The Raisonnier-Shelah construction of a non-measurable set

Brian Semmes

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1 Introduction

In this paper[1] we will give a proof of the following theorem of S. Shelah [7].

Theorem 1. In Zermelo Fraenkel set theory (ZF) plus the axiom of dependent choice (DC), we can prove that there is a non-measurable set if there is an uncountable well-ordered set of reals.

Shelah's proof uses rather sophisticated meta-mathematical arguments that may not be accessible to the general mathematician. As our principle goal is to reach a relatively wide audience, we will use the ideas of J. Raisonnier [5], who has given a simpler and less meta-mathematical proof of Theorem 1. However we will not follow Raisonnier's proof exactly. We will make some simplifications to Raisonnier's arguments and we will also follow to a certain

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extent the exposition of Raisonnier's proof presented in M. Bekkali's lecture notes [1] for a seminar taught by S. Todorcevic.

Although this paper isn't as extensive as the other papers on the subject, it has the advantage of being more or less self-contained for those who are familiar with measure theory on the level of J. Oxtoby's book *Measure and Category*. However the theorem does have important implications in set theory; for those readers with a sufficient background we provide a brief summary here.

We begin by noting the result of Solovay [6], who constructed a model of $ZF+DC+LM$ given a model of $ZFC+I$, where LM is the statement "every set of reals is Lebesgue measurable" and I is the statement "there exists an inaccessible cardinal". The natural question posed by Solovay is whether this large cardinal assumption can be omitted; i.e. whether $ZF+DC+LM$ is equiconsistent with ZFC. The surprising answer, due to Shelah, is "no". Shelah proved the following theorem.

Theorem 2. Assume $ZF+DC+LM$. Then $\aleph_1$ is an inaccessible cardinal in the constructible universe L. (Actually Shelah used a weaker assumption, namely $ZF+DC+$ "every Σ^1_3 set of reals is measurable". But in this paper we will not consider such complexity issues.)

Theorems 1 and 2 are closely related. In fact the proof Shelah gives of Theorem 1 is only a short list of modifications to be made to his proof of Theorem 2. We will explicitly derive Theorem 2 from Theorem 1 below.

From Theorem 2 we may infer that ZFC and $ZF+DC+LM$ are not equiconsistent. Let $\vdash$ be provability over ZFC and assume for contradiction that $Con(ZFC) \vdash Con(ZF+DC+LM)$. By Theorem 2 we have $Con(ZFC) \vdash Con(ZFC+I)$. Recall that from $ZFC+I$ we may derive $Con(ZFC)$. Indeed if κ is an inaccessible then V_κ, the κ-th stage of the cumulative hierarchy, is a model of ZFC. We have shown modulo our assumption that $Con(ZFC) \vdash Con(ZFC+Con(ZFC))$. By Gödel's second incompleteness theorem if our assumption were true then $ZFC + Con(ZFC)$ would be inconsistent, and therefore ZF would be inconsistent as well. Therefore we may "safely" conclude that our assumption is false, namely we may conclude that ZFC is not equiconsistent with $ZF + DC + LM$.

It is immediate from Theorem 2 that $ZFC + I$ and $ZF + DC + LM$ are equiconsistent. One direction is already given to us by Solovay's result. For the other direction, by Theorem 2 we can restrict any model of $ZF{+}DC{+}LM$ to its constructible part to obtain a model of $ZFC + I$.

The reader may be curious as to whether the analogs of these results hold for sets with the Baire property. Indeed, in many instances the notions of measurability and the Baire property play similar roles. However this is not the case here, as it is a known fact that $ZF + DC +$ "every set of reals has the Baire property" is equiconsistent with ZF. Consequently we may not replace the statement "every set of reals is measurable" with the statement "every set of reals has the Baire property" in the hypothesis of Theorem 2.

We finish the introduction with a proof of Theorem 2, using Theorem 1.

Proof of Theorem 2. First we show that $\aleph_1^{L[a]} < \aleph_1$ for all $a \in 2^\omega$. Suppose not, then there is an $a \in 2^\omega$ such that $\aleph_1^{L[a]} = \aleph_1$. Then $2^\omega \cap L[a]$ has cardinality $\aleph_1$. By Theorem 1 there is a non-measurable set of reals, contradicting our assumption.

Now we show that $\aleph_1$ is inaccessible in L. Let $\kappa = \aleph_1$. It is easy to see that $cf(\kappa) = \kappa$ in L. It remains to be shown that $\lambda < \kappa$ implies $|P(\lambda)|^L < \kappa$. Let $\lambda < \kappa$. As λ is a countable ordinal, a well-ordering of length λ is coded by some $a \in 2^\omega$. It follows that λ is countable in $L[a]$ so $\lambda < \aleph_1^{L[a]}$. Since GCH holds in the constructible universe we have that $|P(\lambda)|^{L[a]} = \aleph_1^{L[a]}$. Now if $|P(\lambda)|^L \geq \kappa$ then $\aleph_1^{L[a]} = |P(\lambda)|^{[L(a)]} \geq |P(\lambda)|^L \geq \kappa$, a contradiction. Therefore $|P(\lambda)|^L < \kappa$ as desired. $\square$

2 Notation and Preliminaries

As stated in the introduction, we do not require any special knowledge of set theory. If the reader is so inclined, he may consult any introductory book on set theory for a review of the axioms of ZF. The axiom of dependent choice (DC), which we will use implicitly, is a "much" weaker version of the axiom of choice. Indeed by the result of Solovay, DC is consistent with "every set of reals is Lebesgue measurable".

Axiom of Dependent Choice (DC). Let A be a non-empty set and let R be a binary relation on A. Furthermore assume for all $x \in A$ there is a $y \in A$ such that xRy. Then there is a function $f : \omega \to A$ such that $f(n)Rf(n+1)$ for all $n \in w$.

We continue with notation. For $u \in 2^{<\omega}$ let $[u]$ denote the corresponding basic open set of 2^ω. For $\breve{A} \subset 2^{<\omega}$ let $A \subset 2^\omega$ denote the open set $\bigcup_{u \in \breve{A}} [u]$. For $a \in 2^\omega$ and $n \in \omega$ let a_n denote the finite sequence $a \restriction n$. For u, $v \in 2^{<\omega}$ let $u \cdot v$ denote u concatenated with v. Similarly define $u \cdot a$ for $a \in 2^\omega$ and $u \in 2^{<\omega}$. We may view the elements of 2^ω as subsets of ω. Namely for $a \in 2^\omega$, $n \in a$ is synonymous with $a(n) = 1$. For $A \subset 2^\omega$ let A^c denote the complement of A in 2^ω. Let $X \bigtriangleup Y$ denote the symmetric difference of any two sets X and Y.

We assume that the reader is familiar with Lebesgue measure. For a measurable set $A \subset 2^\omega$ let $\mu(A)$ denote the measure of A. For $u \in 2^{<\omega}$ and $A \subset 2^\omega$ let

$$\mu_u(A) = \frac{\mu(A \cap [u])}{\mu([u])}.$$

For $A \subset 2^\omega$, $a \in 2^\omega$, and $d \in [0, 1]$, if A is measurable then we say that A has *density* d at a if

$$\lim_{n \to \infty} \mu_{a_n}(A)$$

exists and is equal to d. Let $\Phi(A) = \{a : A \text{ has density } 1 \text{ at } a\}$.

Fact 1 (Lebesgue Density Theorem). For any measurable set A, we have that $\mu(A \triangle \Phi(A)) = 0$.

Proof. A proof of this fact is found in [4], or in any other textbook on measure theory. $\square$

Let $S : 2^\omega \to 2^\omega$ denote the shift function, i.e. if $b = S(a)$ then $b(n) = a(n+1)$ for all n. Furthermore let $S^{(k)}$ denote k repeated applications of S. We are now ready to define the notion of a tail set. We say that $A \subset 2^\omega$ is a *tail set* if $a \in A$ and $S^{(n)}(a) = S^{(n)}(b)$ for some n implies $b \in A$.

Fact 2 (Kolmogorov Zero-One Law). If A is a measurable tail set then $\mu(A) = 1$ or $\mu(A) = 0$.

Proof. Assume $\mu(A) < 1$. Then there is an open set $U \supset A$ such that $\mu(U) < 1$. By the density theorem there is an a such that U has density 0 at a. Let $\epsilon > 0$. Then there is an n such that $\mu_{a_n}(U) < \epsilon$. Let

$$U_\epsilon = \{u \cdot c \ : \ a_n \cdot c \in U \text{ where } u \in 2^n \text{ and } c \in 2^\omega\}.$$

So $\mu(U_\epsilon) = 2^n \mu(U \cap [a_n]) = \mu_{a_n}(U) < \epsilon$. It will be shown that $A \subset U_\epsilon$. Let $b \in A$. Then $a_n \cdot S^{(n)}(b) \in A$ since A is a tail set. Therefore $a_n \cdot S^{(n)}(b) \in U$ and $b \in U_\epsilon$. $\square$

For $a, b \in 2^\omega$ let $a \oplus b \in 2^\omega$ be defined by $(a \oplus b)(2n) = a(n)$ and $(a \oplus b)(2n+1) = b(n)$. In other words $a \oplus b$ uses a for its even entries and b for its odd entries.

Fact 3 (Fubini's Theorem). We state Fubini's Theorem in the following context. Let A be a null set. Then both the set of a such that $\mu(\{b : a \oplus b \in A\}) > 0$ and the set of b such that $\mu(\{a : a \oplus b \in A\}) > 0$ are null.

Proof. A proof of Fubini's Theorem can be found in [4]. $\square$

Fact 4. Assume every set of reals is measurable and let $\{A_\alpha\}_{\alpha \in \aleph_1}$ be a sequence in which each $A_\alpha \subset 2^\omega$ is null. Then $\bigcup_{\alpha \in \aleph_1} A_\alpha$ is also null.

Proof. We begin by noting that any null set A can be enlarged to a null tail set. Indeed for $n \in \omega$ and $s \in 2^n$ let $f_s(a) = s \cdot S^{(n)}(a)$. Then it is easy to see that $\bigcup_{s \in 2^{<\omega}} f_s[A]$ is a null tail set. Therefore we may assume that each A_α is a tail set, so we have that $\bigcup_{\alpha \in \aleph_1} A_\alpha$ is also a tail set. Moreover we may also assume that the A_α's are pairwise disjoint. If they are not we simply use the sets $A_\alpha - \bigcup_{\lambda \in \alpha} A_\lambda$. To show that $\bigcup_{\alpha \in \aleph_1} A_\alpha$ is null, by the Zero-One Law it suffices to show that it does not have measure 1. So assume for contradiction that $\bigcup_{\alpha \in \aleph_1} A_\alpha$ has measure 1. Let

$$B = \{a \oplus b : \exists \alpha, \beta \ \alpha < \beta, \ a \in A_\alpha, \text{ and } b \in A_\beta\}.$$

It is easy to check that B is a tail set. It will be shown that B cannot have measure 0 or 1, contradicting the Zero-One Law. Let $a \in \bigcup_{\alpha \in \aleph_1} A_\alpha$ so $a \in A_\alpha$ for some $\alpha \in \aleph_1$. Because $\mu(\bigcup_{\lambda \leq \alpha} A_\lambda) = 0$ we have that $\mu(\bigcup_{\lambda > \alpha} A_\lambda)$ $= \mu(\{b : a \oplus b \in B\}) = 1$. We have shown that

$$\forall a \in \bigcup_{\alpha \in \aleph_1} A_\alpha \quad \mu(\{b : a \oplus b \in B\}) = 1.$$

By Fubini's Theorem if $\mu(B) = 0$ then the set of a such that $\mu(\{b : a \oplus b \in B\})$ > 0 is null. Therefore B cannot have measure 0. In a similar manner it can be shown that

$$\forall b \in \bigcup_{\alpha \in \aleph_1} A_\alpha \quad \mu(\{a : a \oplus b \in B\}) = 0.$$

Taking the complement we get

$$\forall b \in \bigcup_{\alpha \in \aleph_1} A_\alpha \quad \mu(\{a : a \oplus b \in B^c\}) = 1.$$

By Fubini's Theorem B^c cannot have measure 0. Therefore B cannot have measure 1, contradicting the Zero-One Law applied to B. We have derived a contradiction by assuming that $\bigcup_{\alpha \in \aleph_1} A_\alpha$ has measure 1. By the Zero-One Law it has measure 0. $\square$

3 Rapid Filters and Non-Measurable Sets

We now continue with the main theorem of the paper. Let us state it again in a more convenient form.

Theorem 1. In $ZF + DC$ if there is a $K \subset 2^\omega$ with order type $\aleph_1$ then there is a non-measurable set.

Classical constructions of non-measurable sets typically use either the axiom of choice or the existence of a non-trivial ultrafilter. Since the existence of a non-trivial ultrafilter is not provable under the assumptions of Theorem 1, these constructions are not available to us. However in place of ultrafilters we may use rapid filters, a notion introduced by G. Mokobodski [3] and defined below. We prove two lemmas from which Theorem 1 follows immediately.

Lemma 1, due to M. Talagrand [8], says that rapid filters are non-measurable. Actually Talagrand proved some additional results in this vein, namely that rapid filters do not have the Baire property and are not K_σ-regular. Additionally M. Mathias [2] proved that rapid filters can be transformed into non-Ramsey sets. These results we mention only in passing as they will not be used in the proof.

It is not difficult to prove that the existence of rapid filters follows from the axiom of choice and either the continuum hypothesis or Martin's axiom. Lemma 2, due to Raisonnier, says something much less intuitive; to have rapid filters we need only assume the existence of an uncountable well-ordered set of reals and that the union of any sequence of $\aleph_1$ null sets is null. In the proof of Lemma 2 we will make some simplifications to Raisonnier's arguments and also take inspiration from the exposition of Raisonnier's proof in [1].

Theorem 1 follows immediately from Lemmas 1 and 2. Indeed if every set of reals were measurable then by Fact 4 we would have that the union of $\aleph_1$ null sets is null. Then from the two lemmas we get a contradiction.

Proof of Theorem 1. We begin with the proof that rapid filters are non-measurable.

A filter $F \subset 2^\omega$ is *rapid* if for every increasing sequence $\{m_k\}_{k\in\omega}$ there is an $a \in F$ such that $|a \cap m_k| \leq k$ for each k. Here we are identifying a natural number with the set of its predecessors.

Lemma 1. Rapid filters are non-measurable.

Proof. The proof we give is slightly different from that of Talagrand. We begin by showing that a measurable filter F cannot have measure 1. Let $O : 2^\omega \to 2^\omega$ be defined by $O(a) = \omega - a$. In other words, O is the operation on infinite sequences that flips the bits. Note that O is measure-preserving and $F \cap O[F] = \emptyset$. It follows that $\mu(F) \neq 1$ since otherwise $\mu(F) + \mu(O[F]) > 1$.

Let F be a rapid filter. It will be shown that F meets every closed set of positive measure. Then $\mu(F) = 1$ if F is measurable since otherwise $\mu(F^c) > 0$ implies there is a closed set of positive measure contained in F^c. This contradicts what we have shown above.

Let A be a closed set of positive measure. It will be shown that $A \cap F \neq \emptyset$. A sequence $\{\breve{T}_i\}_{i\in\omega}$ will be defined in which each $\breve{T}_i \subset 2^{<\omega}$ is finite and the length of the longest element of $\breve{T}_i$ is less than the length of the shortest element of $\breve{T}_{i+1}$. Each $\breve{T}_i$ will have the following additional properties, where n_i is the length of the longest element in $\breve{T}_i$. (Recall that for $\breve{A} \subset 2^{<\omega}$ we use $A \subset 2^\omega$ to denote the open set $\bigcup_{u\in\breve{A}}[u]$.)

1. $\mu(A - T_i) < 2^{-(n_{i-1}+i+1)}$
2. $u \in \breve{T}_i \Rightarrow \mu_u(A) > 1 - 2^{-(i+2)}$

We proceed now with the construction of $\{\breve{T}_i\}_{i\in\omega}$. Given n_{i-1}, the length of the longest string in $\breve{T}_{i-1}$, we define $\breve{T}_i$. If $i = 0$ then we take $n_{i-1} = 0$. Let

$$\breve{T} = \{u \ : \ |u| > n_{i-1} \text{ and } \mu_u(A) > 1 - 2^{-(i+2)}\}.$$

By the density theorem $\mu(A - \Phi(A)) = 0$. Since $\Phi(A) \subset T$ it follows that $A - T \subset A - \Phi(A)$ and therefore $\mu(A - T) = 0$. Let $\breve{T}_i \subset \breve{T}$ be finite such that

$\mu(T-T_i) < 2^{-(n_{i-1}+i+2)}$. Then $\mu(A-T_i) = \mu(A-T)+\mu(T-T_i) = \mu(T-T_i)$. This completes the construction of $\{\breve{T}_i\}_{i\in\omega}$.

Next a sequence $\{s_i\}_{i\in\omega}$ of elements of $2^{<\omega}$ will be constructed such that $s_i \subset s_{i+1}$ and $s_i \in \breve{T}_i$ for each i. Then we will show that the limit of this sequence is in $A \cap F$. Assume s_i has already been defined. Since F is rapid there is an $a \in F$ such that $|a \cap n_i| \leq i$ for each i. Let

$$H_i = \{c \in 2^\omega \ : \ c(n) = 1 \text{ for each } n \in a \text{ s.t. } |s_i| \leq n < n_{i+1}\}.$$

Since a contains at most $i+1$ elements $< n_{i+1}$ and H_i is independent of $[s_i]$ we have that $\mu_{s_i}(H_i) \geq 2^{-(i+1)}$. It is immediate from property 2 of the $\breve{T}_i$ that $\mu_{s_i}(A^c) < 2^{-(i+2)}$. It follows that

$$\begin{aligned}
\mu_{s_i}(H_i \cap A) &= \mu_{s_i}(H_i) - \mu_{s_i}(H_i \cap A^c) \\
&\geq \mu_{s_i}(H_i) - \mu_{s_i}(A^c) \\
&\geq 2^{-(i+1)} - 2^{-(i+2)} \\
&= 2^{-(i+2)}.
\end{aligned}$$

Therefore $\mu([s_i] \cap H_i \cap A) \geq 2^{-(n_i+i+2)}$ and T_{i+1} intersects $[s_i] \cap H_i \cap A$. Choose any $s_{i+1} \in \breve{T}_{i+1}$ such that $[s_{i+1}] \cap [s_i] \cap H_i \cap A$ is non-empty.

Let b be the limit of $\{s_i\}_{i\in\omega}$. Since A is closed and b is a limit point of A it follows that $b \in A$. Now we show that $b \supset a$. Let $n \in a$ and let i be such that $|s_i| \leq n < s_{i+1}$. By definition $c(n) = 1$ for each c in H_i. Since $[s_{i+1}] \cap H_i \neq \emptyset$ it follows that $s_{i+1}(n) = 1$ and therefore that $n \in b$. This shows that $b \supset a$ and therefore $b \in F$.

We have shown that F meets every closed set A of positive measure. It follows that $\mu(F) = 1$, a contradiction. Therefore a measurable rapid filter cannot exist. This completes the proof of Lemma 1. $\square$

Lemma 2. If there is a $K \subset 2^\omega$ with order type $\aleph_1$, and the union of $\aleph_1$ null sets is null, then a rapid filter exists.

Proof. Before we construct the rapid filter, we first give some definitions. Recall that we use a_m to denote $a \in 2^\omega$ truncated to the first m bits.

- For $a \in 2^\omega$ and $W \subset 2^{<\omega}$ say that W *captures* a if $\exists n\, \forall m \geq n\ a_m \in W$.
- W *captures* $K \subset 2^\omega$ if W captures each $a \in K$.
- W *splits on* n if there is an $s \in W$ of length n such that $s0, s1 \in W$.
- Let $W^{\text{at } k} = \{s \in W \ :\ s \text{ has length } k\}$.

Now we are ready to define F, using the K given to us by hypothesis.

$$a \in F \quad \Leftrightarrow \quad \text{there is a } W \subset 2^{<\omega} \text{ that captures } K \text{ and splits only on elements of } a.$$

We will show that F is rapid, but first we must show that F is a filter.

Lemma 2.1. *F is a filter.*

Proof. It is clear that $a \in F$ and $b \supset a$ implies $b \in F$, and that F is non-empty. We show that $a, b \in F$ implies $a \cap b \in F$. By assumption we have W_1 and W_2 such that W_1 splits only on elements of a, W_2 splits only on elements of b, and both W_1 and W_2 capture K. Then $W_1 \cap W_2$ splits only on elements of $a \cap b$ and $W_1 \cap W_2$ captures K. This shows that $a \cap b \in F$.

We also must show $\emptyset \notin F$. In other words we must show that any W that never splits cannot capture K. This follows easily from the fact that K is uncountable. Indeed if W never splits then any $s \in W$ can contribute to the capturing of at most one sequence. Therefore W can capture at most a countable set. □

To show that F is rapid we will show for any increasing sequence $\{m_k\}_{k\in\omega}$ there is a $W \subset 2^{<\omega}$ that captures K and $|W^{\text{at } m_k}| \leq k$ for all k. Then the result will follow from the following lemma.

Lemma 2.2. Suppose for any increasing sequence $\{m_k\}_{k\in\omega}$ there is a $W \subset 2^{<\omega}$ that captures K and $|W^{\text{at } m_k}| \leq k$ for all k. Then F is rapid.

Proof. Let $\{m_k\}_{k\in\omega}$ be an increasing sequence and let W be given by hypothesis. Let $a = \{k : W \text{ splits on } k\}$, so $a \in F$. We may assume that $u \in W^{\text{at } k}$ implies there is a $v \in W^{\text{at } k+1}$ such that $u \subset v$. Indeed removing any u that doesn't satisfy this property will not affect whether W captures K. It follows that W can only split on k if $|W^{\text{at } k}| < |W^{\text{at } k+1}|$. Therefore $|a \cap m_k| \leq k$ for each k. □

It remains to show the hypothesis of Lemma 2.2. This task is actually the most essential part of this paper; in this sense the work up to this point may be viewed as preliminary.

Lemma 2.3. Suppose that the union of any sequence of $\aleph_1$ null sets is null. Then for any increasing sequence $\{m_k\}_{k\in\omega}$ there is a $W \subset 2^{<\omega}$ that captures K and $|W^{\text{at } m_k}| \leq k$ for all k.

Proof. For $k \in \omega$ and $s \in 2^{<\omega}$ we define $\breve{B}_{k,s} \subset 2^{<\omega}$ such that for any $v \in 2^{<\omega}$ and any $k > |v|$ we have that $\mu_v(B_{k,s}) = 2^{-k}$ and $\{B_{k,s}\}_{s\in 2^{<\omega}}$ are μ_v-independent. Any choice of $\breve{B}_{k,s}$ that satisfies these two requirements will suffice, e.g.

$$\breve{B}_{k,s} = \{u \cdot 1^k : |u| = k \cdot z(s)\}$$

where z is a bijection between $\omega - \{0\}$ and $2^{<\omega}$. Note that for finite $A \subset 2^{<\omega}$ and $k > |v|$ we have

$$\mu_v(\bigcup_{s\in A} B_{k,s}) = 1 - (1 - 2^{-k})^{|A|}.$$

If $|A| \geq 2^k$ it follows that

$$\begin{aligned}(1 - 2^{-k})^{|A|} &\leq (1 - 2^{-k})^{2^k} \\ &\leq (e^{-2^{-k}})^{2^k} \\ &= e^{-1} \\ &< 1/2.\end{aligned}$$

Hence if $k > |v|$ then $\mu_v(\bigcup_{s\in A} B_{k,s}) > 1/2$ for each A with cardinality at least 2^k. To simplify the notation, for $a \in K$ we use $\sigma(k)$ to stand for $a_{m_{2^{2k}}}$. Now for $a \in K$ let

$$U_n^a = \bigcup_{k>n} B_{k,\sigma(k+1)}.$$

Note that $\mu(U_n^a) \leq 2^{-n}$. It follows that $\bigcap_{n\in\omega} U_n^a$ is a null set. By assumption we have that $\bigcup_{a\in K}\bigcap_{n\in\omega} U_n^a$ is null. Let V be an open set of measure $< 1/2$ containing $\bigcup_{a\in K}\bigcap_{n\in\omega} U_n^a$.

For $v \in 2^{<\omega}$ and $k \in \omega$ we define $A(v,k)$ to be the set of those s such that $\mu_v(B_{k,s} - V) = 0$ and $\mu_v(V) < 1/2$. Note that $A(v,k) = \emptyset$ if $\mu_v(V) \geq 1/2$. We have that

$$\begin{aligned}\sum_{s\in A(v,k)} \mu_v(B_{k,s} - V) \;&\geq\; \mu_v \bigcup_{s\in A(v,k)} (B_{k,s} - V)\\ &>\; \mu_v \bigcup_{s\in A(v,k)} B_{k,s} \;-\; 1/2.\end{aligned}$$

From our previous observation it follows that $|A(v,k)| < 2^k$ if $k > |v|$. Otherwise $\mu_v(\bigcup_{s\in A(v,k)} B_{k,s}) > 1/2$ and $\sum_{s\in A(v,k)} \mu_v(B_{k,s} - V) > 0$, contradicting our assumption. Define

$$W = \{s \in 2^{<\omega} : \exists k \exists v(|v| < k, s \in A(v,k), \text{ and } |s| = m_{2^{2(k+1)}})\}.$$

Note that this W only has strings of length $m_{2^{2(k+1)}}$ for each $k \in W$. Since we wish for W to capture K we must fill in the gaps, so to speak. We do this in a straightforward way. For any $k \in \omega$ if there is an $s \in W$ of length $m_{2^{2(k+1)}}$ and a $t \in W$ of length $m_{2^{2(k+2)}}$ that extends s, then we add to W all intermediate sequences that lie between s and t. It is easy to check that this new W satisfies $|W^{\text{at } m_k}| < k$ for all k. To show that W captures K we first prove the following claim.

Claim. For every $a \in K$ there is an $n \in \omega$ and a $v \in 2^{<\omega}$ such that $\mu_v(V) < 1/2$ and $\mu_v(U_n^a - V) = 0$.

Proof. Suppose not, then there is an a such that $\mu_v(V) \geq 1/2$ or $\mu_v(U_n^a - V) > 0$ for every n and v. We construct $\{s_n\}_{n\in\omega}$ inductively to satisfy $\mu_{s_n}(V) < 1/2$ and $[s_{n+1}] \subset U_n^a$ for all n. Let $s_0 = \emptyset$ and assume s_n is given such that $\mu_{s_n}(V) < 1/2$. Then $\mu_{s_n}(U_n^a - V) > 0$. Since U_n^a is open there is a $u \supset s_n$ such that $[u] \subset U_n^a$ and $\mu_{s_n}([u] - V) > 0$. By the density lemma there is a $b \in ([u] - V) \cap \Phi([u] - V)$. Hence there is a $t \supset u$ such that $\mu_t(V) < 1/2$. Let $s_{n+1} = t$. This completes the construction of $\{s_n\}_{n\in\omega}$.

Let c be the limit of $\{s_n\}_{n\in\omega}$. Since $[s_{n+1}] \subset U_n^a$ for each n it follows that $c \in \bigcap_{n\in\omega} U_n^a \subset V$. But $[s_n] \not\subset V$ for all n, contradicting V open. $\square$

Now we are ready to complete the proof of Lemma 2.3 by showing that W captures K. Let $a \in K$, let n and v be given by the claim, and let $k > |v|, n$. By the way W was defined (namely since we have filled the gaps) it suffices to show that $\sigma(k+1) \in W$, in other words that $\mu_v(B_{k,\sigma(k+1)} - V) = 0$. But this follows easily from the fact that $B_{k,\sigma(k+1)} \subset U_n^a$. $\square$

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