathfrak{F}$, (14) is obtained. □

Intuitively, the above lemma means that there are actually not so many descriptive frames. However, with the same domain and relation, if the set of admissible sets are incomparable, two intuitionistic general frames can still both be descriptive.

Example 35. *There exist two descriptive frames $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{F} = \langle W, R, \mathcal{Q} \rangle$ with $\mathcal{P}$ and $\mathcal{Q}$ incomparable.*

Proof. Define

$$W = \{0, 1, 2, ..., \omega\}, \; R = \emptyset,$$

$\mathcal{P} = \{U \subseteq W : U$ is finite and does not contain ω, or cofinite in ω and contains $\omega\}$,

$\mathcal{Q} = \{U' \subseteq W : U'$ is finite and does not contain ω or 0, or cofinite in $\{2, 4, ...\}$ and contains 0, or cofinite in $\{1, 3, 5, ...\}$ and contains $\omega\}$. □

The next proposition shows that (P4) and (P5) are necessary conditions.

Proposition 36. *If a descriptive frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is p-morphic image of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ via f, then $\mathfrak{G}$ must be a descriptively p-morphic image of $\mathfrak{F}$.*

Proof. Take

$$\mathcal{Q}' = \{f(U) : U \in \mathcal{P}, E_f(U) = U\}.$$

By Note 29, $\mathcal{Q}'$ is a set of admissible sets.

Claim: $\mathfrak{G}' = \langle V, S, \mathcal{Q}' \rangle$ is descriptive.

Clearly, f is also a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}'$. Then, by Proposition 26, $\mathfrak{G}'$ is compact. Since $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is refined, to show $\mathfrak{G}' = \langle V, S, \mathcal{Q}' \rangle$ is refined, it suffices to show $\mathcal{Q} \subseteq \mathcal{Q}'$.

Actually, for any $U \in \mathcal{Q}$, it holds that

$$U = f(f^{-1}(U)), \; f^{-1}(U) \in \mathcal{P} \text{ and } E_f(f^{-1}(U)) = f^{-1}(U).$$

Thus, $U \in \mathcal{Q}'$, i.e., $\mathcal{Q} \subseteq \mathcal{Q}'$.

Then, since both $\mathfrak{G}' = \langle V, S, \mathcal{Q}' \rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ are descriptive, by Lemma 34, we are forced to conclude $\mathcal{Q}' = \mathcal{Q}$.

In view of the definition of $\mathcal{Q}'$, obviously f satisfies (P5). Now we show that f satisfies (P4) as well.

Suppose $\neg f(u) S f(v)$. Then by the refinedness of $\mathfrak{G}'$, there exists $f(U) \in \mathcal{Q}'$, such that $f(u) \in f(U)$ and $f(v) \notin f(U)$. It follows that

$$U \in \mathcal{P},\ E(U) = U,\ u \in U \text{ and } v \notin U.$$

Hence, $\mathfrak{G} = \mathfrak{G}'$ is a descriptively p-morphic image of $\mathfrak{F}$. □

So far, we have proved that (P4) and (P5) are sufficient and necessary conditions for a p-morphic image of a descriptive frame to be descriptive. Then, with these two conditions, we can prove some other properties of p-morphic images. Let us first define the isomorphism between general frames.

Definition 37. General frames $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ are *isomorphic* (in symbols $\mathfrak{F} \cong \mathfrak{G}$), if there is an isomorphism f from $\langle W, R, \rangle$ onto $\langle V, S \rangle$ such that $X \in \mathcal{P}$ iff $f(X) \in \mathcal{Q}$.

Proposition 38. *If a descriptive frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is a p-morphic image of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ via f, then $\mathfrak{F}/E_f$ is isomorphic to $\mathfrak{G}$.*

Proof. Define a map $h : W_{E_f} \to V$ by taking

$$h(E_f(w)) = f(w).$$

First we show that h is well-defined. For any u such that $E_f(u) = E_f(w)$, according to the definition of E_f, $f(u) = f(w)$, i.e., $h(E_f(u)) = h(E_f(w))$. Next, we show that h is an isomorphism from $\mathfrak{F}/E_f$ to $\mathfrak{G}'$.

Clearly, h is surjective, since f is surjective. Assume $h(E_f(w)) = h(E_f(u))$. Then $f(w) = f(u)$, which means $E_f(w) = E_f(u)$. Thus, h is injective.

Suppose $E_f(w) R_E E_f(u)$. Then there exist $w' \in E_f(w)$ and $u' \in E_f(u)$ such that $w' R u'$. Since f is a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$, we have $f(w') S f(u')$. Then from $f(w) = f(w')$ and $f(u) = f(u')$, we obtain $f(w) S f(u)$. That is $h(E_f(w)) S h(E_f(u))$.

Suppose $h(E_f(w)) S h(E_f(u))$, i.e. $f(w) S f(u)$. By (P2), there exists $v \in W$ such that wRv and $f(u) = f(v)$. Thus, by the definition of R_E, $E(w) R_E E(u)$.

Suppose $U_{E_f} \in \mathcal{P}_{E_f}$. Then $U \in \mathcal{P}$ and $E_f(U) = U$. By Proposition 36, f satisfies (P5), hence

$$\begin{aligned} h(U_{E_f}) &= \{h(E_f(w)) : w \in U\} \\ &= \{f(w) : w \in U\} \\ &= f(U) \in \mathcal{Q}. \end{aligned}$$

Suppose $h(U_{E_f}) \in \mathcal{Q}$, i.e. $f(U) \in \mathcal{Q}$. Put $U' = f^{-1}(f(U))$. Since f is a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$, by (P3), $U' \in \mathcal{P}$. Obviously it holds that

$$U' = f^{-1}(f(U')), \text{ i.e., } E_f(U') = U',$$

hence $U'_{E_f} \in \mathcal{P}_{E_f}$. Since $f(U') = f(U)$ and h is injective, we have $U'_{E_f} = U_{E_f}$. Hence, $U_{E_f} \in \mathcal{P}_{E_f}$ as required. This completes the proof. □

Corollary 39. *Suppose $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is a descriptive frame.*

(i) *For any $\mathcal{Q} \subseteq \wp(V)$, if an intuitionistic Kripke frame $\langle V, S\rangle$ is a p-morphic image of $\langle W, R\rangle$ via f, and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is a descriptively p-morphic image of $\mathfrak{F}$, then $\mathfrak{G}$ is unique.*

(ii) *If descriptive frames $\mathfrak{G}_1 = \langle V_1, S_1, \mathcal{Q}_1\rangle$ and $\mathfrak{G}_2 = \langle V_2, S_2, \mathcal{Q}_2\rangle$ are p-morphic images of $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ via f_1 and f_2 respectively, and $E_{f_1} = E_{f_2}$, then $\mathfrak{G}_1 \cong \mathfrak{G}_2$.*

Proof. (i) Suppose $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ and $\mathfrak{G}' = \langle V, S, \mathcal{Q}'\rangle$ are both descriptive. Then by Proposition 36, $\mathcal{Q} = \mathcal{Q}' = \{f(U) : U \in \mathcal{P}, E_f(U) = U\}$. Thus, $\mathfrak{G} = \mathfrak{G}'$.

(ii) By Proposition 38, we have $\mathfrak{F}/E_{f_1} \cong \mathfrak{G}_1$ and $\mathfrak{F}/E_{f_2} \cong \mathfrak{G}_2$. Since $E_{f_1} = E_{f_2}$, $\mathfrak{G}_1 \cong \mathfrak{G}_2$. □

The above corollary actually means that the set of admissible sets of a p-morphic image of a descriptive frame is determined uniquely by the p-morphism.

We now prove that there is a one-to-one correspondence between bisimulation equivalences on $\mathfrak{F}$ and descriptively p-morphic images of $\mathfrak{F}$.

Theorem 40. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a descriptive frame. The following hold:*

(i) *If a descriptive frame $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$ via f, then the kernel E_f of f is a bisimulation equivalence relation on $\mathfrak{F}$;*

(ii) *If E is a bisimulation equivalence relation on $\mathfrak{F}$, then the natural map $f : \mathfrak{F} \to \mathfrak{F}/E$ defined by*

$$f(w) = E(w)$$

is a p-morphism, and the quotient frame $\mathfrak{F}/E$ is a descriptive frame.

Proof. (i) First, we check (B1). Suppose $wE_f v$ and vRu. Then $f(w) = f(v)$ and therefore $f(w)Sf(u)$. Since f is a p-morphism, there exits $z \in W$ such that wRz and $f(z) = f(u)$, which means $zE_f u$.

By proposition 36 f satisfies (P4), thus E_f satisfies (B2). For (B3), suppose $f(x) = f(u)$, $f(y) = f(w)$, wRx and uRy. By (P1), $f(w)Sf(x)$ and $f(u)Sf(y)$. Then since S is antisymmetric, $f(w) = f(u)$. Hence, E_f is a bisimulation equivalence on $\mathfrak{F}$.

(ii)(P1) follows immediately from the definition of R_E. To prove f satisfies (P2), suppose $E(w)R_E E(u)$. Then by the definition, there exists w' and u' such that wEw', $u'Eu$ and $w'Ru'$. So by (B1), there exists $v \in W$ such that wRv and vEu'. Since E is an equivalence relation, it follows that vEu, i.e., $f(v) = f(u)$. Finally for (P3), suppose $U_E \in \mathcal{P}_E$. Note that $\mathcal{P}_E = \{U_E : U \in \mathcal{P}, E(U) = U\}$ and $E(w) = f^{-1}(\{E(w)\})$ for any $w \in W$. Thus,

$$\begin{aligned} f^{-1}(U_E) &= \bigcup\{f^{-1}(\{E(w)\}) : E(w) \in U_E\} \\ &= \bigcup\{E(w) : w \in E(U) = U \in \mathcal{P}\} \\ &= U \in \mathcal{P}. \end{aligned}$$

This proves that f is a p-morphism.

Note that E is the kernel of f, and so f satisfies (P4). By the definition of $\mathcal{P}_E$, (P5) is satisfied. By Note 29, $\mathfrak{F}/E$ is an intuitionistic general frame. Thus, $\mathfrak{F}/E$ is actually a descriptively p-morphic image of $\mathfrak{F}$. Then, by Proposition 33, $\mathfrak{F}/E$ is descriptive. □

4.4 Duality Theorems for Descriptive Frames

The next theorem reveals a nice algebraic property of descriptive frames. It means that we can go back and forth between descriptive frames and Heyting algebras. For the proof, see Theorem 8.51 in [5].

Theorem 41. *$\mathfrak{F}$ is a descriptive frame iff $\mathfrak{F} \cong (\mathfrak{F}^+)_+$.*

Corollary 42. *For any Heyting algebra $\mathfrak{A}$, $\mathfrak{A}_+$ is a descriptive frame, i.e., $\mathfrak{A}_+ \cong ((\mathfrak{A}_+)^+)_+$.*

Proof. By Theorem 9, we have $\mathfrak{A} \cong (\mathfrak{A}_+)^+$, so $\mathfrak{A}_+ \cong ((\mathfrak{A}_+)^+)_+$. □

Theorem 41 gives the following generalized duality theorems for descriptive frames as follows.

Theorem 43. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras, and $\mathfrak{F}$ and $\mathfrak{G}$ descriptive frames. Then*

1. **(a)** *$\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$ iff $\mathfrak{A}_+$ is isomorphic to a generated subframe of $\mathfrak{B}_+$*

   (b) *$\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$ iff $\mathfrak{A}_+$ is a p-morphic image of $\mathfrak{B}_+$*

2. **(a)** *$\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{G}$ iff $\mathfrak{F}^+$ is a homomorphic image of $\mathfrak{G}^+$*

   (b) *$\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$ iff $\mathfrak{F}^+$ is isomorphic to a subalgebra of $\mathfrak{G}^+$*

Proof. It follows from Theorem 11,13,15,16 and 41. □

4.5 The Generalized Version of Theorem 5

In the last section, we generalize Theorem 5 to the descriptive frames case.

Recognizing the facts in Example 21 and 30, we state and prove the generalized descriptive version of Theorem 5 as follows:

Proposition 44. *An intuitionistic general frame $\mathfrak{G}'$ is a descriptively generated subframe of a descriptively p-morphic image of a descriptive frame $\mathfrak{F}$ iff $\mathfrak{G}'$ is a descriptively p-morphic image of a descriptively generated subframe of $\mathfrak{F}$.*

Proof. For "⇒": Suppose $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is a descriptively p-morphic image of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ via f, and $\mathfrak{G}' = \langle V', S', \mathcal{Q}'\rangle$ is a descriptively generated subframe of $\mathfrak{G}$.

Define $\mathfrak{F}' = \langle f^{-1}(V'), R \restriction f^{-1}(V'), \mathcal{P}'\rangle$, by taking

$$\mathcal{P}' = \{U \cap f^{-1}(V') : U \in \mathcal{P}\}.$$

And define the map $g = f \restriction f^{-1}(V') : f^{-1}(V') \to V'$. We prove the following:

(i) $\mathfrak{F}'$ is a descriptively generated subframe of $\mathfrak{F}$.
(ii) $\mathfrak{G}'$ is a descriptively p-morphic image of $\mathfrak{F}'$ via g.

For (i): By Theorem 5, $\langle f^{-1}(V'), R \restriction f^{-1}(V')\rangle$ is a generated subframe of $\langle W, R\rangle$. Clearly, $\mathcal{P}'$ is a set of admissible sets and $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$.

Since $\mathfrak{G}' = \langle V', S', \mathcal{Q}'\rangle$ is a descriptively generated subframe of $\mathfrak{G}$, by (S3) we have $V' = \bigcap\{U \in \mathcal{Q} : V' \subseteq U\}$. Hence by (P3) we obtain

$$f^{-1}(V') = \bigcap\{U \in \mathcal{P} : f^{-1}(V') \subseteq U\}.$$

Therefore $\mathfrak{F}'$ is a descriptively generated subframe of $\mathfrak{F}$.

For (ii): From (i), by Proposition 25, $\mathfrak{F}'$ is a descriptive frame. By Theorem 5, g is a p-morphism from $\langle f^{-1}(V'), R \restriction f^{-1}(V')\rangle$ onto $\langle V', S'\rangle$.

Next, we show that g satisfies (P3). For any $X' \in \mathcal{Q}'$, by the definition of $\mathcal{Q}'$, there exists $X \in \mathcal{Q}$ such that $X' = X \cap V'$. Thus

$$g^{-1}(X') = f^{-1}(X) \cap f^{-1}(V').$$

Since f is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$, $f^{-1}(X) \in \mathcal{P}$. Therefore, by the definition of $\mathcal{P}'$, $g^{-1}(X') \in \mathcal{P}'$ i.e., g satisfies (P3).

Then, $\mathfrak{G}'$ is a p-morphic image of $\mathfrak{F}'$. By Proposition 36, $\mathfrak{G}'$ is also a descriptively p-morphic image of $\mathfrak{F}'$.

For "$\Leftarrow$": Assume $\mathfrak{G}' = \langle V', S', \mathcal{Q}'\rangle$ is a descriptively p-morphic image of $\mathfrak{F}' = \langle W', R', \mathcal{P}'\rangle$ via g, where $\mathfrak{F}'$ is a descriptive generated subframe of $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$.

Without loss of generality, we may assume $W \cap V' = \emptyset$. First, we define $V = V' \cup (W - W')$ and $S = S' \cup S_1 \cup S_2$, where $S_1 = R \restriction (W - W')$ and

$$S_2 = \{(w_1, g(w_2)) : w_1 \in W - W', w_2 \in W' \text{ and } w_1 R w_2\}.$$

Next, we define a map $f : W \to V$ by taking

$$f(w) = \begin{cases} g(w), & \text{if } w \in W'; \\ w, & \text{if } w \in W - W'. \end{cases}$$

Finally, we define a frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ by putting

$$\mathcal{Q} = \{f(U) : U \in \mathcal{P}, E_f(U) = U\},$$

where E_f is the kernel of f.

Then by Theorem 5, $\langle V', R'\rangle$ is a generated subframe of $\langle V, R\rangle$, and f is a p-morphism from $\langle W, R\rangle$ onto $\langle V, R\rangle$.

Clearly, $\mathcal{Q}$ is a set of admissible sets. For any $f(U) \in \mathcal{Q}$, since $E_f(U) = U$, we have $f^{-1}(f(U)) = U \in \mathcal{P}$, i.e., f satisfies (P3). Hence f is a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$.

Next, we show that $\mathfrak{G}'$ satisfies (S2). Note that, since $\mathfrak{G}'$ is a descriptively p-morphic image of $\mathfrak{F}'$,

$$\mathcal{Q}' = \{g(X) : X \in \mathcal{P}', E_g(X) = X\}.$$

For any $g(X) \in \mathcal{Q}'$, since $\mathfrak{F}'$ is a descriptively generated subframe of $\mathfrak{F}$, there exists $Y \in \mathcal{P}$ such that $X = Y \cap W'$. Note that $E_f(Y - X) = Y - X$ for $f \restriction (W - W') = id$. Then we have

$$E_f(Y) = E_f(X \cup (Y - X)) = E_g(X) \cup E_f(Y - X) = X \cup (Y - X) = Y.$$

Thus, $f(Y) \in \mathcal{Q}$. Since $E_f(Y) = Y$ and $E_f(W') = W'$, we have

$$g(X) = f(X) = f(Y \cap W') = f(Y) \cap V.$$

So, $\mathfrak{G}'$ is a generated subframe of $\mathfrak{G}$.

Lastly, we show that $\mathfrak{G}$ is descriptive. Since $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$, by Proposition 26, $\mathfrak{G}$ is compact. We now show that $\mathfrak{G}$ is refined. For any $w', u' \in V$, suppose $\neg w' S u'$.

Case 1: $w', u' \in V'$. For any $w \in f^{-1}(w')$ and $u \in f^{-1}(u')$, by (P5') of $g = f \restriction V'$, there exists $U' \in \mathcal{P}'$ such that

$$f^{-1}(f(U')) = U',\ w \in U' \text{ and } u \notin U'.$$

Since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$, there exists $U \in \mathcal{P}$ such that $U' = U \cap W'$. Then

$$U = U' \cup (U - W') = f^{-1}(f(U')) \cup (U - W') = f^{-1}(f(U)).$$

Hence, by the definition, $f(U) \in \mathcal{Q}$.

Clearly, $w \in U$ which implies $w' = f(w) \in f(U)$. Since $u \in W'$, $u \notin U$, and hence $u' = f(u) \notin f(U)$ since $f^{-1}(f(U)) = U$.

Case 2: $w', u' \in V - V'$. Note that $w' = f^{-1}(w')$ and $u' = f^{-1}(u')$. By (P4'), there exists $U \in \mathcal{P}$ such that

$$f^{-1}(f(U)) = U,\ w' \in U \text{ and } u' \notin U.$$

Thus, by the definition, $f(U) \in \mathcal{Q}$. Clearly, $w' \in U$ implies $w' = f(w') \in f(U)$. Since $u' \notin U$ and $f^{-1}(f(U)) = U$, $u' = f(u') \notin f(U)$.

Case 3: $w' \in V'$ and $u' \in V - V'$. For any $w \in f^{-1}(w') \subseteq W'$ and $u' = f^{-1}(u')$, since $\mathfrak{F}'$ is a descriptively generated subframe of $\mathfrak{F}$, by (S3) there exists $U \in \mathcal{P}$ such that

$$u' \notin U \text{ and } U \supseteq W', \text{ which means } w \in U.$$

Since $U \supseteq W'$, we have

$$f^{-1}(f(U)) = f^{-1}(f(W' \cup (U - W'))) = f^{-1}(V' \cup (U - W')) = W' \cup (U - W') = U.$$

Thus, by the definition, $f(U) \in \mathcal{Q}$. Clearly, $w \in U$ implies $w' = f(w) \in f(U)$. Since $u' \notin U$ and $f^{-1}(f(U)) = U$, $u' = f(u') \notin f(U)$.

Case 4: $w' \in V - V'$ and $u' \in V'$. First note that $f^{-1}(w') = w'$. We prove this case by proving some claims.

Claim 1: There exists $U \in \mathcal{P}$, such that $w' \in U$ and $U \cap f^{-1}(u') = \emptyset$.

Proof of Claim 1. Suppose otherwise. Then, for any $U \in \mathcal{P}$ such that $w' \in U$, we have $U \cap f^{-1}(u') \neq \emptyset$. Consider the set

$$\mathcal{X} \cup \mathcal{Y} = \{X \cap W' : w' \in X, X \in \mathcal{P}\} \cup \{\overline{f^{-1}(Y)} : Y \in \mathcal{Q}', u' \notin Y\} \subseteq \mathcal{P}' \cup \overline{\mathcal{P}'}.$$

For any two elements $X_1 \cap W', X_2 \cap W' \in \mathcal{X}$, since $w' \in X_1 \cap X_2 \in \mathcal{P}$, we have

$$(X_1 \cap W') \cap (X_2 \cap W') \in \mathcal{X},$$

which implies that $\mathcal{X}$ is closed under finite intersection.

For any two sets $\overline{f^{-1}(Y_1)}, \overline{f^{-1}(Y_2)} \in \mathcal{Y}$, from $u' \notin Y_1 \in \mathcal{Q}'$ and $u' \notin Y_2 \in \mathcal{Q}'$, it follows that $u' \notin Y_1 \cup Y_2 \in \mathcal{Q}'$. Observe that

$$\begin{aligned} \overline{f^{-1}(Y_1)} \cap \overline{f^{-1}(Y_2)} &= (W' - f^{-1}(Y_1)) \cap (W' - f^{-1}(Y_2)) \\ &= W' - f^{-1}(Y_1 \cup Y_2) \\ &= \overline{f^{-1}(Y_1 \cup Y_2)}. \end{aligned}$$

This means $\mathcal{Y}$ is closed under finite intersections.

For any $X \cap W' \in \mathcal{X}$ and $\overline{f^{-1}(Y)} \in \mathcal{Y}$, since $X \cap f^{-1}(u') \neq \emptyset$ and $f^{-1}(u') \subseteq \overline{f^{-1}(Y)} \subseteq W'$, we have

$$X \cap W' \cap \overline{f^{-1}(Y)} = X \cap \overline{f^{-1}(Y)} \neq \emptyset,$$

which implies that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

Since $\mathfrak{F}'$ is compact, there exists $v \in \bigcap(\mathcal{X} \cup \mathcal{Y})$. From $v \in \bigcap \mathcal{X}$, we get

$$\forall X \in \mathcal{P}(w' \in X \to v \in X),$$

which by the refinedness of $\mathfrak{F}$ implies $w'Rv$, and so

$$f(w')Sf(v),$$

since f is a p-morphism. In the meantime, from $v \in \bigcap \mathcal{Y}$ we get

$$\forall Y \in \mathcal{Q}'(u' \notin Y \to v \notin f^{-1}(Y)),$$

i.e.

$$\forall Y \in \mathcal{Q}'(f(v) \in Y \to u' \in Y),$$

which by the refinedness of $\mathfrak{G}'$ means $f(v)S'u'$, and so

$$f(v)Su'.$$

Thus,

$$f(w')Su', \text{ i.e. } w'Su',$$

which contradicts the assumption that $\neg w'Su'$. □

Claim 2: Let U and $\mathcal{Y}$ be the two sets in Claim 1, then there exists $Y \in \mathcal{Q}'$ such that $U \cap W' \subseteq f^{-1}(Y)$ and $u' \notin Y$.

Proof of Claim 2. Suppose otherwise. Then, for any $Y \in \overline{\mathcal{Q}'}$ such that $u' \notin Y$, we have $U \cap W' \nsubseteq f^{-1}(Y) \subsetneq W'$. Hence, for any $\overline{f^{-1}(Y)} \in \mathcal{Y}$, it holds that

$$(U \cap W') \cap \overline{f^{-1}(Y)} \neq \emptyset.$$

Consider the set

$$\{U \cap W'\} \cup \mathcal{Y} \subseteq \mathcal{P}' \cup \overline{P'}.$$

Since $\mathcal{Y}$ is closed under finite intersection, $\mathcal{Y}$ has the finite intersection property. Thus $\{U \cap W'\} \cup \mathcal{Y}$ has the finite intersection property.

Hence, from $\mathfrak{F}'$ being compact, it follows that there exists $v \in \bigcap(\{U \cap W'\} \cap \mathcal{Y})$. From $v \in \bigcap \mathcal{Y}$, by a similar argument to the one in the proof of Claim 1, we can prove that $f(v)Su'$. By (P2),

$$vRu \text{ for some } u \in f^{-1}(u').$$

In the meantime, since $v \in U \cap W'$, $v \in U$. From U being upward closed, we conclude $u \in U$, which contradicts Claim 1. □

Let U and Y be the sets in Claim 2. Since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$, there exists $X \in \mathcal{P}$ such that $X \cap W' = f^{-1}(Y)$. By Claim 2, we have $U \cap W' \subseteq f^{-1}(Y) = X \cap W'$. Furthermore

$$f^{-1}(Y) = X \cap W' = (U \cup X) \cap W',$$

which leads to

$$\begin{aligned} f(U \cup X) &= f(((U \cup X) \cap W') \cup ((U \cup X) - W')) \\ &= f(f^{-1}(Y)) \cup ((U \cup X) - W')) \\ &= Y \cup ((U \cup X) - W'). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} f^{-1}(f(U \cup X)) &= f^{-1}(Y \cup ((U \cup X) - W')) \\ &= f^{-1}(Y) \cup ((U \cup X) - W') \\ &= U \cup X. \end{aligned}$$

Clearly $U \cup X \in \mathcal{P}$, hence by the definition $f(U \cup X) \in \mathcal{Q}$.

Since $w' \in U$, $w' \in U \cup X$, which implies $w' = f(w') \in f(U \cup X)$. By Claim 2, $u' \notin Y$, and clearly $u' \notin (U \cup X) - W'$, thus by (15),

$$u' \notin Y \cup (U \cup X - W') = f(U \cup X).$$

This completes the proof of Case 4. □

References

[1] G. Bezhanishvili. Varieties of Monadic Heyting Algebras. Part III. *Studia Logica,* 64(2): 215-256, 2000.

[2] N. Bezhanishvili. *Lattices of intermediate and cylindric modal logics,* PhD Thesis, University of Amsterdam, 2006.

[3] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic,* Cambridge University Press, 2001.

[4] S. Burris, and H.P. Sankappanavar. *A Course in Universal Algebra,* Springer-Verlag, 1981.

[5] A. Chagrov and M. Zakharyaschev. *Modal Logic,* Clarendon Press, 1997.

[6] T. Greene. Frame-based Completeness of Intermediate Logics, MoL project in ILLC Technical Notes X.2006-03, University of Amsterdam, January 2006.