# Three recursion theoretic concepts of genericity

Walter Dean *†‡

November 23, 1999

**Abstract**

This paper develops and contrasts three recursion-theoretic notions of genericity : one for sets below $\emptyset'$ and two for the recursive enumerable (r.e.) sets. The first is the classical notion of 1-genericity, developed in the context of the context of the finite extension method first developed by Kleene and Post [4] to study degrees below $\emptyset'$. We review several established facts about 1-generic sets showing that they are bi-immune, generalized low and have Turing incomparable even and odd halves. We then generalize the definition of 1-genericity to obtain a non-effective version of a result of Sacks which states that for any r.e. non-recursive $C$ there is a simple set $A$ such that $C \leq_T A$. We next develop a new notion of genericity based on the Ellentuck-Mathias [EM] topology on $2^\omega$. EM-generic sets are shown to be simple and have Turing incomparable even and odd halves. Finally we introduce and generalize a concept of Jockusch [3] known as e-genericity and show how it can be modified so as to obtain the full result of Sacks stated above.

## 1 Introduction

In this paper we consider several notions of genericity applicable to recursion theory. The application of the concept of genericity to the study of the Turing degrees seeks to unify results originally obtained through individual injury arguments into more general constructions of sets which simultaneously possess

*wdean@ruccs.rutgers.edu / wdean@wins.uva.nl

†This paper is submitted in partial fulfillment of the requirements for the MSc degree from the Institute for Logic Language and Computation, University of Amsterdam.

‡I gratefully acknowledge the following parties for helping to bring about the conjunction of conditions which made this work possible. First and foremost, thanks go to Domenico Zambella without whose patience and unflagging taste for clarity, the results herein would be both fewer in number and significantly more muddled in presentation. My stay in the Netherlands has been made possible both by a grant from the Netherlands America / Fulbright Foundations and a Spinoza Fellowship from the ILLC. I would also like to acknowledge the moral and logistical support I received Susan Schweitzer and from Robert Matthews, Brian Loar and Ginny Mayer at my home institution, Rutgers University. Finally thanks are owed to Clemens Grabmayer for many hours of stimulating conversation on both mathematical and non-mathematical subjects.

several properties of interest. There are a number of motivations for taking this approach: a) to better understand the common aspects of distinct injury arguments; b) to demonstrate that there is a single injury construction which is universal in the sense that it can yield a set possessing multiple properties of interest to the study of the r.e. degrees; and c) to deepen our understanding of the relationship between the priority method and forcing method from which the concept of genericity was derived. Here we will introduce a new r.e. genericity concept as well as reviewing and expanding upon two existing approaches familiar from the literature.

Genericity concepts were first employed in recursion theory when several analogies between priority methods and the forcing method in set theory were noted. In set theoretic forcing (cf. [2], [5]), a Cohen generic set may intuitively be thought of as possessing all of the properties which are "sufficiently simple" so as be recognizably dense in their extension within a countable ground model $M$ of $ZFC$. This is to say that in generic extension $M[G]$ over the ground model $M$ via a $M$-generic set $G$, we ultimately obtain $M[G] \models \varphi$ just in case $\forall q \exists q \leq p[q \Vdash \varphi]$ and this fact can be recognized in $M$. Intuitively, if we were to think of constructing $G$ "by hand" in countably many stages $s$, $G$ will be constructed so that $M[G]$ ultimately satisfies $\varphi$, just in case for every stage $s$, there exists a stage $t$ such that $t > s$ and $t \in G \rightarrow M[G] \models \varphi$.

A priority or finite extension construction of a set $A \subset \omega$ can also be interpreted as a process by which an object is constructed so as to possess a pre-specified list of properties, each of which can be satisfied by actions taken arbitrarily late in the construction. Perhaps the simplest example of the such a construction is that of Post's simple set $S$ (recall that an r.e. set $A$ is simple just in case $|\overline{A}| = \omega$ and $\forall e[|W_e| = \omega \rightarrow A \cap W_e \neq \emptyset]$). In this case, we wish to construct $S$ such that it has the property of intersecting the $e^{th}$ r.e. set but at the same time ensuring that for all $e$, $|\overline{A}| > e$. Since we are only concerned with intersecting *infinite* $W_e$, the former requirements can alway be satisfied by waiting for sufficiently large elements to enter each $W_e$ which may then be added to $A$ without filling up its complement. A similar task is to construct an r.e. set $A$ whose characteristic function can be primitive recursively be split into disjoint components $B$ and $C$ of incomparable Turing degree. In the course of this construction we require that the requirements $\exists x[\chi_A(s) \neq \varphi_e^B(x)]$ and $\exists x[\chi_B(s) \neq \varphi_e^A(x)]$ are made true, again without filling up the complement of either $A$ or $B$.

The study of genericity in recursion theory has passed through two distinct phases. During the first, it was noted that a topological interpretation could be given of the so-called finite extension method in degree theory (cf. [7]) . The resulting concept of genericity, usually known as **1-genericity**, proved to be a useful tool for studying the degrees below $\emptyset'$. However, no 1-generic object is r.e. and hence this method cannot be used to produce results about the structure of the r.e. degrees, such as a positive solution to Post's problem.

The second phase has been aimed at effectivizing the former sort of construction so as yield r.e. generic objects. In [6], Maass introduced the concept the notion of **r.e. genericity**, which amounts to genericity over models very small models of weak set theoretic axioms such as the herditarily finite sets. Maass showed that an r.e. generic object $A$ was r.e. low (i.e. $A' \leq_T \emptyset'$) and could be split into even and odd halves with incomparable Turing degree. He also showed that all r.e. generic degrees are automorphic in $\mathcal{E}^*$, the lattice of r.e. sets by inclusion modulo a finite set.

Despite these positive findings, Jockusch [3] points out two disadvantages with the r.e. generic sets. First, the definition of r.e. genericity is relative to a particular indexing of the formulas of a fixed formal language and is it is easily seen that no r.e. set can be r.e. generic with respect to all such indexings. Second, the definition of r.e. generic is cumbersome to work with and obscures common aspects of constructions involving r.e. generic sets. As a remedy, Jockusch defines another notion of genericity known as *e-genericity*. e-generic sets are also low, promptly simple and have Turing incomparable even and odd halves. Their definition, however, is independent of a fixed formal language. Perhaps more importantly, the construction of a single e-generic shows how a single finite injury construction can be used to derive an object which has most of the properties which can be ensured via basic injury constructions.

In this paper we proceed as follows. In section 2) we review the concept of 1-genericity. We show that 1-generic object possess non-effective analogues of a number of properties originally constructed by the finite extension method. We will also show how the concept of 1-genericity can be strengthened so as derive a non-effective version of a result of Sacks' which states that for every r.e. non-recursive set $C$, there is a simple set $A$ such that $C \not\leq_T A$. In section 3) we develop a new concept of genericity based on the so-called Ellentuck-Mathias topology on $2^\omega$. We will show that Ellentuck Mathias sets are simple, have and Turing incompatible halves. Finally, in section 4) we will present a modified form of Jockusch's definition of e-genericity and show how it can be strengthened so as to reproduce the full result of Sacks mentioned above.

## 2 1-genericity

In this section, we examine the 1-generic sets. The study of 1-generic sets arose out of an examination of the finite extension method for studying the structure of the r.e. degrees which was first employed by Kleene and Post [4] in order to construct incomparable r.e. degrees below $\emptyset'$. In applications of this technique, a set $A$ is constructed by building increasing longer approximations to its characteristic function, $\chi_A$, so as to meet a (typically infinite) list of conditions. Henceforth we adhere to a standard abuse of notation and confuse

$A$ with its characteristic function writing, e.g., $\sigma \subset A$ to mean that $\sigma \subset \chi_A$. The conditions we wish $A$ to satisfy will be finitary in the sense that they can be obtained by ensuring that some initial segment of $A$ has a prescribed property. Since once a bit of $\sigma \subset A$ is decided upon, it cannot be changed, our only hope of ensuring that $A$ will satisfy all the conditions we demand is that the extension of each condition be sufficiently dense in $2^\omega$ so as to satisfiable by some $\tau \supseteq \sigma$, regardless of the earlier steps encoded in $\sigma$. These observations suggest the following topological approach to the finite extension method.

**Definition 2.1** Let $\Omega$ be a topological space, $P$ a basis for $\Omega$, $U \subseteq \Omega$ and $X \in \Omega$. We say that $X$ **decides** $U$ **positively** or that $X$ **meets** $U$ if there is a basic open set set $u \in P$ such that $X \in u$ and $u \subset U$. Conversely, $X$ **decides** $U$ **negatively** if there is a basic open set $u \in P$ which isolates $X$ from $U$ – i.e. $X \in u$ and $u \cap U = \emptyset$. We will say $X$ **decides** $U$ just in case it either decides $U$ positively or decides $U$ negatively. A set $U \subseteq \Omega$ is **dense** just in case it intersects every non-empty $u \in P$. Finally, we say that $U$ is **dense along** a set $X$ if $U$ intersects every $u \in P$ containing $X$.

**Remark** Note that if $X$ is dense along $U$ then $U$ cannot decide $X$ negatively, as otherwise there would exist some $u$ such that $X \in u$ and $u \cap U = \emptyset$. Conversely, if $U$ does not decide $X$ negatively, then $U$ is dense along $X$.

**Definition 2.2** By $2^\omega$ we denote the **Cantor space**, i.e. the set of all total functions $\{f \mid f : \omega \to \{0,1\}\}$. Note that to each such $f$ is associated a set $X \subseteq \omega$ such that $X = \{x \mid f^{-1}(x) = 1\}$. We denote initial initial segments of such total functions by lower case Greek letters $\rho$, $\sigma$, $\tau$, $\ldots$ and the set of such initial functions by $2^{<\omega}$. We denote the set of all possible extensions of a such an initial segment $\sigma$ by $(\sigma) = \{f \in 2^\omega \mid \sigma \subset f\}$.

**Definition 2.3** A basis for a topology on $2^\omega$ is given by $2^{<\omega}$ where the basic open sets are of the form $(\sigma)$ for $\sigma \in 2^{<\omega}$. We shall refer to this topology as the **classical** or **Cantor** topology on $2^\omega$ and the basic open sets as **cylinders** or **conditions**. Open sets in this topology are of the form $(U)$ where $U$ is an arbitrary subset of $2^{<\omega}$ and

$$(U) = \bigcup_{\sigma \in U} (\sigma)$$

The Cantor topology is routinely verified to be compact and separable.

Note that since it is possible to enumerate $2^{<\omega}$ in the canonical order $\emptyset, 0, 1, 01, 10$, $\ldots$, we may assign to each $\sigma$ an index $i$, known as a **strong** or **canonical indices**. When it is necessary to refer to an initial segment by index, we will denote by $\sigma^i$ the $i^{th}$ such initial segment in this ordering.

**Definition 2.4** A set $U \subset 2^{<\omega}$ is called **r.e. open** just in case for some r.e. set $W$ of canonical indices $U = \{\sigma^i \mid i \in W\}$. Note that there is a universal enumeration of r.e. open sets. We denote the $e^{th}$ such set as $U_e$ and the universal family of all r.e. open sets as $\mathcal{U}$.

**Definition 2.5** We say that $A \subset \omega$ is **1-generic** just in case it decides every r.e. open set $U \subset 2^{<\omega}$ either positively of negatively.

**Remark** Note that $A$ is 1-generic if and only if it decides positively every r.e. open set $U$ which is dense along $A$. A similar remark applies to the other two genericity concepts considered in this paper. It will often be more convenient to apply the definition of genericity in this form.

**Theorem 2.1** There exists a 1-generic set $A \leq_T \emptyset'$.
**Proof** To make $A$ 1-generic it is sufficient to satisfy requirements of the following form:
$R_s$ : $A$ decides $U_s$
We construct the (characteristic function of) $A$ in stages $\sigma_s$ so that upon setting $A = \bigcup_{s\in\omega} \sigma_s$, $A$ satisfies all of the $R_s$.

> **Stage 0** $\sigma_0 = \emptyset$
> **Stage s+1** Ask a $\emptyset'$ oracle whether there exists a $\tau \in U_s$ such that $\tau$ is compatible with $\sigma_s$. If so let $\sigma_{s+1} = \sigma \cup \tau$. Otherwise, let $\sigma_{s+1} = \sigma$.

Note that at stage $s+1$, we either extend $\sigma_s$ such that $(\sigma_{s+1}) \subseteq (U_s)$ or $\sigma_s$ is incompatible with all $\tau$ in $U_s$ and hence $(\sigma_s) \cap (U_s) = \emptyset$. Hence $A$ satisfies all the $R_s$.$\square$

1-generic sets have several properties which may be seen as non-effective analogues to those yielded by basic finite injury arguments. Recall that a set is **immune** if it does not contain any infinite r.e. subset and is **bi-immune** if both it and its complement are immune. We now show that 1-generic sets are bi-immune.

**Theorem 2.2** Every 1-generic set $A$ is bi-immune.
**Proof** Suppose for a contradiction that there was an infinite r.e. set $W_e$ such that $W_e \subseteq A$. Consider the set

$$U_e = \{\sigma \mid \exists x[\sigma(x)\downarrow = 0 \wedge x \in W_e]\}$$

$U_e$ is clearly r.e. and hence $A$ decides $U_e$ either positively or negatively. Since $W_e \subseteq A$, $A$ cannot decide $U_e$ positively. Note, however, that under the assumption that $W_e$ is infinite, for any $\sigma$ such that $A \in (\sigma)$, we can find an $n$ such $n \in W_e$ and $n > |\sigma|$. Hence $\tau = A \upharpoonright (n-1) \cup \{\langle n, 0\rangle\} \in U_e$ and $\tau \supset \sigma$. Thus $U_e$ is dense along $A$ and by the Remark above cannot decide $A$ negatively, contradicting its genericity. To see that $\overline{A}$ is also immune, we show that if $A$ 1-generic, $\overline{A}$ is also 1-generic. We argue by contraposition : supposed $A$ is 1-generic and $\overline{A}$

is not. Then by the Remark there is a r.e. $U$ which is dense along $\overline{A}$ such that $\overline{A}$ does not decide $U$ positively. But then note that $V = \{\overline{\sigma} \mid \sigma \in U\}$ (where $\overline{\sigma}(n) = 1 - \sigma(n)\downarrow$) is r.e. dense along $A$, but $A$ not decided positively by $A$, contra to its genericity. Thus $\overline{A}$ is immune by the preceding argument. $\square$

**Corollary 2.1** No 1-generic set is r.e. $\square$

Recall that a set $X \leq_T \emptyset'$ is **low** just in case $X' \leq_T \emptyset'$. We may define a slightly weaker property in the same vein which is possessed by 1-generic sets. We say that $X$ is **generalized low** just in case $X' \leq_T X \oplus \emptyset'$. We then have the following :

**Theorem 2.3** Every 1-generic set is generalized low.

**Proof** Let $A$ be 1-generic. It suffices to show that for every $e$, the truth of $\varphi_e^A(e)\downarrow$ can be decided using an $A \oplus \emptyset'$ oracle. Consider the set

$$U_e = \{\sigma \mid \varphi_e^\sigma(e)\downarrow\}$$

Since $U$ is r.e. $A$ must decide $(U_e)$ positively or negatively. We use this fact in an algorithm to decide $\varphi_e^A(e)\downarrow$ as follows. Using an oracle for $A$, we enumerate initial segments of $A$, $\sigma_s = A \restriction s$, and ask a $\emptyset'$ oracle at each stage whether there is an extension of $\sigma_s$ in $U_e$. More precisely, the $s^{th}$ stage of the algorithm is defined as follows :

$$\exists \tau \supseteq \sigma_s \text{ such that } \tau \in U_e \begin{cases} \text{no} & \text{halt answering "no"} \\ \text{yes and } \sigma_s \in U_e & \text{halt answering "yes"} \\ \text{yes and } \sigma_s \notin U_e & \text{proceed with } \sigma_{s+1} = A \restriction (s+1) \end{cases}$$

Since $A$ is 1-generic, we know that either $A \in (U)$, in which case this procedure will eventually terminate answering "yes", or $A \notin (U)$ and there exists some finite $s$ such that $(A \restriction s) \cap U_e = \emptyset$, in which case the procedure will terminate answering "no". $\square$

As mentioned above, 1-generic sets also provide a non-effective solution to Post's problem of finding two sets of incomparable Turing degrees. As we now see, any means of effectively splitting a 1-generic set into infinite halves yields incomparable degrees.

**Theorem 2.4** Suppose $X$ is 1-generic and let $X = A \oplus B$. Then $A$ and $B$ are of incomparable Turing degree, i.e. $A \not\leq_T$ and $B \not\leq_T A$.

**Proof** Suppose, for a contradiction, that $A = \varphi_e^B$ and consider the set

$$U_e = \{\sigma \oplus \tau \mid |\sigma| = |\tau| \wedge \exists x[\sigma(x) \neq \varphi_e^\tau(x)\downarrow]\}$$

Again $U_e$ is clearly r.e. and hence $X$ must decide it either positively or negatively. Note that if $A = \varphi_e^B$ then, $X$ cannot decide $U_e$ positively. We derive a contradiction by showing that $U_e$ is dense along $X$ and hence cannot be decided negatively. It suffices to show that for every $n$, we can find an extension $\sigma \oplus \tau$ of $X \restriction n$ such that $\sigma \oplus \tau \in U_e$. Note that under the assumption that $A = \varphi_e^B$ we can find $\tau \subset B$ such that $\varphi_e^\tau(x)\downarrow$ for arbitrarily large $x$. (This follows from the fact that $A$ cannot be finite, as otherwise $\{2n \mid n > \max\{A\}\}$ would be an infinite r.e. subset of $\overline{A}$, contrary to its immunity.) Let $\tau$ be such that $\varphi_e^\tau(x)\downarrow$ for some $x > n$ and define $\sigma$ such that $|\sigma| = |\tau|$, $\sigma \oplus \tau$ is compatible with $X \restriction n$ and $\sigma(x) \neq \varphi_e^\tau(x)$. Hence $\sigma \oplus \tau \in U_e$, and thus $U$ is dense along $X$. A symmetric argument yields $B \neq \varphi_e^A$. $\square$

We next turn our attention to the final application taken by Soare [9] as a benchmark of the finite injury method. If we consider an arbitrary set r.e. non-recursive set $C$, one can ask whether there exists a simple set $A$ such that $C \not\leq_T A$. Using a finite injury argument and the Sacks [8] preservation strategy, it can be shown that such sets exist. We will return to this result in context of e-genericity in section 4. For the moment, however, we concern ourselves with producing a non-effective analogue to this result using 1-genericity. In order to do so, we need to work with a broader notion of r.e. open sets. Thus far, r.e. open sets have been given by unions of fixed classical cylinders and hence may be thought of as being comprised of **strong** or **canonical** codes of finite objects. We may generalize this idea to consider sets comprised of r.e. indices of finite objects, or **weak** codes. In this case a code, rather than completely specifying a finite object, denotes a procedure which is guaranteed to terminate producing some $\sigma \in 2^{<\omega}$. Such sets are formally defined as follows.

**Definition 2.6** An sequence of finite initial segments $\tau_s$ is said to be **monotone** if whenever $\tau_{s+1}$ is defined, $\tau_s$ is defined and $\tau_{s+1} \supseteq \tau_s$. There is a universal recursive enumeration of such monotone sequences $\upsilon_{e,s}$ – i.e. for every $e$, $\upsilon_{e,s}$ is monotonic in $s$, and for every monotone recursive sequence $\tau_s$ there exists an $e$ such that $\bigcup_s \tau_s = \bigcup_s \upsilon_{e,s}$. We say that $e$ is an index of the finite initial segment $\sigma \in 2^{<\omega}$ if $\sigma = \bigcup_s \upsilon_{e,s}$. We say that a set $V \subset 2^{<\omega}$ is a **weak r.e. open** set just in case there is a r.e. set $W$ such that $V = \{\bigcup_s \ \upsilon_{e,s} \mid e \in W\}$.

Note that not every $e$ is the index of a sequence with a *finite* limit, nor is this property decidable (the index set of finite sequence is in fact $\emptyset''$-complete). As a consquence, there is no universal enumation of weak r.e. open sets. This does not exclude, however, that some families of weak r.e. sets are uniformly enumerable. We will call an r.e. set $\mathcal{V}$ of sets of weak r.e. codes a **uniform family**. Note in particular, that if that for every r.e set $U$ of strong codes, there is a set $V$ of weak codes such that $U = V$, and hence there is uniform family $\mathcal{U}' = \mathcal{U}$, where $\mathcal{U}$ is the set containing all r.e. open sets of strong codes. If $\mathcal{V}$ is a uniform familty, we say that $A \subset \omega$ is **$\mathcal{V}$-generic** if it decides every $V \in \mathcal{V}$. We now show that $\mathcal{V}$-generic sets exist for arbitrary $\mathcal{V}$.

**Theorem 2.5** Let $\mathcal{V}$ be a uniform family. Then there exists a $\mathcal{V}$-generic set $A \leq_T \emptyset'$.

**Proof** Fix an enumeration of $\mathcal{V}$ and let $V_n$ denote the $n^{th}$ set in this enumeration. As in Theorem 2.1, we construct $A$ in stages $\sigma_n$ so that at stage $n$, we extend $\sigma_n$ to decide $V_n$. Assume that $W_n$ enumerates a set of indices for $V_n$. Note that for any $\sigma_n$, the question of whether there exist $i$ and $t$ such that $i \in W_n$ and and $\upsilon_{i,t} \supset \sigma$ may be decided in $\emptyset'$. Now consider the set $S_i = \{s \mid \exists t[t \geq s \ \wedge \ |\upsilon_{i,t+1}| > |\upsilon_{i,t}|]\}$. Since $|\bigcup_s \upsilon_{i,t}| < \omega$, $S_i$ will always be finite and we can $\emptyset'$ recursively find $m(i) = \mu s[s \notin S_i]$. Hence we can compute $\bigcup_s \upsilon_{i,s} = \upsilon_{i,m(i)}$ in $\emptyset'$. We use these observations to define the sequence $\{\sigma_n\}$ as follows:

**Stage 0** $\sigma_n = \emptyset$

**Stage n+1** Using a $\emptyset'$ oracle, determine if there are $i, t$ such that $i \in W_n$ and $\upsilon_{i,t} \supset \sigma_n$. If so, set $\sigma_{s+1} = \upsilon_{i,m(i)}$.

If we let $A = \bigcup_n \sigma_n$, then note that for all $s$, $A$ will decide $V_n$ since at stage $n+1$, we either define $\sigma_{n+1} = \tau$ for some $\tau \in V_n$, or for no $i, t$ does $\upsilon_{i,t} \supset \sigma$, and hence $(\sigma_n) \cap (V_n) = \emptyset$. $\square$

In order to show obtain the result mentioned above, we need to prove the following technical lemma demonstrating that a certain set defined in terms of a fixed non-recursive set $C$ is weak r.e. .

**Lemma 2.1** Let $C$ be such that $\emptyset <_T C \leq_T \emptyset'$. Then for every $e$ the following set is weak r.e.

$$V_e = \{\sigma \mid \exists x[\varphi_e^\sigma(x)\downarrow \neq C(x)] \ \vee \ \exists x \forall \tau \supseteq \sigma[\varphi_e^\tau(x)\uparrow]\}$$

And thus $\{V_e \mid e \in \omega\}$ is a uniform family.

**Proof** By the limit lemma, fix a recursive sequence of sets $\{C_s\}_{s\in\omega}$ such that $\lim_{s\to\infty} C_s(x) = C(x)$. In order to show that $V_e$ is weak r.e. we demonstrate that for every pair $\langle \tau, t\rangle \in 2^{<\omega} \times \omega$ we can construct a recursive monotonic sequence $\langle \sigma_s, r_s\rangle$ such that $\langle \sigma_0, r_0\rangle = \langle \tau, t\rangle$ and such that if $\langle \sigma, r\rangle = \lim_{s\to\infty}\langle \sigma_s, r_s\rangle$, then $|\sigma|, r < \omega$ and $\sigma \in V_e$. Conversely, we will have that for every $\sigma \in V_e$, there exists a pair $\langle \tau, t\rangle$ such that the limit of such a sequence starting from this pair is $\sigma$. For any $\langle \tau, t\rangle$ inductively define $\langle \sigma_s, r_s\rangle$ as follows :

> **Stage 0** $\langle \sigma_0, r_0\rangle = \langle \tau, t\rangle$
>
> **Stage s+1** If $\exists x \leq r_s[\varphi_e^{\sigma_s}\downarrow \neq C_{r_s}(x)]$, then let $\langle \sigma_{s+1}, r_{s+1}\rangle = \langle \sigma_s, r_s\rangle$. Otherwise consider the function defined as
>
> $$lh(\sigma, e, s) = \begin{cases} \max\{x \mid \forall y < x[\varphi_{e,s}^\sigma(y)\downarrow]\} & \exists x[\varphi_{e,s}^\sigma(y)\uparrow] \\ \infty & \varphi_{e,s}^\sigma \text{ is total} \end{cases}$$
>
> If there is a $\tau \in 2^{<|\sigma|+r_s}$ such that $lh(\tau, e, r_s) > lh(\sigma, e, r_s)$ then, then let $\sigma_{s+1} = \tau$ and $r_{s+1} = |\tau|$. Otherwise let $\sigma_{s+1} = \sigma_s$ and $r_{s+1} = r_s + 1$.

Denote the sequence obtained in this manner by setting $\langle \sigma_0, r_0\rangle = \langle \tau, t\rangle$ as

$\sigma_s^{\tau,t}$. We claim first that for every $\langle\tau, t\rangle$, if $\lim_{s\to\infty}\sigma_s^{\tau,t} = \langle\sigma, r\rangle$, then $|\sigma| < \omega$. Suppose for a contradiction, that $|\sigma| = \omega$. In this case note that $\varphi_e^\sigma$ is total, and that $\forall x[\varphi_e^\sigma(x) = C(x)]$. In order to compute $C(p)$ for any $p \in \omega$, it suffices to recursively search for an $s$ such that $\varphi_e^{\sigma_s}(p)\downarrow = C_s(p)$ as then $\varphi_e^{\sigma_s}(p) = \varphi_e^\sigma(p)$, and hence $C_s(p) = C(p)$. Hence $C$ is recursive contra to $\emptyset <_T C$. We now show that for all $\sigma \in V_e$, there exists a $\langle\tau, t\rangle$ such that $\sigma$ is the limit of the sequence $\sigma_s^{\tau,t}$. If $\sigma \in V_e$, then by definition there is either an $x$ such that $\varphi_e^\sigma(x)\downarrow \neq C(x)$ or all $\tau \supseteq \sigma$ are such that $\varphi_e^\tau$ is *not* total. In the former case, there must exist a $t$ and an $x$ such that $C_t(x) = C(x)$ and $\varphi_e^\sigma(x)\downarrow \neq C_t(x)$, in which case the limit of $\sigma_s^{\sigma,t}$ is equal to $\sigma$. In the latter case the limit of $\sigma_s^{\sigma,0}$ is equal to $\sigma$. Conversely, note that if $\sigma$ is the limit of the sequence $\sigma_s^{\tau,t}$, then $\sigma$ is in $V_e$ by construction.$\square$

If we now fix a set $C \leq_T \emptyset'$ and let $\mathcal{V}_C$ be a uniform family containing the sets $V_e$ of the previous lemma and the family $\mathcal{U}'$ extensionally equivalent to the universal enumeration of canonically indexed open set, then a $\mathcal{V}_C$-generic set $A$ has the properties which we desire.

**Theorem 2.6** For every set $C$ such that $\emptyset <_T C <_T \emptyset'$, there exists a bi-immune set $A$ such that $C \not\leq_T A$.

**Proof** Let $A$ be $\mathcal{V}_C$-generic and suppose for a contradiction that there existed an $e$ such that $C = \varphi_e^A$. Consider the set $V_e$ from Lemma 2.1. $A$ cannot decide $V_e$ positively, as otherwise $\varphi_e^A \neq C$. We derive a contradiction by showing that $A$ cannot decide $V_e$ negatively. Suppose for a contradiction that there existed a $s$ such that $(A \upharpoonright s) \cap (V_e) = \emptyset$. We claim that in this case $C$ is recursive. To decide $C(p)$ for arbitrary $p$, note that there must exist an $\tau \supset A \upharpoonright s$ such that $\varphi_e^\tau(p)\downarrow$, as otherwise all extensions of $A \upharpoonright s$ would fail to be total and thus $(A \upharpoonright s) \cap (V_e) = \emptyset$. Moreover, we must have $\varphi_e^\tau(p) = C(p)$, as otherwise $\tau \in V_e$, also contradicting $(A \upharpoonright s) \cap (V_e) = \emptyset$. Hence $C$ is recursive, contrary to $\emptyset <_T C$. $\square$

The properties of 1-generic sets canvassed in this section were all originally shown to hold of individual objects via priority arguments without injury or the finite extension method. They may also be seen as non-effective analogues of the basic properties constructed via finite injury arguments in chapter VII of [9]. The construction of a single object possessing these properties simultaneously is a typical application of the concept of genericity.

# 3 Ellentuck-Mathias generic sets

In this section we define an alternative topology on $2^\omega$ and then define a notion of genericity with respect to this topology for which it can be shown that there exist r.e. generic sets. Since the new topology is still defined in on the underlying space $2^\omega$, we will still be able to construct a generic object by constructing its characteristic functions, as opposed to constructing an arbitrary recursive enumeration as will be necessary in the next section. A finite injury argument will, however, be required in order to show the existence of r.e. generic objects. The existence of such objects is sufficient to prove an effective version of the Kleene-Post theorem about the existence of incomparable Turing-degrees below $\emptyset'$ equivalent to the classical Friedberg-Muchnik theorem.

**Definition 3.1** Let $2^{\subset\omega}$ denote the class of partial functions from $\omega$ into $\{0,1\}$. Consider the class of functions $f \in 2^{\subset\omega}$ such that $f^{-1}(0)$ is finite and $f^{-1}(1)$ is co-infinite. We induce a topology on $2^\omega$ by considering for each such $f$ the basic open set $(f) = \{X \in 2^\omega \mid f \subseteq X\}$. We refer to the resulting topology as the **Ellentuck-Mathias [EM] topology**.[1]

In the applications of EM-topology to be considered, we will only be interested in forcing with EM condition on classical open sets determined by $\sigma \in 2^{<\omega}$. Hence an arbitrary open set will again be of the form $(U) = \bigcup_{\sigma\in U}\{f \in 2^\omega \mid \sigma \subset f\}$ for $U \subseteq 2^{<\omega}$ . We will refer to such sets as classical open sets. We introduce the notation $\langle D, B\rangle$ to denote $f \in 2^{\subset\omega}$ such that $f^{-1}(0) = D$ and $f^{-1}(1) = B$, where $D$ is finite, $B$ is co-infinite and $D \cap B = \emptyset$. Similarly, $\langle\sigma, B\rangle$ will be used to denote $f \in 2^{\subset\omega}$ such that $f^{-1}(1) = B$ and $\sigma \subset f$.

**Definition** A set $A \subset \omega$ is **EM -generic** if it decides every r.e. classical open set $U$ in the EM-topology – i.e. for all classical open sets $U$, either $A \in (U)$ or there exists a finite $D$ and a co-infinite $B$ such that $A \in (\langle D, B\rangle)$ and $(\langle D, B\rangle) \cap U = \emptyset$.

Note that if $A$ is EM-generic, then $A$ must be coinfinite (as otherwise $U = \{\sigma\cdot 0 \mid \sigma \subset A \wedge |\sigma| \in A\}$ is a r.e. classical open set dense along $A$ which $A$ does not decide positively). In order to demonstrate the existence of an r.e. EM-generic $A$, we must hence construct $A$ so that its complement remains infinite. This is accomplished via the following finite injury construction.

**Theorem 3.1** There exists a r.e. $EM$-generic set $A$.

**Proof** We construct such an $A$ by a finite injury argument. Note that there exists an effective universal enumeration of all all r.e. classical open sets, and let $U_e \subseteq 2^{<\omega}$ denote the $e^{th}$ set in such an enumeration. During the construction we attempt to satisfy two sets of requirements :

$P_e$ : $A$ decides $(U_e)$

$N_e$ : $|\overline{A}| \geq e$

If we fail to construct $A$ so as to satisfy a positive requirement $R_e$ positively (i.e. $\exists\tau \in U_e[A \in (\tau)]$) we will show that $A$ will decide $U_e$ negatively. Our strategy to meet the negative requirements, and thus to ensure that $A$ is co-infinite, is to associate with each positive requirement $R_e$ a "gap", representing an element

[1] The Ellentuck-Mathias topology was introduced in the study of infinitary versions of the Ramsey Theorem (cf., e.g. , [1]) . Note that our definition reverses the role played by 0 and 1 in the original.

which we attempt to keep from being enumerated into $A$ with priority $e$. We will show that such gaps get filled in during the construction at most finitely often. Define the following bookkeeping functions for use in the construction :

- $L_s$ : the set of currently satisfied requirements at stage $s$
- $R(e, s)$ : restraint set for the $e^{th}$ requirement at stage $s$ (to aid in satisfying $P_e$)
- $\rho_{i,s}$ : witness to the satisfaction of the $R_i$ at stage $s$
- $g(e, s)$ : the gap associated with the $e^{th}$ requirement at the $s^{th}$ stage (to aid is satisfying $N_e$)

**Stage 0** Initialize the various functions as follows : $\forall e[g(e, 0) = \langle e, 0\rangle]$, $\forall e[R(e, 0) = \emptyset]$; $L_0 = \emptyset$, $\forall e[\rho_{0,0} = \emptyset]$, $A_0 = \emptyset$.
**Stage s+1** We say that a requirement $R_e$ **requires attention** at stage $s + 1$ if the following conditions obtain :

1. $e \notin L_s$
2. there exists $\tau \in U_{e,s}$ s.t.

   - $A_s \cap \tau_0 = \emptyset$
   - $A_s \not\supseteq \tau_1$
   - $\tau_1 \cap R(e, s) = \emptyset$

where $R(e, s) = \bigcup_{j \in L_s \restriction e}(\rho_{j,s}^{-1}(0)) \cup \{g(j, s)\}$ – i.e. the set consisting of all the 0s contributed by currently satisfied requirements plus the gaps for these requirements. Note that since $U_e$ is r.e. these conditions may be effectively checked. If there is an $i \leq s$ such that $R_i$ requires attention at $s + 1$, let $\tau$ be the least element in $U_e$ which satisfied the above conditions. We proceed as follows :

- $A_{s+1} = A_s \cup \tau_1$
- $L_{s+1} = (L_s \cup \{i\}) - \{j \mid \tau_1 \cap R(s, j) \neq \emptyset\}$
- $\rho_{i,s} = \tau$

We now update all of the lower priority gaps which have been filled in by this action as follows : for all $j \geq i$ such that $g(s, j) \in A_{s+1}$, let $g(s + 1, j) = \langle j, n\rangle$ where $n = \mu m[\langle m, j\rangle \in \omega^{[e]} \ \wedge \ m > \max(A_{s+1}^{[e]})]$ (where $X^{[e]}$ denotes the $e^{th}$ column of $X$).

If there does not exist an $i \leq s$ such that $R_i$ requires attention at stage $s$, then we set $A_{s+1} = A_s$, $L_{s+1} = L_s$ and $\forall e[g(e, s+1) = g(e, s)]$. This completes the construction.

In the course of the construction, we say that a requirement $R_e$ is **injured** at stage $s$ just in case $e \in L_s$ and $\exists x \in (A_{s+1} - A_s)[x \in \rho_{e,s}^{-1}(0)]$. As is typical in finite injury constructions, the crucial observation is that a given requirement is injured at most finitely often. We now verify that $A$ has the desired properties via three lemmas.

**Lemma 3.1** For all $e$, $R_e$ receives attention at most finitely often.

**Proof** We proceed by induction. Consider requirement $R_0$. Note that if $\forall s \forall \tau \in U_{0,s}[\tau_0 \cap A_s \neq \emptyset]$ then $R_0$ never receives attention. Otherwise, suppose that $\exists s \exists \tau \in U_{0,s}[\tau_0 \cap A_s = \emptyset]$. Note that there are no requirements of priority greater than $R_0$, $\forall s[R(0, s) = \emptyset]$ and hence stage $s$, $R_0$ requires attention and is acted on by setting $A_{s+1} = A_s \cup \tau_1$. Note also that for all subsequent stages $t > s$ and lower priority requirements $e > 0$, we have $\tau_0 \subseteq R(e, t)$, and thus $R_0$ is never injured after stage s. Now consider $R_n$. By the induction hypothesis, choose a stage $s$ after which $R_i$ is no longer injured for all $i < n$. If $\forall t \geq s \forall \tau \in U_{n,t}[\tau_1 \cap R(s, n) \neq \emptyset \vee \tau_0 \cap A_t \neq \emptyset]$, then $R_n$ never receives attention after stage $s$. Otherwise, assume that there is some $t \geq s$ such that there is a $\tau \in U_{n,t}$ such that $\tau_1 \cap R(s, n) = \emptyset$ and $\tau_0 \cap A_t = \emptyset$. In this case, $R_n$ requires attention at stage $t$ and is acted upon by setting $A_{t+1} = A_t \cup \tau_1$. Note that for all $u \geq t$, $n \in L_u$ since no higher priority requirements are injured after $t$ by our choice of $s$. Hence $R_n$ is not injured after $t$. $\square$

**Lemma 3.2** $A$ is co-infinite.

**Proof** Note that each gap $g(e, s)$ represents an attempt to constrain a given element from entering $A_s$. It is hence sufficient to show that for all $e$, $g(e) = \lim_{s \to \infty} g(e, s) < \infty$ as then we will have $\forall e[g(e) \notin A]$. As the gaps are chosen from different columns of $\omega$, note that the gap associated with one requirement cannot interfere with those of another. Note also that a given gap $g(e, s)$ can only be filled in by an action taken in an attempt to satisfy $R_i$ for some $i < e$. By the previous lemma, each requirement receives attention at most finitely often and hence each gap is upwardly displaced at most finitely far. In particular, if $R_e$ is last acted upon at stage $s$, then $g(e) \leq \max(A_{s+1}^{[e]}) + 1$. $\square$

**Lemma 3.3** For all $e$, $A$ decides $R_e$.

**Proof** We proceed by induction. Note that Lemma 3.2, $A$ is co-infinite, and hence $\langle D, A \rangle$ is a EM-open set for any finite set $D$. By Lemma 3.1, we hence have $A \in (U_0)$ or $(\langle \emptyset, A \rangle) \cap U_0 = \emptyset$. For $R_n$, let $s$ be such that no $R_i$ such that $i < n$ is injured after $s$. Note that $R(s, n)$ is finite, as it consists of at most finitely many finite sets of zeros and finitely many gaps contributed by higher priority requirements. Again as in Lemma 3.1, either $A \in (U_n)$ or $\langle R(s, n), A \rangle \cap U_n = \emptyset$. $\square$

Note that in the construction of $A$ we have some control over where the gaps left during the construction come to rest. In particular, by assigning the initial placement of the gaps to infinitely many different disjoint sets, we can guarantee that $\overline{A}$ will intersect each of these sets in at least one element. For instance, consider partitioning $\omega$ into $\omega$ columns of the form $C_{2n} = \{\langle 2n, m\rangle \mid m = 2x \wedge x \in \omega\}$ and $C_{2n+1} = \{\langle 2n+1, m\rangle \mid m = 2x+1 \wedge x \in \omega\}$. In the construction of $A$, we may then set $g(n,0) = C_n[0]$ (where $C_n[i]$ denotes the $i^{th}$ element in $C_n[0]$). If we then always update the $i^{th}$ gap by choosing an element from $C_i$ larger than the largest element of $A_s^{[i]}$, then we will be able ensure that $\overline{A}$ meets $C_n$ in at least one element. More generally, for any primitive recursive partition of $\omega$ into $\omega$ pairwise disjoint sets, we can required that $\overline{A}$ contains at least one element from each set. In particular, we have the following :

**Corollary 3.1** There exists an r.e. EM-generic set $A = B \oplus C$ such that $B$ and $C$ are both co-infinite.$\square$

Note also that in the construction of $1A$ it is possible to start the construction with $A_0 = N$ for any co-infinite recursive set $N$. We thus additionally have the following :

**Corollary 3.2** For any co-infinite recursive $N$, there exists a co-infinite r.e. EM-generic $A$ such that $A \supset N$.$\square$

This property may be taken as demonstrating a weakness in the definition of EM-genericity since, as we shall see in the next section, an e-generic set $B$ is such that $|B \cap \overline{M}| = \omega$ for any infinite primitive recursive $M$.

We now catalogue two additional properties of a r.e. EM-generic set $A$.

**Theorem 3.2** Let $A$ be a co-infinite r.e. EM generic set. Then $A$ has the following properties:

1. $A$ is simple;
2. If $A = B \oplus C$, and $A$ and $B$ are both co-infinite, then $B \not\leq_T C$ and $C \not\leq_T B$.

**Proof**

1. $\overline{A}$ is infinite by hypothesis. Consider the following set of conditions :

$$U_e = \{\sigma \mid \exists x[\sigma(x) = 1 \wedge x \in W_e]\}$$

We claim that if $W_e$ is infinite, then $A$ is dense along $U_e$. Let $(\langle \sigma, B\rangle)$ be any EM cylinder such that $A \in (\langle \sigma, B\rangle)$. Under the assumption that $W_e$ is infinite, there is a $x \in W_e$ such that $x > |\sigma|$. Then $(\langle \sigma, B \cup \{x\}\rangle) \subset (\langle \sigma, B\rangle)$

and $(U_e) \cap (\langle\sigma, B\rangle) \neq \emptyset$. As $U_e$ is thus dense along $A$, $A$ decides $U_e$ positively and hence intersects $W_e$.$\Box$

2. Let $A = B \oplus C$ and assume that $B$ and $C$ are co-infinite (such an $A$ exists by Corollary 3.1). Assume for a contradiction that there exists $e$ such that $B = \varphi_e^C$. Consider the set $U_e$ defined as follows :

$$U_e = \{\sigma \oplus \tau \mid |\sigma| = |\tau| \quad \wedge \; \exists x[\sigma(x) \neq \varphi_e^\tau(x)\downarrow]\}$$

Since $B = \varphi_e^C$ by assumption, $A \notin (U_e)$. We derive a contradiction by showing that $A$ cannot decide $U_e$ negatively. Suppose that $A$ did decide $U_e$ negatively and let $(\langle\sigma, S\rangle)$ be such that $A \in (\langle\sigma, S\rangle)$ and $(\langle\sigma, S\rangle) \cap (U_e) = \emptyset$. Note that without loss of generality we may assume that the cylinder $(\langle\sigma, S\rangle)$ omitting $(U_e)$ is as wide as possible – i.e. that $S = A$. We may also clearly assume and that $|\sigma|$ is even. We may hence write $(\langle\sigma, S\rangle) = (\langle B, \rho\rangle) \oplus (\langle C, \tau\rangle)$. Note also that since $\varphi_e^C = B$, $\varphi_e^C$ is total. Hence for every co-infinite set $X$ such that $X \supseteq B$, we have $X = \varphi_e^C = B$ (as otherwise $X \oplus C \in (U_e)$). But since $\overline{B}$ is infinite, there exists $n > |\rho|$ such that $n \notin B$. Noting that $B \cup \{n\} \supset B$, we have that $B \cup \{n\} = \varphi_e^C = B$, a contradiction. We may hence conclude that $B \neq \varphi_e^C$. A symmetrical argument yields $C \neq \varphi_e^B$. $\Box$

# 4 e-genericity

In this section we explore a notion of genericity first developed by Jockusch in [3]. While 1-genericity and EM-genericity were both developed using conditions consisting of fixed segments of the characteristic function of an r.e. set, the conditions employed in e-generic forcing are determined by finite initial segments of a recursive enumeration $\{A_s\}$ of an r.e. set $A$. By speaking about enumerations, e-generic forcing is able to mimic the techniques used in injury arguments more transparently than EM-forcing. For instance, in a typical finite injury construction, the action taken to satisfy a requirement $R_e$ at stage $s$ is designed such that if no elements smaller than the value of some restraint function $R(A_s, e, s)$ enter $A$ after $s$, then $R_e$ remains satisfied throughout the rest of the construction. What characterizes such constructions as injury arguments, however, is that elements less than $R(A_s, e, s)$ are allowed to enter $A$ after $s$ in order to satisfy requirements of priority higher than $e$. In such arguments, $R_e$ will be satisfied at the end of the construction if it can both be contrived that there is a stage $s$ after which no action is taken to satisfy $R_i$ for $i < e$ and that $R_e$ is still capable of being satisfied by actions available $s$.

Note that in order to characterize such constructions with forcing conditions, it is no longer sufficient to work with initial segments of the characteristic function of an r.e. set $A$, since in addition to controlling what elements are enumerated into $A$, the injury technique also depends on controlling *when* elements enter $A$. We will hence consider the space of recursive enumerations of infinite subset of $\omega$. A recursive enumeration of a r.e. set $A$ can be conceived as a sequence of finite sets natural numbers $A_0, A_1, \ldots$ such that $A_i \subseteq A_{i+1}$ and $\bigcup_s A_s = A$.

With this conceptual framework in place, we now modify Jockusch's notation somewhat by introducing an explicit topology on enumerations.

**Definition 4.1** By $\omega^\omega$ we denote the **Baire space**, i.e. the set of all total functions $\{f \mid f : \omega \to \omega\}$. We will henceforth use $\rho, \sigma, \tau, \ldots$ to denote initial segments of such functions and $\omega^{<\omega}$ to denote the set of such initial segments. Note that a recursive function $f \in \omega^\omega$ can be considered as enumerating a set $\mathrm{ran}(f) = A \subseteq \omega$ by taking $A_s = \{x \mid \exists t \leq s[x = f(t)]\}$.

This notation is both more consonant with that of earlier sections and will facilitate the generalization of e-genericity to additional applications of the injury method.

We also wish to incorporate the notion of a lower bound on how a requirement can be met directly into forcing conditions on $\omega^\omega$. To this end, we introduce a new topology on $\omega^\omega$. Let $\mathcal{R}$ be a class of functions $r \in \omega^\omega$ closed under finite iterations of the operation $\mathbf{max}(r, r')$ – the pointwise maximum of $r$ and $r'$ – for all $r, r' \in \mathcal{R}$. We will refer to $\mathcal{R}$ as a **restraint class**. We induce a topology on $\omega^\omega$ by considering the basis $P$ consisting of all pairs $\langle \sigma, r \rangle$ such that $\sigma \in \omega^{<\omega}$ and $r \in \mathcal{R}$. Basic open sets on this space will be of the form $(\langle \sigma, r \rangle) = \{f \in \omega^\omega \mid f \supset \sigma \wedge \forall s[f(s) > r(s)]\}$. Arbitrary open sets will again be of the form

$$(U) = \bigcup_{\langle \sigma, r \rangle \in U} (\langle \sigma, r \rangle)$$

**Lemma 4.1** For any class of function $\mathcal{R}$ which is closed under the operation $\mathbf{max}(r, r')$, for all $r, r' \in \mathcal{R}$, $P$ forms the basis for a topology on $\omega^\omega$.

**Proof** Immediate upon noting that for any two pairs $\langle \sigma, r \rangle$ and $\langle \tau, r' \rangle$, $(\langle \sigma, r \rangle) \cap (\langle \tau, r' \rangle) = (\langle \sigma \cup \tau, \mathbf{max}(r, r') \rangle)$. $\square$

The members of a restraint class will be referred as **restraint functions**. A restraint function is a function which provide a lower bound on the elements which may be enumerated at a given stage by $f \in \omega^\omega$. We say that the enumeration $f$ **respects** the restraint function function $r(s)$ just in case $\forall s[f(s) > r(s)]$. We will be interested in forcing with primitive recursive conditions consisting of sets of initial segments $\sigma \in \omega^{<\omega}$ and primitive recursive recursive restraint functions $r \in \mathcal{R}$.

The idea motivating these definitions is as follows. As stated above, in order to demonstrate that a set $A$ given as the range of a recursive enumeration $f$ satisfies a certain list of properties, it is generally sufficient to guarantee that an initial segments of the characteristic function $A$ remains fixed after a certain point in $A$'s enumeration by $f$. We can force this property by demanding that $f$ respect a restraint function $r$ . In finite injury arguments, it can always be shown that the restraint functions $r(s)$ involved are such that $\lim_{s\to\infty} r(s) < \infty$. This ensures in a straightforward manner that any property which we wish $A$ to possess, and which can be obtained by steps taken arbitrarily late in an enumeration of $A$, can also be satisfied in a manner that does not interfere with the restraints established by higher priority requirements. The simplest useful restraint class is generated by what we will call **step functions**. Step functions are sigmoid-like functions of the form

$$r(s) = \begin{cases} 0 & s < s_0 \\ n & s \geq s_0 \end{cases}$$

for some $n, s_0 \in \omega$. The class generated by these functions will be denoted as $\mathcal{S}$. Note that restraints imposed by step functions are relatively easy to respect since once met they do not grow arbitrarily high. In particular, for all step functions $r$, $\lim_{s\to\infty} r(s) < \infty$ as required. As will be significant below, also note that any step function can be given a simple parametric definitions in terms of a finite number of steps "heights" and "widths". There is hence a universal enumerations of step functions.

Basic results originally derived through finite injury arguments such as the existence of incomparable Turing degrees and the existence of low simple sets can be derived from forcing conditions using only the class of step functions. Results originally obtained using the so-called Sacks' preservation strategy require a more complex restraint class defined below and exploits the greater generality of forcing with conditions in $\omega^{<\omega} \times \mathcal{R}$ over the conditions used by Jockusch. We now proceed to adapt Jockusch's definition of e-genericity to conditions given by sets $U$ consisting of pairs $\langle \sigma, r \rangle$.

In order to effectivize the construction of an e-generic object, we must restrict attention restraint classes $\mathcal{R}$ considered as a primitive recursive set of indices to recursive functions. When no confusion can result, we will write $r \in \mathcal{R}$ as a shorthand for the fact there exists an index $i \in \mathcal{R}$ such that $\varphi_i = r$. It should be kept in mind, however, that the restraint classes we consider here are sets of indices which can be uniformly enumerated. Conversely, we will alway be able to effectively construct an index for a restraint function in the classes we consider form a finite list of parameters.

**Definition 4.2** If $f \in (U)$ we will say that $f$ **meets** $U$ – i.e. $f$ meets $U$ just in case there exists some $\langle \sigma, r \rangle \in U$ such that $\sigma \subset f$ and $\forall s[f(s) > r(s)]$.

**Definition 4.3** Let $q$ be a recursive function and $\mathcal{R}$ a restraint class. We say that $f$ is **$q$-dense along** $U \subseteq \omega^{<\omega} \times \mathcal{R}$ with respect to $\mathcal{R}$ if for every pair $\langle \sigma, r \rangle$ such that $f \in (\langle \sigma, r \rangle)$ (i.e. $f \supset \sigma$ and $f$ respects $r$), there exists $\langle \rho, r' \rangle \in \omega^{<\omega} \times \mathcal{R}$

such that $(\langle\rho, r'\rangle) \subset (\langle\sigma, r\rangle)$, $f \in (\langle\rho, r'\rangle)$ and there exists $(\langle\tau, r''\rangle) \subset (\langle\rho, r'\rangle)$ such that $|\tau|, k < q(|\rho|, j)$ (where $j$ is the index of $r'$ and $k$ is the index of $r''$) and $(\langle\tau, r'\rangle) \in U$.

Several points bear mentioning about this definition. Intuitively we wish to characterize the fact that a set of conditions $U$ is dense along a given enumeration $f$ just in case for any open set $(\langle\sigma, r\rangle)$ containing $f$, it is always possible to find some smaller open set $(\langle\tau, r'\rangle) \subseteq (\langle\sigma, r\rangle)$ such that $\langle\tau, r'\rangle \in U$. These conditions will guarantee that no matter what an initial segment $f \upharpoonright s_0$ looks like, there is a way of extending $f \upharpoonright s_0$ to $f \upharpoonright s$ such that $f \upharpoonright s$ has the property described by $U$ and also retains the properties of $f \upharpoonright s_0$ which are preserved given the fact that $f$ respects $r$. In order to effectivize the construction of e-generic objects, however, we must put a bound on the length of extensions in a set which can be recognized as being dense in this sense. This can be achieved by a relatively mild constraint : the step from $\langle\sigma, r\rangle$ to $\langle\rho, r'\rangle$ along $f$ may be take ineffectively, but once this leap is made, it must be possible to primitive recursively bound the length of an additional extension $\langle\tau, r''\rangle$ needed to extend $\rho$ into $U$. This requirement will ensure that in the construction of an e-generic object, the action required at a given stage to satisfy a requirement can be carried out in a bounded length of time – an essential characteristic if such $f$ is to be r.e. .

There is an additional subtlety in the definition which requires comment. Note that the primitive recursive function $q$ must bound not only the length of the extension $\tau$ but also the length of a index for a restraint function $r \in \mathcal{R}$. Again, this requirement is needed to ensure that the search for an $\langle\tau, r\rangle$ in $U$ can be effectively bounded. In practice, however, in order to meet any of the sets with which we will be concerned will require extensions which will all have obvious primitive recursive upper bounds on both the length of $\tau$ and on a description of $r$ . The latter fact follows since both the class of steps functions and the class of preservation functions to be defined below are described by simple (in fact primitive recursive) algorithms and parameters that are either fixed or grow linearly with the stage in the construction. The former fact follows since we will be concerned only with conditions which arise in finite injury constructions, and they can be met by a "simple" set of actions (canonically by adding a single element to $A$). Note in particular that if $p$ majorizes $q$, then $q$-dense along implies $p$-dense along. Hence we will take $q$ to be a sufficiently fast growing function, such as the Ackerman function, and henceforth write "dense along" for "q-dense along".

**Definition 4.4** Let $\mathcal{R}$ be a restraint class. $f \in \omega^\omega$ is e-generic with respect to $R$ if for every primitive recursive set $U \subseteq \omega^\omega \times \mathcal{R}$, if $U$ is dense along $f$, then $f$ meets $U$.

Again, several points should be noted about this definition. First, note that while it is generally the range $A$ of a given recursive enumeration $f$ (which we will denote as $[f]$) which bears the recursion-theoretic properties in which we are interested, we will speak directly of the enumeration as being e-generic. To say that a set $A$ is e-generic will mean that $A$ has an e-generic enumeration. Second, the notion of e-genericity we consider is relative to a given class of restraint functions. This step is taken because we want to capture the fact that the properties of an e-generic object depend on the class of restraint functions with respect to which it is generic.

The existence proof for e-generic sets with respect to a restraint set whose members have finite limits is a translation of Jockusch's original argument about argument enumerations into one involving the space $\omega^\omega$ and conditions $U \subseteq \omega^{<\omega} \times \mathcal{R}$.

**Theorem 4.1** Let be $\mathcal{R}$ be a primitive recursive restraint class such that $\forall r \in \mathcal{R}[\lim_{s\to\infty} r(s) < \infty]$. Then there exists an e-generic $f$ with respect to $\mathcal{R}$.
**Proof** Note that there is a universal enumeration for all primitive recursive $U \subseteq \omega^{<\omega} \times \mathcal{R}$. Let $U_e$ denote the $e^{th}$ set in this enumeration. We will recursively construct a function $f$ so that for all $e$ the following requirement are satisfied :
$R_e$ : If $U_e$ is dense along $f$ with respect to $\mathcal{R}$, then $f$ meets $U_e$.
Let $F(s)$ be a recursive function which bounds every primitive recursive function on all but finitely many values. Our strategy for meeting $R_e$ will be to search for a $\langle \tau, r\rangle \in U_e$ such that $\tau \supset f \upharpoonright s$ and $|\tau|, j \leq F(s+1)$, where $j$ is the index of $r$, and such that $r$ respects all the restraint functions for higher priority requirements which are currently satisfied. If such a pair is found, say with $\tau = f \upharpoonright s \cup \langle s+1, x_1\rangle \cup \cdots \cup \langle t, x_t\rangle$, we attempt to successively extend $f \upharpoonright s$ with $f \upharpoonright (u+1) = f \upharpoonright u \cup \langle u+1, x_{u+1}\rangle$ for $1 \leq u < t$ given that $\tau$ does not injure a restraint established by a higher priority requirement. The attempt will be successful if no requirement of priority greater than $e$ is acted on between stages $s$ and $t$. As soon as we begin to take action to satisfy $R_e$, we restrain all other elements which have a chance of being enumerated by $f$ by the restraint function $r$ in order to achieve $\forall s[f(s) > r(s)]$. We define several bookkeeping functions to aid in the construction : $L_s$ will denote the list of apparently satisfied requirements at stage $s$; $e_s$ the index of the stage (if any) that receives attention at $s$; and $\tau_s$ and $r_{e,s}$ respectively the sequence via which an attempt is made to satisfy $R_e$ at stage $s$ and its associated restraint function.

**Stage 0** Initialize $f_0 = \emptyset$, $e_0$, $\tau_0$ undefined, and $r_{e,0}(x) = 0$ for all $e, x$.
**Stage s+1** We say that $R_e$ **requires attention** at stage s+1 if $e \notin L_s$ and either $e = e_s$ or there exists a pair $\langle \tau, r\rangle$ satisfying the following requirements:

i) $\tau \supset f_s$

ii) $\langle \tau, r\rangle \in U_{e,s}$

iii) $\langle |\tau|, j\rangle \leq F(s+1)$ where $j$ is the index of $r$

iv) $\forall i < e \forall s \leq |\tau|[\forall i < e[\tau(s) > r_{i,s}(s)]]$

Let $e_{s+1}$ be the least $e < s+1$ such that $R_e$ requires attention at $s+1$. If $e_{s+1} = e_s$, then set $\tau_{s+1} = \tau_s$. Otherwise, let $\langle \tau, r\rangle \in U_{e,s+1}$ be the least pair satisfying i) - iv). Update as follows:

- $f_{s+1} = f_s \cup \langle s+1, \tau(s+1)\rangle$
- $L_{s+1} = \begin{cases} L_s \cup e_{s+1} - \{i \mid i > e_{s+1}\} & \text{if } f_{s+1} = \tau_{s+1} \\ L_s - \{i \mid i > e_{s+1}\} & \text{if } f_{s+1} \subsetneq \tau_s \end{cases}$
- $r_{i,s+1}(s) = \begin{cases} 0 & \text{if } i > e_{s+1} \\ r(s) & \text{if } i = e_{s+1} \\ r_{i,s}(s) & \text{if } i < e_{s+1} \end{cases}$

If there is no $e < s+1$ such that $R_e$ requires attention at $s$, then update as follows :

- $f_{s+1} = f_s \cup \langle s+1, f_s(s)\rangle$
- $L_{s+1} = L_s$
- $r_{e,s+1} = r_{e,s}$
- $e_{s+1}$ and $\tau_{s+1}$ undefined

This completes the construction. We now verify that the construction works via two lemmas. The first states that an attempt to satisfy a requirement is successful unless it is interfered with by a requirement of higher priority, while the second shows that all requirements are ultimately satisfied.

**Lemma 4.2** If $e_{s+1} = e$, then either there is some $t > s+1$ such that $e_t < e$, or else there exists $t \geq s+1$ such that $\langle \tau, r\rangle \in U_{e,t}$, $f_t = \tau$, $\forall x[f(x) > r(x)]$, for all $u > t$, $e \in L_u$, and for all $v > s+1$, $r_{e,v} = r$.

**Proof** This follows directly from the construction. Let $e_{s+1} = e$ and assume inductively that there does not exist $t > s+1$ such that $e_t < e$. $R_e$ thus receives attention at stage $s+1$ and there hence exists a $\langle \tau, r\rangle \in U_{e,s+1}$ such that $\tau \supset f \restriction s$ and $\forall x \leq |\tau|[\forall i < e[\tau(x) > r_{i,s}(x)]]$ and we set $r_{e,s+1} = r$ and $\tau_{s+1} = \tau$. Since no higher priority requirements act between stages $s$ and $t = |\tau|$, we may conclude that at stage $t$, $f_t = \tau_{s+1}$, $\forall x[f(x) > r(x)]$, $e \in L_u$ for all $u > t$, and similarly that all $v > s+1$, $r_{e,v} = r$. $\square$

**Lemma 4.3** For each $e$ the following hold :

1) $e_s = e$ for only finitely many $s$;

2) $\lim_{s\to\infty} \lim_{x\to\infty} r_{e,s}(x)$ exists and is finite;

3) $R_e$ is satisfied.

**Proof**
1) By Lemma 4.2, the initial attempt (if any) to satisfy $R_0$ will be successful. Hence assume inductively that $s$ is such that for all $i < n$, $\neg\exists t > s[e_t = i]$. If there exists $t > s$ such that $e_t = n$, then by Lemma 4.2 we have that there exists $t' > t$ such that $n \in L_{t'}$, and hence $e_u \neq n$ for all $u > t'$. $\Box$

2) By 1), for every $e$ we can choose a stage $s$ such that for all $t > s$, $e_t > e$ and thus $r_{e,t} = r_{e,s}$. Note since $r_{e,s} \in \mathcal{R}$, it has a finite limit and hence there exists a $y$ such that there exists a $x_0$ such that all $x > x_0$, $r_{e,s}(x) \leq y$. Hence $\lim_{s\to\infty} \lim_{x\to\infty} r_{e,s}(x) \leq y < \infty$. $\Box$

3) Suppose $U_e$ is dense along $f$, and let $q$ be a primitive recursive witnessing this fact. We can choose a stage $s_0$ such that $s_0 > e$, $e_s < e$ holds for no $s \geq s_0$ (by 1)), $s_0 > \lim_{s\to\infty} \lim_{x\to\infty} r_{i,s}(x)$ for all $i < e$ (by 2)), and $F(s+1) \geq q(s+1)$ for all $s \geq s_0$ (by our choice of dominating $F$). Define the function $r$ as $r(x) = \max\{r_{i,s_0}(x) \mid i < e\}$ and note that $r \in \mathcal{R}$ since the latter is closed under $\mathbf{max}$. Clearly $f \in ((\langle f \upharpoonright s_0, r\rangle))$. Hence by density, there exists a $s > s_0$ and $r' \in \mathcal{R}$ such that there exists a $\langle \tau, r\rangle \in U_e$ such that $f \in ((\langle f \upharpoonright s, r'\rangle))$, $|\tau| < F(s+1)$ and $((\langle \tau, r''\rangle)) \subset ((\langle f \upharpoonright s, p'\rangle))$. Note that we hence must have $\forall x[r''(x) > r'(x)]$, and hence $\forall x[r''(x) > r_{i,s_0}(x)]$ for all $i < e$. Finally, note that at stage $s$ it must already be the case that $e \in L_s$, as otherwise $R_e$ would require attention at stage $s+1$ and hence $e_{s+1} \leq e$, contrary to our choice of $s_0$. Hence $e \in L_s$ and there thus exists a maximal $s' < s$ such that $R_e$ receives attention at $s'$. Since $e \in L_s$, and $s'$ is the maximal (i.e. last) stage at which $R_e$ receives attention, there can be no stage $s''$ between $s'$ and $s$ such with $e_{s''} < e$. Hence we may conclude that $f$ meets $U_e$ by Lemma 4.2 applied to stage $s'$. $\Box$

Having shown the existence of a generic object with respect to any restraint class of primitive recursive functions with finite limits, we now investigate the properties of an e-generic $f$ with respect to the class of step functions $\mathcal{S}$.

**Theorem 4.2** [Jockusch] Let $f$ be an e-generic enumeration with respect to $S$ and let $A = \mathrm{ran}(f)$. $A$ has the following properties :

i) $A$ is low – i.e. $A' \leq_T \emptyset'$.

ii) Let $M$ be any infinite primitive recursive set. Then $\overline{A} \cap M$ is infinite.

iii) $A$ is promptly simple – i.e. there a recursive function $p$ such that for some enumeration of $\{A_s\}$ of $A$ and every infinite r.e. set $W_e$ there exists $x$ such that for some stage $s$, $x \in (W_{e,s+1} - W_{e,s}) \cap A_{p(s)}$.

iv) If $A = B \oplus C$, then $B$ and $C$ have Turing incomparable degrees.

**Proof**

i) By the Limit Lemma (cf. [9] III.3.3), it suffices to show that for every $e$, if $\exists^\infty s[\varphi_e^{[f\restriction s]}(e)\downarrow]$, then $\varphi_e^{[f]}(e)\downarrow$. Consider the following set

$$U_e = \{\langle \sigma, r\rangle \mid \varphi_e^{\sigma}(e)\downarrow \wedge r \in \mathcal{S} \wedge \forall s > |\sigma|[r(s) \geq \max\{\mathrm{ran}(\sigma)\}]\}$$

Assume that $\exists^\infty s[\varphi_e^{f\restriction s}(e)\downarrow]$. We claim that $U_e$ is dense along $f$. Choose any $s_0$ and $r \in \mathcal{S}$ such that $f \in (\langle f \restriction s_0, r\rangle)$. Since $r \in \mathcal{S}$, there exist $t_0, y \leq \omega$ such that $y = \max\{\mathrm{ran}(r)\}$ and $t_0 = \mu t[r(t) = y]$. Under the assumption $\exists^\infty s[\varphi_e^{[f\restriction s]}(e)\downarrow]$, there exists a $u > s_0, t_0$ such that $\varphi_e^{[f\restriction u]}(e)\downarrow$. Let $z = \max\{[f \restriction u]\}$ and Define the function

$$r'(s) = \begin{cases} 0 & s < u \\ \max\{[f \restriction u]\} & s \geq u \end{cases}$$

and let $r'' = \mathbf{max}(r, r')$. Then $(\langle f \restriction u, r''\rangle)(\subset \langle f \restriction s_0, r\rangle)$ and $\langle f \restriction u, r''\rangle \in U_e$. Hence $U_e$ is dense along $f$. By the e-genericity of $f$, we may conclude that $f$ meets $U_e$. There thus exists some $\langle \sigma, r\rangle \in U_e$ such that $f \supset \sigma$ and $\forall x[f(x) > r(x)]$. From the definition of $U_e$ we have that there is some $s$ such that $\varphi_e^{[f\restriction s]}(e)\downarrow$ and such that $\forall x > s[f(x) \geq \max\{[f \restriction s]\}]$. Hence the computation $\varphi_e^{[f\restriction s]}(e)\downarrow$ is preserved by $f$ and, by the use principle, we may conclude that $\varphi_e^{[f]}(e)\downarrow$. $\square$

ii) Suppose $M$ is infinite and primitive recursive. Consider the following set :

$$U_e = \big\{\langle \sigma, r\rangle \bigm| |\{x \,|x < |\sigma| \;\wedge\; x \in M \;\wedge\; x \notin \mathrm{ran}(\sigma)\}| \geq e \\ \wedge\; r \in \mathcal{S} \;\wedge\; \forall s \geq |\sigma|[r(s) \geq m(\max\{\mathrm{ran}(\sigma)\}, e)]\big\}$$

where $m(y, e) =_{\mathrm{df}} \mu z[\exists^{=e} x[x > y \;\wedge\; x \in M \;\wedge\; x < z]]$. Note that since $M$ is primitive recursive and infinite, $m(y, e)$ is also primitive recursive and total. Consider any $\langle r, f \restriction s_0\rangle$ such that $f \in (\langle r, f \restriction s_0\rangle)$. Since $r \in \mathcal{S}$, there exist $t_0, y \leq \omega$ such that $y = \max\{\mathrm{ran}(r)\}$ and $t_0 = \mu t[r(t) = y]$. Define the function

$$r'(s) = \begin{cases} 0 & s < u \\ m(\max\{[f \restriction s_0]\}, e) & s \geq u \end{cases}$$

and let $r'' = \mathbf{max}(r, r')$. We have that $(\langle f \restriction u, r''\rangle) \subset (\langle f \restriction s_0, r\rangle)$ and $\langle f \restriction u, r''\rangle \in U_e$. Hence $U_e$ is dense along $f$. Hence $f$ meets $U_e$ and as such fails to enumerate at least $e$ elements of $M$. Since $U_e$ is dense along $f$ for every $e$, we thus have $|\overline{A} \cap M| = \infty$. $\square$

iii) By ii), $\overline{A}$ is infinite. Consider the following set :

$$U_e = \{\langle \sigma, r\rangle \mid \exists s \exists x[|\sigma(x)| \leq s \;\wedge\; \sigma(x) \in (W_{e,s+1} - W_{e,s})]\}$$

Consider any $(\langle f \restriction s_0, r\rangle)$ containing $f$. Since $r \in \mathcal{S}$, there exist $t_0, y \leq \omega$ such that $y = \max\{\mathrm{ran}(r)\}$ and $t_0 = \mu t[r(t) = y]$. Under the assumption that $W_e$ is infinite, there is an $u > s_0, t_0$ and a $z > y$ such that $z \in (W_{e,u} - W_{e,u\dot{-}1})$. Hence $\langle f \restriction u \cup \{\langle u+1, z\rangle, r\rangle\} \in U_e$. Thus $U_e$ is dense along $f$, and so $f$ meets $U_e$. Hence for every infinite $W_e$, $\exists s[\exists x \in (W_{e,s} - W_{e,s\dot{-}1}) \cap [f \restriction s]]$. Thus $f$ is an enumeration of $A$ witnessing its promptly simplicity via the identity function. □

iv) Let $[f] = [g] \oplus [h]$ and consider the set

$$U_e = \big\{\langle \sigma \oplus \tau, r\rangle \;\big|\; |\tau| = |\sigma| \;\wedge\; \exists x < |\sigma|[x \in [\sigma] \;\wedge\; \varphi^{\tau}_{e,|\sigma|}(x)\!\downarrow = 0] \\ \wedge\; r \in \mathcal{S} \;\wedge\; \forall s > |\sigma \oplus \tau|[r(s) \geq \max[\sigma \oplus \tau]]\big\}$$

Note that if $f$ meets $U_e$, then $[g] \neq \varphi_e^{[h]}$ by the use principle (since the computation $\varphi^{\sigma}_{e,|\sigma|}(x)\!\downarrow = 0$ is preserved if $f$ respects a restraint $r(s)$ such that $\forall s > |\sigma|[r(s) \geq \max[\sigma \oplus \tau]]$). Hence it suffices to assume that $[g] = \varphi_e^{[h]}$ (i.e. $\chi_{[g]} = \chi_{\varphi_e^{[h]}}$) and show that $U_e$ is dense along $f$, and hence that $f \in (U_e)$, a contradiction. Consider any $(\langle f \restriction s_0, r\rangle)$ containing $f$. Since $r \in \mathcal{S}$, there exist $t_0, y \leq \omega$ such that $y = \max\{\mathrm{ran}(r)\}$ and $t_0 = \mu t[r(t) = y]$. Note that by ii), $\{y \mid y \notin [f]\} \cap \{2n \mid n \in \omega\}$ is infinite, and thus we can find $z \in \{y \mid y \notin [g]\}$ such that $z > y$. Since $[g] = \varphi_e^{[h]}$ by assumption, we can also find $u > s_0, t_0$ such that $\varphi_e^{[h \restriction u]}(z)\!\downarrow = 0$. Now define $\sigma = g \restriction u \cup \{\langle u+1, z\rangle\}$ and $\tau = h \restriction u \cup \langle u+1, h(u)\rangle$ and $r'$ and

$$r'(s) = \begin{cases} 0 & s \leq u \\ \max([\sigma \oplus \tau]) & s > u \end{cases}$$

Finally let $r'' = \mathbf{max}(r', r'')$. Note that we then have $z \in [\tau(u+1)]$ and $\varphi_e^{\sigma}(z) = 0$. Hence $(\langle \sigma \oplus \tau, r''\rangle) \in (U_e)$. Thus $U_e$ is dense along $f$, and consequently $f$ meets $U_e$, contrary to our assumption that $[g] = \varphi_e^{[h]}$. A symmetric arguments yields $[h] \neq \varphi_e^{[g]}$.□

The preceding theorem gives only a sampling of the properties of e-generic enumerations. As remarked by Jockusch, it is often the case that when a property can be shown to hold of a set via a finite injury argument, this property is shared by the sets given as the ranges of enumerations e-generic with respect to $\mathcal{S}$. For instance it may be shown that if $A$ is the range of such an enumeration and $A = A_0 \oplus A_1$, then the difference set $A_0 - A_1$ does not have r.e. degree. This demonstration is very much in the vein of as those of Theorem 4.2, but involves the successive use of two conditions $U, V \subset \omega^\omega \times \mathcal{S}$ to show that a single requirement is satisfied by $A$. A number of other results have been obtained about the degrees of $f$ e-generic with respect to $\mathcal{S}$, many of them relating to the fact that if $f$ e-generic and $A = \mathrm{ran}(f)$, then $A$ is promptly simple and low. For instance, we immediately have that if $\mathbf{a}$ is a degree containing an e-generic $A$, and $\mathbf{b_0} \leq_T \mathbf{a} \leq_T \mathbf{b_1}$, then $\mathbf{b_0}$ must be low and $\mathbf{b_1}$ must be promptly simple as the low degrees are closed downward, and the promptly simple degrees upward (cf. [6]). The preceding construction of a e-generic enumeration, can also be supplemented with prompt permitting requirements (cf. [6],[9] XIII.1) so as to obtain an e-generic degree below any promptly simple degree.

Evidently, however, the results that can be obtained via e-generic forcing with the class of step functions $\mathcal{S}$ does not exhaust the results which can be derived with the finite injury method in general. In particular, it is reasonable to wonder whether there is a class of restraint functions capable of reproducing finite injury results derived through the so-called Sacks' [8] preservation strategy. This technique is used to construct an object (or a class of objects) with certain properties relative to a fixed r.e. non-recursive $C$. The restraint functions $r(e,s)$ used in these constructions are designed in such a way that if $f$ meets conditions involving $r(e,s)$, *agreement* between $C_s$ and $\varphi_e^{[f\restriction s]}$ will be preserved for as long as possible (where $\{C_s\}$ is a recursive enumeration of $C$). Step functions are insufficient to model this behavior because agreement between $C$ and $\varphi_e^{[f\restriction s]}$ may grow indefinitely after a requirement is initially meet, meaning that we may not know how high a restraint function will grow when an initial attempt to respect it is made. In finite injury applications of the preservation strategy, such as the Sacks Splitting Theorem, however, the construction is so contrived that the restraint functions remain finite.

To adapt this technique to the forcing method, we require a slightly different class of functions defined, which we refer to the **preservation functions** and are defined as follows.

**Definition 4.5** Let $C$ be an arbitrary r.e. set and $\{C_s\}$ a fixed recursive enumeration of C. For each r.e. index $e$ and $i$, we define the **length function** $l(i,e,s)$, the **restraint function** $r(i,e,s)$ and **preservation function** $p(i,e,s_0,s)$ as follows :

- $l(i,e,s) = \max\{x \mid \forall y < x[\varphi_e^{W_{i,s}}(y)\downarrow = C_s(y)]\}$
- $r(i,e,s) = \max\{u(W_{i,s},e,x,s) \mid x \leq l(i,e,s)\}$
  (Recall that the **use function** $u(X,e,x,s)$ is defined to be the largest query to oracle $X$ made by the computation $\varphi_{e,s}^X(x)$.)
- $$p(i,e,s_0,s) = \begin{cases} 0 & s < s_0 \\ r(i,e,s) & s \geq s_0 \wedge \neg\exists s_o \leq t < s \exists x \in (W_{i,t+1} - W_{i,t})[x \leq r(i,e,t)] \\ 0 & \text{otherwise} \end{cases}$$

The length function $l(i,e,s)$ should be thought of as keeping track of the length of the longest initial segment on which $C_s$ and $\varphi_e^{W_{i,s}}$ agree, up to and including their first disagreement. The restraint function $r(i,e,s)$ keeps track of the usages of computations $\varphi_e^{W_{i,s}}$ in $W_{i,s}$ which witness this agreement. The preservation function $p(i,e,s_0,s)$ is defined to be 0 on an initial segment up to $s_0$ and then mimic the restraint function as long as the elements enumerated by $W_i$ respect it. If at any stage $s > s_0$ the enumeration of $W_{i,s}$ ceases to respect $r(i,e,s)$, the preservation function is reset to 0. We will ultimately be interested in forcing with conditions defined in terms of the restraint class generated by preservation functions of this form. Note that as defined, $p(i,e,s_0,s)$ is primitive recursive and we may effectively find an index for it from the parameters $i,e$ and $s_0$. We now establish that for a fixed r.e. *non-recursive* set $C$, that all preservation functions have finite limits.

**Lemma 4.4** Let $C$ be a r.e. non-recursive set and $\{C_s\}$ any recursive enumeration of $C$. Then for all enumerations $W_i$, and indices $e$, $\forall s_0[\lim_{s\to\infty} p(i,e,s_0,s) < \infty]$.

**Proof** Suppose not and let $i$ and $s_0$ be such that $\lim_{s\to\infty} p(i,e,s_0,s) = \infty$. Note by the definition of $p(i,e,s_0,s)$, we must then have $\forall s \geq s_0[p(i,e,s_0,s) = r(i,e,s)]$. Hence we also have that $\lim_{s\to\infty} r(i,e,s) = \infty$ and $\neg\exists t \geq s_0 \exists x \in (W_{i,t+1} - W_{i,t})[x \leq r(i,e,t)]$. These facts allow us to use a preservation argument, similar to that employed in the original Sacks construction, twice over : 1) to show that $\lim_{s\to\infty} l(i,e,s) = \infty$; 2) to show that given 1), $C$ must be recursive contrary to assumption.

We show that $\lim_{s\to\infty} l(i,e,s) = \infty$ inductively. Choose any stage $s > s_0$ and suppose $l(i,e,s) = n$. We claim that there is a $t > s$ such that $\forall u \geq t[l(i,e,u) > n]$. Let $r(i,e,s) = m$. Since $\lim_{s\to\infty} r(i,e,s) = \infty$, we can find $t > s$ such that $r(i,e,t) > m$. Note that by definition of $r(i,e,s)$ we must have that $\max\{u(W_{i,t},e,x,t) \mid x \leq l(i,e,t)\} > \max\{u(W_{i,s},e,x,s) \mid x \leq l(i,e,s)\}$. Since $\neg\exists t \geq s_0 \exists x \in (W_{i,t+1} - W_{i,t})[x \leq r(i,e,t)]$, we must hence have $\forall x \leq l(i,e,s)[\varphi_{e,s}^{W_{i,s}}(x)\downarrow = \varphi_{e,t}^{W_{i,t}}(x)\downarrow]$ and also $\forall x \leq n[u(W_{i,s},e,x,s) = u(W_{i,t},e,x,t)]$. We claim that $l(i,e,t) > l(i,e,s)$. Suppose not. Since $r(i,e,t) > r(i,e,s)$ and the "$\varphi_e^{W_{i,s}}$ side" of the equality $\varphi_e^{W_{i,s}}(x) = C_s(x)$ is fixed, there must exist a $\exists y \leq l(i,e,s)[y \notin C_s \ \wedge \ y \in C_t]$ and also $u(W_{i,t},e,z,t) > r(i,e,s)$ for some $z \leq l(i,e,t)$. Note that in this case, however, the *disagreement* $C_s(y) \neq \varphi^{W_{i,t}}(y)$ will be preserved throughout the rest of the enumeration of $W_i$ by the fact that $\neg\exists t \geq s_0 \exists x \in (W_{i,t+1} - W_{i,t})[x \leq r(i,e,t)]$. This is the case because $r(i,e,s)$ is defined so as to preserve all computations $\varphi_e^{W_{i,s}}(x)\downarrow$ up to *and including* the first disagreement between $\varphi_e^{W_{i,s}}$ and $C_s$. Since $\{C_s\}$ is a recursive enumeration, the value of $C_s(x)$ can only change from 0 to 1 and never back again. Hence once a disagreement is found between $\varphi_e^{W_{i,s}}$ and $C_s$, it is either preserved forever or replaced (finitely often) with a smaller one. Hence $\exists z \leq l(i,e,s)\exists u \geq t[\lim_{s\to\infty} r(i,e,s) = u(W_{i,u},e,y,u)] < \infty$, contrary to assumption.

Now assume that $\lim_{s\to\infty} l(i,e,s) = \infty$. We show that $C$ is recursive. In order to decide $C(y)$, choose $s > s_0$ such that $l(i,e,s) > y$. We claim that $\varphi_e^{W_{i,s}}(y) = \varphi_e^{W_i}(y) = C_s(y) = C(y)$. This follows from the fact that (*) $\forall t \geq s[l(i,e,t) > y \ \wedge \ r(i,e,t) \geq \max\{u(W_{i,s},e,x,s) \mid x \leq y\}]$ which we establish inductively. (*) holds at $t = s$ by the definition of $r(i,e,s)$. For $t+1 > s$, note that "$\varphi_e^{W_{i,t}}$ side" is again preserved by all $x \in (W_{i,t+1} - W_{i,t})$. Hence (*) fails only if there is an $x < l(i,e,t)$ such that $C_{t+1}(x) \neq C_t(x)$. As above, if such a disagreement were to be found, it would be preserved throughout the rest of the enumeration of $W_i$, contrary to $\lim_{s\to\infty} l(i,e,s) = \infty$. $\square$

By this lemma, all preservation function for an arbitrary enumeration $W_{i,s}$ and index $e$ have finite limits with respect to a given non-recursive set $C$ and an enumeration of $\{C_s\}$. It is important to realize, however, that the argument used to establish the lemma does not go through if we do not assume the non-recursiveness of $C$. As this fact cannot be effectively established for an arbitrary r.e. set, Lemma 4.4 does not provide a uniform mechanism for producing a class of preservation functions with finite limits for any r.e. set. And hence the following theorem does not provide a (paradoxical) mechanism of producing $A$ such that $A \not\leq_T A$.

For any given r.e non-recursive set $C$ and enumeration $\{C_s\}$, we now define $\mathcal{P}$ as the restraint class generated by the set of the functions $\{p(i,e,s_0,s) \mid i, s_0 \in \omega\}$ together with the step functions $\mathcal{S}$. Since all of the restraint functions in this class are primitive recursive and have finite limits, by Theorem 4.1, there exists a e-generic enumeration $f$ with respect to $\mathcal{P}$ for every such r.e. non-recursive $C$. Let $B = [f]$. We we now catalogue several of $B$'s properties.

**Theorem 4.3** Let $C$ be a r.e. non-recursive and $\{C_s\}$ a recursive enumeration of $C$. If $B$ is e-generic by enumeration $f$ with respect to the resulting restraint class $\mathcal{P}$, then $B$ has the following properties :

1) $C \not\leq_T B$

2) $B$ is low.

3) Let $M$ be any infinite primitive recursive set. Then $\overline{A} \cap M$ is infinite.

4) $A$ is promptly simple.

5) If $B = B_0 \oplus B_1$, then $B_0 \not\leq_T B_1$ and $B_1 \not\leq_T B_0$.

**Proof**
1) Note that since the construction of Theorem 4.2 is effective, $f$ is r.e.Hence there is some r.e. index $i$ such that $\forall s[W_{i,s} = [f \upharpoonright s]]$. Consider the following set :

$$U_{e,i} = \{\langle \sigma, r\rangle \mid r \in \mathcal{P} \ \wedge \ \exists s_0 \forall s > s_0[r(s) \geq p(i,e,s_0,s)]\}$$

We claim that $U_{e,i}$ is dense along $f$. To see this, take any pair $\langle f \restriction s_0, r\rangle \in \mathcal{P}$ such that $f \in (\langle f \restriction s_0, r\rangle)$. Since $r \in \mathcal{P}$, it must have a finite limit by Lemma 4.4. Let $s' = \mu t[\forall u[r(t) \geq r(u)]]$ and $s_1 = \max\{s_0, s'\}$. We define the function $r'$ as follows :

$$r'(s) = \begin{cases} r(s) & s \leq s_1 \\ \max\{p(i,e,s_1,s), r(s)\} & s > s_1 \end{cases}$$

Clearly we have $\forall s[r'(s) \geq r(s)]$, and hence $(\langle f \restriction s_0, r'\rangle) \subset (\langle f \restriction s_0, r\rangle)$. Since $\langle f \restriction s_0, r'\rangle \in U_{e,i}$, we may conclude that $U_{e,i}$ is dense along $f$. Hence $f$ meets $U_{e,i}$. Let $\langle \sigma, r\rangle \in U_{i,e}$ be such that $f \supset \sigma$ and $\forall s[f(s) > r(s)]$. Since $\exists s_0 \forall s > s_0[r(s) > p(i,e,s_0,s)]$, we have that $\forall s > s_0[f(s) > p(i,e,s_0,s)]$. Note that by the definition of $p(i,e,s_0,s)$, this means that $\forall s \geq s_0[f(s) > r(i,e,s)]$. If we now assume for a contradiction that $C = \varphi_e^{[f]}$, then $\lim_{s\to\infty} l(i,e,s) = \infty$ and a preservation argument identical to that of the Lemma 4.4 establishes that $C$ is recursive. Thus $C \neq \varphi_e^{[f]}$.
2) - 5) Since $\mathcal{S} \subset \mathcal{C}$, 2)-5) hold by analogous arguments analogous to Theorem 4.2.1-4. $\square$

Theorem 4.3 demonstrates the added generality obtainable by forcing with conditions $U \subseteq \omega^{<\omega} \times \mathcal{P}$ over the conditions in $U \subseteq \omega^{\omega}$ used by Jockusch. Forcing of the former variety is capable of encompassing a wider range of finite injury arguments since the conditions employed can grow in a manner which is not known at the time they are established – this fact being a distinguishing feature of the Sacks' preservation strategy. By requiring that $f \in (\langle \sigma, r\rangle)$, we wish to ensure that that elements enumerating by $f$ late in the construction do not destroy agreement achieved at earlier stages. This requirement is still finitary because $r$ has a finite limit. But since we do not know initially how long agreement between $\varphi_e^{[f\restriction s]}$ and $C_s$ will continue to grow, agreement cannot be preserved indefinitely by merely requiring that $f$ ends up in appropriate cylinder $(\sigma)$.

It is our current belief that the technique of forcing with finitary restraint functions can be naturally extended to cover infinite injury results such as the Thickness Lemma. The hallmark of such results is the infinitary nature of the positive requirements occurring in the constructions. Such requirements lead to positive requirements which cannot be satisfied all at once, but rather whose satisfaction must be deferred to later stages which cannot be predicted in any bounded manner. As such infinite requirements can also be decomposed into infinitely finite requirements, it appears to be possibly to force their satisfaction using a class of functions related to the preservation functions introduced above. In this case, however, rather then establishing a lower bound on $f$, these functions would be charged with requiring that $f$ enumerate certain elements at finitely many particular stages. Conditions of the form $\langle \sigma, b_e, r\rangle$ are now required, where $b_e \in \omega^{\subset\omega}$ is a partial function making finitely many demands on $f$ in $\omega^{[e]}$. Results in this direction would reaffirm the general premise of this paper that injury constructions may always be reformulated as forcing arguments on the appropriate topology.

## References


[1] B. Bollobas *Combinatorics*, Cambridge University Press, Cambridge, 1986.

[2] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[3] C. Jockusch, "Genericity for recursively enumerable sets" in H. Ebbinghaus, G. Miller and G. Sacks eds. *Recursion Theory Week*, Lecture Notes in Mathematics No. 1141, Springer-Verlag, Berlin, 1985.

[4] S. Kleene and E. Post "The uppersemilattice of degrees of recursive unsolvability," *Annals of Mathematics*, 59, 379-407, 1954.

[5] K. Kunen, *Set Theory : An Introduction to Independence Proofs*, North Holland, Amsterdam, 1980.

[6] W. Maass, "Recursively enumerable generic sets," *Journal of Symbolic Logic*, 47, p. 809-823, 1982.

[7] P. Odifreddi, *Classical Recursion Theory*, North Holland, Amsterdam, 1989.

[8] G. Sacks, "On the degrees less than $\emptyset'$," *Annals of Mathematics*, 77, p. 211-231, 1963.

[9] R. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1989.