Independence Weakening in Judgment Aggregation

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

The fields of preference aggregation and judgment aggregation have strong parallels. In both fields, given certain plausible conditions, there are no aggregators that can universally output consistent and complete results. Several methods of avoiding these impossibility results have been proffered. In preference aggregation, weakening the condition known as Independence of Irrelevant Alternatives has been suggested by Campbell & Kelly (2007), among others. In judgment aggregation, the analogue to Independence of Irrelevant Alternatives is called Independence. Despite the fact that judgment aggregation has this analogue to Independence of Irrelevant Alternatives, weakenings of Campbell & Kelly's type have not been explored.

We argue that weak independence conditions are more defensible in judgment aggregation contexts than in preference aggregation contexts and show that the results of weakening independence allow for further possibilities. In line with Campbell & Kelly's approach, we propose several weakenings of Independence, and characterize these weakenings in terms of dependent sets. We show that, under a weak version of Neutrality, the implications which hold between Campbell & Kelly's (2007) conditions fail to transfer to the judgment aggregation framework. We demonstrate possibility with several of these weakenings and show that the aggregators are applicable. However, a stronger version of Neutrality leads to an impossibility theorem and recreates one of Campbell & Kelly's implications. We also determine that Self-Dependence leads to impossibility over non-simple agendas.

Contents

1 Introduction **1**
1.1 Preference and Judgment Aggregation 1
1.2 Motivations for this Thesis 3
1.3 Structure of the Thesis 4

2 Preference Aggregation **7**
2.1 Condorcet's Paradox 7
2.2 Preference Aggregation Framework 8
2.3 Arrow's Conditions 10
2.3.1 PA Unanimity 10
2.3.2 Independence of Irrelevant Alternatives 11
2.3.3 Non-Dictatorship 11
2.4 Arrow's Theorem 12
2.5 Weakenings of Arrow's Conditions 13

3 Judgment Aggregation **17**
3.1 Discursive Dilemma 18
3.2 Judgment Aggregation Framework 19
3.3 Introduction of the Conditions 21
3.3.1 Anonymity 21
3.3.2 Neutrality 21
3.3.3 Independence 22
3.4 List & Pettit's Impossibility Theorem 22
3.5 Weakening the Conditions 24
3.5.1 Surveying Recent Work 24
3.5.2 Defending Weakenings 25

4 Independence Weakenings **31**
4.1 The Campbell & Kelly Independence Conditions 32
4.1.1 PA Independence of Some Alternatives 32
4.1.2 PA Weakest Independence 33

4.2 Campbell & Kelly Implications . . . 33
4.2.1 Weak Unanimity . . . 33
4.2.2 PA Neutrality . . . 34
4.2.3 Sufficiency . . . 34
4.2.4 PA Independence Implications . . . 35
4.2.5 Discussion . . . 36
4.3 Dependence Characterizations of JA Conditions . . . 37
4.3.1 Independence . . . 39
4.3.2 Some Independence . . . 39
4.3.3 Independence of Some Alternatives . . . 40
4.3.4 Strengthened Independence of Some Alternatives . . . 41
4.3.5 Weakest Independence . . . 42
4.3.6 Self Dependence . . . 43
4.3.7 Trivial Implications . . . 44

5 Independence Results **47**
5.1 Independence Implication Counterexamples . . . 48
5.1.1 Counterexample to (SI) and (SD) implying (I) . . . 48
5.1.2 Counterexample to (WI) and (SD) implying (ISA) . . . 50
5.1.3 Counterexample to (ISA+) and (SD) implying (I) . . . 52
5.2 Neutrality . . . 54
5.3 Possibility Theorems . . . 56
5.3.1 An Example with Weak Independences and Neutrality . . . 57
5.3.2 Possibility with Weak Independences . . . 59
5.4 Impossibility Theorems . . . 60
5.4.1 The Strength of (SD) . . . 60
5.4.2 Impossibility with (N-PERM) . . . 61
5.5 Summary of Independence Implications . . . 63

6 Discussion and Conclusion **65**
6.1 Difference between PA and JA . . . 65
6.1.1 Neutrality . . . 66
6.1.2 Independence and Applications . . . 67
6.2 Possibility Results . . . 70
6.3 Impossibility Results . . . 71
6.4 Conclusion . . . 72

List of Tables

2.1 Condorcet's Paradox 8

3.1 A Discursive Dilemma 19
3.2 List & Pettit's Impossibility 23

5.1 An (SI) and (SD) aggregator which fails (I) 49
5.2 A (WI) and (SD) aggregator which fails (ISA) 51
5.3 An (ISA+) and (SD) aggregator which fails (I) 53
5.4 An (N) aggregator which fails (N-PERM) 55
5.5 Possibility over (WI), (SI) and (ISA) aggregators 58
5.6 Summary of Independence Implications Under (N) and (A) 63
5.7 Summary of Independence Implications Under (N-PERM) 64

Chapter 1

Introduction

1.1 Preference and Judgment Aggregation

In an increasingly interconnected global society, there is greater and greater value in synthesizing and combining ideas from different groups. These ideas can relate to the structure of society, choice of leaders, political ideology, or normative behaviour. With multicultural issues and viewpoints, it is important to know how to come to consensus in a procedural way. It may turn out that different methods of aggregation may be appropriate for different types of societies, and this is the reason for examining the formal properties of these different methods.

However, there happen to be mathematical limits on the types of aggregators that are possible. In particular, in both the fields of *preference aggregation* (PA) and *judgment aggregation* (JA), given certain constraints, there are no aggregators that can universally output consistent and complete results. These constraints include, for example, treating all alternatives equally and not having a dictator whose preferences are unilaterally reflected by the social output. Ideal aggregation may be mathematically implausible, so other options must be explored.

PA and JA model different types of social aggregation. PA considers ordered preferences of alternatives, which can reflect any items which individuals may have to vote upon.[1] Ordered preferences are used to show which alternatives are preferred to others, although orderings need not exclude ties. This makes PA a tool which can reflect the process of voting for mutually exclusive alternatives. The voting method, represented by a function called a *social welfare function*, maps a society composed of individuals with preference orders to a social ordering

[1]PA was born from Arrow's seminal work in (Arrow 1950, Arrow 1951/1963). These are generated by binary ordering relations. Some subsequent key results in PA can be found in (Black 1948), (Gibbard 1973), (Inada 1964), (May 1952), and (Sen 1966, Sen 1970). An overview of the field is contained in the handbook by Arrow, Sen & Suzumura (2002). For a recent introductory text, see (Gaertner 2006).

relation.

JA is concerned with aggregating propositions.[2] Individuals hold different positions (such as agreement or disagreement) on these statements. Since propositions can have logical interconnections, logical consistency constrains the positions that society can hold. This field of aggregation is sometimes reminiscent of decision-making in terms of social argumentation: Certain propositions can act as premises and others as conclusions in the aggregation process. For this reason, JA can be representative of consensus on complex interrelated issues. On a macro level, this is of use for political and normative theory, although it can model corporate agreements or jury decision-making. These examples require aggregation over logically dependent issues. The key is that JA traditionally models the aggregation of reasoning or beliefs. The aggregation method, represented by a function called a *judgment aggregator*, maps a profile of positions to social positions.

The relationship between social welfare functions and judgment aggregators has both historical and mathematical roots. It is natural that there is a family resemblance; both are concerned with aggregating individual opinions into social judgments. In a mathematical context, both began with impossibility results—PA from Arrow (1950) and JA from List & Pettit (2002).[3] Arrow showed that, under certain plausible conditions, voting rules which always generate consistent and complete outputs are not possible and List and Pettit showed a similar result for judgment aggregators. However, the impossibility results from both fields are not corollaries of each other; even mapping the frameworks onto each other is not a trivial matter (List & Pettit 2004).[4]

The historical connection is rooted in the expansion of social choice theory. This marriage between economics, mathematics and logic followed from Arrow's seminal work (List & Pettit 2004). List & Pettit (2002) concern themselves with social judgments which have logical interconnections. In this thesis, we intend to continue to explore the connections between PA and JA. We argue, firstly, that the differences between the two fields lead to different conditions being plausible and, secondly, that weakening independence in JA allows for interesting possibilities.

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[2] A selection of the JA literature is (Dietrich & List 2007*a*, Dietrich & List 2007*b*), (Dokow & Holzman 2009), (Gärdenfors 2006), (List & Pettit 2002), (Pauly & van Hees 2006), and (van Hees 2007). List & Polak (2010) have produced an overview of JA and the surrounding literature.

[3] Although Arrow's Theorem is now considered an "impossibility result" (since it shows the impossibility of certain type of aggregation functions), he labelled it the general *possibility* theorem, due to the fact that imposed or dictatorial functions satisfy all of his conditions, except for non-dictatorship (Arrow 1950, p. 342). We refer to it as an impossibility result; it bears a family resemblance to other modern social choice impossibility results.

[4] Work by Dietrich & List (2007*a*) suggests that the PA framework and Arrow's Theorem is easier to embed into a JA framework than the converse.

1.2 Motivations for this Thesis

This leads to the primary motivations for the work in this thesis. By examining modern conditions proffered in the PA literature, we intend to contribute to understanding the distinctions between the JA and PA frameworks. In particular, we wish to demonstrate that certain weakenings of independence are more defensible and allow for more interesting possibilities in JA than in PA. In order to do so, we consider Campbell & Kelly's (2007) work in weakening a PA condition which is called Independence of Irrelevant Alternatives and adapt these weakenings to the JA framework.

Independence of Irrelevant Alternatives is meant to constrain the social welfare function by considering only the individual opinions on a pair of alternatives when aggregating that pair. In other words, when determining the social position on a pair of alternatives, no voting on any other pair influences in the aggregator.

One of the primary reasons that Independence of Irrelevant Alternatives is stipulated is in order to prevent strategic voting. Consider a situation where a given alternative x is dependent on the positions of other alternatives. If so, then one may misrepresent one's own preferences by changing positions on non-x alternatives in order to change the social position taken over x. This condition prevents such maneuvering since, if this independence condition is assumed, changing unrelated alternatives cannot affect the given alternative.

Our intention is to determine whether results in the PA framework are replicable in the JA framework. The impossibility results lend themselves towards both questioning the conditions for impossibility and weakening them in order to generate possibility. Thus, the first motivation is to examine different weakenings of the JA condition of Independence in response to Campbell & Kelly's (2007) similar weakenings for PA. As a strictly formal issue, it is of value to transfer the conditions of independence in order to test impossibilities from PA into JA and to determine if JA impossibility results can be strengthened, as well as to introduce the associated concept of dependence in the JA framework. Although full Independence of Irrelevant Alternatives can be derived from the weakest of Campbell & Kelly's (2007) conditions with unanimity, JA does not support this implication. This is somewhat surprising, due to the similarity in structure of JA and PA.

The other motivation is to examine the dichotomy between PA and JA. In particular, we examine how the differences between JA and PA have implications for Neutrality and Independence. We argue that the weak independence conditions in JA are more conceptually defensible and that Independence for JA is less warranted than Independence of Irrelevant Alternatives for PA. For Neutrality, we argue that profiles are neutral with respect to alternatives in PA whereas they are not so in JA. For Independence, we argue that the logical implications that hold between propositions should (or at least plausibly could) be reflected by an aggregator's social position on a particular proposition φ taking the individual positions on the propositions which φ entails.

Our findings include possibility theorems under the weaker Neutrality condition. Furthermore, we prove that only one of the Campbell & Kelly implications is replicable in JA, even under a strong Neutrality condition, which is quite a surprising result. We also show two impossibility theorems which are of interest for different reasons. The first requires only a weak dependence condition, which is obtainable via Self-Dependence. The proof takes advantage of the fact that JA profiles cannot freely exchange dependent elements; in particular, conjunctions cannot be separated from the truth-values of their conjuncts. The second shows that the stronger Neutrality condition can spread independence in a symmetric fashion, forcing full Independence from a single independent element.

However, we argue that the stronger Neutrality condition does not accurately reflect the structure of propositions. In other words, while we take some versions of neutrality to be plausible within the PA framework, we argue that the subject matter of JA, i.e. propositions, indicates that neutrality should be jettisoned. In this way, we intend to show that the differences between PA and JA are material for the aptness of both aggregation conditions: Independence and Neutrality. This allows us to claim that the impossibility results we find should be resisted.

1.3 Structure of the Thesis

In Chapter 2, we begin by discussing Condorcet's Paradox. Then, we map out Kenneth Arrow's Impossibility Theorem, focussing on describing the conditions of social welfare functions which are needed to generate the impossibility.

In Chapter 3, we consider the framework of JA. We motivate this framework by examining the Doctrinal Paradox and then move on to explain the role of the conditions necessary to generate impossibility. Finally, we present Pettit & List's Impossibility Theorem.

We argue that, in JA contexts, Neutrality and Independence are less justifiable than in PA contexts and that the disadvantages of weakening independence have less impact than could be assumed.

Drawing on the shortcomings in the independence conditions, in Chapter 4, we explore weakenings in PA and JA. We explain Campbell & Kelly's (2007) conditions, discussing the implications that follow in the PA framework. Then we present our own JA versions, transferring these conditions from PA. We characterize our independence conditions, along with the traditional JA Independence condition, in terms of our explanation of dependent sets.

In Chapter 5, we show that the PA relationships between the Campbell & Kelly conditions fail in JA. We present counterexamples to the type of relationships that the Campbell & Kelly conditions exhibit and summarize the implications which hold and which fail. Several failures follow partially from the weakness of the Neutrality condition in JA. We demonstrate this weakness by showing that the Neutrality condition in JA is strictly weaker than the JA analogue of Campbell & Kelly's Neutrality.

Then we turn to how List & Pettit's (2002) Impossibility Theorem fares when Independence is replaced with the weakened independence conditions. This allows for certain premise-based types of aggregation. However, the stronger dependence condition—which we call Self-Dependence—does regenerate impossibility. Also, one of the weaker independence conditions, together with a stronger Neutrality condition, implies Independence, strengthening List & Pettit's (2002) Impossibility Theorem.

Chapter 6 is concerned with diagnosing the difference between the JA and PA systems. This leads to a discussion of the distinction between these systems, including a discussion on the role of Neutrality and attendant weakenings of Independence. We suggest epistemic applications to illustrate the possibility theorem obtained with weak independence conditions and then argue that the difference between the JA and PA systems manifests itself through weakenings of Neutrality and Independence. Finally, we conclude with suggestions for future avenues of research.

Chapter 2

Preference Aggregation

The study of judgment aggregation grew out of questions originally posed by Arrow (1950). The body of research that has grown out of this seminal work shows that under some plausible conditions on social welfare functions, it is impossible to universally generate socially consistent orderings. These conditions include the absence of a dictator and conditions on the responsiveness of the aggregator. In other words, given any social welfare function which fulfils these conditions, it is possible to find a profile of voters which generates an intransitive or incomplete social ordering. This is so despite the fact that profiles are composed of individuals whose orderings are themselves transitive and complete.

In this chapter, we begin in Section 2.1 with an explanation of Condorcet's Paradox. Then, in Section 2.2, we describe the PA framework. In Section 2.3, we move on to enumerate the revised preference aggregation conditions. We pay attention to rationality assumptions. These assumptions have been questioned as unduly restrictive, and this is one way of addressing Arrow's impossibility result. We prove Arrow's Theorem from these conditions in Section 2.4, using a streamlined proof by Geanakoplos (2005).

After doing so, we turn in Section 2.5 to discuss other possible ways of addressing Arrow's result. One way we focus on foreshadows the weakenings of independence in Chapter 4. Of especial interest is Campbell & Kelly's (2000) suggestion to generate weaker independence conditions to find the space between dictatorship and independent preference aggregators.

2.1 Condorcet's Paradox

Arrow's work was stimulated by Condorcet's Paradox, developed by the Marquis de Condorcet. When three alternatives are to be decided among between three voters (*i.e.* $\{1, 2, 3\}$) and three alternatives (*i.e.* $\{a, b, c\}$), each individual can have a transitive (consistent) order of preferences while majoritarian voting generates intransitivity.[1] Transitivity means that preferring a to b and b to c implies preferring a to c. We label a preference of a over b as aPb.

Such a situation is illustrated by Table 2.1:

Individual	Preference order
1	$aPbPc$
2	$bPcPa$
3	$cPaPb$

Table 2.1: Condorcet's Paradox

In Condorcet's Paradox, each individual has transitive complete preferences. However, when we apply a pairwise majoritarian approach, we see a cyclic result. When comparing a to b, a majority (Individuals 1 and 3) prefer a to b while only 2 disagrees. So the social ordering should have aPb. Similarly, it should have bPc and cPa. But given transitivity, this social ordering is inconsistent: cPa and aPb yield cPb but we have that bPc. Also, we can get that aPa, when we should not have a preference of an alternative over itself.

This example shows that majoritarian approaches to aggregating social welfare can generate intransitivity.[2] Arrow's theorem generalizes this observation to show that there is *no* non-dictatorial function which always generates transitivity and completeness under certain plausible conditions.

2.2 Preference Aggregation Framework

The framework requires a population and individual opinions on alternative options. Begin with a population N, a set of individuals $\{1, 2, \ldots, n\}$ such that $|N| \geq 2$. There are a finite set of alternatives or issues X. These can interpreted as, for instance, candidates or alternative states of society.

Then, we introduce the idea of a weak ordering relation R, a binary relation.[3] For any individual $i \in N$, we have an ordering relation R_i. A similar relation is used to describe the output of a social welfare function (*e.g.* R). This is interpreted as "weak preference," or liking at least as much.

[1]This is a minimal version: The paradox cannot be recreated with less than three alternatives or less than three voters. The former follows from May's theorem (May 1952).

[2]This issue can be reformulated in very different fields. For instance, Schoenmakers (1986) applies this to database knowledge acquisition.

[3]I sometimes refer to R as an "ordering" or an "order" although, strictly speaking, the order is what is *generated* by R.

The individuals are assumed to have rational preferences; the social relations R and the individual relations R_i satisfy transitivity and completeness.

Definition 2.2.1 (Transitivity)**.** An ordering relation R is *transitive* if, for all x, y, z, if xRy and yRz, then xRz.

As above, transitive orderings do not contain cycles. If x is at least as good as y and y is at least as good as z, transitivity is maintained by holding x at least as good as z. Furthermore, whenever x is indifferent to y and y is indifferent to z, there must be indifference between x and z.[4]

Definition 2.2.2 (PA Completeness)**.** An ordering relation R is *complete* if for all x, y, at least one of xRy or yRx.

A complete ordering relation means that any pair of alternatives are comparable: For all x, y in the set of alternatives, either x is preferred to y, y to x, or both are indifferent. These constraints on the ordering relations of both individuals and societies are consequences of traditional readings of economic rationality.

There are also strict equivalents of the ordering, which indicate strict preference. Strict equivalents for individual and social orderings are given by the binary relation P (for strict "preference"). So xPy means x is strictly preferred to y. The indifference relation is I, where xIy indicates no preference between x and y:

Definition 2.2.3 (Ordering Relations)**.**

- An ordering relation is a binary relation R which is transitive and complete.
- xPy is defined as xRy and not yRx.
- xIy is defined as xRy and yRx.

The preference orderings satisfy certain properties, which follow from Transitivity and PA Completeness (Arrow 1950, p. 332):

Fact 2.2.4.

- For all x, xRx.

[4] Tversky (1969) demonstrates that in some scenarios, consistently *intransitive* preferences can be generated in humans. In one experiment, he showed that for a preset selection of lotteries, higher payoffs were preferred even when the probability decreased slightly. However, at the extreme, when the highest payoff lottery had significantly less probability than the lowest payoff lottery (and less expected value), the lowest payoff lottery would be preferred.

If we infer an ordinal utility function ranking each of these lottery conditions, the ordinal function would be intransitive. Thus, there are some situations in which an individual can be induced to demonstrate intransitive preference orderings. These types of results put pressure on the assumption of transitive orderings for individuals.

- If xPy, then xRy.
- If xPy and yPz, then xPz.
- If xIy and yIz, then xIz.
- For all x and y, either xRy or yPx.
- If xPy and yRz, then xPz.

Proof. These follow directly from the preceding definitions. □

A social profile, i.e. a set of ordering relations for every individual $i \in N$ is denoted $\{R_i\}$. If considering the profile preference relations only among a restricted set of alternatives, such as the pair x, y, we denote this $\{R_i\}|\{x, y\}$. In the preference literature, the individuals in the society are assumed to have rational preferences and the social ordering relation is intended to be rational.

A *social welfare function* is denoted F. It maps profiles to a social ordering relation. This intuitively represents that, for a given set of individual ordering relations, the aggregator produces a *social ordering relation*.

The intention is to generate functions which order the alternatives being selected for in a manner which is responsive to individual opinions. Originally, Arrow was concerned with the theoretical underpinnings of capitalist democracy. However, the applications generalize: Determining social orderings from individual preferences is a process that is explicitly or implicitly required from the boardroom and the voting booth to the cinematheque and the cafeteria. Whether choosing movies or governors, individuals may have different preferences, but these have to be aggregated to find social preference orders. Choosing from a given set of individual preferences is an important social activity.

2.3 Arrow's Conditions

Arrow's Impossibility Theorem states that the certain conditions are inconsistent. In this section, we briefly introduce the conditions before turning to the theorem itself. This version of the proof is adapted from (Geanakoplos 2005), which updates and simplifies the proof of the theorem from Arrow's (1950) original.

2.3.1 PA Unanimity

Definition 2.3.1 (PA Unanimity (P-U))**.** A social welfare function F satisfies *PA Unanimity* if, for every pair of alternatives $x, y \in X$, that xP_iy holds for every individual $i \in N$ implies that the social ordering relation satisfies xPy.

This principle states that, for any pair of alternatives x and y, if all members of the society strictly prefer x to y, the social ranking generated by F should also strictly prefer x to y. The plausibility of this principle arises from its minimal responsiveness; a social welfare function need not have many requirements on how it aggregates, but if all members of society prefer one alternative to another, any reasonable social welfare function should reflect this preference.

The negation of (P-U) is quite unintuitive. This is because the antecedent is satisfied when there is no individual in the society who (even weakly) prefers y to x; thus, the society is better off with x over y from everyone's perspective. Having a social ordering where it is not true that xPy is satisfied would fail to reflect any of the individual preferences.

2.3.2 Independence of Irrelevant Alternatives

Definition 2.3.2 (Independence of Irrelevant Alternatives (IIA))**.** A social welfare function F satisfies *Independence of Irrelevant Alternatives* if, for any two alternatives $x, y \in X$, the social position on x and y depends solely on $\{R_i\}|\{x, y\}$.

In other words, (IIA) means that, for any two $x, y \in X$, restricted the profile to only the binary relations between x and y is sufficient to determine the social relation on x, y, whether this takes the form of weak preference, strict preference or indifference. In other words, which non-x, y preferences individuals have do not affect the choice between x and y.

To illustrate (IIA), it is useful to consider a method that violates it. The Borda Count is one such method. This method assigns points to the alternatives reflecting the strength of a voter's preference for that candidate. The typical Borda Count method for m alternatives ranked in order of preference gives $m - 1$ points to a given candidate for each first-place ranking that alternative receives, $m - 2$ for each second-place, and so forth, with 0 points being given for each last place ranking.

Since Borda Count violates (IIA), it is susceptible to strategic voting. Roughly, one such strategy is to insincerely lower one's ranking of strong candidates which are in the middle of preference rankings and raise higher candidates which are close to winning. In response to this possibility, M. de Borda stated: "My scheme is only for honest men" (Black 1958, p. 182).

2.3.3 Non-Dictatorship

Definition 2.3.3 (Non-Dictatorship (ND))**.** A social welfare function satisfies *Non-Dictatorship* if there is no $i \in N$ such that, in every profile $\{R_i\}$, for all $x, y \in X$, xRy just in case xR_iy.

This is the key to social welfare results. Having one individual whose preferences are identical to the social preference order in every profile seems to run heavily counter to notions of fairness and democracy. Also, if we allow functions which copy the dictator's preferences, then preference aggregation is no longer aggregation, it is simply copying an individual ordering.

2.4 Arrow's Theorem

This proof of Arrow's Theorem is adapted from (Geanakoplos 2005). Although it is expressed in terms of R, it holds for preferences in terms of P or I since, after all, both are definable in terms of R.

It is important to keep in mind the scope of profiles throughout the proof. Initially, it is shown that there *exist* two profiles where some individual d can change the position of b, but it is by repeated application of (IIA) and the *arbitrary* arrangements of other pairs that dictatorship is generated.

Theorem 2.4.1 (Arrow's Theorem)**.** *No social welfare function which generates complete transitive orderings satisfies (P-U), (IIA), and (ND).*

Proof. Assume, towards a contradiction, that there is such a function. We show that it is a dictatorship, contradicting (ND).

This proof requires showing that such a social welfare function satisfies extremity.

Note that this proof is stated in terms of weak ordering relations R, but it holds similarly for strict preference.

Lemma 2.4.2 (Extremal Lemma)**.** *Consider arbitrary $b \in X$. In any profile $\{R_i\}$ where for all voters, b is either at the very top or the very bottom of their orderings, either b is at the very bottom of R or at the very top of R. (This holds even if the same number of individuals put b at the top as the bottom of their individual orderings.)*

Proof. Suppose that this is false. So for distinct $a, b, c \in X$ we have that aRb and bRc for a profile $\{R_i\}$. By (IIA), this holds even with a new profile $\{R'_i\}$ which is formed by each $i \in N$ moving c strictly above a. This can be done since this does not disturb any ab or bc preferences (recall that b is extremal).

By (P-U), $\{R'_i\}$ yields $cP'a$. But we have $aR'b$ and $bR'c$ which yields $aR'c$, since we assumed that the social welfare function generates complete transitive preferences. This contradiction proves the lemma. □

Next, we show that there is a voter $d \in N$ who is extremely pivotal, in that his changing b from the very bottom to the very top changes the social ranking of

b from the bottom to the top. After showing this, we prove d is dictatorial over combinations not involving b and also all combinations that involve b.

To find d, consider a sequence of profiles where all the voters have b at the very bottom of their rankings and sequentially (*i.e.* voter by voter) move b from the very bottom to the very top. By (P-U), the initial profile must have b ranked lowest. Similarly, by (P-U) the final profile must have b ranked highest. Between two profiles, let them be $\{R_i^1\}$ and $\{R_i^2\}$, a particular voter changes to social ranking of b from the least to the most liked preference. In other words, in $\{R_i^1\}$, b was socially ranked lowest and $\{R_i^2\}$, b was socially ranked highest, and this corresponds directly to the change in some particular voter. We let d be that voter.

Now, we show that d is dictatorial over any pair ac not involving b. Generate a new profile $\{R_i^3\}$ from $\{R_i^2\}$ where aP_d^3b indicating that d strictly prefers a to b. Also, bP_d^3c, since b was the topmost element for d in profile $\{R_i^2\}$. Let other voters arbitrarily change preferences between a and c, although b stays at the very top or the very bottom of each individual ranking as in $\{R_i^2\}$. In $\{R_i^3\}$, by (IIA), aP^3b since we have all the same voters who had b ranked lowest in $\{R_i^1\}$ now have aP_i^3b (as well as d) and we also have, by (IIA), bP^3c because the same voters who had b ranked highest in $\{R_i^2\}$ now have bP_i^3c (as well as d). By transitivity of R and P, aP^3c, and by (IIA) this occurs whenever aP_dc. Note that this holds no matter what the preferences over a and c are issued by $i \in N$, $i \neq d$ since these were arbitrarily assigned while b was held fixed, and they are the only determinants of aPc. Thus, d is dictatorial over the ac pair inter-profile.

It remains to argue that d also dictates every pair ab, *i.e.* pairs involving b. By generating profiles parallel to $\{R_i^1\}$ and $\{R_i^2\}$ where c is at the bottom of every ranking, one can argue that there is some $i \in N$ who is dictatorial over any pair not involving c (by the same reasoning as the preceding paragraph). But we know that, at profiles $\{R_i^1\}$, $\{R_i^2\}$, d from the previous paragraph controls social preferences between ab. So $i = d$, meaning that d is also dictatorial over ab pairs. □

2.5 Weakenings of Arrow's Conditions

In order to avoid the impossibility of Arrow's Theorem, we briefly discuss challenges to Arrow's conditions. This foreshadows similar weakenings for the conditions of List & Pettit's conditions in Section 3.5. I begin with several less interesting weakenings, before turning to (IIA).

The condition Unrestricted Domain can be weakened by defining the social welfare function for a smaller domain. Not surprisingly, if we do restrict our domain to social profiles $\{R_i\}$ with certain more predictable characteristics, then possibility results do obtain.

Black (1948) shows that there exists a social welfare function which satisfies Arrow's conditions when the preference curves form single-peaked curves. Such patterns may occur in several plausible real-world scenarios. For instance, this would occur when the alternatives are ideal heights or weights or when the alternatives are ranked single-dimensionally along a political spectrum. In such cases, preference would presumably peak with one's ideal, and fall on either side. This social welfare function is simple majoritarian voting. In other words, when we relax the universality of an aggregator F, it is possible to generate possibility results. However, it is preferable to not limit the domain of the social welfare function: Many applications, such as elections for leaders or ballots on issues, need not reflect any single-peaked curve.

Another way to generate possibility is to weaken the "rationality" conditions on either the input profiles or the output orderings. For instance, weakening transitivity to transitivity of strict preferences only allows for oligarchic social welfare functions (Sen 1993).[5] However, as Sen notes, impossibility results reoccur when removing further undesirable social welfare functions (i.e. removing more than just dictatorship, and excluding social welfare functions such as oligarchies).

Campbell & Kelly (2000) suggest trading off some independence in favour of staying far from oligarchies and dictators. In other words, by weakening independence, they determined whether there there are more social welfare functions which are moderately independent. They view it as information trade-offs. When (IIA) is satisfied, only the individual orderings on a pair are necessary to aggregate the pair. This is (nearly) minimal information; however, aggregators which require somewhat more information may be useful.

If (IIA) is dropped from Arrow's Theorem, for instance, possibility is obtained for rules such as the Borda Count (cf. 2.3.2 for a description). However, the Borda Count is *maximally* dependent—determination of social preferences on any pair requires the complete profile (to calculate the Borda score of each alternative)—so this is not valuable when trying to minimize the information necessary to aggregate.

As Blau (1971) notes, after Arrow's theorem, (IIA) was considered by some to be too strong a condition.[6] Blau considers a different method from Campbell & Kelly. Campbell & Kelly consider conditions whereby subsets of the agenda are available dependent on each individual proposition.

Blau's conditions, in contrast, are formed in terms of making every n-ary subset of alternatives independent. In other words, (IIA) makes each set of alternatives independent. Weakenings include making every doubleton independent, or every

[5] In other words, weakening Transitivity (cf. Section 2.2) such that aPb and bPc implies aPc but aIb and bIc do not imply aIc.

[6] For instance, Baumol (1952), in the very first review of (Arrow 1950), suggests targeting both (IIA) and transitivity as possible candidates for weakening.

triple independent. Blau finds considerably weaker independence claims lead to (IIA). In particular, $|X| > m > 1$, then m-ary independence implies (IIA). In other words, some weakenings of (IIA) lead to impossibility and some weak independences have iterative implications which, taken together, can resuscitate Arrow's Impossibility Theorem.

These issues in JA similarly lead us to the weakenings in Chapter 4. Our intention is to explore what types of weaker independence conditions are conducive to possibility and impossibility results. However, before introducing such weakenings, it is necessary to develop a JA framework.

Chapter 3

Judgment Aggregation

As we explored in the previous section, Arrow generalized Condorcet's Paradox to show that all social welfare functions generate inconsistent preferences under plausible conditions. The field of JA also stems from a paradoxical case, and List & Pettit (2002) generalize this into an impossibility theorem. In Section 3.1, we begin by presenting this case. Then, in Section 3.2, we explain the JA framework. Before proving List & Pettit's Impossibility Theorem, we detail List & Pettit's conditions in Section 3.3. We give particular attention to the condition Independence, since this is the condition which we will focus on weakening.

Once we have enumerated the conditions, in Section 3.4, we move on to prove List & Pettit's Impossibility Theorem. In Section 3.5, we consider ways of avoiding this result. These methods include weakening the conditions on the aggregator, as well as relaxing the input and output conditions on the aggregator.

It is useful to begin with an overview of the intention of JA. Traditional PA assumes that the alternatives or candidates have no logical connections. Judgment aggregation changes the framework of social choice by making the alternatives (logical) propositions.

JA is widely applicable. It can be employed in any situation where opinions, beliefs, desires or positions are aggregated and there are logical connections between these disparate positions.[1] This contrasts with a PA framework, since adopting one alternative in PA precludes choosing any other. When judging propositions or other claims, however, taking certain positions imply others. A simple example is that if a society accepts a and $a \rightarrow b$, then the society should accept b on pain of

[1]It is usually taken that the position is either for or against (true or false) and that degrees of belief, for instance, are inadmissible. Relaxing this condition is possible: Dokow & Holzman (2009) have investigated allowing abstentions and Pauly & van Hees (2006) have examined different degrees of belief. In the former case, oligarchic aggregators satisfy the List & Pettit's conditions. In the latter case, Pauly & van Hees showed, e.g. that with multiple truth-values, a independent aggregator which satisfies minimal responsiveness must be a veto-dictatorship.

inconsistency. This inconsistency is distinct from PA inconsistency. In the latter, inconsistency occurs at the level of ordering preferences, by holding intransitive preferences. In JA, inconsistency is holding positions on propositions which are *logically* inconsistent, i.e. unsatisfiable.

This is highly applicable; there are many propositions which societies need to aggregate. For example, if a group adopts the proposition that suspicion of nuclear recalcitrance is sufficient for preemptive war and that a given situation suggests nuclear armament, then the society, on pain of inconsistency, should accept preemptive war. Of course, if either premise turns out, in fact, to be false, then the truth of the conclusion is not required by logical consistency.

It may be thought that a distinction between facts (atomic propositions) and procedural or recommended rules (molecular propositions) can be invoked to solve this issue. For instance, then aggregation can occur on the atomic propositions, and then, once the truth-values are established on said atomic sentences (as premises), this would force truth-values on molecules.[2] However, such a strategy will not always work. Facts need not correspond to atomic propositions; for instance, conditional/dependent claims can express factual truths (e.g. "If I press this lever, the ball will be released").

Such reasoning occurs not only at the national level, but by judges, business-people, parents and anyone who has to make group decisions with some complexity. The results from the JA literature, beginning with (List & Pettit 2002), suggest that—like Arrow's Impossibility—under plausible conditions aggregation is impossible.

Before turning to Pettit & List's Impossibility, it is important to illustrate how aggregation of propositions suffers in a similar manner to aggregating preferences. JA was inspired by the judicial theories of Kornhauser, who discussed the issues involved in courtroom decisions with multiple judges (Kornhauser & Sager 1986). He distinguished JA from preference aggregation in that judges need to decide on the truth of certain claims. This can lead to a situation in which inconsistent majority opinions are rendered. Since it motivates the JA framework, this class of cases invites consideration.

3.1 Discursive Dilemma

The paradoxical case in JA is called the "discursive dilemma" (or the "doctrinal paradox") (Kornhauser & Sager 1993). This occurs when aggregating a group of individuals, each of whom holds complete and consistent positions, generates inconsistent social positions. The motivating case comes from theory of multiple

[2] For simplicity, I assume here that the agenda is atomically closed.

judges. In a case, they hear several claims and have to opine on the truth of each. Kornhauser views this as a dilemma because a stalemate sometimes occurs, in such cases, unless the judges decide to aggregate over the rationale (i.e. premises) or the verdict (i.e. the conclusion).

We present a simple example. In this case, the society includes three individuals $\{1,2,3\}$ and the agenda being decided upon includes three propositions $\{a, b, a \wedge b\}$ as in Table 3.1.

Individual	a	b	$a \wedge b$
1	0	1	0
2	1	0	1
3	1	1	1
Group	1	1	0

Table 3.1: A Discursive Dilemma

Each individual evaluation of these three propositions is complete and consistent. However, as with Condorcet's Paradox, if we take a simple issue-by-issue majoritarian approach, we find inconsistency. On a, both 2 and 3 agree. On b, both 1 and 3 agree. Finally, a majority disagrees on $a \wedge b$. So we have that the majority assigns $a = 1$, $b = 1$, and $a \wedge b = 0$. This is a logically inconsistent evaluation.

Kornhauser's solution is, in the context of judging, to aggregate either on the rationale for the verdict or on the verdict itself. Here, the decision of the individual's guilt (*i.e.* b) is determined by the premises (*i.e.* $\{a, a \to b\}$). In this manner, certain elements of the agenda are designated the premises and certain one(s) conclusion(s). This is sensical for some situations, but does require prejudicing certain elements of the agenda. In other words, it prevents speaking neutrally about the elements of the agenda. This point will turn out to be important when speaking of different independence conditions.

As we will show, this dilemma generalizes given an agenda of sufficient complexity.

3.2 Judgment Aggregation Framework

Let the society $N = \{1, 2, \ldots, n\}$ be a set of individuals such that $n \geq 2$. Consider a set of propositions under evaluation, which is called the "agenda." The agenda is denoted X.

Let L, a standard propositional logic, be a pair $(L, \vDash)$, where L contains all atomic propositions $a, b, c, \ldots$, and recursively defined molecular propositions in the usual manner: If L contains φ and ψ, then L contains $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$. We take a set of propositions S to satisfy consistency if and only if there is a truth-value assignment such that all the propositions $\varphi \in S$ are true.

The agenda X consists of positive propositions from L, with elements labelled $a, b, c, \ldots$ as needed and metavariables $\varphi, \psi, \chi \ldots$ as needed. An agenda $X \subset L$, where X contains only non-negated propositions (meaning that they can have negation, but not over the scope of the rest of the proposition). Call an agenda X *non-simple* if it contains at least two propositions a, b and their conjunction $a \wedge b$.

Let a partial function $A : X \rightarrow \{0, 1\}$ be a *judgment function*:

Definition 3.2.1. A *judgment function* A partially or completely maps elements of the agenda X to $\{0, 1\}$.

The partiality allows for incomplete evaluations of the agenda, making completeness nontrivial. For each individual $i \in N$, we have an *individual judgement function* A_i. This function indicates which formulas of the agenda the individual accepts. Similarly, a *social judgment function* A indicates the social evaluation of the agenda.

Assent to a positive proposition is given by assigning 1 and dissent by 0. More specifically, for an individual $i \in N$, we write $A_i(\varphi) = 1$ to indicate that i assents to φ and dissents to $\neg\varphi$ and $A_i(\varphi) = 0$. Since X contains only nonnegated propositions, for a positive proposition φ, $A(\varphi) = 0$ on the positive proposition can be treated analogously to $A(\neg\varphi) = 1$. We keep the agendas non-wide-scope-negated to simplify the combinatorial presentation of the counterexamples in Section 5; it does not affect the generality of our claims.[3]

A *profile* is an n-tuple of individual judgment functions, *i.e.* $(A_1, \ldots, A_n)$, also denoted $\{A_i\}$. Primes are used to distinguish profiles when multiple profiles are referred to. In other words, $\{A_i\}, \{A_i'\}, \{A_i''\}, \ldots$ are different profiles with associated social judgment functions A, A', A'',..., respectively.

Definition 3.2.2 (Consistency)**.** A judgment function A is *consistent* if the set which is constructed of all φ such that $A(\varphi) = 1$ and all $\neg\psi$ such that $A(\psi) = 0$ is satisfiable.

Definition 3.2.3 (Completeness)**.** A judgment function A is *complete* if, for every $\varphi \in X$, either $A(\varphi) = 1$ or $A(\varphi) = 0$.

A *judgment aggregation function* F (the "aggregator") assigns to each profile $\{A_i\}$ a social judgment function A. In contrast, the individual judgment functions

[3]This would be material if aggregators did not generate consistent or complete judgment functions. In such a case, "$A(\varphi) = 1$ if and only if not $A(\varphi) = 0$" might fail for some $\varphi \in X$. However, we admit only consistent and complete evaluations both for individual and social judgment functions, so this issue does not arise.

are always indexed to an individual $i \in N$, such as A_1. For instance, considering a $\varphi \in X$, $F(A_1, \dots, A_n)(\varphi) = F_\varphi(A_1, \dots, A_n) = A(\varphi)$.[4]

The set of profiles that F is defined for is called the $Dom(F)$. The social judgment function can be consistent and complete in the same way as individual judgment functions. One can impose conditions on the function F making it: *universal* if $Dom(F)$ includes all profiles which are composed of consistent and complete individual judgment functions; *consistent* or *complete*, if for each $(A_1, \dots, A_n) \in Dom(F)$, the social judgment function A is, respectively, consistent or complete.

3.3 Introduction of the Conditions

With the JA framework in place, we can introduce the conditions governing aggregators. The first two deal with general treatment of individuals in the society and the elements of the agenda. The latter operates across profiles, making sure that agreement on propositions in different profiles results in social agreement.

3.3.1 Anonymity

Definition 3.3.1 (Anonymity (A))**.** An aggregator F satisfies *Anonymity* if, given a permutation $\sigma : N \to N$ and an admissible profile $\{A_i\}$, then $F(\{A_i\}) = F(\{A_{\sigma(i)}\})$.

(A) is satisfied when permuting the individuals taking the positions on the elements of the agenda does not affect the social judgment function.

This condition is strongly desirable, especially in a democratic context. Each member of the society should have equal say. Viewing JA in a similar vein to voting theory makes this clear. Denying certain individuals influence is a rejection of our modern ideal of universal suffrage. Furthermore, allowing certain individuals extra impact may take forms, such as oligarchies or dictatorships, where the aggregation will not reflect all of the members of the society (or systems where certain individuals have greater power over the social judgment function than others). (A) prevents both dictatorships and oligarchies, which is why JA frameworks do not include a specific non-dictatorship condition.

3.3.2 Neutrality

Definition 3.3.2 (Neutrality (N))**.** An aggregator F satisfies *Neutrality* if, given any propositions in the agenda $\varphi, \psi \in X$ and admissible profile $\{A_i\}$, if for all individuals $i \in N$, $A_i(\varphi) = A_i(\psi)$, then $A(\varphi) = A'(\varphi)$.

[4] For expediency, we suppress the extra parentheses, writing $F((A_1, \dots, A_n))$ as $F(A_1, \dots, A_n)$ or just $F(\{A_i\})$. The latter two are both shorthand for the former.

(N) is satisfied when, given a particular profile where all members of the society agree on two separate propositions (*i.e.* φ and ψ), the aggregator takes the same value for each of these propositions. In other words, there is a symmetry whenever individuals all agree on two propositions *within* profiles.

This is considerably weaker than it initially appears. The intuitive interpretation is that this makes an aggregator treat each proposition in the same manner. However, as our examples in Chapter 5 show, aggregators can be designed so that they do not treat propositions in the same way. Such aggregators treat propositions the same within profiles as long as the individual judgment functions coincide, but diverge when the individual judgment functions do not coincide.

3.3.3 Independence

Definition 3.3.3 (Independence (I))**.** An aggregator F satisfies *Independence* if, given any proposition in the agenda $\varphi \in X$ and admissible profiles $\{A_i\}$, $\{A'_i\}$ such that, if, for all individuals $i \in N$, $A_i(\varphi) = A'_i(\varphi)$, then $A(\varphi) = A'(\varphi)$.

(I) is patterned on PA (IIA).[5] Like (IIA), the point is that, when the individual opinions on the agenda concur restricted to some proposition (*i.e.* φ), the aggregation should produce the same result for that proposition φ. The intuition is that the aggregation on φ should be independent of all the other non-φ elements of this proposition. Furthermore, (I) is intended to prevent strategic voting. We discuss this issue further in Section 3.5.

3.4 List & Pettit's Impossibility Theorem

The conditions we have called (N) and (I) together are implied by (their conjunction is not necessarily logically equivalent with) what List & Pettit (2002) call "Systematicity." Given Systematicity and (A), we can generate an impossibility result for JA. The original impossibility is found in (List & Pettit 2002), and was supplemented in (Pauly & van Hees 2006).

It roughly suggests that doctrinal paradoxes can recur with any agenda of sufficient complexity, so it can be viewed as a generalization of such paradoxes.

Note that $\neg(\varphi \wedge \psi)$ is implicit since this framework makes assent to negated propositions dissent to positive propositions.

[5] For this reason, it is preferable not to refer to (I) as Independence of Irrelevant Alternatives in JA. This can lead to conflation, especially because (I) and (IIA) are independent of each other and neither straightforwardly implies the other. However, it should be noted that in the JA literature, sometimes this condition is called Independence of Irrelevant Alternatives, e.g. (van Hees 2007). However, in such situations, these conditions are identical to (I).

Theorem 3.4.1 (List & Pettit's Impossibility Theorem)**.** *Let X be a non-simple agenda. Then there is no universal aggregator F over X producing consistent and complete judgments such that F satisfies (A), (N), and (I).*

Proof. By Systematicity (*i.e.* (N) and (I)), it follows that there is a function $f : \{0,1\}^n \to \{0,1\}$ such that, for any given profile $\{A_i\} \in Dom(F)$, $F_\varphi(\{A_i\}) = 1$ if and only if $f(A_1(\varphi), \dots, A_n(\varphi)) = 1$. In other words, the function f accepts or rejects the profile in that proposition φ based solely on the acceptance patterns on φ.

By (A), $f(A_1(\varphi), \dots, A_n(\varphi)) = f(A_{\sigma(1)}(\varphi), \dots, A_{\sigma(n)}(\varphi))$ for any permutation $\sigma : N \to N$. This means that only the value of the vector matters, and not the coordinates. Thus, for any $(A_1, A_2, \dots, A_n)$, $(A'_1, A'_2, \dots, A'_n) \in \{0,1\}^n$ we have that $f(A_1, A_2, \dots, A_n) = f(A'_1, A'_2, \dots, A'_n)$ if $|\{A_i : A_i(\varphi) = 1\}| = |\{A'_i : A'_i(\varphi) = 1\}|$.

We define the number of assents to φ which we call $|\varphi|$, as follows:

$$|\varphi| \coloneqq |\{A_i : A_i(\varphi) = 1\}|.$$

The number of dissents to φ we call $|\varphi^-|$ which we define analogously:

$$|\varphi^-| \coloneqq |\{A_i : A_i(\varphi) = 0\}|.$$

Generate a profile as follows (cf. Table 3.2) for which no aggregator cannot consistently aggregate.

	$A_i(\varphi)$	$A_i(\psi)$	$A_i(\varphi \wedge \psi)$
$i = 1$	1	1	1
$i = 2$	1	0	0
$i = 3$	0	1	0
$i > 3$ and i is even	1	1	1
$i > 3$ and i is odd	0	0	0

Table 3.2: List & Pettit's Impossibility

By assumption, $|N| \geq 2$. There are two cases:

Case 1: $|N|$ is even. In this case, $|\varphi \wedge \psi| = |(\varphi \wedge \psi)^-|$. Thus, $A(\varphi \wedge \psi) = 1$ if and only if $A(\varphi \wedge \psi) = 0$. Since A is complete, at least one of these must hold. This contradicts the consistency of A.

Case 2: $|N|$ is odd. In this case, $|\varphi| = |\psi| = |(\varphi \wedge \psi)^-|$. Since the number of acceptances determines acceptance, we have $A(\varphi) = A(\psi) = 1 - A(\varphi \wedge \psi)$, that is, these all take 1 or 0. In the former case, A is inconsistent. In the latter, by A's completeness and the fact that $A(\varphi \wedge \psi) \neq 0$, $A(\varphi \wedge \psi) = 1$. So we have that $A(\varphi) = A(\psi) = 0$ and $A(\varphi \wedge \psi) = 1$. This contradicts the consistency of A. □

3.5 Weakening the Conditions

3.5.1 Surveying Recent Work

Since List & Pettit (2002) showed that doctrinal paradox-style issues can stymie any aggregation functions which satisfy certain plausible conditions, the aptness of these conditions has come under question. The doctrinal paradox prevents universal consistent and complete aggregation outputs, even given consistent and complete input profiles. So there are two primary ways to attempt to avoid List & Pettit's impossibility results: To weaken the conditions on the input/output of the aggregator F and to weaken the conditions necessary for the impossibility. Both have been considered, and we proceed to survey a few of the interesting results.

One way of changing the input of F is relaxing domain universality. van Hees (2007) weakens the universality of F to "Agenda richness," which requires a significant amount of closure on the agenda. Under this, if F satisfies (I), then there is a restricted dictator, i.e. a dictator who is able to force any social valuation which falls within the range of F.

Gärdenfors (2006) weakens both the input and the output of F by not requiring completeness of either the individual or the social judgment functions. Instead, he only requires consistency and deductive closure. This is motivated by his influential theory of belief revision (cf. (Gärdenfors 1992)). Thus, with the key condition that the Boolean algebra of the agenda is atomless, and a form of unanimity, he shows that the oligarchies satisfy a weakened (I).

Dietrich & List (2008) and Dietrich & List (2007*b*) examined other weakenings of the input/output by relaxing consistency and completeness. In (Dietrich & List 2008), they consider judgment sets which are not complete, but deductively closed and consistent. They determine that this is not very much more permissive; it allows oligarchies instead of dictatorships, so some subset of the society controls the evaluation. In (Dietrich & List 2007*b*), they weaken these constraints to just consistency (although deductive closure seems like a plausible minimal requirement of judgment functions). This leads to impossibilities when the agenda has certain minimal requirements and the aggregator is not biased towards either rejection or acceptance of propositions.

These results are not encouraging for possibility since oligarchies and restricted dictatorships are not very desirable types of possibilities. It is worth considering alternative relaxations of the impossibility criteria to see if more robust possibilities are available.

In order to see what strength of conditions are required in order to generate such impossibility, it is also necessary to weaken them and see what conclusions can be drawn. Pauly & van Hees (2006) considered characterizations of dictatorship by examining Systematicity. Since Systematicity is composed of (I) and

(N), Pauly & van Hees considered weakening (N) to "Responsiveness," which enforces the existence of profiles that change the truth values of two literals. With Responsiveness and (I), Pauly & van Hees characterized dictatorships in multiple truth-valued contexts.

Weakening Systematicity further than Pauly & van Hees (2006) do is possible as well. Since Weak Responsiveness with (N) characterized dictatorships, it is not surprising that further weakening generates possibility. In particular, van Hees (2007, p. 661) weakens Systematicity to (I) and possibility is obtained on certain agendas. He further characterizes dictatorships as Systematicity and Non-Imposition, which requires that for some φ, for every truth value of φ (working within multi-valued contexts), some profile provides that truth-value.

Pauly & van Hees (2006) and van Hees (2007) have shown that weakening Systematicity does allow for certain possibility results which we take to be more fruitful than weakening the input/output conditions on aggregators. We also weaken the conditions of Systematicity. For this reason, it is valuable to argue that weakening Systematicity is defensible in JA contexts.

3.5.2 Defending Weakenings

We consider weakening Systematicity in line with Pauly & van Hees (2006) and van Hees (2007) to be defensible. Before explaining why, it is worth contrasting PA with JA. When presented with any instance of PA or a JA, a specified agenda X is included. In PA, this agenda is composed of alternatives $x, y, z \dots$ and in JA, it is composed of propositions. This subsection focusses on the implications of this difference.

In preference aggregation, the alternatives are considered to be both mutually inconsistent and structurally indistinguishable. In contrast, JA agenda elements hold entailment relations and are structurally distinguishable. We hold that these differences are consequential for weakening both Neutrality and Independence conditions.

Let us first consider the structural similarity of PA alternatives. This is a result of the fact that, for Arrow, preferences ranged over social and economic complete states of society (Arrow et al. 2002, p. 4). The Arrovian framework was developed to find the logical structure of welfare economics, where it was not clear which type of economic structure best satisfies the needs of multiple autonomous individuals. So in this sense, the alternatives being aggregated are—prior to the aggregation—(structurally) equal.

This manifests itself in PA admissible profiles. Simply put, any permutation of an admissible R relation where pairs of alternatives are uniformly exchanged yields a new admissible admissible R relation. This is true for any individual R relation in a profile. So exchanging pairs of alternatives in an admissible profile yields a new admissible profile. This occurs even if multiple pairs are exchanged in the profile or if different individual R relations experience different alternative exchanges. This is because (individually uniform) exchange of alternative pairs does not change an ordering's transitivity and, hence, it's admissibility.[6] In an intuitive sense, therefore, PA treats its initial alternatives "neutrally."

It may be objected that the primitives in R relations are not the relata but, instead, the atoms which compare pairs, such as xRy. But my point holds at this level as well. PA treats xRy and $yR'x$ "neutrally" given that x and y are exchanged uniformly from R to R'. Exchanging all x and y in a given R relation for some individual in an admissible profile will yield a new admissible profile.

On the other hand, JA begins by considering propositions with structural dissimilarity. This is evident at an intuitive level, since certain propositions in an agenda are more "complex" than others in that, for instance, molecular propositions have more logical connectives than atomic propositions. But this intuitive dissimilarity has concrete implications for aggregation in JA as well. Unlike PA, exchanging pairs of agenda elements in JA in an admissible profile does not lead to a new admissible profile.

A simple example suffices to show this. Let $X = \{a, b, a \wedge b\}$ and an individual i be such that $A_i(a) = 1$, $A_i(b) = 0$ and $A_i(a \wedge b) = 0$. This evaluation is complete and consistent. Now consider what occurs when we permute the agenda in such a way that we replace a with $a \wedge b$ and vice versa. Then $A_i(a \wedge b) = 1$, $A_i(b) = 0$, and $A_i(a) = 0$. No longer do we have an admissible profile. Furthermore, it is intuitively clear why—a and $a \wedge b$ are structurally different.[7] They are not treated neutrally.

It may be objected that this argument would fail if it were the logical constants—instead of the entire propositions—which were being permuted. (In these two instances, that would occur if a and b were uniformly exchanged in each given proposition for i.) In this case, the evaluations would remain consistent. An objector could press this remark by suggesting that this type of permutation is more similar to exchanging the individual alternatives $x, y, z, \ldots$ in PA.

We would reply that these would be disanalogous since the logical constants in JA do not play the same role as the individual alternatives in PA. Why is this? This is because the set X in a given PA instance is composed of alternatives $x, y, z \ldots$ but the set X in a given JA instance is composed of *propositions*. When defining a given JA problem, the analogue of a PA set X is a set of propositions,

[6]Completeness of R relations remain, naturally, unchanged.

[7]In fact, this can occur even when the elements being exchanged have the same number of connectives. Let $X = \{a, b, a \to b\}$ and an individual i be such that $A_i(a) = 0$, $A_i(b) = 1$ and $A_i(a \to b) = 1$. This is an admissible evaluation. However, an agenda permutation which exchanges a with b yields $A_i(b) = 0$, $A_i(a) = 1$ and $A_i(a \wedge b) = 1$. This is no longer admissible.

not logical constants.[8]

Since PA treats alternatives neutrally in profiles and JA does not treat agenda elements neutrally in profiles, we argue that strong neutrality is less desirable as a restriction on JA aggregators than for PA social welfare functions.

The other key difference is that PA alternatives are often assumed to be mutually exclusive, in that selection of one alternative precludes any other.[9] JA agenda elements, in contrast, have logical dependency relations. Not only does acceptance of $\alpha \wedge \beta$ not exclude acceptance of α, it actually *requires* acceptance on α. In line with the consideration that α depends on $\alpha \wedge \beta$, it is not surprising for the social positions on α to be in a direct sense determined by (and dependent upon) the individual positions on both $\alpha \wedge \beta$ (or vice versa). Since JA is concerned with propositional knowledge and beliefs, it must respect the logical structure of propositions, and the stricter constraints, on complex molecular propositions.

To illustrate, consider a group deciding a simple atom p and a complex formula such as $(q \wedge r) \to s$. We take it to be plausible that the former should be decided independently in the sense that individual positions on p alone are sufficient to decide the group decision on p. However, we also take it be plausible that the latter proposition resides not only upon individual positions, but also their "reasons," including their positions on the components of the molecular proposition.

It could be objected that weakening (I) would make it easier to strategically "vote" with weakened independence contexts. In particular, if a given element which one wishes to affect depends on others, there may be ways to manipulate a position on the given element by falsely representing the positions on the other elements. This is a non-trivial concern, but we argue that there are a few reasons for sidelining it.

The first is that—unlike PA—there is an intuitive justification in JA for introducing elements of the agenda in order. Recall from Subsection 2.3.2 that the introduction of (IIA) into PA was to prevent strategic pairwise comparisons: In particular, when considering the Condorcet Paradox, if pairwise comparisons were performed sequentially, *which order* the alternatives were introduced determined the winner. The introduced sequence is immaterial for an aggregator satisfying (IIA). In other words, there was the possibility of manipulation.

However, given the logical structure of propositions, there *is* an intuitive order upon which propositions should be aggregated and, thus, which dependencies should hold. In particular, often atoms should be aggregated "first" and then simple molecular propositions and subsequently more complex molecular proposi-

[8] Generally speaking, this is true. Of course, in special cases such as very trivial agendas composed solely of atoms, this would hold. Our point is that these cases do not generalize.

[9] This is not always the case; approval voting, for instance, maps profiles to sets of candidates which may be empty or may be non-singleton.

tions.[10] Once again, the assumption in (IIA) that the alternatives are equal does not do the same work in JA.

The second is that formal manipulation requires knowledge of the entire profile which is a more complex structure in JA than in PA. In actual (e.g. political) life, this presents a much larger problem with manipulating in JA than PA. An example makes this intuition clearer. We select a topical issue and give a simplified example. In a certain constituency, suppose there are three possibilities: Gay marriage, gay civil unions, or neither of these. Assume also that certain pairwise comparisons change preferences in some way. It is not very difficult to determine the approximate level of support for each of the three possibilities if one is so inclined in the given constituency. Thus, finding the agenda (and possibly manipulation) is not as difficult for a motivated block of voters. Suppose that we formulate the problem in JA terms. A plausible agenda would include reasons for one or the other (e.g. the divorce rates for opposite-sex parents, religious convictions, impact on children, etc). It is far more complex to gather the required information needed to influence the social positions by manipulating the reasons, especially if the reasons had logical interconnections as well. The more data that would be required would be far greater in the JA case than in the PA case.

Finally, even provided the JA profile information, a few results suggest that formal manipulation is far more difficult in JA than PA in terms of computational complexity. Dietrich & List (2007*c*) showed that Monotonicity and (I) imply strategy-proofness, i.e. non-manipulability. For those which fail both of these conditions, Endriss, Grandi & Porello (2010*b*) have shown that the complexity of some JA manipulation problems are intractable for majoritarian premise-based procedures (which are similar, although not equivalent, to the types of aggregators I consider in Section 5.3).[11] Since this procedure are computationally intractable, it is presumable that other judgment aggregators are similarly difficult to manipulate.

If so, it is plausible that both conditions of Systematicity could be weakened in JA contexts. Neutrality could be weakened because the profiles in JA treat agenda elements in structurally distinct ways when compared to alternatives in PA. Inde-

[10] Naturally, one could alternately argue that certain special propositions, i.e. "conclusions" should be aggregated first and then their constituent propositions should be aggregated based on the social positions on the conclusion. In this case, the aggregator operates in roughly the reverse way. We find this less appealing for modelling social reasoning. For a discussion of these issues, see (Nurmi 2005).

[11] Consistency requires that the agenda X satisfies the "median property" which requires inconsistent subsets to be formed with inconsistent subsets (Endriss, Grandi & Porello 2010*a*). Endriss et al. (2010*b*) find that manipulation of a version of the premise-based procedure to be NP-complete. This should be compared with PA manipulation, where, for many voting rules, manipulation is computationally easy, excepting the notable case of "single-transferrable voting" (Bartholdi, Tovey & Trick 1989).

pendence could be weakened because the logical interconnections of propositions lends itself to considering the individual positions on some propositions when determining the social judgments of other propositions. These are the considerations that lead us to weaken (I) and to consider the weakenings in PA of (IIA).

Chapter 4

Independence Weakenings

Since JA's (I) and PA's (IIA) are quite demanding conditions, it is of technical and conceptual interest to consider weakening them. In Section 4.1, we begin by introducing Campbell & Kelly's (2007) weakened independence conditions. For Campbell & Kelly's conditions, instead of considering two alternatives and only the individual preferences between those two alternatives, the preference aggregator can include information on individual preferences beyond those of the two alternatives.

Campbell & Kelly prove that certain relationships hold between their conditions. In Section 4.2, we use their proofs to show that the PA conditions—Weakest Independence and Neutrality—jointly imply Independence of Some Alternatives, and that under Universal Domain, Weak Unanimity, Weakest Independence and Neutrality jointly imply Independence of Irrelevant Alternatives. In other words, the weakest of their independence conditions together with Neutrality rules out weakly unanimous social welfare functions. This is disheartening; in PA, Weak Unanimity is a very desirable condition.

In Section 4.3, after presenting the Campbell & Kelly conditions, we explain our JA analogues. These weakenings of (I) allow for (limited) dependence. The main intuition is that, instead of aggregating each element of the agenda individually, aggregation of each element may include information from the profile on other elements (although not all of them—that would indicate full dependence and no independence).

After introducing each independence condition, we then turn to characterize that independence condition in terms of dependent and independent sets. Before characterizing our own independence conditions, we also characterize (I) in terms of dependent sets. Given an element φ, these dependent sets are subsets of the agenda which indicate the elements which are required in order to determine the social position on φ.

We end by summarizing the implications which are made conspicuous by these characterizations.

4.1 The Campbell & Kelly Independence Conditions

Campbell & Kelly's (2000) suggestions for weakening (IIA) are based on the idea of minimizing the information which is required in order to aggregate preferences. For any given pair, the information required to aggregate by the original (IIA) is the set of individual preferences for that pair. However, we can expand this set to more individual preferences in the profile. By allowing for different sets of alternatives which are needed to aggregate, the Campbell & Kelly conditions relax (IIA).

4.1.1 PA Independence of Some Alternatives

Definition 4.1.1 (PA Independence of Some Alternatives (P-ISA))**.** A social welfare function F satisfies *PA Independence of Some Alternatives* if, for *every pair* of alternatives x, $y \in X$, there exists $Y \subset X$ (which may depend on x, y) such that, for any two profiles $\{R_i^1\}$, $\{R_i^2\}$, if $\{R_i^1\}|Y = \{R_i^2\}|Y$, then $\{R^1\}|\{x,y\} = \{R^2\}|\{x,y\}$.

By the original (IIA), the dependent set of x, y is $\{x,y\}$. In other words, the only determinants of the social preference between any two alternatives are the individual preferences on precisely those alternatives. (P-ISA) relaxes this condition by allowing the dependent set be *any* strict subset of X. These dependent sets may differ depending on which pair one is considering.

A small note: In (Campbell & Kelly 2000), it is demonstrated that (P-U) and (P-ISA) preference aggregation can be nondictatorial. Their examples of aggregation (viz. the "gateau rule" and the "signaling rule") are difficult to defend, however, as they require identifying distinguished elements. In PA, the intuition is against having distinguished elements.[1]

[1] Addressing this concern for gateau rules, Campbell & Kelly (2000, p. 10) write:

> We have deliberately left open the the issue of selecting a distinguished element. That can depend on context, and is an opportunity for introducing further axioms. In a legislative context, the status quo alternative is one possibility. In an Edgeworth-Bowley box context, the middle point, the equal-division outcome, is a possibility.

Choosing such distinguished elements, however, is the *purpose* of preference aggregation, since, before aggregation, the alternatives have no distinguishing structure or prior claim on acceptance. We argue that the JA framework treats elements of the agenda in a different light.

4.1.2 PA Weakest Independence

Definition 4.1.2 (PA Weakest Independence (P-WI))**.** A social welfare function F satisfies *PA Weakest Independence* if, for *at least one pair* of alternatives x, $y \in X$, there exists $Y \subset X$ (which may depend on x, y) such that, for any two profiles $\{R_i^1\}$, $\{R_i^2\}$, if $\{R_i^1\}|Y = \{R_i^2\}|Y$, then $\{R^1\}|\{x,y\} = \{R^2\}|\{x,y\}$.

(P-WI) requires that there is some pair which has a decisive subset of the agenda. This subset is decisive in the sense that knowing the individual preferences on the elements of the subset is sufficient to output the preference on the original pair.

This second Campbell & Kelly independence condition is a weaker version of (P-ISA)—in fact, it is the existential version. Instead of requiring that there a be a strict subset $Y \subset X$ for *every pair* of alternatives that being considered in the aggregation, (P-WI) just requires there be at least one pair of alternatives which can be decided by individual preferences on a subset of the agenda.

4.2 Campbell & Kelly Implications

Campbell & Kelly (2007) prove that, under universal domain, weak unanimity and neutrality, (P-WI) implies (IIA). This is done in multiple parts. The first lemma shows that (P-WI) and neutrality imply (ISA). The second part consists of the intersection principle, which shows that the sufficient sets are closed under intersection. Then, full domain and weak unanimity are used to prove that $\{x,y\}$ are in the sufficiency set for x and y. Finally, the intersection principle is used to show that $\{x,y\}$ constitute the entire x, y sufficiency set.

We begin by defining the Campbell & Kelly conditions and then move to the proof, which requires several lemmas.

4.2.1 Weak Unanimity

Definition 4.2.1 (Weak Unanimity (WU))**.** A social welfare function F satisfies *Weak Unanimity* if, given arbitrary x, for every profile $\{R_i\}$, if every individual $i \in N$ for all $y \neq x$ satisfies xP_iy, then for all $y \neq x$, xPy holds for that x.

This condition is weaker than (P-U). It requires that if all agree on the *topmost* alternative in individual orderings, then the social order also agrees on that alternative as the top of the social ordering. In comparison, (P-U) makes the social order on any pair match unanimous pair comparisons, not just those where all agree on the most preferred alternative.

4.2.2 PA Neutrality

Given any permutation $\mu : X \to X$ which permutes the available alternatives X, we can define the permutation of the relation R, *i.e.* $\mu(R)$, defined as follows:

$$(x, y) \in \mu(R) \text{ if and only if } (\mu^{-1}(x), \mu^{-1}(y)) \in R.$$

Let $\mu(\{R_i\}) \coloneqq \{\mu(R_i)\}$, meaning that the profile is composed of individual permuted orders.

With such permutations in mind, Campbell & Kelly (2007) define PA neutrality:

Definition 4.2.2 (PA-Neutrality (P-N)). A social welfare function F satisfies *PA-Neutrality* if for every profile $\{R_i\}$ and every permutation $\mu : N \to N$ on X, $\mu(\{R_i\})$ is in in the domain of F and $F(\mu(\{R_i\})) = \mu(F(\{R_i\}))$.

Note that Campbell & Kelly's (P-N) differs considerably in formulation from (N). Instead of considering only pairs of elements where each takes the same acceptances intra-profile, this (P-N) makes the aggregator treat all permutations of elements the same way.

In Section 5.2, we show that the JA equivalent of (P-N) is strictly stronger than (N) by itself.

4.2.3 Sufficiency

In order to make the proofs clear, it is necessary to introduce the idea of sufficient sets. In non-weak independence contexts, sufficient sets are unnecessary because the sufficient set for any given pair xy is always the set $\{x, y\}$. In other words, under (IIA), the only information about individual preferences that you need to determine a given pair xy are the preferences over that pair.

However, in contexts where independence is weakened, it is valuable to have a shorthand to be to address such sets. In other words, it is important to be able to easily identify which information about pairs is sufficient in order to aggregate.

In Section 4.3, we will introduce our own notion which helps specify dependency ("dependent sets"), but our notion is significantly different from sufficiency sets, and these should not be conflated.

Definition 4.2.3 (Sufficiency). Given a social welfare function F and a set $Y \subset X$, Y is *sufficient* for $\{x, y\}$ if for any two admissible profiles $\{R_i\}$, $\{R'_i\}$, if $\{R_i\}|Y = \{R'_i\}|Y$, then $R|\{x, y\} = R'|\{x, y\}$.

A set Y which is sufficient for the pair $\{x, y\}$, by this definition, is such that the following condition is satisfied: If two profiles are equal when restricted to Y only, then they produce the same social preferences on the xy-pair. The intuition is that the set Y over individual preferences encodes all the information necessary to determine the social preference between $\{x, y\}$.

Note that this definition does not prejudice whether or not x and y themselves are members of Y.

4.2.4 PA Independence Implications

The proofs in this subsection are by Campbell & Kelly (2007). For these proofs, we assume that $|X| \geq 3$.

Lemma 4.2.4. *If F satisfies (P-WI) and (P-N), then F also satisfies (P-ISA).*

Proof. By (P-WI), there are x, y and z in X such that $X \setminus z$ is sufficient to determine the pair (x, y). For any given pair u and v in X, let μ be a permutation such that $\mu(x) = u$, $\mu(y) = v$ and $\mu(z) = w$ for some w. Then we will show $X \setminus w$ is sufficient to determine the pair (u, v).

Assume, towards a contradiction, that it is not. Then there are profiles $\{R_i\}|X \setminus w = \{R'_i\}|X \setminus w$ where uRv and $vP'u$. This follows because w is necessary to fix a social ranking for (u, v). Then invert the permutation: Since $\{R_i\}|X \setminus w = \{R'_i\}|X \setminus w$, we have that $\mu^{-1}\{R_i\}|X \setminus z = \mu^{-1}\{R'_i\}|X \setminus z$. But then, by (P-N), xRy at $\mu^{-1}\{R_i\}$ and yPx at $\mu^{-1}\{R'_i\}$, which contradicts the sufficiency of $X \setminus z$ in determining (x, y). □

This lemma shows that (WI), together with (P-N), implies (P-ISA). This is natural: Since there is some weak level of independence for some element in the agenda, by permutation, this weak level of independence spreads. Note that it is not necessary to use neutrality across profiles, since intra-profile, the weak independence is enough to generate weak independence for different elements.

Next, we move to a principle which allows for simplifying the sufficient sets. It is highly intuitive that, when two sets are both sufficient for a pair, their intersection is also sufficient.

Lemma 4.2.5 (Intersection Principle)**.** *If F has full domain, and Y and Z are each sufficient for $\{x, y\}$, then so is $Y \cap Z$.*

Proof. Suppose $\{R_i\}$, $\{R'_i\}$ are admissible profiles such that $\{R_i\}|Y \cap Z = \{R'_i\}|Y \cap Z$, where Y, Z are both sufficient for $\{x, y\}$. Choose $\{R''_i\}$ such that $\{R''_i\}|Y = \{R_i\}|Y$ and $\{R''_i\}|Z = \{R'_i\}|Z$. Then $R''|Y = R|Y$ and $R''|Z = R'|Z$.

Thus, $R|Y \cap Z = R'|Y \cap Z$. □

This lemma allows the sufficient sets to be simplified, but it is not specified how they are constituted. Since neutral aggregators can treat each alternative as dependent on some strict subset of the agenda, in order to show (IIA), it is necessary to guarantee that the strict subset at least includes the given alternative itself. In other words, the sufficiency set of a given alternative a must include a in order to satisfy (IIA). A very weak unanimity condition—such as (WU)—suffices to show this.

If all individuals agree on a preference between two alternatives, then these alternatives must lie within their own sufficiency set.

Lemma 4.2.6. *If F has full domain and satisfies (WU), then $\{x, y\} \subset Y$ where Y is the sufficiency set of x and y.*

Proof. Construct $\{R_i\}$, a profile where for each $i \in N$, xP_iy, yP_iz, for all $z \neq x \neq y$. Now construct $\{R'_i\}$ from $\{R_i\}$ where x and y are interchanged. This yields xPy but $yP'x$. If we consider any $Y \subset X$ which does not include x, y, $\{R_i\}|S = \{R'_i\}|S$. □

With these lemmata in place, (IIA) follows easily. (P-N) and (P-WI) imply (P-ISA). (P-ISA) together with (WU) forces each sufficiency set for a given alternative to have its own alternative in its sufficiency set. Then, by repeated application of the Intersection Principle, (IIA) is obtained.

Lemma 4.2.7. *If F has full domain, satisfies (WU), (P-N), then (P-WI) implies (IIA).*

Proof. By Lemma 4.2.4, F satisfies (P-ISA). Consider arbitrary x and y, two distinct alternatives in X. To satisfy this lemma, it is required to show that $\{R_i\}|\{x, y\}$ is sufficient for $\{x, y\}$. By (P-ISA), there is some $z \in X$ such that $X \setminus \{z\}$ is sufficient for $\{x, y\}$.

By Lemma 4.2.6, $\{x, y\} \subset Y$ where Y is the sufficiency set of $\{x, y\}$; thus, $z \neq x \neq y$. In other words, $X \setminus \{x, y\}$ contains a non-x, non-y member z. By (P-N), for every z in $X \setminus \{x, y\}$, the set $X \setminus \{z\}$ is sufficient for $\{x, y\}$.

Then using Lemma 4.2.5 repeatedly over these sets,

$$\{x, y\} = \bigcap_{z \in X \setminus \{x,y\}} X \setminus \{z\}$$

is sufficient for $\{x, y\}$. Since this holds for arbitrary $x, y \in X$, (IIA) follows. □

4.2.5 Discussion

Since the weakest PA weakening that Campbell & Kelly (2007) considers, (P-WI), under the assumption of (W-U), implies full (IIA), weakening (IIA) in PA does not allow for much more possibility with respect to aggregators. In the succeeding section, we will introduce our own weakenings of (I), which do not imply (I) in such a simple manner.

We will argue that the JA weakenings inspired by Campbell & Kelly's work lead to more interesting and plausible aggregators which take advantage of the structural properties of propositions, as opposed to alternatives.

4.3 Dependence Characterizations of Judgment Aggregation Conditions

Now that we have introduced the two Campbell & Kelly independence conditions, we introduce our JA versions, along with a few other interesting independence conditions.[2]

Since, as we will explain, our definition of dependence is not a direct analogue of Campbell & Kelly's, our weak independence conditions do not directly correlate with their weakened conditions. Intuitively, the difference is that Campbell & Kelly's independence conditions are geared towards independence of *sets*, while ours are geared towards *element-by-element* independence in the agenda.

To characterize these independence conditions, it is useful to refer to the dependent sets of agenda elements. We introduce our notions of dependence and independence in order to demonstrate the dependency restrictions on these weakened independence conditions.

Definition 4.3.1 (Dependence)**.** Given an agenda X, an element $\psi \in X$ is *dependent* on $\varphi \in X$, if and only if there exist two admissible profiles $\{A_i\}, \{A'_i\}$ such that the following three conditions are all met:

1. For all $\chi \neq \varphi$, $A_i(\chi) = A'_i(\chi)$;
2. $A(\psi) = 1$; and
3. $A'(\psi) = 0$.

Definition 4.3.2 (Independence)**.** Given an agenda X, an element $\psi \in X$ is *independent* of $\varphi \in X$, if and only if for any two admissible profiles $\{A_i\}, \{A'_i\}$, if, for all $\chi \neq \varphi$, $A_i(\chi) = A'_i(\chi)$, then $A(\psi) = A'(\psi)$.

A small remark that the definition of independence is just the negation of the definition of dependence. To see this, note that, instead of having condition 1

[2] A small remark: Since the conditions in this section are weakenings of (I), trivially (I) implies all of these conditions, although this does not hold for Single Dependence, which is not a weakening of (I).

and condition 2 *and* condition 3, this definition has *if* condition 1, *then either* condition 2 *and not* condition 3 *or* condition 3 *and not* condition 2; thus, we have that $\varphi \in D(\chi)$ if and only if not $\varphi \in I(\chi)$.

Once we have dependency and independency, we gather all the elements that ψ depends on into a set called $D(\psi)$ and the independent elements into a set called $I(\psi)$:

Definition 4.3.3 (Dependent Set)**.** $D(\psi)$, ψ's *Dependent Set*, is the set of formulas φ such that $\psi \in X$ is dependent on φ.

Definition 4.3.4 (Independent Set)**.** $I(\psi)$, ψ's *Independent Set*, is the set of formulas φ such that $\psi \in X$ is independent of φ.

Alternately, we could have defined a given independent set as the agenda minus the appropriate dependent set, i.e. for arbitrary $\varphi \in X$, $I(\varphi) = X \setminus D(\varphi)$, since these sets do not overlap, and together they form the entire agenda.

Dependent sets play a similar role to Campbell & Kelly's (2007) PA sufficiency sets, but dependent sets differ from sufficiency sets and should not be conflated. The key is that JA dependence sets are determined element-by-element from the agenda. In other words, to say that $D(\psi) = X$, this means that there are two profiles showing independence for *each* $\varphi \in X$. Furthermore, to say that $D(\psi) = \varnothing$, this means that for *any* $\varphi \in X$, one cannot produce two profiles $\{A_i\}$, $\{A'_i\}$ satisfying the above conditions, i.e. all the non-φ elements are fixed between two profiles (i.e. for each $\chi \neq \varphi$, $A_i(\chi) = A'_i(\chi)$) and the two profiles differ on ψ (i.e. $A(\psi) \neq A'(\psi)$).

This is to be contrasted with Campbell & Kelly's (2007) sufficiency sets; the sufficiency sets operate as sets where *all* the alternatives in the set are fixed, instead of keeping all except some element fixed, and testing this dependence element-by-element. So for some specific alternatives with sufficiency set $Y \subset X$, if we keep all elements of Y fixed, then our specific alternatives will take the same value. Our version of dependence reflects that certain elements of the agenda depend on *specific* others.

Among other advantages, characterizing the following weak independence conditions in this way makes it simple to determine the implications between them. These characterizations show how several implications are trivial. We list them at the end of the section, and summarize all the implications in Table 5.6.

Another advantage of dependent sets is that they make it easier to check the results in Chapter 5, since we explain our aggregators in terms of dependent sets and also use these sets to check which independence conditions they satisfy.

We begin with (I), follow with Some Independence and then introduce the JA versions of Independence of Some Alternatives, a strengthened Independence of Some Alternatives, and then Weakest Independence. After introducing each condition, we characterize it in terms of dependent sets.

4.3.1 Independence

If an aggregator satisfies (I), then, for any given element of the agenda, that element is not non-self-dependent (cf. Definition 3.3.3). What we mean by this admittedly complex locution is that, for that given element, it cannot be dependent on any element *distinct from itself*. This does not imply that it is self-dependent, since it could satisfy (I) by being, for instance, a constant function. So it is not true that an arbitrary element under an (I) aggregator need be self-dependent. It could have an empty dependent set (meaning that this the social judgment function on this element does not change over any profile).

So there are two ways a given element can not be non-self-dependent. It can be self-dependent, or it can be independent of the entire agenda.

Lemma 4.3.5 (I)**.** *An aggregator F satisfies (I) if and only if, for all $\chi \in X$ and for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$.*

Proof.

⇒ Assume, towards a contradiction, that F satisfies (I) and the second condition fails, i.e. that there exists $\chi \in X$ such that for some $\varphi \neq \chi$, $\varphi \in D(\chi)$.

Since $\varphi \in D(\chi)$, then, by definition of dependence, there are $\{A_i\}$, $\{A'_i\}$ such that for all $\psi \neq \varphi$, $A_i(\psi) = A'_i(\psi)$, $A(\chi) = 1$ and $A'(\chi) = 0$. But, since $\chi \neq \varphi$, for all $i \in N$, $A_i(\chi) = A'_i(\chi)$. So, by (I), $A(\chi) = A'(\chi)$. Contradiction.

⇐ Assume the second condition holds, i.e., for all $\chi \in X$ and for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$. Assume, towards a contradiction, that F fails (I).

By (I)'s failure, for some $\chi \in X$, there exist profiles $\{A_i\}$, $\{A'_i\}$ such that $A_i(\chi) = A'_i(\chi)$, $F_\chi(\{A_i\}) = 1$ and $F_\chi(\{A'_i\}) = 0$. By the functionality of F, $\{A_i\} \neq \{A'_i\}$. Since $\{A_i\} \neq \{A'_i\}$, there exists $\varphi \in X$, $\varphi \neq \chi$ such that $A_i(\varphi) \neq A'_i(\varphi)$. So, by definition of dependence, $\varphi \in D(\chi)$. Contradiction. □

4.3.2 Some Independence

If an aggregator satisfies (SI), then there is an element which is not non-self-dependent. Again, this does not imply that it *is* self-dependent: The element could either be self-dependent or independent of all elements in the agenda.

Definition 4.3.6 (Some Independence (SI))**.** An aggregator F satisfies *Some Independence* if there exists $\varphi \in X$ such that, for any two profiles $\{A_i\}$, $\{A'_i\}$, if, for all individuals $i \in N$, $A_i(\varphi) = A'_i(\varphi)$, then $A(\varphi) = A'(\varphi)$.

(SI) is the existential version of (I), meaning that there is some element of the agenda which is not other-dependent (i.e. there is some φ which satisfies either $D(\varphi) = \{\varphi\}$ or $D(\varphi) = \varnothing$). Thus, (SI) is weaker than (I).

Lemma 4.3.7 (SI)**.** *An aggregator F satisfies (SI) if and only if, for some $\chi \in X$, for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$.*

Proof.

⇒ Assume, towards a contradiction, that F satisfies (SI) and the second condition fails, i.e., that for all $\chi \in X$, there is some $\varphi \neq \chi$ such that $\varphi \in D(\chi)$.

By (SI), there is some $\chi \in X$ such that for any two profiles $\{A_i\}$, $\{A_i'\}$: If, for all $i \in N$, $A_i(\chi) = A_i'(\chi)$, then $A(\chi) = A'(\chi)$.

Since the second condition fails, for $\chi \in X$, there is some $\varphi \neq \chi$ such that $\varphi \in D(\chi)$.

By the definition of dependence, there exist two profiles $\{A_i''\}$, $\{A_i'''\}$ such that for all $\alpha \neq \varphi$ and all $i \in N$, $A_i''(\alpha) = A'''(\alpha)$, $A''(\chi) = 1$ and $A'''(\chi) = 0$. But then, since $\varphi \neq \chi$, for all $i \in N$, $A_i''(\chi) = A_i'''(\chi)$. So, by (SI), $A''(\chi) = A'''(\chi)$. Contradiction.

⇐ Assume the second condition holds, i.e., for some $\chi \in X$ and for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$. Assume, towards a contradiction, that F does not satisfy (SI).

By (SI)'s failure, for all $\psi \in X$, there exist some profiles $\{A_i\}$, $\{A_i'\}$ such that $A_i(\psi) = A_i'(\psi)$ and $A(\psi) \neq A'(\psi)$. Since this holds for all $\psi \in X$, it holds for χ. Thus, applied to χ, since $\{A_i\} \neq \{A_i'\}$ and $A_i(\chi) = A_i'(\chi)$, there is some $\varphi \neq \chi$ such that, for some $i \in N$, $A_i(\varphi) \neq A_i'(\varphi)$. But then, by definition of dependence, $\varphi \in D(\chi)$. Contradiction. □

4.3.3 Independence of Some Alternatives

If an aggregator satisfies (ISA), then for each element of the agenda, there is some measure of independence. In particular, for any given element, there is at least one (not necessarily distinct) element upon which it does not depend.

Definition 4.3.8 (Independence of Some Alternatives (ISA))**.** An aggregator F satisfies *Independence of Some Alternatives* if, for all $\chi \in X$, there exists $\varphi \in X$ (which may be dependent on χ) such that, for any two profiles $\{A_i\}$, $\{A_i'\}$, if, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$.

Note that (ISA) does not imply (SI), nor does (SI) imply (ISA).

Lemma 4.3.9 (ISA)**.** *An aggregator F satisfies (ISA) if and only if, for all $\chi \in X$, there exists some $\varphi \in X$ such that $\varphi \in I(\chi)$.*

Proof.

⇒ Assume, towards a contradiction, that F satisfies (ISA) and the second condition fails, i.e., that for some $\chi \in X$, for all $\varphi \in X$, $\varphi \in D(\chi)$.

Since $\chi \in X$, χ is subject to (ISA). Thus, there exists $\varphi \in X$ satisfying the following: For any two profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$.

By the failure of the second condition, for this φ, $\varphi \in D(\chi)$; that is, there exist two profiles $\{A''_i\}$ and $\{A'''_i\}$ such that, for all $\beta \neq \varphi$, $A''(\beta) = A'''(\beta)$, and $A''(\chi) \neq A'''(\chi)$. But then this satisfies the antecedent of (ISA), so $A''(\chi) = A'''(\chi)$. Contradiction.

⇐ Assume the second condition holds, i.e., for all $\chi \in X$, there exists some $\varphi \in X$ such that $\varphi \in I(\chi)$. Assume, towards a contradiction, that F does not satisfy (ISA).

By the failure of (ISA), there exists some $\chi \in X$ such that, for all $\varphi \in X$, there exist two profiles $\{A_i\}$, $\{A'_i\}$ such that, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'_i(\alpha)$ and $A(\chi) \neq A'(\chi)$.

But since $\chi \in X$, the antecedent of the second condition is satisfied by $\{A_i\}$, $\{A'_i\}$ for some φ. So, since for all $i \in N$ and for all $\alpha \neq \varphi$, $A_i(\alpha) = A'_i(\alpha)$, by the definition of independence, $A(\chi) = A'(\chi)$. Contradiction. □

4.3.4 Strengthened Independence of Some Alternatives

It will be of interest to consider a strengthening of (ISA) to (ISA+). If an aggregator satisfies (ISA+), then for each element of the agenda, there is some measure of independence. In particular, for any given element there is at least one *distinct* element upon which it does not depend.

Definition 4.3.10 (Strengthened Independence of Some Alternatives (ISA+))**.** An aggregator F satisfies *Strengthened Independence of Some Alternatives* if, for all $\chi \in X$, there exists $\varphi \in X$ (which may be dependent on χ), $\varphi \neq \chi$, such that, for all profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$.

Lemma 4.3.11 (ISA+)**.** *An aggregator F satisfies (ISA+) if and only if, for all $\chi \in X$, there exists some $\varphi \in X$ with $\varphi \neq \chi$, such that $\varphi \in I(\chi)$.*

Proof.

⇒ Assume, towards a contradiction, that F satisfies (ISA+) and the second condition fails, i.e., that for some $\chi \in X$, for all $\varphi \in X$ with $\varphi \neq \chi$, $\varphi \in D(\chi)$.

Since $\chi \in X$, χ is subject to (ISA+). Thus, there exists $\varphi \in X$ with $\varphi \neq \chi$, satisfying the following: For any two profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$.

By the failure of the second condition, for this $\varphi \neq \chi$, $\varphi \in D(\chi)$; that is, there exist two profiles $\{A''_i\}$ and $\{A'''_i\}$ such that, for all $\beta \neq \varphi$, $A''(\beta) = A'''(\beta)$, and $A''(\chi) \neq A'''(\chi)$. But then this satisfies the antecedent of (ISA+), so $A''(\chi) = A'''(\chi)$. Contradiction.

$\Leftarrow$ Assume the second condition holds, i.e., for all $\chi \in X$, there exists some $\varphi \in X$ with $\varphi \neq \chi$, such that $\varphi \in I(\chi)$. Assume, towards a contradiction, that F does not satisfy (ISA+).

By the failure of (ISA+), there exists some $\chi \in X$ such that, for all $\varphi \in X$ such that $\varphi \neq \chi$, there exist two profiles $\{A_i\}$, $\{A'_i\}$ such that, for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'_i(\alpha)$ and $A(\chi) \neq A'(\chi)$.

But since $\chi \in X$, the independence of the second condition is satisfied by $\{A_i\}$, $\{A'_i\}$ for some $\varphi \neq \chi$. So, since for all $i \in N$, for all $\alpha \neq \varphi$, $A_i(\alpha) = A'_i(\alpha)$, by the definition of independence, $A(\chi) = A'(\chi)$. Contradiction. □

4.3.5 Weakest Independence

If an aggregator satisfies (WI), then for *some* element of the agenda, there is some measure of independence. In particular, for that particular element, there is at least one (not necessarily distinct) element upon which it does not depend.

Definition 4.3.12 (Weakest Independence (WI)). An aggregator F satisfies *Weakest Independence* if, for some $\chi \in X$, there exists $\varphi \in X$ such that, for any two profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$ and for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$.

This condition is extremely weak: It eliminates the possibility that all elements in the agenda are dependent on all the elements in the agenda. (WI) has the same relation to (ISA) as (SI) has to (I). It is the existential version of (ISA); instead of holding that every element has some independence, (WI) holds that *some* element has some independence. Thus, any aggregator which satisfies (ISA) trivially satisfies (WI) and (SI) also implies (WI).

Lemma 4.3.13 (WI). *An aggregator F satisfies (WI) if and only if, for some $\chi \in X$, there exists some $\varphi \in X$ such that $\varphi \in I(\chi)$.*

Proof.

$\Rightarrow$ Assume, towards a contradiction, that F satisfies (WI) and the second condition fails, i.e., that for all $\chi \in X$, for all $\varphi \in X$, $\varphi \in D(\chi)$.

By (WI), for some $\chi \in X$, there exists $\varphi \in X$ such that, for any two profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$ and for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$, then $A(\chi) = A'(\chi)$. But since $\chi, \varphi \in X$, by the failure of the second condition, $\varphi \in D(\chi)$. By the definition of dependence, there exist two profiles $\{A''_i\}$, $\{A'''_i\}$ such that for all $i \in N$, for all $\beta \neq \varphi$, $A''_i(\beta) = A'''_i(\beta)$ and $A''(\chi) \neq A'''(\chi)$. But, by (WI), $A''(\chi) = A'''(\chi)$. Contradiction.

$\Leftarrow$ Assume the second condition holds, i.e., for some $\chi \in X$, there exists some $\varphi \in X$ such that $\varphi \in I(\chi)$. Assume, towards a contradiction, that F does not satisfy (WI).

For some χ, φ, $\varphi \in I(\chi)$. By the failure of (WI), for some profiles $\{A_i\}$, $\{A'_i\}$, for all $i \in N$ and for all $\alpha \neq \varphi$, $A_i(\alpha) = A'(\alpha)$ but $A(\chi) \neq A'(\chi)$. But since $\varphi \in I(\chi)$, by the definition of dependence, $A(\chi) = A'(\chi)$. Contradiction. □

4.3.6 Self Dependence

(ISA+) requires that, for a given element, there is some distinct element which it is independent of. However, this does not force the given element to be self-dependent. To specify that an element has itself in its own dependent set, it is necessary to define a further condition.

This condition, (SD), requires that an element is responsive to changes to individual positions on itself. This is a very natural condition since it is natural to assume that some individual evaluation changes on a given element affect the social evaluation of that element, but we were unable to find it in this form in the JA literature.[3]

Definition 4.3.14 (Self Dependence (SD))**.** An aggregation F satisfies *Self Dependence* if, for every $\chi \in X$, there exist two profiles $\{A_i\}$, $\{A'_i\}$ such that, for all $i \in N$ and for all $\alpha \neq \chi$, $A_i(\alpha) = A'_i(\alpha)$ and $A(\chi) \neq A'(\chi)$.

Note that (SD) is logically independent of every other condition listed here. It neither implies, or is implied by any of them. For the former, consider an aggregator which takes the full agenda to determine every position. It may satisfy (SD), but it will not satisfy any of the other weak independence conditions in this chapter. For the latter, consider constant aggregators. Constant aggregators satisfy all of the other weak independence conditions, but do not satisfy (SD).

[3](WI) is bears an interesting connection to Pauly & van Hees's (2006, p. 578) condition of Weak Responsiveness. Weak Responsiveness specifies a more minimal dependence. Weak Responsiveness is the claim that there exists $\chi \in X$ such that, for some $\{A_i\}$, $\{A'_i\}$, $A(\chi) \neq A'(\chi)$. However, this condition does not specify that the profiles, except for the element χ, are equal, as (SD) does. In other words, Weak Responsiveness guarantees different social aggregation values, but (SD) is stronger: (SD) guarantees that the different social aggregation values will come from profile changes in χ itself.

Lemma 4.3.15 (SD)**.** *An aggregator F satisfies (SD) if and only if, for every $\chi \in X$, $\chi \in D(\chi)$.*

Proof.

⇒ Assume, towards a contradiction, that F satisfies (SD) and the second condition fails, i.e., that for some $\chi \in X$, $\chi \notin D(\chi)$.

Since the second condition fails, for some $\chi \in X$, $\chi \notin D(\chi)$. Thus, $\chi \in I(\chi)$. Since $\chi \in X$, χ is subject to (SD). Thus, there exist two profiles $\{A_i\}$, $\{A'_i\}$ such that, for all $i \in N$ and for all $\alpha \neq \chi$, $A_i(\alpha) = A'_i(\alpha)$ and $A(\chi) \neq A'(\chi)$. But $\{A_i\}$, $\{A'_i\}$ satisfy the antecedent of independence. Thus, by the definition of independence, $A(\chi) = A'(\chi)$. Contradiction.

⇐ Assume the second condition holds, i.e., for every $\chi \in X$, $\chi \in D(\chi)$. Assume, towards a contradiction, that F does not satisfy (SD).

Since F fails (SD), there exists $\chi \in X$ such that for every two profiles $\{A_i\}$, $\{A'_i\}$, if, for all $i \in N$, and for all $\alpha \neq \chi$, $A_i(\alpha) = A'_i(\alpha)$, then $A(\chi) = A'(\chi)$.

We have that $\chi \in D(\chi)$. By the definition of dependence, there exist two profiles $\{A''_i\}$, $\{A'''_i\}$ such that, for all $\alpha \neq \chi$, $A''_i(\alpha) = A'''_i(\alpha)$, and $A''(\chi) \neq A'''(\chi)$. But then $\{A''_i\}$, $\{A'''_i\}$ satisfy the antecedent of the previous paragraph. So $A''(\chi) = A'''(\chi)$. Contradiction. □

4.3.7 Trivial Implications

In the succeeding chapter, it will be of value to recall the implications which are evident from their dependency characterizations. We list them here for the reader's ease:

Fact 4.3.16. The following implications hold:

- (I) ⇒ (SI),
- (I) ⇒ (ISA+),
- (ISA+) ⇒ (ISA),
- (SI) ⇒ (WI), and,
- (ISA) ⇒ (WI).

Proof. These follow directly from Lemmas 4.3.5, 4.3.7, 4.3.9, 4.3.11 and 4.3.13. □

Note that (SD) is an outlier. It neither implies nor is implied by any of these other conditions; this is because (SD) is the only condition we list which requires certain elements are *within* dependent sets. All of the other conditions are characterized as *excluding* certain types of elements from dependency sets. For instance, when all dependent sets are empty (i.e. the aggregator is a constant function), all of the conditions except (SD) are satisfied.

Chapter 5

Independence Results in Judgment Aggregation

In this chapter, we consider the relationships that hold between the JA weakened independence conditions. Two overarching themes are that (i) (N) is a weak neutrality condition, which is demonstrated by its inability to generate implications between independence conditions while a stronger neutrality condition is sufficient to do so; and, (ii) the distinction between PA and JA mean that isolating elements for self-dependence is easier in PA than in JA. The latter discovery leads to an interesting impossibility result.

Our counterexample results in Section 5.1 show that, in contrast to Campbell & Kelly's (2007) results, weak independence conditions are separable under (N). In other words, the implications that Campbell & Kelly find do not hold in JA under (N). These counterexamples hold when $|X| = 2$. Subsection 5.1.3 shows that (ISA) does not imply (I), but the proof requires that $|X| \geq 3$. Thus, the proposition in Subsection 5.1.3 does not imply the earlier two propositions, although it happens to subsume the first one when given an agenda with cardinality at least three.

In Section 5.2, we consider why (N) is insufficient to generate such implications. The answer is that (N) is a weak neutrality condition: Since the aggregators can be specified to satisfy (N) by considering only intra-profile neutrality, (N) does not require that all elements of the agenda are handled symmetrically. Thus, (N) is not the JA equivalent of Campbell & Kelly's (2007) (P-N). We generate a counterpart for (P-N) in terms of agenda permutations and demonstrate that it is strictly stronger than (N).

After establishing the comparative weakness of (N), we demonstrate a (N) aggregator over non-simple agendas in Section 5.3. First, we illustrate with an example of such an aggregator with a society of cardinality two. Then we prove that its properties continue to hold for larger societies. This aggregator satisfies all of the JA weakened independence conditions from Chapter 4, except (SD) and

(ISA+).

In Section 5.4, we prove impossibility theorems, beginning with (SD). It turns out that (SD) *alone* generates impossibility over non-simple agendas. When aggregating over non-simple agendas, conjunctions cannot be checked in isolation from their conjuncts, and (SD) is strong enough to require that they can. This creates an interesting and surprising impossibility result.

Next, we turn to the stronger neutrality condition, which is the analogue of Campbell & Kelly's (2007) (P-N). This condition operates in terms of permutations of the agenda, and show that this generates implications that (N) was too weak for. Furthermore, since one of these implications is (SI) implying (I), we generate a stronger impossibility theorem than List & Pettit's (2002) Impossibility Theorem.

Finally, in Section 5.5, we summarize the implications that hold between the weakened independence conditions that have been determined, using Tables 5.6 and 5.7.

5.1 Independence Implication Counterexamples

In this section, we show that, under the neutrality condition (N) (and (A)), none of the weak independences have new implications (other than the trivial ones listed in Fact 4.3.16). This will lead to an alternative stronger neutrality condition in the succeeding section.

In Subsection 5.1.1, we provide a counterexample showing that both (SI) $\not\Rightarrow$ (I) and (SD) $\not\Rightarrow$ (I). In Subsection 5.1.2, we provide a counterexample showing that both (WI) $\not\Rightarrow$ (ISA) and (SD) $\not\Rightarrow$ (ISA). Both of these hold for societies of cardinality two. Finally, in Subsection 5.1.3, we provide a counterexample showing that, for societies of cardinality at least three, (SI) $\not\Rightarrow$ (I), (SD) $\not\Rightarrow$ (I), and (ISA+) $\not\Rightarrow$ (I). Thus, for societies with at least three members, the third counterexample subsumes the first one.

5.1.1 Counterexample to (SI) and (SD) implying (I)

Proposition 5.1.1. *There exist aggregators which satisfy (A), (N), (SD), and (SI), but fail (I).*

Proof. The following is a minimal example:

$X = \{a, b\}$, $N = \{1, 2\}$.

Let $|\varphi| := |\{\, i \in N \mid A_i(\varphi) = 1 \,\}|$, the size of the accepting group. Let F be defined as follows:

$$A(a) = \begin{cases} 1 & |a| \geq 1 \\ 0 & |a| < 1 \end{cases}$$

For any profile which satisfies $|a| = |b|$, b is accepted iff a is. In any profile where $|a| \neq |b|$, then b is accepted by the social judgment function A iff a is not accepted:

$$A(b) = \begin{cases} 1 & |b| = |a| \geq 1 \\ 0 & |b| = |a| < 1 \\ 0 & |b| \neq |a| \text{ and } |a| \geq 1 \\ 1 & |b| \neq |a| \text{ and } |a| < 1 \end{cases}$$

By this aggregator, we have that a is such that $b \notin D(a)$ and b is such that $D(b) = \{a, b\} = X$.

This aggregator is presented in the Table 5.1. Note that, in the table, the social pattern of acceptance 01 on b means that $A_1(b) = 0$ and $A_2(b) = 1$. Note also that, due to (A), only the *size* of the accepting group matters (and these are indicated in brackets).

Social Judgments		Individual Judgments	
$A(a)$	$A(b)$	$A_1(a), A_2(a)$ ($\|a\|$)	$A_1(b), A_2(b)$ ($\|b\|$)
0	0	00(0)	00(0)
1	1	01(1)	01(1)
1	1	10(1)	10(1)
1	1	11(2)	11(2)
1	1	01(1)	10(1)
1	1	10(1)	01(1)
0	1	00(0)	01(1)
0	1	00(0)	10(1)
0	1	00(0)	11(2)
1	0	01(1)	00(0)
1	0	01(1)	11(2)
1	0	10(1)	00(0)
1	0	10(1)	11(2)
1	0	11(2)	00(0)
1	0	11(2)	01(1)
1	0	11(2)	10(1)

Table 5.1: An (SI) and (SD) aggregator which fails (I)

1. (N) is satisfied: In every profile where, for all $i \in N$, $A_i(a) = A_i(b)$, we have $A(a) = A(b)$, namely in the top four profiles between the horizontal lines;

2. (A) is satisfied: Only the cardinality of the accepting groups bears on the $A(a)$ and $A(b)$;

3. (SI) is satisfied since there exists $\chi \in X$ such that $D(\chi) = \{\chi\}$, namely, $\chi = a$,[1] a depends only on the number of acceptances of a;

4. (SD) is satisfied since, for all $\chi \in X$, $\chi \in D(\chi)$. $a \in D(a)$ and $b \in D(b)$.

   To see the former, consider the first row (call it $\{A_i\}$) and the fourteenth row (third from the bottom) (call it $\{A'_i\}$). Both take the same positions on b, i.e. $A_i(b) = A'_i(b)$ for all $i \in N$, but $A(a) \neq A'(a)$.

   To see the latter, consider the first row (call it $\{A_i\}$) and the seventh row (call it $\{A'_i\}$). Both take the same positions on a, i.e. $A_i(a) = A'_i(a)$ for all $i \in N$, but $A(b) \neq A'(b)$; but,

5. (I) is not satisfied since there exists $\chi \in X$ and $\varphi \neq \chi$ such that $\varphi \in D(\chi)$, namely, $\chi = b$ and $\varphi = a$. b requires knowing what the acceptance pattern on a is (and then doing the opposite of a in the bottom twelve cases). For instance, consider the fifth profile and the sixteenth (bottom) profile: Although the individual positions on b are the same, the social positions on b differ since individuals differ on a. □

5.1.2 Counterexample to (WI) and (SD) implying (ISA)

Proposition 5.1.2. *There exist aggregators which satisfy (A), (N), (SD), and (WI), but fail (ISA).*

Proof. The following is a minimal example:
$X = \{a, b\}$, $N = \{1, 2\}$. Let F be defined as follows:

$$A(a) = \begin{cases} 1 & |a| \geq 1 \\ 0 & |a| = 0 \end{cases}$$

$$A(b) = \begin{cases} 1 & |a| + |b| \geq 2 \\ 0 & |a| + |b| < 2 \end{cases}$$

By this aggregator, $D(a) = \{a\}$ and $D(b) = \{a, b\} = X$. Such an aggregator is presented in Table 5.1.2:

1. (N) is satisfied: In every profile where for all $i \in N$, $A_i(a) = A_i(b)$, we have $A(a) = A(b)$, namely in the top four profiles between the horizontal lines;

2. (A) is satisfied since only the cardinality of accepting individuals is needed to determine social acceptance;

[1] By equality, we actually intend an instantiation relation (e.g. (SI) is true because a is a witness), but trust that this is sufficiently intuitive. I also slightly abuse notation in this manner throughout the following sections.

Social Judgments		Individual Judgments	
$A(a)$	$A(b)$	$A_1(a), A_2(a)$ ($\lvert a\rvert$)	$A_1(b), A_2(b)$ ($\lvert b\rvert$)
0	0	00 (0)	00 (0)
1	1	01 (1)	01 (1)
1	1	10 (1)	10 (1)
1	1	11 (2)	11 (2)
1	1	01 (1)	10 (1)
1	1	10 (1)	01 (1)
0	0	00 (0)	01 (1)
0	0	00 (0)	10 (1)
0	1	00 (0)	11 (2)
1	0	01 (1)	00 (0)
1	1	01 (1)	11 (2)
1	0	10 (1)	00 (0)
1	1	10 (1)	11 (2)
1	1	11 (2)	00 (0)
1	1	11 (2)	01 (1)
1	1	11 (2)	10 (1)

Table 5.2: A (WI) and (SD) aggregator which fails (ISA)

3. (SD) is satisfied since for every $\chi \in X$, $\chi \in D(\chi)$. In particular, $a \in D(a)$ and $b \in D(b)$. The former is clear from inspecting F. The latter can be shown by considering the first row (call it $\{A_i\}$) and the ninth row (call it $\{A_i'\}$). We have that $A_i(a) = A_i'(a)$ for all $i \in N$, but $A(b) \neq A'(b)$;

4. (WI) is satisfied since there exists $\chi \in X$ and $\varphi \in X$ such that $\varphi \in I(\chi)$, since $D(a) = \{a\}$, and $I(a) = \{b\}$; but,

5. (ISA) fails because there exists $\chi \in X$ such that $I(\chi) = \varnothing$, namely $\chi = b$.

   To see that $a \in D(b)$, consider the first row (which we call $\{A_i\}$) and the fourteenth row—third from the bottom (which we call $\{A_i'\}$). We have that $A_i(b) = A_i'(b)$ for each $i \in N$ [and that $A_i(a) \neq A_i'(a)$ for each $i \in N$] but $A(b) \neq A'(b)$. b's acceptance in the social set depends on a, not just on the social positions of b.

   This is so because in (A_1, A_2), $|a| + |b| \geq 2$ but in (A_1', A_2'), $|a| + |b| < 2$. □

5.1.3 Counterexample to (ISA+) and (SD) implying (I)

Proposition 5.1.3. *Suppose that $|X| \geq 3$. Then, there exist aggregators which satisfy (A), (N), (SD), and (ISA+), but fail (I).*

Proof. The following is a minimal example:

$X = \{a, b, c\}$, $N = \{1, 2\}$. Let F be defined as follows:

$$A(a) = \begin{cases} 1 & |a| = 1 \\ 0 & |a| \neq 1 \end{cases}$$

$$A(b) = \begin{cases} 1 & |a| = |b| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A(c) = \begin{cases} 1 & |a| = |c| = 1 \\ 0 & \text{otherwise} \end{cases}$$

By this aggregator, $D(a) = \{a\}$, $D(b) = \{a, b\}$, and $D(c) = \{a, c\}$.

In order to simplify the table for F, we offer shorthand measures. Whenever all of $|a| \neq 1$, $|b| \neq 1$ and $|c| \neq 1$, then all social judgment functions A take 0, so these rows are suppressed from the table.

Note that 0 on $|\varphi|$ indicates *any* $\{A_i\}$ such that the acceptance on φ is *not* singly accepted; that is, any profile which does not have either $A_1(\varphi) = 1$ and $A_2(\varphi) = 0$ or $A_1(\varphi) = 0$ and $A_2(\varphi) = 1$.

Furthermore, we only express the cardinalities of the acceptances of the individual positions. Thus, some rows represent more than one profile but, by (A), permuting individuals does not change the social judgment function. Thus, each "individual position" (such as $|a| = 1$) actually represents different permutations of individual judgment functions.[2]

The aggregator is presented in part in Table 5.3.

1. (A) is satisfied since only the cardinality of accepting individuals is needed to determine social acceptance;

2. (N) is satisfied since, in every profile (row), when the individual positions are identical, then the social judgment function must take the same value. This is indicated by the underlined positions, which must take the same value intra-profile;

[2] For instance, $|a| = 1$ represents (a) $\{A_i\}$ where $A_1(a) = 1$, $A_2(a) = 0$ and (b) $\{A'_i\}$ where $A'_1(a) = 0$, $A_2(a) = 1$, but the rest of the row would not change between these two profiles $\{A_i\}$ and $\{A'_i\}$, so we represent both profiles with one row.

Social Judgments			Individual Judgments		
$A(a)$	$A(b)$	$A(c)$	$\lvert a \rvert$	$\lvert b \rvert$	$\lvert c \rvert$
⋮	⋮	⋮	⋮	⋮	⋮
$\underline{0}$	$\underline{0}$	0	0	0	1
$\underline{0}$	0	$\underline{0}$	0	1	0
0	$\underline{0}$	$\underline{0}$	0	1	1
1	$\underline{0}$	$\underline{0}$	1	0	0
$\underline{1}$	0	$\underline{1}$	1	0	1
$\underline{1}$	$\underline{1}$	0	1	1	0
$\underline{1}$	$\underline{1}$	$\underline{1}$	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮

Table 5.3: An (ISA+) and (SD) aggregator which fails (I)

3. (SD) is satisfied since for all $\chi \in X$, $\chi \in D(\chi)$. $D(a) = \{a\}$, $D(b) = \{a,b\}$, and $D(c) = \{a,c\}$. These dependent sets can easily be checked from the definitions of $A(a)$, $A(b)$ and $A(c)$;

4. (ISA+) is satisfied since for all $\chi \in X$, there exists $\varphi \in X$ with $\varphi \neq \chi$ such that $\varphi \in I(\chi)$. This is so because $I(a) = \{b,c\}$, $I(b) = \{c\}$, and $I(c) = \{b\}$; and,

5. (I) is clearly not satisfied, there exists $\chi \in X$ and some $\varphi \neq \chi$ such that $\varphi \in D(\chi)$. This is satisfied by $\varphi = b$ or $\varphi = c$ for $\chi = a$, since $D(b) = \{a,b\}$ and $D(c) = \{a,c\}$.

   To show a couple of these dependencies (i.e. $a \in D(b)$ $a \in D(c)$), consider the third row and the final row. Call these rows $\{A_i\}$ and $\{A'_i\}$, respectively. We have that, for all $\gamma \neq a$, $A_i(\gamma) = A'_i(\gamma)$. But $A(b) = A(c) = 0$ and $A'(b) = A'(c) = 1$. Thus, both b and c depend on a. □

There are two things to note about this aggregator. The first is that stronger claims follow. The aggregator suggested in this lemma does satisfy (ISA+) and (SD), the strongest of the weakened independence conditions, showing that neither (ISA+) nor (SD) imply (I).[3] But stronger claims can be made of this aggregator: In fact, it satisfies *all* of our weakened JA independence conditions from Section 4.3 while not implying full independence, (I). Consider the following:

- (WI) is satisfied trivially since any (ISA+) aggregator is (WI);

[3] Note that Theorem 5.4.1 does preclude a (SD) aggregator over a *non-simple* agenda, but this agenda is simple since a, b and c are logically independent.

- (SI) is satisfied since there is a $\chi \in X$ such that for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$, namely χ itself since $D(\chi) = \{\chi\}$; and,
- (ISA) is satisfied trivially since any (ISA+) aggregator is (ISA).

The other thing to note is that the agenda X must satisfy $|X| \geq 3$, unlike the previous lemmata. When $|X| < 3$, (ISA+) and (SD) do imply (I). This is trivial when $|X| = 1$. When $|X| = 2$, by (SD), arbitrary $\varphi \in X$ satisfies $\varphi \in D(\varphi)$. For $\psi \neq \varphi$, we cannot have $\psi \in D(\varphi)$, or else $D(\varphi) = \{\psi, \varphi\} = X$, contradicting (ISA+), so $D(\varphi) = \{\varphi\}$. Thus, (I) holds.

5.2 Neutrality

Since the implications that Campbell & Kelly find do not hold with (N) and (A), it is of interest to consider why. The key is that the neutrality condition (P-N) which Campbell & Kelly employ is not equivalent to the JA (N) condition we consider. In this section, we construct the analogous condition in JA terms and then show that (N) is strictly weaker than this new strengthened condition. We begin by defining a condition of neutrality which is the JA analogue to (P-N), and then proceed to show that (N) does not imply it.

Given a permutation of the agenda: $\mu : X \to X$, permuting the judgment function A as applied to φ is defined as applying the judgment function to the permutation of φ:

$$\mu(A)(\varphi) \coloneqq A(\mu(\varphi)).$$

Then we can define a profile of such permuted relations as follows:

$$\mu(\{A_i\}) \coloneqq \{\mu(A_i)\}.$$

In other words, permuting a profile means generating a profile where for each $i \in N$, each individual judgment function A_i is permuted. With this concept in hand, one can define neutrality to mean invariance under permutation:

Definition 5.2.1 (Neutrality by Permutation (N-PERM))**.** An aggregator F satisfies *Neutrality by Permutation* if, for every profile $\{A_i\}$ and every permutation $\mu : X \to X$, $\mu(\{A_i\})$ is in the domain of F and $F(\mu(\{A_i\})) = \mu(F(\{A_i\}))$.

(N-PERM) implies (N).[4] To see this, consider an arbitrary (N-PERM) aggregator F. Given any profile $(A_1, \dots, A_n)$ such that for all $i \in N$, $A_i(\varphi) = A_i(\psi)$,

[4](N-PERM) is the same concept of neutrality that van Hees (2007, p. 661) considers. By van Hees's (2007) Proposition 4.1, we see that Systematicity implies (N-PERM) and (I). From the claim that (N-PERM) implies (N), Systematicity implies (N).

construct a permutation μ which maps φ to ψ and vice versa but leaves the rest of the agenda alone. By (N-PERM),

$$F_\varphi(\{A_i\}) = F_\varphi(\mu(\{A_i\})) \text{ [since } A_i(\varphi) = A_i(\psi)] = \mu(F_\varphi(\{A_i\})) = F_\psi(\{A_i\}),$$

so F is (N).

Although (N-PERM) implies (N), we demonstrate that the converse does not hold:

Proposition 5.2.2. *(N) does not imply (N-PERM).*

Proof. $X = \{a, b\}$, $N = \{1, 2\}$.

Let F be defined as follows:

$$A(a) = \begin{cases} 1 & |a| \geq 1 \\ 0 & |a| = 0 \end{cases}$$

$$A(b) = \begin{cases} 1 & |a| + |b| \geq 2 \\ 0 & |a| + |b| < 2 \end{cases}$$

Such an aggregator is presented in Table 5.4.

Social Judgments		Individual Judgments	
$A(a)$	$A(b)$	$A_1(a), A_2(a)$ ($\vert a\vert$)	$A_1(b), A_2(b)$ ($\vert b\vert$)
0	0	00 (0)	00 (0)
1	1	01 (1)	01 (1)
1	1	10 (1)	10 (1)
1	1	11 (2)	11 (2)
1	1	01 (1)	10 (1)
1	1	10 (1)	01 (1)
0	0	00 (0)	01 (1)
0	0	00 (0)	10 (1)″
0	1	00 (0)	11 (2)
1	0	01 (1)	00 (0)
1	1	01 (1)	11 (2)
1	0	10 (1)	00 (0)′
1	1	10 (1)	11 (2)
1	1	11 (2)	00 (0)
1	1	11 (2)	01 (1)
1	1	11 (2)	10 (1)

Table 5.4: An (N) aggregator which fails (N-PERM)

1. (N) is satisfied: In every profile where for all $i \in N$, $A_i(\varphi) = A_i(\psi)$, we have $A(\varphi) = A(\psi)$, namely in the top four profiles between the horizontal lines.

2. (N-PERM) fails. *E.g.* consider the twelfth row $\{A'_i\}$ (which we indicate with a prime): In this row, 10(1) for a and 00(0) for b. This yields $A'(a) = 1$, $A'(b) = 0$.

   Permuting with μ, $\mu(a) = b$, $\mu(b) = a$ (the only permutation), we generate the profile $\{A''_i\}$ in the eighth row (which we indicate with a double prime): 00(0) for a and 10(1) for b. This yields $A''(a) = 0$, $A''(b) = 0$. Clearly, A'' is not a permutation of A'. □

(A) is not necessary for this counterexample to work (although of course it is satisfied). The antecedent of (N) is satisfied for the first four rows where each individual judgment function coincides, i.e. $A_i(a) = A_i(b)$. The cardinality of acceptances are included in the table only because it is simpler to check the values of $A(a)$ and $A(b)$.

Now that we have established that (N) is weaker than (N-PERM), we show that it is weak enough to sustain possibility theorems under our weakened independence conditions.

5.3 Possibility Theorems

In this section, we present two possibility results under (N). As shown already in this chapter, the conditions (WI), (SI), (ISA) and (ISA+) do not imply the stricter (I). However, this does not mean that JA impossibility results will not succeed (they may not require conditions as strong as (I) plus (N)). In this section, we test the associated claims in terms of weaker independence conditions.

Recall, by the Propositions 5.1.1, 5.1.2, and 5.1.3 in the previous sections, that (WI) and (SI) do not imply stronger forms of independence. We now turn to how List & Pettit's (2002) Impossibility Theorem fares when these weaker forms of independence replace the strong (I).

In Subsection 5.3.1, we introduce an example of an aggregator which satisfies all of our weakened independence conditions over a non-simple agenda for a society of cardinality two which demonstrates possibility. Then, in Subsection 5.3.2, we prove that this aggregator suffices over any finite cardinality society and, thus, show possibility.

5.3.1 An Example with Weak Independences and Neutrality

We begin by introducing a premise-based aggregator[5] over a non-simple agenda X as follows:

$X = \{a, b, a \wedge b\}$, $N = \{1, 2\}$. Let F be defined as follows:

$$A(a) = \begin{cases} 1 & |a| \geq 1 \\ 0 & |a| = 0 \end{cases}$$

$$A(b) = \begin{cases} 1 & |b| \geq 1 \\ 0 & |b| = 0 \end{cases}$$

$$A(a \wedge b) = \begin{cases} 1 & |a| \geq 1 \text{ and } |b| \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

By this aggregator, $D(a) = \{a\}$, $D(b) = \{b\}$ and $D(a \wedge b) = \{a, b\}$. [Furthermore, $I(a) = \{b, a \wedge b\}$, $I(b) = \{a, a \wedge b\}$, and $I(a \wedge b) = \{a \wedge b\}$.] The aggregator is presented in Table 5.5.

Note that not every combination in the profile is *logically* possible because, for instance, an individual $i \in N$ cannot have $A_i(a \wedge b) = A_i(a) = 1$ and $A_i(b) = 0$. Thus, the number of admissible profiles is restricted by the logical constraints on the agenda.

Each social judgment function is complete and consistent. The completeness is evident. The consistency follows from $A(a \wedge b) = 1$ if and only if $A(a) = 1$ and $A(b) = 1$ for each profile $\{A_i\}$. Again, this is because this is a premise-based aggregator.

Now consider which conditions this aggregator satisfies. It is useful to refer to the dependent sets in order to confirm the weak independence conditions are satisfied. We reproduce them here: $D(a) = \{a\}$, $D(b) = \{b\}$ and $D(a \wedge b) = \{a, b\}$. [Furthermore, $I(a) = \{b, a \wedge b\}$, $I(b) = \{a, a \wedge b\}$, and $I(a \wedge b) = \{a \wedge b\}$.]

1. (A) is satisfied since only the cardinality of accepting individuals is needed to determine social acceptance;

[5] Although this may appear to be different from a normal premise-based aggregator, which takes a majoritarian approach over premise agenda elements and then takes their logical implications, it does encompass such aggregation with premises a, b and conclusion $a \wedge b$. Firstly, the level of acceptance (here, one) is a weak majority in a society of size two. This level, as I mention in Section 6.2, is adjustable. This is actually a class of aggregators since uniformly replacing 1 with any other natural number yields an aggregator which also gives consistent complete outputs. Secondly, although most premise-based aggregators just specify that the consequences of the premises hold, this actually does the same. It just does so for $a \wedge b$ in an explicit manner.

Social Judgment Functions			Individual Judgment Functions		
$A(a)$	$A(b)$	$A(a \wedge b)$	$A_1, A_2(a)$ $(\lvert a \rvert)$	$A_1, A_2(b)$ $(\lvert b \rvert)$	A_1, A_2 $(\lvert a \wedge b \rvert)$
$\underline{0}$	$\underline{0}$	$\underline{0}$	00(0)	00(0)	00(0)
$\underline{0}$	1	$\underline{0}$	00(0)	01(1)	00(0)
$\underline{0}$	1	$\underline{0}$	00(0)	10(1)	00(0)
$\underline{0}$	1	$\underline{0}$	00(0)	11(2)	00(0)
1	$\underline{0}$	$\underline{0}$	01(1)	00(0)	00(0)
$\underline{1}$	$\underline{1}$	$\underline{1}$	01(1)	01(1)	01(1)
1	1	1	01(1)	10(1)	00(0)
$\underline{1}$	1	$\underline{1}$	01(1)	11(2)	01(1)
1	$\underline{0}$	$\underline{0}$	10(1)	00(0)	00(0)
$\underline{1}$	$\underline{1}$	$\underline{1}$	10(1)	10(1)	10(1)
1	1	1	10(1)	01(1)	00(0)
$\underline{1}$	1	$\underline{1}$	10(1)	11(2)	10(1)
1	$\underline{0}$	$\underline{0}$	11(2)	00(0)	00(0)
1	$\underline{1}$	$\underline{1}$	11(2)	01(1)	01(1)
1	$\underline{1}$	$\underline{1}$	11(2)	10(1)	10(1)
$\underline{1}$	$\underline{1}$	$\underline{1}$	11(2)	11(2)	11(2)

Table 5.5: Possibility over (WI), (SI) and (ISA) aggregators

2. (N) is satisfied since, in every profile, when the individual positions are identical, then the social judgment function must take the same value. This is indicated by the underlined positions, which indicate where in the profiles the antecedent of (N) is satisfied for that pair (or triple) of elements, and where, therefore, the social aggregation function must take the same value;

3. (SI) is satisfied since there is a $\chi \in X$ such that for all $\varphi \neq \chi$, $\varphi \notin D(\chi)$ (i.e. $\varphi \in I(\chi)$). This holds for both $\chi = a$ and $\chi = b$ since neither are dependent on other elements of the agenda, i.e $I(a) = \{b, a \wedge b\}$, $I(b) = \{a, a \wedge b\}$;

4. (ISA) is satisfied since for any given $\chi \in X$, there exists φ such that $\varphi \in I(\chi)$. For $\chi = a$, this holds for $\varphi = b$ ($b \in I(a)$); for $\chi = b$, this holds for $\varphi = a$ ($a \in I(b)$), and for $\chi = a \wedge b$, this holds for $\varphi = a \wedge b$, *i.e.* $a \wedge b \in I(a \wedge b)$;

5. (WI) is satisfied trivially, since (ISA) implies (WI); and,

6. (I) is not satisfied, there exists $\chi \in X$ such that for some $\varphi \neq \chi$, $\varphi \in D(\chi)$, namely, $\chi = a \wedge b$, and $\varphi = a$ (or $\varphi = b$). This holds because $D(a \wedge b) = \{a, b\}$.

5.3.2 Possibility with Weak Independences

Now, we show that the example in Subsection 5.3.1, which holds for $|N| = 2$, holds more generally for larger societies, allowing for two possibility theorems.

Theorem 5.3.1. *Let X be a non-simple agenda. There exist universal (A), (N), and (SI) aggregators F over X which produce consistent and complete social judgment functions.*

Proof. The aggregator F in Subsection 5.3.1 satisfies these conditions for $|N| = 2$. It remains to show that when $|N| \geq 3$, (A), (N), (SI) and the completeness and consistency of the output still holds. We treat these in turn.

Completeness is trivial: $A(a)$, $A(b)$ and $A(a \wedge b)$ are defined for any profile of complete individual judgment functions.

Consistency is also maintained for larger societies, due to our definitions of $A(a)$, $A(b)$ and $A(a \wedge b)$. We demonstrate why. There are two situations in which the social judgment function would be inconsistent over X. We will show that neither of these is possible for any profile of any finite[6] cardinality of N:

1. $A(a \wedge b) = 0$, $A(a) = 1$ and $A(b) = 1$. In this case, by the definition of $A(a)$ and $A(b)$, there exists $i \in N$ such that $A_i(a) = 1$ and there exists $i' \in N$ such that $A_{i'}(b) = 1$. But then, by the definition of $A(a \wedge b)$, $A(a \wedge b) = 1$. Contradiction.

2. $A(a \wedge b) = 1$, $A(a) = 0$ or $A(b) = 0$. In this case, by the definition of $A(a \wedge b)$, there exists some $i \in N$ such that $A_i(a) = 1$ and some $i' \in N$ such that $A_{i'}(b) = 1$. But then, by the definition of $A(a)$ and $A(b)$, $A(a) = 1$ and $A(b) = 1$. Contradiction.

(A) still holds, it does not depend on the size of $|N|$.

It is a simple exercise to show that (N) continues to hold for $|N| \geq 3$. Note that any time $A_i(a) = A_i(b)$, $A(a) = A(b)$ by the symmetric specifications of $A(a)$ and $A(b)$. Now consider a profile $\{A_i\}$, for $|N| \geq 3$, where (N) fails for a and $a \wedge b$ (b and $a \wedge b$ is handled similarly).

In such a profile, for all $i \in N$, $A_i(a) = A_i(a \wedge b)$ and either $A(a) = 1$ and $A(a \wedge b) = 0$ or $A(a) = 0$ and $A(a \wedge b) = 1$. The second is impossible because $A(a \wedge b) = 1$ requires that $|a| \geq 1$ and, thus, $A(a) = 1$. The first fails because then $|b| = 0$, but since some individual $i \in N$ has $A_i(a \wedge b) = 1$ (recall that $A_i(a) = A_i(a \wedge b)$), by consistency and completeness of A_i, $A_i(b) = 1$. Contradiction.

(SI) still holds as it is defined in terms of the dependent sets, which do not depend on $|N|$. □

[6] Technically, this also holds both for ω-sized societies and $|N| = 1$.

Theorem 5.3.2. *Let X be a non-simple agenda. There exist universal (A), (N), and (ISA) aggregators F over X which produce consistent and complete social judgment functions.*

Proof. The preceding example, which holds for all $|N| \geq 2$, once again suffices.

Note that (ISA), like (SI), continues to hold for higher cardinality societies since, the dependent sets of a, b and $a \wedge b$ are do not vary as $|N|$ changes. □

However, these theorems do not exhaust our list of weak independence conditions. In particular, this aggregator does not satisfy (ISA+), since there is no distinct element that $a \wedge b$ is independent of (recall that (ISA+) requires that some independent element be distinct). Furthermore, it does not satisfy (SD), since, once again, $a \wedge b$ is not self-dependent. We will turn to (SD) in the next section. However, we postpone discussion of (ISA+) until the conclusion of the thesis, for future work, since we were unable to prove our conjecture that (ISA+) implies (I) under (N-PERM). We were, however, able to show that (SD) is sufficient for impossibility, and we turn to this in Subsection 5.4.1.

5.4 Impossibility Theorems

I begin in Subsection 5.4.1 by mentioning an interesting impossibility that was discovered while trying to find aggregators which are self-dependent. I show that (SD) by itself generates impossibility, since, over a non-simple agenda, a conjunction cannot be excavated from the truth-value of its conjuncts.

In Subsection 5.4.2, I use (N-PERM) to prove implications between existential and universal versions of weak independence conditions. This leads to an impossibility theorem using (SI) and (N-PERM).

5.4.1 The Strength of (SD)

Since one condition that the aggregators in the possibility theorems of Subsection 5.3.2 fail is (SD), it is natural to consider whether (SD) generates impossibility. We show here that under very mild conditions, (SD) does so. This shows that (SD) is a significantly stronger condition than the other weakenings.

We prove the theorem for the weak technical condition $\varphi \wedge \psi \in D(\varphi \wedge \psi)$,[7] and then leave the version with (SD), which is significantly more intuitive, as a corollary.

[7]Note that this does not mean $\varphi \in D(\varphi \wedge \psi)$ and $\psi \in D(\varphi \wedge \psi)$; instead, the element of the agenda $\varphi \wedge \psi$ is in its own dependent set.

Theorem 5.4.1. *Let X be a non-simple agenda. Then there is no universal aggregator F over X, satisfying $\varphi \wedge \psi \in D(\varphi \wedge \psi)$ and generating consistent and complete social judgment functions.*

Proof. We have that $\varphi \wedge \psi \in D(\varphi \wedge \psi)$.

By the definition of dependent sets, there exist two profiles $\{A_i\}$ and $\{A'_i\}$ such that for all $\alpha \neq \varphi \wedge \psi$; for all $i \in N$, $A_i(\alpha) = A'_i(\alpha)$; and $A(\varphi \wedge \psi) \neq A'(\varphi \wedge \psi)$.

Consider any admissible profile such that, for all $i \in N$, $A_i(\varphi) = A'_i(\varphi)$ and $A_i(\psi) = A'_i(\psi)$. We will show that they must coincide on $\varphi \wedge \psi$ as well. By the consistency and completeness of the profiles, we have that $A_i(\varphi \wedge \psi) = 1$ if and only if both $A_i(\varphi) = 1$ and $A'_i(\psi) = 1$.

There are four possibilities over the individual judgment functions with regard to φ and ψ. By assumption, for all $i \in N$ $A_i(\varphi) = A'_i(\varphi)$ and $A_i(\psi) = A'_i(\psi)$; so, in each of the following four cases, what applies to $\{A'_i\}$ also applies to $\{A_i\}$:

1. $A_i(\varphi) = A_i(\psi) = 0$,

2. $A_i(\varphi) = 1$, $A_i(\psi) = 0$,

3. $A_i(\varphi) = 0$, $A_i(\psi) = 1$, and

4. $A_i(\varphi) = A_i(\psi) = 1$.

If any of the first three cases obtain, then by the completeness and consistency of $\{A_i\}$, $\{A'_i\}$, $A_i(\varphi \wedge \psi) = A'_i(\varphi \wedge \psi) = 0$. If the final case obtains, then $A_i(\varphi \wedge \psi) = A'_i(\varphi \wedge \psi) = 1$.

Either way, for all $i \in N$ and for all $\varphi \in X$, $A_i(\varphi) = A'_i(\varphi)$. Thus, $A(\varphi \wedge \psi) = A'(\varphi \wedge \psi)$. But, as previously shown, by the definition of dependence, $A(\varphi \wedge \psi) \neq A'(\varphi \wedge \psi)$. This contradiction proves the theorem. □

Corollary 5.4.2. Let X be a non-simple agenda. Then there is no universal aggregator F over X, satisfying (SD) and generating consistent and complete social judgment functions.

5.4.2 Impossibility with (N-PERM)

As shown in Section 5.2, (N-PERM), the analogue of Campbell & Kelly's (P-A), is strictly stronger than (N). So it is worth considering whether independence implications using (N-PERM) succeed for JA as they do in PA.

Since Systematicity is generated by (I) together with (N-PERM), the following theorem also represents a strengthening of List & Pettit's (2002) Impossibility Theorem.

Lemma 5.4.3. *If F has full domain, satisfies (N-PERM) and (SI), then F satisfies (I).*

Proof. By Lemma 4.3.7, if F satisfies (SI), then for some $\varphi \in X$, for all $\alpha \neq \varphi$, $\alpha \notin D(\varphi)$. Consider arbitrary $\psi \in X$. We will show, for all $\beta \neq \psi$, $\beta \notin D(\psi)$.

Suppose not. Construct $\mu : X \to X$ such that $\mu(\varphi) = \psi$ (μ maps all the non-φ elements to non-ψ elements as well). By the supposition, there is some $\beta \neq \psi$ such that $\beta \in D(\psi)$. For simplicity, let this be β, and call the element α (where $\alpha \neq \varphi$) such that $\mu(\alpha) = \beta$.

By the definition of dependency, there exists profiles $\{A_i\}$, $\{A'_i\}$ such that for all $i \in N$, and each $\gamma \neq \beta$, $A_i(\gamma) = A'_i(\gamma)$, but $A(\psi) \neq A'(\psi)$.

Since $A(\psi) \neq A'(\psi)$, by (N-PERM), $\mu^{-1}(A(\psi)) \neq \mu^{-1}(A'(\psi))$. This yields $A(\mu^{-1}(\psi)) \neq A'(\mu^{-1}(\psi))$; thus, $A(\varphi) \neq A'(\varphi)$. On the profiles, we get $\mu^{-1}(\{A_i\})$ and $\mu^{-1}(\{A'_i\})$. It follows that for all $\gamma \neq \alpha$, $A_i(\gamma) = A'_i(\gamma)$. By the definition of dependence, $\alpha \in D(\varphi)$. This contradiction completes the proof. □

Theorem 5.4.4. *Let X be non-simple. There is no universal aggregator F over X producing consistent and complete judgment functions such that F satisfies (A), (N-PERM), and (SI).*

Proof. By Lemma 5.4.3, F satisfies (I). Since F satisfies (I) and (N-PERM), F satisfies Systematicity. The theorem then follows from Proof 3.4.1. □

Proposition 5.4.5. *If F satisfies (WI) and (N-PERM), then F also satisfies (ISA).*

Proof. By (WI) and Lemma 4.3.13, for some element $\chi \in X$, there exists some $\varphi \in X$ such that $\varphi \notin D(\chi)$. So for arbitrary $\psi \in X$, let $\mu : X \to X$ be a permutation satisfying $\mu(\chi) = \psi$, and let α be the element of the agenda such that $\mu(\varphi) = \alpha$. We will show that $\alpha \notin D(\psi)$, that is, F satisfies (ISA).

Assume, towards a contradiction, that this is not so. Then $\alpha \in D(\psi)$; thus, by the definition of dependence, there are profiles $\{A_i\}$, $\{A'_i\}$ such that for all $\gamma \neq \alpha$, $A_i(\gamma) = A'_i(\gamma)$, but $A(\psi) \neq A'(\psi)$. Now we invert μ.

Since $A(\psi) \neq A'(\psi)$, by (N-PERM), $\mu^{-1}(A(\psi)) \neq \mu^{-1}(A'(\psi))$. This yields $A(\mu^{-1}(\psi)) \neq A'(\mu^{-1}(\psi))$; thus, $A(\chi) \neq A'(\chi)$. On the profiles, we get $\mu^{-1}(\{A_i\})$ and $\mu^{-1}(\{A'_i\})$. It follows that for all $\gamma \neq \varphi$, $A_i(\gamma) = A'_i(\gamma)$. By the definition of dependence, $\varphi \in D(\chi)$. This contradiction completes the proof. □

You can see that strengthening (N) to (N-PERM) allows stronger results. (WI) with (N-PERM) does imply (ISA), although, as shown previously in 5.1.1, (WI) with (N) does not.

5.5 Summary of Independence Implications

In this section, we summarize the implications between the weakened JA independence conditions. We begin with implications under weak neutrality (N) in Table 5.6. The implications are read in rows, so the fourth cell in the second row is read "(I) together with (N) implies (ISA)." This cell is trivial, since, by Fact 4.3.16, (I) implies (ISA) by itself.

Note that, under (N), the *only* implications that hold are those which hold by Fact 4.3.16.

$\Rightarrow$	(I)	(ISA+)	(ISA)	(SI)	(WI)	(SD)
(I)	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
(ISA+)	$\times^3$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$
(ISA)	$\times^3$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$
(SI)	$\times^1$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\times$
(WI)	$\times^3$	$\times^2$	$\times^2$	$\times$	$\checkmark$	$\times$
(SD)	$\times^3$	$\times^2$	$\times^2$	$\times^1$	$\times$	$\checkmark$

Table 5.6: Summary of Independence Implications Under (N) and (A)

$\checkmark$**:** Implication under (N).

$\times$**:** No implication under (N).

$\times^1$**:** (SI) and (SD), together with (A) and (N) does not imply (I), by Proposition 5.1.1.

$\times^2$**:** (WI) and (SD), together with (A) and (N), does not imply (ISA), by Proposition 5.1.2. Thus, (WI), (A), and (N) do not imply (ISA+).

$\times^3$**:** None of (ISA+), (ISA), (SD) and (WI) (together with (A) and (N)) imply (I). This follows from the counterexample in Proposition 5.1.3.

Then we move to implications under strengthened neutrality (N-PERM) in Table 5.7:

$\checkmark$**:** Implication under (N-PERM).

$\times$**:** No implication under (N-PERM).

$\checkmark^1$**:** (SI), together with (N-PERM), implies (I), by Lemma 5.4.3.

$\checkmark^2$**:** (WI), together with (N-PERM), implies (ISA), by Proposition 5.4.5.

$\Rightarrow$	(I)	(ISA+)	(ISA)	(SI)	(WI)	(SD)
(I)	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
(ISA+)	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$
(ISA)	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$
(SI)	$\checkmark^1$	$\checkmark^1$	$\checkmark^1$	$\checkmark$	$\checkmark$	$\times$
(WI)	$\times$	$\times$	$\checkmark^2$	$\times$	$\checkmark$	$\times$
(SD)	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$

Table 5.7: Summary of Independence Implications Under (N-PERM)

Chapter 6

Discussion and Conclusion

In this chapter, we argue that, due to the differences between PA and JA, the weakenings that we defend in this thesis should be adopted for certain types of agendas. In doing so, we address possible applications for our results.

In Section 6.1, we return to the arguments made in Subsection 3.5.2. We reiterate why the distinctive aims of PA and JA suggest that certain conditions which are plausible in PA contexts are less so in JA contexts. In particular, we discuss neutrality conditions, both the weak (N) and strong (N-PERM) forms, and the weakened JA independence conditions from Section 4.3. After discussing the plausibility of (I), we discuss applications for the weakened independence conditions, assessing which types of agendas they fit.

In Section 6.2, we discuss the implications of the possibility results we have obtained, focussing on epistemic applications and the plausibility of the required conditions.

Section 6.3 applies the arguments about the differences between the PA and JA frameworks to argue against the requirements that generate impossibility. Our first impossibility theorem's use of (SD) is instructive since (SD) is significantly different from other weak independence conditions. In particular, it is surprising that (SD) would generate impossibility. The other impossibility results obtained are heavily dependent upon (N-PERM), which we argue to be overly demanding.

Finally, we conclude in Section 6.4 and suggest future related avenues of research.

6.1 Difference between PA and JA

In this section, I revisit and expand the arguments about the distinctions between PA and JA frameworks from Subsection 3.5.2. These issues are worth revisiting here since they bear directly on the applicability of our possibility and impossibility results.

PA had its historical roots in economics and social welfare.[1] In social welfare, the purpose was to determine the right candidate or social state from individual preference rankings. Thus, in the aggregation, the aggregator should favour no initial social states, nor should one alternative imply another (in fact, PA alternatives are usually taken to be mutually exclusive). Certain preferences on *pairs* can (and should) imply others (e.g. xRy and yRz imply xRz), but this is a distinct claim from one alternative itself implying another.

JA, on the other hand, demands social evaluations which are consistent and presumably deductively closed, if not complete. Thus, the implications holding between elements of the agenda are integral to the framework in that holding certain positions on some elements logically compels holding certain others. So not only do certain types of acceptances imply each other, but also the structure of the propositional agenda elements should be reflected in the aggregation process. The most natural way to do so is to make the *social* positions on propositions (which have logical dependence on other agenda elements including, but not limited to, atomic sentences) dependent on the *individual* judgment functions of those other elements. This not only reflects the propositional structures, but also honours the original application of JA, namely, (judicial) decision-marking and argumentation. In other words, since JA models *social* decision-making, it is plausible that some elements of the agenda depend on the individual rationales for decisions (not just on the individual judgment functions values of the agenda elements themselves).

This can be viewed as a trade-off between the aggregation of individual preferences and the structure of the elements being aggregated over. In PA, there should be maximal aggregation, with minimal prejudice for or against different alternatives; in JA, there should be *some* reflection of the logical structure of the elements of the agenda.

6.1.1 Neutrality

Our claim that Neutrality (N-PERM) in JA should be weakened from Neutrality (P-N) in PA is meant to mirror the fact that agenda elements in JA are not treated neutrally, while they are in PA. It is important to note that the conditions (N-PERM) and (P-N) are not the same as “neutral treatment.” Neutral treatment in this context means that exchanging the agenda elements in an admissible profile yields a new admissible profile. Neutral treatment, therefore, is *pre-aggregation* whereas (N-PERM) and the other neutrality conditions *constrain aggregators*. We are not arguing that (N-PERM) is inconsistent with the JA framework, just that it presents conceptual friction. If JA includes pre-aggregation non-neutral treatment,

[1] For a summary of the history of PA, cf. (Arrow et al. 2002, Ch. 1).

why should JA aggregators be neutral?

Another way of approaching this issue is by considering the following: The implications from JA do bear some similarity to transitivity between alternatives in PA, but the difference is that such implications are built into the agenda elements in JA, so they constrain the profiles in asymmetric (and, thus, non-neutral) ways. For an agenda $X = \{a, b, a \wedge b\}$, an individual judgment function $A_i(a) = A_i(a \wedge b) = 0$, $A_i(b) = 1$ is admissible but $A_i(a) = A_i(b) = 0$, $A_i(a \wedge b) = 1$ is not. In PA, one can always permute the alternatives in an admissible ordering and generate an admissible ordering. Since, in JA, the profiles themselves respect the logical connections of the agenda, we suggest that aggregators should as well.

This example shows that JA need not demonstrate such neutral treatment towards agenda elements under aggregation (in fact, it usually does not). In contrast, PA always treats alternatives neutrally. Thus, we suggest that JA Neutrality is less justified than PA Neutrality. How does this play out?

(N-PERM), which requires that, intra-profile, all elements of the agenda are accepted with symmetric patterns of individual acceptance, violates this. If an aggregator F satisfies (N-PERM), then each permutation of the agenda delivers the same (permuted) evaluation. But then, for the appropriate permutations, the pairs a, $a \wedge b$ are treated the same as the pairs a, b. This is so even though the example above shows non-neutral pre-aggregation treatment of a, $a \wedge b$.

In contrast, in a set of PA alternatives, there is reason to treat them impartially or to resist prejudice the truth of one as opposed to the other. This is why the application of neutrality is much more plausible in the case of PA.

The other form of JA neutrality that we consider—(N)—is somewhat more difficult to grasp intuitively. This holds that when, in a profile, all the same individuals have the same positions on two propositions, the aggregator should give the same truth-value to the pair. By Proposition 5.2.2, (N) is weaker than (N-PERM), but it still allows cases we take to be unreasonable. When the same individuals accept and reject two propositions, they take the same value while disregarding the differences of the propositions.

For these reasons, we take these neutrality conditions to be implausible in a JA context, although they are fitting in a PA context. The moral is that the logical structure of agenda elements should (or at least arguably could) act as a basis for the way that the aggregator treats it, while (N-PERM) preempts such possibilities.

6.1.2 Independence and Applications

The second consequence of the differences between JA and PA is that aggregator independence loses its plausibility when aggregating in JA contexts. In PA, the two strongest arguments for (IIA) are its role in preventing manipulation, and the intuitive independence of certain alternatives from others. Neither of these conditions are as persuasive as JA contexts as they are in PA contexts.

On the first point, as we argued in Subsection 3.5.2, the mere *possibility* of JA manipulation is far from manipulation being an issue in actual instances of JA. The reasons for this are twofold: (a) Gathering the data required to manipulate an agenda when there are logically interconnected propositions is far more difficult than determining the level of support for different candidates in a set (à la PA); and, (b) even if this information is successfully obtained, manipulation problem of JA premise-based procedures aggregators comparable to the one in our possibility results are computationally intractable, as Endriss et al. (2010*b*) have shown. In contrast, for common PA voting rules (e.g. majoritarian voting), manipulation is computationally easy (with the notable exception of single-transferrable vote) (Bartholdi et al. 1989).

On the second point, the intuitive independence of pairs of alternatives in PA relates to their traditional interpretation as mutually exclusive states. Whereas it is somewhat perplexing for the choice of one pair of mutually exclusive state of affairs to be dependent on the individual positions on a completely distinct pair (PA), it is very plausible for the social acceptance of a molecular proposition to be dependent on the individual positions on constitutive propositions (JA). So, while PA alternatives are mutually exclusive, JA elements need not always be. This parallels the fact that, in JA, there is not mutual exclusivity; in fact, one element can logically imply another.

For these reasons, we take our weakenings of (I) to be readily admissible in a JA framework. Here, we consider how these weakenings might be interpreted and what types of applications they can be used for. Although all of these (except (SD)) are satisfied by constant aggregators, we do not discuss such aggregation here since it is trivial (and, in application, does not function as aggregation at all).

(SI) holds that there are elements of the agenda which are not dependent on the rest of the agenda. Since literals are the most plausibly independent, we take (SI) to be applicable to agendas where some elements of the agenda are foundational.

Although this is true in many instances which JA models, one especial application of interest is modelling aggregation in foundationalist epistemic systems. In such theories, certain propositions are taken to be epistemically privileged, and other propositions are dependent upon the privileged atoms.[2] Theoretically, the foundational beliefs are independent, and then there is dependence among other beliefs.

[2] A classic source of foundationalist epistemological theories is the British empiricists. Russell (1910), for instance, took the foundational literals to be sense-data. In JA, an aggregator which satisfies that some elements of the agenda (i.e. the sense-data) are independent of other elements and other elements of the agenda are dependent on these sense-data could model an epistemic system.

(SI) could also reflect any balloting situation where some issues are dependent on others. This is easier to illustrate with an example: Suppose a city is determining municipal placement of public buildings. One question they might ask is "Should we have a swimming pool" ($p \in X$) and then there would be a follow-up: "Should it be in location a, b or c?" ($p \wedge a$, $p \wedge b$, $p \wedge c \in X$). The second question is unimportant if, for instance, only 5% of individuals want a pool. It is plausible that p should be independent—satisfying (SI)—but the aggregator on the three conjunctions might require a certain level of individual acceptance of p or else return 0 for each conjunction, disregarding the individual positions on the second question. In fact, this could make the aggregation simpler for the city, since it could avoid gathering the data on the follow-up.

Of course, most judgment aggregation applications have foundational elements in their agendas in this sense (especially in reasoning contexts, where premises are foundational). Aggregators which satisfy (SI) include those which make only some (foundational) elements in the agenda independent of other alternatives. For instance, the discursive dilemma (cf. Table 3.1) would be avoided by having a and b be independent, and have $a \wedge b$ not be independent. In other words, by requiring the aggregator satisfy (SI) instead of (I), the aggregator both more accurately reflects the reasoning process and may also avoid impossibility.

(ISA) holds that, for any given element of the agenda, some elements are independent of it. This holds for agendas which have interconnections but permit self-dependence. It excludes agendas where some elements depend on the entire agenda. This appears to be a minimal condition to prevent rampant (and trivial) manipulation, since having certain elements depend on the entire profile makes control of those elements available to the individual(s) for whom changing their positions forces a change on the social judgment.

To avoid trivial manipulation, in other words, (ISA) is desirable. Furthermore, it is plausible when modelling reasoning with conclusions, as in Subsection 5.3.1. This possibility class of aggregators ostensibly reflects premise-based procedures since certain foundational agenda elements require a certain level of support and then other elements are determined with direct appeal to the level of support on the foundational elements.

However, like (SI), (ISA+) is applicable to an epistemic position, namely, coherentism.[3] Coherentist epistemology holds a given belief is revisable based on

[3]In the philosophy of science, one particularly influential coherentist position is Quine & Ullian's (1970/1978) theoretical holism. This picture has social knowledge in a "web" where certain extremal beliefs are revisable due to experiential data, but the central beliefs are firmly entrenched. However, there are no beliefs, on this view, which can be ruled out as unrevisable, even the central beliefs can be changed with enough challenging data.

Such a view resembles (ISA+) aggregation, since no element is dependent upon the entire agenda, but each element has some non-self-dependence. In particular, the truth-value of some certain changes in other beliefs, but that there are no privileged beliefs which are entirely independent (or foundational). (ISA+) is actually weaker than the type of dependence that would best model coherentism. A more apt condition would also require that each element is dependent on distinct elements (such a condition would be roughly the reverse of (SD), i.e. with other-dependence instead of self-dependence).

Note that both of these epistemological theories are poorly modelled by fully (I) contexts. In such contexts, there is no way that the aggregation of a given element can depend on others, which prevents the explicit modelling of any epistemic dependence.

(SD) we take to be overly strong. The reason is that we wish to allow non-foundational propositions to be dependent on the individual positions taken on foundational propositions. Although (SD) does not formally prevent this, it does practically over non-simple agendas (as Theorem 5.4.1 shows). Moreover, the *purpose* of exploring weak independence contexts was to allow for more unusual dependence sets. From a methodological perspective, we wish to examine aggregators where dependence sets are not simply self-dependent.

Finally, we consider (WI). This is the most difficult condition to apply; it allows some elements of the agenda to have full dependence (i.e. requiring the entire profile evaluation of the agenda in order to socially determine a single element). In fact, it allows *all but one* of the agenda elements to be fully dependent. It is difficult to conceive of applications for (WI), so we will just consider it as a technical condition. Keep in mind that any (SI) or (ISA) aggregator is trivially (WI), but we do not see a reason for weakening either of these to (WI) for a particular application.

In short, we find (SI) and (ISA) to both be desirable conditions for an aggregator with an agenda containing literals and molecular propositions which are composed with other elements of the agenda. In practice, this would include any agenda of minimal complexity. An agenda with literals allows for independence on some (atomic) propositions and an agenda with molecular propositions allows for some level of dependence on the acceptance patterns of their constituent parts.[4]

6.2 Possibility Results

We have demonstrated two possibility results in Subsection 5.3.2. The first has conditions (A), (N), and (SI) and the second has (A), (N) and (ISA) over a non-

propositions can change given the different profiles which change adjacent beliefs.

[4]Note that (ISA) can be satisfied by fairly trivial dependence. For instance, φ may depend on ψ with an aggregation where $\varphi = 1$ in all but one profile (one in which ψ takes a different value).

simple agenda. In fact, the same aggregator, from Subsection 5.3.1, satisfies both (ISA) and (SI). We take this to be applicable to general social reasoning contexts: The society determines acceptance over the atomic propositions and the individual positions on the atoms may partially determine the acceptance on the conjunction. It would be easy to extend the aggregator to treat more complex logical connectives (e.g. $\vee$, $\rightarrow$, etc.).

We take (A) to be a desirable condition but (N), as we have argued, is not as plausible, especially when more complex molecular sentences are introduced. However, this possibility is meant to see what happens when the conditions of Systematicity are weakened, so it is one of the conditions here as an artifact of Systematicity.

Furthermore, we think such an aggregator does model social reasoning in a premise-based manner. The idea is that a society which has a sufficient level of agreement on certain literals must accept the consequences of these literals.

One objection that could be made to our aggregator is the trivial amount of agreement (e.g. $|\varphi| \geq 1$) necessary to secure social assent, especially when the size of the society is very large. However, the level of assent can be adjusted to reflect a more skeptical society. One was chosen as the level required because the initial example has a society of size two so $|\varphi| \geq 1$ represents a relatively high level of assent, but there is actually a class of aggregators since this value could be replaced with any natural number, and all members of this class similarly demonstrate possibility.

6.3 Impossibility Results

Theorem 5.4.1 is quite interesting as an impossibility result. It is generated by the requirement, for dependence, that one can change one element, while keeping the rest of the agenda fixed. As we have shown in Section 6.1, consistent and complete profiles in JA cannot be permuted in the same way they can in PA. This result comes from taking advantage of this fact.

Theorem 5.4.4 is a straightforward strengthening of List & Pettit's Impossibility. In particular, it demonstrates how powerful an assumption (N-PERM) is; (N-PERM) is capable of spreading independence in a symmetric fashion, strengthening weaker independence conditions to sufficiently strong conditions. As I have argued, (N-PERM) should be avoided in JA, so this impossibility should be resisted in applications of JA.

6.4 Conclusion and Future Work

Characterizing and identifying these weakenings of independence is of significant utility for JA. In particular, we have argued that weak independence contexts can reflect several properties of agendas in which reasoning is modelled (i.e. where some elements have logical dependence on others). Since JA was formulated to model and address argumentation, it appears that such agendas are widespread in the intended realm of application.

We have singled out epistemic contexts as particularly reflective of weakened dependence. The implications holding between different beliefs are well-modelled by aggregators satisfying weak forms of dependence, and poorly modelled by aggregators satisfying Systematicity.

The demonstrated possibility results allow one to sidestep impossibility with aggregators that have certain desirable features. Furthermore, they can be extended to allow for more complex agendas with relative ease, and can be adjusted to reflect conclusion-based aggregation instead of premise-based.

Our reminders about the distinctions between PA and JA are intended to slow the rush of transferring plausible conditions in PA into JA frameworks without questioning whether they fit into such a new environment. We take neutrality conditions such as (P-N) to (N-PERM) to be particularly egregious examples of this issue.

There remain several interesting lines of research from the point of view of weak independence. Since (I) was intended to forestall manipulation, it remains to be seen whether the aggregators that I have presented are easy to manipulate—it may be that dropping (I) is too costly. Another is considering the applicability of this new neutrality (N).

We conjectured at the close of Section 5.3 that (ISA+) is strong enough to sustain impossibility. We conjectured that (ISA+) with (N) does generate (I), but were unable to prove this either inductively or by cases. In the former case, the issue was applying an inductive hypothesis that allowed the aggregator to "grow" while retaining the structure of dependent sets from smaller societies. In the latter case, continuous new cases with larger agendas had to be added and we could not see a systematic way to reduce their number. If (ISA+) with (N) does generate (I), we think that this is due to asymmetrical dependence patterns. To illustrate, consider arbitrary φ. By (ISA+), φ is independent of some $\psi \neq \varphi$. But then changing ψ cannot change the value of φ, even when, for all $i \in N$, $A_i(\varphi) = A_i(\psi)$, which, by (N), guarantees that $A(\varphi) = A(\psi)$.[5] If true, this would be of significant interest and it would be of both technical and conceptual interest to determine

[5] Of course, (N) is necessary for this implication; we were able to find an (A), (ISA+) aggregator which failed (I) but also did not satisfy (N).

why (ISA) is weak enough but (ISA+) strong enough.

Finally, it would be of interest to characterize the types of (ISA), (SI), (N) and (A) aggregators which are possible; we have found a class of examples, but it would be of interest to see which other aggregators can occur. While we consider our class of aggregators to be highly plausible and applicable, there may be further interesting aggregators satisfying this set of weakened independence conditions.

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