Multi-Player Logics

MSc Thesis (*Afstudeerscriptie*)

written by

Loes Olde Loohuis
(born May 15th, 1985 in Oss, Netherlands)

under the supervision of **Dr Yde Venema**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:	**Members of the Thesis Committee:**
September 5, 2008	Prof. dr. Johan van Benthem
	Prof. dr. Peter van Emde Boas
	Prof. dr. Jouko Väänänen

Institute for Logic, Language and Computation

Contents

Chapter 1

Introduction

During the second half of the twentieth century, two-player games have become an important tool in many branches of logic. Examples of these games are back-and-forth games to compare structures (such as the Ehrenfeucht-Fraïssé games attributed to Ehrenfeucht and Fraïssé [7, 8, 5]), dialogue games to express formal proofs (originally due to Lorenzen [13]) and semantic games to define truth [9].

In (classical) game semantics, the truth of a formula is defined via a game between two players, commonly known as 'Eloise' and 'Abelard' (or 'Myself' and '(Evil) Nature' [10]). It was Hintikka who, in [9], first introduced a two-person semantical game G to define truth of propositional formulas in terms of winning strategies of the players. This game can be summarized as follows.

Given a propositional formula ϕ and a propositional valuation V, the game $G(\phi, V)$ starts at position ϕ. At this initial position, Eloise has the role of Verifier and Abelard that of Falsifier. At each position of the form $\phi_1 \vee \phi_2$ it is Verifier who has a choice between ϕ_1 and ϕ_2, and the game continues at the chosen subformula. At a conjunction $\phi_1 \wedge \phi_2$ it is the Falsifier who chooses one of the conjuncts. If a negation, $\neg\phi_1$, is encountered, the game continues at ϕ_1 except that the roles of the players are reversed: Verifier becomes Falsifier and vice versa. The play ends when no further moves can be made, that is, when the game has reached an atomic subformula. At this point the current Verifier is deemed the winner if the resulting proposition is true, and the current Falsifier is considered to be the winner if it is false. The original propositional formula ϕ is defined to be true given a valuation in case Eloise has a winning strategy for the game, and false whenever Abelard has one.

Later, Hintikka extended his original idea to natural language semantics and developed game semantics for imperfect information [11]. Semantic games, or sometimes called evaluation games, have also been applied to many other branches of logic. Examples are first and second order logic, infinitary logic, modal logic and fixpoint logics like the modal μ-calculus (see cf. [6]).

The simplest game-semantic setting for propositional logic provides us with a new vantage point regarding the boolean connectives: we can think of the disjunction $\vee$ as a disjunction - or a choicepoint - for Eloise ($\vee_\exists$) and the conjunction $\wedge$ as a disjunction for Abelard, ($\vee_\forall$). The negation $\neg$ can be seen as a role-switch operator permuting the roles of Eloise and Abelard ($\neg_{\exists\forall}$).

Having realized this, it takes only a small leap to enter a world of multi-player game semantics. A question that naturally arises in this context is: what would the game look like if there are more than two players? What if there are three, or maybe even an arbitrary finite set players? In other words, can we generalize the game semantics for various logics to a multiple-player setting? And, more importantly, what kind of logic arises in this context?

The first steps towards generalizing game semantics to the setting of multiple-player games were taken by Abramsky [1] and Tulenheimo and Venema [14]. In [1], a logic is developed, whose semantics is defined in terms of strategies for n-person games. This logic, called $\mathcal{L}_\mathcal{A}$, allows for a compositional analysis of partial information constructs like Henkin-style branching quantifiers. In [14], two formulations of propositional logic are developed, whose semantics is described not by a two-player, but by a three-player game. Both [1] and [14] address the question: what kind of logic has a natural semantics in multi-player games? The two papers share the same starting point; they introduce connectives indexed by agents that allow for choices of that agent and negations as role-switch operators. However, methods and objectives quickly diverge. In [1], the main aim is to represent partial information constraints in a multi-player setting so as to obtain a compositional account of constructs of partial information. In [14], the main focus lies in studying multi-player systems from a logical point of view by revealing properties of the three-player logic and comparing these to properties of classical propositional logic. The pioneering contributions of [1] and [14] have opened doors towards studying logics in a multi-player setting.

In this thesis we continue the work of Tulenheimo and Venema [14] by developing a multi-player propositional logic (*MPL*) and a multi-player modal logic (*MML*) for any finite set of n players. To see how this works in practice let us introduce our most elementary multi-player logic, the logic *MPL*.

Let A be a finite set of agents, or players. Intuitively, each player $i \in \mathsf{A}$ has his own connective $\vee_i$, which in the game represents a choice for the player who currently has the role of player i. Also, for every two players $i, j \in \mathsf{A}$ there is a negation, or role-switch operator, permuting the roles of i and j. Valuations of the logic *MPL* assign to each proposition letter a subset of players who, at the end of the play, will be the winners of the game. The semantics of the logic is defined by an n-player game $G(\phi, V)$. This game is very similar to Hinitkka's original game. Initially, all the players have their own role (that is, every player i plays the role of player i). Each move consists of allowing the owner of the dominant connective to pick one of its branches, after which the play continues in that subformula. Moreover, when a negation $\neg_{ij}$ is encountered the players i and j switch roles. When a propositional formula p is reached, no more moves can be made and the play has come to an end. The players whose current roles are winning for p given the valuation V, are the winners of the match. The others lose.

In our multi-player logics, truth, satisfiability and validity are defined relative to the agent. We say that the *MPL*-formula ϕ is true for player i given a valuation V, if player i has a winning strategy for the game $G(\phi, V)$. We say that a *MPL*-formula ϕ is i-satisfiable when there is a valuation V such that ϕ is true for player i given V. A *MPL*-formula ϕ is i-valid iff there is a valuation V such that ϕ is true for player i given V.

Note that in case there are only two agents, the logic *MPL* is similar to the propositional logic *PL*. However, an important difference arises from the fact that we allow valuations to assign all (or none) of the players as the set of winners given a proposition letter p. Thus, in the most general case, given a *MPL*-formula ϕ, there are four different possible outcomes of the game: all the players win the game, either one of the players wins the game, and none of the players win the game. It turns out that if we identify these outcomes with the values in Belnap's four-valued logic, the semantics of the two 'disjunctions' of *MPL* agrees with that of the conjunction and disjunction in Belnap's system, but the semantics of the negation differs [2].

We can extend *MPL* to a (basic) multi-player modal logic *MML* by introducing a modal operator $\Diamond_i$ for every player $i \in \mathsf{A}$. Formulas of *MML* are interpreted over classical Kripke frames with multi-player valuations assigning to each proposition letter and each state a set of winners. These models will be called 'multi-player models'. The semantics of *MML* semantics is defined using an extension of the game defining the semantics of *MPL*.

Because in our multi-player setting it no longer makes sense to talk about 'valid' formulas we should pause and reflect for a moment on what we actually mean when we refer to a multi-player *logic*. Given any multi-player logic we cannot simply think of it as a set of valid formulas. We can, however, think of the multi-player logic as the collection of i-valid formulas for each agent i. Moreover, we can think of two formulas ϕ and ϕ' as being i-equivalent if for player i, the formulas are i-satisfied in exactly the same situations. In case of *MPL* this is the case if for every valuation V, ϕ is i-satisfied by V iff ϕ' is i-satisfied by V. Two formulas are said to be equivalent if they are i-equivalent for all players $i \in \mathsf{A}$. A natural question that arises in this context is whether we can axiomatize the notion of (i-)equivalence. One of the main aims of this thesis is to solve exactly this issue. We will do so by studying the logics algebraically and developing a (quasi-)equational axiom system.

The first contribution of this thesis is to extend the game theoretical semantics of propositional logic, modal logic and modal μ-calculus to a n-player setting, for arbitrary finite n. We also develop some general insights about these logics, such as invariance of the i-satisfiablility in *MML* under various operations on models or a coalgebraic perspective on multi-player models.

The second and main contribution are two algebraic representation theorems; a multi-player analogue of Stone's representation theorem for *MPL*, and a multi-player analogue of Jónsson Tarski theorem for *MML*. Regarding the first theorem, we proceed as follows. On the one hand, we construct concrete multi-player algebras that exactly capture the semantics of *MPL*. On the other, we introduce a (quasi-)equational axiom system for *MPL* and define 'multi-Boolean algebras' - abstract algebras satisfying these axioms. In the theorem we show that every multi-Boolean algebra is isomorphic to a concrete multi-player algebra.

Regarding the second theorem, we first give an axiomatic definition of the notion of abstract 'multi-modal algebras'. Secondly, we show that every Kripke frame can be characterized as an algebra. We show that for every abstract multi-modal algebra $\mathfrak{A}$, there exists a frame (its 0-Prime Filter Frame) such that $\mathfrak{A}$ is embeddable in the algebraic representation of this frame. With this result we establish a deep relation between game semantics of *MML* defined over relational structures and abstract multi-modal algebras.

Thirdly, we show decidability and prove some complexity results for the logics *MPL* and *MML*. We show that in the most general case, that is, in case valuations are arbitrary, the i-satisfiability problem of *MPL* can be solved in polynomial time. Moreover, we show that *MML* with restricted valuations (meaning that for each proposition letter, there have to be both winners and losers at each state) lacks the polysize model property. We conclude this last chapter by proving that the i-satisfiability problem of *MML* is in *PSPACE*.

The structure of the thesis is as follows. In the next chapter, we will introduce and discuss the logics *MPL*, *MML* and a multi-modal μ-calculus, μ*MML*, in formal detail. In chapter 3 we will study the logic *MPL* from an algebraic perspective and prove the analogue of Stone's representation theorem in our multi-player setting. In chapter 4 we will study the logic *MML* by algebraic means and show a multi-player version of the Jónsson-Tarski theorem. Finally, in chapter 5 we will take first steps towards analyzing computability and complexity of the logics *MPL* and *MML*. We establish decidability of both logics and inquire about time and space complexity.

Chapter 2

Multi-Player Logics

2.1 Introduction

In this chapter we will introduce multi-player propositional logic (*MPL*) and extend it to multi-player modal logic (*MML*) and multi-player μ-calculus μMML. In [14], Tulenheimo and Venema already introduced two logics for three players, PL_0^3 and PL^3. The main difference between the two approaches is the following. In the first, PL_0^3, each play is won by some players and lost by the others. In general, any subset of the set of players can win the game. In line with Hintikka's interpretation, negations of PL_0^3 represent role-switch operations between two players. In the second generalization of propositional logic, PL^3, payoff is defined in terms of a ranking of the agents. Moreover, negations are slightly more complicated: they do not just permute roles of two players, but of all the players at the same time. In this chapter, we will define the syntax and semantics of *MPL* (and later *MML* and μMML) along the lines of PL_0^3. That is, valuations of the logic *MPL* assign to each proposition letter a subset of players. Moreover, for every two players $i, j \in \mathsf{A}$ there is a role-switch operator, i.e. a 'negation', permuting the roles of i and j.

2.2 Multi-Player Propositional Logic

In this section we will introduce the Multi-Player Propositional Logic (*MPL*).

Let A be a finite set of agents or players and P the set of proposition letters. The *syntax* of *MPL* is defined as follows:

$$\phi ::= \bot_i \mid p \mid (\phi \vee_i \psi) \mid \neg_{ij}\phi,$$

where $i, j \in \mathsf{A}$ and $p \in \mathsf{P}$. Throughout the thesis, the symbol $\neg_{ij}$ will be referred to as a 'negation'. Also, we sometimes refer to $\vee_i$ as a 'disjunction' for player i. It turns out that *MPL* is a syntactical fragment of Abramsky's logic $\mathcal{L}_{\mathcal{A}}$. A *valuation* V is defined as follows:

$$V : \mathsf{P} \to (\mathsf{A} \to \{\mathbf{w},\mathbf{l}\}))$$

where $\mathbf{w}$ and $\mathbf{l}$ stand for *win* and *lose* respectively. Intuitively, V assigns to each proposition variable $p \in \mathsf{P}$ a subset of A of winners for p. We say that player i wins p if $V(p)(i) = \mathbf{w}$ and loses otherwise. To the constant $\bot_i$ each valuation assigns $\mathbf{l}$ to player i and $\mathbf{w}$ to all players other then i. That is, $V(\bot_i)(i) = \mathbf{l}$ and $V(\bot_i)(j) = \mathbf{w}$ when $i \neq j$. For convenience, we will assume that there is one player, player 0, that is always in A.

With n denoting the number of players, i.e., $|\mathsf{A}| = n$, we define the *semantics* of *MPL* using n-player games $G(\phi, V)$, where V is a valuation and ϕ is a formula. Positions of the game are of the form (ψ, ρ), where ψ is a subformula of ϕ and ρ a role distribution. A role distribution ρ is a bijection from the set of agents to the set of agents; $\rho : \mathsf{A} \to \mathsf{A}$. If ρ is a role distribution then we define ρ_{ij} such that $\rho_{ij}(i) = \rho(j)$, $\rho_{ij}(j) = \rho(i)$, and for all the other $k \in \mathsf{A}$ such that $k \notin \{i, j\}$ $\rho_{ij}(k) = \rho(k)$. Thus, ρ_{ij} is the same function as ρ, except for the fact that the values (the roles) for players i and j are switched. Given a position (ψ, ρ) we say that a player i assumes the role of player $\rho(i)$. Equivalently, $\rho^{-1}(i)$ denotes the player that assumes role i. The initial position of the game $G(\phi, V)$ is (ϕ, Id), where Id denotes the initial role distribution, the identity function. The instantiation of the game $G(\phi, V)$ with starting position (ϕ, Id) will be denoted $G(\phi, V)@(\phi, Id)$. The rules of the game are the following:

At any position (ψ, ρ)

- If $\psi \in \mathsf{P}$ then the game has come to an end. A player i is a winner if and only if $V(\psi)(\rho(i)) = \mathbf{w}$.
- If $\psi = \bot_i$ then the game has come to an end. All players except the player that assumes the role of i, i.e. $\rho^{-1}(i)$, win the game.
- If $\psi = \chi_1 \vee_i \chi_2$ then player $\rho^{-1}(i)$, the player who currently assumes the role of i, chooses χ_j with $j \in \{1, 2\}$ and the next position of the game is (χ_j, ρ).
- If $\psi = \neg_{ij}\chi$ then nobody plays and the game continues at position (χ, ρ_{ij}).

The rules of the game as described above are summarized in the following table:

Position	Player	Admissible moves	Winners
(p, ρ)	-	-	$\{i \in \mathsf{A} \mid V(p)(\rho(i)) = \mathbf{w}\}$
$(\bot_i, \rho)$	-	-	$\{j \in \mathsf{A} \mid j \neq \rho^{-1}(i)\}$
$(\chi_1 \vee_i \chi_2, \rho)$	$\rho^{-1}(i)$	$\{(\chi_j, \rho) \mid j \in \{1, 2\}\}$	-
$(\neg_{ij}\chi, \rho)$	-	$\{(\chi, \rho_{ij})\}$	-

Table 2.1: The game G for *MPL*

We say that a player i has a *winning strategy* for $G(\phi, V)@(\phi, Id)$ if player i can play in such a way that he/she is guaranteed to be one of the winners of the game. An arbitrary position (ϕ, ρ) is said to be *winning* for player i if she has a winning strategy for the game $G(\phi, V)@(\phi, \rho)$. We say that a formula ϕ of *MPL* is *i-satisfied* by a valuation V, if player i has a winning strategy for $G(\phi, V)@(\phi, Id)$. Following up on this we say that a formula ϕ is *i-satisfiable* if there is a valuation V such that ϕ is i-satisfied by V. Similarly, ϕ is called *i-valid* if every valuation V i-satisfies ϕ. We define two formulas ϕ and ϕ' of *MPL* to be *i-equivalent* if for every valuation V, ϕ is i-satisfied by V iff ϕ' is i-satisfied by V. ϕ and ϕ' are *equivalent* if for every agent i the two formulas are i-equivalent.

Note that the definition of valuation puts no restrictions on the number of winners a valuation assigns to a proposition letter. In particular, given a propositional variable p it might be the case that $V(p)(i) = \mathbf{w}$ (or $\mathbf{l}$) for all players. For such valuations and appropriate formulas ϕ it could be that all (or none) of the players have a winning strategy for the game $G(\phi, V)@(\phi, Id)$. In order to prevent all of the players from having a winning strategy, we could make use of *restricted valuations.* A valuation V is called *restricted* if for every $p \in \mathsf{P}$ there exist $i, j \in \mathsf{A}$ such that $V(p)(i) = \mathbf{w}$ and $V(p)(j) = \mathbf{l}$. The use of restricted valuations does not rule out the existence of formulae for which no player has a winning strategy, see for example 2.2.1(vi) below. In the sequel, unless stated otherwise, we will assume valuations are arbitrary. In chapter 5 it will turn out that the choice for arbitrary versus restricted valuations greatly influences the computational properties of the logic. However, this is jumping ahead. Let us first develop a feeling for the semantics of *MPL.*

Example 2.2.1. Let A be a set of n players with $0, 1 \in \mathsf{A}$ and let V an arbitrary valuation. We will have a look at some examples of formulas ϕ of *MPL.*

(i) Let $\phi = p$. Only the players i such that $V(p)(i) = \mathbf{w}$ have a winning strategy.

(ii) Let $\phi = \bot_0$. Every player except for 0 has a winning strategy. Note that $\bot_0$ is not 0-satisfiable and i-valid for all $i \neq 0$.

(iii) Let $\phi = \neg_{01}\psi$. Player 0 has a winning strategy for $G(\phi, V)$ iff 1 has a winning strategy for $G(\psi, V)$, and vice versa. For all the other players $j \notin \{0, 1\}$ j has a winning strategy for $G(\phi, V)$ iff j has a winning strategy for $G(\psi, V)$. Observe that negations $(\neg_{jk})$ are permutations of the set A. Note that in case $\phi = \neg_{ij}\psi$ with $i = j$, $\rho_{ij} = \rho$ and each player has a winning strategy for $G(\phi, V)$ iff he/she has a winning strategy for $G(\psi, V)$. Negations of this form can be seen as identity or empty permutations.

(iv) Let $\phi = \psi_1 \vee_1 \psi_2$. Since player 0 is not in the position to make the first move she has a winning strategy for $G(\phi, V)$ iff she has a winning strategy for both $G(\psi_1, V)$ and for $G(\psi_2, V)$. Player 1, on the other hand, can choose one of the ψ_is in his first move and therefore has a winning strategy for $G(\phi, V)$ iff he has a winning strategy for either one of $G(\psi_i, V)$, with $i \in \{1, 2\}$.

(v) Let $\phi = \bigvee_0\{\bot_j \mid j \in \mathsf{A}\}$. The only player that has a winning strategy is 0; she can choose to play any $\bot_j$ such that $j \neq 0$ and win. Any other player k risks the chance to lose because 0 can always choose to play $\bot_k$. From now on, instead of writing the formula $\bigvee_0\{\bot_j \mid j \in \mathsf{A}\}$ we will make use of the (obvious) short-hand notation $\top_0$.

(vi) Let $\phi = \bigvee_0\{\top_j \mid j \neq 0\}$. In this case no player has a winning strategy. Clearly, 0 cannot win because for any of the $\top_j$ with $j \neq 0$ only j will win. As for the other players, they don't have a guarantee that 0 will play 'their' $\top$. Hence they do not have a winning strategy. From now on, instead of writing the formula $\bigvee_0\{\top_j \mid j \neq 0\}$ we will make use of the short-hand notation $\bot$. It follows that there is no player $i \in \mathsf{A}$ such that $\bot$ is i-satisfiable.

The abbreviated connectives that were introduced in the preceding example are summarized as follows:

Symbol	Abbreviation for
$\top_i$	$\bigvee_i\{\bot_j \mid j \in \mathsf{A}\}$
$\bot$	$\bigvee_0\{\top_j \mid j \neq 0 \in \mathsf{A}\}$

Table 2.2: Abbreviated Symbols

Note that the player 0 used in the definition of $\bot$ is a fixed player, a player always assumed to be in A.

There are correspondences between standard boolean/propositional constructs and multi-player propositional operators. For player i the connective $\vee_i$ is like a classical disjunction since she can win the game for a $\vee_i$-formula iff she can win one of the disjuncts. For all the other players however, $\vee_i$ is like a conjunction. That is, they would have to win both 'conjuncts' in order to win a game of the form $\phi_1 \vee_i \phi_2$. Also, for i, $\bot_i$ is like $\bot$ and $\top_i$ like $\top$. For all the players other then i however, $\bot_i$ is like $\top$ and $\top_i$ is like $\bot$.

MPL in the case of two players

MPL is developed for any set of agents of a finite size n. It is interesting, however, to pause for a moment and have a look at the special case in which $n = 2$. Let $\mathsf{A} = \{0, 1\}$. It is not difficult to see that in the case of restricted valuations the semantics of *MPL* boils down to that of classical propositional logic (*PL*).

Given a (multi-player) valuation V we can define a propositional valuation V' as follows: $V'(p) = 1$ iff $V(p)(1) = \mathbf{w}$. If we now translate a formula ϕ of *MPL* with $\mathsf{A} = \{0, 1\}$ into a formula ϕ' of *PL* in the following way

- replace $\bot_0$ with $\bot$ and $\bot_1$ with $\top$,

- replace $\vee_0$ with $\wedge$ and $\vee_1$ with $\vee$,
- replace $\neg_{01}$ and $\neg_{01}$ with $\neg$,
- delete $\neg_{00}$ and $\neg_{11}$,

we obtain that for any formula ϕ and (multi-player) valuation V, 0 has a winning strategy for $G(\phi, V)$ iff $V'(\phi) = 0$ and 1 has a winning strategy for $G(\phi, V)$ iff $V'(\phi) = 1$. This can be proved by an easy induction and is left to the reader.

We can obtain the converse direction by a similar construction. Given a formula ϕ' and a valuation V' of *PL* we can define a *MPL* valuation V as follows: $V(p)(1) = \mathbf{w}$ iff $V'(p) = 1$. We can construct the corresponding formula ϕ by replacing $\perp$ with $\perp_0$, $\top$ with $\perp_1$, $\wedge$ with $\vee_0$, $\vee$ with $\vee_1$ and $\neg$ with $\neg_{01}$. In section 5.3 we will again discuss the implications of allowing only restricted valuation in a slightly different setting. In this section we will show that allowing only restricted valuations implies that satisfiability of *PL* is polytime reducible to i-satisfiability of *MPL*. But, again, let us try not to jump ahead to chapter 5.

In the case of arbitrary valuations the semantics of *MPL* with $\mathsf{A} = \{0, 1\}$ can be captured by the following 'truth tables'

$\vee_0$	$\emptyset$	0	1	01
$\emptyset$	$\emptyset$	0	$\emptyset$	0
0	0	0	0	0
1	$\emptyset$	0	1	01
01	0	0	01	01

$\vee_1$	$\emptyset$	0	1	01
$\emptyset$	$\emptyset$	$\emptyset$	1	1
0	$\emptyset$	0	1	01
1	1	1	1	1
01	1	01	1	01

$\neg_{01}$	$\emptyset$	0	1	01
	$\emptyset$	1	0	01

Here $\emptyset$ denotes that the set of player with a winning strategy for ϕ is empty, 0 (or 1) denotes that only 0 (or 1) has a winning strategy, and 01 denotes that both players have a winning strategy. Thus, in case of two players, *MPL* with arbitrary valuations can be viewed as a four-valued logic (with values $\emptyset, 0, 1, 01$, the four possible combinations of players with a winning strategy).

It is interesting to observe that the semantics of the two 'disjunctions' directly corresponds to the conjunction and disjunction in Belnap's four-valued logic. In his original paper, [2], Belnap developed his four-valued logic as a practical reasoning-guide for a computer that is threatened by receiving contradictory information. For the computer to be able to receive and reason about inconsistent information, Belnap suggested the following system: each basic item in the database of the computer should be marked with one of the following four values: **None** (our $\emptyset$); there is no information about the item, **F** (our 0); the computer is told that the item is false, **T** (our 1); the computer is told that the item is true, **Both** (our 01); the computer is both told that the item is false and that it is true. The computer should not only be able to store information, but also answer questions. If the computer is faced with a question whether a basic item is true, it returns the mark for that item. But the computer should also be able to reason about compound items: that is, decide about the truth-value of a conjunction $\wedge$, or disjunction $\vee$, of two items. Belnap suggested inference tables for $\wedge$ and $\vee$. These tables correspond directly to our 'truth value'-tables above: our $\vee_0$ corresponds to Belnap's $\wedge$ and $\vee_1$ to $\vee$. It is interesting to observe that despite the different motivations behind them, both formalisms, *MPL* and Belnap's four-valued logic, give rise to the semantics for the two-placed connectives. Belnap also has negation ($\sim$) in his system. This negation, however, does not correspond to our $\sim_{01}$ since in our scheme $\sim_{01} 01 = 01$ and $\sim_{01} \emptyset = \emptyset$ whereas in Belnap's system $\sim_{01} 01 = \emptyset$ and $\sim_{01} \emptyset = 01$.

This observation, regarding the correspondence between Belnap's four valued logic and the logic *MPL* with two players, touches upon a more general (open) question: what is the relation between the logic *MPL* and various multi-valued logics (like for example Kleene's three-valued logic)?

2.3 Multi-Player Modal Logic

As was already promised to the reader in the introduction, the logic *MPL* also has a modal extension, Multi Modal Logic (*MML*). As before, we start by defining the syntax of *MML* and then define its semantics using a multi-player game G.

Let A, with $|\mathsf{A}| = n$, be a finite set of agents, or players, and P the set of proposition letters. The *syntax* of *MML* is defined as follows:

$$\phi ::= \bot_i \mid p \mid (\phi \vee_i \psi) \mid \neg_{ij}\phi \mid \Diamond_i\phi,$$

where $i, j \in \mathsf{A}$ and $p \in \mathsf{P}$. On a standard Kripke frame $\mathcal{F} = (W, R)$ a *valuation* V is a function:

$$V : \mathsf{P} \to (W \to (\mathsf{A} \to \{0, 1\}))$$

Intuitively, V assigns to each proposition letter $p \in \mathsf{P}$ and each state $s \in W$ a subset of A of winners for p at s. We say that an agent $i \in \mathsf{A}$ wins p at s if $V(p)(s)(i) = \mathbf{w}$, and loses otherwise. A (multi-player) model M is a standard Kripke frame (W, R) with a (multi-player) valuation V over W (notation $M = (W, R, V)$).

We define the semantics of *MML* using an n-player game $G(\phi, M)$, relative to the formula ϕ and model M. Positions of the game $G(\phi, M)$ will be of the form

(ψ, t, ρ) where ψ is a subformula of ϕ, t is a state in W and ρ is a role distribution. The initial position is (ϕ, w, Id), where w is the state of evaluation in the model and Id, as before, is the identity function. The instantiation of the game $G(\phi, M)$ with starting position (ϕ, w, Id) will be denoted $G(\phi, M)@(\phi, w, Id)$. The rules of the game are essentially the same as for *MPL* plus an additional rule for the modal operators $(\Diamond_i)$:

At any position (ψ, t, ρ)

- If $\psi \in \mathsf{P}$ then the game has come to an end. A player i is a winner if $V(\psi)(t)(\rho(i)) = \mathbf{w}$.
- If $\psi = \bot_i$ then the game has come to an end. All players except for $\rho^{-1}(i)$ win the game.
- If $\psi = \chi_1 \vee_i \chi_2$ then player $\rho^{-1}(i)$ chooses χ_j with $j \in \{1, 2\}$ and the next position of the game is (χ_j, t, ρ).
- If $\psi = \neg_{ij}\chi$ then the game continues at position (χ, t, ρ_{ij}).
- If $\psi = \Diamond_i\chi$ then the player $\rho^{-1}(i)$ chooses a $u \in W$ such that Rtu in M and the next position is (χ, u, ρ). It is possible that player $\rho^{-1}(i)$ cannot choose such a u because t has no successors in M. In this case we say that player $\rho^{-1}(i)$ gets stuck and the game has come to an end. The winners will be all the players other then $\rho^{-1}(i)$, that is, the winners will be all players in $\{j \in \mathsf{A} | j \neq \rho^{-1}(i)\}$.

The rules of the 'semantics'-game G for formulas of *MML* are summarized in the following table:

Position	Player	Admissible moves	Winners
(p, t, ρ)	-	-	$\{i \in \mathsf{A} \mid V(p)(t)(\rho(i)) = \mathbf{w}\}$
$(\bot_i, t, \rho)$	-	-	$\{j \in \mathsf{A} \mid j \neq \rho^{-1}(i)\}$
$(\chi_1 \vee_i \chi_2, t, \rho)$	$\rho^{-1}(i)$	$\{(\chi_j, t, \rho) \mid j \in \{1, 2\}\}$	-
$(\neg_{ij}\chi, t, \rho)$	-	$\{(\chi, t, \rho_{ij})\}$	-
$(\Diamond_i\chi, t, \rho)$	$\rho^{-1}(i)$	$\{(\chi, u, \rho) \mid Rtu\}$	-

Table 2.3: The game G for *MML*

In accordance with our previous use of terminology, an arbitrary position (ϕ, t, ρ) is said to be winning for player i if she has a winning strategy for the game $G(\phi, M)@(\phi, t, \rho)$. We say that a formula ϕ of *MML* is *i-satisfied* in M at w (notation $M, w \vDash_i \phi$) if player i has a winning strategy for $G(\phi, M)@(\phi, w, Id)$. Often we will make use of the shorthand notation $M, w \vDash_i \phi$ instead of writing that 'i has a winning strategy for $G(\phi, M)@(\phi, w, Id)$'. However, we would like to stress that the reader should be aware of the game-theoretic interpretation behind the use of this notation. We say that a formula ϕ is *i-satisfiable* if there is a model M and a state w in the model such that $M, w \vDash_i \phi$. The formula ϕ

is defined to be *i-valid* if for every model M and every state $w \in M$, $M, w \vDash_i \phi$. We can also define i-satisfiability and i-validity with respect to a frame or with a state in a frame. That is, given a Kripke frame $F = (W, R)$ we say that a formula ϕ is i-satisfiable in F iff there is a state $w \in W$ and a valuation V such that $(W, R, V), w \vDash_i \phi$ and ϕ is i-valid in F iff for every state $w \in W$ every valuation V, (W, R, V), $w \vDash_i \phi$. Lastly, a formula ϕ is i-satisfiable in F at w iff there is a valuation V such that $(W, R, V), w \vDash_i \phi$ and ϕ is i-valid in F at w iff for every valuation V, (W, R, V), $w \vDash_i \phi$. We define two formulas ϕ and ϕ' of *MML* to be *i-equivalent* if every multi-player model M and state w, ϕ is i-satisfied in M at w iff ϕ' is i-satisfied in M at w. And ϕ and ϕ' are *equivalent* if for every agent ϕ and ϕ' are i-equivalent. Next, we will have a look at some examples of formulas of *MML*.

Example 2.3.1. Let A be a set of n players with $0, 1 \in \mathsf{A}$, let $M = (W, R, V)$ be a multi-player model and w a state in W.

(i) Let $\phi = \Diamond_0 \psi$. Then $M, w \vDash_0 \phi$ iff there is a successor t of w such that $M, t \vDash_0 \psi$. Any other player $j \neq 0$ has a winning strategy for $G(\phi, M)@(\phi, w, Id)$ iff for all successors t of w, $M, t \vDash_j \psi$.

(ii) Let $\phi = \Diamond_0 \bot_1$. In this case $M, w \vDash_0 \phi$ iff w has a successor. If w has a successor w', then 0 can play this w' in her first move. Since she has a trivial winning strategy for $\bot_1$ at any state in any model, it follows that playing w' provides her with a winning strategy for $G(\phi, M)@(\phi, w, Id)$. If w has no successor 0 loses and all the other players win by definition of the game. For player 1, $M, w \vDash_1 \phi$ iff w has no successor. If w has a successor 1 is guaranteed to lose the game for $\bot_1$. All the other players $j \notin \{0, 1\}$ are guaranteed to win, whether w has successors or not. To see why, let j be an agent distinct from 0 and 1 and suppose w has no successors. In this case 0 will get stuck in the initial position $(\Diamond_0 \bot_1, w, Id)$. According to the rules of the game, this implies that all the other players (including j) win the game. In case 0 can pick a successor w' of w, then j will win at the final position $(\bot_1, w', Id)$. In other words, for $j \notin \{0, 1\}$ the formula ϕ is j-valid.

The parallels between basic modal logic (*ML*) and multi- player modal logic should be clear by now: for a player i, $\Diamond_i$ is like a regular $\Diamond$ and for all players $j \neq i$, $\Diamond_i$ is like a $\Box$.

Given the logic *MML* we can define analogous concepts of bisimulation, bounded morphism and generalized submodel. It will turn out that *MML* is invariant under all these constructs. First we will define the concept of a *subframe.*

Definition 2.3.2. *Given two multi-player models $M = (W, R, V)$ and $M' = (W', R', V')$, we say that M' is a* subframe *of M iff $W' \subseteq W$ and R' is the restriction of R to W' (that is, $R' = R \cap (W' \times W')$).*

Definition 2.3.3. *Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two multi-player models. A nonempty binary relation $Z \subseteq W \times W'$ is called a* bisimulation *between M and M' if it satisfies the following conditions:*

(i) If wZw' then $V(p)(w)(i) = \boldsymbol{w}$ iff $V'(p)(w')(i) = \boldsymbol{w}$.

(ii) If wZw' and Rwv then there is a v' such that vZv' and $Rw'v'$.(The forth *condition)*

(iii) If wZw' and $Rw'v'$ then there is a v such that vZv' and Rwv.(The back *condition)*

The following two concepts of generated submodel and bounded morphism, are special cases of bisimulations.

Definition 2.3.4. *Given two multi-player models $M = (W, R, V)$ and $M' = (W', R', V')$, we say that M' is a* generated submodel *of M iff M' is a submodel of M and the identity function $Id : W' \to W$ is a bisimulation.*

Definition 2.3.5. *Given two multi-player models $M = (W, R, V)$ and $M' = (W', R', V')$, a functional bisimulation between M and M' is called a* bounded morphism.

Claim 2.3.6. *i-Satisfiability of formulas of MML is invariant under generated submodels, bounded morphism and bisimulation.*

Proof. Each fact can be proved by an easy induction on the complexity of the formula ϕ. Note that each concept is defined in exactly the same way as for the basic modal logic except for some modification of the clause taking care of the propositional case. Given the parallels observed between *MML* and standard *ML* it is not surprising that the proofs of the various claims will structurally similar to the proofs of invariance results of (standard) modal logic that can be found in [3]. In the appendix 7.1, we provide the proof of invariance of i-satisfaction of *MML* under one of these concepts in some detail: we prove invariance of i-satisfiablity of formulas of *MML* under bisimilar models.

Odds and Ends

We would like to conclude this section on *MML* by briefly discussing two topics related to the logic. The first is about an analogue to the cover modality ∇ used in *ML* and in modal μ-calculus. Secondly, we will briefly shed some light on multi-player models from a coalgebraic perspective.

A multi-player cover modality

Instead of using a model operators $\Diamond_i$ for every agent, we could use a different connective: the multi-player cover modality ∇_{ij}. This alternative connective, indexed by two agents, ranges over finite sets of formulas. It can be defined from *MML* as follows. Let Φ be a finite set of formulas, $i, j \in \mathsf{A}$ such that $i \neq j$, then

$$\nabla_{ij}\Phi = (\Diamond_j \bigvee_i \Phi) \vee_j (\bigvee_j \Diamond_i \Phi)$$

Given this definition, when is a ∇_{ij}- formula k-satisfied by at a state w in a model M? We will make a distinction between the various agents. For player i, $M, w \vDash_i \nabla_{ij}\Phi$ if for every successor w' of w there is a $\phi \in \Phi$ such that $M, w' \vDash_i \phi$ *and* for every formula $\phi \in \Phi$ there is a successor w'' such that $M, w'' \vDash_i \phi$. Note that truth for player i corresponds to truth of $\nabla\Phi$ in the case of basic modal logic. For player j, $M, w \vDash_j \nabla_{ij}\Phi$ if *either* there is a successor w'' of w such that $M, w'' \vDash_j \phi$ for all $\phi \in \Phi$ *or* there is a $\phi \in \Phi$ such that for every successor w' of w, $M, w' \vDash_j \phi$. Note that it follows from this that the semantic interpretation of ∇_{ij} is different from that of ∇_{ji}. For any player k other then i, j, $M, w \vDash_k \nabla_{ij}\Phi$ iff $M, w \vDash_k \phi$ for every successor w'' of w and for every formula $\phi \in \Phi$.

A question that arises, is whether we can express the operators $\Diamond_i$ with ∇_{jk}. The answer is yes. If we let $\Phi = \{\phi, \perp_j\}$ with $j \neq i$ then for all players k, $M, w \vDash_k \Diamond_i\phi$ iff $M, w \vDash_k \nabla_{ij}\Phi$. To see why this is so, let us make a distinction between the agents. If $k = i$, it follows that $M, w \vDash_k \Diamond_i\phi$ iff there is a successor w' of w such that $M, w' \vDash_i \phi$. Since all succesors of w k-satisfy $\perp_j$ when $k \neq j$, this holds iff $M, w \vDash_k \nabla_{ij}\Phi$. Since $\perp_j$ is never j-satisfied it follows that for $k = j$, $M, w \vDash_j \nabla_{ij}\Phi$ iff for every successor w' of w, $M, w' \vDash_j \phi$ iff $M, w \vDash_j \Diamond_i\phi$. For $k \notin \{i, j\}$ the result is easily obtained since for every successor w' of w, $M, w' \vDash_k \phi$ and $M, w' \vDash_k \perp_j$ iff $M, w \vDash_k \Diamond_i\phi$.

A coalgebraic perspective

We would briefly like to elaborate on multi-player models $M = (W, R, V)$ from a coalgebraic perspective.

We know that binary relations in a Kripke frame can be characterized by a function $R[\_] : W \to \mathcal{P}(W)$ mapping a point $w \in W$ to the set of its successors. Given that a multi-player valuation is a function $V : \mathsf{P} \to \{\mathbf{w},\ \mathbf{l}\}^{\mathsf{A}^W}$ mapping a proposition letter to a function assigning to each state a set of winners, it can also be seen as a function $V : W \to \{\mathbf{w},\ \mathbf{l}\}^{\mathsf{A}^\mathsf{P}}$ assigning to each state and each proposition letter a set of winners. It follows that multi-player models can be identified with coalgebras of the functor Ω such that

$$\Omega : X \to \{\mathbf{w},\ \mathbf{l}\}^{\mathsf{A}^\mathsf{P}} \times \mathcal{P}(X),$$

and a function $f : W \to W'$ is mapped to the image $\Omega(f)$ as follows

$$\Omega(f)(v \times X) = v \times f[X],$$

where $f[X] = \{f(x) \mid x \in X\}$. The transition map $\sigma : W \to \Omega(W)$, assigns to every state w an element $v \times S$ where $v : \mathsf{P} \to (\mathsf{A} \to \{\mathbf{w},\mathbf{l}\})$ is the valuation at w and $S = \{w' \mid Rww'\}$ the set of successors of w.

It is not very difficult to see that a bounded morphism between two multi-player models coincides with an Ω-coalgebraic morphism between their coalgebraic representations. To see why this is so, let $M = (W, R, V)$ and $M' =$

(W', R', V') be two multi-player models and (W, σ) and (W', σ') their coalgebraic interpretations. We need to show that for a function $f : W \to W'$,

$$f \text{ is a bounded morphism iff } \Omega(f) \circ \sigma(w) = \sigma' \circ f(w).$$

Let $f(w) = w'$, $R[w] = T$, $R'[w'] = T'$. Then $\sigma(w) = v \times T$ and $\sigma'(w') = v' \times T'$. Moreover, $\Omega(f)(v \times T) = v \times f[T]$. Proving the claim boils down to showing that

$$v \times f[T] = v' \times T'.$$

Assume that f is a bounded morphism. Then, by condition (i), $v = v'$. What remains to be shown is that $f[T] = T'$. Suppose that $f(t) \in f[T]$, then t is a successor of w in M. By the forth condition (ii), it follows that $f(t) \in R'[w']$, hence $f[T] \subseteq T'$. We can show that $T' \subseteq f[T]$ by using the back-condition (iii). The other direction - that $\Omega(f) \circ \sigma(w) = \sigma' \circ f(w)$ implies that f is a bounded morphism - follows by similar reasoning. Hence, homomorphisms of Ω-coalgebras correspond to bounded morphism between multi-player models.

2.4 Multi-Player μ-Calculus (μMML)

In this section we propose an extension of *MML* to μ*MML* which will be a multi-player analogue of the modal μ-calculus. For each player there will be one 'fixpoint operator' μ_i.

Let A be the set of agents and P the set of proposition letters. The syntax of μ*MML* is defined as follows:

$$\phi ::= \bot_i \mid p \mid X \mid (\phi \vee_i \psi) \mid \neg_{ij}\phi \mid \Diamond_i\phi \mid \mu_i X.\phi,$$

where $i, j \in \mathsf{A}$, $T \subseteq \mathsf{A}$, $p \in \mathsf{P}$, and X is a variable. The following restriction applies to the formulae of the form $\mu_i X.\phi$; within ϕ, the variable X cannot occur under the scope of any multi-player negation $\neg_{ik}$ or $\neg_{ki}$ with $k \neq i$. In order to make a distinction between the two types of variables, we will call a variable $p \in \mathsf{P}$ a *propositional variable*, or a *proposition letter*, and a variable $X \notin \mathsf{P}$ simply a *variable*.

Observe formulas like $\neg_{ij}\mu_j X.(p \vee_i \Diamond_k X)$ and $\mu_j X.(p \vee_i \Diamond_k \neg_{ik} X)$ with $j \notin \{i, k\}$ are μ*MML*-formulas. However, we claim that the syntactical restriction we put on formulas of the form $\mu_i X.\phi$ is still a bit too restrictive. In remark 2.4.9, at the end of this chapter, we will briefly elaborate on this point.

On a Kripke frame $\mathcal{F} = (W, R)$ a valuation V is defined as in the *MML*-case:

$$V : \mathsf{P} \to (W \to (\mathsf{A} \to \{\mathbf{w}, \mathbf{l}\}))$$

Next we will define some syntactical properties of μ*MML*-formulas. All definitions are standard in the area of ('normal') modal μ-calculus.

Definition 2.4.1. *A formula ϕ of μMML is called* clean *if there are no two distinct fixpoint operators binding the same variable and no variables have both bound an free occurrences in ϕ. If X is a bound variable of a clean formula ϕ, we denote with $\phi_X = \mu_i X \delta_X$ the unique subformula of ϕ where X is bound by μ_i. We say that X is an i-variable.*

Definition 2.4.2. *Given a clean formula ϕ, we define a* dependency order *on the set of bound variables of ϕ as follows: the variable Y ranks higher than X (notation, $X \leq_\phi Y$) iff ϕ_X is a subformula of ϕ_Y.*

The intuition behind the notion of 'dependency order' is that $X \leq_\phi Y$ if the meaning of ϕ_X is dependent on that of ϕ_Y. Observe that given a *μMML*-formula ϕ, there can be two variables X, Y that are incomparable in terms of dependency order, that is, $X \nleq_\phi Y$ and $Y \nleq_\phi X$. An example of such a formula is $\mu_i X.(p \vee_i \Diamond_j X) \vee_k \mu_j Y.(q \vee_j \Diamond_i X)$. Before extending the game G to define the semantics of *μMML*, we will define one more concept, that of guardedness.

Definition 2.4.3. *We say that variable X in a μMML-formula ϕ is* guarded *with respect to ϕ if every occurrence of X in ϕ is in the scope of a modal operator (a $\Diamond_i$ for some player i). We say that the formula ϕ is guarded when for every subformula $\mu_i X.\delta_X$, the variable X is guarded with respect to δ_X.*

The semantics *μMML* is defined using an extension of the game G defined to evaluate *MML*-formulas. The (initial) positions and role distributions are defined in the same way as before. All the rules of the game stay the same, but we have two additional ones:

At any position (ψ, t, ρ)

- If $\psi = \mu_i X \chi$ then nobody makes a move and the game continues at position (χ, t, ρ). (The fixpoint rule)
- If $\psi = X$ and X is a bound variable in ϕ with binding definition δ_X, nobody makes a move and the next position of the game is (δ_X, t, ρ). (The unfolding rule)

Adding these two rules to the game G we now obtain the following table describing the semantic game for *μMML*:

Position	Player	Admissible moves
(p, t, ρ)	-	-
$(\bot_i, t, \rho)$	-	-
$(\chi_1 \vee_i \chi_2, t, \rho)$	$\rho^{-1}(i)$	$\{(\chi_j, t, \rho) \mid j \in \{1, 2\}\}$
$(\neg_{ij}\chi, t, \rho)$	-	$\{(\chi, t, \rho_{ij})\}$
$(\Diamond_i \chi, t, \rho)$	$\rho^{-1}(i)$	$\{(\chi, u, \rho) \mid Rtu\}$
$(\mu_i X.\chi, t, \rho)$	-	$\{(\chi, t, \rho)\}$
(X, t, ρ)	-	$\{(\phi_X, t, \rho)\}$

Table 2.4: The game G for *μMML*

Note that contrary to the cases of *MPL* and *MML* we have not specified the winners of the game in table 2.4. The reason for this is that matches of formulae of μMML are no longer guaranteed to be finite because variables can be unfolded (infinitely often). Before, when describing the game G for *MML* (and also *MPL*) we were imprecise about the definitions of a match of G and winning conditions for matches of G. Given a *MML*-formula ϕ, a model M and a state $w \in M$, a match was simply a 'play' of $G(\phi, M)$ - a sequence of positions starting at a position (ϕ, w, Id). Because all matches were finite, for each match the winning conditions were fully specified at the last position of each match. In our current setting - that of μMML - we are faced with infinite matches and therefore we need to be a bit more precise about who wins the game in which situations.

Definition 2.4.4. *Given a clean μMML formula ϕ and a Kripke model $M = (W, R, V)$ and $w \in W$,* a match *of $G(\phi, M)@(\phi, w, Id)$ is an (in)finite sequence of positions*

$$\pi = (\phi, w, Id), (\phi_1, w_1, \rho_1), (\phi_2, w_2, \rho_2), \ldots,$$

which is in accordance with the rules of the game $G(\phi, M)$ defined above. Given an infinite match π, we define $Unf^{\infty}(\pi)$ as the set of variables that are unfolded infinitely often in π. With $max(Unf^{\infty}(\pi))$ we denote the variable X in $Unf^{\infty}(\pi)$ that is highest in terms of dependency order, that is, for X it holds that $Y \leq_{\phi} X$ for all $Y \in Unf^{\infty}(\pi)$.

Remark 2.4.5. Observe that for every μMML-formula ϕ, any match π of $G(\phi, M)@(\phi, w, Id)$, and any two variables $X, Y \in Unf^{\infty}(\pi)$ there is a $Z \in Unf^{\infty}(\pi)$ such that either $X \leq_{\phi} Z$ or $Y \leq_{\phi} Z$. The proof of this little claim is essentially the same as for the standard modal μ-calculus and can be found in [15].

There are a few things that we will note before getting to the winning conditions for G-matches. Given an infinite match π, there must be some finite k and a position $(\mu_i X.\phi, w_k, \rho_k)$ after which only variables from $Unf^{\infty}(\pi)$ are unfolded. It follows that after this point the role distribution will be 'stable'. That is, for all later positions (ϕ_n, w_n, ρ_n) with $k \leq n$ we have $\rho_n^{-1}(i) = \rho_k^{-1}(i)$. We call this ρ_k the *stable role distribution* of the (infinite) match π (notation ρ_{π}). Finally, we are ready to define the winning conditions for matches of μMML-formulas.

Definition 2.4.6. *Given a clean μMML-formula ϕ, a multi-player model M, a state $w \in M$, and a match π of $G(\phi, M)@(\phi, w, Id)$ we define the winning conditions for π as follows:*

Here, $last(\pi)$ denotes the last position of the (finite) sequence π.

	player a wins
π is finite	$(last(\pi))$ is a winning position for a
π is infinite	$max(Unf^{\infty}(\pi)) = X$, X is a j-variable and $\rho_\pi(a) = j$

Table 2.5: The winning conditions for the μMML-game G

In words, the winning conditions state the following: for finite matches, the winning conditions are exactly the same as for *MML* and are specified at the final position of the game. An infinite match π is won by a player j if the highest variable unfolded infinitely often in π is a i-variable, and j assumes the role of i according to the stable role distribution ρ_π [1].

As before, we say that a formula ϕ of μMML is *i-satisfied* in M at w, $M, w \vDash_i \phi$, if player i has a winning strategy for $G(\phi, M)@(\phi, w, Id)$. Now let us have a look at some examples of μMML-formulas.

Example 2.4.7. Let A be a set of n players with $0, 1, 2 \in \mathsf{A}$, let $M = (W, R, V)$ be a multi-player model and w a state in W. We will have a look at some formulas ϕ of μMML.

(i) Let $\phi = \mu_0 X.(\bot_0 \vee_0 \Diamond_0 X)$. Then $M, w \vDash_0 \phi$ iff there is an infinite path leading from w. All the other players have a winning strategy iff there is no infinite path leading from w: first of all, note that since there are no negations involved, the role distribution at each position of the game will be Id. Note also that player 0 makes all the moves. Moreover, when forced to choose between $\bot_0$ and $\Diamond_0 X$ she will never prefer to play $\bot_0$ since this results in an immediate loss for her. If the model has an infinite path, 0 can always choose $\Diamond_0 X$ and play in such a way that the game remains on the path. By definition of the winning conditions, player 0 is the only player that will win if the game is infinite. If there is no infinite path, then 0 is guaranteed to lose since she loses both in case of any position of the form $(\bot_0, t, Id)$ and of $(\Diamond_0 X, t, \rho)$ when t has no successors. In the last case, the rest of the players are guaranteed to win the game.

(ii) Let $\phi = \mu_0 X.(p \vee_1 \Diamond_0 X)$. In this case 0 wins iff there is an infinite path leading from w such that for every state t on the path $V(p)(t)(0) = \mathbf{w}$. Note that, as before, 0 will be the only winner of an infinite game. Moreover, at any position of the form $(p \vee_1 \Diamond_0 X, t, Id)$, 0 is guaranteed to 'survive the round' only if $V(p)(t)(0) = \mathbf{w}$. Thus, when choosing a successor at position $(\Diamond_0 X, t, Id)$, 0 has to make sure that p is satisfied at this successor for 0. For player 1, $M, w \vDash_1 \phi$ iff every path that can be reached from w passes through some w' such that $V(p)(w')(1) = \mathbf{w}$. The

[1] We realize that the winning conditions more resemble that of the greatest fixpoint ν, rather than that of the least fixpoint μ. Therefore, it could be argued that the use of ν_i would be more appropriate in this context as opposed to the current use of μ_i.

reason for this is that at a position $(p \vee_1 \Diamond_0 X, t, Id)$, 1 will only be willing to play (p, t, Id) if $V(p)(w')(1) = \mathbf{w}$. Otherwise, he is forced to play $\Diamond_0 X$. For all the other players k, $M, w \vDash_k \phi$ iff there is no infinite path leading from w and for all states t reachable from w, $V(p)(t)(k) = \mathbf{w}$. If there is an infinite path leading from w player k risks that player 1 will play $\Diamond_0 X$ at each position $(p \vee_1 \Diamond_0 X, t, Id)$. If there is a state t finitely reachable from w such that $V(p)(t)(k) \neq \mathbf{w}$ player k risks that the game will end up here.

(iii) Let $\phi = \neg_{01} \Diamond_2 \mu_0 X.(p \vee_2 X)$. In this case 0 has a winning strategy iff w has no successors in M. The direction from right to left should be clear: if w has no successor, player 2 will get stuck and all the other players are guaranteed winners (note that $2 \notin \{0, 1\}$ hence the negation before the modal operator does not change this fact). For the other direction, suppose there is a successor w' of w, that 2 chooses to play. In this case we end up in a position $(p \vee_2 X, w', \rho_{01})$. Since $\rho_{01}(0) = 1$ it follows that all infinite matches are won by 1. Now player 0 risks that player 2 will play (X, w', ρ_{01}) at all positions $(p \vee_2 X, w', \rho_{01})$, in which case 0 looses the game. For player 1, $M, w \vDash_1 \phi$ iff w has no successors or for all successors w' of w, $V(p)(w')(0) = \mathbf{w}$. This should be clear. Lastly, $M, w \vDash_2 \phi$ iff there is a successor w' of w such that $V(p)(w')(2) = \mathbf{w}$. If there is no such a successor, she will lose in all possible scenarios: either because she cannot pick a successor in the first place, or because she plays p at some point, or because she never plays p.

It should be observed that the semantics of μMML is only defined over clean formulas. At this point, there are μMML-formulas like $\mu_0 X.(\mu_1 X.(X))$ that have no semantic interpretation.

Proposition 2.4.8. *Just like the standard modal μ-calculus μML, the multi-player version μMML is bisimulation invariant and has the (bounded)tree model property.*

Proof. The proof of these facts is left to the reader. □

In this section, we have made a first attempt to formulate a multi-player variant of the modal μ-calculus. The formalism, as it was introduced here, can still be improved in many ways. We would like to conclude by making two remarks regarding possible improvements of μMML.

Remark 2.4.9. First of all, we have defined formulas of the form $\mu_i X.\phi$ in such a was that within ϕ, X cannot occur under the scope of any multi-player negation $\neg_{ik}$ such that $i \neq k$. We have put this condition to ensure that at a position of the form $(\mu_i X.\phi, w, \rho)$, the player that assumes the role of i, that is $\rho^{-1}(i)$, wins all the infinite matches. However, the formula constraint as formulated here is a bit too restrictive. To see why, take for example the formula $\mu_i X.(\neg_{ij} \neg_{ij} X)$. According to our definition, this is not a μMML formula, whereas clearly we would like it to be. We leave it for some other occasion to find a neat way to define the notion of i-positive occurrences and to improve on this syntactic restriction regarding the negations occurring within fixpoint formulas.

Remark 2.4.10. Secondly, we would like to stress that other variants of μMML are possible and interesting to consider as well. We could, for example, index a fixpoint operator with a set of agents rather than with a single agent. One could even argue that this variant is more in line with our other multi-player logics, since subsets of agents, rather than single players, will win infinite matches.

Chapter 3

Multi-Player Algebras: The Propositional Case

3.1 Introduction

In the next two chapters we will study the logics *MPL* and *MML* algebraically. In this chapter, we will first construct concrete multi-player algebras embodying the semantics of *MPL*. After that, we define an abstract (quasi-)equational axiom system capturing the notion of i-equivalence in *MPL*: algebras satisfying these axioms will be called 'multi-Boolean algebras'. One of our main results is a representation theorem, analogue to Stone's representation theorem that will link these two concepts together: we show that every multi-Boolean algebra is isomorphic to a concrete multi-player algebra. In the next chapter we will study the logic *MML* by algebraic means and define 'multi-modal algebras' axiomatically. By showing a multi-player version of the Jónsson-Tarski theorem, we establish a deep relation between game semantics of *MML* defined over relational structures and abstract multi-modal algebras.

3.2 Concrete Multi-Player Algebras

Definition 3.2.1. *Given a nonempty finite set* A *we define the similarity type* MBool(A) *as the algebraic similarity type having one constant symbol* $\bot_i$*, one binary function symbol* $\vee_i$ *for every* i *such that* $i \in \mathsf{A}$*, and one unary function* $\neg_{ij}$ *for each pair of* $i, j \in \mathsf{A}$*. Algebras of the MBool(*A*) type will often be denoted as* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$.

Throughout the thesis we will assume that one element, 0, is always contained in A. We will make heavy use of abbreviations $\top_i$ and $\bot$. Here $\top_i$ is defined as $\bigvee_i \{\bot_j | j \in \mathsf{A}\}$ and $\bot$ as $\bigvee_0 \{\top_j | j \neq 0 \in \mathsf{A}\}$ (and can also be found in table 3.2). Note that $\bot$ is defined using a fixed player 0.

Given a finite set of players or agents A and a set S [1], we define the *full concrete multi-player algebra* over S as the structure

$$\mathfrak{C}(S) = (E, \cup_i, \sim_{ij}, \emptyset_i)_{i,j \in \mathsf{A}}.$$

where E denotes $\{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^S}$, A the set of all the agents, and $\mathbf{w}$ and $\mathbf{l}$ stand for *win* and *lose* respectively. Thus, an element f of $\mathfrak{C}(S)$ is a function $f : S \to (\mathsf{A} \to \{\mathbf{w},\mathbf{l}\})$. That is, the function f assigns to every state $s \in S$ and every agent $i \in \mathsf{A}$ the value *win* or the value *lose.* If $f(s)(i) = \mathbf{w}$ we say that player i *wins at* s *given* f, and *loses* otherwise. The symbol $\emptyset_i$ denotes the following function: for each $s \in S$ $\emptyset_i(s)(i) = \mathbf{l}$ and $\emptyset_i(s)(j) = \mathbf{w}$ if $j \neq i$. That is, for all states $s \in S$, player i loses given $\emptyset_i$ and all the players distinct from i win. Moreover, the functions $f \cup_i g$ and $\sim_{ij} f$ are given by:

$$\begin{array}{lll} (f \cup_i g)(s)(i) & = \mathbf{w} & \text{iff } f(s)(i) = \mathbf{w} \text{ or } g(s)(i) = \mathbf{w} \\ (f \cup_i g)(s)(j) & = \mathbf{w} & \text{iff } f(s)(j) = \mathbf{w} \text{ and } g(s)(j) = \mathbf{w} \text{ for } j \neq i. \end{array}$$

and,

$$\sim_{ij} f(s)(k) = \begin{cases} f(s)(i) & \text{if } k = j, \\ f(s)(j) & \text{if } k = i, \\ f(s)(k) & \text{if } k \neq i \text{ and } k \neq j. \end{cases}$$

Note that this algebra is of the MBool(A) similarity type (with $\cup_i$ corresponding to $\vee_i$, $\sim_{ij}$ to $\neg_{ij}$ and $\emptyset_i$ to $\bot_i$). Any subalgebra of $\mathfrak{C}(S)$, that is, a subset of E that contains $\emptyset_i$ for all $i \in \mathsf{A}$ and is closed under $\cup_i$ and $\sim_{ij}$ for all $i, j \in \mathsf{A}$, will be called a *concrete multi-player algebra over* S. With **CcMA** we denote the class of all concrete multi-player algebras.

Note that in the special case of $\sim_{ii}$, we have that for all $f \in E$ $\sim_{ii} f = f$. We will use the element $\natural_i$ to denote the function $\natural_i(s)(i) = \mathbf{w}$ and $\natural_i(s)(j) = \mathbf{l}$ if $j \neq i$ for all $s \in S$. In words, for every state s the only player that wins at s given $\natural_i$ is player i, the rest loses. The function $\natural_i$ can be defined from $\cup_i$ and $\emptyset_j$'s as follows: $\natural_i = \bigcup_i \{\emptyset_j | j \in \mathsf{A}\}$. We can also consider the elements $\emptyset$ and $\natural$ without indices. At each state s, all players lose at s given $\emptyset$ and all players win at s given $\natural$. In symbols, for all $i \in \mathsf{A}, s \in S$ $\emptyset(s)(i) = \mathbf{l}$ and $\natural(s)(i) = \mathbf{w}$. Note that $\emptyset$ equals $\bigcup_0 \{\natural_j | j \neq 0 \in \mathsf{A}\}$ for any $i \in \mathsf{A}$. Observe that $\emptyset$ is defined using the fixed player 0, whereas the abbreviations of the other symbols like $\emptyset_i$ and $\natural_i$ are defined relative to an agent. Note also that $\natural_i$ corresponds to $\top_i$ and that $\emptyset$ corresponds to $\bot$. In the following table we give a summary of the abbreviations just introduced.

We do not give a definition in terms of the signature for the function $\natural$. The reason for this is that do not know how to define $\natural$ from the signature. In fact, we believe that $\natural$ is not definable from the signature at all. Because we do not prove this claim here, we will not state it as a fact.

[1] S can be thought of as the set of states of a Kripke frame. But in principle, this can be any set.

Symbol	Abbreviation for	Interpretation
$\emptyset_i$	$\emptyset_i$	$\emptyset_i(s)(i) = \mathbf{l}$ iff $j = i$
$\natural_i$	$\bigcup_i\{\emptyset_j \mid j \in \mathsf{A}\}$	$\natural_i(s)(j) = \mathbf{w}$ iff $j = i$
$\emptyset$	$\bigcup_0\{\natural_j \mid j \neq 0 \in \mathsf{A}\}$	$\emptyset(s)(i) = \mathbf{l}$ for all $j \in \mathsf{A}$
$\natural$	-	$\natural(s)(i) = \mathbf{w}$ for all $j \in \mathsf{A}$

Table 3.1: Abbreviated Symbols

For any two functions $f, g \in E$ we say that $f \leq_i g$ if $f \cup_i g = g$ (that is, if for all $s \in S$, $j \in \mathsf{A}$ $f \cup_i g(s)(j) = g(s)(j)$). From this it follows that $f \leq_i g$ iff for all $s \in S$, $f(s)(i) = \mathbf{w}$ implies $g(s)(i) = \mathbf{w}$ and $f(s)(j) = \mathbf{l}$ implies $g(s)(j) = \mathbf{l}$ for all $j \neq i$. Intuitively, $f \leq_i g$ means that for all states $s \in S$, g is at least as good as f for i, and for all $j \neq i$ g is at least as bad as f.

Remark 3.2.2. For any concrete multi algebra $\mathfrak{C}(S) = (E, \cup_i, \sim_{ij}, \emptyset_i)_{i,j \in \mathsf{A}}$ with $E = \{\mathbf{w}, \mathbf{l}\}^{\mathsf{A}^S}$ the following, maybe somewhat counterintuitive, inequalities hold: $\emptyset_i \leq_i \emptyset$ and $\natural \leq_i \natural_i$. To see why this is so, let us have a closer look at $\emptyset_i \leq_i \emptyset$. We know that for all $s \in S$ player i gets assigned the same value by $\emptyset$ and $\emptyset_i$: $\emptyset_i(s)(i) = \emptyset(s)(i) = \mathbf{l}$. Thus for all s, $\emptyset_i$ is at least as good as $\emptyset$ for i. Also, $\emptyset_i(s)(j) = \mathbf{l}$ implies $\emptyset(s)(j) = \mathbf{l}$ for all $j \neq i$. The reason for this is that $j \neq i$, $\emptyset_i(s)(j) = \mathbf{w}$ for all $s \in S$ and secondly, $\emptyset(s)(j) = \mathbf{l}$ for all $s \in S$. Thus, for all $j \neq i$, the function $\emptyset$ is at least as bad as $\emptyset_i$. More generally, for all $f \in E$, $\emptyset_i \leq_i f$ and $f \leq_i \natural_i$. It does not, however, generally hold that $\emptyset_i \leq_j f$ or $f \leq_j \natural_i$. Also, the other directions, $f \leq_j \emptyset_i$ and $\natural_i \leq_j f$ are not valid.

We say that an equation $f = g$ *is valid* in the full concrete multi-player algebra $\mathfrak{C}$ over S (notation: $\mathfrak{C}(S) \vDash f = g$) if for each variable assingment each player has the same value assigned to f and g at each state $s \in S$. Similarly, $f =_i g$ is valid in the full concrete multi-player algebra if $f \cup_i \emptyset = g \cup_i \emptyset$ is valid in $\mathfrak{C}(S)$. This implies that for player i, the value of f equals that of g at each state. Generalizing these concepts, we say that the class of concrete multi-player algebras **CcMA** validates $f = g$ (notation: **CcMA** $\vDash f = g$) if for every set S, the algebra $\mathfrak{C}(S)$ validates $f = g$. And similarly, **CcMA** validates $f =_i g$ if for every S, the $\mathfrak{C}(S)$ validates $f =_i g$.

It follows from the definition of concrete multi-player algebras that:

Lemma 3.2.3. *$\mathfrak{C}(\{s\}) \vDash f =_i g$ iff **CcMA** $\vDash f =_i g$.*

Proof. The direction from right to left is immediate. For the other direction: suppose **CcMA** $\nvDash f =_i g$. Then there is a set S such that $\mathfrak{C}(S) \nvDash f =_i g$. This implies that there is an $t \in S$ and a variable-assignment such that $f(t)(i) \neq g(t)(i)$. It follows that there is a variable assignment at s such that $f(s)(i) \neq g(s)(i)$, hence $\mathfrak{C}(\{s\}) \nvDash f =_i g$. □

At this point, we would like to make the observation that the class of concrete multi-player algebras exactly captures the semantics of *MPL* in the following sense:

Lemma 3.2.4. *For any MPL formula ϕ,*

$$\phi \text{ is } i\text{-valid iff } \boldsymbol{CcMA} \vDash \phi' =_i \natural_i$$

where ϕ' is the algebraic term equal to ϕ using the corresponding concrete-multi-player symbols.

Proof. Both directions can be proved by induction on ϕ. For the direction from right to left, we use the above lemma 3.2.3. The key point to observe here is that $\mathbf{CcMA} \vDash \phi' =_i \natural_i$ iff for each set of states S, each variable assignment and $s \in S$, we have $\phi'(s)(i) = \mathbf{w}$. □

Next, we will define the 'small' algebra $\widetilde{\mathsf{A}}$.

Definition 3.2.5. *Given a finite set A, the algebra $\widetilde{\mathsf{A}} = (\mathcal{P}(\mathsf{A}), +_i, -_{ij}, 0_i)_{i,j\in\mathsf{A}}$ is defined as follows: for any two $X, Y \in \mathcal{P}(\mathsf{A})$*

$$X +_i Y = \begin{cases} (X \cap Y) \cup \{i\} & \textit{if } i \in X \cup Y \\ X \cap Y & \textit{if } i \notin X, i \notin Y \end{cases}$$

Also,

$$-_{ij} X = \begin{cases} (X - \{i\}) \cup \{j\} & \textit{if } i \in X, j \notin X \\ (X - \{j\}) \cup \{i\} & \textit{if } j \in X, i \notin X \\ X & \textit{otherwise} \end{cases}$$

and $0_i = \mathsf{A} - \{i\}$

Note that for a fixed A the algebras $\mathfrak{C}(S)$ and $\widetilde{\mathsf{A}}$ are of the same signature. In the next proposition it will be shown that the small algebra $\widetilde{\mathsf{A}}$ corresponds to concrete multi-player algebras in exactly the same way as the algebra **2** of truth values corresponds to concrete set algebras.

Proposition 3.2.6. *Given an arbitrary set S, $\mathfrak{C}(S)$ is isomorphic to $\widetilde{\mathsf{A}}^S$*

Proof. Fix an arbitrary set S and consider the following function

$$\alpha : \{\mathbf{w,l}\}^{\mathsf{A}^S} \to \widetilde{\mathsf{A}}^S,$$

mapping elements from $\{\mathbf{w,l}\}^{\mathsf{A}^S}$ to functions from S to $\mathcal{P}(\mathsf{A})$:

$$\alpha(f)(s) = \{i \in \mathsf{A} | f(s)(i) = w\}.$$

First we will show that α is a bijection. To show injectivity, suppose $f, g \in \{\mathbf{w,l}\}^{\mathsf{A}^S}$ with $f \neq g$, then there is an $s \in S$ and an $i \in \mathsf{A}$ such that $f(s)(i) \neq g(s)(i)$. From this it follows that $\alpha(f)(s) \neq \alpha(g)(s)$, and thus we obtain injectivity of α. To show surjectivity, let $u \in (\widetilde{\mathsf{A}})^S$ and define $f \in \{\mathbf{w,l}\}^{\mathsf{A}^S}$ as follows

$$f(s)(i) = \mathbf{w} \Leftrightarrow i \in u(s)$$

It follows from the definition of α that $\alpha(f) = u$.

Next, we need to show that α is a homomorphism from $\{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^S}$ to $\widetilde{\mathsf{A}}^S$. Let us consider the $+_i$ operator. We need to show that for any $j \in \mathsf{A}$, $s \in S$ and $f, g \in \{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^S}$, $j \in \alpha(f \cup_i g)(s)$ if and only if $j \in (\alpha(f) +_i \alpha(g))(s)$. We distinguish two cases: $j = i$, and $j \neq i$.

Case $j = i$.

$$\begin{array}{rcl} i \in \alpha(f \cup_i g)(s) & \Leftrightarrow & (f \cup_i g)(s)(i) = \mathbf{w} \\ & \Leftrightarrow & f(s)(i) = \mathbf{w} \text{ or } g(s)(i) = \mathbf{w} \\ & \Leftrightarrow & i \in \alpha(f)(s) \text{ or } i \in \alpha(g)(s) \\ & \Leftrightarrow & i \in (\alpha(f) +_i \alpha(g))(s) \end{array}$$

Case $j \neq i$.

$$\begin{array}{rcl} j \in \alpha(f \cup_i g)(s) & \Leftrightarrow & (f \cup_i g)(s)(j) = \mathbf{w} \\ & \Leftrightarrow & f(s)(j) = \mathbf{w} \text{ and } g(s)(j) = \mathbf{w} \\ & \Leftrightarrow & j \in \alpha(f)(s) \text{ and } j \in \alpha(g)(s) \\ & \Leftrightarrow & j \in (\alpha(f) +_i \alpha(g))(s) \end{array}$$

Considering the $-_{ij}$ operator, we need to show that $k \in \alpha(\sim_{ij} f)(s)$ if and only if $k \in -_{ij}\alpha(f)(s)$. Again, we distinguish two cases: $k \in \{i, j\}$ and $k \notin \{i, j\}$.

Case $k \in \{i, j\}$. wlog assume $k = i$.

$$\begin{array}{rcl} i \in \alpha(\sim_{ij} f)(s) & \Leftrightarrow & \sim_{ij} f(s)(i) = \mathbf{w} \\ & \Leftrightarrow & f(s)(j) = \mathbf{w} \\ & \Leftrightarrow & j \in \alpha(f)(s) \\ & \Leftrightarrow & i \in -_{ij}\alpha(f)(s) \end{array}$$

Case $k \notin \{i, j\}$.

$$\begin{array}{rcl} k \in \alpha(\sim_{ij} f)(s) & \Leftrightarrow & \sim_{ij} f(s)(k) = \mathbf{w} \\ & \Leftrightarrow & f(s)(k) = \mathbf{w} \\ & \Leftrightarrow & k \in \alpha(f)(s) \\ & \Leftrightarrow & k \in -_{ij}\alpha(f)(s) \end{array}$$

Thus we may conclude that every concrete multi algebra for A is isomorphic to a power of $\widetilde{\mathsf{A}}$.

In fact, we may also conclude the converse direction, viz. that every power of $\widetilde{\mathsf{A}}$ is isomorphic to a concrete multi algebra for A, since it suffices that the inverse of α, α^{-1}, is an isomorphism from $\widetilde{\mathsf{A}}^S$ to $\{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^S}$. This concludes the proof of the proposition.

$\square$

3.3 Multi-Boolean Algebras (*MBA*)

Definition 3.3.1. *Let* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ *be an algebra of the similarity type* MBool(A). *For each agent* $i \in \mathsf{A}$ *we define two orderings of the elements of* A*:* $\leq_i$ *and* $\preceq_i$*. We define*

$$a \leq_i b \text{ if } a \vee_i b = b$$

And,

$$a \preceq_i b \text{ if } a \vee_i b \vee_i \bot = b \vee_i \bot$$

That is, $a \preceq_i b$ *if* $a \vee_i \bot \leq_i b \vee_i \bot$*. If* $a \preceq_i b$ *and* $b \preceq_i a$ *we get* $a \vee_i \bot = a \vee_i b \vee_i \bot = b \vee_i \bot$ *and thus* $a \vee_i \bot = b \vee_i \bot$*. In this case we say* a *is* i*-equal to* b*. Notation:* $a =_i b$*.*

At this point, we would like to make the reader aware of the fact that in the sequel, we will be constantly switching between the two equivalent notations: $a =_i b$ and $a \vee_i \bot = b \vee_i \bot$.

The intuition behind these two orderings is the following. The first, $\leq_i$ can be interpreted as follows: given two elements a, b we have that $a \leq_i b$ iff for player i the element a is at least as good as b and for all other players $j \neq i$ the element a is at least as bad as b. The other relation $\preceq_i$ is a less restricted relation: given two elements a, b, $a \preceq_i b$ iff a is at least as good as b for i. It follows that $a \leq_i b$ implies that $a \preceq_i b$, but the two relations are not the same.

Definition 3.3.2. *Let* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ *be an algebra of the multi-Boolean similarity type. The algebra* $\mathfrak{A}$ *is called a* multi-Boolean algebra *(MBA) iff it satisfies the (quasi-)equations:*

(MB0) $x \vee_i y = y \vee_i x$
(MB1) $x \vee_i (y \vee_i z) = (x \vee_i y) \vee_i z$
(MB2) $x \vee_i (y \vee_j z) = (x \vee_i y) \vee_j (x \vee_i z)$
(MB3) $\bot_i \vee_i x = x$
(MB4) $\top_i \vee_i x = \top_i$
(MB5) $(x \vee_j y) \vee_i x =_i x$ *for* $j \neq i$
(MB6) $x \preceq_i y$ *and* $y \preceq_j x$ *for all* $j \neq i$ *implies* $x \leq_i y$

(MBn0) $\neg_{ij}x = \neg_{ji}x$
(MBn1) $\neg_{ii}x = x$
(MBn2) $\neg_{ik}\bot_i = \bot_k$, $\neg_{jk}\bot_i = \bot_i$ *if* $i \notin \{j,k\}$ *and* $\neg_j k\bot = \bot$
(MBn3) $\neg_{ij}\neg_{jk}x =_i \neg_{ik}x$
(MBn4) $\neg_{ij}\neg_{lk}x =_i \neg_{ij}x$ *if* $j \notin \{l,k\}$
(MBn5) $x =_i \neg_{jk}x$ *for* $i \notin \{j,k\}$
(MBn6) $\neg_{ij}\neg_{lk}x = \neg_{lk}\neg_{ij}x$ *if* $i \notin \{l,k\}$ *and* $j \notin \{l,k\}$
(MBn7) $\neg_{ij}(x \vee_i y) = \neg_{ij}x \vee_j \neg_{ij}y$
(MBn8) $\neg_{ij}(x \vee_k y) = \neg_{ij}x \vee_k \neg_{ij}y$ *for* $k \notin \{i,j\}$

All (quasi-)equations are assumed to be universally quantified over both the variables and the agents. As mentioned before, $\top_i$ *is a shorthand for* $\bigvee_i\{\bot_j \mid j \in \mathsf{A}\}$*. Also,* $\bot$ *is short for the formula* $\bigvee_0\{\top_j \mid j \neq 0\}$ *for any* $j \in \mathsf{A}$*.*

We call the class of all multi-Boolean algebras MBA.

The following proposition highlights some properties of multi-Boolean algebras.

Proposition 3.3.3. *Every multi-Boolean algebra* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in\mathsf{A}}$ *has the following properties*

(i) *The relation* $\leq_i$ *is a partial order on* A *for each* i*.*

(ii) *The relation* $\preceq_i$ *is a quasi-order, i.e., it is reflexive and transitive.*

(iii) $\neg_{ik}\top_i = \top_k$ *and* $\neg_{jk}\top_i = \top_i$ *when* $i \notin \{j,k\}$*.*

(iv) $\top_j \vee_i \bot = \bot$ *for* $j \neq i$*.*

(v) $a = b \Leftrightarrow \neg_{jk}a = \neg_{jk}b$*.*

(vi) $a \preceq_i b \Leftrightarrow \neg_{ik}a \preceq_k \neg_{ik}b$*.*

(vii) $a \vee_j b \preceq_i a$ *for all* $j \neq i$*.*

(viii) $a \leq_i b$ *implies* $a \preceq_i b$ *and* $b \preceq_j a$ *for all* $j \neq i$*.*

(ix) $a \vee_j b =_i a \vee_k b$ *when* $i \notin \{j,k\}$

Proof. For the proof of this proposition we refer to the appendix 7.2. $\square$

Note that from the definition of $\leq_i$ and the above proposition it follows that, if for some agent $i \in \mathsf{A}$ and for some elements $a, b \in A$, $a \leq_i b$ and $b \leq_i a$, then $a = b$. From $a = b$ we infer that for all $j \in \mathsf{A}$ $a \leq_j b$ and $b \leq_j a$. Of course, the stronger statement, $a \leq_i b$ implies $a \leq_j b$ does not hold.

Proposition 3.3.4 (Soundness)**.** *Every concrete multi-player algebra is a multi-Boolean algebra.*

Proof. Let A be a set of agents and S a set of states. Consider the concrete multi-player algebra $\mathfrak{C}(S)$:

$$\mathfrak{C}(S) = (E, \cup_i, \sim_{ij}, \emptyset_i)_{i,j\in\mathsf{A}}.$$

Here E denotes $\{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^S}$. We need to show that the concrete multi-player algebra $\mathfrak{C}(S)$ satisfies all the axioms of multi-Boolean algebras. Most axioms are easily verifiable. We will go through some of the axioms in detail.

- (MB2). To show that for any $s \in S$ and $k \in \mathsf{A}$, $(x \cup_i (y \cup_j z))(s)(k) = ((x \cup_i y) \cup_j (x \cup_i z))(s)(k)$. We need to distinguish three cases: $k = i$, $k = j$ and $k \notin \{i, j\}$. If $k = i$ we get

$$\begin{array}{rll} (x \cup_i (y \cup_j z))(s)(i) = \mathbf{w} & \text{iff} & x(s)(i) = \mathbf{w} \text{ or } (y \cup_j z)(s)(i) = \mathbf{w} \\ & \text{iff} & x(s)(i) = \mathbf{w} \text{ or} \\ & & \quad (y(s)(i) = \mathbf{w} \text{ and } (z)(s)(i) = \mathbf{w}) \\ & \text{iff} & (x(s)(i) = \mathbf{w} \text{ or } y(s)(i) = \mathbf{w}) \text{ and} \\ & & \quad (x(s)(i) = \mathbf{w} \text{ or } y(s)(i) = \mathbf{w}) \\ & \text{iff} & ((x \cup_i y) \cup_j (x \cup_i z))(s)(i) = \mathbf{w}. \end{array}$$

  The cases $k = j$ and $k \notin \{i, j\}$ can be proved by similar reasoning.

- (MB5). In order to show that for any $s \in S$ and $k \in \mathsf{A}$, $((x \cup_i y) \cup_j x \cup_j \emptyset)(s)(k) = (x \cup_j \emptyset)(s)(k)$ for $j \neq i$ we need to distinguish three cases: $k = j$, $k = i$ and $k \notin \{i, j\}$. If $k = j$ we get

$$\begin{array}{rll} ((x \cup_i y) \cup_j x \cup_j \emptyset)(s)(j) = \mathbf{w} & \text{iff} & x(s)(j) = \mathbf{w} \text{ or } \emptyset(s)(j) = \mathbf{w} \text{ or} \\ & & \quad (x \cup_i y)(s)(j) = \mathbf{w} \\ & \text{iff} & x(s)(j) = \mathbf{w} \text{ or } \emptyset(s)(j) = \mathbf{w} \text{ or} \\ & & \quad (x(s)(j) = \mathbf{w} \text{ and } y(s)(j) = \mathbf{w}) \\ & \text{iff} & x(s)(j) = \mathbf{w} \\ & \text{iff} & x(s)(j) = \mathbf{w} \text{ or } \emptyset(s)(j) = \mathbf{w} \\ & \text{iff} & (x \cup_j \emptyset)(s)(j) = \mathbf{w}. \end{array}$$

  For all the other players $((x \cup_i y) \cup_j x \cup_j \emptyset)(s)(k) = \mathbf{w}$ iff $((x \cup_i y) \cup_j x)(s)(k) = \mathbf{w}$ *and* $\emptyset(s)(k) = \mathbf{w}$. By definition of $\emptyset$, this is never the case. Thus, for every $s \in S$ and $k \neq j \in \mathsf{A}$, $((x \cup_i y) \cup_j x \cup_j \emptyset)(s)(k) = \mathbf{l}$. For the same reason also $(x \cup_j \emptyset)(s)(k) = \mathbf{l}$ for every $s \in S$. Hence we may conclude that $((x \cup_i y) \cup_j x \cup_j \emptyset)(s)(k) = (x \cup_j \emptyset)(s)(k)$.

- (MB6). Recall that $a \preceq_i b$ iff $a \cup_i b \cup_i \emptyset = b \cup_i \emptyset$. This means that if player i wins $a \cup_i b$ at s, then i also wins b at s. Hence $a \preceq_i b$ iff $a(s)(i) = \mathbf{w}$ implies $b(s)(i) = \mathbf{w}$. For all the other players $j \neq i$, we cannot conclude anything from $a \preceq_i b$, since j loses at any s given both $a \cup_i b \cup_i \emptyset$ and $b \cup_i \emptyset$. Now suppose $x \preceq_i y$ and $y \preceq_j x$ for all $j \neq i$. Then $a(s)(i) = \mathbf{w}$ implies $b(s)(i) = \mathbf{w}$ (b is at least as good for i as a) and for all $j \neq i$ $b(s)(j) = \mathbf{w}$ implies $a(s)(j) = \mathbf{w}$ (a is at least as good for j as b). As we have discussed in chapter 2, these are exactly the conditions for $a \leq_i b$.

- (MBn3). In order to show that $\sim_{ij}\sim_{jk} x \cup_i \emptyset = \sim_{ik} x \cup_i \emptyset$ it suffices to show that for any $s \in S$, $(\sim_{ij}\sim_{jk} x)(s)(i) = \mathbf{w}$ iff $(\sim_{ik} x)(s)(i) = \mathbf{w}$ (see the discussion of soundness of axiom (MB6)). This follows from the definition of $\sim$:

$$\begin{aligned}(\sim_{ij}\sim_{jk} x)(s)(i) = \mathbf{w} \quad &\text{iff} \quad (\sim_{jk} x)(s)(j) = \mathbf{w}\\ &\text{iff} \quad x(s)(k) = \mathbf{w}\\ &\text{iff} \quad (\sim_{ik} x)(s)(i) = \mathbf{w}.\end{aligned}$$

- (MBn6). Assume $\{i, j\} \cap \{l, k\} = \emptyset$. We will show that for any $s \in S$ and $m \in \mathsf{A}$, $(\sim_{ij}\sim_{lk} x)(s)(m) = \mathbf{w}$ iff $(\sim_{lk}\sim_{ij} x)(s)(m) = \mathbf{w}$. Distinguish cases: $m \in \{i, j, k, l\}$ and $m \notin \{i, j, k, l\}$. Case $m \in \{i, j, k, l\}$. Assume wlog that $m = i$:

$$\begin{aligned}(\sim_{ij}\sim_{lk})x(s)(i) = \mathbf{w} \quad &\text{iff} \quad (\sim_{lk} x)(s)(j) = \mathbf{w}\\ &\text{iff} \quad x(s)(j) = \mathbf{w}\\ &\text{iff} \quad (\sim_{ij} x)(s)(i) = \mathbf{w}\\ &\text{iff} \quad (\sim_{jk}\sim_{ij} x)(s)(i) = \mathbf{w}.\end{aligned}$$

If $m \notin \{i, j, k, l\}$, the proof is even more straightforward:

$$\begin{aligned}(\sim_{ij}\sim_{lk} x)(s)(m) = \mathbf{w} \quad &\text{iff} \quad (\sim_{lk} x)(s)(m) = \mathbf{w}\\ &\text{iff} \quad x(s)(m) = \mathbf{w}\\ &\text{iff} \quad (\sim_{ij} x)(s)(m) = \mathbf{w}\\ &\text{iff} \quad (\sim_{jk}\sim_{ij} x)(s)(m) = \mathbf{w}.\end{aligned}$$

□

3.4 Representation theorem

Definition 3.4.1. *An i-*filter *of a multi-Boolean algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ is a nonempty subset F of A satisfying*

(F1) $\top_i \in F$,
(F2) if $a, b \in F$ then $a \vee_j b \in F$ for all $j \neq i$,
(F3) if $a \in F$ and $a \preceq_i b$ then $b \in F$.

An i-filter is called proper *if $\bot_i \notin F$. An i-*prime filter *is a proper i-filter such that*

(F4) for every pair of elements $a, b \in A$, $a \vee_i b \in F$ implies $a \in F$ or $b \in F$.

The collection of i-filters of $\mathfrak{A}$ is denoted with $\mathcal{F}_i(\mathfrak{A})$. With $\mathcal{F}_i^p(\mathfrak{A})$ we denote the collection of i-prime filters of $\mathfrak{A}$. Next, we will define the concept of an i-(prime) ideal.

Definition 3.4.2. *An* i-ideal *of a multi-Boolean algebra* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ *is a nonempty subset* I *of* A *satisfying*

(I1) $\bot_i \in I$,
(I2) if $a, b \in I$ *then* $a \vee_i b \in I$,
(I3) if $a \in I$ *and* $b \preceq_i a$ *then* $b \in I$.

An i*-ideal is called* proper *if* $\top_i \notin I$. *An* i-prime ideal *is a proper* i*-ideal such that*

(I4) For every pair of elements $a, b \in A$, $a \vee_j b \in I$ *for some* $j \neq i \in \mathsf{A}$ *implies* $a \in I$ *or* $b \in I$

The collection of i-ideals of $\mathfrak{A}$ is denoted with $\mathcal{I}_i(\mathfrak{A})$. With $\mathcal{I}_i^p(\mathfrak{A})$ we denote the collection of i-prime ideals of $\mathfrak{A}$. It is important to observe that given an arbitrary i-filter F of $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ and two elements $a, b \in A$

$$a =_i b \text{ implies } a \in F \text{ iff } b \in F. \tag{3.1}$$

Remember that $a =_i b$ is shorthand notation for $a \preceq_i b$ and $b \preceq_i a$. Thus whenever a or b are in F it follows immediately from (F3) that the other element is in F as well. The same also holds for i-ideals. Note that, contrary to the standard 'two-player' filters and ideals, i-ideal and i-filters are not order-duals. That is, we cannot exchange $\top_i$ with $\bot_i$ and $\vee_i$ with some $\vee_j$ in the conditions of i-filters to obtain the conditions of i-ideals. Because of this fact, we will not be able to restrict ourselves to the use of i-filters or i-ideals only, we will need both concepts to prove our desired representation theorem.

Just like in case of classical ideals and filters, the complement of an (proper) i-filter is not necessarily an (proper) i-ideal. However, the complement of an i-*prime* ideal is an i-*prime* filter. We will prove this fact in the next proposition.

Proposition 3.4.3. *For any multi-Boolean algebra* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$, $I \subset A$ *is an* i*-prime ideal if and only if* $A \setminus I$ *is an* i*-prime filter.*

Proof. For the direction from left to right, assume that $I \subset A$ is and i-prime ideal. We have to show that $F = A \setminus I$ does not contain $\bot_i$ and satisfies (F1)-(F4). (F1) follows immediately from the assumption that I is proper. Let us have a look at (F2). Suppose that $a, b \in F$. We need to show that $a \vee_j b \in F$ for all $j \neq i$. Suppose for contradiction that for some $j \neq i$, $a \vee_j b \notin F$. This implies that $a \vee_j b \in I$. But since I is assumed to be a proper i-*prime* filter it follows from (I4) that $a \in I$ or $b \in I$, which contradicts our assumption that both a and b are in F. In a similar fashion, (F3) follows from (I3) and (F4) can be proved using (I2). By our assumption that I is an i-ideal and thus contains $\bot_i$ by definition, it follows that F is proper.

The proof for the other direction is similar. (I1) follows from the fact that F is proper, (I2) from (F4), (I3) from (F3) and (I4) from (F2). Also, I is proper by (F1).

□

Lemma 3.4.4. *Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in \mathsf{A}}$ be a multi-Boolean algebra. Let F be an i-filter of $\mathfrak{A}$ and a an element of F. Then for all $b \in A$ $a \vee_i b \in F$. Also, from $a \vee_j b \in F$ it follows that both a and b are in F.*

Proof. In order to show the first part, assume $a \in F$ and let b be any element of A. We will show that $a \preceq_i a \vee_i b$ from which it follows by (F3) that $a \vee_i b \in F$. This result is easily obtained, since $a \vee_i (a \vee_i b) \vee_i \bot = (a \vee_i a) \vee_i b \vee_i \bot = a \vee_i b \vee_i \bot$. In order to show the second part of the lemma, assume that $a \vee_j b \in F$. By proposition 3.3.3 (vii) it follows that $a \vee_j b \preceq_i a$ and $a \vee_j b \preceq_i b$. Again, by (F3) it follows that both a and b are in F. □

Proposition 3.4.5. *If F is an i-prime filter, then $\{\neg_{ij} a \mid a \in F\}$ is a j-prime filter.*

Proof. Let F be an i-prime filter of a multi-Boolean algebra and let $G = \{\neg_{ij} a | a \in F\}$. We need to check that G is a j-prime filter.

(F1) $\neg_{ij} \top_j = \top_i \in F$ by 3.3.3 (iii) thus $\top_j \in G$.

(F2) Assume $a, b \in G$, then $\neg_{ij} a$ and $\neg_{ij} b$ are in F and thus $\neg_{ij} a \vee_j \neg_{ij} b \in F$. It follows by (MBn7) that $\neg_{ij}(a \vee_i b) \in F$ and thus $a \vee_i b \in G$. For $k \notin \{i, j\}$ we have $\neg_{ij} a \vee_k \neg_{ij} b \in F$ and by (MBn8) $\neg_{ij}(a \vee_k b) \in F$ and thus $a \vee_k b \in G$.

(F3) Assume $a \in G$ and $a \preceq_j b$. We know that $\neg_{ij} a \in F$. By 3.3.3(vi) it follows that $\neg_{ij} a \preceq_i \neg_{ij} b$ and thus we obtain $b \in G$.

(Pr) $\bot_i \notin F$. It follows by 3.3.3 (iii) that $\neg_{ij} \bot_j \notin F$ and thus $\bot_j \notin G$. The filter G is proper.

(F4) Suppose $a \vee_j b \in G$, then $\neg_{ij}(a \vee_j b) \in F$ and thus $\neg_{ij} a \vee_i \neg_{ij} b \in F$ from which it follows by (F4) that $\neg_{ij} a \in F$ or $\neg_{ij} b \in F$. Thus either a or b must be contained in G.

□

Definition 3.4.6. *Given a multi-Boolean algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in \mathsf{A}}$, $a \in A$ and $i \in \mathsf{A}$ we define the sets $\downarrow_i a$ and $\uparrow_i a$ as follows:*

$$\downarrow_i a = \{b \in A \mid b \preceq_i a\}$$

and,

$$\uparrow_i a = \{b \in A \mid a \preceq_i b\}.$$

Given a set $C \subseteq A$ the sets $\downarrow_i C$ and $\uparrow_i C$ are defined as

$$\downarrow_i C = \{b \in A \mid b \preceq_i c \text{ for some } c \in C\}$$

and,

$$\uparrow_i C = \{b \in A \mid c \preceq_i b \text{ for some } c \in C\}.$$

These sets will play an important role in proving our desired representation theorem.

Proposition 3.4.7. *For any multi-Boolean algebra* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in\mathsf{A}}$ *and* $a \in A$*:*

(i) The set $\downarrow_i a$ *is an i-ideal of* $\mathfrak{A}$*.*

(ii) The set $\uparrow_i a$ *is an i-filter of* $\mathfrak{A}$*.*

(iii) For any i-ideal I *of* $\mathfrak{A}$*, the set* $\downarrow_i \{a \vee_i c \mid c \in I\}$ *is an i-ideal.*

(iv) For any i-filter F *of* $\mathfrak{A}$ *and* $j \in \mathsf{A}$ *such that* $j \neq i$ *the set* $\uparrow_i \{a \vee_j b | b \in F\}$ *is an i-filter.*

Proof. Fix a multi-Boolean algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in\mathsf{A}}$ and $a \in A$. It will turn out that in each of the four cases, showing the second clause (That is, (F2) in (ii) and (iv), and (I2) in (i) and (iii)) is the hardest part. For proofs of (i), (ii) and (iii) we refer to the appendix. Here, we will focus on proving item (iv), since it involves the most laborious effort.

Let F be an arbitrary i-filter of $\mathfrak{A}$ and $j \in \mathsf{A}$ such that $j \neq i$. We will show that $\uparrow_i \{a \vee_j b | b \in F\}$ is an i-filter. Let G to be the set

$$G = \uparrow_i \{a \vee_j b \mid b \in F\}$$

We need to show that G satisfies the properties (F1)-(F3). Proving (F1) and (F3) are relatively easy, whereas showing that (F2) is satisfied involves a fair amount of effort.

(F1) By assumption $\top_i \in F$. From this it follows that $a \vee_j \top_i \in G$. By (MB4) it follows that $a \vee_j \top_i \leq_i \top_i$, hence $a \vee_j \top_i \preceq_i \top_i$ and thus $\top_i \in G$.

(F2) Assume e and e' are in G, we need to show that $e \vee_k e' \in G$ for any $k \neq i$. Our strategy will be to find an element d such that there is a $b \in F$ such that $a \vee_j b \preceq_i d$, and $d \preceq_i e \vee_k e'$. By transitivity of $\preceq_i$ it then follows that $a \vee_j b \preceq_i e \vee_k e'$ and hence $e \vee_k e' \in G$.

By assumption that e, e' are in G it follows that there is are two elements c, c' contained in F such that $a \vee_j c \preceq_i e$ and $a \vee_j c' \preceq_i e'$. In order to improve readability of what there is to come, let's denote $v = a \vee_j c$ and $v' = a \vee_j c'$. Now, define d as follows:

$$d = (v \vee_k v') \vee_i (v \vee_k e') \vee_i (e \vee_k v').$$

We start by showing that there is a $b \in F$ such that $a \vee_j b \preceq_i d$. Our candidate for b will be $c \vee_j c'$.

Claim 3.4.8. $a \vee_j b \preceq_i d$

Proof. Since both c and c' are in F, it follows that $b = c \vee_j c'$ is in F as well. Moreover, it follows that

$$\begin{array}{rcl} a \vee_j b & = & a \vee_j (c \vee_j c') \\ & = & (a \vee_j c) \vee_j (a \vee_j c') \\ & = & v \vee_j v' \end{array}$$

By proposition 3.3.3 we know that $v \vee_j v' =_i v \vee_k v'$ since $i \notin \{j, k\}$. Hence, $a \vee_j b =_i v \vee_k v'$, from which we obtain that $a \vee_j b \preceq_i v \vee_k v'$. Since $a \vee_j b \in G$ it follows that $v \vee_k v'$ is as well. For any x and y, we know that $x \leq_i y$, since $x \vee_i x \vee_i y = x \vee_i y$. Hence we know that $x \preceq_i x \vee_i y$ as well (by property 3.3.3 (viii)). It follows from this fact that $v \vee_k v' \preceq_i d$. Hence, $a \vee_j b \preceq_i d$ □

Now that we have shown that there is a b such that $a \vee_j b \preceq_i d$ (and hence that $d \in G$), it remains to be shown that $d \preceq_i e \vee_k e'$.

Claim 3.4.9. $d \preceq_i e \vee_k e'$

Proof. By definition of $\preceq_i$ and by assumption that $a \vee_j c \preceq_i e$ and $a \vee_j c' \preceq_i e'$, we have the following equalities:

$$\begin{array}{rcccl} (a \vee_j c) \vee_i e \vee_i \bot & = & v \vee_i e \vee_i \bot & = & e \vee_i \bot, \\ (a \vee_j c') \vee_i e' \vee_i \bot & = & v' \vee_i e' \vee_i \bot & = & e' \vee_i \bot. \end{array}$$

From these equalities we obtain:

$$\begin{array}{rcl} (e \vee_k e') \vee_i \bot & = & (e \vee_i \bot) \vee_k (e' \vee_i \bot) \\ & = & (v \vee_i e \vee_i \bot) \vee_k (v \vee_i e' \vee_i \bot) \\ & = & (e \vee_k v') \vee_i (v' \vee_k e') \vee_i (v \vee_k \bot) \\ & & \quad \vee_i (e \vee_k v') \vee_i (e \vee_k e') \vee_i (e \vee_k \bot) \\ & & \quad \vee_i (\bot \vee_k v') \vee_i (\bot \vee_k e') \vee_i \bot \\ & \stackrel{*}{=} & (v \vee_k v') \vee_i (v \vee_k e') \vee_i (e \vee_k v') \vee_i (e \vee_k e') \vee_i \bot \end{array} \tag{3.2}$$

Here the last equality (*) follows by application of (MB5): by reflexivity of $\leq_i$, a formula of the form $(\bot \vee_k v) \vee_i \bot$ is equivalent to $(\bot \vee_k v) \vee_i \bot \vee_i \bot$ and hence by (MB5) to $\bot$. It follows that we can eliminate all the 'conjuncts' $(v \vee_k \bot)$ from

$$\begin{array}{l} (v \vee_k v') \vee_i (v \vee_k e') \vee_i (v \vee_k \bot) \\ \quad \vee_i (e \vee_k v') \vee_i (e \vee_k e') \vee_i (e \vee_k \bot) \\ \quad \vee_i (\bot \vee_k v') \vee_i (\bot \vee_k e') \vee_i \bot \end{array}$$

and obtain that this formula equals

$$(v \vee_k v') \vee_i (v \vee_k e') \vee_i (e \vee_k v') \vee_i (e \vee_k e') \vee_i \bot.$$

We may conclude from (3.2) that $(v \vee_k v') \vee_i (v \vee_k e') \vee_i (e \vee_k v') \preceq_i (e \vee_k e')$ and hence, $d \preceq_i (e \vee_k e')$. □

To summarize, we have shown that there is an element d and an element $b \in F$ such that $a \vee_j b \preceq_i d \preceq_i e \vee_k e'$. Hence, $e \vee_k e' \in G$ as desired.

(F3) Let e be in G and assume $e \preceq_i f$, then for some $c \in F$ we have $a \vee_j c \preceq_i e$ and since $e \preceq_i f$ it follows by transitivity that $a \vee_j c \preceq_i f$. Hence, $f \in G$.

□

Note that, in general, the i-ideals and i-filters described in the above proposition are not prime. Next we will prove a multi-player analogue of the prime-filter theorem.

Theorem 3.4.10. *Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ be a multi-Boolean algebra. Let J be an i-ideal and G an i-filter of A such that $J \cap G = \emptyset$. Then there exists $I \in \mathcal{I}_i^p(\mathfrak{A})$ and $F = A \setminus I \in \mathcal{F}_i^p(\mathfrak{A})$ such that $J \subseteq I$ and $G \subseteq F$.*

Proof. The proof of this theorem will be structurally similar to the proof of Theorem 9.13 in [4].

Define

$$\mathcal{E} = \{K \in \mathcal{I}(\mathfrak{A}) \mid J \subseteq K \text{ and } K \cap G = \emptyset\}$$

We will show that $(\mathcal{E}, \subseteq)$ has a maximal element I. First of all, $\mathcal{E}$ contains J and so is nonempty. Let $\mathcal{C} = \{K_\alpha \mid \alpha \in \lambda\}$ be a chain in $\mathcal{E}$. We need to show that $K = \bigcup_{\alpha \in \lambda} K_\alpha \in \mathcal{E}$. Certainly $J \subseteq K$ and $K \cap G = \emptyset$ (for if not, then there would be some $\alpha \in \lambda$ such that $K_\alpha \cap G \neq \emptyset$). What remains to be shown is that $K \in \mathcal{I}(\mathfrak{A})$.

Claim 3.4.11. *K is an i-filter*

Proof. (I1) follows is immediately since $\bot_i \in K_\alpha$ for each $K_\alpha \subseteq K$. Also (I3) is easy to obtain since if $a \in K$, then $a \in K_\alpha$ for some $\alpha \in \lambda$. If in addition and $a \preceq_i b$, then $b \in K_\alpha$ as well and hence $b \in K$. As for (I2), assume that $a, b \in K$. We need to show that $a \vee_i b \in K$. By assumption, there are $\alpha, \beta \in \lambda$ such that $a \in K_\alpha$ and $b \in K_\beta$. Since $\mathcal{C}$ is a chain, we may assume, without loss of generality that $K_\alpha \subseteq K_\beta$. Thus it follows that $b \in K_\beta$ and also $a \vee_i b \in K_\beta$, from which we conclude that $a \vee_i b \in K$. It follows that K is an i-filter. □

Hence, $K \in \mathcal{E}$. By Zorn's lemma we may now conclude that $\mathcal{E}$ has a maximal element I. This I will be our candidate to prove the theorem.

Claim 3.4.12. *I is an i-prime ideal.*

Proof. We know know that I is an i-ideal since it is in $\mathcal{E}$. The only thing that there is left to be shown is that it is an *i-prime* ideal. Clearly, I is proper for we know that G is nonempty. (I4) Suppose that for some $j \neq i \in \mathsf{A}$ we have $a \vee_j b \in I$ but $a, b \notin I$. Because I is assumed to be the maximal element of $\mathcal{E}$, any ideal properly containing I intersects with G. Therefore the i-ideal $I_a = \downarrow_i \{a \vee_i c \mid c \in I\}$ intersects with G. From this we may conclude that there exists a $c_a \in I$ such that $g \preceq_i a \vee_i c_a$ for some $g \in G$. Since G is assumed to be an i-filter, we know that $a \vee_i c_a \in G$. By similar reasoning we can find a $c_b \in I$ such that $a \vee_i c_b \in G$. The following equivalence is valid in $\mathfrak{A}$:

$$(a \vee_j b) \vee_i (c_a \vee_i c_b) = ((a \vee_i c_a) \vee_i c_b) \vee_j ((b \vee_i c_b) \vee_i c_a) \tag{3.3}$$

Since G is a filter, and since $a \vee_i c_a$ is in G, it follows by lemma 3.4.4 that $((a \vee_i c_a) \vee_i c_b) \in G$. Similarly, we know that $((b \vee_i c_b) \vee_i c_a) \in G$. By (F2) it follows that $((a \vee_i c_a) \vee_i c_b) \vee_j ((b \vee_i c_b) \vee_i c_a)$ is in G. However, the left-hand side of the equation 3.3 is in I. This is because both c_a and c_b are in I, and thus $(c_a \vee_i c_b) \in I$. Since $(a \vee_j b)$ was assumed to be in I we have $(a \vee_j b) \vee_i (c_a \vee_i c_b)$ in I. This contradicts the fact that $I \cap G = \emptyset$. Thus we obtain that I satisfies (I4), hence it is a i-prime ideal. $\square$

We may now conclude, (by proposition 3.4.3), that $F = A \setminus I$ an i-prime filter. $\square$

Corollary 3.4.13 (i-Prime Filter Theorem)**.** *Given a multi-Boolean algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$, an i-filter $G \subset A$ and an element $a \in A$ such that $a \notin G$, there is a i-prime filter F such that $G \subseteq F$ and F does not contain a.*

Proof. Consider the set $I = \downarrow_i a$. By proposition 3.4.7, I is an i-ideal. Moreover, $I \cap G = \emptyset$. Suppose otherwise, then there is a b such that $b \in I$ and $b \in G$, hence by definition of I, $b \preceq_i a$. From (F3) it thus follows that $a \in G$ which contradicts our assumption that $a \notin G$. By theorem 3.4.10 it follows that there is an i-prime filter F extending G such that $a \notin F$. $\square$

Theorem 3.4.14 (Representation Theorem)**.** *Every multi-Boolean algebra is isomorphic to a concrete multi-player algebra.*

Proof. Fix a multi-Boolean algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$. We will fix one player, player $0 \in \mathsf{A}$ and embed $\mathfrak{A}$ into the concrete multi algebra $\mathfrak{C}(\mathcal{F}_0^p(A))$. Remember that $\mathcal{F}_0^p(A)$ is the collection of 0-prime filters of A. Consider the map: $\rho : \mathfrak{A} \to \mathfrak{C}(\mathcal{F}_0^p(A))$ defined as follows:

$$\rho_a(F)(j) = \begin{cases} \mathbf{w} \text{ if } \neg_{0j} a \in F \\ \mathbf{l} \text{ otherwise} \end{cases}$$

We need to show that ρ is an injective homomorphism. First we will show that ρ is a homomorphism. Because the proof of this fact involves some convoluted arguments combining many of the propositions proved in this section, we will go through the proof in great detail.

Let us first consider the $\vee_l$ operator. We need to show that $\rho(a \vee_l b) = \rho(a) \cup_l \rho(b)$. This boils down to showing that

$$\rho_{a \vee_l b}(F)(j) = \mathbf{w} \text{ iff } (\rho_a \cup_l \rho_b)(F)(j) = \mathbf{w}.$$

We distinguish two cases, $l = 0$ and $l \neq 0$. That is, we distinguish between $\vee_0$ and $\vee_l$ with $l \neq 0$.

Case $\vee_0$.

For $j \neq 0$,

$$\begin{array}{rcl}
\rho_{a \vee_0 b}(F)(j) = \mathbf{w} & \Leftrightarrow & \neg_{0j}(a \vee_0 b) \in F \\
& \Leftrightarrow & \neg_{0j} a \vee_j \neg_{0j} b \in F \\
& \overset{*}{\Leftrightarrow} & \neg_{0j} a \in F \text{ and } \neg_{0j} b \in F \\
& \Leftrightarrow & \rho_a(F)(j) = \mathbf{w} \text{ and } \rho_b(F)(j) = \mathbf{w} \\
& \Leftrightarrow & (\rho_a \cup_0 \rho_b)(F)(j) = \mathbf{w}
\end{array}$$

Note that the crucial step is the equality (*). The direction from left to right follows from 3.3.3 (vii) and the other direction follows by clause (F2) of the definition of 0-filter. The equality of the last two lines follows from the definition of concrete multi-player set algebras in section 3.2.

For $j = 0$,

$$\begin{array}{rcl}
\rho_{a \vee_0 b}(F)(0) = \mathbf{w} & \Leftrightarrow & \neg_{00}(a \vee_0 b) \in F \\
& \Leftrightarrow & \neg_{00} a \vee_0 \neg_{00} b \in F \\
& \overset{*}{\Leftrightarrow} & \neg_{00} a \in F \text{ or } \neg_{00} b \in F \\
& \Leftrightarrow & \rho_a(F)(0) = \mathbf{w} \text{ or } \rho_b(F)(0) = \mathbf{w} \\
& \Leftrightarrow & (\rho_a \cup_0 \rho_b)(F)(0) = \mathbf{w}
\end{array}$$

Again, the crucial step is the (*)-equality. Here, the direction from left to right follows from the assumption that F is a 0-*prime* filter. The other direction follows from 3.4.4. Again, the equality of the last two lines follows from the definition of concrete multi-agent powerset algebras in section 3.2.

Case $\vee_l$, with $l \neq 0$.

For $j \neq l$ (possibly $j = 0$),

$$\begin{array}{rcl}\rho(a \vee_l b)(F)(j) = \mathbf{w} & \Leftrightarrow & \neg_{0j}(a \vee_l b) \in F \\ & \Leftrightarrow & \neg_{0j} a \vee_l \neg_{0j} b \in F \\ & \Leftrightarrow & \neg_{0j} a \in F \text{ and } \neg_{0j} b \in F \\ & \Leftrightarrow & \rho_a(F)(j) = \mathbf{w} \text{ and } \rho_b(F)(j) = \mathbf{w} \\ & \Leftrightarrow & (\rho_a \cup_l \rho_b)(F)(j) = \mathbf{w}\end{array}$$

and for $j = l$,

$$\begin{array}{rcl}\rho(a \vee_l b)(F)(l) = \mathbf{w} & \Leftrightarrow & \neg_{0l}(a \vee_l b) \in F \\ & \Leftrightarrow & \neg_{0l} a \vee_0 \neg_{0l} b \in F \\ & \Leftrightarrow & \neg_{0l} a \in F \text{ or } \neg_{0l} b \in F \\ & \Leftrightarrow & \rho_a(F)(l) = \mathbf{w} \text{ or } \rho_b(F)(l) = \mathbf{w} \\ & \Leftrightarrow & (\rho_a \cup_l \rho_b)(F)(l) = \mathbf{w}\end{array}$$

Now, let us have a look at the other connective, $\neg_{ij}$.We will show that $\rho(\neg_{ij} a) = \sim_{ij} \rho(a)$

This boils down to showing that

$$\rho_{\neg_{ij} a}(F)(k) = \mathbf{w} \text{ iff } \sim_{ij} \rho_a(F)(k) = \mathbf{w}.$$

Again, we will consider two cases $0 \in \{i, j\}$ and $0 \notin \{i, j\}$. If $0 \in \{i, j\}$, we can need to distinguish between two cases $\neg_{0j}$ and $\neg_{i0}$. Here, we will only treat the first case and leave the second to the reader. Thus, here we will discuss the following two negations: $\neg_{0j}$ with j can be equal to 0 or not, and $\neg_{ij}$ with both i and j distinct from 0.

Case $\neg_{0j}$.

For $k = 0$,

$$\begin{array}{rcl}\rho_{\neg_{0j} a}(F)(0) = \mathbf{w} & \Leftrightarrow & \neg_{00} \neg_{0j} a \in F \\ & \Leftrightarrow & \neg_{0j} a \in F \\ & \Leftrightarrow & \rho_a(F)(j) = \mathbf{w} \\ & \Leftrightarrow & \sim_{0j} \rho_a(F)(0) = \mathbf{w}\end{array}$$

Here, the equivalence between the first and the second line follows from (MBn1).

For $k = j$,

$$\begin{array}{rcl}\rho_{\neg_{0j}a}(F)(j)=\mathbf{w} & \Leftrightarrow & \neg_{0j}\neg_{0j}a\in F\\ & \Leftrightarrow & \neg_{0j}\neg_{j0}a\in F\\ & \Leftrightarrow & \neg_{00}a\in F\\ & \Leftrightarrow & \rho_a(F)(0)=\mathbf{w}\\ & \Leftrightarrow & \sim_{0j}\ \rho_a(F)(j)=\mathbf{w}\end{array}$$

Here, the equivalence between the second and third line follows from (MBn3).

For $k\notin\{0,j\}$,

$$\begin{array}{rcl}\rho_{\neg_{0j}a}(F)(k)=\mathbf{w} & \Leftrightarrow & \neg_{0k}\neg_{0j}a\in F\\ & \Leftrightarrow & \neg_{0k}a\in F\\ & \Leftrightarrow & \rho_a(F)(k)=\mathbf{w}\\ & \Leftrightarrow & \sim_{0j}\ \rho_a(F)(k)=\mathbf{w}\end{array}$$

Here the equivalence between the first and the second line follows from (MBn4) and the observation 3.1. The last step follows from the definition of $\sim$ in the concrete multi-player algebra.

Case $\neg_{ij}$, with $0\notin\{i,j\}$.

For $k=i$,

$$\begin{array}{rcl}\rho_{\neg_{ij}a}(F)(i)=\mathbf{w} & \Leftrightarrow & \neg_{0i}\neg_{ij}a\in F\\ & \Leftrightarrow & \neg_{0j}a\in F\\ & \Leftrightarrow & \rho_a(F)(j)=\mathbf{w}\\ & \Leftrightarrow & \sim_{ij}\ \rho_a(F)(i)=\mathbf{w}\end{array}$$

and for $k\notin\{i,j\}$,

$$\begin{array}{rcl}\rho_{\neg_{ij}a}(F)(k)=\mathbf{w} & \Leftrightarrow & \neg_{0k}\neg_{ij}a\in F\\ & \Leftrightarrow & \neg_{ij}\neg_{0k}a\in F\\ & \Leftrightarrow & \neg_{0k}a\in F\\ & \Leftrightarrow & \rho_a(F)(k)=\mathbf{w}\\ & \Leftrightarrow & \sim_{ij}\ \rho_a(F)(k)=\mathbf{w}\end{array}$$

Here the equivalence between the first and second line follows from (MBn6) since all four $0,i,j,k$ are different. The next step follows from (MBn2) since $0\neq i$ and $0\neq j$. The last equality holds since $k\neq i$ and $k\neq j$.

We may conclude that ρ is a homomorphism. What remains to be shown is that ρ is injective. Consider two elements a,b such that $a\neq b$. By antisymmetry of $\leq_i$, it follows from this that either $a\not\leq_0 b$ or $b\not\leq_0 a$. Without loss of generality we may assume that $b\not\leq_0 a$. By (MB6) it follows that either $b\not\leq_0 a$ or $a\not\leq_j b$ for some $j\neq 0$. In the first case $b\not\leq_0 a$ we know that $a\notin{\uparrow_0}b$. By theorem 3.4.13 it follows that there exists an 0-prime filter F containing ${\uparrow_0}b$ (and thus also the element b) but not containing a. From this it follows that

$\rho_a(F)(0) = \mathbf{l}$ whereas $\rho_b(F)(0) = \mathbf{w}$, and thus $\rho_a \neq \rho_b$.

In the second case, when $a \not\leq_j b$ for some $j \neq 0$, it follows by proposition 3.3.3(vi) that $\neg_{0j} a \not\leq_0 \neg_{0j} b$ and thus $\neg_{0j} b \notin \uparrow_0 \neg_{0j} a$. Again, by theorem 3.4.13 it follows that there exists an 0-prime filter F containing $\uparrow_0 \neg_{0j} a$ (and thus also the element $\neg_{0j} a$) but not containing $\neg_{0j} b$. From this it follows that $\rho_a(F)(j) = \mathbf{w}$ but $\rho_b(F)(j) = \mathbf{l}$, hence $\rho_a \neq \rho_b$. We may conclude that ρ is an injective homomorphism. We obtain that $\mathfrak{A}$ is isomorphic to a concrete multi-player algebra.

□

Corollary 3.4.15. *Every multi-Boolean algebra of the similarity type* $Mbool(\mathsf{A})$ *is isomorphic to a subalgebra of a power of* $\widetilde{\mathsf{A}}$.

Proof. This follows immediately from the result just obtained and proposition 3.2.6. □

Finally, we would briefly like to mention an implication of the above representation theorem that establishes a deep relation between the logic *MPL* and the class of abstract multi-Boolean algebras. Firstly, the representation theorem guarantees us that

$$\mathbf{CcMA} \vDash f' =_i g' \text{ implies } \mathbf{MBA} \vDash f =_i g,$$

where f' equals the term f using the corresponding concrete multi-player algebraic symbols and similarly g' equals g.

By lemma 3.2.4 we knew already that:

$$\phi \text{ is } i\text{-valid iff } \mathbf{CcMA} \vDash \phi' =_i \natural_i$$

and hence we obtain:

$$\phi \text{ is } i\text{-valid (implies } \mathbf{CcMA} \vDash \phi' =_i \natural_i) \text{ implies } \mathbf{MBA} \vDash \phi =_i \top_i.$$

Chapter 4

Multi-Player Algebras: The Modal Case

4.1 Introduction

In this chapter we will algebraize multi-player modal logic (*MML*). Our strategy is to expand the multi-Boolean algebras, subject of the previous chapter, to multi-modal algebras *MMA*s. Then we generalize the representation theorem from chapter 3 to the case of *MMA*s. In this way we obtain a multi-player analogue of the celebrated Jónsson-Tarski theorem.

4.2 Multi-Modal Algebras (*MMA*)

Definition 4.2.1. *Given a finite set* A*, We define* MBoolO(A) *as the algebraic similarity type MBool(*A*) plus one unary operator* $\Diamond_i$ *for each* $i \in \mathsf{A}$*. Algebras of the MBoolO(*A*) type will be denoted as* $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i, \Diamond_i)_{i,j \in N}$*.*

Definition 4.2.2. *A* multi-modal algebra *(MMA) is an algebra*

$$\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i, \Diamond_i)_{i,j \in \mathsf{A}}$$

of the MBoolO(A*) similarity type such that* $(A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ *is a multi-Boolean algebra and the following axioms are satisfied*

(MBO1) $\Diamond_i x \vee_j \Diamond_j y \leq_i \Diamond_i(x \vee_j y)$
(MBO2) $\Diamond_i x \vee_i \Diamond_i y = \Diamond_i(x \vee_i y)$
(MBO3) $\neg_{ij}\Diamond_j x = \Diamond_i \neg_{ij} x$ *and* $\neg_{ik}\Diamond_j x = \Diamond_j \neg_{ik} x$ *when* $j \notin \{i, k\}$
(MBO4) $\Diamond_i \bot_i = \bot_i$
(MBO5) $\Diamond_j \top_i =_i \top_i$ *for* $j \neq i$ *and* $\Diamond_i \bot =_i \bot$
(MBO6) $\Diamond_j x =_i \Diamond_k x$ *if* $i \notin \{j, k\}$
(MBO7) $x \preceq_i y$ *implies* $\Diamond_j x \preceq_i \Diamond_j y$ *for all* $j \neq i$

We call the class of all multi-modal algebras MMA.

Definition 4.2.3. *Let $\mathfrak{F} = (W, R)$ be a Kripke frame. The* full complex multi-player algebra of $\mathfrak{F}$*, denoted as $\mathfrak{F}^+$ is the expansion of the full concrete multi-player algebra $\mathfrak{C}(W)$ with one operator m_i for every $i \in \mathsf{A}$. The operator m_i is defined as follows*

$$m_i(f)(s)(i) = \boldsymbol{w} \Leftrightarrow \textit{there exists a } t \in W \textit{ such that } Rst \textit{ and } f(t)(i) = \boldsymbol{w}$$

and for $j \neq i$,

$$m_i(f)(s)(j) = \boldsymbol{w} \Leftrightarrow \textit{for all } t \in W \textit{ such that } Rst,\ f(t)(i) = \boldsymbol{w}$$

A subalgebra of a full complex multi-player algebra is a complex multi-player algebra. *Note that $\mathfrak{F}^+$ is of the MBoolO(A) similarity type where m_i corresponds to $\Diamond_i$. The class of all complex multi-player algebras is denoted* ***CpMA****.*

As before, we say that an equation $f = g$ *is valid* in the full complex multi-player algebra $\mathfrak{F}^+$ of $\mathfrak{F}$ (notation: $\mathfrak{F}^+ \vDash f = g$) if for each variable assigment each player has the same value assigned to f and g at each state s of the frame. Similarly, $f =_i g$ is valid in $\mathfrak{F}^+$ if $f \cup_i \emptyset = g \cup_i \emptyset$ is valid in $\mathfrak{F}^+$. This implies that for player i, the value of f equals that of g at each state. Generalizing these concepts, we say that the class of complex multi-player algebras **CpMA** validates $f = g$ (notation: **CpMA** $\vDash f = g$) if every complex multi-player algebra validates $f = g$. And similarly, **CpMA** validates $f =_i g$ if every complex multi-player algebra validates $f =_i g$.

Observe that $\mathfrak{F}^+$ exactly encodes all the information about the frame $\mathfrak{F}$ in the following sense:

Lemma 4.2.4. *There is a valuation such that i-satisfies ϕ in $\mathfrak{F}$ at s iff there is an assigment such that $\phi'(s)(i) = \boldsymbol{w}$ in $\mathfrak{F}^+$. Here ϕ' denotes the algebraic term equal to ϕ using the corresponding concrete-multi-player symbols.*

Proof. Both directions follow from an easy induction on ϕ. Let us look at the $\Diamond_i$-case for the direction from left to right. Assume $\mathfrak{F}, V, s \vDash_i \Diamond_i\phi$. Then there exists a successor t of s such that $\mathfrak{F}, V, t \vDash_i \phi$. By induction hypothesis it follows that $\phi'(t)(i) = \mathbf{w}$ for some variable assignment. By definition of the m_i-operator, we obtain that $m_i(\phi)(s)(i) = \mathbf{w}$. □

From this follows that

$$\phi \text{ is } i\text{-valid in } \mathfrak{F} \text{ at } s \text{ iff } \phi'(s)(i) = \mathbf{w} \text{ for all variable assignments,}$$

and,

$$\phi \text{ is } i\text{-valid in } \mathfrak{F} \text{ iff } \mathfrak{F}^+ \vDash \phi' =_i \natural_i.$$

Hence:

$$\phi \text{ is } i\text{-valid in the class of all frames iff } \mathbf{CpMA} \vDash \phi' =_i \natural_i. \tag{4.1}$$

Proposition 4.2.5. *For any Kripke frame $\mathfrak{F}$, the algebra $\mathfrak{F}^+$ is a multi-modal algebra.*

Proof. We need to show that $\mathfrak{F}^+$ validates $(MBO1) - (MBO7)$ Let us have a look at some of the axioms.

(MBO1) In order to show that $m_i(f) \cup_j m_j(g) \leq_i m_i(f \cup_j g)$ holds in $\mathfrak{F}^+$ we need to show two things: (1) $(m_i(f) \cup_j m_j(g))(s)(i) = \mathbf{w}$ implies $m_i(f \cup_j g)(s)(i) = \mathbf{w}$, and (2) $m_i(f \cup_j g)(s)(k) = \mathbf{w}$ implies $(m_i(f) \cup_j m_j(g))(s)(k) = \mathbf{w}$ for all $k \neq i$.

(1) Suppose that $(m_i(f) \cup_j m_j(g))(s)(i) = \mathbf{w}$. Then $m_i(f)(s)(i) = \mathbf{w}$ and $m_j(g)(s)(i) = \mathbf{w}$. By definition of m it follows that there is a $t \in W$ such that Rst and $f(t)(i) = \mathbf{w}$. Moreover, for all $u \in W$ such that Rsu, $g(u)(i) = \mathbf{w}$. From this it follows that there is a $t \in W$ such that Rst, $f(t)(i) = \mathbf{w}$ and $g(t)(i) = \mathbf{w}$. Hence, $m_i(f \cup_j g)(s)(i) = \mathbf{w}$.

(2) Suppose that for some $k \neq i$, $m_i(f \cup_j g)(s)(k) = \mathbf{w}$. This implies that for all $t \in W$ such that Rst, $(f \cup_j g)(t)(k) = \mathbf{w}$. We need to distinguish two cases: $k = j$ and $k \neq j$

- If $k = j$, the fact that $(f \cup_j g)(t)(k) = \mathbf{w}$ for all $t \in W$ such that Rst implies that for all t such that Rst $f(t)(j) = \mathbf{w}$ or $g(t)(j) = \mathbf{w}$. Since $k \neq i$, we can infer from this that either for all t such that Rst $f(t)(j) = \mathbf{w}$, or there is a t such that Rst and $g(t)(j) = \mathbf{w}$. This is equivalent to $m_i(f)(s)(j) = \mathbf{w}$ or $m_j(g)(s)(j) = \mathbf{w}$ and thus $(m_i(f) \cup_j m_j(g))(s)(j) = \mathbf{w}$. Since $j = k$ we have $(m_i(f) \cup_j m_j(g))(s)(k) = \mathbf{w}$.
- If $k \neq j$, the fact that for all $t \in W$ such that Rst, $(f \cup_j g)(t)(k) = \mathbf{w}$ implies that that for all t such that Rst $f(t)(k) = \mathbf{w}$ and $g(t)(k) = \mathbf{w}$. From this we obtain that $m_i(f)(s)(k) = \mathbf{w}$ and $m_j(f)(s)(k) = \mathbf{w}$. Hence, $(m_i(f) \cup_j m_j(g))(s)(k) = \mathbf{w}$.

(MBO3) In order to show the first part, that $\sim_{ij} m_j(f) = m_i(\sim_{ij} f)$ it suffices to show that for all $k \in \mathsf{A}$ and $s \in W$ $\sim_{ij} m_j(f)(s)(k) = \mathbf{w} \Leftrightarrow m_i(\sim_{ij} f)(s)(k) = \mathbf{w}$. We distinguish two cases $k \in \{i, j\}$ and $k \notin \{i, j\}$.

– Case $k \in \{i, j\}$. We will treat only the case $k = i$. The other, $k = j$, is left to the reader.

$$\begin{aligned}
\sim_{ij} m_j(f)(s)(i) = \mathbf{w} \quad &\Leftrightarrow \quad m_j(f)(s)(j) = \mathbf{w} \\
&\Leftrightarrow \quad \exists t \text{ such that } Rst \text{ and } f(t)(j) = \mathbf{w} \\
&\Leftrightarrow \quad \exists t \text{ such that } Rst \text{ and } \sim_{ij} f(t)(i) = \mathbf{w} \\
&\Leftrightarrow \quad m_i(\sim_{ij} f)(s)(i) = \mathbf{w}.
\end{aligned}$$

– Case $k \notin \{i, j\}$.

$$\begin{aligned}
\sim_{ij} m_j(f)(s)(k) = \mathbf{w} \quad &\Leftrightarrow \quad m_j(f)(s)(k) = \mathbf{w} \\
&\Leftrightarrow \quad \forall t \text{ such that } Rst,\ f(t)(k) = \mathbf{w} \\
&\Leftrightarrow \quad \forall t \text{ such that } Rst,\ \sim_{ij} f(t)(k) = \mathbf{w} \\
&\Leftrightarrow \quad m_i(\sim_{ij} f)(s)(k) = \mathbf{w}.
\end{aligned}$$

The second part, that $\sim_{ik} m_j(f) = m_j(\sim_{ik} f)$ in case $j \notin \{i, k\}$ follows by similar reasoning. Details can be found in the appendix.

(MBO5) We need to show that $m_j(\natural_i)(s)(i) = \mathbf{w} \Leftrightarrow \natural_i(s)(i) = \mathbf{w}$ if $j \neq i$. Since for any $s \in W$, $\natural_i(s)(i) = \mathbf{w}$, showing that (MBO5) holds in $\mathfrak{F}^+$ boils down to showing that for any $s \in W$ $m_j(\natural_i)(s)(i) = \mathbf{w}$. This follows easily since for all t such that Rst, $\natural_i(s)(i) = \mathbf{w}$ and thus $m_j(\natural_i)(s)(i) = \mathbf{w}$. The second part of (MBO5) is left to the reader.

(MBO7) Assume that $f \preceq_i g$ for some $f, g \in \mathfrak{C}(W)$. That is, for all $s \in W$ $f(s)(i) = \mathbf{w}$ implies $g(s)(i) = \mathbf{w}$. We need to show that $m_j(f)(s)(i) = \mathbf{w}$ implies $m_j(g)(s)(i) = \mathbf{w}$ for $j \neq i$. Suppose $m_j(f)(s)(i) = \mathbf{w}$. That is, for all $t \in W$ such that Rst, $f(t)(i) = \mathbf{w}$. From this we infer that for all $t \in W$ such that Rst, $g(t)(i) = \mathbf{w}$. In other words $m_j(g)(s)(i) = \mathbf{w}$.

□

4.3 i-Prime Filter Frames

Definition 4.3.1. *Given the collection of i-prime filters of a multi-modal algebra $\mathfrak{A}$, $\mathcal{F}_i\mathfrak{A}$, we define the i-prime filter frame $\mathfrak{A}^i_\bullet$ of $\mathfrak{A}$, as the structure $(\mathcal{F}_i\mathfrak{A}, Q^i)$, where Q^i is a binary relation defined as follows:*

$$\begin{aligned}
Q^i FG \Leftrightarrow \quad & a \in G \textit{ implies } \Diamond_i a \in F, \textit{ and} \\
& \Diamond_j a \in F \textit{ implies } a \in G \textit{ for all } j \neq i.
\end{aligned}$$

Given a multi-modal algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i, \Diamond_i)_{i,j \in \mathsf{A}}$ and its i-prime filter frame $\mathfrak{A}^i_\bullet = (\mathcal{F}_i\mathfrak{A}, Q^i)$ we call the full complex multi-player algebra of $\mathfrak{A}^i_\bullet$, that is $(\mathfrak{A}^i_\bullet)^+$ the *i-embedding algebra* of $\mathfrak{A}$ and denote it $\mathfrak{Em}_i\mathfrak{A}$.

Proposition 4.3.2. *Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i, \Diamond_i)_{i,j \in \mathsf{A}}$ be any multi-modal algebra. Then the two prime-filter frames $\mathfrak{A}^i_\bullet$ and $\mathfrak{A}^j_\bullet$ are isomorphic.*

Proof. Define the function $\gamma : \mathcal{F}_i\mathfrak{A} \to \mathcal{F}_j\mathfrak{A}$:

$$\gamma(F) = \{\neg_{ij} a \mid a \in F\}$$

Proposition 3.4.5 guarantees that γ is indeed a function from $\mathcal{F}_i\mathfrak{A}$ to $\mathcal{F}_j\mathfrak{A}$. We start by showing that γ is a bijection. Let F, G be two i-prime filters such that $F \neq G$. Wlog we assume that there is an $a \in A$ such that $a \in F$ but $a \notin G$. It follows by definition of γ that $\neg_{ij} a \in \gamma(F)$ but $\neg_{ij} a \notin \gamma(G)$ since

$a \notin G$. Hence, γ is injective. Surjectivity is easy as well: let G be an arbitrary j-prime filter. Then by proposition 3.4.5 it follows that the set $F = \{\neg_{ij}a \mid a \in G\}$ is an i-prime filter. By definition of γ; $\gamma(F) = \{\neg_{ij}\neg_{ij}a \mid a \in G\}$. It follows from the (MBn)-axioms that for each agent k and each $a \in G$, $\neg_{ij}\neg_{ij}a =_k a$. Hence by (MB6) it follows that $\neg_{ij}\neg_{ij}a = a$. Thus, $\gamma(F) = G$.

What remains to be shown is that Q^iFG iff $Q^j\gamma(F)\gamma(G)$ in $\mathfrak{A}^j_\bullet$ holds for any two states $F, G \in \mathfrak{A}^i_\bullet$. For the direction from left to right, assume Q^iFG and let $a \in \gamma(G)$. Then $a = \neg_{ij}b$ for some $b \in G$. It follows by assumption that $\Diamond_i b \in F$. Hence, $\neg_{ij}\Diamond_i a \in \gamma(F)$ and thus $\Diamond_j\neg_{ij}a \in \gamma(F)$ by (MBAO3), as required. Secondly, we need to show that for every $k \neq j$ that $\Diamond_k a \in \gamma(F)$ implies $a \in \gamma(G)$. Let $\Diamond_k a \in \gamma(F)$. By definition of γ we know that a is of the form $\neg_{ij}b$. Thus, we need to show that $\neg_{ij}b \in \gamma(G)$. We distinguish cases: $k = i$ and $k \neq i$. If $k = i$, we obtain that $\Diamond_i\neg_{ij}b \in \gamma(F)$. This implies that $\neg_{ij}\Diamond_j b \in \gamma(F)$ and hence $\Diamond_j b \in F$. From this we conclude that $b \in G$ and hence $\neg_{ij}b \in \gamma(G)$. If, on the other hand, $k \neq i$ we get $\Diamond_k\neg_{ij}b \in \gamma(F)$. This implies $\neg_{ij}\Diamond_k b \in \gamma(F)$ and thus $\Diamond_k b \in F$. Hence, $b \in G$ and $\neg_{ij}b \in \gamma(G)$. We conclude that $Q^j\gamma(F)\gamma(G)$. The other direction follows by symmetry, since it can easily be verified that $\gamma(\gamma(F)) = F$. $\square$

We have just shown that for any two player $i, j \in \mathsf{A}$ the frames $\mathfrak{A}^i_\bullet$ and $\mathfrak{A}^j_\bullet$ are isomorphic. Because of this result, we feel it is justified to no longer index i-prime filter frames by the agent i, and will denote it simply $\mathfrak{A}_\bullet = (\mathcal{F}_i\mathfrak{A}, Q)$.

4.4 Representation theorem; the modal case

Proposition 4.4.1. *Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j\in\mathsf{A}}$ be any multi-modal algebra and $a, b \in A$. Then $a \preceq_i b$ implies $\Diamond_i a \preceq_i \Diamond_i b$. Also, $a \leq_i b$ implies $\Diamond_i a \leq_i \Diamond_i b$.*

Proof. In order to show the first part, assume $a \preceq_i b$, that is, $a \vee_i b \vee_i \bot = b \vee_i \bot$. Then

$$\Diamond_i(a \vee_i b \vee_i \bot) = \Diamond_i(b \vee_i \bot)$$

It follows by (MBO2) that

$$\Diamond_i a \vee_i \Diamond_i b \vee_i \Diamond_i\bot = \Diamond_i b \vee_i \Diamond_i\bot$$

and thus

$$(\Diamond_i a \vee_i \Diamond_i b \vee_i \Diamond_i\bot) \vee_i \bot = (\Diamond_i b \vee_i \Diamond_i\bot) \vee_i \bot$$

Since by (MBO5) $\Diamond_i\bot \vee_i \bot = \bot$ we obtain

$$\Diamond_i a \vee_i \Diamond_i b \vee_i \bot = \Diamond_i b \vee_i \bot$$

from which we may conclude that $\Diamond_i a \preceq_i \Diamond_i b$. The second part of the statement follows from axiom (MBO2). To see why, suppose $a \leq_i b$, that is $a \vee_i b = b$. It follows that $\Diamond_i(a \vee_i b) = \Diamond_i b$. By (MBO2) we obtain $\Diamond_i a \vee_i \Diamond_i b = \Diamond_i b$, and thus $\Diamond_i a \leq_i \Diamond_i b$.

□

Proposition 4.4.2. *For any multi-modal algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \perp_i)_{i,j \in \mathsf{A}}$ and any $a, b \in A$, it holds that $\Diamond_i(a \vee_j b) \preceq_i \Diamond_i a \vee_j \Diamond_i b$.*

Proof. If $j = i$ the result is immediately obtained by (MBO2) and the fact that $x \leq_i y$ implies $x \preceq_i y$ (proposition 3.3.3(vii)). Let us assume that $j \neq i$. We know that $a \vee_j b \preceq_i a$ and $a \vee_j b \preceq_i b$. From this it follows by the previous proposition that $\Diamond_i(a \vee_j b) \preceq_i \Diamond_i a$ and similarly for b, $\Diamond_i(a \vee_j b) \preceq_i \Diamond_i b$. It generally holds that if $x \preceq_i y$ and $x \preceq_i z$ then $x \preceq_i y \vee_j z$ for $j \neq i$: suppose that $x \vee_i y \vee_i \perp = y \vee_i \perp$ and $x \vee_i z \vee_i \perp = z \vee_i \perp$. Then

$$\begin{aligned}
x \vee_i (y \vee_j z) \vee_i \perp &= (x \vee_i y \vee_i \perp) \vee_j (x \vee_i z \vee_i \perp) \\
&= (y \vee_i \perp) \vee_j (z \vee_i \perp) \\
&= (y \vee_j z) \vee_i (y \vee_i \perp) \vee_i (z \vee_i \perp) \vee_i \perp \\
&= (y \vee_j z) \vee_i \perp
\end{aligned}$$

Here, the last step follows from (MB5).

Applying this result to $\Diamond_i(a \vee_i b), \Diamond_i a$ and $\Diamond_i b$, we may conclude that $\Diamond_i(a \vee_j b) \preceq_i \Diamond_i a \vee_j \Diamond_i b$ for $j \neq i$. □

Lemma 4.4.3. *Given the i-prime filter frame $\mathfrak{A}_\bullet$ of any multi-modal algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \perp_i, \Diamond_i)_{i,j \in \mathsf{A}}$ and i-prime filter F in $\mathcal{F}_i\mathfrak{A}$,*

$$\Diamond_i a \in F \textit{ implies that there is a } G \in \mathcal{F}_i(\mathfrak{A}) \textit{ such that } QFG \textit{ and } a \in G$$

Proof. Let F be an i-filter of $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \perp_i)_{i,j \in \mathsf{A}}$ such that $\Diamond_i a \in F$. We will construct a filter G such that $a \in G$ and QFG. First we define two sets X and Y as follows

$$X = \{b \mid \Diamond_i b \notin F\},$$

and

$$Y = \{b \mid \Diamond_j b \in F \text{ for some } j \neq i\}.$$

Claim 4.4.4. *(i) X is an i-ideal;*

(ii) Y is an i-filter.

Proof. (i) We need to check (I1)-(I3). (I1) $\perp_i \notin F$, since F is proper. Thus it follows by (MBO4) that $\Diamond_i \perp_i \notin F$ and thus $\perp_i \in X$. (I2) Assume that $b, b' \in X$. We need to show that $b \vee_i b' \in X$. By assumption $\Diamond_i b \notin F$ and $\Diamond_i b' \notin F$ and so $\Diamond_i(b \vee_i b') \notin F$ by (MBO2) and thus $b \vee_i b' \in X$. (I3) Assume $b \in X$ and $b' \preceq_i b$, then by proposition 4.4.1 it follows that

$\Diamond_i b' \preceq_i \Diamond_i b$. Since $\Diamond_i b \notin F$ it follows by (F3) that $\Diamond_i b' \notin F$ and thus $b' \in X$.

(ii) We need to check (F1)-(F3). (F1) We know that $\top_i \in F$, therefore, by (MBO5), $\Diamond_j \top_i \in F$ for all $j \neq i$ and thus $\top_i \in Y$. (F2) Assume that $b, b' \in Y$. We need to show that for any $j \neq i$, $b \vee_j b' \in Y$. By assumption $\Diamond_k b$ and $\Diamond_l b'$ are in F for some l and k different from i. It follows from (MBO6) that both $\Diamond_k b =_i \Diamond_j b$ and $\Diamond_l b' =_i \Diamond_j b'$. From this it follows by (F3) that both $\Diamond_j b$ and $\Diamond_j b'$ are in F, and hence $\Diamond_j(b \vee_j b') \in F$ by (MBO2). We obtain that $b \vee_j b' \in Y$. (F3) assume $b \in Y$ and $b \preceq_i b'$. By definition of Y, $\Diamond_j b \in F$ for some $j \neq i$. By (MBO7) it follows that $\Diamond_j b \preceq_i \Diamond_j b'$. Since $\Diamond_j b \in F$ we immediately get that $\Diamond_j b' \in F$ and thus $b' \in Y$.

□

Since Y is an i-filter it follows by proposition 3.4.7 that the set $H = \uparrow_i \{a \vee_j c | c \in Y\}$ for $j \neq i$ is an i-filter as well. Also, from axiom (MB5) it follows that for all $c \in Y$ $a \vee_j c \preceq_i c$ and also $a \vee_j c \preceq_i a$. Thus $Y \subseteq H$ and $a \in H$. Moreover, we claim that $X \cap H = \emptyset$. Suppose for contradiction that there is a $b \in X$ such that $a \vee_j c \preceq_i b$ for some $c \in Y$. Since X is an i-ideal it follows that $a \vee_j c \in X$ which means that $\Diamond_i(a \vee_j c) \notin F$. However, by definition of Y, $\Diamond_k c \in F$ for some $k \neq i$ and thus $\Diamond_j c \in F$ as well by (MBO6). Also, by assumption $\Diamond_i a \in F$. It follows, since F is an i-filter (by clause (F2)), that $\Diamond_i a \vee_j \Diamond_j c \in F$. From this we infer by (F3) and (MBO1) that $\Diamond_i(a \vee_j c) \in F$ and we arrive at a contradiction. Thus, $X \cap H = \emptyset$. If we apply theorem 3.4.10 to X and H, it follows that there is an i-prime filter G and an i-prime ideal $I = A \setminus G$ such that $X \subseteq I$ and $H \subseteq G$. Since $H \subseteq G$ we know $a \in G$. Also, $b \in G$ for all $\Diamond_j b \in F$. Aditionally, because $I \cap G = \emptyset$ and thus $G \cap X = \emptyset$, it follows by modus tollens that for all $b \in G$, $\Diamond_i a \in F$. Thus we obtain that QFG and $a \in G$. This concludes the proof of the lemma.

□

Lemma 4.4.5. *Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i, \Diamond_i)_{i,j \in \mathsf{A}}$ be an arbitrary multi-modal algebra and let $\mathfrak{A}_\bullet$ be its i-prime filter frame. Let $j \neq i \in \mathsf{A}$ and $F \in \mathcal{F}_i(\mathfrak{A})$. Then the following holds:*

$\Diamond_j a \notin F$ implies that there is a $G \in \mathcal{F}_i(\mathfrak{A})$ such that QFG and $a \notin G$.

Proof. Let F be an i-prime filter in $\mathfrak{A}_\bullet$ such that $\Diamond_j a \notin F$ for some $a \in A$ and agent $j \neq i$. By (MBO6) this implies that there is no $k \in \mathsf{A}$ such that $k \neq i$ and $\Diamond_k a \in F$. We will construct an i-filter G such that $a \notin G$. By the previous proposition it follows that the set $X = \{b \mid \Diamond_i b \notin F\}$ is an i-ideal. Moreover, it follows by proposition 3.4.7(ii) that the set

$$J = \downarrow \{a \vee_i c \mid c \in X\}$$

is an i-ideal as well. Let $Y = \{b \mid \Diamond_j b \in F \text{ for some } j \neq i\}$. In the previous proposition we have shown that Y is an i-filter. Moreover, we can show that

$Y \cap J = \emptyset$: suppose by the way of contradiction that there is $b \in A$ such that $b \in J$ and $b \in Y$. From this it follows that there is a $c \in X$ such that $b \preceq_i a \vee_i c$. Since Y is a filter it follows that $a \vee_i c \in Y$. By definition of Y this means that $\Diamond_j(a \vee_i c) \in F$ for some $j \neq i$. However, by (MBO1) it follows that $\Diamond_j a \vee_i \Diamond_i c \leq_j \Diamond_j(a \vee_i c)$ and hence by proposition 3.3.3 (vii) that $\Diamond_j(a \vee_i c) \preceq_i \Diamond_j a \vee_i \Diamond_i c$. Thus $\Diamond_j a \vee_i \Diamond_i c \in F$. Since F is assumed to be an i-prime filter we must have that either $\Diamond_j a$ or $\Diamond_i c$ is in F. But by assumption $\Diamond_j a \notin F$ and since $c \in X$, it also follows that $\Diamond_i c \notin F$. We arrive at a contradiction and we may conclude that $Y \cap J = \emptyset$. As before, we may apply theorem 3.4.10 to Y and J. It follows that there exists an i-prime filter G and an i-prime ideal $I = A \setminus G$ such that $Y \subseteq G$ and $J \subseteq I$. By construction, for all $\Diamond_j b \in F$ such that $j \neq i$, b is contained in Y and thus also in G. Also, as before, $b \in G$ implies $\Diamond_i b \in F$. Hence, QFG. Also, $a \notin G$ since $a \in J$ and thus in I. This concludes the proof of the lemma. □

Theorem 4.4.6. *For all players $i \in \mathsf{A}$, any multi-modal algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, 0_i, \Diamond_i)_{i,j \in \mathsf{A}}$ can be embedded into $\mathfrak{Em}_0\mathfrak{A}$.*

Proof. Fix a multi-modal algebra $\mathfrak{A} = (A, \vee_i, \neg_{ij}, 0_i, \Diamond_i)_{i,j \in \mathsf{A}}$ and a player $0 \in \mathsf{A}$. We will embed $\mathfrak{A}$ into $\mathfrak{Em}_0\mathfrak{A}$. In order to prove the theorem we will make use of the same representation function as used in the proof of theorem 3.4.14. That is, we define $\rho : \mathfrak{A} \to \mathfrak{Em}_0\mathfrak{A}$ as follows

$$\rho_a(F)(j) = \begin{cases} \mathbf{w} \text{ if } \neg_{0j} a \in F \\ \mathbf{l} \text{ otherwise} \end{cases}$$

By the representation theorem in the previous section 3.4.14, it follows that the function ρ is a multi-Boolean embedding. Thus, in order to prove the theorem it suffices to show that ρ is also a homomorphism with respect to the operators $\Diamond_i$. That is, $\rho(\Diamond_i a) = m_i(\rho_a)$. This boils down to showing that for any $i, j \in \mathsf{A}$

$$\rho_{\Diamond_i a}(F)(j) = \mathbf{w} \Leftrightarrow m_i(\rho_a)(F)(j) = \mathbf{w}$$

We distinguish two cases: $i = 0$ and $i \neq 0$.

Case $i = 0$.

We need to show that for any $j \in \mathsf{A}$ the following holds: $\rho_{\Diamond_0 a}(F)(j) = \mathbf{w} \Leftrightarrow m_0(\rho_a)(F)(j) = \mathbf{w}$. Again, we need to distinguish between $j = 0$ and $j \neq 0$.

For $j = 0$,

$$
\begin{aligned}
\rho_{\Diamond_0 a}(F)(0) = \mathbf{w} &\Leftrightarrow \neg_{00}(\Diamond_0 a) \in F \\
&\Leftrightarrow \Diamond_0 a \in F \\
&\overset{*}{\Leftrightarrow} \text{ there exists a } G \text{ such that } QFG \text{ and } a \in G \\
&\Leftrightarrow \text{ there exists a } G \text{ such that } QFG \text{ and } \neg_{00} a \in G \\
&\Leftrightarrow \text{ there exists a } G \text{ such that } QFG \text{ and } \rho_a(G)(0) = \mathbf{w} \\
&\Leftrightarrow m_0(\rho_a)(F)(0) = \mathbf{w}
\end{aligned}
$$

Here, (*) denotes the crucial equality. The implication from left to right holds by lemma 4.4.3. The other direction follows from the definition of the relation Q. The last equivalence holds by definition of m_i.

For $j \neq 0$,

$$
\begin{aligned}
\rho_{\Diamond_0 a}(F)(j) = \mathbf{w} &\Leftrightarrow \neg_{0j}(\Diamond_0 a) \in F \\
&\Leftrightarrow \Diamond_j \neg_{0j} a \in F \\
&\overset{*}{\Leftrightarrow} \text{ for all } G \text{ such that } QFG \text{ and } \neg_{0j} a \in G \\
&\Leftrightarrow \text{ for all } G \text{ such that } QFG \text{ and } \rho_a(G)(j) = \mathbf{w} \\
&\Leftrightarrow m_0(\rho_a)(F)(j) = \mathbf{w}
\end{aligned}
$$

Again, the crucial step is the equality (*). The implication from left to right follows by definition of Q. The other direction follows by contraposition from lemma 4.4.5.

The other case, $0 \neq i$ is similar to the one for $i = 0$.

Case $i \neq 0$.

For $j = i$,

$$
\begin{aligned}
\rho_{\Diamond_i a}(F)(i) = \mathbf{w} &\Leftrightarrow \neg_{0i}(\Diamond_i a) \in F \\
&\Leftrightarrow \Diamond_0 \neg_{0i} a \in F \\
&\Leftrightarrow \text{ there exists a } G \text{ such that } QFG \text{ and } \neg_{0i} a \in G \\
&\Leftrightarrow \text{ there exists a } G \text{ such that } QFG \text{ and } \rho_a(G)(i) = \mathbf{w} \\
&\Leftrightarrow m_i(\rho_a)(F)(i) = \mathbf{w}
\end{aligned}
$$

For $j \neq i$ (note that j can be equal to 0),

$$
\begin{aligned}
\rho_{\Diamond_i a}(F)(j) = \mathbf{w} &\Leftrightarrow \neg_{0j}(\Diamond_i a) \in F \\
&\Leftrightarrow \Diamond_i \neg_{0j} a \in F \\
&\Leftrightarrow \text{ for all } G \text{ such that } QFG \text{ and } \neg_{0j} a \in G \\
&\Leftrightarrow \text{ for all } G \text{ such that } QFG \text{ and } \rho_a(G)(j) = \mathbf{w} \\
&\Leftrightarrow m_i(\rho_a)(F)(j) = \mathbf{w}
\end{aligned}
$$

It follows that ρ is a modal homomorphism. This concludes the proof of the theorem. □

Now, we are ready to relate this algebraic result to our logic *MML*. The above theorem implies that:

$$\mathbf{CpMA} \vDash f' =_i g' \text{ implies } \mathbf{MMA} \vDash f =_i g,$$

where f' equals the term f using the corresponding concrete multi-player algebraic symbols and similarly g' equals g.

By 4.1 we know that:

$$\phi \text{ is } i\text{-valid iff } \mathbf{CpMA} \vDash \phi' =_i \natural_i$$

and hence we obtain the following relation between the logic *MML* and the algebraic class **MMA**:

$$\phi \text{ is } i\text{-valid (implies } \mathbf{CpMA} \vDash \phi' =_i \natural_i\text{) implies } \mathbf{MMA} \vDash \phi =_i \top_i.$$

Chapter 5

Computational Complexity

5.1 Introduction

In this chapter we will explore some computability and complexity issues concerning the logics *MPL* and *MML*. We will start by proving decidability for both logics and continue with a discussion about the complexity of the i-satisfiability problem. We will show that in the case when the valuations are arbitrary the i-satisfiability problem of *MPL* can be solved in polynomial time. Moreover, we will show that *MML* with restricted valuations lacks the polysize model property. Whether *MML* in case of arbitrary valuations lacks the polysize model property as well, we leave as an open problem. We conclude this last chapter by showing that there is an algorithm that solves the i-satisfiability problem of *MML* using only polynomial space. This algorithm called i-$Witness$ is an analogue of the algorithm $Witness$ used to prove that modal logic is in $PSPACE$.

5.2 Decidability via Filtrations

We say that a multi-player logic is decidable if the i-satisfiability problem of the logic is decidable. The crucial result for the decidability of *MML* is the following theorem.

Theorem 5.2.1. *MML has the finite model property. More precisely, if ϕ is an i-satisfiable formula of MML with the number of subformulas of ϕ equal to n, then ϕ is satisfiable on a finite model of size at most $2^{|\mathsf{A}|^n}$.*

Proof. In order to prove this lemma we need to define filtrations of multi-player models analogous to filtrations for standard (that is; two player-) Kripke models. Also, we need to modify some of the well-known results regarding them.

Let $M = (W, R, V)$ be a multi-player model and Σ be a subformula closed set of formulas. We define the relation $\leftrightsquigarrow_\Sigma$ on the states of M as follows:

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } i \in \mathsf{A} \text{ and for all } \phi \text{ in } \Sigma\text{: } M, w \vDash_i \phi \text{ iff } M, v \vDash_i \phi.$$

It is not difficult to see that the relation $\leftrightsquigarrow_\Sigma$ is an equivalence relation. With $|w|_\Sigma$ we denote the equivalence class of w with respect to Σ. Often, when no confusion arises, we drop the subscript Σ. Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. A model $M^f_\Sigma = (W^f, R^f, V^f)$ is called an *i-filtration of M through* Σ, if the following conditions are satisfied:

(i) $W^f = W_\Sigma$.

(ii) If Rwv then $R^f|w||v|$.

(iii) If $R^f|w||v|$ then for all $\Diamond_i\phi \in \Sigma$, $M, v \vDash_i \phi$ implies $M, w \vDash_i \Diamond_i\phi$.

(iv) If $R^f|w||v|$ then for all $\Diamond_j\phi \in \Sigma$ such that $i \neq j$, $M, w \vDash_i \Diamond_j\phi$ implies $M, v \vDash_i \phi$.

(v) $V^f(|w|)(p)(i) = \mathbf{w}$ iff $M, w \vDash_i p$.

For the proof we will need the following two propositions.

Proposition 5.2.2. *Let Σ be a finite subformula closed set of multi-player modal formulas of size n, M be a multi-player model and M^f any i-filtration through Σ. Then M^f has at most $2^{|\mathsf{A}|^n}$ states.*

Proof. We know that the states in W^f are equivalence classes in W_Σ. Define a function $g : W_\Sigma \to \{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^\Sigma}$ as follows:

$$g(|w|)(\phi)(i) = \begin{cases} \mathbf{w} & \text{if } M, w \vDash_i \phi \\ \mathbf{l} & \text{if } M, w \nvDash_i \phi \end{cases}$$

The function g is injective: suppose $|w| \neq |v|$, then by definition of $\leftrightsquigarrow_\Sigma$ there is a $\phi \in \Sigma$ and a player $i \in \mathsf{A}$ such that $M, w \vDash_i \phi$ and $M, v \nvDash_i \phi$ (or the other way around). It follows from the definition of g that $g(|w|)(\phi)(i) = \mathbf{w}$ but $g(|v|)(\phi)(i) = \mathbf{l}$. Hence, g is injective. Since the size of $\{\mathbf{w},\mathbf{l}\}^{\mathsf{A}^\Sigma}$ is exactly $2^{|\mathsf{A}|^n}$ it follows that the size of W_Σ (and thus the size of W^f) is at most $2^{|\mathsf{A}|^n}$. □

Proposition 5.2.3. *Let $M^f = (W^f, R^f, V^f)$ be an i-filtration of a multi-player model $M = (W, R, V)$ through a subformula closed set Σ. Then for all formulas ϕ, all players $i \in \mathsf{A}$ and all nodes $w \in M$, we have $M, w \vDash_i \phi$ if and only if $M^f, |w| \vDash_i \phi$.*

Proof. By induction on ϕ. The base case is immediate from the definition of V^f. Also, the multi-Boolean cases follow immediately from the induction hypothesis. Let us have a look at the modal case. Suppose $\Diamond_i\phi \in \Sigma$ and $M, w \vDash_i \Diamond_i\phi$. Then there is a $v \in W$ such that Rwv and $M, v \vDash_i \phi$. Since M^f is an i-filtration, $R^f|w||v|$. By induction hypothesis $M^f, |v| \vDash_i \phi$, whence $M^f, |w| \vDash_i \Diamond_i\phi$. For the other direction, assume $M^f, |w| \vDash_i \Diamond_i\phi$. Then there is a state $|v| \in W_f$ such that $R^f|w||v|$ and $M^f, |v| \vDash_i \phi$. By induction hypothesis $M, v \vDash_i \phi$ and by clause (iii) of the definition of i-filtration it follows that $M, w \vDash_i \Diamond_i\phi$.

In order to show the other modal case, $\Diamond_j$ with $j \neq i$, assume for the direction from left to right that $M^f, |w| \nvDash_i \Diamond_j\phi$. It follows that there is a $|v| \in W^f$ such that $R^f|w||v|$, and $M^f, |v| \nvDash_i \phi$. By induction $M, v \nvDash_i \phi$. By clause (iv) of the definition of i-filtration it follows that $M, w \nvDash_i \Diamond_j\phi$. For the other direction suppose that there is a $M, w \nvDash_i \Diamond_j\phi$. This implies that there is a $v \in W$ such that Rwv (and thus $R|w||v|$) and $M, v \nvDash_i \phi$. By induction $M^f, |v| \nvDash_i \phi$, whence $M^f, |w| \nvDash_i \Diamond_j\phi$.

□

Next, we will show that for any given model M and subformula closed set Σ, there actually exist i-filtrations of the model over Σ.

Definition 5.2.4. *Let M be any multi player model, Σ any subformula closed set, and W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$. We define two models (W_Σ, R^l, V^f) and (W_Σ, R^s, V^f) with R^l and R^s defined as follows:*

$R^l|w||v|$ iff for all formulas $\Diamond_i\phi \in \Sigma$:

- *$M, v \vDash_i \phi$ implies $M, w \vDash_i \Diamond_i\phi$,*
- *$M, w \vDash_j \Diamond_i\phi$ with $j \neq i$ implies $M, v \vDash_j \phi$.*

$R^s|w||v|$ iff there exists a $w' \in |w|$ and a $v' \in |v|$ such that $Rw'v'$.

Lemma 5.2.5. *Let M be any multi player model, Σ any subformula closed set, and W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$. Then both (W_Σ, R^l, V^f) and (W_Σ, R^s, V^f) are i-filtrations of M through Σ.*

Proof. We will start by showing that (W_Σ, R^s, V^f) is an i-filtration. It suffices to show that conditions (ii), (iii) and (iv) are satisfied.

(ii) Suppose Rwv. It follows immediate from the definition of R^s that $R^s|w||v|$.

(iii) Assume $R^s|w||v|$. Then there exists a $w' \in |w|$ and a $v' \in |v|$ such that $Rw'v'$. Let $\Diamond_i\phi \in \Sigma$ and suppose that $M, v \vDash_i \phi$, by definition of W_Σ it follows that $M, v' \vDash_i \phi$ and hence $M, w' \vDash_i \Diamond_i\phi$.

(iv) Assume $R^s|w||v|$. Then there exists a $w' \in |w|$ and a $v' \in |v|$ such that $Rw'v'$. Let $\Diamond_j\phi \in \Sigma$ with $j \neq i$ and suppose that $M, w \vDash_i \Diamond_j\phi$. Then by definition of W_Σ we have $M, w' \vDash_i \Diamond_j\phi$. Since $Rw'v'$ it follows that $M, v' \vDash_i \phi$, and hence $M, v \vDash_i \phi$.

In order to show that (W_Σ, R^l, V^f) is an i-filtration, it suffices to show that the condition (ii) is satisfied (since conditions (iii) and (iv) follow immediately from the definition of R^l).

(ii) Suppose Rwv. It follows that for all $\Diamond_i\phi \in \Sigma$: $M, v \vDash_i \phi$ implies $M, w \vDash_i \Diamond_i\phi$. Moreover, $M, v \vDash_j \Diamond_i\phi$ with $j \neq i$ implies $M, v \vDash_i \phi$. Hence, $R^s|w||v|$.

□

Let us now conclude the proof of theorem 5.2.1 - the finite model property of *MML*. Let ϕ be an i-satisfiable *MML*-formula. Assume that ϕ is i-satisfiable on the model M. Now take any i-filtration of M through the set of subformulas of ϕ. The fact that ϕ is i-satisfied by the i-filtration follows immediately from proposition 5.2.3. Moreover, the bound on the size of the i-filtration is $2^{|\mathsf{A}|^n}$ (where n equals the number of subformulas of ϕ) by proposition 5.2.2. We may conclude that *MML* has the finite model property. $\square$

Now we are prepared to prove the following:

Theorem 5.2.6. *MML is decidable*

Proof. From theorem 5.2.1 we know that every i-satisfiable formula is satisfiable on a finite model. We also know the bound on the model: with the number of subformulas of ϕ equal to n this bound equals $2^{|\mathsf{A}|^n}$. Moreover, since the class of all models is clearly a recursive set we can built a (Turing) machine that given a bound n generates all distinct models with at most n states. So, given a formula ϕ with n subformulas we can use this machine to generate all models up to size $2^{|\mathsf{A}|^n}$ and for each model M we can check whether ϕ is i-satisfiable on M. If ϕ is i-satisfiable on at least one of them, it is an i-satisfiable formula. If not, it is not i-satisfiable. We would like to remark that the same reasoning as it is presented here can also be found in a little bit more detail in the proof of theorem 6.7 [3]. $\square$

Unfortunately, the bound on the models that we have found in this section grows exponentially with the size of the formula. In the next two sections we will focus on the degree of computability of our multi-player logics. That is, we will discuss the complexity of the logics *MPL* and *MML*. We will make a distinction between allowing arbitrary or only restricted valuations.

5.3 Complexity: *MPL*

Definition 5.3.1. *Given two propositional valuations $V, V' : \mathsf{P} \to (\mathsf{A} \to \{\boldsymbol{w},\boldsymbol{l}\})$, we say that V'* extends V *(notation $V \subseteq V'$) if for all $p \in \mathsf{P}, i \in \mathsf{A}$*

$$V(p)(i) = \boldsymbol{w} \textit{ implies } V'(p)(i) = \boldsymbol{w}.$$

Similarly, given a frame (W, R) and two valuations over that frame $V, V' : W \to (\mathsf{P} \to (\mathsf{A} \to \{\boldsymbol{w},\boldsymbol{l}\}))$, V' extends V if for all $w \in W$, $p \in \mathsf{P}$ and $i \in \mathsf{A}$

$$V(w)(p)(i) = \boldsymbol{w} \textit{ implies } V'(w)(p)(i) = \boldsymbol{w}.$$

Intuitively, in case of *MPL*, the valuation V' extends V if for every proposition letter p, the valuation V' makes p true for all the agents for which V makes p true and possibly more. Or, in other words, for every p, the set of winners of p given V is a subset of the set of winners for p given V'. In the modal case, V' extends V if for every proposition letter p and every state w, the set of players that win p given V at w is a subset of the set of winners for p given V' at w.

Definition 5.3.2. *Consider the smallest and largest valuations V^s and V^l, respectively, defined as follows. In the propositional case,*

$$V^s(p)(i) = \boldsymbol{l} \text{ for all } p \in \mathsf{P} \text{ and } i \in \mathsf{A},$$

and,

$$V^l(p)(i) = \boldsymbol{w} \text{ for all } p \in \mathsf{P} \text{ and } i \in \mathsf{A}.$$

Given any Kripke frame (W, R) define,

$$V^s(w)(p)(i) = \boldsymbol{l} \text{ for all } w \in W,\ p \in \mathsf{P} \text{ and } i \in \mathsf{A},$$

and

$$V^l(w)(p)(i) = \boldsymbol{w} \text{ for all } w \in W,\ p \in \mathsf{P} \text{ and } i \in \mathsf{A}.$$

Because usually no confusion will arise we use the same notation for both valuations defined over the set of propositional letters P and valuations defined over frames. It follows immediately from the above definition that for any valuation V: $V^s \subseteq V \subseteq V^l$. It will be convenient to extend valuations to functions over arbitrary formulas.

Definition 5.3.3. *For any MPL formula ϕ define:*

$V(\phi)(i) = \boldsymbol{w} \Leftrightarrow$ *player i has a winning strategy for the game $G(\phi, V)@(\phi, Id)$,*

and in the case of a MML formula ϕ, given a multi-player model $M = (W, R, V)$ and $w \in W$,

$V(w)(\phi)(i) = \boldsymbol{w} \Leftrightarrow$ *player i has a winning strategy for the game $G(\phi, M)@(\phi, w, Id)$, that is $M, w \vDash_i \phi$.*

Proposition 5.3.4. *Let ϕ be a MPL formula and V, V' two valuations such that V' extends V. If for some $i \in \mathsf{A}$, $V(\phi)(i) = \boldsymbol{w}$, then $V'(\phi)(i) = \boldsymbol{w}$.*

Proof. Let V and V' be as in the statement. The proposition can be proved by an easy induction on the complexity of ϕ. We will only discuss the case of $\phi = \neg_{jk}\psi$. Assume $V(\neg_{jk}\psi)(i) = \mathbf{w}$. Distinguish cases: $i \in \{j, k\}$ and $i \notin \{j, k\}$. In the first case, assume wlog that $i = j$. By assumption $V(\neg_{jk}\psi)(j) = \mathbf{w}$. That is, player j has a winning strategy for the game $G(\neg_{jk}\psi, V)$. From this we may conlcude that k has a winning strategy for the game $G(\psi, V)$ by mimicking j's strategy during the game. By induction hypothesis $V'(\psi)(k) = \mathbf{w}$ and hence $V'(\neg_{jk}\psi)(j) = \mathbf{w}$. Also, in the second case $i \notin \{j, k\}$, the result is easily obtained: $V(\neg_{jk}\psi)(i) = \mathbf{w}$ implies that $V(\psi)(i) = \mathbf{w}$. By induction hypothesis $V'(\psi)(i) = \mathbf{w}$ and hence $V'(\neg_{jk}\psi)(i) = \mathbf{w}$. □

Remark 5.3.5. From the above proposition and the observation that V^l extends all valuations, it follows that if a *MPL* formula ϕ is i-satisfiable then $V^l(\phi)(i) = \mathbf{w}$. Similarly, if $V^s(\phi)(i) = \mathbf{w}$, it follows that ϕ is i-valid.

Theorem 5.3.6. *i-satisfiability of MPL is in P.*

Proof. By the above remark 5.3.5 it follows that in order to determine whether a given *MPL*-formula ϕ is i-satisfiable it suffices to evaluate $V^l(\phi)(i)$. This can be computed in very little (read: polynomial) time, since it involves only one check per connective. $\square$

Theorem 5.3.7. *i-validity of MPL is in P.*

Proof. We apply similar reasoning as in the above theorem. In order to determine whether a given *MPL*-formula ϕ is i-valid it suffices to evaluate $V^s(\phi)(i)$. Again, this can be done in polynomial time, since it involves only one logical operation per connective. $\square$

At this point we can hear the reader wonder: what about i-satisfiability (or i-validity) of *MML*? Can it be decided in polynomial time as well? Unfortunately, at present we do not have any decicive answer to this question. The reason being that we cannot prove (or disprove) that every i-satisfiable formula ϕ of *MML* with arbitrary valuations is satisfiable on a model that is polynomial in the size of ϕ. We will come back to this so-called *polysize model property* in section 5.4.

Restricted Valuations

We have just discussed the complexity of the i-satisfiability for *MPL* in case of arbitrary valuations. It seems natural however, to consider the case of restricted valuations as well. Remember that restricted valuations were defined as follows: for every proposition letter p there have to be two players i and j such that $V(p)(i) = \mathbf{w}$ and $V(p)(j) = \mathbf{l}$. Similarly, in case of *MML*, for every proposition letter p and every state w there have to be two players i and j such that $V(p)(w)(i) = \mathbf{w}$ and $V(p)(w)(j) = \mathbf{l}$. In chapter 2 we already briefly discussed some of the significant implications of moving from arbitrary to restricted valuations. In this part, we will see that if we confine ourselves to restricted valuations we lose the nice computational properties of *MPL* as discussed in above. That is, we will show that *MPL* with arbitrary valuations is in NP.

Lemma 5.3.8. *The satisfiability problem of PL is polytime reducible to the i-satisfiability problem of MPL with restricted valuations.*

Proof. Let ϕ be a formula of *PL*. Fix a player $0 \in \mathsf{A}$. In order to check whether ϕ is satisfiable we can translate it into a formula ϕ' of *MPL* and check whether ϕ' is 0-satisfiable. The translation will be as follows: Let $1 \neq 0 \in \mathsf{A}$, then in the following order

- distribute $\neg$ over $\vee$ and $\wedge$ and apply double negation elimination whenever possible (that is, rewrite ϕ into negation normal form),
- replace proposition letters p that are not under the scope of $\neg$ with $\bigvee_1\{\neg_{0k}p \mid k \in \mathsf{A} - \{1\}\}$,

- replace $\bot$ with $\bot_0$, $\top$ with $\top_0$,
- replace $\neg\psi$ with $\neg_{01}\psi$ (note that ψ can only be a proposition letter p, $\bot_0$ or $\top_0$),
- replace $\vee$ with $\vee_0$,
- replace $\wedge$ with $\vee_1$.

This algorithm, translating a *PL* formula ϕ into a *MPL* formula ϕ' can be executed in polynomial time. What remains to be shown is that the above translation really reduces satisfiability of *PL* to 0-satisfiabilility of *MPL*.

Claim 5.3.9. *A formula ϕ of PL is satisfiable iff the MPL formula ϕ' is* 0-*satisfiable.*

Proof. Without loss of generality we may assume that ϕ is in negation normal form.

For the direction from left to right, assume ϕ is satisfiable in *PL*. Let V be the valuation evaluating ϕ to $\mathbf{1}$. Define a *MPL* valuation V' as follows:

$V'(p)(k) = \mathbf{w}$ if $V(p) = \mathbf{1}$, for all players $k \neq 1$,
$V'(p)(1) = \mathbf{w}$ if $V(p) = \mathbf{0}$.

First of all, note that since V is a standard propositional valuation, it follows that V' is a valuation in the restricted sense. By induction on ϕ we will show that player 0 has a winning strategy for the game $G(\phi', V')$. The inductive argument can be found in the appendix 7.4.

In order to show the direction from right to left assume ϕ' is 0-satisfiable using restricted valuations. Let V' be the valuation such that player 0 has a winning strategy for the game $G(\phi', V')$. Define a propositional valuation V as follows:

$V(p) = \mathbf{1}$ if there is a $k \neq 1$ such that $V'(p)(k) = \mathbf{w}$,
$V(p) = \mathbf{0}$ if $V'(p)(1) = \mathbf{w}$.

By induction on ϕ we show that $V(\phi) = 1$. The proof can be found in the appendix (7.4). We may conclude that the satisfiability problem of *PL* is polytime reducible to the i-satisfiability problem of *MPL*. □

□

Theorem 5.3.10. *i-Satisfiability of MPL with restricted valuations is NP-complete.*

Proof. By the above lemma it follows that i-satisfiability of *MPL* is NP-hard. What remains to be shown is that it is in NP. This boils down to showing that if we are given a (restricted) valuation by a nondeterministic Turing machine we can deterministically compute in polynomial time whether the given valuation i-satisfies ϕ or not. Just like in case of classical propositional logic this can easily be done since we have to do only one logical operation per connective. □

5.4 Complexity: *MML*

We have just shown that, if we allow for only restricted valuations, we can reduce the satisfiability problem of *PL* to the i-satisfiability problem of *MPL*. In a similar fashion we will show that the satisfiability problem of classical modal logic *ML* is reducible to the i-satisfiability problem of *MML*.

Lemma 5.4.1. *The satisfiability problem of ML is polytime reducible to the i-satisfiability problem of MML with restricted valuations.*

Proof. We adopt the same strategy as before: we will translate a *ML*-formula ϕ into a *MML* formula ϕ' and show that ϕ is satisfiable at a state w in a frame F iff ϕ' is i-satisfiable at w in F. Let ϕ be a formula of *ML*. Fix a player $0 \in \mathsf{A}$. In order to check whether ϕ is satisfiable we translate it into a formula ϕ' of *MML* and check whether ϕ' is 0-satisfiable. The translation will be similar as before: Let $1 \neq 0 \in \mathsf{A}$, then in the following order we

- distribute $\neg$ over $\vee$, $\wedge$, $\Diamond$ and $\Box$, and apply double negation elimination whenever possible,
- replace proposition letters p, that are not under the scope of $\neg$ with $\bigvee_1\{\neg_{0k} p \mid k \in \mathsf{A} - \{1\}\}$,
- replace $\bot$ with $\bot_0$ and $\top$ with $\top_0$,
- replace $\neg\psi$ with $\neg_{01}\psi$ (note that ψ can only be a proposition letter p, $\bot_0$ or $\top_0$),
- replace $\vee$ with $\vee_0$,
- replace $\wedge$ with $\vee_1$,
- replace $\Diamond$ with $\Diamond_0$,
- replace $\Box$ with $\Diamond_1$.

In order to show that this translation really is a reduction from the satisfiability problem of *ML* to the i-satisfiability problem of *MML* it would be sufficient to show that given ϕ, a *ML*-formula, ϕ is satisfiable at some state w in a frame F iff ϕ' is 0-satisfiable at some state w' in some frame F'. We will prove something stronger:

Claim 5.4.2. *Let ϕ be a formula of ML and ϕ' its translation into MML. Then, given a Kripke frame $F = (W, R)$ and w, a state in W, the following holds:*

$$\phi \textit{ is satisfiable at } w \textit{ iff } \phi' \textit{ is } 0\textit{-satisfiable at } w.$$

Proof. As for the direction from left to right: assume ϕ is satisfiable in *ML*. Let V be the valuation such that $(F, V), w \vDash \phi$ Define a *MML* valuation V' as follows:

$V'(p)(w)(k) = \mathbf{w}$ iff $V(w)(p) = \mathbf{1}$ for all players $k \neq 1$,
$V'(p)(w)(1) = \mathbf{w}$ iff $V(w)(p) = \mathbf{0}$.

Denote $M = (F, V)$ and $M' = (F, V')$. By induction on ϕ we will show that $M, w \vDash_0 \phi'$. We only treat the modal cases, since the others follow from the proof of 5.4.1.

- $\phi = \Diamond\psi$. In this case $\phi' = \Diamond_0\psi'$ where ψ' is the translation of ψ. By assumption there is a w' such that Rww' and $M, w' \vDash \psi$ and hence by induction hypothesis $M', w' \vDash_0 \psi$. Since Rww' it follows that player 0 can choose to play w' as a first move of $G(\phi', M, w)$ and continue to play her winning strategy for $G(\psi', M, w')$.

- $\phi = \Box\psi$. In this case $\phi' = \Diamond_1\psi'$ where ψ' is the translation of ψ. By assumption, for all w' such that Rww', $M, w' \vDash \psi$ and hence by induction hypothesis $M', w' \vDash_0 \psi$ for all w' such that Rww' as well. It follows that whichever state w' player 1 chooses to play at the first move of $G(\phi', M, w)$, player 0 may continue to play her winning strategy for $G(\psi', M, w')$.

The other direction follows by similar reasoning and is therefore omitted. $\Box$

The proof of the above claim concludes the proof of the lemma. $\Box$

Definition 5.4.3. *A normal modal logic has the* polysize model property *if there is a polynomial function f such that every satisfiable formula (ϕ) is satisfiable on a model containing at most $f(|\phi|)$ states. Moreover, we say that a multi-modal logic has the* polysize model property *if there is a polynomial function f such that every i-satisfiable formula (ϕ) is i-satisfiable on a model containing at most $f(|\phi|)$ states.*

Proposition 5.4.4. *MML with restricted valuations lacks the polysize model property.*

Proof. We know that there is a formula $\phi(m)$ of *ML* that forces the existence of binary trees of depth m. This formula $\phi(m)$ grows polynomial in the size of m, whereas its models grow exponentially. Hence, *ML* lacks the polysize model property. (For a thorough discussion of the formula and its models we refer to [3] chapter 6 section 7.) Let $\phi'(m)$ be the translation of $\phi(m)$ into *MML*. By 5.4.2 we know that if $\phi'(m)$ is 0-satisfiable in a frame F at state w, then $\phi(m)$ is satisfiable at w in F. From this it follows that $\phi'(m)$ forces the existence of binary trees just like $\phi(m)$. Moreover, from the way the translation procedure is defined it follows that the formula $\phi'(m)$ grows in the same degree as $\phi(m)$ with the size of m: the proposition letters occurring only positively in ϕ are replaced by a string $\bigvee_1\{\neg_{0k}p \mid k \in \mathsf{A} - \{1\}\}$, all the other symbols are replaced by maximally one other symbol. From this it follows that given a *ML*-fomula ϕ its translation ϕ' is linear in the size of ϕ. Hence, $\phi'(m)$ grows polynomial in the size of m. If we combine these two facts together we obtain that $\phi'(m)$ is not satisfiable in a model of polynomial size with respect the size of the formula and thus *MML* lacks the polysize model property. $\Box$

The above proof guarantuees that there is a formula witnessing that *MML* lacks the polysize model property. Because we have a concrete algorithm to translate *ML*-formulas into *MML*-formulas we could also have a look at the actual witness $\phi'(m)$. This $\phi'(m)$ is the translation of the *ML*-formula $\phi(m)$ mentioned in the above proof (i.e. the formula that forces the existence of binary trees of fig. 6.7 in [3]). According to the above lemma, this formula $\phi'(m)$ should force its models to be exponentially large. For those who are interested, the formula $\phi'(m)$ is defined in the appendix.

Space complexity of *MML*

Because *MML* with restricted valuation lacks the polysize model property there is no obvious way of designing an *NP*-algorithm that solves the i-satisfiablility problem of *MML*. If we focus on space-complexity rather then time-complexity some positive results may be obtained. In the last part of this chapter we will define a *PSPACE*-algorithm that we can use to determine whether a *MML*-formula with restricted valuations is i-satisfiable or not. We will do this by modifying the so-called *Witness* algorithm that is used to prove that satisfiability of *MML*. Because the proof will be structurally similar to the one presented in [3] (chapter 6), we will not go into too much detail here. Whenever applicable, we will refer to [3] for more details. Moreover, we also claim that i-satisfiability of *MML* with arbitrary valuations can be solved in polynomial space as well.

Before we start, let us first introduce some new notation. Given a set of *MML*-formulas H, we denote

$$M, w \vDash_i H \text{ iff } M, w \vDash_i \phi \text{ for all } \phi \in H.$$

We start by defining two new concepts: i-closure of a set and i-Hintikka sets. These are (not very surprisingly) multi-player analogues of closure and Hintikka sets respectively.

Definition 5.4.5. *If Γ is a set of MML-formulas, then $CL_i(\Gamma)$, the* i-closure *of Γ, is defined in the following two steps. First, let Γ' be the set of subformulas of formulas of Γ. Given Γ', the set $CL_i(\Gamma)$ is defined as follows:*

$$CL_i(\Gamma) = \Gamma' \cup \{\neg_{ij}\phi \mid \phi \in \Gamma' \textit{ and } j \in \mathsf{A}\}$$

In words, $CL_i(\Gamma)$ is the smallest set of formulas containing Γ that is closed under subformulas and, for the set of subformulas of Γ, closed under single negations of the form $\neg_{ij}$. Note that, if ϕ is in Γ', then $\neg_{ii}\phi$ will be in $CL_i(\Gamma)$ as well. If $\Gamma = \{\phi\}$ then we write $CL_i(\phi)$ instead of $CL_i(\{\phi\})$ and call this set the i-closure of ϕ.

Definition 5.4.6. *Given a set of MML-formulas Σ, we say that Σ is an i-*subformula closed set of formulas *iff there is a set Γ such that $\Sigma = CL_i(\Gamma)$.*

Definition 5.4.7. *Let Σ be an i-subformula closed set of formulas. An i-*Hintikka set *H over Σ is a subset of Σ such that the following conditions are satisfied:*

(i) $\perp_i \notin H$ *and for all* $k \in \mathsf{A}$, $\neg_{ik}\perp_k \notin H$.

(ii)
- *If* $\phi \vee_i \psi \in \Sigma$, *then* $\phi \vee_i \psi \in H$ *implies* $\phi \in H$ *or* $\psi \in H$.
- *If* $\neg_{ij}(\phi \vee_i \psi) \in \Sigma$ *with* $j \neq i$, *then* $\neg_{ij}(\phi \vee_i \psi) \in H$ *implies* $\neg_{ij}\phi \in H$ *and* $\neg_{ij}\psi \in H$.
- *If* $\phi \vee_j \psi \in \Sigma$ *with* $j \neq i$, *then* $\phi \vee_j \psi \in H$ *implies* $\phi \in H$ *and* $\psi \in H$.
- *If* $\neg_{ij}(\phi \vee_j \psi) \in \Sigma$ *with* $j \neq i$, *then* $\neg_{ij}(\phi \vee_j \psi) \in H$ *implies* $\neg_{ij}\phi \in H$ *or* $\neg_{ij}\psi \in H$.
- *If* $\neg_{ik}(\phi \vee_j \psi) \in \Sigma$ *with* $j \notin \{i,k\}$, *then* $\neg_{ik}(\phi \vee_j \psi) \in H$ *implies* $\neg_{ik}\phi \in H$ *and* $\neg_{ik}\psi \in H$.

(iii)
- *If* $\neg_{jj}\phi \in \Sigma$, *then* $\neg_{jj}\phi \in H$ *implies* $\phi \in H$.
- *If* $\neg_{ji}\phi \in \Sigma$, *then* $\neg_{ji}\phi \in H$ *implies* $\neg_{ij}\phi \in H$
- *If* $\neg_{jk}\phi \in \Sigma$ *with* $i \notin \{j,k\}$, *then* $\neg_{jk}\phi \in H$ *implies* $\phi \in H$.
- *If* $\neg_{ij}\neg_{jk}\phi \in \Sigma$, *then* $\neg_{ij}\neg_{jk}\phi \in H$ *implies* $\neg_{ik}\phi \in H$.
- *If* $\neg_{ij}\neg_{kj}\phi \in \Sigma$, *then* $\neg_{ij}\neg_{kj}\phi \in H$ *implies* $\neg_{ik}\phi \in H$.
- *If* $\neg_{ij}\neg_{kl}\phi \in \Sigma$ *with* $j \notin \{k,l\}$, *then* $\neg_{ij}\neg_{kl}\phi \in H$ *implies* $\neg_{ij}\phi \in H$.

(iv) For any $p \in \Sigma$, *we cannot have that* $p \in H$ *and* $\neg_{ij}p \in H$ *for all* $j \neq i \in \mathsf{A}$.

There are a few observations that we would like to make regarding this definition. First of all, note that for all i-subformula closed set of formulas Σ, the set $\emptyset$ is an i-Hintikka set over Σ. Note moreover, that not all i-Hintikka sets are satisfiable. (One could, for example think of an i-Hintikka set containing both the formulas $\Diamond_i\top_i$ and $\Diamond_j\perp_i$ with $j \neq i$.) An i-satisfiable i-Hintikka is called an *i-Atom*. Since Σ is assumed to be an i-closed set of formulas, it is easily verified that there is no clause in the definition of i-Hintikka set that can force a formula $\phi \notin \Sigma$ to be in and i-Hintikka set over Σ.

Remark 5.4.8. We claim that a formula ϕ is i-satisfiable in some multi-player model with restricted valuations iff there is an i-Atom A over $CL_i(\phi)$ containing ϕ. The direction from right to left follows immediately. For the other direction, assume $M, w \vDash_i \phi$ and let $H = \{\psi \mid M, w \vDash_i \psi \text{ with } \psi \in CL_i(\phi)\}$. It is left for the reader to verify that H is indeed an i-Atom over $CL_i(\phi)$. The intuitive reason for this is that all the clauses in the definition of i-Hintikka set reflect i-valid properties of *MML*. Moreover, the same fact holds for sets of formulas Γ. That is, a set of formulas Γ is i-satisfiable in some multi-player model with restricted valuations iff there is an i-Atom A over the $CL_i(\Gamma)$ containing Γ. To prove the direction from left to right assume that $M, w \vDash_i H$, one needs to show that the set $H = \{\psi \mid M, w \vDash_i \psi \text{ with } \psi \in CL_i(\Gamma)\}$ is an i-Atom over Σ.

The analogue of the $Witness$-algorithm, that we will define in this section will (not very originally) be called i-$Witness$. The algorithm i-$Witness$ takes two finite sets H and Σ and checks whether H is an i-Atom over Σ. It does so by checking whether all the demands that H makes can be satisfied. What we mean when using the word demand, is made precise in the following definition.

Definition 5.4.9. *Let H be an i-Hintikka set over Σ and suppose $\phi = \Diamond_i\chi \in H$. Then, the* demand *that ϕ creates in H (notation: $Dem(H, \phi)$) is the following set:*

$$\begin{array}{lll} \{\chi\} & \cup & \{\psi \mid \Diamond_j\psi \in H \text{ for some } j \neq i \in \mathsf{A}\} \\ & \cup & \{\neg_{ij}\psi \mid \neg_{ij}\Diamond_i\psi \in H \text{ for some } j \neq i \in \mathsf{A}\} \\ & \cup & \{\neg_{ik}\psi \mid \neg_{ik}\Diamond_j\psi \in H \text{ for some } j \notin \{i,k\}\} \end{array}$$

Similarly, for a formula $\phi = \neg_{ij}\Diamond_j\chi \in H$ with $j \neq i$, the demand *that ϕ creates in H is the following set:*

$$\begin{array}{lll} \{\neg_{ij}\chi\} & \cup & \{\psi \mid \Diamond_k\psi \in H \text{ for some } k \neq i \in \mathsf{A}\} \\ & \cup & \{\neg_{ik}\psi \mid \neg_{ik}\Diamond_i\psi \in H \text{ for some } k \neq i \in \mathsf{A}\} \\ & \cup & \{\neg_{ik}\psi \mid \neg_{ik}\Diamond_l\psi \in H \text{ for some } l \notin \{i,k\}\} \end{array}$$

In line with the notation used in [3], we use H_ϕ to denote the set of i-Hintikka sets over $CL_i(Dem(H, \phi))$ that contain $Dem(H, \phi)$.

Remark 5.4.10. At this point, we would like to make the following observation. If A is an i-Atom over Σ and $\phi = \Diamond_i\chi \in A$, then there is a multi-player model M with restricted valuations and a state $w \in M$ such that $M, w \vDash_i A$. It follows from this that there is a w' related to w such that $M, w' \vDash_i Dem(A, \Diamond_i\phi)$. Hence, $Dem(A, \phi)$ is i-satisfiable and thus there has to be at least one atom in H_ϕ. The same reasoning applies to formulas of the form $\phi = \neg_{ij}\Diamond_j\chi$ with $j \neq i$.

Next, we will define an important concept that (later) turns out to be a syntactic criterion for i-satisfiability of i-Hintikka sets.

Definition 5.4.11. *Let H and Σ be two finite sets such that H is an i-Hintikka set over Σ. Then $\mathcal{H} \subseteq \mathcal{P}(\Sigma)$ is an* i-witness set generated by H on Σ *if the following conditions are satisfied:*

(i) $H \in \mathcal{H}$.

(ii) If $I \in \mathcal{H}$, then for each $\phi = \Diamond_i\chi \in I$ and for each $\phi = \neg_{ij}\Diamond_j\chi \in I$ with $j \neq i$, there is a $J \in I_\phi$ such that $J \in \mathcal{H}$.

(iii) If $J \in \mathcal{H}$ and $J \neq H$, then for some $n > 0$ there are $I^0, \ldots, I^n \in \mathcal{H}$ such that $H = I^0$, $J = I^n$ and for each $0 \leq l < n$, there is some formula $\phi \in I^l$ such that $\phi = \Diamond_i\chi$ or $\phi = \neg_{ij}\Diamond_j\chi$ with $j \neq i$, and $I^{l+1} \in I^l_\phi$.

Definition 5.4.12. *Given a formula ϕ of MML, we define the* degree *of ϕ (notation: deg(ϕ)) in the standard way:*

$$
\begin{aligned}
deg(p) &= 0,\\
deg(\bot_i) &= 0,\\
deg(\neg_{ij}\phi) &= deg(\phi),\\
deg(\phi \vee_i \psi) &= max\{deg(\phi), deg(\psi)\},\\
deg(\Diamond_i\phi) &= deg(\phi) + 1.
\end{aligned}
$$

Moreover, we define the degree *of a finite set of formulas* Σ *to be the maximum of the degrees of the formulas contained in* Σ.

Given an arbitrary i-Hintikka set H over an arbitrary finite i-subformula closed set Σ, we can make the following observations regarding any i-witness set $\mathcal{H}$ generated by H. First of all, it must be finite since $\mathcal{P}(\Sigma)$ is finite. Secondly, for all $I, J \in \mathcal{H}$ such that $J \in I_\phi$, the degree of J is strictly less than that of I. Thirdly, condition (iii) guarantees that all i-Hintikka sets in $\mathcal{H}$ are there for a reason, that is, each $J \in \mathcal{H}$ different from H, is ultimately generated by H.

We are now ready to prove one of the most crucial lemmas leading to our desired result (that is, the i-satisfiability problem of *MML* with restricted valuations can be solved in *PSPACE*). The following lemma will link the concepts of i-satisfiability of an i-Hintikka set to the existence of an i-witness set generated by the i-Hintikka set.

Lemma 5.4.13. *Suppose that H and Σ are finite sets of formulas such that H is an i-Hintikka set over Σ. Then H is an i-Atom of Σ iff there is an i-witness set generated by H on Σ.*

Proof. We start by proving the easy direction: the direction from left to right. Assume H is an i-Atom of Σ. By induction on the degree of Σ, we will prove the claim. Let $deg(\Sigma) = 0$. Trivially, $\{H\}$ is an i-witness set generated by H. For the induction step, assume the claim holds for all H', Σ' such that H' is an i-Atom of Σ' and $deg(\Sigma') < n$. Assume $deg(\Sigma) = n$. By remark 5.4.10 it follows that for every $\phi = \Diamond_i\psi \in H$, there exists at least one i-Atom I^ψ in $H_{\Diamond_i\psi}$. By induction hypothesis I^ψ generates an i-witness set $\mathcal{I}^\psi$ on $CL_i(Dem(H, \Diamond_i\psi))$. Similarly, for every $\phi = \neg_{ij}\Diamond_j\psi \in H$ with $j \neq i$, there exists at least one i-Atom $I^{\neg_{ij}\psi}$ in $H_{\neg_{ij}\Diamond_j\psi}$. Again, by induction hypothesis, $I^{\neg_{ij}\psi}$ generates an i-witness set $\mathcal{I}^{\neg_{ij}\psi}$ on $CL_i(Dem(H, \neg_{ij}\Diamond_j\psi))$. We define

$$\mathcal{H} = \{H\} \cup \bigcup\{\mathcal{I}^\phi \mid \Diamond_i\phi \in H\} \cup \bigcup\{\mathcal{I}^{\neg_{ij}\phi} \mid \neg_{ij}\Diamond_j\phi \in H \text{ with } j \neq i\}.$$

It follows that $\mathcal{H}$ is an i-witness set generated by H on Σ.

For the other direction, assume that $\mathcal{H}$ is an i-witness set generated by H on Σ. We need to show that H is i-satisfiable. We will prove something stronger, namely, that if $\mathcal{H}$ is an i-witness set generated by Σ, H can be satisfied at the root of a model (F, V) where F is a finite tree with depth at most $deg(\Sigma)$ and V a restricted valuation.

Let $W = \{w_0, w_1, \ldots\}$ be a countably infinite set of new entities. Finitely many of these $w_i's$ will be the states from which we are going to built F. The method that we are going to apply in order to construct the frame F, is called the 'step-by-step' method and explained in section 4.6 of [3]. In a nutshell, we will construct the tree F step by step by going from its root down the tree. At each step i of the construction, the set W_i denotes the states of F up to depth i, R_i denotes the relation of F up to depth i, and f_i is a function that assigns to each state w in W_i the (minimal) set of formulas that w has to satisfy. Define $W_0 = \{w_0\}$, $R_0 = \emptyset$ and $f_0(w_0) = H$. Suppose W_n, R_n and f_n have been defined. We can halt the step by step construction if the following two conditions hold:

(1) For all $w \in W_n$, such that $\Diamond_i\phi \in f_n(w)$, there is a state $w' \in W_n$ such that: (i) $\phi \in f_n(w')$, and (ii)$f_n(w') \in f_n(w)_{\Diamond_i\phi}$ (we refer to 5.4.9 for the definition of this shorthand notation).

(2) For all $w \in W_n$, such that $\neg_{ij}\Diamond_j\phi \in f_n(w)$ with $j \neq i$, there is a state $w' \in W_n$ such that: (iii) $\neg_{ij}\phi \in f_n(w')$, and (iv)$f_n(w') \in f_n(w)_{\neg_{ij}\Diamond_j\phi}$.

If this is not the case, then there is a $w \in W_n$ and a formula $\psi \in f_n(w)$ such that:

- either $\psi = \Diamond_i\phi$ while for no w' the conditions (i) and (ii) are satisfied,

- or $\psi = \neg_{ij}\Diamond_j\phi$ with $j \neq i$, while for no $w' \in W_n$, (iii) and (iv) are satisfied.

In this case, we proceed by going to stage $n+1$ and define:

$$\begin{aligned} W_{n+1} &= W_n \cup \{w_{n+1}\}, \\ R_{n+1} &= R_n \cup \{(w, w_{n+1})\}, \\ f_{n+1} &= f_n \cup \{w_{n+1}, I\}. \end{aligned}$$

where $I \in \mathcal{H}$ such that $I \in f_n(w)_\psi$. Note that since $\mathcal{H}$ is assumed to be an i-witness set, we can always find such an I. Because each $I \in \mathcal{H}$ contains only finitely many (modal) formulas, this construction halts after finitely many steps. Moreover, the depth of F is at most $deg(H)$ since whenever R_nww'

$$deg(f_n(w')) < deg(f_n(w)).$$

Let m be the stage at which the construction halts. We define $F = (W_m, R_m)$. What remains is to find a suitable restricted valuation V such that $(F, V), w_0 \vDash_i H$. We define the valuation V to be *any* restricted valuation satisfying:

(i) $V(p)(w)(i) = \mathbf{w}$ if $p \in f_m(w)$, and

(ii) $V(p)(w)(j) = \mathbf{w}$ if $\neg_{ij}p \in f_m(w)$.

We claim that such a V exists. First of all, there are no $w \in W_m$ and $p \in \mathsf{P}$ such that the conditions on V force $V(p)(w)(j) = \mathbf{w}$ for all players. It is exactly clause (iv) of the definition of an i-Hintikka set that guarantees this fact. Secondly, since V is assumed to be any restricted valuation it cannot be that there exists a state $w \in W_m$ and a $p \in \mathsf{P}$ such that $V(p)(w)(j) = \mathbf{l}$ for all $j \in \mathsf{A}$. Hence, such a V exists. We claim that $(F, V), w_0 \vDash_i H$ and leave for to the reader to verify this fact. □

We are now ready to define the i-$Witness$ algorithm. This algorithm is structurally similar to the one defined in [3]. i-$Witness$ takes two finite sets H and Σ as inputs and returns **true** if and only if there is a i-witness set generated by H on Σ.

i-$Witness(H, \Sigma)$

begin
  if H is an i-Hintikka set over Σ
    and for each formula $\Diamond_i \phi \in H$, there is a set of formulas $I \in H_{\Diamond_i \phi}$ such that i-$Witness(I, Cl_i(Dem(H, \Diamond_i \phi)))$
    and for each formula $\neg_{ij} \Diamond_j \phi \in H$ with $j \neq i$, there is a set of formulas $I \in H_{\neg_{ij} \Diamond_j \phi}$ such that i-$Witness(I, Cl_i(Dem(H, \neg_{ij} \Diamond_j \phi)))$
  then return **true**
  else return **false**
end

The algorithm i-$Witness$ is computable. Computability of the original algorithm $Witness$ is discussed in [3]. Since i-$Witness$ is structurally similar to $Witness$, this reasoning also applies to i-$Witness$. We leave verification of this fact to the reader. Moreover, the algorithm is correct. That is, i-$Witness(H, \Sigma)$ returns **true** iff H is an i-Hintikka set generating an i-Witness set in Σ. This can easily be obtained by induction on the degree of Σ and is left to the reader.

Now, we are all set to prove the desired result.

Theorem 5.4.14. *i-satisfiability of MML with restricted valuations is in PSPACE.*

Proof. From our earlier observation 5.4.8 that ϕ is i-satisfiable iff there is an i-Atom over the i-closure of $CL_i(\phi)$ containing ϕ, combined with lemma 5.4.13 and the correctness of i-$Witness$, we obtain the following result. For any *MML*-formula ϕ: ϕ is i-satisfiable iff there is an $H \subseteq CL_i(\phi)$ such that $\phi \in H$ and i-$Witness(H, CL_i(\phi))$ returns the value **true**.

It follows that if we can show that there is a *PSPACE* implementation of i-$Witness$, we are done. For the original $Witness$ algorithm developed for classical modal logic, this is discussed in [3]. And we are lucky, since the reasoning in this discussion is not restricted to the two-player case but can also be adapted to our multi-player version i-$Witness$. Of crucial importance in the proof is that the model we have constructed in the direction from right to left in proof of lemma 5.4.13 is shallow. That is, it has depth at most $deg(CL_i(\phi))$. Once again, we refer to [3] and leave the details to the reader. □

Corollary 5.4.15. *i-satisfiability of MML with arbitrary valuations in PSPACE.*

Proof. This result is in fact easier to obtain than the case of restricted valuations. The basic idea is that we can drop the condition (iv) of the definition of an i-Hintikka set. The intuitive reason for this is that now that we allow arbitrary valuations, we could have an (i-satisfiable) i-Hintikka set containing p and $\neg_{ij}p$ for all agents $j \neq i$. We only need to modify this definition and the remainder of the discussion turns out to apply to *MML* in case of arbitrary valuations. □

Chapter 6

Conclusion and Further Research

In this thesis we have studied multi-player logics. We developed several logics whose semantics is captured by an n-person game of perfect information. The thesis resolved around two such multi-player logics, multi-player propositional logic (*MPL*) and multi-player modal logic (*MML*). After having introduced and discussed some basic properties of these logics, we studied them algebraically. Our main results are two representation theorems: an analogue of Stone's representation theorem for *MPL* and a multi-player version of the Jónsson-Tarski theorem for *MML*. In the last part of this thesis we have obtained some results regarding the computability and complexity of *MPL* and *MML*. Using filtrations we showed that both logics are decidable. Moreover, we have shown that in case of arbitrary valuations, the i-satisfiability problem of *MPL* can be solved in polynomial time. In case of restricted valuations, however, i-satisfiability is *NP*-complete. Lastly, we showed that *MML* with restricted valuations lacks the polysize model property and that i-satisfiability of *MML* is in *PSPACE*.

Because our terrain had been previously nearly unexplored (with the exceptions of [1] and [14]), many issues presented in this thesis were merely touched upon rather than developed in great detail. It is therefore not surprising that at this point there are more questions arising than being answered. We have organized the discussion of some of these open issues in two parts. We will start by discussing some of the issues that immediately arise from the thesis. Hereafter, we will discuss some broader themes that are of interest in relation to multi-player game semantics.

Let us start by considering some of the open problems that directly suggest themselves from the issues discussed in this thesis. We would like to begin by mentioning multi-player modal μ-caluclus, μMML, for which we only gave some preliminary syntactical and semantical definitions. In the section about μMML we already mentioned some possible ways of improving μMML, such as considering a multi-player version of the modal μ-calculus with fixpoint operators indexed by sets of agents, rather than by single agents. But even μMML in its current form gives rise to many questions. For example: can we find an algebraic semantics that is adequate in describing the semantics of μMML? And, how can μMML be studied using automata?

Secondly, in chapter 2 we briefly mentioned a coalgebraic perspective on multi-player models. Studying multi-player logics more thoroughly in a colagebraic context might provide us with helpful insights regarding the logics.

From the algebraic perspective, a natural question is whether the representation theorems of the chapters 3 and 4 can be extended to some nice full-blown categorical dualities. This would involve the definition of some natural topological structure on the concrete multi-Boolean algebras.

Many other questions that emanate from the thesis are related to computational properties of the logics *MPL*, *MML* and μMML. Firstly, we would like to recall the question that arose in chapter 5: what is the complexity of i-satisfiability of *MML* when we allow for valuations to be arbitrary? Secondly, to which complexity class does i-validity of *MML* belong? Of course, we are also interested in the time and space complexity of μMML.

Solving the problems mentioned above might provide us with illuminating new insights regarding the multiple-player logics as developed in this thesis. However important, we think that at this point it is at least as vital to look at multi-player semantics from a broader perspective.

First of all, it is essential to provide a context for this research by relating it to other, already existing, formalisms. In chapter 2, for example, we briefly compared multiple-player propositional logic with arbitrary valuations in the case of two players with Belnap's four-valued logic. We claim that this, and other links between multi-player logics and many-valued logics, could be established more thoroughly. It would also be interesting to compare our multi-player setting to the multi-player logic as developed by Abramsky in [1]. This immediately brings us to our next point.

Multi-player games of imperfect and/or partial information present another research field that would be worth exploring. In [1], a multi-player logic that can express branching quantifiers and other partial-information constructs compositionally was already developed. It would be interesting to see if we can apply similar methods in our context to account for imperfect information. Next to this, we think it would be worth exploring multi-player logics in setting of (dynamic) epistemic logic.

Thirdly, we think it would be an interesting enterprise to establish an even more intimate connection between logic and games by importing game theoretical concepts into the logic. We will briefly elaborate on this by mentioning three possible leads to doing so: assume common knowledge of rationality, allow cooperation and attribute social (or anti-social) behavior to the agents.

(1) The first example concerns the assumption of *common knowledge of rationality*. Let $0, 1 \in \mathsf{A}$ and consider the *MPL*-formula

$$\phi = \bot_0 \vee_1 \top_0$$

Regardless of the valuation V, the only player with a winning strategy for the game $G(V, \phi)$ is player 1. Clearly, 0 does not have a winning strategy, since 1 might choose to play $\bot_0$. No player k distinct from 0 and 1 has a winning strategy, since 1 might choose $\top_0$. This result fits well with the formal rules of the game and the notion of a winning strategy. However, if we imagine this game to be played in a real world situation, we know that player 1, if he is smart (i.e. rational), will never play $\top_0$ since this guarantees his loss. He would rather choose to play $\bot_0$ since this move leads him to victory. Thus, it will turn out that whenever a match of the game for ϕ is played, all the players except for 0 will win. Moreover, each player expects this to happen knowing that player 1 is rational. Assuming common knowledge of rationality, it seems 'unfair' that ϕ is only true for player 1 and not for all players other then 0. In order to account for common knowledge of rationality we could change the semantics of the multi-player logics by defining truth for an agent relative to his expected payoff. This allows us to study multi-player logics using various game-theoretical solution concepts like for example rationalizability, backward induction or elimination of dominated strategies etc.

(2) Secondly, we would like to mention the possibility of forming *coalitions*. Let $0, 1, 2 \in \mathsf{A}$ and consider the following formula

$$\phi = \top_0 \vee_1 (\top_0 \vee_2 \bot_0)$$

According to our formulation of the game, there is no player with a winning strategy for this formula. However, if player 1 and 2 form a coalition they have the power to force the game to end in the basic position $\bot_0$ and hence they can both win the game. Thus, allowing for cooperation or coalition forming could change the game and hence the semantics of the logic *MPL*. The formation of coalitions could be incorporated into a logic by for example allowing the indexing of operators by groups of agents. Another way would be to represent negations as arbitrary permutation functions that are not necessarily injective. That is, two or more agents could be mapped to the same agent and assume the same role.

(3) The last example we discuss involves the *social behavior* of the players. Consider the formula

$$\top_1 \vee_1 \bot_0$$

Clearly, player 1 will win the play no matter what he chooses. Also, player 0 has no chance of winning the game. All the other players however, are dependent on the generosity of player 1: if he is *selfish* he will be the only winner of the game. If, on the other hand, he is what we call *weakly altruistic*, he will share his happiness with all the other players (except for 0) by choosing $\perp_0$. Continuing along the same line of thought, we could define concepts like *friends* and *enemies*; a friend being somebody whom you would like to win, and an enemy someone whom you would like to lose. Agents that are selfish, weakly altruistic, have many friends or many enemies all have 'second agendas': first and foremost they want themselves to win, but in addition they care about the payoffs of other agents. It would potentially be interesting to integrate some of these social concepts into the logic. A possible way to do this is by assuming that each player has a utility function that partially orders the possible outcomes of the game. If, for example, we assume that all players are weakly altruistic, the outcome of the game, and thus the semantics of the logic will be affected. Note that if we think in terms of social behavior, it follows that according to our definition of the game all the players assume a worst-case scenario: a player has a winning strategy if and only if he can play in such a way that he is guaranteed to win the game, even if all the other players are selfish and his enemies. Thus, a player has a winning strategy only if his guaranteed minimum payoff, or his *security level* is winning. (This was already observed in [14]).

We would like to conclude by stressing that our interpretation of multi-player games is not the only nor the right one. There are many more possible ways to generalize classical, two-player logics to a multi-player setting. In [14], for example, two formalizations of multi-player propositional logic were developed. In this thesis we elaborated on only one of the two. The other, involving valuations describing rankings between players, has been left completely unexplored. Also, as was already mentioned before, we have not established any connection between Abramsky's n-player logic and our own multi-player setting. One could think of establishing a framework, in which various multi-player logics could be compared and evaluated.

Acknowledgements

I would like to thank my supervisor Yde Venema for his devoted guidance and much needed support at all stages of this project. Without him inspiring me with the topic, and his detailed comments later on . . . there would be no thesis. I am especially grateful for his patience with my clumsy mathematical writing. Also, I would like to thank Tomasz Sadzik for his help during the last (most stressful) stage of writing the thesis.

Chapter 7

Appendix

In the following sections (details of) proofs that were skipped over in the chapters 3 4 and 5 can be found.

7.1 Proofs of Chapter 2: Multi-Player Logics

Proof. **of part of 2.3.6**: i Satisfiability of *MML* is bisimulation invariant.

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two multi-player models, and let Z be a bisimulation between M and M'. We need to show that for any $w \in W$ and $w' \in W'$ such that Zww', for any *MML*-formula ϕ and for any player $i \in \mathsf{A}$; $M, w \vDash_i \phi$ iff $M', w' \vDash_i \phi$. We will prove this claim by induction on the formula ϕ.

- Base case.
  - $\phi = p$. In this case, by item (i) of the definition of bisimulation, it immediately follows that $V(p)(w)(i) = \mathbf{w}$ iff $V'(p)(w')(i) = \mathbf{w}$. Hence, $M, w \vDash_i \phi$ iff $M', w' \vDash_i \phi$.
  - $\phi = \bot_j$. By definition of the game G, $M, w \vDash_i \phi$ iff $j \neq i$. The same holds for M', w'. Hence, $M, w \vDash_i \phi$ iff $M', w' \vDash_i \phi$.
- Induction step.
  - $\phi = \psi_1 \vee_j \phi_2$. We treat one case: $j \neq i$. Using the induction hypothesis we get

$$
\begin{array}{lll}
M, w \vDash_i \psi_1 \vee_j \phi_2 & \text{iff} & \text{player } i \text{ has a winning strategy for the} \\
 & & \quad \text{game } G(\phi, M)@(\phi, w, Id), \\
 & \text{iff} & M, w \vDash_i \psi_1 \text{ and } M, w \vDash_i \phi_2, \\
 & \text{iff} & M', w' \vDash_i \psi_1 \text{ and } M', w' \vDash_i \phi_2, \\
 & \text{iff} & \text{player } i \text{ has a winning strategy for the} \\
 & & \quad \text{game } G(\phi, M')@(\phi, w', Id) \\
 & \text{iff} & M', w' \vDash_i \psi_1 \vee_j \phi_2.
\end{array}
$$

- $\phi = \sim_{jk} \psi$. Again we treat only one case: $i \in \{j,k\}$. Without loss of generality we assume that $i = j$. Suppose $M, w \vDash_j \sim_{jk} \psi$. From the rules of the game it follows that player k has a winning strategy for $G(\psi, M)@(\psi, w, Id)$ from which follows by induction hypothesis that $M', w' \vDash_j \psi$ and hence $M, w \vDash_j \sim_{jk} \psi$. In case $i \notin \{j,k\}$, the equality follows immediately from the observation that for any model and any state $M, w \vDash_i \sim_{jk} \psi$ iff $M, w \vDash_i \psi$.
- $\phi = \Diamond_j \phi$. We discuss the case $j = i$. Suppose $M, w \vDash_j \Diamond_j \psi$ then player i has a winning strategy for the game $G(\Diamond_j \psi, M)@(\Diamond_j \psi, w, Id)$. This implies that there is a successor t of w, such that j has a winning strategy for $G(\psi, M)@(\psi, t, Id)$. By the forth condition of the definition of bisimulation it follows that there exists a t' such that $R'w't'$ and Ztt'. By induction hypothesis $M, t' \vDash_j \psi$, whence $M, w' \vDash_j \Diamond_j \psi$. For the other direction, the back condition of the bisimulation definition is used to prove the implication.

$\square$

7.2 Proofs of Chapter 3: Multi-Player Algebras: The Propositional Case

Proof. **of proposition 3.3.3.**
Let $\mathfrak{A} = (A, \vee_i, \neg_{ij}, \bot_i)_{i,j \in \mathsf{A}}$ be a multi-Boolean algebra.

(i) First, we will show that $\leq_i$ is a partial order. Reflexivity follows from (MB0)-(MB4): Let $a \in A$. We need to show that $a \vee_i a = a$.

$$\begin{aligned} a &= a \vee_i \bot_i \\ &= (a \vee_i \bot_i) \vee_i \bot_i \\ &= a \vee_i (\bot_i \vee_i \bot_i) \\ &= (a \vee_i \bot_i) \vee_i (a \vee_i \bot_i) \\ &= a \vee_i a. \end{aligned}$$

In order to prove transitivity, assume $a \leq_i b$ and $b \leq_i c$ for some $a, b, c \in A$, that is $a \vee_i b = b$ and $b \vee_i c = c$. We need to show that $a \vee_i c = c$.

$$\begin{aligned} a \vee_i c &= a \vee_i (b \vee_i c) \\ &= (a \vee_i b) \vee_i c \\ &= b \vee_i c \\ &= c. \end{aligned}$$

Finally, we need to show that $\leq_i$ is antisymmetric, that is $a \leq_i b$ and $b \leq_i a$ implies $a = b$. Assume $a \leq_i b$ and $b \leq_i a$, from this it follows immediately that $a = b$:

$$a = a \vee_i b = b.$$

It follows that $\leq_i$ is a partial order.

(ii) The second relation, $\preceq_i$, satisfies reflexivity and transitivity for every $i \in \mathsf{A}$. Reflexivity follows from refexivity of $\leq_i$:

$$a \vee_i a \vee_i \bot = a \vee_i \bot.$$

In order to show transitivity assume that $a \preceq_i b$ and $b \preceq_i c$, that is, $a \vee_i b \vee_i \bot = b \vee_i \bot$ and $b \vee_i c \vee_i \bot = c \vee_i \bot$. Then

$$\begin{array}{rcl} a \vee_i c \vee_i \bot & = & a \vee_i (c \vee_i \bot) \\ & = & a \vee_i (b \vee_i c \vee_i \bot) \\ & = & (a \vee_i b \vee_i \bot) \vee_i c \\ & = & (b \vee_i \bot) \vee_i c \\ & = & b \vee_i c \vee_i \bot \\ & = & c \vee_i \bot. \end{array}$$

(iii) We need to show that $\neg_{ik}\top_i = \top_k$ and $\neg_{jk}\top_i = \top_i$ when $i \notin \{j,k\}$. In order to show the fist part, remember that $\top_i = \bigvee_i\{\bot_j \mid j \in \mathsf{A}\}$. It follows that $\neg_{ik}\top_i$ equals $\neg_{ik}\bigvee_i\{\bot_j \mid j \in \mathsf{A}\}$, which, by (MBn7) equals $\bigvee_k\{\neg_{ik}\bot_j \mid j \in \mathsf{A}\}$. For all agents j distinct from i and k it follows by (MBn2) that $\neg_{ik}\bot_j = \bot_j$. For player i we have $\neg_{ik}\bot_i = \bot_k$ and for k, $\neg_{ik}\bot_k = \bot_i$. It follows that the set $\{\neg_{ik}\bot_j \mid j \in \mathsf{A}\}$ equals $\{\bot_j \mid j \in \mathsf{A}\}$ and thus $\bigvee_k\{\neg_{ik}\bot_j \mid j \in \mathsf{A}\}$ equals $\bigvee_k\{\bot_j \mid j \in \mathsf{A}\}$ which is the definition of $\top_k$. We obtain that $\neg_{ik}\top_i = \top_k$. Knowing how to prove the first part, we easily obtain the second part of the statement: $\neg_{jk}\bigvee_i\{\bot_l \mid l \in \mathsf{A}\} = \bigvee_i\{\neg_{jk}\bot_l \mid l \in \mathsf{A}\}$ by (MBn8). As before, we obtain that the set $\{\neg_{jk}\bot_l \mid l \in \mathsf{A}\} = \{\bot_l \mid l \in \mathsf{A}\}$ and hence, $\neg_{jk}\top_i = \top_i$ whenever $i \notin \{j,k\}$.

(iv) We need to show that $\top_j \vee_i \bot = \bot$ when $j \neq i$. Let $j \neq i$. By (MBn2) the formula $\top_j \vee_i \bot$ equals $\top_j \vee_i \neg_{0i}\bot$. If we rewrite the definitions we obtain

$$\begin{array}{rcl} \top_j \vee_i \bot & = & \top_j \vee_i \neg_{0i}\bot \\ & = & \top_j \vee_i \neg_{0i} \vee_0 \{\top_k \mid k \neq 0\} \\ & = & \top_j \vee_i \bigvee_i\{\neg_{0i}\top_k \mid k \neq 0\} \\ & \stackrel{1}{=} & \top_j \vee_i \bigvee_i\{\top_k \mid k \neq i\} \\ & = & \bigvee_i\{\top_k \mid k \neq i\} \\ & = & \neg_{0i}\bigvee_0\{\top_k \mid k \neq 0\} \\ & \stackrel{2}{=} & \neg_{0i}\bot \\ & = & \bot \end{array}$$

Here, equality (1) follows from item (iii) of this definition. Equality (2) follows from the axiom (MBn2).

(v) The direction from left to right is immediate. The other direction requires an argument. We know that $\neg_{jk}a = \neg_{jk}b$ implies $\neg_{jk}\neg_{jk}a = \neg_{jk}\neg_{jk}b$. From this it follows by (MBn0) and (MBn3) that $\neg_{jj}a \vee_j \bot = \neg_{jj}b \vee_j \bot$, and thus $a \vee_j \bot = b \vee_j \bot$ by (MBn1). From this we conclude that $a \preceq_j b$ and $b \preceq_j a$. By similar reasoning we obtain the same result for k. For all other $i \in \mathsf{A}$, that is $i \notin \{j, k\}$, it follows by (MBn5) that $a \vee_i \bot = b \vee_i \bot$.We have that for all $l \in \mathsf{A}$, $a \preceq_l b$ and $b \preceq_l a$, from which it follows by (MB6) that $a \leq_l b$ and $b \leq_l a$ for each $l \in \mathsf{A}$. Hence by antisymmetry (proposition 3.3.3 (i)), $a = b$.

(vi) From (MBn7), (iii) and (v) it follows that

$$\begin{array}{rcl} a \preceq_i b & \Leftrightarrow & a \vee_i b \vee_i \bot = b \vee_i \bot \\ & \Leftrightarrow & \neg_{ik}(a \vee_i b \vee_i \bot) = \neg_{ik}(b \vee_i \bot) \\ & \Leftrightarrow & \neg_{ik}a \vee_k \neg_{ik}b \vee_k \neg_{ik}\bot = \neg_{ik}b \vee_k \neg_{ik}\bot \\ & \Leftrightarrow & \neg_{ik}a \vee_k \neg_{ik}b \vee_k \bot = \neg_{ik}b \vee_k \bot \\ & \Leftrightarrow & \neg_{ik}a \preceq_k \neg_{ik}b. \end{array}$$

(vii) We need to show that $a \vee_j b \preceq_i a$. This follows immediately by (MB5):

$$(a \vee_j b) \vee_i a \vee_i \bot = a \vee_i \bot$$

(viii) Assume $a \leq_i b$. That is, $a \vee_i b = b$ for some $a, b \in A$. We need to show that (1) $a \preceq_i b$ and (2) $b \preceq_j a$ for all $j \neq i$. But (1) follows immediately from the assumption that $a \leq_i b$:

$$(a \vee_i b) \vee_i \bot = b \vee_i \bot,$$

while (2) follows from (MB5),

$$\begin{array}{rcl} b \vee_j a \vee_j \bot & = & (b \vee_i a) \vee_j a \vee_j \bot \\ & = & a \vee_j \bot. \end{array}$$

(ix) By symmetry it suffices to show only $a \vee_k b \preceq_i a \vee_j b$. The result can be obtained as follows:

$$\begin{array}{rcl} (a \vee_j b) \vee_i (a \vee_k b) \vee_i \bot & = & (a \vee_i (a \vee_k b) \vee_i \bot) \vee_j \\ & & \qquad (b \vee_i (a \vee_k b) \vee_i \bot) \\ & \overset{*}{=} & (a \vee_i \bot) \vee_j (b \vee_i \bot) \\ & = & (a \vee_j b) \vee_i (a \vee_j \bot) \vee_i (b \vee_j \bot) \vee_i \bot \\ & = & (a \vee_j b) \vee_i \bot \end{array}$$

The first equality holds by (MB1). The equality (*) is obtained by a double application of (MB5). That is, by (MB5) we have that $(a \vee_i (a \vee_k b) \vee_i \bot) = a \vee_i \bot$ and $(b \vee_i (a \vee_k b) \vee_i \bot) = b \vee_i \bot$. Hence, the equality (*) holds.

$\square$

Proof. **of part of 3.4.7**: items (i), (ii) and (iii).

(i) We need to show that $\downarrow_i a = \{b \mid b \preceq_i a\}$ is an i-ideal. This is in fact very easy. (I1) $\perp_i \vee_i a = a$, thus $\perp_i \leq_i a$ and by proposition 3.3.3 (viii) it follows that $\perp_i \preceq_i a$. Hence, $\perp_i \in \downarrow_i a$. (I2) If $b \in \downarrow_i a$ and $c \in \downarrow_i a$, then $b \vee_i a \vee_i \perp = a \vee_i \perp$ and $c \vee_i a \vee_i \perp = a \vee_i \perp$. It follows that

$$\begin{aligned}(b \vee_i c) \vee_i a \vee_i \perp &= b \vee_i (c \vee_i a \vee_i \perp) \\ &= b \vee_i a \vee_i \perp \\ &= a \vee_i \perp.\end{aligned}$$

and thus $b \vee_i c \preceq_i a$ and it follows that $b \vee_i c \in \downarrow_i a$. (I3) This follows by transitivity of $\preceq_i$. Let $b \in \downarrow_i a$, that is $b \preceq_i a$, and $c \preceq_i b$. It follows that $c \preceq_i a$ whence $c \in \downarrow_i a$.

(ii) We will show that $\uparrow_i a = \{b \mid a \preceq_i b\}$ is an i-filter. (F1) follows immediately from (MB4) and (F3) follows from transitivity. Here, we will focus on (F2). Let $b, c \in \uparrow_i a$, then $a \vee_i b \vee_i \perp = b \vee_i \perp$ and $a \vee_i c \vee_i \perp = c \vee_i \perp$. Let $j \neq i$, then

$$\begin{aligned}a \vee_i (b \vee_j c) \vee_i \perp &= (a \vee_i b \vee_i \perp) \vee_j (a \vee_i c \vee_i \perp) \\ &= (b \vee_i \perp) \vee_j (c \vee_i \perp) \\ &= (b \vee_j c) \vee_i (b \vee_j \perp) \vee_i \perp \vee_i (c \vee_j \perp) \\ &= (b \vee_j c) \vee_i ((b \vee_j \perp) \vee_i \perp \vee_i \perp) \vee_i \\ &\qquad ((c \vee_j \perp) \vee_i \perp \vee_i \perp) \\ &\stackrel{*}{=} (b \vee_j c) \vee_i \perp \vee_i \perp \\ &= (b \vee_j c) \vee_i \perp.\end{aligned}$$

and thus it follows that $b \vee_j c \in \uparrow_i a$. Note that the equality (*) between the fourth and the fifth line follows by two applications of (MB5): by this axiom it follows that $((b \vee_j \perp) \vee_i \perp \vee_i \perp) = \perp \vee_i \perp$, which again equals $\perp$. Similar reasoning can be applied to the clause $((c \vee_j \perp) \vee_i \perp \vee_i \perp)$.

(iii) Fix I, an arbitrary i-ideal of $\mathfrak{A}$. (I1) For all $c \in I$ we have $\perp_i \leq_i a \vee_i c$. Therefore $\perp_i \preceq_i a \vee_i c$ and $\perp_i \in \downarrow_i \{a \vee_i c \mid c \in I\}$. (I2) Let $e, f \in \downarrow_i \{a \vee_i c \mid c \in I\}$. This means that $e \vee_i (a \vee_i c) \vee_i \perp = a \vee_i c \vee_i \perp$ and $f \vee_i (a \vee_i c') \vee_i \perp = a \vee_i c' \vee_i \perp$ for some $c, c' \in I$. It follows that

$$\begin{aligned}e \vee_i f \vee_i (a \vee_i (c \vee_i c')) \vee_i \perp &= e \vee_i f \vee_i (a \vee_i c) \vee_i (a \vee_i c') \vee_i \perp \\ &= e \vee_i (a \vee_i c) \vee_i \perp \vee_i f \vee_i (a \vee_i c') \vee_i \perp \\ &= (a \vee_i c) \vee_i \perp \vee_i (a \vee_i c') \vee_i \perp \\ &= (a \vee_i (c \vee_i c')) \vee_i \perp\end{aligned}$$

and since c and c' are both in the ideal I, it follows by (I2) that $c \vee_i c' \in I$. Thus we obtain that $e \vee_i f \in \downarrow_i \{a \vee_i c | c \in I\}$. (I3) follows almost immediately by transitivity of $\preceq_i$.

$\square$

7.3 Proofs of Chapter 4: Multi-Player Algebras: The Modal Case

Proof. **of part of 4.2.5**: the second part of (MBO3).
We need to show that $\sim_{ik} m_j(f) = m_j(\sim_{ik} f)$ in case $j \notin \{i,k\}$ holds in $\mathfrak{F}^+$ for a fixed Kripke frame F. In showing that $\sim_{ik} m_j(f)(s)(l) = \mathbf{w} \Leftrightarrow m_j(\sim_{ik} f)(s)(l) = \mathbf{w}$ we need to distinguish three cases $l \in \{i,k\}, l = j$ and $l \notin \{i,j,k\}$

- In case $l \in \{i,k\}$ we assume without loss of generality that $l = i$. In this case we have

$$\begin{array}{rcl} \sim_{ik} m_j(f)(s)(i) = \mathbf{w} & \Leftrightarrow & m_j(f)(s)(k) = \mathbf{w} \\ & \Leftrightarrow & \forall t \text{ such that } Rst,\ f(t)(k) = \mathbf{w} \\ & \Leftrightarrow & \forall t \text{ such that } Rst,\ \sim_{ik} f(t)(i) = \mathbf{w} \\ & \Leftrightarrow & m_j(\sim_{ik} f)(s)(i) = \mathbf{w}. \end{array}$$

- In case $l = j$,

$$\begin{array}{rcl} \sim_{ik} m_j(f)(s)(j) = \mathbf{w} & \Leftrightarrow & m_j(f)(s)(j) = \mathbf{w} \\ & \Leftrightarrow & \exists t \text{ such that } Rst \text{ and } f(t)(j) = \mathbf{w} \\ & \Leftrightarrow & \exists t \text{ such that } Rst \text{ and } \sim_{ik} f(t)(j) = \mathbf{w} \\ & \Leftrightarrow & m_j(\sim_{ik} f)(s)(j) = \mathbf{w}. \end{array}$$

- In case $l \notin \{i,j,k\}$,

$$\begin{array}{rcl} \sim_{ik} m_j(f)(s)(l) = \mathbf{w} & \Leftrightarrow & m_j(f)(s)(l) = \mathbf{w} \\ & \Leftrightarrow & \forall t \text{ such that } Rst,\ f(t)(l) = \mathbf{w} \\ & \Leftrightarrow & \forall t \text{ such that } Rst,\ \sim_{ik} f(t)(l) = \mathbf{w} \\ & \Leftrightarrow & m_j(\sim_{ik} f)(s)(l) = \mathbf{w}. \end{array}$$

$\square$

7.4 Proofs of Chapter 5: Computational Complexity

Proof. **of part of claim 5.3.9**:

- For the direction from left to right.

  Assume ϕ is satisfiable in PL. Let V be the valuation evaluation ϕ to $\mathbf{1}$. Define a MPl valuation V' as follows:

  $V'(p)(k) = \mathbf{w}$ iff $V(p) = \mathbf{1}$ for all players $k \neq 1$,
  $V'(p)(1) = \mathbf{w}$ iff $V(p) = \mathbf{0}$.

  By induction on ϕ we will show that player 0 has a winning strategy for the game $G(\phi', V')$.

  – Base case.

    - $\phi = p$. In this case $\phi' = \vee_1\{\sim_{0k} p \mid k \in \mathbf{Ag} - \{1\}\}$. Since $V(p)$ must be $\mathbf{1}$ $V'(p)(k) = \mathbf{w}$ for all $k \neq 1$. Thus, whichever conjunct player 1 will choose, 0 will win. Hence, 0 has a winning strategy for the game $G(\phi', V')$.
    - $\phi = \top$, then $\phi' = \top_0$ and 0 has a winning strategy for the game $G(\phi', V')$.
    - $\phi = \neg p$, then $\phi' =\sim_{ij} p$. Since $V(p) = \mathbf{0}$ it follows that $V'(p)(1) = \mathbf{w}$, and 0 has a winning strategy.
    - $\phi = \neg\bot$, then $\phi' =\sim_{01} \bot_0$. It follows immediately that 0 has a winning strategy for the game $G(\phi', V')$.

  – Induction step.

    - $\phi = \psi_1 \vee \psi_2$, then ϕ' is of the form $\psi_1' \vee_i \psi_2'$. Since $V(\phi) = \mathbf{1}$ it follows that there is an $l \in \{1, 2\}$ such that $V(\psi_l) = \mathbf{1}$. By induction hypothesis it follows that 0 has a winning strategy for $G(\psi_l', V')$ and thus she also has a winning strategy for $G(\phi', V')$. (She can simply choose ψ_l' in her first move, then play according to her winning strategy).
    - $\phi = \psi_1 \wedge \psi_2$, then ϕ' is of the form $\psi_1' \vee_1 \psi_2'$.Since $V(\phi) = \mathbf{1}$ it follows that for both $l \in \{1, 2\}$, $V(\psi_l) = \mathbf{1}$. By induction hypothesis it follows that 0 has a winning strategy for $G(\psi_l', V')$ for both $l = 0$ and $l = 1$. Thus she also has a winning strategy for $G(\phi', V')$. (Whichever conjunct 1 chooses she can use her winning strategy for that conjunct).

- For the direction from right to left.

  Assume ϕ' is 0-satisfiable using restricted valuations. Let V' be the valuation such that player 0 has a winning strategy for the game $G(\phi', V')$. Define a *PL* valuation V as follows:

  $V(p) = \mathbf{1}$ iff there is a $k \neq 1$ such that $V'(p)(k) = \mathbf{w}$,
  $V(p) = \mathbf{0}$ iff $V'(p)(1) = \mathbf{w}$.

By induction on ϕ we will show that $V'(\phi')(0) = \mathbf{w}$.

– Base case.

- $\phi = p$. In this case $\phi' = \vee_1\{\sim_{0k} p \mid k \in \mathbf{Ag} - \{1\}\}$. Since 0 has a winning strategy for $G(\phi', V')$ it follows that for all agents $k \neq 0$, $V'(p)(k) = \mathbf{w}$. Hence, $V(p) = \mathbf{1}$.
- $\phi = \top$, then $\phi' = \top_0$ and $V(\top) = \mathbf{1}$.
- $\phi = \bot$, then $\phi' = \bot_0$, which is not 0-satisfiable, hence ϕ cannot equal $\bot$.
- $\phi = \neg p$, then $\phi' =\sim_{ij} p$. Since 0 has a winning strategy it follows that $V'(p)(1) = \mathbf{w}$, hence $V(p) = \mathbf{1}$ and $V(\neg\phi) = \mathbf{1}$.
- $\phi = \neg\top$, then $\phi' =\sim_{01} \top_0$ which equals $\top_1$. Since the formula $\top_1$ is not 0-satisfiable ϕ cannot equal $\neg\top$.
- $\phi = \neg\bot$, then $\phi' =\sim_{01} \bot_0$. It follows immediately that $V(\neg\bot) = \mathbf{1}$.

– Induction step.

- $\phi = \psi_1 \vee \psi_2$, then ϕ' is of the form $\psi'_1 \vee_i \psi'_2$. Since 0 has a winning strategy for $G(\phi', V')$ it follows that there is an $l \in \{1, 2\}$ such that 0 has a winning strategy for $G(\psi'_l, V')$. By induction hypothesis it follows $V(\psi_l) = \mathbf{1}$, whence $V(\phi) = \mathbf{1}$.
- $\phi = \psi_1 \wedge \psi_2$, then ϕ' is of the form $\psi'_1 \vee_1 \psi'_2$.Since 0 has a winning strategy for ϕ' it follows that for both $l \in \{1, 2\}$, 0 has a winning strategy. By induction hypothesis it follows that both $V(psi_1)$ and $V(\psi_2)$ evaluates to $\mathbf{1}$, hence $V(\phi) = \mathbf{1}$.

□

Definition 7.4.1 (Translation of the formula $\phi(m)$ forcing the existence of binary trees)**.** *For any proposition letter p define $\mathbf{p} = \bigvee_1\{\sim_{0k} p \mid k \in \mathsf{A} - \{1\}\}$.*

The formula $\phi'(m)$ will be the $\vee_1$-conjunction of the following formulas:

(i) q_0

(ii) $\Diamond_1^m(\sim_{01} q_i \vee_0 (\bigvee_1\{\sim_{01} q_j \mid \text{ for } j \neq i\}))$ *for* $0 \leq i \leq m$

(iii) $B_0 \vee_1 \Diamond_1 B_1 \vee_1 \Diamond_1^2 B_2 \vee_1 \ldots \vee_1 \Diamond_1^{m-1} B_{m-1}$

(iv)

$$\begin{aligned} \Diamond_1 S(\boldsymbol{p}_1, \sim_{01} p_1) \vee_1 \Diamond_1^2 S(\boldsymbol{p}_1, \sim_{01} p_1) \ldots \vee_1 \Diamond_1^{m-1} S(\boldsymbol{p}_1, \sim_{01} p_1) \\ \vee_1 \Diamond_1^2 S(\boldsymbol{p}_2, \sim_{01} p_2) \ldots \vee_1 \Diamond_1^{m-1} S(\boldsymbol{p}_2, \sim_{01} p_2) \\ \vdots \\ \Diamond_1^{m-1} S(\boldsymbol{p}_{m-1}, \sim_{01} p_{m-1}), \end{aligned}$$

where the two auxilary predicates B_i and $S(\boldsymbol{p}_i, \sim_{01} p_i)$ are defined as follows:

$$B_i = \sim_{01} q_i \vee_0 (\Diamond_0(\boldsymbol{q}_{i+1} \vee_1 \boldsymbol{p}_{i+1}) \vee_1 \Diamond_i(\boldsymbol{q}_{i+1} \vee_1 \sim_{01} p_{i+1})),$$

and,

$$S(\boldsymbol{p}_i, \sim_{01} p_i) = (\sim_{01} p_i \vee_0 \Diamond_1 \boldsymbol{p}_i) \vee_1 (\boldsymbol{p}_i \vee_0 \Diamond_1 \sim_{01} p_i).$$

This formula $\phi'(m)$ is 0-satisfiable. However, it forces its models to be exponentially large.

Bibliography

[1] S. Ambramsky: *A Compositional Game Semantics for Multi-Agent Logics of Partial Information*, pp 11-47 in Interactive Logic, edited by J. van Benthem, D. Gabbay, B. Löwe. Amsterdam University Press, (2007).

[2] N.D. Belnap: *A useful four-valued logic*, pp. 8-37 in Modern Uses of Multiple-Valued Logics, edited by J. M. Dunn and G. Epstein. Reidel, Dordrecht, (1977).

[3] P. Blackburn, M. de Rijke and Y. Venema: *Modal Logic*, Cambridge University Press, (2001)

[4] B.A. Davey and H.A. Priestley: *Introduction to Lattices and Order*, Cambridge University Press, (1990)

[5] A. Ehrenfeucht: *An application of games to the completeness problem for formalized theories*, pp. 129141, Fundamenta Mathematicae 49, (1961).

[6] E.A. Emerson and C.S. Jutla: *Tree-automata, mu-calculus and determinacy* pp. 368-496 in Proceedings 32th Anual IEEE Symp. on Foundations of Computer Science. IEEE Computer Society Press, (1999).

[7] R. Fraïssé: *Sur une nouvelle classification des systmes de relations* pp. 1022-1024, Comptes Rendus 230, (1950).

[8] R. Fraïssé: *Sur quelques classifications des systmes de relations* thesis, Paris, 1953; published in Publications Scientifiques de l'Universit d'Alger, series pp. 35182 A 1, (1954).

[9] J. Hintikka: *Language-games for quantifiers* pp. 46-72 in Americal Philosophical Quarterly Monograph Series 2: Studies in Logical Theory. Blackwell, (1968).

[10] J. Hintikka: *Logic, Language Games and Information* Clarendon, Oxford, (1973).

[11] J. Hintikka and G. Sandu: *Game-Theoretical Semantics* pp. 361-410 in J. van Benthem and A. ter Meulen (eds.), Handbook of Logic and Language. Elsevier, Amsterdam, (1997).

[12] W. Hodges: *A shorter model theory*, Cambridge University Press, (1997).

[13] P. Lorenzen: *Logik und Agon*, pp.187-194 in Atti del XII Congresso Internazionale di Filosofia. Sansoni, (1977).

[14] T. Tulenheimo and Y. Venema: *Propositional Logics for Three*, manuscript, ILLC, (2007)

[15] Y. Venema: *Lecture notes on modal μ-calculus*, manuscript, ILLC, (2007)