Bitopological Vietoris spaces and positive modal logic*

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Abstract

Using the isomorphism from [3] between the category **Pries** of Priestley spaces and the category **PStone** of pairwise Stone spaces we construct a Vietoris hyperspace functor on the category **PStone** related to the Vietoris hyperspace functor on the category **Pries** [4, 18, 24]. We show that the coalgebras for this functor determine a semantics for the positive fragment of modal logic. The sole novelty of the present work is the explicit construction of the Vietoris hyperspace functor on the category **PStone** as well as the phrasing of otherwise well-known results in a bitopological language.

*This constitutes the final report for a individual project under the supervision of Nick Bezhanishvili at the Institute of Logic, Language and Computation.

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1 Introduction

A celebrated theorem by Marshall Stone [22] from 1936 states that the category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of *Stone spaces*, i.e. zero-dimensional compact Hausdorff spaces, and continuous maps[1]. We refer the reader to [13] for an introduction to the theory of Stone duality.

From 1936 onwards a plethora of *Stone like dualities* between algebras and topologies was discovered. Already in 1937 Stone [23] showed that the category **bDist** of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category **Spec** of spectral spaces and spectral maps.

In 1970 Priestley [19] showed that the category **bDist** and the category **Pries** of *Priestley spaces*, i.e. compact and totally order-disconnected spaces are dually equivalent, and in 1975 Cornish [7] showed that the category **Pries** and the category **Spec** are not only equivalent but in fact isomorphic. Finally, in 1994 Picado [17] showed that the category **Pries** is also isomorphic to the category **PStone** of pairwise Stone spaces and bi-continuous maps. We refer the reader to [3] for a good and coherent exposition of all the different equivalences and isomorphisms between the above categories.

Abramsky [1] showed that not only can the well-known Stone duality be expressed in coalgebraic terms but also the Jónsson-Tarski duality between descriptive general Kripke frames and modal algebras can be given a coalgebraic formulation[2], see also [15] for more details. More precisely the category of descriptive general frames is isomorphic to the category of coalgebras for the Vietoris hyperspace functor on the category **Stone**. This connection was already noted by Esakia [9]. Thus we can represent a sound and complete semantics for modal logic by coalgebras for the Vietoris functor on **Stone**. In 2004 Palmigiano [18] obtained a similar result for the positive fragment of basic modal logic in terms of coalgebras of a Vietoris hyperspace functor on the category **Pries**. See also [4, 24].

Lastly we mention that in Johnstone's work [14] from 1985 on Vietoris locales we already find a construction of a Vietoris hyperspace functor on **Spec**. This leaves only the bitopological approach to positive modal logic untouched.

The aim of this report is to combine the idea that the semantics of positive modal logic can be seen as coalgebras for the hyperspace functor on **Pries** with the fact that the category **Pries** is isomorphic to the category **PStone**, to obtain a semantics for positive modal logic in terms of coalgebras for an endofunctor on

[1] Of course Stone did not formulate his result in terms of an equivalence of categories, for the simple reason that category theory was first introduced by Eilenberg and Mac Lane in the mid Nineteen-Forties.

[2] Abramsky had already presented this work at several occasions during the period 1988-1989.

PStone. Although the construction of a hyperspace functor on the category of pairwise Stone spaces does not exist in the literature the report at hand does not as such contain any new results, but only provides a bitopological perspective to the already known fact that the relational semantics for positive modal logic can be obtained as coalgebras for the Vietoris hyperspace functor on a category dually equivalent to **bDist**.

The structure of this report is as follows: In section 2 we recall the definitions of the categories **PStone** and **Pries** as well as the isomorphism between them. In section 3 we construct a *Vietoris hyperspace functor* on the category **Pries**, which is very similar to the one found in [18, 4, 24]. Section 4 contains the construction of the Vietoris hyperspace functor on the category **PStone**, and we show that this functor is the analogue of the hyperspace functor on the category **Pries**. In section 5 we introduce positive modal logic and its various semantics and in section 6 we show that the coalgebras for the hyperspace functor on **PStone**, encode a sound and complete semantics for the minimal positive modal logic $\mathbf{K}^+$, which is well behaved with respect to axiomatic extensions. Finally section 7 contains a proof that the hyperspace functor on **Pries** restricts to a functor on the category **Esa** of Esakia spaces, as well as a short discussion on intuitionistic modal logic.

We assume the readers familiarity with basic notions from topology and category theory. Finally to appreciate the connection to modal logic it might be helpful if the reader is familiar with the Jónsson-Tarski duality for which we refer the reader to chapter 5 of [2].

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2 Pairwise Stone spaces and Priestley spaces

Definition 1. A *Priestley space* is triple $(X, \tau, \leq)$ such that (X, τ) is a Stone space and $\leq$ is a partial order of X, i.e. reflexive, transitive and anti-symmetric, satisfying the *Priestley Separation Axiom:*

$x \nleq y$ implies that there exists a clopen upset U, such that $x \in U$ and $y \notin U$.

If $(X, \leq)$ is a partial ordered space and $U \subseteq X$ we define

$$\uparrow U = \{x \in X : \exists u \in U \ u \leq x\} \quad \downarrow U = \{x \in X : \exists u \in U \ x \leq u\}.$$

If U a subset of X we call $\downarrow U$ the *downset of* U and $\uparrow U$ the *upset of* U. Furthermore we say that U is a *downset* if $\downarrow U = U$ and an *upset* if $\uparrow U = U$.

Proposition 1. *Let $(X, \tau, \leq)$ be a Priestley space. If $F \subseteq X$ is closed then $\uparrow F$ and $\downarrow F$ are both closed, i.e. upsets and downsets of closed sets are closed.*

Proof. Let $F \subseteq X$ be closed and suppose that $x \notin \uparrow F$. Then for all $y \in F$ it must be the case that $y \nleq x$ and so by the Priestley separation axiom we have for all $y \in F$ a clopen upset U_y satisfying:

$$y \in U_y \quad \text{and} \quad x \notin U_y.$$

It follows that $\bigcup_{y \in F} U_y$ is an open cover of F. Now since (X, τ) is compact and F is closed, F is compact and we must therefore have a finite subcover $\bigcup_{i=1}^{n} U_{y_i}$ of F. So letting $U := \bigcap_{i=1}^{n} U_{y_i}^{\mathrm{c}}$ we get a clopen downset satisfying $x \in U$ and $U \subseteq F^{\mathrm{c}}$. Moreover if we have $x' \in U$ and $y \in F$ such that $y \leq x'$, then since U is a downset $y \in U$, which contradicts the fact that $U \subseteq F^{\mathrm{c}}$. It follows that $U \subseteq (\uparrow F)^{\mathrm{c}}$, hence U is an open neighbourhood of x contained in $(\uparrow F)^{\mathrm{c}}$. So as $x \notin \uparrow F$ was arbitrary we conclude that $(\uparrow F)^{\mathrm{c}}$ is open and hence that $\uparrow F$ is closed.

The argument to the fact that $\downarrow F$ is closed is completely similar and is therefore omitted. ■

The following theorem due to Priestley explains why we are interested in Priestley spaces.

Theorem 1 ([19])**.** *The category* **Pries** *of Priestley spaces and continuous order-preserving functions is dually equivalent to the category* **bDist** *of bounded distributive lattices and bounded lattice homomorphisms.*

By a *bitopological space*[3] we understand a triple (X, τ_1, τ_2) such that (X, τ_1) and (X, τ_2) are topological spaces. Given a bitopological (X, τ_1, τ_2) we denote by δ_1 and δ_2 the collections of τ_1 and τ_2 closed sets, respectively. Finally we let $\beta_1 := \tau_1 \cap \delta_2$ and $\beta_2 := \tau_2 \cap \delta_1$.

Definition 2 ([21])**.** Let (X, τ_1, τ_2) be a bitopological space.

i) We say that (X, τ_1, τ_2) is *pairwise Hausdorff* if for every $x, y \in X$ with $x \neq y$ we have disjoint sets $U \in \tau_1$ and $V \in \tau_2$ separating x and y.

ii) We say that (X, τ_1, τ_2) is *pairwise zero-dimensional* if β_1 constitutes a basis for τ_1 and β_2 constitutes a basis for τ_2.

iii) We say that (X, τ_1, τ_2) is *pairwise compact* if $(X, \tau_1 \vee \tau_2)$ is compact, where $\tau_1 \vee \tau_2$ is the smallest topology containing both τ_1 and τ_2.

[3] Bitopological spaces were first introduced by Kelly in [11], although his motivation for studying these space was very different from ours.

Note that by The Alexander Subbasis Lemma 8, definition 2iii) is equivalent to the conditions that any cover of X consisting of elements of $\tau_1 \cup \tau_2$ has a finite subcover.

We call a bitopological space which is pairwise Hausdorff, pairwise zero-dimensional and pairwise compact a *pairwise Stone space*. Moreover we let **PStone** denote the category of pairwise Stone spaces and bi-continuous maps between them, where a *bi-continuous* map $f\colon (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ between bitopological spaces is a map which is continuous as a map from (X, τ_i) to (X', τ_i'), for $i \in \{1, 2\}$.

The following three proposition establish the connection between Priestley spaces and pairwise Stone spaces.

Proposition 2 (Lem. 3.3, [3])**.** *Let (X, τ_1, τ_2) be a pairwise zero-dimensional space and let $\leq_i$ be the specialization preorder of (X, τ_i), $i \in \{1, 2\}$. Then $\leq_1 = \geq_2$.*

Proposition 3 (Prop. 3.4, [3])**.** *If (X, τ_1, τ_2) is a pairwise Stone space then $(X, \tau, \leq)$ is a Priestley space, where $\tau = \tau_1 \vee \tau_2$ and $\leq$ is the specialisation preorder of (X, τ_1). Furthermore with the obvious notation we have that:*

$$\beta_1 = \mathrm{ClpUp}(X, \tau, \leq) \qquad \beta_2 = \mathrm{ClpDo}(X, \tau, \leq)$$
$$\tau_1 = \mathrm{OpUp}(X, \tau, \leq) \qquad \tau_2 = \mathrm{OpDo}(X, \tau, \leq)$$
$$\delta_1 = \mathrm{ClUp}(X, \tau, \leq) \qquad \delta_2 = \mathrm{ClDo}(X, \tau, \leq)$$

Proposition 4 (Prop. 3.6, [3])**.** *If $(X, \tau, \leq)$ is a Priestley space, then (X, τ_1, τ_2) is a pairwise Stone space, where $\tau_1 = \mathrm{OpUp}(X, \tau, \leq)$ and $\tau_2 = \mathrm{OpDo}(X, \tau, \leq)$. Furthermore we have that:*

i) $\beta_1 = \mathrm{ClpUp}(X, \tau, \leq)$

ii) $\beta_2 = \mathrm{ClpDo}(X, \tau, \leq)$

iii) $\leq = \leq_1 = \geq_2$.

The above actually gives an isomorphism between categories.

Theorem 2. *The categories* **Pries** *and* **PStone** *are isomorphic*

Sketch. Given a pairwise Stone space (X, τ_1, τ_2), then one constructs a Priestley space $\Phi((X, \tau_1, \tau_2)) = (X, \tau, \leq)$ by letting $\tau = \tau_1 \vee \tau_2$ and $\leq$ be the specialization pre-order of (X, τ_1). We let Φ act as the identity on functions.

Conversely given a Priestley space $(X, \tau, \leq)$ one constructs the corresponding pairwise Stone space $\Psi(X, \tau, \leq) = (X, \tau_1, \tau_2)$ by letting $\tau_1 = \mathrm{OpUp}_\tau(X)$ and $\tau_2 = \mathrm{OpDo}_\tau(X)$, i.e. the open upsets and open downsets of $(X, \tau, \leq)$, respectively. We let Ψ act as the identity on functions.

By Propositions 3 and 4 we see that these two functors are well-defined on objects and that they are each others inverses on the object level. Thus one only needs to check that the functors are well-defined on morphisms, but this is straightforward given Propositions 3 and 4, and is therefore omitted. ∎

In the following we will often refer back to the functors $\Phi\colon \mathbf{PStone} \to \mathbf{Pries}$ and $\Psi\colon \mathbf{Pries} \to \mathbf{PStone}$.

3 Hyperspaces of Priestley space

We here briefly recall the well-known Vietoris functor $K\colon \mathbf{KHaus} \to \mathbf{KHaus}$, on the category of compact Hausdorff spaces and continuous maps between them.

If (X, τ) is a compact Hausdorff space, then we let $K(X)$ denote the set of closed sets of X. We can then define the *hit-and-miss* or *Vietoris* topology τ_v on $K(X)$ to be the topology generated by the subbasis $\{\Box U, \Diamond U\}_{U \in \tau}$, where

$$\Box U = \{F \in K(X) \colon F \subseteq U\} \quad \text{and} \quad \Diamond U = \{F \in K(X) \colon F \cap U \neq \varnothing\}.$$

The action of K on arrow is given by $K(f) = f_*$, where f_* denotes the direct image function obtained from f. Now we can think of $\Box U$ as the set of all the closed sets that miss the closed set U^{c} and the set $\Diamond U$ as the set of all the closed sets that hit the open set U.

Note that a basis for $(K(X), \tau_v)$ is given by

$$\nabla(U_1, \dots, U_n) = \{F \in K(X) \colon F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \varnothing\},$$

Where $U_1, \dots U_n$ ranges over all finite collections of open subsets of X.

One can now prove the following:

Theorem 3 ([16])**.** *If (X, τ) is a Stone space then so is $(K(X), \tau)$.*

In fact this give rise to a endofunctor on the category of Stone spaces with the following actions on morphisms:

Proposition 5. *The assignment $(X, \tau) \mapsto (K(X), \tau_v)$ determines an endofunctor $K\colon \mathbf{Stone} \to \mathbf{Stone}$ on the category of Stone spaces and continuous functions, with $K(f\colon X \to Y)$ the direct image function $f_*\colon K(X) \to K(Y)$ given by $F \mapsto f[F]$.*

Proof. Clearly $K(\mathrm{id}\colon X \to X) = \mathrm{id}_{K(X)}$ and

$$K(g\colon Y \to Z) \circ K(f\colon X \to Y) = K(g \circ f\colon X \to Z).$$

Thus we only need to show that for $f\colon (X, \tau) \to (Y, \tau')$ continuous, the assignment $F \mapsto f[F]$ determines a well-defined function from $K(X)$ to $K(Y)$. If F is a closed subset of X, then as (X, τ) is compact we have that F is compact as well and hence $f[F]$ is a compact subset of Y as f is continuous. Finally as (Y, τ') is Hausdorff we have that $f[F]$ is closed in Y i.e. $f[F] \in K(Y)$. ∎

Now given a Priestley space $(X, \tau, \leq)$ we want to find a way to lift the partial order $\leq$ to partial order $\leq_v$ on $K(X)$ such that $(K(X), \tau_v, \leq_v)$ is a Priestley space. Moreover we want such a construction to determine an endofunctor on the category of Priestley spaces. It turns out that for this to work we must modify the above construction slightly.

Definition 3. Let R be any relation on a set X. Then the *Egli-Milner lift* of R is the relation R^{EM} on $\mathscr{P}(X)$ given by

$$W R^{\mathrm{EM}} V \iff W \subseteq R^{-1}(V) \quad \text{and} \quad V \subseteq R(W),$$

where

$$R(W) = \{x \in X \colon \exists w \in W \ xRw\} \quad \text{and} \quad R^{-1}(V) = \{x \in X \colon \exists v \in V \ vRx\}.$$

Now if $\leq$ is a partial order it is easy to check that $\leq^{\mathrm{EM}}$ is reflexive and transitive. But $\leq^{\mathrm{EM}}$ will not in general be antisymmetric. However if we restrict it to the class of *convex sets* then any partial order on a set X will lift to a partial order $\leq^{\mathrm{EM}}$ on the set of convex sets of X.

Definition 4. Let X be a set with a relation R. We say that $U \subseteq X$ is *convex* if for all $y \in X$ and all $x, z \in U$

$$xRy \wedge yRz \implies y \in U.$$

Proposition 6. *Let $(X, \leq)$ be a partially ordered set and let $U \subseteq X$, then the following are equivalent:*

1. *U is a convex subset of X.*
2. *$U = \uparrow U \cap \downarrow U$.*
3. *$U = V \cap W$, for some upset V and some downset W of X.*

Proof. We first show that if U is a convex set then $\uparrow U \cap \downarrow U \subseteq U$. This suffices to establish item 2 since $U \subseteq \uparrow U \cap \downarrow U$ as $\leq$ is a partial order and as such reflexive.

If $y \in \uparrow U \cap \downarrow U$ then since $y \in \uparrow U$ we have $x \in U$ such that xRy and since $y \in \downarrow U$ we have $z \in U$ such that yRz, and so by the convexity of U we must have that $y \in U$.

Conversely if $U = \uparrow U \cap \downarrow U$ then as

$$\uparrow U \cap \downarrow U = \{y \in X \colon \exists x \exists y \ x, y \in U \ \& \ xRy \wedge yRz\}$$

we can immediately conclude that U is convex.

Evidently item 2 implies item 3. On the other hand if U is the intersection of an upset V and a downset W then if $x, z \in U$ and $y \in X$ is such that $x \leq y$ and $y \leq z$, then we must have that $y \in \uparrow V$ as $x \in V$ and $y \in \downarrow W$ as $z \in W$. Hence

$$y \in \uparrow V \cap \downarrow W = V \cap W = U.$$

∎

3.1 The construction of the functor V: Pries → Pries

Our aim in this subsection is to define an endofunctor V: **Pries** → **Pries** analogous to the Vietoris functor K: **Stone** → **Stone** of section 2. The definition of the functor V seems to be folklore. It can be found in [4, 24] but no explicit proof of the fact that this is indeed a well-defined functor is given there. The definition is also very similar to the one found in [18] but simpler due to the fact that we can prove that the set $\mathrm{Conv}(X)$ of closed and convex sets of a Priestley space is compact when topologized by the topology given below. This is not done in [18] which instead considers a quotient space construction to make the Egli-Milner lift of the order on a Priestley space a partial order on the set of (equivalence classes of) closed subsets.

If $(X,\tau,\le)$ is a Priestley space we let $\mathrm{Conv}(X)$ denote the set of closed and convex sets of $(X,\tau,\le)$. Moreover for any Priestley space $(X,\tau,\le)$ we let $V(X) = (\mathrm{Conv}(X),\tau_v,\le^{\mathrm{EM}})$ where with abuse of notation we let τ_v be the topology on $\mathrm{Conv}(X)$ generated by the subbasis $\{\Box U,\Diamond U\}_{U\in\mathrm{ClpUp}(X)\cup\mathrm{ClpDo}(X)}$ with

$$\Box U = \{F\in\mathrm{Conv}(X): F\subseteq U\} \quad\text{and}\quad \Diamond\, U = \{F\in\mathrm{Conv}(X): F\cap U\neq\varnothing\}.$$

In the remainder of this section we show that the above assignment determines an endofunctor V: **Pries** → **Pries**.

Proposition 7. *If $(X,\tau,\le)$ is a Priestley space then $(\mathrm{Conv}(X),\tau_v)$ is compact.*

Proof. To show that $(\mathrm{Conv}(X),\tau_v)$ is compact it suffices by the Alexander Subbasis Lemma to show that every cover by elements of the subbasis $\{\Box U,\Diamond U\}$ with $U\in\mathrm{ClpUp}(X)\cup\mathrm{ClpDo}(X)$ has a finite subcover. Therefore let

$$\mathrm{Conv}(X) = \bigcup_{i\in I}\Box U_i \cup \bigcup_{j\in J}\Diamond U_j \bigcup_{k\in K}\Box V_k \cup \bigcup_{l\in L}\Diamond V_l \tag{$\star$}$$

be such a cover, with U_i and U_j τ-clopen upsets for all $i\in I$ and $j\in J$, and with V_k and V_l τ-clopen downsets for all $k\in K$ and $l\in L$. We claim that

$$X = \bigcup_{i\in I}U_i \cup \bigcup_{j\in J}U_j \bigcup_{k\in K}V_k \cup \bigcup_{l\in L}V_l. \tag{‡}$$

To see this let $x\in X$ then as (X,τ) is a Stone space it is in particular T_1 whence $\{x\}$ is closed and therefore $\{x\}\in\mathrm{Conv}(X)$, as it is clearly also convex. Thus $\{x\}\in\Box W$ for some $W\in\{U_i,V_k\}_{i\in I,k\in K}$ or $\{x\}\in\Diamond W'$ for some $W'\in\{U_j,V_l\}_{j\in J,l\in L}$. In the former case we obtain that $x\in W$ and in the latter case we obtain that $x\in W'$. Hence we see that (‡) follows.

Now since $\bigcup_{j\in J}U_j\cup\bigcup_{l\in L}V_l$ is a τ-open set we have that

$$F := X - (\bigcup_{j\in J}U_j \cup \bigcup_{l\in L}V_l)$$

is a τ-closed set. Moreover as $\bigcup_{j\in J} U_j$ is an upset and $\bigcup_{l\in L} V_l$ is a downset we must have that F is the intersection of an downset and an upset and therefore convex. Hence $F \in \mathrm{Conv}(X)$. So by $(\star)$ we must have that $F \in \Box U_{i'}$ for some $i' \in I$ or $F \in \Box V_{k'}$ for some $k' \in I$. We may without loss of generality assume that the former is the case. It then follows from (‡) that $X = U_{i'} \cup \bigcup_{j\in J} U_j \cup \bigcup_{l\in L} V_l$. So by the compactness of X we have that $X = U_{i'} \cup \bigcup_{i=1}^{n} U_{j_i} \cup \bigcup_{i=1}^{m} V_{l_i}$. We claim that

$$\mathrm{Conv}(X) = \Box U_{i'} \cup \bigcup_{i=1}^{n} \Diamond U_{j_i} \cup \bigcup_{i=1}^{m} \Diamond V_{l_i}.$$

To see this let $G \in \mathrm{Conv}(X)$ then either $G \subseteq U_{i'}$ in which case $G \in \Box U_{i'}$ or $G \nsubseteq U_{i'}$ in which case we get that $G \cap (\bigcup_{j\in J} U_j \cup \bigcup_{l\in L} V_l) \neq \varnothing$ and thus either $G \cap U_{j_i} \neq \varnothing$ for some $i \leq n$, in which case $G \in \Diamond U_{j_i}$ or $G \cap V_{l_i} \neq \varnothing$ for some $i \leq m$, in which case $G \in \Diamond V_{l_i}$. ∎

Lemma 1. *If $(X, \tau, \leq)$ is a Priestley space then $V(X) = (\mathrm{Conv}(X), \tau_v)$ is zero-dimensional.*

Proof. To see that $\mathrm{Conv}(X)$ is zero-dimensional we note that

$$\Box U = (\Diamond(U^{\mathrm{c}}))^{\mathrm{c}} \quad \text{and} \quad \Diamond U = (\Box(U^{\mathrm{c}}))^{\mathrm{c}},$$

whence $\mathrm{Conv}(X)$ has a subbasis for clopen and hence a basis of clopens. ∎

Lemma 2. *If $(X, \tau, \leq)$ is a Priestley space then $V(X) = (\mathrm{Conv}(X), \tau_v)$ is Hausdorff.*

Proof. Let $F, G \in \mathrm{Conv}(X)$ with $F \neq G$. Then as $\leq^{EM}$ is a partial order on $\mathrm{Conv}(X)$ we have that $F \nleq^{\mathrm{EM}} G$ or $G \nleq^{\mathrm{EM}} F$. We may without loss of generality assume that the former is the case. Therefore we either have i) $x \in F$ such that $x \nleq y$ for all $y \in G$ or we have have ii) $y \in G$ such that $x \nleq y$ for all $x \in F$.

We consider first the case i). By the Priestley Separation Axiom we have clopen upsets U_y for all $y \in G$ such that $x \in U_y$ and $y \notin U_y$. It follows that $\bigcup_{y\in G} U_y^{\mathrm{c}}$ is an open cover of G. Now G is a closed subset of X and therefore compact, hence we have a finite subcover $\bigcup_{i=1}^{n} U_{y_i}^{\mathrm{c}}$ of G. So if we let $U = \bigcap_{i=1}^{n} U_{y_i}$ then we have that $x \in U$ and $U \cap G = \varnothing$. So as $x \in F$ we get that $F \cap U \neq \varnothing$. It follows that $F \in \Diamond U$ and $G \notin \Diamond U$. So as U is a clopen upset we have that $\Diamond U$ is open in τ_v and as moreover τ_v is zero-dimensional with subbasis $\{\Box U, \Diamond U\}_{U \in \mathrm{ClpUp}(X) \cup \mathrm{ClpDo}(X)}$ we have that $\Diamond U$ is also closed. Hence $(\Diamond U)^c = \Box U^{\mathrm{c}}$ is an open set containing F disjoint from $\Diamond U$ containing G.

In the case ii) a similar argument allows us to find clopen upset U such that $F \in \Box U$ and $G \notin \Box U$.

Hence in both cases we may conclude that $\mathrm{Conv}(X)$ is Hausdorff. ∎

Lemma 3. *If $(X, \tau, \leq)$ is a Priestley space then so is $V(X) = (\mathrm{Conv}(X), \tau_v, \leq^{EM})$.*

Proof. By Proposition 7 and Lemma 1 and 2 above it is enough to show that $\leq^{\mathrm{EM}}$ satisfies that Priestley Separation Axiom.

Therefore let $F, G \in \mathrm{Conv}(X)$ such that $F \nleq^{\mathrm{EM}} G$. Then we must find clopen upset W of $\mathrm{Conv}(X)$ such that $F \in W$ and $G \notin W$.

If $F \nleq^{\mathrm{EM}} G$ the either $F \nsubseteq \downarrow G$ or $G \nsubseteq \uparrow F$. In the former case we must have $x \in F$ such that $x \notin \downarrow G$. So for all $y \in G$ we have that $x \nleq y$. Now as $(X, \tau, \leq)$ is a Priestley space we must have clopen upsets U_y for each $y \in G$ such that $x \in U_y$ and $y \notin U_y$. It follows that $\bigcup_{y \in G} U_y^{\mathrm{c}}$ is an open cover of G. Now G being a closed subset of the compact Hausdorff space X must also be compact. Consequently we have $y_1, \ldots, y_n \in G$ such that $\bigcup_{i=1}^n U_{y_i}^{\mathrm{c}}$ covers G. Letting $U = \bigcap_{i=1}^n U_{y_i}$ we see that U is a clopen upset with $x \in U$ and $U \cap G = \varnothing$. So as $U \cap F \neq \varnothing$ we see that $F \in \Diamond U$ and $G \notin \Diamond U$.

Finally we show that $\Diamond U$ is upwards closed. Therefore let $H \in \Diamond U$ and $H \leq^{\mathrm{EM}} H'$. If $H' \notin \Diamond U$ then $H' \cap U = \varnothing$ and consequently $H' \subseteq U^{\mathrm{c}}$ and as U is an upset U^{c} is a downset. Hence we must have that $\downarrow H' \subseteq U^{\mathrm{c}}$, and as $H \leq^{\mathrm{EM}} H'$ we have that $H \subseteq \downarrow H'$ contradiction the fact that $H \in \Diamond U$. Thus $\Diamond U$ must be an upset, and we may take $W = \Diamond U$.

The argument for the latter case is similar and is therefore omitted. ∎

Let $f\colon (X, \tau, \leq) \to (X', \tau', \leq')$ be a morphism between Priestley spaces. Then we show that $F \mapsto \downarrow f[F] \cap \uparrow f[F]$ determines a mapping $V(f)\colon V(X) \to V(Y)$ making V into an endofunctor on the category **Pries**. For this the following proposition suffices.

Proposition 8. *If $f\colon (X, \tau, \leq) \to (X', \tau', \leq')$ is a morphism between Priestley space, then $F \mapsto \downarrow f[F] \cap \uparrow f[F]$ determines a continuous and order preserving function $V(f)\colon V(X) \to V(Y)$ between the Priestley spaces $V(X)$ and $V(Y)$.*

Proof. We first show that $V(f)$ is well-defined. As X is compact every closed subset of X is compact and hence every $F \in \mathrm{Conv}(X)$ will be compact. The image of a compact set under a continuous function is again compact and so as Y is Hausdorff $f[F]$ will be closed. Since in any Priestley space the upwards and downwards closure of closed set is again closed we have that $V(f)[F]$ is closed. Hence as $V(f)[F]$ is obviously convex we see that the function $V(f)\colon V(X) \to V(Y)$ is indeed well-defined.

To see that it is also order preserving assume that $F \leq^{EM} F'$, i.e.

$$F \subseteq \downarrow F' \quad \text{and} \quad F' \subseteq \uparrow F.$$

Then if $y \in V(f)[F]$ we have $x_1, x_2 \in F$ such that $f(x_1) \leq' y \leq' f(x_2)$. As $F \subseteq \downarrow F'$ we must have $x_3 \in F$ such that $x_2 \leq x_3$, consequently we have that $f(x_2) \leq f(x_3)$ so $f(x_2) \in \downarrow f[F'] \cap \uparrow f[F']$ and hence $y \in \downarrow(\downarrow f[F'] \cap \uparrow f[F'])$. Thus we have that

$$V(f)[F] \subseteq \downarrow(\downarrow f[F'] \cap \uparrow f[F']) = \downarrow(V(f)(F')).$$

The argument for the fact that $V(f)(F') \subseteq {\uparrow}V(f)(F)$ is completely similar.

Finally to see that it is continuous it suffices to show that the preimages of elements of the subbasis are open. Therefore let $U \in \tau'$ be a clopen upset then we show that $V(f)^{-1}(\Box U) = \Box f^{-1}U$ and $V(f)^{-1}(\Diamond U) = \Diamond f^{-1}U$. From which we may conclude that $V(f)$ is continuous.

$$\begin{aligned} V(f)^{-1}(\Box U) &= \{F \in \mathrm{Conv}(X) \colon V(f)(F) \in \Box U\} \\ &= \{F \in \mathrm{Conv}(X) \colon {\downarrow}f[F] \cap {\uparrow}f[F] \subseteq U\} \\ &= \{F \in \mathrm{Conv}(X) \colon f[F] \subseteq U\} && \text{(since } U \text{ is an upset)} \\ &= \{F \in \mathrm{Conv}(X) \colon F \subseteq f^{-1}(U)\} \\ &= \Box f^{-1}(U). \end{aligned}$$

$$\begin{aligned} V(f)^{-1}(\Diamond U) &= \{F \in \mathrm{Conv}(X) \colon V(f)(F) \in \Diamond U\} \\ &= \{F \in \mathrm{Conv}(X) \colon {\downarrow}f[F] \cap {\uparrow}f[F] \cap U \neq \varnothing\} \\ &= \{F \in \mathrm{Conv}(X) \colon f[F] \cap U \neq \varnothing\} && \text{(since } U \text{ is an upset)} \\ &= \{F \in \mathrm{Conv}(X) \colon F \cap f^{-1}(U) \neq \varnothing\} \\ &= \Diamond f^{-1}(U). \end{aligned}$$

Similar argument applies to the situation where U is a clopen downset. ■

Remark 1. Note that taking $V(f)$ to be the mapping $F \mapsto f[F]$ would not be well-define as $f[F]$ is not necessarily convex when f is non-surjective.

The above shows that we have a functor $V \colon \mathbf{Pries} \to \mathbf{Pries}$.

4 Hyperspace of Pairwise Stone space

In this section we want to define a notion of a *bitopological Vietoris space* for a bitopological Stone space. More precisely what we want is an endofunctor $\mathrm{V}_{\mathrm{bi}} \colon \mathbf{PStone} \to \mathbf{PStone}$ such that the following diagram

$$\begin{array}{ccc} \mathbf{PStone} & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & \mathbf{Pries} \\ \big\downarrow \mathrm{V}_{\mathrm{bi}} & & \big\downarrow V \\ \mathbf{PStone} & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & \mathbf{Pries} \end{array} \tag{1}$$

commutes. By standard category theoretic argumentation this will ensure that the category of coalgebras for the functor V_{bi} will be isomorphic to the category of the coalgebras for the functor V.

4.1 The construction of the functor V_{bi}

Given a bitopological space (X, τ_1, τ_2) we define

$$\kappa(X) := \{F \cap G : F \in \delta_1,\ G \in \delta_2\},$$

that is $\kappa(X)$ is the set of all intersections of τ_1-closed and τ_2-closed sets.

Furthermore we let τ_1^v be the topology on $\kappa(X)$ generated by the subbasis $\{\Diamond U, \Box U\}_{U \in \beta_1}$ where as usual

$$\Diamond U = \{H \in \kappa(X) : H \cap U \neq \varnothing\} \quad \text{and} \quad \Box U = \{H \in \kappa(X) : H \subseteq U\}.$$

Similarly τ_2^v will be the topology on $\kappa(X)$ generated by $\{\Diamond V, \Box V\}_{V \in \beta_2}$.

We then define the *bitopological Vietoris space* $\mathrm{V}_{\mathrm{bi}}(X)$ of a Pairwise Stone space (X, τ_1, τ_2) to be the bitopological space $(\kappa(X), \tau_1^v, \tau_2^v)$. In what follows we will show that this constitutes the object part of an endofunctor on **PStone**.

Remark 2. One could also have defined τ_1^v to be generated by $\{\Diamond U\}_{U \in \beta_1 \cup \beta_2}$ and τ_2^v to be generated by $\{\Box U\}_{U \in \beta_1 \cup \beta_2}$. This also determines an endofunctor on the category **PStone**. However this will not make the diagram (1) commute.

Lemma 4. *If (X, τ_1, τ_2) is a Pairwise Stone space and $\leq_1$ is the specialization order of (X, τ_1) then $\leq_1^{EM}$ is a partial order when restricted to $\kappa(X)$.*

Proof. As (X, τ_1, τ_2) is pairwise zero-dimensional and pairwise Hausdorff we have that (X, τ_1) is a T_0-space, whence $\leq_1$ is a partial order. Therefore $\leq_1^{\mathrm{EM}}$ is reflexive and transitive. Now suppose that we have $H, H' \in \kappa(X)$, i.e. $H = F \cap G$ and $H' = F' \cap G'$ for some $F, F' \in \delta_1$ and $G, G' \in \delta_2$, such that $H \leq_1^{\mathrm{EM}} H'$ and $H' \leq_1^{\mathrm{EM}} H$. As $H \subseteq {\downarrow} H'$ we have that for all $x \in H$ there is $y_x \in H'$ such that $x \leq_1 y_x$. So we must have that

$$\begin{aligned} H &\subseteq \bigcup_{x \in H} \mathrm{cl}_1(y_x) \\ &\subseteq \bigcup_{y \in H'} \mathrm{cl}_1(y) \\ &\subseteq \mathrm{cl}_1(\bigcup_{y \in H'} y) \\ &\subseteq \mathrm{cl}_1(H') \\ &\subseteq \mathrm{cl}_1(F') = F'. \end{aligned}$$

As $H' \subseteq {\uparrow} H$ we have that for all $y \in H'$ an element $x_y \in H$ such that $x_y \leq_1 y$ and therefore by Proposition 2 such that $y \leq_2 x_y$. So by a similar argument we see that $H' \subseteq G$. Finally using that $H' \leq_1^{\mathrm{EM}} H$ we see that $H \subseteq G'$ and $H' \subseteq F$, whence from we conclude that $H = H'$ as $H = F \cap G$ and $H' = F' \cap G'$.

∎

Lemma 5. *If (X, τ_1, τ_2) is a pairwise Stone space then $\mathrm{V}_{\mathrm{bi}}(X) = (\kappa(X), \tau_1^v, \tau_2^v)$ is pairwise zero-dimensional.*

Proof. We show that $\beta_1^v = \tau_1^v \cap \delta_2^v$, is a basis for τ_1^v where δ_2^v is the set of τ_2^v-closed sets.

By definition we have that if $W \in \tau_1^v$ then

$$W = \bigcup_{i\in I}(\bigcap_{j=1}^{m_i} \Diamond U_j \cap \bigcap_{k=1}^{n_i} \Box U_k), \quad U_j, U_k \in \beta_1.$$

Thus to show that β_1^v is a basis for τ_1^v it suffices to show that

$$\bigcap_{j=1}^{m} \Diamond U_j \cap \bigcap_{k=1}^{n} \Diamond U_k \in \beta_1^v$$

when $U_j, U_k \in \beta_1$. Since β_1^v is closed under finite intersections it is enough to show that $\Diamond U, \Box U \in \beta_1^v$ whenever $U \in \beta_1$ We immediately see from the definition of τ_1^v that $\Diamond U, \Box U \in \tau_1^v$, whenever $U \in \beta_1$. Now $(\Diamond U)^c = \Box U^c$ and $(\Box U)^c = \Diamond U^c$. So as $U \in \beta_1$ implies that $U^c \in \beta_2$ we see that $\Diamond U, \Box U \in \delta_2^v$ whenever $U \in \beta_1$. Hence we may conclude that $\Diamond U, \Box U \in \beta_1^v$ when $U \in \beta_1$ and so β_1^v is indeed a basis for τ_1^v.

The argument to the fact that $\beta_2^v = \tau_2^v \cap \delta_1^v$, where δ_1^v is the set of τ_1^v-closed sets, is a basis for τ_2^v is completely similar and thus omitted. ∎

Lemma 6. *If (X, τ_1, τ_2) is a pairwise Stone space then $\mathrm{V_{bi}}(X) = (\kappa(X), \tau_1^v, \tau_2^v)$ is pairwise Hausdorff.*

Proof. Let $H, H' \in \kappa(X)$ with $H \neq H'$. So $H = F \cap G$ and $H' = F' \cap G$ for some $F, F' \in \delta_1$ and some $G, G' \in \delta_2$. By Lemma 4 we then have that $H \nleq_1^{EM} H'$ or $H' \nleq_1^{EM} H$. We therefore first assume that $H \nleq_1^{EM} H'$.

Thus either $H \nsubseteq {\downarrow}H'$ or $H' \nsubseteq {\uparrow}H$. We assume first that $H \nsubseteq {\downarrow}H'$. It immediately follows from this assumption that we must have $x \in H$ such that $x \nleq_1 y$ for all $y \in H'$, i.e for each $y \in H'$ we have that $x \notin \mathrm{cl}_1(y)$. So as (X, τ_1, τ_2) is a pairwise Stone space we have that β_1 is a basis for τ_1 and hence we must have for each $y \in H'$ a basic τ_1-open $U_y \in \beta_1$ witnessing that $x \notin \mathrm{cl}_1(y)$. This means that $x \in U_y$ and $y \notin U_y$. It follows that $\bigcup_{y\in H'} U_y^c$ is a τ_2-open cover of H'. In particular it is a $\tau_1 \vee \tau_2$ open cover of H'. Now by proposition 13 we have that H' is pairwise compact, being the intersection of a δ_1 and a δ_2 set. So we have a finite subcover $\bigcup_{i=1}^n U_{y_1}^c$ of H'. Therefore if we let $U := \bigcap_{i=1}^n U_{y_i}$ we have that $H \cap U \neq \varnothing$, as $x \in H \cap U$ and $H' \cap U = \varnothing$, as $U \subseteq H'^c$. It follows that $H \in \Diamond U$ and $H' \notin \Diamond U$. Finally since β_1 is closed under finite intersections we have that $U \in \beta_1$ and thereby that $U^c \in \beta_2$. Hence we have that $\Diamond U \in \tau_1^v$ and $(\Diamond U)^c = \Box U^c \in \tau_2^v$, thus $\Diamond U$ and $\Box U^c$ are disjoint τ_1^v and τ_2^v sets respectively which separates H and H'.

If $H' \nsubseteq {\uparrow}H$ then we have $y \in H'$ such that $x \nleq_1 y$ for all $x \in H$. So as β_1 is a basis for τ_1 we obtain by a similar argument as above a set $V \in \beta_1$ such that $H \in \Box V$ and $H' \notin \Box V$. So $\Box V$ and $\Diamond V^c$ will be disjoint τ_1^v and τ_2^v sets respectively which separates H and H'.

Finally if $H' \nleq_1^{EM} H$, then by a completely similar argument as above we obtain disjoint τ_1^v and τ_2^v sets separating H and H'. ∎

Lemma 7. *If* (X, τ_1, τ_2) *is a pairwise Stone space then* $\mathrm{V_{bi}}(X) = (\kappa(X), \tau_1^v, \tau_2^v)$ *is pairwise compact.*

Proof. By Proposition 14 we have that $\{\Box V, \Diamond V\}_{V \in \beta_1 \cup \beta_2}$ is a subbasis for $\tau_1^v \vee \tau_2^v$ and so by the Alexander Subbasis Lemma it suffices to show that any cover of $\kappa(X)$ by elements from the subbasis $\{\Box V, \Diamond V\}_{V \in \beta_1 \cup \beta_2}$ has a finite subcover.

Therefore assume that

$$\kappa(X) = \bigcup_{i \in I} \Box U_i \cup \bigcup_{j \in J} \Diamond V_j$$

is an open cover with $U_i, V_j \in \beta_1 \cup \beta_2$.

We claim that

$$X = \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} V_j.$$

For suppose that we have $x \notin \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} V_j$. Then if we let $C_x = \mathrm{cl}_1(x) \cap \mathrm{cl}_2(x)$ we have that $C_x \in \kappa(X)$, and therefore $C_x \in \Box U_i$, for some $i \in I$ or $C_x \in \Diamond V_j$, for some $j \in J$. Now as $x \in C_x$ we have that $C_x \in \Box U_i$ implies $x \in U_i$ contradicting the assumption that $x \notin \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} V_j$. On the other hand, if $C_x \in \Diamond V_j$ for some $j \in J$, then $V_j \cap C_x$ is non-empty. But as $x \notin \bigcup_{j \in J} V_j$ we have that $x \in V_j^{\mathrm{c}}$ for all $j \in J$. Therefore if $V_j \in \beta_1$ then $V_j^{\mathrm{c}} \in \beta_2$ and so V_j^{c} is a τ_1-closed set containing x, whereby it follows that $\mathrm{cl}_1(x) \subseteq V_j^{\mathrm{c}}$ and thereby that $C_x \subseteq V_j^{\mathrm{c}}$, contradicting the fact that $C_x \cap V_j \neq \varnothing$. A similar argument applies if $V_j \in \beta_2$.

Now let $F := X \setminus \bigcup_{j \in J} V_j$. Since $V_j \in \beta_1 \cup \beta_2$ we have that

$$F = \bigcap_{k \in K} W_k \cap \bigcup_{l \in L} W'_l$$

with $W_k \in \beta_1$ and $W'_l \in \beta_2$. So since $\beta_1 = \tau_1 \cap \delta_2$ and $\beta_2 = \tau_2 \cap \delta_1$ we must have that $W := \bigcap_{k \in K} W_k$ is in δ_2 and $W' := \bigcup_{l \in L} W'_l$ is in δ_1. Therefore we have that F is the intersection of a δ_1 set and a δ_2 set and as such $F \in \kappa(X)$. Thus as $\bigcup_{i \in I} \Box U_i \cup \bigcup_{j \in J} \Diamond V_j$ covers $\kappa(X)$ we must have that $F \in \bigcup_{i \in I} \Box U_i$, as F is disjoint from all V_j. Let $i' \in I$ be such that $F \in \Box U_{i'}$. Then $F \subseteq U_{i'}$, and therefore

$$X = U_{i'} \cup \bigcup_{j \in J} V_j,$$

and so as (X, τ_1, τ_2) is pairwise compact we have that

$$X = U_{i'} \cup \bigcup_{j=1}^{m} V_j.$$

We finally claim that

$$\kappa(X) = \Box U_{i'} \cup \bigcup_{j=1}^{m} \Diamond V_j.$$

To see this, let $G \in \kappa(X)$. Then either $G \subseteq U_{i'}$ in which case $F \in \Box U_{i'}$ or $G \nsubseteq U_{i'}$ in which case $G \cap \bigcup_{j=1}^{m} V_j$ most be non-empty. From this it follows that $G \cap V_j \neq \varnothing$ for some $j \leq m$ and therefore that $G \in \Diamond V_j$ for some $j \leq m$. ∎

The three above lemmas establish:

Theorem 4. *If (X, τ_1, τ_2) is a pairwise Stone space, then $\mathrm{V_{bi}}(X) = (\kappa(X), \tau_1^v, \tau_2^v)$ is also a pairwise Stone space.*

We define how $\mathrm{V_{bi}}$ acts on maps: Given a bi-continuous map $f\colon (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ we define a function $\mathrm{V_{bi}}(f)\colon \mathrm{V_{bi}}(X) \to \mathrm{V_{bi}}(X')$ by $H \mapsto \mathrm{cl}_1'(f[H]) \cap \mathrm{cl}_2'(f[H])$, where $\mathrm{cl}_i'(-)$ is the closure operator associated to the topology τ_i' for $i \in \{1, 2\}$.

Proposition 9. *If $f\colon (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ is bi-continuous then $\mathrm{V_{bi}}(f)$ given by*

$$H \mapsto \mathrm{cl}_1'(f[H]) \cap \mathrm{cl}_2'(f[H]),$$

is a bi-continuous function from $(\kappa(X), \tau_1^v, \tau_2^v)$ to $(\kappa(Y), (\tau_1')^v, (\tau_2')^v)$.

Proof. It suffices to show that preimages of the elements of the subbasis are open. We simply observe that if $U' \in \beta_1'$ then

$$\begin{aligned} V(f)^{-1}(\Diamond U') &= \{H \in \kappa(X) \colon V(f)(H) \in \Diamond U'\} \\ &= \{H \in \kappa(X) \colon \mathrm{cl}_1'(f[H]) \cap \mathrm{cl}_2'(f[H]) \in \Diamond U'\} \\ &= \{H \in \kappa(X) \colon (\mathrm{cl}_1'(f[H]) \cap \mathrm{cl}_2'(f[H])) \cap U' \neq \varnothing\} \\ &= \{H \in \kappa(X) \colon f[H] \cap U' \neq \varnothing\} \qquad \text{(as } U' \text{ is } \delta_2'\text{)} \\ &= \{H \in \kappa(X) \colon H \cap f^{-1}(U') \neq \varnothing\} = \Diamond f^{-1}(U'). \end{aligned}$$

A similar argument applies when $V' \in \beta_2$. Furthermore an analogous argument shows that $V(f)^{-1}(\Box W') = \Box f^{-1} W'$, for $W' \in \beta_1 \cup \beta_2$. So as f is assumed to be bi-continuous we have that $f^{-1} W' \in \beta_i$ whenever $W' \in \beta_i'$, for $i \in \{1, 2\}$. Hence $V(f)^{-1}(\Diamond U')$ and $V(f)^{-1}(\Box U')$ are open in τ_1^v for all $U' \in \beta_1'$. Likewise $V(f)^{-1}(\Diamond V')$ and $V(f)^{-1}(\Box V')$ is open in τ_2^v whenever $V' \in \beta_2'$. ∎

4.2 The relation between V and $\mathrm{V_{bi}}$

To justify the claim that $\mathrm{V_{bi}}$ determines the bitopological analogue of the Vietoris functor on **Pries** we must show that the diagram (1) commutes. Now establishing the commutativity of (1) amounts to showing that the following two diagrams commutes:

$$\begin{array}{ccc} \mathbf{PStone} & \xrightarrow{\Phi} & \mathbf{Pries} \\ {\scriptstyle \mathrm{V_{bi}}}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbf{PStone} & \xrightarrow{\Phi} & \mathbf{Pries} \end{array} \qquad \begin{array}{ccc} \mathbf{PStone} & \xleftarrow[\Psi]{} & \mathbf{Pries} \\ {\scriptstyle \mathrm{V_{bi}}}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbf{PStone} & \xleftarrow[\Psi]{} & \mathbf{Pries} \end{array}$$

This is the case because Φ and Ψ are each others inverses.

Proposition 10. *Let V and V_{bi} be the hyperspace functors for Priestley spaces and pairwise Stone spaces and let Φ: **PStone** → **Pries** and Ψ: **Pries** → **PStone** be as in theorem 2 then $V\Phi = \Phi\,\mathrm{V}_{\mathrm{bi}}$.*

Proof. Let (X, τ_1, τ_2) be a pairwise Stone space. Then we must show that

$$(\mathrm{Conv}_\tau(X), \tau_v, (\leq_1)^{\mathrm{EM}}) = (\kappa(X), \tau_1^v \vee \tau_2^v, \leq_{\tau_1^v}),$$

where $\tau = \tau_1 \vee \tau_2$, and $\leq_{\tau_1^v}$ is the specialization preorder of $(\kappa(X), \tau_1^v)$.

By proposition 3 and proposition 6 we see that $\mathrm{Conv}_\tau(X) = \kappa(X)$. By using proposition 3 we see that τ_v is the topology generated by the subbasis $\{\Box U, \Diamond U\}$ for $U \in \beta_1 \cup \beta_2$, which must coincide with the topology $\tau_1^v \vee \tau_2^v$ by proposition 14. Hence we only need to show that $(\leq_1)^{\mathrm{EM}} = \leq_{\tau_1^v}$. Therefore let $H, H' \in \mathrm{Conv}_\tau(X) = \kappa(X)$.

If $H \nleq_{\tau_1^v} H'$, then we have $U \in \beta_1$ such that either

$$H \in \Diamond U \quad \text{and} \quad H' \notin \Diamond U,$$

or

$$H \in \Box U \quad \text{and} \quad H' \notin \Box U.$$

In the former case we have an $x \in H \cap U$ for which U witnesses that $x \notin \mathrm{cl}_1(y)$ for all $y \in H'$. Thus $x \notin {\downarrow}H'$ and therefore $H \nsubseteq {\downarrow}H'$, why $H \nleq_1^{\mathrm{EM}} H'$.

In the latter case we have $y \in H \cap U^{\mathrm{c}}$ for which U witnesses that $x \notin \mathrm{cl}_1(y)$ for all $x \in H$. Thus $y \notin {\uparrow}H$ and therefore $H' \nsubseteq {\uparrow}H$, why $H \nleq_1^{\mathrm{EM}} H'$.

If $H \nleq_1^{\mathrm{EM}} H'$ then either $H \nsubseteq {\downarrow}H'$ or $H' \nsubseteq {\uparrow}H$. In the former case we get $x \in H$ such that $x \notin {\downarrow}H'$ i.e such that for all $y \in H'$ we have $x \notin \mathrm{cl}_1(y)$, which means that for all $y \in H'$ we have $U_y \in \beta_1$ such that $x \in U_y$ and $y \notin U_y$. So by the now standard compactness argument we can find $U \in \beta_1$ such that $x \in U$ and $y \notin U$ for all $y \in H'$, whence from it follows that $H \cap U \neq \varnothing$ and $H' \cap U = \varnothing$, i.e. $H \in \Diamond U$ and $H' \notin \Diamond U$, showing that $H \nleq_{\tau_1^v}$ H'. In the former case we get by a similar argument $U \in \beta_1$ such that $H \subseteq U$ and $H' \nsubseteq H'$. Hence $H \in \Box U$ and $H' \notin \Box U$ which shows that $H \nleq_{\tau_1^v} H'$. ∎

Proposition 11. *Let V, V_{bi} and Ψ, Φ be as in proposition 10. Then $\mathrm{V}_{\mathrm{bi}}\,\Psi = \Psi V$.*

Proof. We show that $\Phi\,\mathrm{V}_{\mathrm{bi}}\,\Psi = V$ and thereby that $\mathrm{V}_{\mathrm{bi}}\,\Psi = \Psi V$ as Ψ is the inverse of Φ.

Let $(X, \tau, \leq)$ be a Priestley space. Then we see that

$$\begin{aligned}\Phi\,\mathrm{V}_{\mathrm{bi}}\,\Psi(X, \tau, \leq) &= \Phi\,\mathrm{V}_{\mathrm{bi}}(X, \mathrm{OpUp}_\tau(X), \mathrm{OpDo}_\tau(X)) \\ &= \Phi(\kappa(X), (\mathrm{OpUp}_\tau(X))^v, (\mathrm{OpDo}_\tau(X))^v)) \\ &= (\kappa(X), (\mathrm{OpUp}_\tau(X))^v \vee (\mathrm{OpDo}_\tau(X))^v, \leq'),\end{aligned}$$

where $\leq'$ is the specialization preorder of $(X, (\mathrm{OpUp}_\tau(X))^v)$, and where now $\kappa(X)$ is obtained from the pairwise Stone space $(X, \mathrm{OpUp}_\tau(X), \mathrm{OpDo}_\tau(X))$, i.e.,

$$\begin{aligned}\kappa(X) &= \{F \cap G : F^{\mathrm{c}} \in \mathrm{OpUp}_\tau(X),\ G^{\mathrm{c}} \in \mathrm{OpDo}_\tau(X)\} \\ &= \{F \cap G : F \in \mathrm{ClDo}_\tau(X),\ G \in \mathrm{ClUp}_\tau(X)\}.\end{aligned}$$

By proposition 4 and proposition 6 we see that $\kappa(X) = \mathrm{Conv}_\tau(X)$. Moreover as $(\mathrm{OpUp}_\tau(X))^v$ is the topology generated by $\{\Diamond U, \Box U\}_{U \in \mathrm{ClpUp}_\tau(X)}$ and $(\mathrm{OpDo}_\tau(X))^v$ is the topology generated by $\{\Diamond V, \Box V\}_{V \in \mathrm{ClpDo}_\tau(X)}$ we see by lemma 14 that $(\mathrm{OpUp}_\tau(X))^v \vee (\mathrm{OpDo}_\tau(X))^v$ is the topology generated by $\{\Diamond W, \Box W\}_{W \in \mathrm{ClpUp}_\tau(X) \cup \mathrm{ClpDo}_\tau(X)}$, which is the topology τ_v, the Vietoris topology on $(X, \tau, \leq)$.

Therefore we now only need to show that $\leq^{\mathrm{EM}} = \leq'$. To see this, we observe that if $F \nleq' F'$, then there exists $U \in \mathrm{ClpUp}_\tau(X)$ such that

$$F \in \Diamond U \text{ and } F' \notin \Diamond U \quad \text{or} \quad F \in \Box U \text{ and } F' \notin \Box U.$$

In the former case we get $x \in F \cap U$ such that $x \leq y$ for all $y \in F'$, witnessing that $F \notin {\downarrow} F'$ and therefore that $F \nleq^{\mathrm{EM}} F'$. In the latter we get $y \in F'$ such that $x \nleq y$ for all $x \in F$, witnessing that $F' \notin {\uparrow} F$, and thereby that $F \nleq^{\mathrm{EM}} F'$.

Conversely if $F \nleq^{\mathrm{EM}} F'$ then by the now standard compactness argument we either obtain $U \in \mathrm{ClpUp}_\tau(X)$ such that $F \in \Diamond U$ and $F' \notin \Diamond U$ or $U' \in \mathrm{ClpUp}_\tau(X)$ such that $F \in \Box U'$ and $F' \notin \Box U'$, which in both cases implies that $F \nleq' F'$.

We have thus shown that

$$\Phi\, \mathrm{V}_{\mathrm{bi}}\, \Psi(X, \tau, \leq) = (\mathrm{Conv}(X), \tau_v, \leq^{EM}) = V(X, \tau, \leq),$$

as desired. ∎

From the commutativity of (1) we immediately obtain that the functors Φ and Ψ induce an isomorphism of categories $\mathbf{CoAlg}(V)$ and $\mathbf{CoAlg}(\mathrm{V}_{\mathrm{bi}})$.

5 Positive modal logic

By *the language of positive modal logic* we understand the $\{\wedge, \vee, \Box, \Diamond, \top, \bot\}$-fragment of the language of basic modal logic.

Given that we do not have implication in our language we cannot present a deductive system for positive modal logic in terms of a Hilbert style calculus as is done in [2] for basic modal logic. Therefore we give a presentation of the minimal positive modal logic $\mathbf{K}^+$ in terms of *consequence pairs*. This presentation is due to Dunn [8] see also [10] for a generalization of positive modal logic to distributive modal logic. A consequence pair is an expression $\varphi \vdash \psi$ where φ, ψ are formulas in the language of positive modal logic.

A positive modal logic Λ is a set of consequence pairs which contains the following axioms:

$$\varphi \vdash \varphi \quad \varphi \vdash \top \quad \bot \vdash \varphi$$

$$\varphi \wedge \psi \vdash \varphi \quad \varphi \wedge \psi \vdash \psi$$

$$\varphi \vdash \varphi \vee \psi \quad \psi \vdash \varphi \vee \psi$$

$$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\top \vdash \Box\top \quad \Diamond\bot \vdash \bot$$

$$\Box(\varphi \wedge \psi) \dashv\vdash \Box\varphi \wedge \Box\psi \quad \Diamond(\varphi \vee \psi) \dashv\vdash \Diamond\varphi \vee \Diamond\psi$$

$$\Diamond\varphi \wedge \Box\varphi \vdash \Diamond(\varphi \wedge \psi) \quad \Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$$

and which moreover is closed under the following inference rules:

$$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi}$$

$$\frac{\varphi \vdash \psi \qquad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

$$\frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

$$\frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

Remark 3. One can also give a presentation of positive modal logic in terms of a genuine Gentzen style sequent calculus. For this see e.g. [5, 6, 8].

We now define $\mathbf{K}^+$ to be the minimal positive modal logic. In [8] it is proven that $\mathbf{K}^+$ is sound and complete with respect to Kripke frames with one accessibility relation.

Theorem 5 ([8]). *Let φ and ψ be formulas in the language of positive modal logic, then we have that*

$$\varphi \vdash \psi \in \mathbf{K}^+ \text{ if and only if } \varphi \vDash \psi.$$

All of the above axioms and inference rules should not be surprising, with the exception of the $(\Box, \Diamond)$-interaction rules i.e.

$$\Diamond\varphi \wedge \Box\varphi \vdash \Diamond(\varphi \wedge \psi) \qquad \Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi.$$

To get an idea of why these axioms are imposed we note that since we do not have negation in the language, $\Box$ and $\Diamond$ are no longer interdefinable. Therefore we can consider the positive modal logic to consist of two separate modalities. Consequently in the Kripke semantics for this language we would have two different accessibility relations with the following clauses:

$$w \vDash \Box\varphi \text{ iff } \forall w' \in W \ wR_{\Box}w' \text{ implies } w' \vDash \varphi,$$

and

$$w \vDash \Diamond\varphi \text{ iff } \exists w' \in W \ wR_{\Diamond}w' \text{ and } w' \vDash \varphi.$$

Now together the $(\Box, \Diamond)$-interaction rules ensures that $R_{\Box} = R_{\Diamond}$, in the sense that $\Diamond\varphi \wedge \Box\varphi \vdash \Diamond(\varphi \wedge \psi)$ is valid on precisely those frames which satisfies $R_{\Diamond} \subseteq R_{\Box}$ and $\Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$ is valid on precisely whose frames which satisfies $R_{\Box} \subseteq R_{\Diamond}$. Therefore we can consider Kripke frames with only one accessibility relation, so that $\Diamond$ will be the dual of $\Box$, at least on the semantic level.

Even though the logic $\mathbf{K}^+$ is sound and complete with respect to the the usual Kripke semantics restricted to the forcing clauses for the positive connectives, it turns out that this semantics is not very well-behaved when it comes to even very simple extensions of minimal positive modal logic. E.g. if the consequent pair $\Box\varphi \vdash \Box\Box\varphi$ is added to $\mathbf{K}^+$ then the resulting logic $\mathbf{K4}^+$ is frame incomplete. More precisely we have that the consequent pair $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ is valid on every frame that validates $\Box\varphi \vdash \Box\Box\varphi$, however $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ is not deducible from $\Box\varphi \vdash \Box\Box\varphi$ over $\mathbf{K}^+$. Thus we need a different semantics for positive modal logic where the frame complete logics are precisely those that are characterized by some class of frames. In order to achieve this Celani and Jansana introduced in [5] a semantics based on quasi ordered Kripke frames $(W, \leq, R)$ satisfying

$$(\leq \circ R) \subseteq (R \circ \leq) \quad \text{and} \quad (\geq \circ R) \subseteq (R \circ \geq).$$

Moreover the valuations on these frames is restricted to those which maps the propositional letters into upsets. In [5] it is shown that $\mathbf{K}^+$ is sound and complete with respect to this semantics and that the frame complete extensions of minimal positive modal logic are precisely the ones characterized by some class of frames.

5.1 Duality theory for positive modal logic

As one could imagine an algebraic semantics for $\mathbf{K}^+$ is based on distributive lattices with two operators interpreting the box and the diamond modalities. More precisely we define:

Definition 5. A *positive modal algebra* is a structure $(A, \wedge, \vee, \Box, \Diamond, 0, 1,)$ such that $(A, \wedge, \vee, 0, 1,)$ is a bounded distributive lattice with operators $\Box, \Diamond : A \to A$ satisfying:

$$\begin{aligned} \Box 1 &= 1 & \Diamond 0 &= 0 \\ \Box(a \wedge b) &= \Box a \wedge \Box b & \Diamond (a \vee b) &= \Diamond a \vee \Diamond b \\ \Box a \wedge \Diamond b &\leq \Diamond(a \wedge b) & \Box (a \vee b) &\leq \Box a \vee \Diamond b. \end{aligned}$$

By **PMA** we will denote the category of positive modal algebras and bounded lattice homomorphisms which commute with the operators $\Box$ and $\Diamond$.

In order to obtain a duality for positive modal logic á al Jónsson-Tarski, Celani and Jansana [6] introduced the notion of a $\mathbf{K}^+$-space which is going to be to positive modal algebras what the general descriptive frames are to normal modal algebras in the well-know Jónsson-Tarski duality theory for full modal logic. In the following we briefly sketch the duality theory for positive modal logic. For details we refer to [6].

Definition 6. A $\mathbf{K}^+$*-space* is a quadruple $(X, \tau, \leq, R)$ such that

i) $(X, \tau, \leq)$ is a Priestley space.

ii) The set $R[x]$ is closed for each $x \in X$.

iii) $R[x] = (R \circ \leq)[x] \cap (R \circ \geq)[x]$ for each $x \in X$

iv) The topology τ is closed under the operations $\langle R \rangle$ and $[R]$,

where $R[x] = \{y \in X : xRy\}$ is the set of R-successors of x.

By a $\mathbf{K}^+$*-morphism* we understand a continuous and order preserving function $f : (X, \tau, \leq, R) \to (X', \tau', \leq', R')$ between $\mathbf{K}^+$-space which are also a p-morphism with respect to the relations R and R'. Recall that a p-morphism is a map $f : (X, R) \to (Y, R')$ between sets with relations such that

i) xRx' implies $f(x)R'f(x')$,

ii) $f(x)Ry$ implies that there exists $x' \in X$ such that xRx' and $f(x') = y$.

Since $(R \circ \leq)[x] = \uparrow R[x]$ and $(R \circ \geq)[x] = \downarrow R[x]$ one readily checks that the $\mathbf{K}^+$-spaces of [6] are the same as the Modal Priestley spaces of [4], where a *modal Priestley space* is defined as a quadruple $(X, \tau, \leq, R)$ such that $(X, \tau, \leq)$ is a Priestley space and R is a binary relation on X satisfying

i) $R[x]$ is closed and convex for all $x \in X$.

ii) The set $\mathrm{ClpUp}(X)$ is closed under the operations $\langle R \rangle$ and $[R]$.

Remark 4. Note that as $\mathrm{ClpDo}(X) = \{U^c : U \in \mathrm{ClpUp}(X)\}$ and $[R]U^c = (\langle R \rangle U)^c$ and $\langle R \rangle U^c = ([R]U)^c$ it follows that if R satisfies ii) then $\mathrm{ClpDo}(X)$ is also closed under the operations $\langle R \rangle$ and $[R]$.

We let **MPS** denote the category of modal Priestley spaces with morphisms continuous order preserving functions which are also p-morphisms with respect to the relation R. Hence **MPS** is the same as the category of $\mathbf{K}^+$-spaces and $\mathbf{K}^+$-morphisms and therefore in what follows we shall refer to this category as the category of modal Priestley spaces rather that the category of $\mathbf{K}^+$-spaces.

Now if $(X, \tau, \leq, R)$ is a modal Priestley space then

$$A(X) = (\mathrm{ClpUp}(X), \cap, \cup, [R], \langle R \rangle, \varnothing, X)$$

will be a positive modal algebra. Moreover if $f\colon (X, \tau, \leq, R) \to (X', \tau', \leq', R')$ is morphism of modal Priestley spaces then $A(f)\colon A(X') \to A(X)$ given by

$$A(f)(U) = f^{-1}(U)$$

is a positive modal algebra homomorphism. In fact this determines a functor $A\colon \mathbf{MPS} \to \mathbf{PMA}$.

Conversely if $(A, \wedge, \vee, \Box, \Diamond, 0, 1)$ is positive modal algebra then

$$F(A) = (\mathsf{pf}(A), \tau, \subseteq, R_A)$$

is a modal Priestley space, where $\mathsf{pf}(A)$ is the set of all prime filters on A and τ is the topology on $\mathsf{pf}(A)$ generated by the subsets

$$\varphi_+(a) = \{P \in \mathsf{pf}(A)\colon a \in P\} \quad \text{and} \quad \varphi_-(a) = \{P \in \mathsf{pf}(A)\colon a \notin P\},$$

with a ranging in A. Finally the relation R_A is given by

$$P R_A Q \iff \Box^{-1}(P) \subseteq Q \subseteq \Diamond^{-1}(P).$$

If we let $R_\Box := (R_A \circ \subseteq)$ and $R_\Diamond := R_A \circ \subseteq$ then on may readily check that $R_A = R_\Box \cap R_\Diamond$.

If $h\colon A \to B$ is an homomorphism between positive modal algebras then $F(h)\colon F(B) \to F(A)$ given by $F(h)(P) = h^{-1}(P)$ will be a morphism of modal Priestley spaces. Thus we have a functor $F\colon \mathbf{PMA} \to \mathbf{MPS}$.

Celani and Jansana has showed that this indeed gives us a Jónsson-Tarski style duality theory for positive modal logic, i.e.

Theorem 6 ([6]). *The functors $F\colon \mathbf{PMA} \to \mathbf{MPS}$ and $A\colon \mathbf{MPS} \to \mathbf{PMA}$ determines a dual equivalence between the categories* **PMA** *and* **MPS**.

Note that this is an extension of the Priestley duality between the categories **bDist** and **Pries** in exactly in the same way that the Jónsson-Tarski duality is an extension of Stone duality.

6 The Coalgebraic view

Let $\mathbf{C}$ be a category and $F{:}\,\mathbf{C} \to \mathbf{C}$ an endofunctor. By a *coalgebra* for F (or an F-*coalgebra*) we shall understand a pair (C, ξ) such that $\xi{:}\,C \to F(C)$ is an arrow in $\mathbf{C}$.

Example 1. *The coalgebras for the covariant powerset functor* $\mathscr{P}(-){:}\,\mathbf{Set} \to \mathbf{Set}$ *can be seen as Kripke frames. More precisely given a coalgebra* $\xi{:}\,W \to \mathscr{P}(W)$ *we define a relation* R *on* W, *by*

$$xRy \iff y \in \xi(x).$$

Conversely given a Kripke frame (W, R) *we get a* $\mathscr{P}$*-coalgebra* $R[-]{:}\,W \to \mathscr{P}(W)$.

A coalgebra morphism between F-coalgebras (C, ξ) and (D, χ) is an arrow $f{:}\,C \to D$ in $\mathbf{C}$ such that the following diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(D) \\ {\scriptstyle\xi}\uparrow & & {\scriptstyle\chi}\uparrow \\ C & \xrightarrow{\quad f \quad} & D \end{array}$$

commutes.

Example 2. *The coalgebra morphisms between coalgebra for the covariant powerset functor corresponds to the p-morphisms between the corresponding Kripke frames.*

Now given an endofunctor $F{:}\,\mathbf{C} \to \mathbf{C}$ we denote by $\mathbf{CoAlg}(F)$ the category of F coalgebras and F-coalgebra morphisms.

Thus if we let $\mathbf{Krp}$ denote the category of Kripke frames and p-morphisms then examples 1 and 2 tell us that the categories $\mathbf{Krp}$ and $\mathbf{CoAlg}(\mathscr{P})$ are isomorphic.

Hence we can see the category $\mathbf{CoAlg}(\mathscr{P})$ as providing a semantics for basic modal logic. By changing the base category we can view coalgebra as a tool for obtaining semantics for different kinds of modal logics. The following two theorems may serve as an example of this.

Theorem 7 ([18])**.** *The category* $\mathbf{CoAlg}(V)$ *of coalgebras for the endofunctor* $V{:}\,\mathbf{Pries} \to \mathbf{Pries}$ *is dually equivalent to the category* $\mathbf{PMA}$ *of positive modal algebras.*

Remark 5. Note that the functor $V{:}\,\mathbf{Pries} \to \mathbf{Pries}$ in [18] is define somewhat differently from the one of the present work. However argument in [18] should easily be adaptable to the functor as define by us. This fact is also noted in [4, 24].

The above theorem is inspired by following:

Theorem 8 ([1, 15]). *The category* $\mathbf{CoAlg}(K)$ *of coalgebras for the endofunctor* $K\colon \mathbf{Stone} \to \mathbf{Stone}$ *is dually equivalent to the category* $\mathbf{MA}$ *of modal algebras.*

Where a modal algebra is a Boolean algebra $\langle A, \wedge, \neg, 0, 1\rangle$ with an operator $\Box\colon A \to A$ which preserves finite meets, i.e $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$. And as morphisms in **MA** we take Boolean algebra homomorphisms which commute with the operator $\Box$.

Our aim is to study the category $\mathbf{CoAlg}(\mathrm{V_{bi}})$ of coalgebras for the bi-Vietoris functor $\mathrm{V_{bi}}\colon \mathbf{PStone} \to \mathbf{PStone}$, which in light of theorem 7 should be dually equivalent to the category of **PML**.

6.1 Coalgebras for the bi-Vietoris functor $\mathrm{V_{bi}}$

We begin with an example

Example 3. *Let* (X, τ_1, τ_2) *be a bitopological space and let* $\xi\colon X \to \kappa(X)$ *be given by* $x \mapsto \mathrm{cl}_1(x) \cap \mathrm{cl}_2(x)$. *Then we see that*

$$\xi^{-1}(\Diamond U) = \xi^{-1}(\Box U) = U$$

for all $U \in \beta_1 \cup \beta_2$. *Therefore we can conclude that* ξ *is a bi-continuous function from* (X, τ_1, τ_2) *to* $\mathrm{V_{bi}}(X)$. *So* $((X, \tau_1, \tau_2), x \mapsto \mathrm{cl}_1(x) \cap \mathrm{cl}_2(x))$ *is a* $\mathrm{V_{bi}}$*-coalgebra for all bitopological spaces* (X, τ_1, τ_2).

We wish to investigate the relational semantics of positive modal logic by characterizing the relations R which give rise to a $\mathrm{V_{bi}}$ coalgebra by $x \mapsto R[x]$. And vice versa we want to determine the relations R defined via a $\mathrm{V_{bi}}$-coalgebra ξ as follows:

$$xRy \iff y \in \xi(x).$$

6.1.1 From relations to coalgebras

If R is a relation on a set X and $U \subseteq X$ we define

$$\langle R\rangle U := \{x \in X\colon R[x] \cap U \neq \varnothing\} \quad \text{and} \quad [R]U := \{x \in X\colon R[x] \subseteq U\}.$$

Definition 7. Let (X, τ_1, τ_2) be a pairwise Stone space, and R a binary relation on X. We say that R is a *positive accessibility relation* if the following holds:

i) $R[x] \in \kappa(X)$, for all $x \in X$, i.e. for all $x \in X$ we have τ_1-closed $F \subseteq X$ and τ_2-closed $G \subseteq X$ such that $R[x] = F \cap G$.

ii) $\langle R\rangle U, [R]U \in \beta_1$ for all $U \in \beta_1$.

Remark 6. Note that as $\beta_2 = \{U^c : U \in \beta_1\}$ and $[R]U^c = (\langle R\rangle U)^c$ and $\langle R\rangle U^c = ([R]U)^c$ it follows that if R satisfies item ii) in the above definition, then β_2 is also closed under the operations $\langle R\rangle$ and $[R]$.

We claim that positive accessibility relations give rise to coalgebras for V_{bi} in the following sense.

Theorem 9. *Let (X, τ_1, τ_2) be a pairwise Stone space, and R a positive accessibility relation on X. Then $\xi_R : X \to \mathrm{V}_{\mathrm{bi}}(X)$ given by $\xi_R(x) = R[x]$ is a coalgebra for the functor V_{bi}.*

Proof. By item i) in the definition of positive accessibility relation we have that $\xi_R(x)$ belongs to $\kappa(X)$ for all $x \in X$, and therefore ξ_R is indeed a well-defined function from X to $\kappa(X)$. Thus in order to show that ξ_R is a coalgebra for V_{bi} we only need to show that ξ_R is bi-continuous, i.e. that ξ_R is continuous as a mapping from $(X, \tau_1) \to (\kappa(X), \tau_1^v)$ and as a mapping $(X, \tau_2) \to (\kappa(X), \tau_2^v)$. As $\{\Diamond U, \Box U\}_{U \in \beta_1}$ is a subbasis for τ_1^v and $\{\Diamond V, \Box V\}_{V \in \beta_2}$ is a subbasis for τ_2^v it suffices to show that $\xi_R^{-1}(\Diamond U)$ and $\xi_R^{-1}(\Box U)$ are τ_1-open for all $U \in \beta_1$ and $\xi_R^{-1}(\Diamond V)$ and $\xi_R^{-1}(\Box V)$ are τ_2-open for all $V \in \beta_2$.

We easily see that

$$\xi_R^{-1}(\Diamond W) = \langle R\rangle W \quad \text{and} \quad \xi_R^{-1}(\Box W) = [R]W,$$

for all $W \in \beta_1 \cup \beta_2$. So by item ii) in the definition of positive accessibility relations we see that

$$\xi_R^{-1}(\Diamond U), \xi_R^{-1}(\Box U) \in \beta_1, \quad \text{for all } U \in \beta_1,$$

and by and remark 6 we also obtain that

$$\xi_R^{-1}(\Diamond V), \xi_R^{-1}(\Box V) \in \beta_2, \quad \text{for all } V \in \beta_2.$$

Therefore we may conclude that ξ_R is bi-continuous. ∎

6.1.2 From coalgebras to relations

Conversely we want to show that any coalgebra $\xi : X \to \mathrm{V}_{\mathrm{bi}}(X)$ induces a positive accessibility relation R^ξ on X such that $\xi_{R^\xi} = \xi$ and $R^{\xi_R} = R$.

Theorem 10. *Let (X, τ_1, τ_2) be a pairwise Stone space and let $\xi : X \to \mathrm{V}_{\mathrm{bi}}(X)$ be a coalgebra for V_{bi}. Then the relation R^ξ given by*

$$xR^\xi y \iff y \in \xi(x)$$

is a positive accessibility relation on X.

Proof. We first observe that for any $x \in X$ we have $R^\xi[x] = \xi(x)$. It follows that $R^\xi[x] \in \kappa(X)$ for all $x \in X$, which is item i) in the definition of a positive accessibility relation. Moreover we see that for $U \in \beta_1$

$$\begin{aligned}\langle R^\xi\rangle U &= \{x \in X : R^\xi[x] \cap U \neq \varnothing\} \\ &= \{x \in X : \xi(x) \cap U \neq \varnothing\} \\ &= \{x \in X : \xi(x) \in \Diamond U\} \\ &= \xi^{-1}(\Diamond U).\end{aligned}$$

Similarly we have that $[R^\xi]U = \xi^{-1}(\Box U)$.

So as ξ is bi-continuous we can therefore conclude that $\langle R^\xi\rangle U, [R^\xi]U \in \tau_1$ for all $U \in \beta_1$ and $\langle R^\xi\rangle V, [R^\xi]V \in \tau_2$ for all $U \in \beta_1 \cup \beta_2$. Moreover as $\beta_2 = \{U^c : U \in \beta_1\}$ and as

$$\langle R^\xi\rangle U = ([R^\xi]U^c)^c \quad \text{and} \quad [R^\xi]U = (\langle R^\xi\rangle U^c)^c,$$

we must also have that $\langle R^\xi\rangle U, [R^\xi]U \in \delta_2$ for $U \in \beta_1$. Therefore we see that $\langle R^\xi\rangle U, [R^\xi] \in \beta_1$, which is the second and final item in the definition of positive accessibility relations. ■

Let **PAR** be the category whose objects are pairwise Stone spaces with a positive accessibility relation and whose morphism are bi-continuous maps

$$f : (X, \tau_1, \tau_2, R) \to (X, \tau_1', \tau_2', R')$$

satisfying

$$R'[f(x)] = \mathrm{cl}_1'(f[R[x]]) \cap \mathrm{cl}_2'(f[R[x]]).$$

The condition on the maps is made to ensure that they are also V_{bi}-coalgebra morphisms between the corresponding V_{bi}-coalgebras.

Proposition 12. *The correspondences $R \mapsto \xi_R$ and $\xi \mapsto R^\xi$ determines an isomorphism between the categories* **PAR** *and* **CoAlg**$(\mathrm{V}_{\mathrm{bi}})$.

Proof. We first show that if $f : (X, \tau_1, \tau_2, R) \to (X', \tau_1', \tau_2', R')$ is a morphism in **PAR** then it is a coalgebra morphism from $(X, \xi_R) \to (X', \xi_{R'})$. For this we simply observe that

$$\begin{aligned}(V(f) \circ \xi_R)(x) &= V(f)(R[x]) \\ &= \mathrm{cl}_1'(f[R[x]]) \cap \mathrm{cl}_2'(f[R[x]]) \\ &= R'[f(x)] \\ &= (\xi_{R'} \circ f)(x).\end{aligned}$$

Conversely if $f : (X, \xi) \to (Y, \chi)$ then we have by a similar argument that f is a **PAR**-morphism from $(X, \tau_1, \tau_2, R^\xi)$ to $(X', \tau_1', \tau_2', R^\chi)$.

It follows from the above that the assignment

$$(X, \tau_1, \tau_2, R) \mapsto ((X, \tau_1, \tau), \xi_R) \quad \text{and} \quad ((X, \tau_1, \tau), \xi) \mapsto (X, \tau_1, \tau_2, R^{\xi})$$

on objects and $f \mapsto f$ on morphism determines functors between **PAR** and **CoAlg**(V_{bi}). We show that these functors are indeed isomorphisms.

Let (ξ, X) be a V_{bi}-coalgebra and let R be a positive accessibility relation on X. We observe that

$$\xi_{R^{\xi}}(x) = R^{\xi}[x] = \{y \in X : y \in \xi(x)\} = \xi(x),$$

for all $x \in X$, and hence $\xi_{R^{\xi}} = \xi$. Moreover we see that for all $x, y \in X$:

$$xR^{\xi_R}y \iff y \in \xi_R(x) \iff y \in R[x] \iff xRy,$$

so $R^{\xi_R} = R$. ■

Theorem 11. *The category* **MPS** *and* **PAR** *are isomorphic.*

Proof. We obtain this immediately from the isomorphism between the categories **PStone** and **Pries**, as one can readily check that R is a positive accessibility relation on the pairwise Stone space (X, τ_1, τ_2) precisely when $(X, \tau_1 \vee \tau_2, \leq_1, R)$ is a Modal Priestley space. ■

Thus we have now esthablished the following theorem

Theorem 12. *The logic* $\mathbf{K}^+$ *is sound and complete with respect to coalgebras for the functor* V_{bi}: **PStone** $\to$ **PStone**.

7 Intuitionistic modal logic

We here briefly touch upon the question of intuitionistic modal logic, which we think arises naturally in the context of this report. For this we return to the setting of Priestley spaces.

Definition 8. An *Esakia space* is a Priestley space $(X, \tau, \leq)$ such that $\downarrow U$ is clopen for every clopen subsets U.

Recall that a map $f : (X, \leq) \to (Y, \leq')$ between partial ordered sets is called a *p-morphism* if f is order preserving and for all $x \in X$ and $y \in Y$:

$$f(x) \leq' y \implies \exists x' \in X \; x \leq x' \wedge f(x') = y.$$

Note that being a p-morphism is equivalent to satisfying the condition that $f[\uparrow x] = \uparrow f(x)$, for all $x \in X$. Moreover the above definition also makes sense for an arbitrary binary relation. The following theorem shows that Esakia spaces are to intuitionistic logic what Stone spaces are to classical logic, i.e. the dual spaces of the algebraic semantics:

Theorem 13 ([9]). *The category* **Esa** *of Esakia spaces and continuous p-morphisms is dually equivalent to the category* **HA** *of Heyting algebras.*

Ideally we would like to extend the coalgebraic techniques so that we can define the category **IMA** of *intuitionitic modal algebras* as the dual of the category of coalgebras for an appropriate functor. The following theorem gives a candidate for such a functor.

Theorem 14. *The functor* V: **Pries** → **Pries** *restricts to an endofunctor on* **Esa**.

Proof. To establish this we need to prove that $(\mathrm{Conv}(X), \tau_v, \leq^{\mathrm{EM}})$ is an Esakia space whenever $(X, \tau, \leq)$ is an Esakia space. And that $V(f)$ is a p-morphism whenever f is.

Therefore let W be a τ_v-clopen set. Then we have that W is a finite intersection of subbasis element, and as $\Box$ distributes over intersection we in fact have that

$$W = \bigcap_{i=1}^{m} \Diamond U_i \cap \bigcap_{j=1}^{n} \Diamond V_j \cap \Box U \cap \Box V,$$

for $U_i, U \in \mathrm{ClpUp}_\tau(X)$ and $V_i, V \in \mathrm{ClpDo}_\tau(X)$.

We show that

$$\downarrow W = \bigcap_{i=1}^{m} \Diamond{\downarrow}(U_i \cap U \cap V) \cap \bigcap_{j=1}^{n} \Diamond{\downarrow}(V_j \cap U \cap V) \cap \Box{\downarrow}(U \cap V)$$

from which it will follow that $\downarrow W$, being an intersection of finitely many clopen, is clopen.

If $F \in \downarrow W$. Then we have $F' \in W$ such that $F \subseteq \downarrow F'$ and $F' \subseteq \uparrow F$. Since $F' \in W$ we must have that

$$F' \cap U_i \neq \varnothing, \quad \text{and} \quad F' \cap V_j \neq \varnothing \quad \text{and} \quad F' \subseteq U \quad \text{and} \quad F' \subseteq V.$$

It follows that for all $i \leq m$ that we have $x_i \in F \cap U_i \cap U \cap V$ and for every $j \leq m$ we have $y_j \in F \cap V_j \cap U \cap V$ so since there for $F' \subseteq \uparrow F$ we must have that

$$F \cap {\downarrow}(U_i \cap U \cap V) \neq \varnothing \quad \text{and} \quad F \cap {\downarrow}(V_j \cap U \cap V) \neq \varnothing,$$

for all $i \leq m$ and all $j \leq n$. As also $F \subseteq U \cap V$ we see that $F \subseteq \downarrow F' \subseteq \downarrow(U \cap V)$. Thus we obtain that $F \in \bigcap_{i=1}^{m} \Diamond{\downarrow}(U_i \cap U \cap V) \cap \bigcap_{j=1}^{n} \Diamond{\downarrow}(V_j \cap U \cap V) \cap \Box{\downarrow}(U \cap V)$, and thereby that

$$\downarrow W \subseteq \bigcap_{i=1}^{m} \Diamond{\downarrow}(U_i \cap U \cap V) \cap \bigcap_{j=1}^{n} \Diamond{\downarrow}(V_j \cap U \cap V) \cap \Box{\downarrow}(U \cap V).$$

If on the other hand

$$F \in \bigcap_{i=1}^{m} \Diamond{\downarrow}(U_i \cap U \cap V) \cap \bigcap_{j=1}^{n} \Diamond{\downarrow}(V_j \cap U \cap V) \cap \Box{\downarrow}(U \cap V).$$

Then we must have that $F \cap {\downarrow}(U_i \cap U \cap V) \neq \varnothing$ for all $i = 1, \ldots m$, and that $F \cap {\downarrow}(V_j \cap U \cap V) \neq \varnothing$ for all $j = 1, \ldots, n$. It follows that ${\uparrow}F \cap U_i \cap U \cap V$ and ${\uparrow}F \cap V_j \cap U \cap V \neq \varnothing$. Now as $F \subseteq {\downarrow}(U \cap V)$ we must have that $F' := {\uparrow}F \cap (U \cap V)$ is non-empty and also that $F \leq^{\mathrm{EM}} F'$. We claim that $F' \in W$, whence from it will follow that $F \in {\downarrow}W$, and thereby that

$$\bigcap_{i=1}^{m} \Diamond{\downarrow}(U_i \cap U \cap V) \cap \bigcap_{j=1}^{n} \Diamond{\downarrow}(V_j \cap U \cap V) \cap \Box{\downarrow}(U \cap V) \subseteq {\downarrow}W.$$

To see that $F' \in W$ we first note that by proposition 1 ${\uparrow}F$ is closed and therefore F' is closed being the intersection of closed set. Moreover as U is an upset and V a downset we have that F' is an intersection of an upset, namely ${\uparrow}F \cap U$, and a downset, namely V, and therefore convex by proposition 6. Thus we have that $F' \in \mathrm{Conv}(X)$. Finally by construction we have that $F' \subseteq U$ and $F' \subseteq V$ and $F' \cap U_i \neq \varnothing$ and $F' \cap V_j \neq \varnothing$ for all $i = 1, \ldots m$ and all $j = 1, \ldots n$. So we indeed have that $F' \in W$.

Finally assume that $f\colon (X, \tau, \leq) \to (Y, \tau', \leq')$ is a continuous continuous p-morphism. Then by proposition 8 we know that $V(f)\colon V(X) \to V(Y)$ is continuous and order preserving and so we only need to show that $V(f)$ is also a p-morphism.

Therefore let $F \in \mathrm{Conv}(X)$ and $G \in \mathrm{Conv}(Y)$ be such that

$$V(f)(F)(\leq')^{\mathrm{EM}} G.$$

Then we must find $F' \in \mathrm{Conv}(X)$ such that $F \leq^{\mathrm{EM}} F'$ and $V(f)(F') = G$. We claim that $F' := {\uparrow}F \cap f^{-1}(G)$ will satisfy these conditions. We first show that $f^{-1}(G) \in \mathrm{Conv}(X)$ from which it will follow that $F' \in \mathrm{Conv}(X)$. As f is a p-morphism we see that

$$f^{-1}({\downarrow}G) = \{x \in X\colon \exists y \in G\ f(x) \leq y\} = \{x \in X\colon \exists x' \in f^{-1}(G)\ x \leq x'\} = {\downarrow}f^{-1}(G).$$

Likewise we obtain that $f^{-1}({\uparrow}G) = {\uparrow}f^{-1}G$. Hence by proposition 6 we must have that

$$f^{1}(G) = f^{-1}({\downarrow}G \cup {\uparrow}G) = f^{-1}({\downarrow}G) \cup f^{-1}({\uparrow}G) = {\downarrow}f^{-1}G \cap {\uparrow}f^{-1}(G).$$

Hence $f^{-1}(G)$ is convex and as f is continuous and G is close in τ' we have that $f^{-1}(G)$ is closed and so $f^{-1}(G) \in \mathrm{Conv}(X)$. Clearly $F' \subseteq {\uparrow}F$. To see that $F \subseteq {\downarrow}F'$, suppose that $x \in F$ then as $f(x) \in V(f)[F]$ we have that $f(x) \leq' y$ for some $y \in G$. Now as f is a p-morphism we know that there exists $x' \in X$ such that $x \leq x'$ and $f(x') = y$. Hence $x' \in F'$ and therefore $x \in {\downarrow}F'$. We have thus shown that $F \leq^{\mathrm{EM}} F'$.

We conclude the proof by showing that $V(f)(F') = G$. For this it suffices to prove that $f[F'] = G$, as G is convex. Evidently we have that $f[F'] \subseteq G$. On the other hand if $y \in G$. Then as $V(f)(F)(\leq')^{\text{EM}}G$ we have $x' \in V(f)(F)$ such that $x' \leq' y$. Now $x' \in V(f)(F)$ implies that $f(x_1) \leq' x'$ for some $x_1 \in F$. It follows that $f(x_1) \leq y$ and so since f is a p-morphism we must have $x_2 \in X$ such that $x_1 \leq x_2$ and $f(x_2) = y$. Hence we have that $x_2 \in \uparrow F \cap f^{-1}(G) = F$ whence y being the image of x_2 under f must belong to the set $f[F']$. ∎

There do not seem to be any consensus in the literature on what intuitionistic modal logic should be. But in light of the above theorem one could argue by way of analogue that since classical modal logic is sound and complete with respect to the coalgebras for the Vietoris functor on the category of Stone space, the coalgebras for the endofunctor $V\colon \mathbf{Esa} \to \mathbf{Esa}$ should determine a sound a complete relational semantics for intuitionistic modal logic - whatever that might be. We refer the reader to [26] and to [20] sec. 3.2. for a nice and brief survey of the different (non-coalgebraic) approaches to intuitionist modal logic. However the problem with saying that intuitionistic modal logic should be determined by $\mathbf{CoAlg}(V)$ is, that since $\mathbf{Esa}$ is not a full subcategory of $\mathbf{Pries}$ it is not clear how to give a characterisation of the relations R on Esakia spaces such that $x \mapsto R[x]$ becomes a coalgebra for $V\colon \mathbf{Esa} \to \mathbf{Esa}$. In the end what we would like is a purely axiomatic description for intuitionistic modal logic, and at present it still seems to be an open problem how to axiomatize the logic with semantics given by the coalgebras for the Vietoris functor on the category of Esakia spaces. Moreover even if we could give such an axiomatization it is not certain that we get a logic that we would like to call intuitionistic modal logic. This is after all a philosophical question. See e.g. [20, 26] for a list of arguably very reasonable desiderata for intuitionistic modal logic. It could very well be that one would have to consider the coalgebras for a functor on a completely different base category to obtain a logic that could be called intuitionistic modal logic.

A General (bi)topological results

Proposition 13. *If (X, τ_1, τ_2) is a bitopological space which is pairwise compact, and $\{F_i\}_{i=1}^n$ is a finite family of subsets of X, with $F_i \in \delta_1 \cup \delta_2$ then $\bigcap_{i=1}^n F_i$ is pairwise compact.*

Proof. It suffices to show the claim for $n = 2$ as the general case then follows from an easy induction argument. Let $F_1 \cap F_2 \subseteq \bigcup_{i \in I} U_i$ be an $\tau_1 \vee \tau_2$ open cover. Then as $F_1^{\text{c}}, F_2^{\text{c}} \in \tau_1 \cup \tau_2$ we have that

$$\bigcup_{i \in I} U_i \cup (F_1 \cap F_2)^{\text{c}} = \bigcup_{i \in I} U_i \cup F_1^{\text{c}} \cup F_2^{\text{c}} \qquad (\clubsuit)$$

is a $\tau_1 \vee \tau_2$-open cover of X, and as X is assumed to be pairwise compact we have a finite subcover of the cover ($\clubsuit$). If $(F_1 \cap F_2)^{\text{c}}$ does not belong to this cover this will also be a finite subcover of $\bigcup_{i\in I} U_i$. If on the other hand $(F_1 \cap F_2)^c$ does belong to this subcover then simply discarding it will yield a finite subcover of $\bigcup_{i\in I} U_i$. ■

It follows from the above proposition that any δ_1 or δ_2 subset of a pairwise compact bitopological space will again be pairwise compact.

Proposition 14. *If $\mathscr{B}_1$ is a subbasis for τ_1 and $\mathscr{B}_2$ is a subbasis for τ_2 then $\mathscr{B}_1 \cup \mathscr{B}_2$ is a subbasis for $\tau_1 \vee \tau_2$.*

Proof. Let τ be the topology generated by the set $\mathscr{B}_1 \cup \mathscr{B}_2$. We show that $\tau = \tau_1 \vee \tau_2$, from which the claim follows.

By definition $\tau_1 \vee \tau_2$ is the least topology containing the topologies τ_1 and τ_2, and as τ_1 is the least topology containing $\mathscr{B}_1$ and τ_2 is the least topology containing $\mathscr{B}_2$ we have that $\tau_1 \vee \tau_2 \supseteq \mathscr{B}_1 \cup \mathscr{B}_2$. So as τ is the least topology containing $\mathscr{B}_1 \cup \mathscr{B}_2$ we must have that $\tau \subseteq \tau_1 \vee \tau_2$.

Conversely as $\tau \supseteq \mathscr{B}_1 \cup \mathscr{B}_2$ we have that $\tau \supseteq \mathscr{B}_1$, whence $\tau \supseteq \tau_1$ as τ_1 is the least topology containing $\mathscr{B}_1$. By a similar argument we see that $\tau \supseteq \tau_2$ and therefore that $\tau \supseteq \tau_1 \cup \tau_2$. So as $\tau_1 \vee \tau_2$ is the least topology which contains the set $\tau_1 \cup \tau_2$ we get that $\tau \supseteq \tau_1 \vee \tau_2$.

We may thus conclude that $\tau = \tau_1 \vee \tau_2$, whence from it follows that $\mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for $\tau_1 \vee \tau_2$. ■

The following classic result in point-set topology, known as the Alexander Subbasis Lemma, can be found most advanced textbooks on topology, e.g [25].

Lemma 8. *Let (X, τ) be a topological space and let $\mathscr{B}$ be a subbasis for τ. Then (X, τ) is compact iff every cover of X by element of the subbasis $\mathscr{B}$ has a finite subcover.*

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