A game theoretic approach to cost allocation in the Dutch electricity grid

MSc Thesis *(Afstudeerscriptie)*

written by

Babette Paping
(born September 5th, 1988 in Schiedam, Netherlands)

under the supervision of **Prof. Dr. Krzysztof Apt (CWI, ILLC), Dr. René van den Brink (VU)** and **Prof. Dr. Annelies Huygen (UvA, TNO)**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:	**Members of the Thesis Committee:**
20 April 2015	Prof. Dr. Krzysztof Apt
	Dr. Stéphane Airiau
	Dr. Alexandru Baltag
	Dr. René van den Brink
	Prof. Dr. Jan van Eijck
	Dr. Ulle Endriss
	Prof. Dr. Annelies Huygen

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

The aim of this thesis is to formalise cost allocation in the Dutch electricity sector by means of concepts from the theory of cost allocation and cooperative games. Consumers of electricity are connected to one of the seven voltage levels in the electricity grid. Most electricity is fed into the grid at the highest voltage level and is transported to lower voltage levels. Research in collaboration with TNO has shown that a heavy burden of the electricity network costs (in particular transmission-related costs) is born by small-scale consumers connected to lowest voltage level. One of the reasons that small-scale consumers are charged this large share of the costs is because they are also charged for the upstream voltage levels by means of the cascade method. In this thesis we provide a formal framework that models the electricity demand problem, where groups of agents with individual electricity demands are connected to a specific voltage level in the electricity grid and are allocated cost shares by the network operators. This framework provides the opportunity to analyse the cascade rule and several other cost allocation rules for our problem, inspired by and in comparison with rules proposed in the literature on other problems. We provide axiomatic characterizations for three rules differing in the properties they obey. Building on the electricity demand problem we introduce a cooperative cost game and simplified versions of the union- and agent-Shapley value, assigning cost shares to groups of agents. Also other union values are discussed and evaluated. Hence, cost allocation from practice and theory are combined and formalised by means of a cost allocation and cooperative game theoretic approach.

Acknowledgements

First and foremost, I would like to thank my supervisors. Krzysztof, many thanks for always helping me and directing me to the right people. You were also always willing to discuss and think with me about other matters, such as my future. I admire your sincerity and cheerful view on life. I am also very thankful to René, if it was not for you, my thesis would not be anything close to what it is today. Right from the start you were enthusiastic and helped me on the right track. Even though you are not a professor at the UvA , I could always count on your help and guidance. My gratitude also goes to my supervisor at TNO, namely Annelies. You are always full of ideas and enthusiasm. I really enjoyed working with and for you at TNO, it taught me a great deal. I would also like to thank my thesis committee: Stéphane Airiau, Alexandru Baltag, Jan van Eijck and Ulle Endriss. I am very pleased that you are interested in my thesis and want to be part of my thesis committee. My brother said to me that no one was ever going to read my thesis, but I am pleased that I can prove him wrong with you as my thesis committee.

Big hugs go to Sanne, Femke, Jasper and Sam for proofreading parts of my thesis and providing me with useful comments. Further I would like to thank my dear friends Femke and Sanne for making the years at the Science Park unforgettable. You made math and logic (and many other things) just so much more fun. I am also very happy that Steef was with me in the library the last months of writing my thesis. Thank you everyone at TNO, in particular Jasper. It was great fun and instructive to work with you. Further I would like to thank my roomies, Rikkert, Floor and Laar, you are the best to come home to after a day of study. Tim, thanks for making me think about other things than Logic and my thesis and for always being so supportive. Finally, I am very grateful to the best parents and brothers and sister I could imagine for always believing in me and supporting me in whatever I do. Special thanks to you mom for all your cards, emails, photos, poems and you-tube songs to encourage me.

Contents

Chapter 1

Introduction

By means of this thesis we consider a real-life situation from the perspective of cost allocation and cooperative game theory. This research was motivated by findings obtained in a research project on the cost allocation in the Dutch electricity sector, performed on behalf of TNO. We therefore believe this thesis contributes to these theoretic fields as well as the practical field. In this thesis we provide a framework that models the situation where groups of agents with individual electricity demands are connected to a specific voltage level in the electricity grid and are allocated cost shares by their network operator.

1.1 Motivation

Electricity prices have been at the centre of debates lately. If you search on the internet or check the newspapers for electricity prices, you find a lot of headlines about changes in this sector or dissatisfied consumer or interest groups. This is partly the result of a changing market. With the transition to more sustainable energy resources a lot is happening in the electricity sector and in the energy sector in general. As a response to these changes we performed a quantitative research, commissioned by TNO, on the current cost allocation in the Dutch electricity sector. This research focused on the network and tax costs. We found that a large part of the income of the regional and national network operators comes from small-scale consumers with only low electricity demands, instead of from large-scale consumers with high electricity demands. This is the result of major differences between tariffs and between tariff heights for different types of consumer groups. These differences in tariff heights are mainly caused by the way the costs are allocated amongst the voltage levels.

The Dutch electricity grid consist of multiple voltage levels, transporting electricity from producers to consumers. This ranges from the Extra-High Voltage level (220 - 380 kV) to the Low Voltage level (less than 1 kV). The Authority for Consumer and Market distinguishes seven voltage levels in the electricity grid (Autoriteit Consument en Markt, 2013). Each voltage level serves ***end-users*** and possibly other voltage levels. For example the Extra-High Voltage level serves large industrial companies as well as the High Voltage level. Each consumer is connected to one of the seven voltage levels, depending on the size of its connection, which in turn is dependent on the peak demand of electricity any time in the year. Most ***small-scale consumers***, like households, are connected to the Low Voltage level. In this thesis we focus on the ***transmission-related costs***, which make up the largest share of the network cost. We elaborate on the transmission-related costs in Chapter 2. The focus of this thesis is not on the tax costs, as we found that the allocation of tax costs is partly established from a political point of view instead of an economical point of view. In the allocation of transmission-related costs to consumers we distinguish the following steps:

1. Each regional and national network operator determines its transmission-related costs per voltage level. In Chapter 2 we specify what these costs include. So in the first step the total costs per voltage level for every operator are determined.

2. The transmission-related costs that are attributed to a voltage level by the network operators are for each operator separately cascaded towards the directly underlying voltage level, referred to as the cascade method, which is depicted in figure 1.1. This method is based on the idea that electricity is fed into the grid at the highest voltage levels by means of large production facilities, resulting in a electricity flow going from high to lower voltage levels.[1] This entails that lower voltage levels make use of higher voltage levels and not the other way round. This idea is somewhat outdated, as decentralized production installation incur bilateral flow between voltage levels, but this is discussed later. Hence, this step entails reallocating the costs obtained in step one to the different voltage levels.

3. The ***final costs*** per voltage level are apportioned amongst consumers connected to the respective voltage level through a combination of tariff carriers. The tariff carriers for the transmission-related tariff per level are set, but the size of the allocated costs to a voltage level determines the height of these tariff carriers. Thus in this final step the resultant costs from step two are allocated to the agents.

[1] We use transmission, transport and flow of electricity all to denote the transportation of electricity over the grid from a source to a consumer.

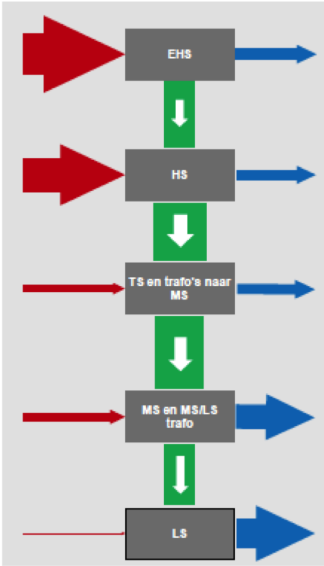

Figure 1.1: **Schematic simplified representation of the electricity grid and the cascade method** Red arrows represent production of electricity, blue arrows consumption and green arrows the net flow between the voltage levels (Hakvoort et al., 2013). The following (Dutch) abbreviations are used: extra hoogspanning (EHS, 220-380 kV), hoogspanning (HS, 110-150 kV), tussenspanning(TS, 25-50 kV), trafo hoogspanning naar tussen-en middenspanning (HS-TS/MS), middenspanning (MS, 1-20 kV), trafo middenspanning naar laagspanning (MS-LS) en laagspanning (LS, <1 kV). Note that some voltage levels in this figure are merged such that only five voltage levels are given instead of seven.

The ***second step*** is an important cause of the great difference in tariffs for different consumer groups, which is highlighted in figure 1.2.[2] In this figure the net electricity flow is compared with the revenues generated by the tariffs of certain consumer groups. Our focus is on this step in the cost allocation process. Much has been written and discussed about appropriate pricing mechanisms for electricity, amongst others in Rodríguez Ortega et al. (2008), since there is still not one overall accepted pricing mechanism for electricity and for electricity transmission in particular.[3] In addition, also here national and international politics play an important role in the price determination. Instead of taking part in this discussion we decided not to focus on an appropriate pricing mechanism, but highlight the crux in the current cost allocation leading to the varying electricity prices, namely step two.

From a historical perspective, the Dutch tariff structure is developed with the idea

[2]The consumer groups are differentiated based on the voltage levels.
[3]The pricing mechanism determines which tariff carriers are employed for different consumer types and/or groups.

that central large production installations feed electricity into the grid at the highest voltage levels. So costs of higher voltage levels are charged to lower levels proportional to the net demand of the lower level network as described in step two above (Autoriteit Consument en Markt, 2013). There is however growing criticism of this method, since by the increasing decentralized production of electricity the production and consumption of transmission are brought closer together (Hakvoort and Huygen, 2012). Also, when the decentralized production exceeds the demand of the respective level, this electricity is transported to a higher level. In Aalbers et al. (2003) is pleaded for other allocation methods, since they claim that the cascade method allows for heavy cross-subsidizing of lower voltage levels over higher voltage levels. Also NMA and SEO economic research (NMA and SEO, 2011) advise to further investigate the cost allocation over the levels with respect to ongoing changes. Around 60% of the revenue generated by the small-scale consumers at the low voltage level is a contribution to costs of the higher level networks (see figure 1.2). These observations and findings are a motivation to perform further research on the uneven allocation of the electricity transmission costs and the fairness and reasoning of the currently used allocation methods.

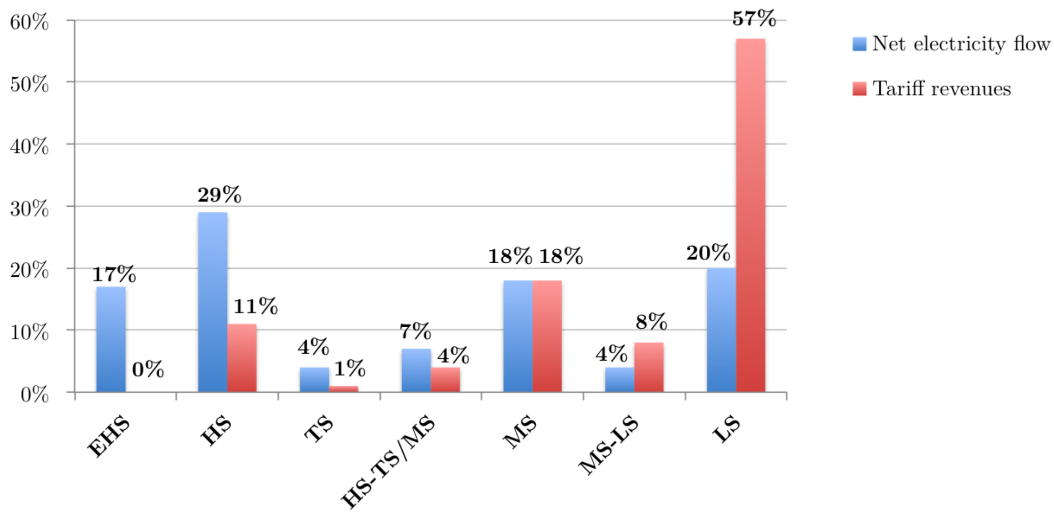

Figure 1.2: **Percentage share total inflow and outflow of electricity over the different voltage levels (blue) and percentage share total tariff revenue (red)** The electricity flow is based on values from 2008 and the tariff revenues are based on the x-factor model of the regional network operators and Tennet of 2009, 2010 respectively (ACM, 2014), (Hakvoort and Huygen, 2012).

We find that cooperative game theory provides appropriate tools to analyse cost allocation. It gives the possibility to approach cost allocation rules in an axiomatic way and argue about their fairness based on these properties. Further does cooperative game theory typically not take personal preferences into account and assumes demands of agents to be inelastic (Koster, 2009).[4] Electricity demand can be considered inelastic, as it is a necessary good for most companies and households, they always have demand for electricity. Moreover, because network operators have a monopoly, consumers have few alternatives to fully foresee themselves in their electricity demands.[5] Further are most consumers also dependent on energy suppliers. So the electricity demand is assumed to be inelastic and preferences of consumer are not explicit, which makes the theory on cooperative cost games well suited. Further there are also a number of nice well-known rules within cooperative game theory with interesting properties, such as the Shapley value, that can be and have been applied to real life cost or profit allocation problems, for example in the Tennessee Valley Authority in Young (1994), the museum pass game in Ginsburgh and Zang (2003), the airport game in Littlechild and Owen (1973) or the river pollution sharing game in Ni and Wang (2007).

1.2 Contribution

Many articles in the literature can be found on setting electricity tariffs, also from a game theoretic perspective, which is highlighted in the next section. However to our knowledge, little research has been done from a perspective of cost allocation and cooperative game theory on firstly, the axiomatic characteristics of the second step in the transmission-related cost allocation mechanism and secondly, on alternative cost allocation methods for this second step in the light of today, and possibly with a view to the future. We define a model that represents a (simplified) version of the real-life situation, where agents are grouped in unions with regard to the voltage levels they are connected to. This model gives us the opportunity to analyse the cascade method and give an axiomatic representation of the associated rule. Also other in the literature proposed rules are for the first time in this context analysed with respect to their properties. Moreover, we define an associated cooperative cost game. For this game we provide a simplified version of the Shapley value and introduce several union values. Finally, we discuss extensions of the model in anticipation of changes in the electricity sector. Hence, this model is the first model, to our knowledge, that strives to analyse the cost allocation over different voltage levels from a perspective of the formal theory of cost allocation and cooperative games.

[4]Demands are inelastic if they are not sensitive to price changes.

[5]The monopolies are regulated by the ACM, however there are no alternative full-fledged electricity operators.

1.3 Background and related literature

In this thesis we explore the problem of cost allocation to unions of agents in an hierarchically ordered electricity network. This research finds its roots in the real-life application and in the theory of cost allocation and cooperative game theory. In Sudhölter (1998) and Moulin (2002) we find an axiomatic approach to cost allocation problems. Koster (2009) and Young (1994) provide a more general overview of cost allocation problems and the induced cooperative cost games. In Moulin and Shenker (1994), amongst others, serial cost sharing and average cost sharing are compared with respect to the properties they obey.

In our problem we have some form of a network structure. In the literature there is a variety of cost allocation problems where a network structure is also present. We briefly discuss the ones that are relevant for this thesis. Littlechild and Owen (1973) discuss the ***airport game***, in which the costs of an airport runway have to be shared amongst different types of airplanes. This corresponds to sharing the fixed costs of voltage levels in our model. In Ni and Wang (2007) the costs of cleaning a polluted river amongst agents alongside a river have to be allocated, in the ***pollution cost sharing problem***. The electricity flow in our model resembles the water flow in this model and as a result unions of agents can be located either upstream or downstream from one another in both models. Another related class of games, is the class of ***infrastructure cost games***, described by Fragnelli et al. (2000). These games are a composition of an airport game and a maintenance cost game and discuss how to share infrastructure costs amongst unions of different types of trains using the infrastructure. The main resemblance with our model is the infrastructure cost structure and the main differences are that we deal with electricity flow and associated demand vectors. The last games were developed on behalf of a company and applied in real-life. In Bergantiños and Martínez (2014) cost allocation to asymmetric agents connected by a tree is discussed, where the asymmetry is due to the different demand and production capacities of the agents. The main asymmetry between agents in our model is imposed by the location of the agents in the grid (e.g. high, medium or low voltage), which is in turn also partly due to the demand vector of the agents.

Also other forms of social asymmetry may arise, for example in cooperative games with restricted cooperation, which represent situations where only specific coalitions can form. In games with communication structures, introduced by Myerson (1977), only agents that are directly connected in the graph may cooperate. Some important values in this context are the Myerson (Myerson, 1977) or Owen value (Owen, 1977). For more on communication structures see (Winter, 1989), (Alonso-Meijide et al., 2009). In Gilles et al. (1992) hierarchical constraints on coalitions are imposed by means of permission structures, such that some agents need permission from others to be able to cooperate. Even though in our model we also have asymmetric agents that are hierarchically ordered, we do not define a restriction on the cooperation between agents, as in theory all of them could decide to cooperate. In Aumann and Drèze (1974) ***games with coalition structures*** are defined and van den Brink and Dietz (2014) discuss union values for games with coalition structure. Our coalition structure partitions the agents in unions corresponding to the voltage levels they are connected to. We are interested in union values allocating costs to these levels.

Some other interesting profit or cost games, without a network structure, that we encountered are the following. The ***museum pass game*** is discussed in Ginsburgh and Zang (2003) and Wang (2011) and defines ways to share the joint income generated by the sales of museum passes over the museums that jointly offer these passes. Visitors do not go to all museums offered by the pass and also some museums are more crucial for the sales than others. The Tennessee Valley Authority, amongst others in Young (1994), discusses the cost allocation problem incurred by building dams and reservoirs along the Tennessee river. This game as well as the museum pass game were applied in real-life.

Many articles exist on the allocation of transmission costs in the electricity network, some methods are discussed in Rodríguez Ortega et al. (2008) and Olmos and Pérez-Arriaga (2009). The allocation of these transmission costs are also analysed by means of cooperative games, see for instance Junqueira et al. (2007), Divya et al. (2012) and Zolezzi and Rudnick (2002). This literature mainly focuses on transmission tariff design and individual cost allocation. These articles concentrate on the applications and not on the axiomatizations of allocation rules. Also the allocation of network losses has gained a lot of attention in the literature, as this does not have a one-to-one correspondence with the amount of electricity transported. As these losses only incur a small part of the total grid costs, we do not dedicate much attention to this subject in this thesis, for more information we refer to Rodríguez Ortega et al. (2008) or NMA and SEO (2011). Pérez-Arriaga et al. (2013) provide a nice and recent overview of transmission pricing methods.

As for the more applied side of this thesis, much research in the electricity sector is done with respect to sustainable energy. Vereniging van Nederlandse Gemeenten (2013) investigated the costs and benefits of sustainable energy initiatives and in Hakvoort and Huygen (2012) local energy productions are discussed. In Hakvoort and Huygen (2012), as well as in Hakvoort et al. (2013) and NMA and SEO (2011) the cascade method is discussed and debated. In Hakvoort et al. (2013) a broader view on the current tariff system in the Netherlands is presented. At the end of this thesis we discuss other models and frameworks that were considered for analysing our problem.

1.4 Outline

This thesis is structured in the following way:

Chapter 2: In this chapter we provide a brief introduction into the electricity sector and the costs associated with the electricity grid. Thereafter we discuss some basic notions of cooperative game theory and cost allocation.

Chapter 3: Subsequently we elaborate on solutions concepts for cost allocation problems and cooperative games. We present various known cost allocation rules and some accompanying properties, on the basis of which we compare the rules. We further discuss solutions for TU games and TU games with coalition structure. For the latter we solely consider union values, which are single-valued solutions allocating cost shares to unions of agents. For the solutions as well as the union values we discuss different properties and use these for a comparison of the solutions.

Chapter 4: After the introductory chapters, we present a formal representation of the electricity demand problem. This problem models the situation in which electricity costs have to be allocated to unions of agents connected to the grid. We study three cost allocation rules, either proposed in the literature or employed in real-life. The rules are formalised by means of axiomatic characterizations.

Chapter 5: Building on the electricity demand problem, we present a cooperative cost game: the electricity demand game. The game assumes that all coalitions are possible and every coalition always needs to make use of the upstream voltage levels. For this game we mainly focus on the agent- and union-Shapley value, but also consider some other appropriate union values.

Chapter 6: As an addition to the electricity demand problem and game, in this chapter we discuss possible extensions of the electricity demand problem. These extensions incorporate production capacities of agents and bilateral flow between voltage levels. We suggest some solution concepts for these extensions, but do not consider them in much detail.

Chapter 7: We end this thesis with a conclusion and discussion, in which we summarise the thesis, provide directions for future work and discuss some other relevant models and frameworks.

Chapter 2

Preliminaries

In this chapter we provide some background knowledge. First we briefly discuss the electricity sector and consider the transmission costs and tariffs. The important actors, as well as the relationship between the costs and the tariffs and the determination of the tariffs are discussed. Thereafter we provide an introduction to the theory of cooperative games and cost allocation.

2.1 Transmission costs and tariffs

In this section we consider how tariffs related to the transmission of electricity are established in the Netherlands. European and national legislation play an important role in this establishment. Consumers that are connected to the electricity grid are charged for supply services, network services and taxes. Our focus is on the network services, which are provided by the network operators. Within the network services another distinction can be made. Tariffs are based on the statutory duties of the network operators, namely providing connection services, transmission services, system services and metering services, together referred to as network services (see figure 2.1). In the Netherlands there is one national network operator (TSO), namely TenneT and eight regional network operators (DSOs), namely Cogas, DNWB, Endinet, Enexis, Liander, RENDO, Stedin and Westland. The regional networks operators and TenneT are both in control of different networks, consisting of different voltage levels. TenneT is in charge of all the Extra-High Voltage (EHS) levels and most of the High Voltage (HS) levels. The regional network operators control some of the HS voltage levels and the lower voltage levels. Providing these services to consumers entail costs, the so-called network costs. These costs of the network operators include capital expenses, with regard to the technical infrastructure and operational expenses for maintenance, operational and management tasks that are incurred by connection to and use of the grid (this includes expenses for congestion management, purchase of reactive power and grid losses). As the technical infrastructure is especially expensive, fixed costs are high and variable costs low. The revenue of a network operator, generated by tariffs, should recover the network costs made by the network operator. This entails that the tariffs should be ***cost-efficient***, which is an important principle in the Dutch tariff system.

In this thesis we focus on the ***allocation of transmission-related costs*** incurred by providing transmission-related services.[1] The transmission tariff consists of a 1. non-transmission-related tariff and 2. transmission-related tariff.

1. The ***non-transmission-related tariff*** is meant for costs of administration tasks, consumer service, billing, etcetera, i.e. costs that are not directly incurred by the transmission of electricity. This tariff is charged to producers as well as consumers.

2. The ***transmission-related tariff*** covers the costs that are incurred by the transmission of electricity, such as the depreciation, investments and maintenance of the infrastructure, but also the costs of grid losses and congestion management (Autoriteit Consument en Markt, 2013). These costs make up the largest part of the network costs and are incurred for the benefit of the grid and therefore socialized over all voltage levels by means of the cascade method. The transmission-related producers tariff is set to zero and thus the producers do not help pay for these costs. The rationale and the estimated effect of introducing a producers tariff is discussed in (Koutstaal et al., 2012). For future research it could be interesting to analyse the effect of this introduction from a game theoretical perspective. So our focus is on this tariff.

The actual determination of the network tariffs is done under supervision of the authority ACM (Autoriteit Consument en Markt) since all network operators have a monopoly in their region. In the Netherlands this is done in the form of a benchmark regulation. A network operator that operates more efficiently than the benchmark makes higher profits than a network operator who is less efficient. In case of the national network operator TenneT, this benchmark is based on foreign national network operators (Str, 2014). The ACM makes use of the codes in the Tarievencode (Autoriteit Consument en Markt, 2013) to determine the tariffs that the network operators may charge their consumers. Also the formulas for the cascade method are officially recorded in this document. The tariff structure determines which tariff carriers (e.g. kW, kWh) are charged to which consumer groups. The actual tariff structure differs between services (e.g. between connection, transmission) and within services (e.g. between transmission-related and non-transmission-related services within the transmission services) for different consumer groups. In figure 2.1 below is presented which services are distinguished and which tariff carriers apply. The focus of this thesis is highlighted in figure 2.1 by a square: the transmission service (also referred to as the transport service).

[1] Thus we do not consider the connection service, system service and metering service costs.

Clearly there are some desirable and EU mandatory ***conditions*** that the tariffs have to satisfy to comply with EU and Dutch law:

1. tariffs should reflect the costs incurred (cost-reflectiveness principle)
2. cover the total costs (cost-efficiency principle)
3. be transparent, unambiguous and verifiable
4. stimulate efficient consumption
5. be non-discriminatory (not biased against a particular group)
6. be distance independent.

Already the first condition is a tricky one, as it is very hard to determine on a detailed level which costs are incurred by whom (Hakvoort et al., 2013). Thus, different consumer groups are charged varying amounts within and between different services and in particular within the transmission-related services.

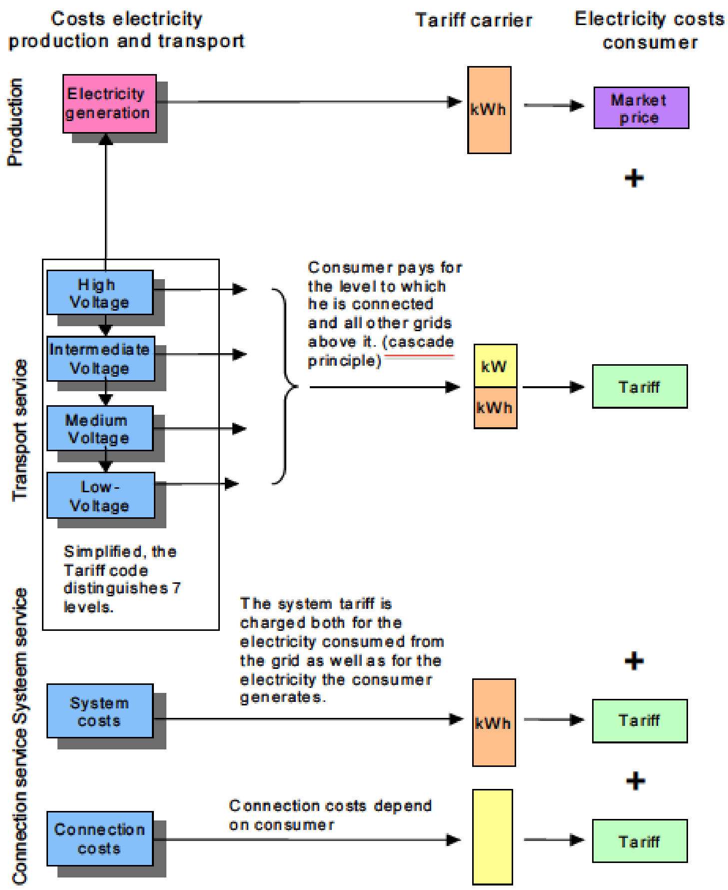

Figure 2.1: **Structure of Dutch electricity tariff system** Overview of the network and production services, with corresponding tariff carriers (Wals et al., 2003, p.19).

2.2 Cooperative game theory

Games in the context of game theory are mathematical models of interactions between rational agents. A rational agent always aims to reach the best possible outcome in a game, taking into account the possible actions of the opponents. In strategic game theory (non-cooperative game theory) every agent wants to maximize its payoff function, which value depends on the actions taken by all the agents simultaneously. In this section we provide a brief introduction on ***cooperative game theory*** and give some important definitions, properties and examples. In contrast to strategic games, in cooperative games agents can cooperate and form coalitions to either reduce costs or increase profits.[2] Sometimes it is given that all agents should cooperate and form the so-called grand coalition. A coalition S is a non-empty subset of N. Each coalition of agents is assigned a value or worth by means of a characteristic function v. Cooperative games were first introduced in Von Neumann and Morgenstern (1947).

Definition 2.2.1. *(Cooperative game) A cooperative game with transferable utility, referred to as a TU game, is a pair (N, v) where N represents a finite non-empty set of agents and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to every coalition $S \subseteq N$ the value $v(S)$, under the condition that $v(\emptyset) = 0$.*

Games with transferable utility are games where side payments are allowed, such that the worth of a coalition can be divided amongst the agents in any possible way. We use the notions cooperative games and TU games interchangeably. As games with non-transferable utilities are not relevant for this thesis, they are not discussed here. The value $v(S)$ may be interpreted as the incurred profit or cost in case the agents in coalition S work together. In case of $v(N)$ we say that the ***grand coalition*** forms, which implies that all agents in N cooperate. We refer to $v(\{i\})$ as the ***stand-alone*** worth or cost of agent i.[3] Let $\mathcal{G}$ denote the class of all TU games with $(N, v) \in \mathcal{G}$ representing the game (N, v). We consider some well-known examples to clarify the notion of a cooperative game.

Example 2.2.1. ***(Glove Game)*** *Consider the set $N = \{1, ..., n\}$, which is the union of two disjoint subsets L and R, i.e. $N = L \cup R$ and $L \cap R = \emptyset$. The agents in L all possess one left glove and the agents in R all posses one right glove. The gloves only have a value when they are paired in a left and right glove. We can model this by a TU game (N, v), such that the value of each coalitions is determined by the number of left-right pairs of gloves. Hence, for each $S \subseteq N$ the characteristic function v is defined by*

$$v(S) = min\{|L \cap S|, |R \cap S|\}.$$

Example 2.2.2. ***(Unanimity game)*** *Consider agent set N and $T \subseteq N \setminus \{\emptyset\}$. The*

[2] In game theory both the notions agent and player are used to designate a participant in the game.

[3] For convenient notation we from now on also use $v(i)$ instead of $v(\{i\})$.

unanimity game (N, v_T) *is defined for all* $S \subseteq N$ *by the characteristic function*

$$v_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Hence a coalition S *is winning if it contains all agents in* T *and losing otherwise.*

Example 2.2.3. ***(Weighted majority voting game)*** *Consider a proposal and a number of people each having a weighted vote. The proposal is accepted if the sum of the weights exceeds a certain threshold value. More formally, a weighted voting game* (N, w_i, q) *or* $[q; w_1, ..., w_n]$ *is a simple game*[4] *with agent set* N *and where* w_i *denotes the weight assigned to each player* $i \in N$. *The required weighted votes for a coalition to win (or a proposal to pass) is given by the threshold value* q. *This situation is modelled by means of the following characteristic function, for all* $S \subseteq N$

$$v(S) = \begin{cases} 1 & if \displaystyle\sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

A weighted voting game is proper if $q > \frac{1}{2}\sum_{i \in N} w_i$. *Then for all* $S \subseteq N : v(S) + v(N \setminus S) \leq 1$. *The weighted majority voting game is a generalization of the well-known regular majority voting game. In this game the weights of all agents are equal to one such that we obtain the game* $[q; 1, ..., 1]$ *with* $q > \frac{|N|}{2}$ *and* $q \in \mathbb{N}$.

2.2.1 Properties of cooperative games

We now introduce some basic properties of cooperative games to classify the characteristic function. A cooperative game (N, v) is ***super-additive*** if for all $S, T \subseteq N$ with $S \cap T = \emptyset$ holds that

$$v(S \cup T) \geq v(S) + v(T). \tag{2.1}$$

In words this property states that a pair of disjoint coalitions always obtain a higher value in case of cooperation. The inverse of a super-additive game is a ***sub-additive*** game, given by the equation

$$v(S \cup T) \leq v(S) + v(T), \tag{2.2}$$

for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A TU game (N, v) is ***monotonic*** if for all S, T with $S \subseteq T \subseteq N$ we have

$$v(S) \leq v(T). \tag{2.3}$$

Monotonicity implies that growing coalitions obtain non-decreasing values. A ***convex*** game satisfies the property

$$v(S \cup T) + v(S \cap T) \geq v(T) + v(S), \tag{2.4}$$

[4] A TU game v is ***simple*** if the worth of a coalition is 1 or 0, i.e. coalitions can either win or lose.

for all $S, T \subseteq N$. The inverse of a convex game is a ***concave*** game, satisfying for all $S, T \subseteq N$

$$v(S \cup T) + v(S \cap T) \leq v(T) + v(S). \tag{2.5}$$

Clearly convex (concave) games are super-additive (sub-additive). When a game is convex the marginal contribution of each player is increasing with respect to larger coalitions. The ***marginal contribution*** of an agent is the extra value an agent brings a coalition by joining it. We denote the marginal contribution of agent $i \in N$ joining coalition $S \subset N$ by $mc_i(S) = v(S \cup \{i\}) - v(S)$.[5] The convex property can be rewritten in terms of marginal contributions as follows

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. A game that is ***additive*** is referred to as an ***inessential*** game.

Definition 2.2.2. *(Inessential game) A game (N, v) is called inessential if for all $S \subseteq N$ holds that $v(S) = \sum_{i \in S} v(i)$.*

So in an inessential game cooperation is not beneficial. Every game that is not inessential, is ***essential***. To put some properties into more concrete terms, we show that the unanimity game is convex and monotonic and the weighted majority game is monotonic.

Example 2.2.4. ***(Unanimity game II)*** *Consider the unanimity game (N, v_T) as defined in example 2.2.2 with T the set of veto agents. Agent i is a veto agent iff for all winning coalitions S we have that $i \in S$. So agent i has the power to veto any coalition. It is easy to show that this unanimity game is convex, i.e. $v_T(S) + v_T(P) \leq v_T(S \cup P) + v_T(S \cap P)$ for all $S, P \subseteq N$. Assume $S \subseteq N$ and $P \subseteq N$, we consider three possible cases:*

- *$T \subseteq S \cap P$: it follows that $T \subseteq S$, $T \subseteq P$ and so $T \subseteq S \cup P$, such that $v_T(S) = v_T(P) = v_T(S \cup P) = v_T(S \cap P) = 1$. Hence, we obtain $1 + 1 \leq 1 + 1$.*
- *$T \nsubseteq S \cap P$ and $T \subseteq S \cup P$: if $T \subseteq P$, then $T \nsubseteq S$, so that $0 + 1 \leq 1 + 0$. The same holds for $T \subseteq S$. If $T \nsubseteq S$ and $T \nsubseteq P$, then $0 + 0 \leq 1 + 0$.*
- *$T \nsubseteq S \cup P$: it follows that $T \nsubseteq S$, $T \nsubseteq P$ and clearly $T \nsubseteq S \cap P$, such that $v(S) = v(P) = v(S \cup P) = v(S \cap P) = 0$. Hence, we obtain $0 + 0 \leq 0 + 0$.*

Also monotonicity of (N, v_T) easily follows. Assume $S \subseteq P \subseteq N$. If $v_T(S) = 1$, then $T \subseteq S$ and hence $T \subseteq P$, whereby $v_T(P) = 1$. Thus $v_T(S) = v_T(P)$. If $v_T(S) = 0$, then either $T \subseteq P$ or $T \nsubseteq P$, such that $v_T(S) < v_T(P)$ or $v_T(S) = v_T(P)$ respectively. Hence, for all $S \subseteq P \subseteq N$ we have $v_T(S) \leq v_T(P)$ and thus (N, v_T) is monotonic.

[5] For convenience, from now on we also use $v(S \cup i)$ instead of $v(S \cup \{i\})$.

Example 2.2.5. ***(Weighted majority voting game II)*** *If we assume for all $i \in N$ that $w_i \geq 0$, then monotonicity for the weighted voting game (N, v) is guaranteed. We show that for all $S \subseteq P \subseteq N$ we have $v(S) \leq v(P)$:*

- *If $v(P) = 1$, then it is always true.*
- *If $v(P) = 0$, then it follows that $\sum_{i \in P} w_i < q$. Since $S \subseteq P$, also $\sum_{i \in S} w_i < q$. Hence $v(S) = 0$.*

The characteristic function attaches a worth to a coalition, but does not specify how this worth should be allocated amongst the agents. A ***payoff distribution*** of a TU game (N, v) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ that allocates payoff x_i to agent i in N. This vector represents how the worth of a coalition is allocated amongst the agents in the coalition. For every payoff distribution x and $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. A payoff distribution is ***efficient*** if it equals the total worth of the grand coalition, thus $x(N) = v(N)$ for $x \in \mathbb{R}^N$. Two properties that provide incentives for individual and groups of agents to voluntarily cooperate, are ***individual rationality*** and ***coalitional rationality***. A payoff distribution that is individual rational ensures that each agent receives at least the worth he or she would realise alone. So for $i \in N$ and $x \in \mathbb{R}^N$, $x_i \geq v(i)$. A coalitional rational payoff distribution allocates each coalition with at least the worth it would have realised on its own, i.e. for $S \subseteq N$ and $x \in \mathbb{R}^N$, $x(S) \geq v(S)$. When both properties are obeyed, the distribution can be considered ***stable***. However, for example Young (1994) pleads that a distribution is only stable if it satisfies efficiency and group rationality. A payoff distribution that is efficient as well as individually rational is referred to as an ***imputation*** for a game.

In many situations one can imagine that certain groups exist within one larger group, e.g. different school classes within a school. In cooperative game theory this idea is defined by a coalition structure. A ***coalition structure*** partitions the set of agents N such that $P = \{P_1, ..., P_m\}$ is a partition with $\cup_{k=1}^{m} P_k = N$ and $P_k \cap P_l = \emptyset$ for $k \neq l$ (Aumann and Drèze, 1974). Given $P = \{P_1, ..., P_m\}$, denote $M = \{1, ..., m\}$. We refer to elements of a partition P as unions. A TU game with coalition structure is a triple (N, v, P). We denote the class of all games with coalition structure by $\mathcal{GP}$. For more background on cooperative game theory, we refer to e.g. Young (1994), Gilles (2010), Feng (2013).

2.3 Cost allocation

A ***cost allocation problem*** defines the problem of allocating total cost $C(q)$, that is incurred by foreseeing in a vector of demands $q \in \mathbb{R}_+^N$, amongst agent set N (Koster, 2009). Each demand vector $q = (q_i)_{i \in N}$ has an associated cost $C(q)$, which has to be paid by the agents in N. Denote the class of all possible demand vectors by Q. A cost allocation problem is defined by the triple (N, q, C), with a non-decreasing cost function $C : Q \to \mathbb{R}_+$ mapping a demand vector to a positive real number.[6] The sum of the demands of all agents is given by $q(N) = \sum_{i \in N} q_i$. We denote the class of all cost allocation problems by $\mathcal{C}$. Similar as a payoff distribution, a ***cost allocation*** is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N_+$ allocating cost share x_i to agent $i \in N$.

A cost allocation problem can be translated into a TU cost game by defining the characteristic cost function $v : 2^N \to \mathbb{R}$ such that $v(S) = C(q_S, 0_{N \setminus S})$ for $S \subseteq N$ and $v(\emptyset) = 0$. Let $z := (q_S, 0_{N \setminus S})$ denote the vector $z \in \mathbb{R}^N$ such that $z_i = q_i$ if $i \in S$ and $z_i = 0$ if $i \in N \setminus S$. The obtained TU game is referred to as ***the induced cost game***, with $v(S)$ corresponding to the cost incurred by foreseeing in the demands of the agents in S. We denote a cost game, the same as any cooperative game, by the pair (N, v). With every cost game $(N, v) \in \mathcal{G}$ we can associate a profit game $p \in \mathcal{G}$ for all $S \subseteq N$ as follows

$$p(S) = \sum_{i \in S} v(i) - v(S).$$

So $p(S)$ represents the cost that coalition S saves by cooperation and therefore is also known as the ***cost-saving*** game. Below we present some well-known cost allocation problems and their induced cooperative games.

Example 2.3.1. ***(Airport problem and game I)*** *A famous cost allocation problem is the airport problem, introduced by Littlechild and Owen (1973).*[7] *The problem is the allocation of maintenance and building costs of one airport runway over different types of airplanes. So each airplane type $i \in N$ demands a runway of length l_i, which has a corresponding cost c_i. For simplicity assume that $l_i = c_i$. The elements of the airport problem (N, l, C) are presented by*

- $N = \{1, ..., n\}$ *denotes the set of airplane types that want to share a runway*
- $l = (l_i)_{i \in N} \in \mathbb{R}^N_+$ *is the demand vector, such that each airplane type $i \in N$ has a demand for a runway of length l_i*
- C *is the cost function defined by $C(l) = \max_{i \in N} l_i = \max_{i \in N} c_i$.*

Without loss of generality we can order the costs of the runway for the corresponding airplane types so that $0 < c_1 \leq c_2 \leq ... \leq c_n$.[8] *In figure 2.2 a visual representation of the problem is presented.*

[6] For some problems only a cost vector $c \in \mathbb{R}^N_+$ is given instead of C and q. In that case the problem is denoted by the pair (N, c).

[7] In the original problem different types of airplanes using an airport runway are charged for every airplane movement (take-off or landing). For the original game we refer to Littlechild and Owen (1973).

[8] Clearly the lengths l_i of the runway are ordered in the same way.

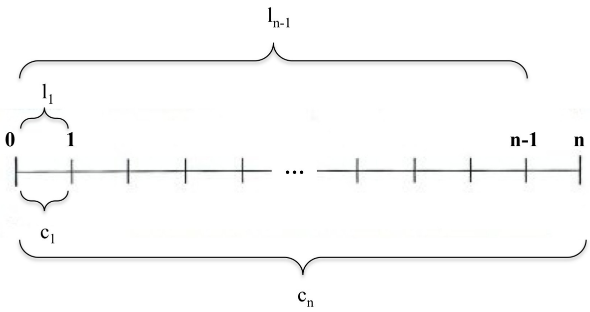

Figure 2.2: **Airport problem** (N, l, C)

The induced airport game is defined by the following characteristic function $(N, v) \in \mathcal{G}$ *for* $S \subseteq N$*:*

$$v(S) = C(l_S, 0_{N \setminus S}) = \max_{i \in S} c_i.$$

So $v(S)$ *represents the costs of building a runway suitable for all types of airplanes in* S*, where the type of airplane with the largest runway requirements is determinative. The characteristic function of the airport game is concave. Note that this game could also be presented by the pair* (N, c)*.*

Example 2.3.2. ***(Tennesee Valley Authority I)*** *The Tennessee Valley Authority (TVA) problem concerns the problem of sharing the cost of building a dam in the Tennessee River to realise a multi-purpose reservoir. This reservoir can be employed for navigation, flood control and hydro-electric power. Let* N *denote the set of purposes, such that* $N = \{1, 2, 3\}$*. This problem actually arose in 1930 for the Tennessee river. Each of the purposes imposes requirements on the dam. The problem is how to allocate the construction cost of the dam amongst the different services. In the original situation the following cost function was established for the three purposes of the reservoir, obtained from Young (1994):*

$\boldsymbol{S}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\boldsymbol{v(S)}$	0	16,3520	140,826	250,096	301,607	378,821	367,370	412,584

Table 2.1: Cost function v for $S \subseteq N$

The Authority considered different game theoretic solution concepts to solve this problem.

In the next chapter solution concepts are discussed and applied to some of the examples discussed in this chapter.

Example 2.3.3. ***(Polluted river sharing)*** *(Ni and Wang, 2007) Consider a river that is divided into* n *segments, ordered such that* $1 < 2 < .. < n$ *where* 1 *represents the* *most upstream segment and n the most downstream segment. Consider n agents, which are established alongside the river, one in each segment according to the above order. The river is polluted by agents in each segment and this pollution flows subsequently to downstream segments. The cost of cleaning segment i to obtain a pollutant free river equals* c_i*. The pollution cost vector is given by* $c = (c_1, .., c_n) \in \mathbb{R}_+^N$*. The problem is how to divide the total cleaning costs amongst the agents. Thus, the polluted river sharing problem is given by the pair* (N, c)*, with agent set N and cost vector c.*

In summary, in this chapter we introduced the electricity sector and electricity transmission costs and provided some basic theory of cooperative games and cost allocation. This background knowledge is important as we in Chapter 4 formally define a cost allocation problem in the electricity sector and introduce in Chapter 5 a corresponding cooperative cost game to this problem.

Chapter 3

Solution concepts and characterizations

Within cooperative game theory one can focus on the selection problem, that is caused by finding a composition of the coalition, or on the allocation problem, that is caused by constructing an allocation of the worth or cost of a coalition, or on both. In this thesis we concentrate on the allocation problem of costs. Resulting allocation vectors for agents are obtained by applying a solution concept to a problem or TU game. Within TU games it is mostly assumed that the grand coalition forms, such that a solution concept provides a rule for allocating the worth or cost of the grand coalition.

A cost allocation vector for a problem can be derived by either directly applying a rule to the problem or by applying a rule to the induced game. So the problem describes the actual situation, whereas the game is a mathematical model based on the problem. We refer to the first type of rules as (cost) ***allocation rules*** and to the second type of rules as ***solutions***. In figure 3.1 these two ways to obtain a cost allocation vector for a problem are displayed. By means of properties solution concepts can be characterized.[1] Solution concepts can either yield a ***set of allocation vectors*** (set of payoff distributions), e.g. the core, or provide a ***unique allocation vector*** (unique payoff distribution), e.g. Shapley value. Many different properties and a large variety of solution concepts are discussed in the literature. In this chapter we first discuss cost allocation rules and corresponding properties for cost allocation problems. Thereafter we define solutions and corresponding properties for TU games and finally we consider solutions (union values) and corresponding properties for TU games with coalition structure. Some properties for cost allocation rules and solutions coincide, but many differ and therefore we define for the cost allocation rules, the solutions and the union values separately a variety of

[1]So the notion of solution concepts refers to both allocation rules for problems and solutions for games.

(sometimes similar) properties.

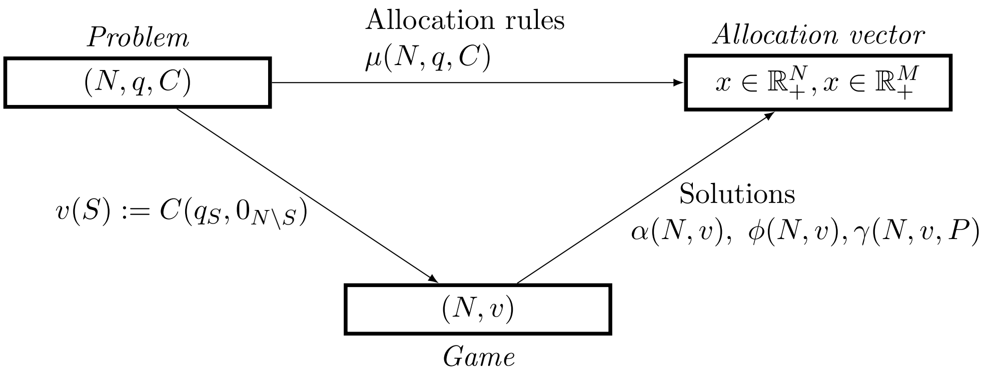

Figure 3.1: **Solution concept scheme** Overview how to obtain a cost allocation vector from a cost allocation problem, directly or indirectly via a cooperative game.

3.1 Cost allocation rules

In the last chapter we defined the triple $P = (N, q, C) \in \mathcal{C}$ representing a cost allocation problem. In this section we discuss several (cost allocation) rules to obtain an allocation vector and provide some important properties. We solely focus on allocation rules that provide ***a single cost allocation*** vector.

Definition 3.1.1. *(Cost allocation rule) A cost allocation rule is a function μ that associates with each problem $P \in \mathcal{C}$ a cost allocation vector $\mu(P)$ which assigns cost $\mu_i(P) \in \mathbb{R}^N_+$ to agent $i \in N$.*

There are many different rules for cost allocation problems. We introduce ***the egalitarian rule*** and ***the average cost pricing rule***, amongst others defined in Koster (2009). We define the cost function as a mapping from a demand vector to a non-negative real. A cost function can also be defined as a mapping on the non-negative real numbers such that $C(q(N))$ has to be shared amongst the agents. This type of cost function maps the sum of all demands of the agents to a non-negative real number. For these problems we refer to Koster (2009) or Sudhölter (1998). The first rule we discuss is the egalitarian rule. This rule does not distinguish between agents and allocates the cost shares equally over the agents.

Definition 3.1.2. *(Egalitarian rule) The egalitarian rule is given by*

$$EG_i(P) = \frac{C(q)}{n}, \tag{3.1}$$

for all $P \in \mathcal{C}$, all $i \in N$ and $|N| = n$.

So this rule only takes into account the total cost and the number of agents. The next rule is the average cost pricing rule. This is a rule that is suggested in the electricity pricing literature to allocate costs based on consumption. By means of this rule consumers are charged their number of units demanded times the average price per unit (Hakvoort and Huygen, 2012).

Definition 3.1.3. *(Average cost pricing rule) The average cost pricing rule is given by*

$$ACP_i(P) = \begin{cases} q_i \cdot \frac{C(q)}{q(N)} & \textit{if } q(N) \neq 0 \\ 0 & \textit{otherwise}, \end{cases} \tag{3.2}$$

for all $P \in \mathcal{C}$ *and all* $i \in N$.

Note that the average cost per unit demand is computed by $\frac{C(q)}{q(N)}$. This rule solely depends on the demand of the agents and the total cost of foreseeing in the demands. Application of this rule for transmission costs is however not straightforward as there still has to be distinguished between different types of consumers, based on voltage level, time, location and so on. Another suggested rule for consumption based cost allocation is ***marginal cost pricing*** (Hakvoort and Huygen, 2012). This rule charges consumers conform the marginal cost of one extra unit of electricity. The disadvantage of this rule however is that in case of a concave cost function, the cost allocation obtained by this rule does not cover all costs. Let us consider an example.

Example 3.1.1. ***(Airport problem II)*** *Consider again the airport problem, as defined in example 2.3.1. We present a numerical example and compute the allocation vectors according to the average cost pricing and egalitarian rule. In figure 3.2 a problem is presented with three airplane types in set* N*, denoted by* 1, 2, 3*. Node* 0 *is known as the source node, which does not represent an agent. The demand vector for the runway lengths is given by* $l = (l_1, l_2, l_3) = (10, 18, 30)$*, with corresponding cost vector* $c = (c_1, c_2, c_3) = (10, 18, 30)$.

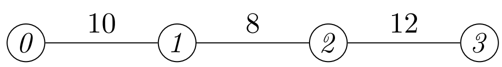

Figure 3.2: **Airport game** Numerical example.

The cost function gives $C(l) = \max_{i \in N} c_i = \max\{c_1, c_2, c_3\} = c_3 = 30$*. So the problem is how to share a total cost of* 30 *amongst airplane types* 1, 2 *and* 3*. The egalitarian rule gives the vector*

$$\begin{aligned} EG(P) &= (\frac{C(l)}{n}, \frac{C(l)}{n}, \frac{C(l)}{n}) \\ &= (\frac{30}{3}, \frac{30}{3}, \frac{30}{3}) \\ &= (10, 10, 10). \end{aligned}$$

The average cost pricing rule gives the vector

$$
\begin{aligned}
ACP(P) &= (c_1 \cdot \frac{C(l)}{l(N)}, c_2 \cdot \frac{C(l)}{l(N)} c_3 \cdot \frac{C(l)}{l(N)}) \\
&= (10 \cdot \frac{30}{58}, 18 \cdot \frac{30}{58}, 30 \cdot \frac{30}{58}) \\
&= (5, 9, 16)
\end{aligned}
$$

We see that both rules give very different cost allocation vectors. The egalitarian rule does not take any individual information into account, whereas the the average cost pricing rule allocates according to individual demands.

These rules give allocation vectors with cost shares for individual agents. However as noted above, application of these rules in the electricity sector still requires extra distinction between consumer types. One possibility is to distinguish between voltage levels. Therefore in Chapter 4 we focus on cost allocation to voltage levels, which are unions of agents. A following step could be the application of one of the rules discussed here, applied separately for the cost allocated to every voltage level, to obtain individual cost shares. We leave this step for future research.

3.1.1 Properties of cost allocation rules

We now discuss some properties associated to cost allocation problems (Sudhölter, 1998). By means of these properties we can characterize the rules. For all properties below we take $P := (N, q, C) \in \mathcal{C}$ a cost allocation problem and μ a cost allocation rule. The first two properties ensure that all costs are recovered.

- FE **Feasibility**: at least the total costs incurred by the demands should be allocated amongst the agents.
  For all $i \in N$ have $\sum_{i \in N} \mu_i(P) \geq C(q)$, for all $P \in \mathcal{C}$.
- EF **Efficiency**: the total costs incurred by the demands are exactly allocated amongst the agents.[2]
  For all $i \in N$ we have $\sum_{i \in N} \mu_i(P) = C(q)$, for all $P \in \mathcal{C}$.

If a cost allocation rule treats similar agents in a similar way, then either one of the two or both of the following properties are desirable.

[2]This condition is also known as the ***budget-balancing condition*** (Koster, 2009).

RAN **Ranking**: an agent with a demand that is at least as high as the demand of another agent, obtains a cost share that is at least as high as the cost share of the other agent.
For all $i, j \in N$, if $q_i \leq q_j$ then $\mu_i(P) \leq \mu_j(P)$, for all $P \in \mathcal{C}$.

ET **Equal Treatment of Equals**: agents with equal demands are allocated equal cost shares.
For all $i, j \in N$ we have that $q_i = q_j$ implies $\mu_i(P) = \mu_j(P)$, for all $P \in \mathcal{C}$.

More properties related to fair treatment of agents are the next two. The first property states that if an agent has no demand he or she is charged no costs and the second property states that a rule is independent of the order in which the agents are arranged.

NP **Null property**: an agent that has a zero demand, gets a zero cost share allocated.
For all $i \in N$ if $q_i = 0$ then $\mu_i(P) = 0$, for all $P \in \mathcal{C}$.

AN **Anonymity**: an allocation rule does not discriminate based on the names of the agents.
Let $P, P^\pi \in \mathcal{C}$ be such that $P = (N, q, C)$ and $P^\pi = (N, \pi q, C)$ for some permutation π of N and $\pi q = (q_{\pi(i)})_{i \in N}$. Then for all $i \in N$ it holds that $\mu_{\pi(i)}(P^\pi) = \mu_i(P)$.

The following two properties state that the allocation rule is monotonic and additive with respect to the cost function.

MON **Monotonicity**: the allocation rule gives increasing vectors for increasing cost functions.
Let $P^1, P^2 \in \mathcal{C}$ be such that $P^1 = (N, q, C^1)$ and $P^2 = (N, q, C^2)$ with $C_1 \leq C_2$. Then we have $\mu(P^1) \leq \mu(P^2)$.

ADD **Additivity**: splitting the cost function in any two parts and sharing the cost of each part separately according to the cost allocation rule results in the same allocation as applying the cost allocation rule to the total cost function.
Let $P^1, P^2, P^3 \in \mathcal{C}$ be such that $P^1 = (N, q, C^1)$, $P^2 = (N, q, C^2)$ and $P^3 = (N, q, C^1 + C^2)$. Then it holds that $\mu(P^1) + \mu(P^2) = \mu(P^3)$.

3.1.2 Comparison of the rules

In table 3.1 below we compare the two rules discussed in this chapter with respect to the above defined properties they obey.

	EG(P)	*ACP(P)*
FE	+	+
EF	+	+
RAN	+	+
ET	+	+
NP	-	+
AN	+	+
MON	+	+
ADD	+	+

Table 3.1: Summary of properties for cost allocation rules

From this table we may conclude that the average cost pricing rule satisfies all the properties discussed in this section. This observation does however not imply that the average cost pricing rule is the best rule. There are many other properties to come up with that are not satisfied by the average cost pricing rule or the egalitarian rule. For each allocation problem separately it should be identified which properties are crucial to be obeyed by the desired rule. So cost allocation rules and properties should always be placed in the context of the problem.

3.2 Solutions for TU games

In this section we focus on solutions of TU games. We first discuss solutions that provide one or more allocation vectors, allocating costs to individual agents. From now on we assume the characteristic function is a cost function, unless mentioned otherwise. We consider solutions for a game $(N, v) \in \mathcal{G}$. We make a distinction between ***multi-valued solutions*** and ***single-valued solutions***, referred to as ***values***. In the first category we consider the core and in the last category we consider the Shapley value, the separable cost remaining benefit solution, the proportional solution and the non-cooperative cost solution, amongst others defined in Koster (2009) and Hoàng (2012). We assume the grand coalition forms.

Definition 3.2.1. *(Multi-valued solution) A multi-valued solution is a function α that associates with each game $(N, v) \in \mathcal{G}$ a set of cost allocation vectors $\alpha(N, v) \subseteq \mathbb{R}^N$.*

The core is a well-known multi-valued solution, first introduced for profit games by Gillies (1953). The ***core*** of a cost game $(N, v) \in \mathcal{G}$ is defined by

$$core(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \leq v(S), \text{ for all } S \subseteq N\}.^3 \qquad (3.3)$$

[3] Note that allocation vector $x \in \mathbb{R}^N$ is in the core iff x is an ***imputation*** and ***group rational***.

Note that in case of a profit game, $x(S) \leq v(S)$ becomes $x(S) \geq v(S)$. So an allocation vector x for a cost game (N, v) is in the core if there is no coalition S or agent i that can do better by forming an alternative coalition. As no group of agents wants to abandon the grand coalition, this coalition can be considered ***stable***.

Definition 3.2.2. *(Value) A single-valued solution, referred to as a value, is a function ϕ that associates with each game $(N, v) \in \mathcal{G}$ exactly one cost allocation vector $\phi(N, v) \in \mathbb{R}^N$.*

One of the most famous values is the Shapley value, introduced by Shapley (1953). The Shapley value gives the average marginal contribution over all possible orders agents may join a coalition. Let $\pi : N \to N$ denote a permutation from the set of all permutation $\Pi(N)$ of N such that $\pi(i)$ indicates the position of agent i. The ***marginal vector*** $mc(\pi) \in \mathbb{R}^N$ is the cost vector such that $mc_i(\pi) = v(\{j \in N | \pi(j) < \pi(i)\} \cup \{i\}) - v(\{j \in N | \pi(j) < \pi(i)\})$.

Definition 3.2.3. *(Shapley value) The Shapley value is given by*

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} mc_i(\pi) \tag{3.4}$$

for all $(N, v) \in \mathcal{G}$ and all $i \in N$. Or equivalently,

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \big(v(S \cup \{i\}) - v(S) \big) . \tag{3.5}$$

So the core is a solution that relies on the ***stability*** of the grand coalition, whereas the Shapley value focusses on the ***fairness*** of a solution with respect to cost shares reflecting agents' marginal contributions. We now consider the Shapley value for two examples: the airport game and the minimum cost spanning tree game.

Example 3.2.1. ***(Airport game III)*** *Consider again the airport game, as defined in example 2.3.1 and 3.1.1. We present the same numerical example and compute the Shapley value of this game. In figure 3.3 the game is again presented with three airplane types in set N, denoted by 1, 2, 3 and 0 the source node. Each type of airplane has a demand for a runway length with associated costs c_1, c_2, c_3, corresponding to 10, 18, 30 respectively. In table 3.2 the corresponding cost game is given.*

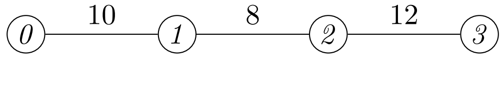

Figure 3.3: **Airport game** Numerical example.

$\boldsymbol{S}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\boldsymbol{v(S)}$	0	10	18	30	18	30	30	30

Table 3.2: Cost function v for $S \subseteq N$

Let us now compute the Shapley value by means of the formulas presented in definition 3.2.3. The marginal vectors and the Shapley value are given in table 3.3 below.

$\boldsymbol{\pi(N)}$	$\boldsymbol{mc_1(\pi)}$	$\boldsymbol{mc_2(\pi)}$	$\boldsymbol{mc_3(\pi)}$
$(1,2,3)$	10	8	12
$(1,3,2)$	10	0	20
$(2,1,3)$	0	18	12
$(2,3,1)$	0	18	12
$(3,1,2)$	0	0	30
$(3,2,1)$	0	0	30
$\sum_{\pi \in \pi(N)} mc_i(\pi)$	20	44	116
$Sh(N,v)$	*3.3*	*7.3*	*19.3*

Table 3.3: Marginal vectors and Shapley value for the airport game

As this is quite an extensive calculation in case N is large, Littlechild and Owen (1973) found a simple expression for the Shapley value of the airport game: for $c_1 \leq c_2 \leq ... \leq c_n$, the Shapley value is given by

$$Sh_i(N,v) = \sum_{j=1}^{i} \frac{c_j - c_{j-1}}{n-j+1}, \tag{3.6}$$

for $(N,v) \in \mathcal{G}$, for all $i \in N$, $j = 1,...,n$ and $c_0 = 0$. Or equivalently,

$$Sh_i(N,v) = Sh_{i-1}(N,v) + \frac{c_i - c_{i-1}}{n-i+1}. \tag{3.7}$$

Note that $c_i - c_{i-1}$ denotes the extra cost for a runway for airplane type i compared to a runway for airplane type $i-1$, corresponding to the cost of an edge in figure 3.3. Below we show that these formulas give the same vector as calculated in table 3.3:

$Sh_1(N,v) = \frac{10}{3} = 3.3$

$Sh_2(N,v) = \frac{10}{3} + \frac{8}{4} = 7.3$

$Sh_3(N,v) = \frac{10}{3} + \frac{8}{4} + \frac{12}{1} = 19.3$

Thus, the Shapley value is given by $Sh(N,v) = (3.3, 7.3, 19.3) \in \mathbb{R}^N_+$. *Note that this vector is efficient, i.e.* $\sum_{i \in N} Sh_i(N,v) = 30$.

Example 3.2.2. ***(Minimum Cost Spanning Tree Game)*** *Consider the graph presented in figure 3.4. Let* 0 *be the source node and* 1, 2, 3 *agents that want to be connected in the cheapest way to the source* 0*, by using the costly edges. For example, the minimum cost to connect agent 1 to the source is 10 and the minimum cost to connect agents 1 and 3 to the source is 12. A spanning tree is a subgraph connecting all the nodes. So a minimum cost spanning tree connects all agents in the least costly way directly or indirectly to the source. Let* (i,j) *denote the edge from agent* i *to* j *for* $i,j \in N$ *and* $c(i,j)$ *the corresponding cost. A minimum cost spanning tree in figure 3.4 is emphasized by the bold lines and given by* $\{(0,1),(1,2),(1,3))\}$ *with corresponding cost vector* $(10,5,2)$.

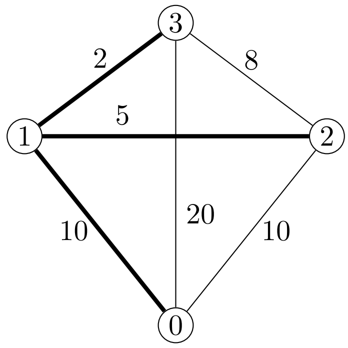

Figure 3.4: **A minimum cost spanning tree** Numerical example.

Let $G_N = (V_N, E_N)$ *be a minimum cost spanning tree connecting the agents in* N *to the source, presented by the set of nodes* $V_N = N \cup \{0\}$*, by means of edges from the set* E_N.[4] *For* $S \subseteq N$ *we have* $G_S = (V_S, E_S)$ *presenting the minimum cost spanning tree for nodes in* $S \cup \{0\}$*, connected by edges from the set* E_S*. Note that* G_S *does not have to coincide with* G_N*. The cost game* $(N,v) \in \mathcal{G}$ *is given by*

$$v(S) = \sum_{(i,j) \in E_S} c(i,j),$$

for $S \subseteq N$ *and* $v(\emptyset) = 0$*. In table 3.4 the characteristic function for the situation presented in figure 3.4 is given.*

The Shapley value of this game is $Sh(N,v) = (2.5, 5.5, 9.0) \in \mathbb{R}^N_+$*. Also this vector is efficient, i.e.* $\sum_{i \in N} Sh_i(N,v) = 17$.

[4]Note that a minimum cost spanning tree does not have to be unique.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	10	10	20	15	12	18	17

Table 3.4: Cost function v for $S \subseteq N$

The next theorem states a nice relation between the Shapley value and the core, given that a game is convex. For a proof of the theorem we refer to Shapley (1971, p.22).

Theorem 3.2.1. *(Shapley, 1971) If $(N, v) \in \mathcal{G}$ is a convex profit game, then the Shapley value $Sh(N, v)$ is in the core.*[5]

This property makes the Shapley value for convex games an appealing solution. A solution that was employed in the Tennessee Valley Authority and is applicable to multi-purpose projects, is the ***separable cost remaining benefit solution***, amongst others defined in Young (1994). The separable cost of agent i is defined by $s_i = v(N) - v(N \backslash \{i\})$ and the remaining benefit by $r_i = v(i) - s_i$. To obtain a positive r_i for all agents the function v should be at least sub-additive. So the separable cost of purpose i is the cost incurred by adding purpose i to the project. This solution concept allocates each agent with its separable cost and the remaining non-separable cost is subsequently shared proportionally to the remaining benefit r_i amongst the agents. More formally,

Definition 3.2.4. *(Separable cost remaining benefit solution) The separable cost remaining benefit solution is defined by*

$$SCRB_i(N, v) = s_i + \frac{r_i}{\sum\limits_{j \in N} r_j} \cdot r_N, \tag{3.8}$$

for all $(N, v) \in \mathcal{G}$, all $i \in N$ and with $r_N = v(N) - \sum\limits_{j \in N} s_j$.

This solution solely considers the coalitions of size 1, $N - 1$ and N. Three other solutions are defined below.

Definition 3.2.5. *(Proportional solution) The proportional solution is defined by*

$$Pr_i(N, v) = \frac{v(i)}{\sum_{j \in N} v(j)} \cdot v(N), \tag{3.9}$$

for all $(N, v) \in \mathcal{G}$ and all $i \in N$.

This solution allocates to each agent a cost share that is proportional to its individual cost. The next solution we have seen in a slightly different form as a cost allocation rule, namely the ***egalitarian rule***. Similar to this rule, does the egalitarian solution not distinguish between agents and allocates to all agents the same cost share.

[5]If (N, v) is a cost game, then $Sh(N, v)$ is in the core if (N, v) is concave.

Definition 3.2.6. *(Egalitarian solution) The egalitarian solution is given by*

$$Eg_i(N,v) = \frac{v(N)}{n}, \tag{3.10}$$

for all $(N,v) \in \mathcal{G}$ *and all* $i \in N$.

The last solution is a ***non-cooperative solution***, allocating each agents its individual cost.

Definition 3.2.7. *(Non-cooperative solution) The non-cooperative solution is defined by*

$$NC_i(N,v) = v(i), \tag{3.11}$$

for all $(N,v) \in \mathcal{G}$ *and all* $i \in N$.

This solution allocates exactly each agent's individual cost. Let us now consider the values defined above for the TVA example.

Example 3.2.3. ***(Tennessee Valley Authority II)*** *Consider again the Tennessee Valley Authority game defined in example 2.3.2. We compute all the single-valued solutions defined above. For the core solution we refer to Young (1994, p.1200). For the set of purposes* $N = \{1,2,3\}$ *we obtain the following cost allocations,*

	1	2	3
Sh(N,v)	117,829	100,757	193,999
SCRB(N,v)	117,476	99,157	195,951
Pr(N,v)	121,682	104,794	186,107
Eg(N,v)	137,528	137,528	137,528
NC(N,v)	163,520	140,826	250,096

Table 3.5: Different cost allocation vectors for TVA. Solutions are the Shapley value, the separable cost remaining benefit solution, the proportional solution, the egalitarian solution and the non-cooperative solution.

3.2.1 Properties of TU games

We consider some basic properties that are satisfied by some of the solutions. Comparing solutions with respect to the properties they obey, provides a way to judge about their fairness. Let $(N,v) \in \mathcal{G}$ be a TU game and ϕ a single-valued solution. Some of the properties were in an adjusted form already presented for cost allocation rules. The first two properties guarantee that at least the total costs are recovered.

EF **Efficiency**: exactly the total cost incurred by the grand coalition is allocated.
For all $i \in N$ we have $\sum_{i \in N} \phi_i(N,v) = v(N)$, for all $(N,v) \in \mathcal{G}$.

FE **Feasibility**: at least the total cost is allocated amongst the agents in N.
For all $i \in N$ we have $\sum_{i \in N} \phi_i(N,v) \geq v(N)$, for all $(N,v) \in \mathcal{G}$.

Note that if a solution is efficient, then it is also feasible. The next two properties are also related and define the impact of a null agent and a dummy agent. Agent $i \in N$ is a ***null agent*** if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus \{i\}$ and a ***dummy agent*** if $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N \setminus \{i\}$. Now consider the following properties.

NA **Null Agent**: if an agent does not inflict cost on any coalition S he or she joins, the agent is charged no cost.
For all $i \in N$ holds that if agent i is a ***null agent***, then $\phi_i(N,v) = 0$, for all $(N,v) \in \mathcal{G}$.

DA **Dummy Agent**: if an agent inflicts exactly its stand-alone cost to any coalition S he or she joins, the agent is charged its stand-alone cost.
For all $i \in N$ holds that if agent i is a ***dummy agent***, then $\phi_i(N,v) = v(i)$, for all $(N,v) \in \mathcal{G}$.

Note that a dummy agent with zero cost ($v(i) = 0$) is a null agent. The next property is a condition for solutions that are non-discriminatory with respect to symmetric agents. Agents $i,j \in N$ are ***symmetric*** if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i,j\}$. One can argue that two agents inflicting the same cost on every coalition, should be allocated equal cost shares. This idea is formalised by the symmetry property.

SYM **Symmetry**: equal cost shares are allocated to symmetric agents.
For all symmetric agents $i,j \in N$ it holds that $\phi_i(N,v) = \phi_j(N,v)$, for all $(N,v) \in \mathcal{G}$.

The final property we discuss here is ***additivity***. By means of additivity different cost parts can be allocated separately, without changing the total cost allocation for each agent. The sum of two games is defined as follows: $v + w(S) = v(S) + w(S)$ for $(N,v),(N,w) \in \mathcal{G}$ and $S \subseteq N$.

ADD **Additivity**: splitting the characteristic function in any two parts and allocating the cost of each part separately according to the solution results in the same allocation as applying the solution to the sum of the characteristic functions.
For all games $(N,v),(N,w) \in \mathcal{G}$ it holds that $\mu(N,v) + \mu(N,w) = \mu(N,v+w)$.

3.2.2 Comparison of the solutions

In table 3.6 below we compare the solutions with respect to the above defined properties they obey.

	Sh(N,v)	*SCRB(N,v)*	*Pr(N,v)*	*Eg(N,v)*	*NC(N,v)*
EF	+	+	+	+	-
FE	+	+	+	+	+
NA	+	+	+	-	+
DA	+	+	-	-	+
SYM	+	+	+	+	+
ADD	+	-	-	+	+

Table 3.6: Summary of properties satisfied by different solutions

From this table we may conclude that the Shapley value satisfies all the properties discussed in this section. However, as discussed before, this observation does not imply that the Shapley value is the best solution, as the context is important. For example, the non-cooperative solution can be argued from this table to be quite desirable, since it satisfies most of the properties. Only, in many situations, efficiency is an essential condition for a fair allocation. Further does this solution give no incentives for cooperation, which can be argued to be the whole point of considering cooperative game theoretic solutions. Consider the following axiomatic characterization of the Shapley value by Shapley. For a complete proof we refer to Shapley (1953) or Gilles et al. (1992, p.97).

Theorem 3.2.2. *(Shapley, 1953) The Shapley value $Sh(N,v)$ is the unique solution that satisfies the properties efficiency, dummy agent, symmetry and additivity.*

In the literature many axiomatic characterizations are discussed for the Shapley value, for example in Gilles (2010) characterizations of Shapley, Young and van den Brink are presented. Above we solely considered the first. Also for the other solutions presented in this section axiomatic characterization exist, but are not discussed in this thesis.

3.3 Union values for TU games with coalition structure

Let us now consider games with ***coalition structure*** Owen (1977). A single-valued solution for a game with coalition structure is a function that assigns to every game $(N,v,P) \in \mathcal{GP}$ one allocation vector that defines the cost for every agent $i \in N$. A game with coalition structure is also referred to as a game with ***a priori unions***. The idea behind a priori unions is that some groups of agents are more likely to cooperate within the grand coalition than others. For example in the electricity grid all agents attached to the same voltage level can be seen as an a priori union. If we want to know how to allocate the cost of the grand coalition amongst the unions instead of the agents, we have to consider ***union values***. A union value is a mapping assigning one cost share to every union of agents. So we now assume that not the agents, but the unions of agents are the decision makers. In van den Brink and Dietz (2014) two Shapley-related union values and corresponding axiomatizations are introduced, which are highlighted in this section. Note that the solutions discussed before can also be translated into union values, by taking the union as the decision making agent. Consider the cost game v with agent set N and partition in a priori unions P such that $(N, v, P) \in \mathcal{GP}$.

Definition 3.3.1. *(Union value) A union value is a function γ that associates with each game $(N, v, P) \in \mathcal{GP}$ exactly one cost allocation vector $\gamma(N, v, P) \in \mathbb{R}^M$, where $M = |P|$, assigning a cost share to every union in a TU game with coalition structure.*

The first value we define is the ***union-Shapley value***. This value interprets a union as one, such that a union enters a coalition with all its agents at once.

Definition 3.3.2. *(The union-Shapley value) The union-Shapley value is given by*

$$Sh^u(N, v, P) = Sh(M, v^p) \tag{3.12}$$

where $v^p(U) = v(\cup_{k \in U} P_k)$ for all $(N, v, P) \in \mathcal{GP}$, for $P = \{P_1, ..., P_m\}$, $M = \{1, ..., m\}$, $k \in M$ and $U \subseteq M$.

The second value we define is the ***agent-Shapley value***. This value first considers the Shapley value of the individual agents in the game and then sums over the Shapley values of the agents belonging to the same union.

Definition 3.3.3. *(The agent-Shapley value) The agent-Shapley value is given by*

$$Sh^a_{P_k}(N, v, P) = \sum_{i \in P_k} Sh_i(N, v) \tag{3.13}$$

for all $(N, v, P) \in \mathcal{GP}$, all $P_k \in P$ and $P = \{P_1, ...P_m\}$.

We clarify these two notions by means of the airport game.

Example 3.3.1. (Airport game - IIII) *Consider again the airport game as presented in examples 2.3.1, 3.1.1 and 3.2.1. Union values for the airport game are also discussed in van den Brink and Dietz (2014). We compute the union- and agent-Shapley value for the game defined before, but now with coalition structure. Let N be the same set of airplane types, but now partitioned into airlines. Airline 1 possesses airplane types 1 and 3 and airline 2 possesses airplane type 2. So the elements of the game are given by:*

- $N = \{1, 2, 3\}$ *is the set of airplane types*
- $P = \{P_1, P_2\} = \{\{1, 3\}, \{2\}\}$ *is the partition into airlines, where* P_k *denotes airline* $k \in M$
- $v(S) = \max_{i \in S} c_i$ *is the characteristic function for all* $S \subseteq N$.

The union-Shapley value is computed by finding the Shapley value of the game

$$v^p(U) = v(\cup_{k \in U} P_k) = max\{c_i | i \in \cup_{k \in U} P_k\},$$

for all $U \subseteq M$. *In words, the largest airplane type* $i \in N$ *in the coalition* $\cup_{k \in U} P_k$ *determines the cost of the coalition. The union game* $(M, v^P) \in \mathcal{GP}$ *is presented in figure 3.5.*

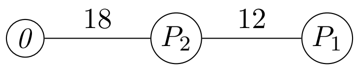

Figure 3.5: **Airport game** (v^P) Numerical example.

With regards to the numerical example we obtain for $v^P(U)$ *for all* $U \subseteq M$, $v^P(1) = 30$, $v^P(2) = 18$ *and* $v^P(1, 2) = 30$. *The resulting allocation vector for the union-Shapley value is* $Sh^u(N, v, P) = (21, 9) \in \mathbb{R}^2_+$.

The agent-Shapley value as defined by formula (2.13) is given by

- $Sh^a_1(N, v, P) = \sum_{i \in P_1} Sh_i(N, v) = Sh_1(N, v) + Sh_3(N, v) = 3.3 + 19.3 = 22.6$
- $Sh^a_2(N, v, P) = \sum_{i \in P_2} Sh_i(N, v) = Sh_2(N, v) = 7.3$

The resulting allocation vector for the agent-Shapley value is $Sh^a(N, v, P) = (23, 7) \in \mathbb{R}^2_+$. *We find that the two values give different allocation vectors. For the union-Shapley value the marginal contribution of an airline is determined by the cost of the largest airplane type. Thus if the size of an airplane is the main cost driver for a runway, this seems a reasonable solution. If we on the other hand also want to take into account the use of the runway, the agent-Shapley seems more reasonable. In this case first the cost for each airplane type is determined by means of the Shapley value, where after an airline pays the sum of these costs, depending on its types of airplanes.*

The difference between the union- and agent-Shapley value can be formalised by means of an axiomatic characterization. Therefore, we first consider some important properties.

3.3.1 Properties of TU games with coalition structure

We provide several properties for union values, based on the properties defined in van den Brink and Dietz (2014). Some properties correspond to properties discussed before, but are adapted to fit union values. Let $(N, v, P) \in \mathcal{GP}$ be a game with coalition structure and γ a union value. The first property is discussed before and implies that all costs are recovered.

EF **Efficiency**: the cost of the grand coalition is exactly allocated amongst the unions in P.
For all $k \in M$ we have that $\sum_{k \in M} \gamma_{P_k}(N, v, P) = v(N)$, for all $(N, v, P) \in \mathcal{GP}$.

In section 3.2 we presented the definition of a ***null agent***. The following property considers the effect of deleting a null agent from a union for the cost allocation. The null agent out property was introduced by Derks and Haller (1999) for TU games.[6]

NAO **Null Agent Out**: deleting a null agent from any union has no effect on the cost shares of the unions.
If agent $i \in P_k$ is a null agent and $|P_k| \geq 2$, then $\gamma(N, v, P) = \gamma(N \setminus \{i\}, c_{N \setminus \{i\}}, (P \setminus \{P_k\}) \cup \{P_k \setminus \{i\}\})$, for all $(N, v, P) \in \mathcal{GP}$.

Note that the union P_k cannot be a singleton, since then the union would not exist once agent i leaves, therefore the condition $|P_k| \geq 2$ is required. Due to this condition, agents can be excluded from the game, but not entire unions. Next we consider symmetric unions. Unions $k, l \in M$ are ***symmetric*** if $|P_k| = |P_l|$ and there exist a permutation $\pi^k = (\pi_1, ..., \pi_{|P_k|})$ on P_k and a permutation $\pi^l = (\pi_1, ..., \pi_{|P_l|})$ on P_l such that $v(S \cup \{\pi_i^k\}) = v(S \cup \{\pi_i^l\})$ for all $i \in 1, ..., |P_k|$ and $S \subseteq N \setminus \{\pi_i^k, \pi_j^l\}$. So two unions are symmetric if the agents of the unions can be ordered such that there is a one to one symmetry correspondence between the agents in one union and the other union in game $(N, v, P) \in \mathcal{GP}$.

SYM **Symmetry**: symmetric unions are allocated the same cost share.
If unions $k, l \in M$ are symmetric, then $\gamma_{P_k}(N, v, P) = \gamma_{P_l}(N, v, P)$, for all $(N, v, P) \in \mathcal{GP}$.

Strong monotonicity is adapted from the definition of Young (1985) and states that if the marginal contributions of all agents of a union in a game v are at least as high as in a game w, then this union gets a cost share in game v that is at least as high as in game w. Marginality is a weaker version of strong monotonicity.

[6] In Derks and Haller (1999) the property is referred to as the null player out property.

SM **Strong Monotonicity**: if the marginal contributions for all agents in union P_k in game v are at least as high as the marginal contributions for all agents in union P_k in game w, then the cost shares for the union in game v should be at least as high as in game w .
For all $i \in P_k$, $k \in M$ and $S \subseteq N \setminus \{i\}$ such that $v(S \cup i) - v(S) \geq w(S \cup i) - w(S)$ it holds that $\gamma_{P_k}(N, v, P) \geq \gamma_{P_k}(N, w, P)$, for all $(N, v, P), (N, w, P) \in \mathcal{GP}$.

MR **Marginality**: if the marginal costs for all agents in union P_k in game v are equivalent to the marginal costs for all agents in union P_k in game w, then they should obtain equal cost shares.
For all $i \in P_k$, $k \in M$ and $S \subseteq N \setminus \{i\}$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ it holds that $\gamma_{P_k}(N, v, P) = \gamma_{P_k}(N, w, P)$, for all $(N, v, P), (N, w, P) \in \mathcal{GP}$.

For games with a priori unions some extra properties with respect to collusion are important. In van den Brink and Dietz (2014) two types of collusion properties with respect to union values are considered, ***collusion between agents*** and ***collusion between unions***. The first concept is based on two agents colluding defined by the proxy agreement described in Haller (1994). If agent $i \in N$ acts as a proxy agent for agent $j \in N \setminus \{i\}$ then we define the characteristic function $(N, v_{ij}) \in \mathcal{G}$ instead of v, as follows

$$v_{ij} = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S \cup \{j\}) & \text{if } i \in S. \end{cases} \tag{3.14}$$

So the meaning of agent i being a proxy agent for an agent j is that an agent j only incurs its cost in a coalition when agent i is also in that coalition. As long as agent j is in a coalition without agent i, agent j can be seen as a null agent. This brings us to the agent collusion neutrality axiom.

ACN **Agent Collusion Neutrality**: collusion of two agents i, j belonging to the same union, does not change the cost share of this union.
For all $i, j \in P_k$ and $k \in M$ it holds that $\gamma_{P_k}(N, v, P) = \gamma_{P_k}(N, v_{ij}, P)$, for every $(N, v, P) \in \mathcal{GP}$.

Now we consider collusion between unions instead of between agents. Collusion between unions P_k and P_l is described by the union of the these two unions, such that we obtain the partition $P^{kl} = (P \setminus \{P_k, P_l\} \cup \{P_k \cup P_l\})$. Without loss of generality we may assume that $k < l$, such that we may reorder the partition P^{kl} with $P^{kl}_k = P_k \cup P_l$ and $P^{kl}_h = P_h$ for all $h \in M \setminus \{k, l\}$. Note that unions P_k, P_l do not have to be consecutive.

UCN **Union Collusion Neutrality**: collusion of two unions P_k, P_l does not change the total cost share of these unions.

For all $k \in M$ it holds that $\gamma_{P_k}(N, v, P) + \gamma_{P_l}(N, v, P) = \gamma_{P_k}(N, v, P^{kl})$, for all $(N, v, P) \in \mathcal{GP}$.

Note that collusion of agents or unions changes the cardinality of the agent set or the union set, respectively.

3.3.2 Comparison of the union values

In table 3.7 an overview is presented of the properties that are obeyed by the union- and the agent-Shapley value.

	$Sh^u(N, v, P)$	$Sh^a(N, v, P)$
FE	+	+
NAO	+	+
SYM	+	+
SM	+	+
MR	+	+
ACN	+	-
UCN	-	+

Table 3.7: Summary of properties satisfied by agent- or union-Shapley value

The two values only differ in the collusion neutrality property they obey. The vector obtained by the union-Shapley value is not affected by collusion of two agents in the same union, whereas for the vector obtained by the agent-Shapley value it holds that after collusion of two unions, the sum of the cost shares of these unions is not affected. We consider for each of the union values a unique characterization, as presented and proved in van den Brink and Dietz (2014).

Theorem 3.3.1. *The union-Shapley value is the unique union value that satisfies efficiency, marginalism, symmetry, the null agent out and the agent collusion neutrality.*

Theorem 3.3.2. *The agent-Shapley value is the unique value that satisfies efficiency, marginality, symmetry and union collusion neutrality.*

In summary, in this chapter we distinguished between cost allocation rules for cost allocation problems, solutions for TU games and union values for TU games with coalition structure. We use the term 'solution concepts' as an umbrella term to refer to all of them. All three types of solution concepts provide at least one allocation vector for either agents or unions of agents. In the following chapters some solution concepts and corresponding properties of this chapter are repeated, but also new solution concepts and properties are introduced.

Chapter 4

Electricity demand problem

The ***electricity demand problem*** concerns the problem of ***reallocation*** of the total transmission-related electricity grid costs, which is the sum of the voltage level costs, over unions of agents that are connected to a specific voltage level in the grid and where the agents are endowed with an electricity demand. For convenience from now on we also refer to the transmission-related costs (as defined in Chapter 2) simply as grid costs or electricity costs.

The model we describe in this chapter represents a non-trivial extension of the airport problem, that includes a fixed cost and variable cost depending on the demands of the agents. The model gives a simplified representation of the current situation and provides the opportunity to analyse the currently employed cascade rule and other cost allocation rules for the electricity demand problem. This model only takes into account the demands of the agents and not the production capacities. We silently assume that the production equals the demand and that electricity is only fed into the highest voltage levels in the grid, where after the excess electricity flows to adjacent lower voltage levels. Before we define the framework of the problem, we give a short introduction on the background of the cost allocation problem. Subsequently we define the ***level paying rule***, the ***equal downstream rule*** and the ***cascade rule***. The first two rules are familiar rules, analysed for other problems (e.g. in Ni and Wang (2007)). They are adapted to fit our problem. The cascade rule is currently used in the electricity sector. This rule is not formally defined in any cost allocation literature, but only explained in electricity research reports, such as in Hakvoort et al. (2013). We point out the difference and similarities between the rules and between the rules as proposed for this problem and for other problems in the literature. Further, we define some relevant properties and compare the rules with respect to the properties they obey. As the cascade rule has the focus, for this rule we prove which properties are satisfied and which not. The main contribution of this chapter is the analysis of several cost allocation rules for our problem in comparison with rules proposed in the literature on other problems, in the theoretical cost allocation literature (e.g. in Ni and Wang (2007)) as well as in the more practice oriented literature (e.g. in Hakvoort et al. (2013)). So suggestions on the cost allocation from practice and theory are combined and formalised.

4.1 The framework

The electricity grid is characterized by high fixed costs and considerably low variable costs. So once the infrastructure is there (i.e. the maintenance and expansion costs are covered) the actual flow of electricity imposes little extra costs. As electricity takes any path available (i.e. the magnitude of the electricity flow over a path depends on the path's voltage and resistance) it is hard to determine for consumers which path the electricity flow followed and thus which source the electricity originated from. Due to the above mentioned reasons a network flow model is, even though applicable, not our primary choice. Also, we do not have a cost minimization or cost optimization problem: the objective of this research is not to find a shortest path or construct the cheapest flow. Assuming the electricity grid is present, we aim to find a ***fair*** way to divide the associated costs over the different unions of agents.[1]

Consider the electricity grid consisting of multiple voltage levels, as depicted in the introduction in figure 1.1. Based on the size of the electricity connection of a consumer, which is on its turn based on the consumer's peak demand during the year, he or she is connected to a particular voltage level. Each consumer has a demand, which is considered to be price inelastic, since we assume that electricity is a necessity good. We further assume that electricity is fed into the grid on the highest voltage level and subsequently flows downstream to foresee in the demand of the agents connected to lower levels. We ignore reactive power and grid losses and assume that no electricity is lost during transmission over the grid. So the electricity fed in to the grid equals the sum of the demands of the agents. Every network operator assesses the costs of each of its voltage levels, which we assume depend on a fixed and variable part. The problem is how to allocate the cost of each voltage level amongst all the unions of agents connected to the voltage levels, taking into account the flow of the electricity through the grid. Before we provide a detailed framework for the problem, we present the elements of the electricity demand problem.

The ***electricity demand problem*** concerns the problem of dividing the total cost of the electricity grid in a fair way over the voltage levels and associated unions of agents. The problem is defined by the quadruple $P = (N, L, d, c)$, such that

- $N = \{1, ..., n\}$ is a finite set of agents
- $L = \{L_1, ..., L_m\}$ is a partition of N

[1]Note that there is not one interpretation of ***fair***, this depends on the properties one considers fair.

- $d = (d_i)_{i \in N}$ is the demand vector of the agents in N
- $c = (c_{L_k})_{k \in M}$ is the level cost vector where $M = \{1, ..., m\}$.

Let $N = \{1, ..., n\} \subset \mathbb{N}$ be a finite set of agents, representing the consumers of electricity in the network. We suppose this set of agents is endowed with a ***partial order*** $\leq$, i.e. a binary relation which is reflexive, antisymmetric and transitive. Without loss of generality we assume the agents are ordered from 1 to n. The agents are partitioned in a priori unions by the different voltage levels in the grid, given by the partition $L = \{L_1, ...L_m\}$. So the highest voltage level has the lowest ranking, i.e. L_1 is the highest voltage level. Hence all agents connected to a particular voltage level belong to the same union. Therefore the notions union and voltage level are used interchangeably and both refer to an element of the partition. Element L_k of L is referred to as the k-th level of the network. The set of levels is given by $M = \{1, ..., m\}$ such that $k \in M$ for L_k.[2] The level to which agent $i \in N$ belongs is presented by $l(i)$. Thus, $l(i) \in M$ is such that $i \in L_{l(i)}$. In figure 4.1 we find a representation of this partition. Let $\mathcal{L}^N$ denote the collection of all partitions of N. Note that by the partial ordering on the agents we have:

- $\forall i \in N$: $l(i) = k \iff i \in L_k$
- $\forall i, j \in N$: $i, j \in L_k$ and $i < h < j \Rightarrow h \in L_k$
- $\forall i, j \in N$: $i < j, i \in L_p$ and $j \in L_s \Rightarrow p \leq s$.

In words:

- The level of agent i defines which a priori union agent i belongs to.
- For each two agents belonging to the same union it holds that any agent in between these two agents in the order $1 < 2 < ... < n$, also belongs to that same union.
- For each two agents i and j so that agent i is smaller than agent j it holds that they either belong to the same union or agent i belongs to a lower labeled union (higher voltage level), i.e. $l(i) \leq l(j)$.

[2]We use both $k \in M$ and $L_k \in L$ to denote the k-th level of L.

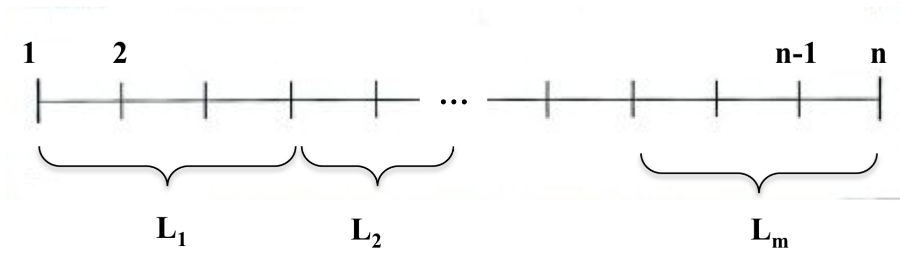

Figure 4.1: **Visual representation of the ordering and partition of the agents**

The cardinality of the sets are $|N| = n$ and $|L| = m$. Each agent in the model has a demand representing its electricity demand.[3] The total demand passing a voltage level L_k is the aggregated demand of the individual demands of all agents connected to that voltage level or lower voltage levels, i.e. the sum of the demands of all agents in L_j with $j \geq k$.[4] Let $d = (d_i)_{i \in N} \in \mathbb{R}_+^N$ be the demand vector where d_i denotes the demand of agent i.[5] The total demand of the agents in L_k is represented by

$$\hat{d}_{L_k} := \sum_{i \in L_k} d_i \tag{4.1}$$

with $\hat{d} = (\hat{d}_{L_k})_{k \in M} \in \mathbb{R}_+^M$. The ***aggregated demand vector*** is denoted by $\bar{d} = (\bar{d}_{L_k})_{k \in M} \in \mathbb{R}_+^M$, where $\bar{d}_{L_k}$ is given by

$$\begin{aligned} \bar{d}_{L_k} &:= \sum_{i \in N, l(i) \geq k} d_i \\ &:= \sum_{j \in M, j \geq k} \hat{d}_{L_j}, \end{aligned} \tag{4.2}$$

such that $\bar{d}_{L_k}$ represents the demand passing level k. Note that by this definition it is assumed that the net electricity flow streams from higher voltage levels to lower voltage levels. Each level has an associated cost, consisting of a fixed and variable part. The ***level cost vector*** $c = (c_{L_k})_{k \in M} \in \mathbb{R}_+^M$, with c_{L_k} representing the cost of voltage level k is defined by

$$c_{L_k} := a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k}, \tag{4.3}$$

with constants $a_{L_k}, b_{L_k} \in \mathbb{R}_+$, such that $a = (a_{L_k})_{k \in M}, b = (b_{L_k})_{k \in M} \in \mathbb{R}_+^M$. However, in this chapter it is not yet relevant that the level cost vector consists of the sum of these

[3] In Chapter 3 the ***demand vector*** is denoted by the letter q, whereas for this problem it is denoted by the letter d.

[4] Level 1 represents the highest voltage level and level m the lowest.

[5] We take $\mathbb{R}_+$ as the set of non-negative reals including zero, i.e. $\mathbb{R}_+ = \mathbb{R}_{\geq 0}$.

two parts. This only becomes significant in the next chapter. For this reason we choose to take c as the given element in the problem in this chapter, instead of the elements a_{L_k}, b_{L_k} for all $k \in M$.

Given a partially ordered set $(N, \leq)$, the element $g \in N$ is a greatest element of N if for all $h \in N$ we have $h \leq g$. We denote a greatest elements of N by $\max N$, i.e. $\max N = \max_{j \in N} j$. The level of the greatest element of N is given by $l(\max N)$, such that $l(\max N) = \max_{j \in N} l(j)$. The total cost of foreseeing in the demands of all agents is given by the function

$$\begin{aligned} C(d) &:= \sum_{k=1}^{l(\max N)} c_{L_k} \\ &:= \sum_{k=1}^{m} c_{L_k}, \end{aligned} \tag{4.4}$$

such that C is the total cost function $C : \mathbb{R}_+^N \to \mathbb{R}_+$, mapping a demand vector to the non-negative reals and $c_{L_k} \in \mathbb{R}_+$ as defined in equation (4.3).

To summarize, the electricity demand problem is in its entirety defined by
$P = (N, \leq, L, d, a, b)$, where N is a finite set of agents, $\leq$ is a partial ordering on agent set N, L is a partition of N representing the voltage levels, d is the demand vector of the agents in N and a, b are vectors of constants for the level cost vector. However, we assume that $\leq$ is implicitly incorporated in N and for simplicity we assume (in this chapter) that c is given instead of the vectors of constants a, b and therefore we denote the problem P from now on by $P = (N, L, d, c)$. Other elements, such as the aggregated demand vector of the voltage levels $\bar{d}$, the level demand vector $\hat{d}$ and the total cost function C can be derived from the problem. Denote the class of electricity demand problems by $\mathcal{C}_{ed}$. Below we clarify the problem by means of a numerical example.

Example 4.1.1. *Consider a situation with eight agents and a partition in three a priori unions. We define the elements of a problem $P \in \mathcal{C}_{ed}$ as follows:*

- $N = \{1, ..., 8\}$
- $L = \{L_1, L_2, L_3\}$ *with* $L_1 = \{1, 2\},\ L_2 = \{3, 4\},\ L_3 = \{5, 6, 7, 8\}$
- $d = (30, 20, 15, 15, 5, 5, 10, 10)$
- $c = (230, 125, 62)$

Now we have defined the problem we can deduce the other elements. In figure 4.2 we find a representation of the situation.

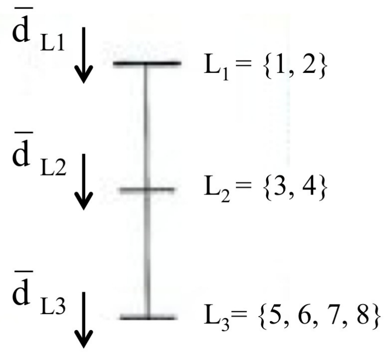

Figure 4.2: **Example with eight agents and three levels**

By means of the demand vector we can obtain the level demand vector $\hat{d}$:

$$\hat{d} = \begin{bmatrix} \hat{d}_{L_1} \\ \hat{d}_{L_2} \\ \hat{d}_{L_3} \end{bmatrix} = \begin{bmatrix} d_1 + d_2 \\ d_3 + d_4 \\ d_5 + d_6 + d_7 + d_8 \end{bmatrix} = \begin{bmatrix} 50 \\ 30 \\ 30 \end{bmatrix}$$

Finally, the aggregated demand vector $\bar{d}$ *is given by,*

$$\bar{d} = \begin{bmatrix} \bar{d}_{L_1} \\ \bar{d}_{L_2} \\ \bar{d}_{L_3} \end{bmatrix} = \begin{bmatrix} d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 \\ d_3 + d_4 + d_5 + d_6 + d_7 + d_8 \\ d_5 + d_6 + d_7 + d_8 \end{bmatrix} = \begin{bmatrix} 110 \\ 60 \\ 30 \end{bmatrix}$$

The total cost of the grid is the sum of the cost of the levels, hence

$$C(d) = \sum_{k=1}^{3} c_{L_k} = 417.$$

The electricity demand problem concerns finding the best allocation of the 417 *amongst the different a priori unions* L_1, L_2 *and* L_3.

Electricity grid costs consist of long-term marginal costs, short-term marginal costs and fixed costs. The long-term marginal costs concern the expansion of the transmission capacity of the grid. The short-term marginal costs concern the transmission losses and capacity shortages. And finally the fixed costs involve maintenance and capital costs of the investments (Huygen, 1999). In this model we assume the grid is already there and the fixed costs are sunk, i.e. these cost are already made. The short-term marginal costs depend mainly on the quantity of the transported electricity, as large quantities at one time may incur for example congestions. Note that quantity electricity can be interpreted in multiple ways, namely quantity in terms of consumption or capacity (electricity units). At different voltage levels different electricity units are directive. For now however we assume each agent has one type of demand and this will be leading for every voltage level.

To be clear, in this thesis we take into account all agents connected to the grid, but assume the ***unions of agents*** are the ***decision making agents***. Consequently, instead of considering rules as mappings assigning cost shares to individual agents, we consider rules as mappings assigning cost shares to unions of agents. For future research it would be highly interesting to consider solutions of the first type, assigning cost shares to individual agents.

One of the difficulties with electricity is that net electricity flow between voltage levels can be different from the actual flow. The flow goes mostly downstream, but sometimes also upstream, meaning that electricity flows from level L_{k+1} to L_k. As in total more electricity streams from L_k to L_{k+1} than the other way around, we in this model only consider the net flow between the voltage levels. This is also what is considered in real-life with the cascade method. It is expected that this might change in the future in view of the increase in distributed generation.[6]

4.2 Rules

A vector of cost shares, dividing the costs amongst unions, can be obtained by applying a cost allocation rule μ, which is a function mapping a problem to a cost vector. Formally, define rule μ as a mapping that assigns to every problem $P = (N, L, d, c) \in \mathcal{C}_{ed}$ a vector of cost shares $y \in \mathbb{R}^M_+$. In this section we discuss three cost allocation rules, including the before-mentioned cascade rule. Some rules are informally suggested in Hakvoort and Huygen (2012) and NMA and SEO (2011) and others are formally, for a different problem, described in Ni and Wang (2007). We use subscripts L_k, L_l to refer to union cost shares and i, j to refer to agent cost shares, thus μ_{L_k} versus μ_i respectively.

Level paying rule

In Hakvoort and Huygen (2012) it is suggested that each voltage level pays for the associated cost of the voltage level. They acknowledge this could cause resistance from the consumers attached to the highest voltage levels since as long as producers are not charged for feeding electricity into the grid, the costs for the few consumers connected

[6]With distributed generation all production realised at the distribution networks is referred, i.e. production plants at lower voltage levels.

to that grid will become extremely high. The level paying rule describes this allocation method, which simply allocates the cost of each voltage level to the union of consumers connected to that voltage level. Formally, the ***level paying rule*** can be defined by the function LP that associates with every $P \in \mathcal{C}_{ed}$ a cost allocation vector $LP(P) \in \mathbb{R}_+^M$ such that

$$LP_{L_k}(P) := c_{L_k}, \tag{4.5}$$

for all $k \in M$. So this rule allocates to each union the cost share corresponding to its level, despite the fact that downstream unions also take advantage of this level. In Ni and Wang (2007) a similar rule is considered for the polluted river sharing problem, which is presented in example 2.3.3, namely the Local Responsibility Sharing method. This rule charges agents in every segment its own local pollution cleaning costs, based on the idea that every segment is responsible for its own pollution and therefore also the cleaning. An important difference is that the cost of a level in our model depends on the demand of that voltage level and on the demand of the downstream voltage levels. Thus, by charging each voltage level with its corresponding level cost, we do not take these dependencies into consideration.

Example 4.2.1. *Consider again the example as described in example 4.1.1. The level paying rule gives the following cost allocation vector*

$$\begin{aligned} LP_{L_1}(P) &= c_{L_1} = 230 \\ LP_{L_2}(P) &= c_{L_2} = 125 \\ LP_{L_3}(P) &= c_{L_3} = 62 \end{aligned}$$

Thus the allocation vector is given by $LP(P) = (230, 125, 62) \in \mathbb{R}_+^3$. *Clearly this rule is efficient, i.e.* $\sum_{k \in M} LP_{L_k}(P) = \sum_{k \in M} c_{L_k}$.

Equal downstream rule

It can be argued that the level paying rule is not fair for the upstream levels, for a brief discussion see Hakvoort and Huygen (2012). In NMA and SEO (2011) it is proposed to use a proportional downstream allocation method, partly because of computational convenience. In this section we discuss the equal downstream rule. This rule is inspired on the Upstream Equal Sharing rule, as discussed in Ni and Wang (2007). The equal downstream rule charges all downstream unions for the upstream level costs proportional to the number of unions using the corresponding level. Formally the ***equal downstream rule*** can be defined by the function ED that associates with every $P \in \mathcal{C}_{ed}$ a cost allocation vector $ED(P) \in \mathbb{R}_+^M$ such that

$$ED_{L_k}(P) := \sum_{h=1}^{k} \frac{1}{m - h + 1} \cdot c_{L_h}, \tag{4.6}$$

for all $k \in M$ and $m = l(\max N)$.

Example 4.2.2. *Consider again the example as described in example 4.1.1. The equal downstream rule gives the following cost allocation vector*

$$\begin{aligned} ED_{L_1}(P) &= \frac{1}{3-1+1} \cdot c_{L_1} \\ &= \frac{1}{3} \cdot c_{L_1} \\ &= 77 \end{aligned}$$

$$\begin{aligned} ED_{L_2}(P) &= \frac{1}{3-1+1} \cdot c_{L_1} + \frac{1}{3-2+1} \cdot c_{L_2} \\ &= \frac{1}{3} \cdot c_{L_1} + \frac{1}{2} \cdot c_{L_2} \\ &= 139 \end{aligned}$$

$$\begin{aligned} ED_{L_3}(P) &= \frac{1}{3-1+1} \cdot c_{L_1} + \frac{1}{3-2+1} \cdot c_{L_2} + \frac{1}{3-3+1} \cdot c_{L_3} \\ &= \frac{1}{3} \cdot c_{L_1} + \frac{1}{2} \cdot c_{L_2} + \frac{1}{1} \cdot c_{L_3} \\ &= 201. \end{aligned}$$

So the allocation vector is given by $ED(P) = (77, 139, 201) \in \mathbb{R}^3_+$. Note that also the downstream allocation rule provides an efficient cost allocation vector: $\sum_{k \in M} ED_{L_k}(P) = \sum_{k \in M} c_{L_k}$.

The main difference between this rule and the LP rule is that the ED rule takes into consideration that downstream unions are partly responsible for upstream level costs. It charges unions for the use of their own voltage level and the upstream voltage levels. The ED rule, likewise to the LP rule, does not take into account the demands of the unions. A difference between the ED rule and the Upstream Equal Sharing rule, presented in Ni and Wang (2007), is that in the latter rule upstream agents are charged for downstream costs.

Demand-proportional downstream allocation: cascade rule

The cascade method makes use of the cascade rule to allocate the costs of the grid to the unions of agents. Due to this rule small scale electricity consumers that are connected to the Low Voltage grid pay around 60% of the total network cost (see figure 1.2, p.4). By the cascade method the costs of higher voltage levels are cascaded to lower voltage levels, on the basis of the assumption that the electricity flows from high to low voltage levels. Currently the cascaded costs are ***proportional*** to the net demand of the lower level network, which is officially recorded in Autoriteit Consument en Markt (2013). Formally the cascade rule can be defined by the function CA that associates with every $P \in \mathcal{C}_{ed}$ a cost allocation vector $CA(P) \in \mathbb{R}^M_+$ such that

$$CA_{L_k}(P) = \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}, \tag{4.7}$$

for all $k \in M$. The cascade rule can be viewed as an iterative allocation method in which first the cost of the highest voltage level are partly allocated to the union corresponding to that level and the rest of the cost of that level are cascaded to the voltage level below. For this lower voltage level we now have its own level cost plus the cascaded cost of the above level. The sum of these costs are then allocated partly to the union corresponding to that level and the rest is again cascaded to the level below and so on. Based on this observation, we prove the following proposition.

Proposition 4.2.1. *The cascade rule can be written as a recursive function as follows,*

$$CA_{L_k}(P) = \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)) \tag{4.8}$$

for every $P \in \mathcal{C}_{ed}$, $k \in M$ *and* $CA_{L_0}(P) = 0$.

Proof. We show that the equations provided for the cascade rule in (4.7) and (4.8) are equivalent. We start with equation (4.8) and deduce from this equation (4.7):

$$\begin{aligned}
CA_{L_k}(P) =& \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)) \qquad (4.8)\\
=& \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot \sum_{h=1}^{k} (c_{L_h} - \sum_{l=1}^{h-1} \frac{\hat{d}_{L_{h-1}}}{\bar{d}_{L_l}} \cdot c_{L_l})\\
=& \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\sum_{h=1}^{k} c_{L_h} - (\sum_{l=1}^{1} \frac{\hat{d}_{L_1}}{\bar{d}_{L_l}} \cdot c_{L_l} + \sum_{l=1}^{2} \frac{\hat{d}_{L_2}}{\bar{d}_{L_l}} \cdot c_{L_l} + ... + \sum_{l=1}^{k-1} \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_l}} \cdot c_{L_l}))\\
=& \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\sum_{h=1}^{k} c_{L_h} - ((\frac{\hat{d}_{L_1}}{\bar{d}_{L_1}} \cdot c_{L_1}) + (\frac{\hat{d}_{L_2}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_2}}{\bar{d}_{L_2}} \cdot c_{L_2}) + ...+\\
& (\frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_2}} \cdot c_{L_2} + ... + \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}})))\\
=& \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\sum_{h=1}^{k} c_{L_h} - (\frac{\hat{d}_{L_1} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_2} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_2}} \cdot c_{L_2} + ...+\\
& \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}}))
\end{aligned}$$

$$=\frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (c_{L_1} - \frac{\hat{d}_{L_1} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_1}} \cdot c_{L_1} + c_{L_2} - \frac{\hat{d}_{L_2} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_2}} \cdot c_{L_2} + ...+$$
$$c_{L_{k-1}} - \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} + c_{L_k})$$
$$=\frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\frac{\bar{d}_{L_1}}{\bar{d}_{L_1}} \cdot c_{L_1} - \frac{\hat{d}_{L_1} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\bar{d}_{L_2}}{\bar{d}_{L_2}} \cdot c_{L_2} - \frac{\hat{d}_{L_2} + ... + \hat{d}_{L_{k-1}}}{\bar{d}_{L_2}} \cdot c_{L_2} + ...+$$
$$\frac{\bar{d}_{L_{k-1}}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} - \frac{\hat{d}_{L_{k-1}}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} + \frac{\bar{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k})$$
$$=\frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\frac{\hat{d}_{L_k} + ... + \hat{d}_{L_m}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_k} + ... + \hat{d}_{L_m}}{\bar{d}_{L_2}} \cdot c_{L_2} + ...+$$
$$\frac{\hat{d}_{L_k} + ... + \hat{d}_{L_m}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} + \frac{\bar{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k})$$
$$=\frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot (\frac{\bar{d}_{L_k}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\bar{d}_{L_k}}{\bar{d}_{L_2}} \cdot c_{L_2} + ... + \frac{\bar{d}_{L_k}}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} + \frac{\bar{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k})$$
$$=\hat{d}_{L_k} \cdot (\frac{1}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{1}{\bar{d}_{L_2}} \cdot c_{L_2} + ... + \frac{1}{\bar{d}_{L_{k-1}}} \cdot c_{L_{k-1}} + \frac{1}{\bar{d}_{L_k}} \cdot c_{L_k})$$
$$=\hat{d}_{L_k} \cdot \sum_{h=1}^{k} \frac{1}{\bar{d}_{L_h}} \cdot c_{L_h}$$
$$=\sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \tag{4.7}$$

□

When the rule is defined in this way it is easy to see that costs of a union are dependent on the costs shares of the upstream unions, namely $CA_{L_k}(P)$ depends on all $CA_{L_h}(P)$ for $k \geq h$.

Example 4.2.3. *Consider again the example as described in example 4.1.1. The cascade rule gives the following cost allocation vector*

$$\begin{aligned} CA_{L_1}(P) &= \frac{\hat{d}_{L_1}}{\bar{d}_{L_1}} \cdot c_{L_1} \\ &= \frac{50}{110} \cdot 230 \\ &= 105 \end{aligned}$$

$$CA_{L_2}(P) = \frac{\hat{d}_{L_2}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_2}}{\bar{d}_{L_2}} \cdot c_{L_2}$$

$$
\begin{aligned}
&= \frac{30}{110} \cdot 230 + \frac{30}{60} \cdot 125 \\
&= 125
\end{aligned}
$$

$$
\begin{aligned}
CA_{L_3}(P) &= \frac{\hat{d}_{L_3}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_3}}{\bar{d}_{L_2}} \cdot c_{L_2} + \frac{\hat{d}_{L_3}}{\bar{d}_{L_3}} \cdot c_{L_3} \\
&= \frac{30}{110} \cdot 230 + \frac{30}{60} \cdot 125 + \frac{30}{30} \cdot 62 \\
&= 187
\end{aligned}
$$

It follows again that $\sum_{k \in M} CA_{L_k}(P) = \sum_{k \in M} c_{L_k} = 417$ *and hence the total costs of the network are allocated, i.e. the CA rule satisfies efficiency.*

For an application of the cascade rule to the Dutch electricity grid, we refer to an illustrative example presented in Appendix I.

4.3 Properties

In this section we define several properties for the level paying rule, the equal downstream rule and the cascade rule. Some properties are standard properties in cost allocation and are, amongst others, described in Sudhölter (1998) and reviewed in Chapter 3. Others are established for this problem and have resemblances with the polluted river sharing problem from Ni and Wang (2007). Finally, some properties for union values are considered and adapted from van den Brink and Dietz (2014). We classify the properties to obtain a clearer overview, we acknowledge however that this classification is not uniquely determined. The classification is based on the ***regulatory objectives*** taken into account when setting transmission tariffs, as described in (Hakvoort et al., 2013):

1. The total tariff revenue must cover the grid costs, so allocations should be ***cost-efficient***.
2. The tariffs should be ***non-discriminatory***, which entails that comparable consumers should pay comparable tariffs.
3. The tariffs should be ***transparent*** for all of its consumers, so that everyone understands the methodology behind it.

The first objective is a basic principle for most tariff systems and the last two are obligatory directives of European regulations (Autoriteit Consument en Markt, 2013).

Also ***cost-reflectiveness*** is a desired principle, stating that the tariffs reflect the costs caused by the consumer. Accordingly we distinguish cost-reflective, comparison and simplifying properties. For all properties in this section we assume $P = (N, L, d, c) \in \mathcal{C}_{ed}$ and μ is a union cost allocation rule.

Cost-reflective properties

We consider properties that convey the principles of cost-reflectiveness and cost-efficiency.

EF **Efficiency**: the total costs are exactly allocated.
It holds that $\sum_{k\in M} \mu_{L_k}(P) = C(d)$, for all $P \in \mathcal{C}_{ed}$.

FE **Feasibility**: at least the total costs should be allocated.
For all $k \in M$ we have $\sum_{k\in M} \mu_{L_k}(P) \geq C(d)$, for all $P \in \mathcal{C}_{ed}$.

The above two properties are basic properties ensuring that at least all costs are covered. The next two properties state that if either the demand of a level is zero or the cost of that level is zero, the cost share for that level should also equal zero.

NDP **Null Demand Property**: if the total demand of a union is zero, then this union gets a zero cost share.
For all $k \in M$ if $\hat{d}_{L_k} = 0$, then $\mu_{L_k}(P) = 0$, for all $P \in \mathcal{C}_{ed}$.

NCP **Null Cost Property**: if the cost of a level equals zero, then the corresponding union receives a zero cost share.
For all $k \in M$ if $c_{L_k} = 0$, then $\mu_{L_k}(P) = 0$, for all $P \in \mathcal{C}_{ed}$.

Note that the null cost property (NCP) ignores the levels, i.e. it states that if a level cost is zero, the corresponding union gets a zero cost share, even if that union makes use of upstream levels. Therefore we consider similar properties that incorporate the fact that unions might use multiple levels. The following properties define the responsibility of lower voltage levels towards higher voltage levels with respect to the costs incurred by their demands. It follows that the cost of a union is independent of the costs of lower levels.

IDC **Independence of Downstream Costs**: the cost share of a union does not depend on the costs of downstream levels.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for any $l \in M$ and $l < h$ holds that $c^1_{L_l} = c^2_{L_l}$. Then for all $k < h$ we have $\mu_{L_k}(P^1) = \mu_{L_k}(P^2)$.

NUC **Null Upstream Costs**: the cost share of a union is zero if the costs of its own level and all upstream levels are zero.
If $P = (N, L, d, c) \in \mathcal{C}_{ed}$ is such that $c_{L_h} = 0$ for all $h \leq k$, then $\mu_{L_k}(P) = 0$ for $h, k \in M$.

Remark that null cost property (NCP) implies null upstream cost (NUC) and null upstream cost (NUC) implies independence of downstream costs (IDC). The next property states that if all upstream level costs are zero, the corresponding unions can be deleted from the problem without consequences for the cost shares of the other unions.

NUCO **Null Upstream Costs Out**: deleting upstream unions with zero level costs, does not change the cost share of the remaining unions.
If $P = (N, L, d, c) \in \mathcal{C}_{ed}$ is such that $c_{L_h} = 0$ for all $h < l$. Then we have for all $k \geq l$ that $\mu_{L_k}(P) = \mu_{L_k}(N \setminus \cup_{h<k} L_h, L \setminus \{L_h\}_{h<k}, (d_i)_{i \in N \setminus \cup_{h<k} L_h}, (c_{L_f})_{L_f \in L \setminus \{L_h\}_{h<k}})$, for all $k \in M$.

The final property of the cost-reflective properties states that the rule is monotonic with respect to the level cost vector.

C-MON **Cost-Monotonicity**: when costs of the levels increase, also the allocated cost shares increase.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for all $k \in M$ holds that $c^1_{L_k} \leq c^2_{L_k}$. Then we have $\mu(P^1) \leq \mu(P^2)$.

So if we have two level cost vectors and one of them is lower with respect to every entry of the vector, then the cost shares obtained by the allocation rule of the lower vector are not higher in every entry (i.e. for every union) than for the higher vector.

Comparison properties

In this section we discuss some properties that allow for comparing of agents and unions and ensure that no discrimination between agents or unions takes place. The first two properties are non-discriminatory properties with respect to unions' demands.

RAN **Ranking**: unions with a larger demand obtain a larger cost share.
For all $h, k \in M$ and $P \in \mathcal{C}_{ed}$ such that $\hat{d}_{L_h} \leq \hat{d}_{L_k}$, it holds that $\mu_{L_h}(P) \leq \mu_{L_k}(P)$.

ET **Equal Treatment of Equal Demands**: unions with equal demands obtain the same cost share.
For all $h, k \in M$ and $P \in \mathcal{C}_{ed}$ such that $\hat{d}_{L_h} = \hat{d}_{L_k}$, it holds that $\mu_{L_h}(P) = \mu_{L_k}(P)$.

Note that ranking (RAN) is a weaker version of equal treatment of equal demands (ET). The above two properties state non-discriminatory properties solely based on equal demands of unions, but do not consider the levels of the unions in the grid. In our situation unions with equal demands should be compared taking into account the voltage level they belong to. Therefore we propose the following property.

RED **Ranking of Equal Demands**: if two unions have equal demands, then the cost share of the downstream union is at least as high as the cost share of the upstream union.
For all $h, k \in M$ and $P \in \mathcal{C}_{ed}$ such that $\hat{d}_{L_h} = \hat{d}_{L_k}$ and $h < k$, it holds that $\mu_{L_h}(P) \leq \mu_{L_k}(P)$.

Note that the condition $h < k$ is of importance here and the main difference with the properties ranking and equal treatment of equal demands. The next three properties imply that downstream unions have a responsibility for their upstream incurred costs. The first two properties state that downstream unions with equal demands and all downstream unions bear an equal responsibility for their upstream incurred costs. The third property states that the magnitude of the responsibility of the downstream unions for their upstream incurred costs is dependent on their demands.

D-SYM **Downstream Symmetry**: in case of a special level cost vector where all level costs are zero, except for the cost of level l, all unions with equal demands and levels lower (downstream) than or equal to l, pay an equal share of the level cost of level l.
Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. Then for all $h, k \geq l$ such that $\hat{d}_{L_h} = \hat{d}_{L_k}$ for $h, k \in M$, it holds that $\mu_{L_h}(P^l) = \mu_{L_k}(P^l)$.

DR **Downstream Responsibility**: in case of a special level cost vector where all level costs are zero, except for the cost of level l, all unions with levels lower (downstream) than or equal to l, pay an equal share of the level cost of level l.
Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. Then for all $l, h, k \in M$ and for all $h, k \geq l$ we have $\mu_{L_h}(P^l) = \mu_{L_k}(P^l)$.

DDPR **Downstream Demand Proportional Responsibility**: all downstream unions pay a share of their upstream incurred costs that is proportional to their demands.
Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. Then for all $l \in M$ and for all $k \geq l$ we have

$$\mu_{L_k}(P^l) = \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l}.$$

Remark that downstream responsibility (DR) implies downstream symmetry (D-SYM). For the coming properties we explore ***collusion***, based on some properties discussed in Chapter 3 and described in van den Brink and Dietz (2014). Collusion entails the merging of either agents or unions. The merging of agents is more straightforward and is therefore considered first. Collusion of two agents within one union can be interpreted as two agents becoming one by adopting the name of the agent lowest in the ordering and adding up the demands. If two agents $i, j \in L_k$ collude, without loss of generality assume $i < j$, then we denote the new set of agents by $N^{ij} = N \setminus \{j\}$. Hence we assume the greatest agent of the two is eliminated from N, since it also adopts the name of agent i. By this merger the demands of the agents collude such that $d_i^{ij} = d_i + d_j$ and $d_g^{ij} = d_g$ for all $g \in N \setminus \{i, j\}$. Further we have $L_k^{ij} = L_k \setminus \{j\}$ if $l(j) = k$ and $L_k^{ij} = L_k$ if $k \in M \setminus \{l(j)\}$. The level cost vector does not change. Let us now consider what happens with the union value of union L_k if agents $i, j \in L_k$ collude.

ACN **Agent Collusion Neutrality**: collusion of two or more agents in the same union does not affect the cost share of that union.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N^{ij}, L^{ij}, d^{ij}, c)$ such that agents $i, j \in L_k$ colluded. Then for $k \in M$ holds that $\mu_{L_k}(P^1) = \mu_{L_k}(P^2)$.

Hereafter we consider two properties with regard to collusion between unions. Collusion between unions is defined by merging two or more elements of the partition, which do not have to be consecutive. Similar as for agent collusion, collusion of unions can be interpreted as merging two unions into one union, where the new union adopts the name of the originally most upstream union, i.e. the union with the lowest ranking. This new union consists of all the agents of the two original unions, hence the collusion entails a merger of the sets of agents of these unions. The new partition after collusion of unions L_k, L_l is such that $L^{L_k L_l} = (L \setminus \{L_k, L_l\}) \cup \{L_k \cup L_l\}$ for $k, l \in M$. Without loss of generality we assume $k < l$, such that the elements of $L^{L_k L_l}$ are $L_k^{L_k L_l} = L_k \cup L_l$ and $L_h^{L_k L_l} = L_h$ for all $h \in M \setminus \{l, k\}$.[7] The entries of the level cost vector after collusion of unions L_k, L_l are such that $c_{L_k}^{L_k L_l} = c_{L_k} + c_{L_l}$ and $c_{L_h}^{L_k L_l} = c_{L_h}$ for $h \in M \setminus \{l, k\}$. Hence, the level costs of the unions that collude are added.

It is debatable what the effect of collusion of two voltage levels would be in real-life on the level costs. On the one hand we can argue that costs would rise. If the collusion entails that the agents of the downstream union have to be connected to the upstream voltage level, then it can become very expensive. Especially if the fixed costs of the most downstream level are still operative, for example the depreciation costs of the grid of all voltage levels continue to exist. On the other hand we can argue that costs would fall, since the amount of electricity that flows through the upstream level of the two unions remains unchanged after collusion, i.e. $\bar{d}_{L_k}^{L_k L_l} = \bar{d}_{L_k}$ where $\bar{d}_{L_k}^{L_k L_l}$ represents the aggregated demand through $L_k^{L_k L_l}$. So assuming that the grid can handle the agents from the other union and given the fact that the same amount of electricity flows through the upstream

[7]Note that after collusion of agents or unions the sets N and M might not have cardinality n and m respectively, anymore.

level as before, the new cost will be less than the sum of the two level costs, since the transportation distance is reduced. For simplicity in this problem we assume that the level costs of two colluded unions are added. Hence, we denote $(N, L^{L_k L_l}, d, c^{L_k L_l})$ for a problem where unions L_k, L_l colluded. We now consider two properties with respect to collusion of two unions.

DUCN-I **Downstream Union Collusion Neutrality I**: collusion of two downstream unions L_k, L_l does not change the cost share of their upstream unions.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_k L_l}, d, c^{L_k L_l})$ a problem where unions L_k and L_l colluded. Then for all $h \in M$ such that $h < k, l$ it holds that $\mu_{L_h}(P^1) = \mu_{L_h}(P^2)$.

UCN-I **Union Collusion Neutrality II**: the cost share of two unions does not change after collusion of these unions.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_k L_l}, d, c^{L_k L_l})$ a problem where unions L_k and L_l colluded with $k < l$. Then it holds that $\mu_{L_k}(P^1) + \mu_{L_l}(P^1) = \mu_{L_k}(P^2)$.

Note that collusion of more than two unions is also possible and can be interpreted as applying collusion of two unions multiple times, until all desired unions are colluded. Let $P = (N, L^{L_l \dots L_m}, d, c^{L_l \dots L_m})$ be a problem where the consecutive unions L_l to L_m colluded, such that $L_l^{L_l \dots L_m} = \cup_{k=l}^{m} L_k$ and $L_h^{L_l \dots L_m} = L_h$ for all $h \in M \setminus \{l, ..., m\}$. Further we have $c_{L_l}^{L_l \dots L_m} = \sum_{k=l}^{m} c_{L_k}$ and $c_{L_h}^{L_l \dots L_m} = c_{L_h}$ for $h \in M \setminus \{l, ..., m\}$. We now consider two properties with respect to collusion of multiple downstream unions.

DUCN-II **Downstream Union Collusion Neutrality II**: collusion of all downstream unions does not change the cost share of these unions.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_l \dots L_m}, d, c^{L_l \dots L_m})$ with $m = l(\max N)$ such that unions L_l to L_m colluded. Then it holds that $\mu_{L_l}(P^2) = \sum_{k=l}^{m} \mu_{L_k}(P^1)$.

UCN-II **Union Collusion Neutrality II**: the cost share of unions does not change after collusion of all downstream unions.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_l \dots L_m}, d, c^{L_l \dots L_m})$ with $m = l(\max N)$ such that unions L_l to L_m colluded. Then for all $P^1, P^2 \in \mathcal{C}_{ed}$ and all $k \in M$ we have

$$\mu_{L_k}(P^2) = \begin{cases} \mu_{L_k}(P^1) & \text{if } k < l \\ \sum_{h=k}^{m} \mu_{L_h}(P^1) & \text{otherwise.} \end{cases}$$

Note that the first two union collusion properties include collusion of any two unions, also if the corresponding levels of the unions are not adjacent. Whereas in the last two properties the collusion is between at least the two highest ranked (lowest in voltage) consecutive unions. Downstream union collusion neutrality-I and -II (DUCN-I, -II) are weaker versions of union collusion neutrality II (UCN-II). The last comparison property is a standard property in cost allocation and ensures that unions are not discriminated based on their names.

AN **Anonymity**: an allocation rule is independent of the names of the unions.
Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, \pi L, \pi d, c)$ for some permutation π of M with respect to the union demands. Then for all $k \in M$ it holds that $\mu_{\pi(L_k)}(P^2) = \mu_{L_k}(P^1)$.[8]

The permutation in anonymity does not permute the level costs, but only permutes the levels and the corresponding level demands and thereby ignores the levels.

Simplifying properties

We aim at properties that simplify a problem. For example splitting a problem in additive parts can simplify its study.

C-ADD **Cost-Additivity**: splitting a cost vector in additive parts does not affect the cost shares.
Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, c^1 + c^2)$. For all $k \in M$ we have that $\mu_{L_k}(P^1) + \mu_{L_k}(P^2) = \mu_{L_k}(P^3)$.

The next property states that the allocation rule is additive and homogeneous of degree one with respect to the cost vector. A homogeneous mapping entails that if a cost vector is multiplied by a factor, then the resulting allocation shares are also multiplied by this factor.[9]

C-LIN **Cost-Linearity**: splitting a cost vector in additive parts and multiplying these parts by a constant does not affect the cost shares
Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, \alpha \cdot c^1 + \beta \cdot c^2)$ for $\alpha, \beta \in \mathbb{R}_+$. For all $k \in M$ we have that $\mu_{L_k}(P^3) = \alpha \cdot \mu_{L_k}(P^1) + \beta \cdot \mu_{L_k}(P^2)$.

Note that cost-linearity is stronger than cost-additivity. The final property states that an allocation rule satisfies the standard property for a two-union problem. This property

[8] $\pi(L) = \{L_{\pi(k)}\}_{k \in M}$ and $\pi d = (\hat{d}_{\pi(L_k)})_{k \in M}$.
[9] When the mapping is homogeneous of degree t, shares are multiplied by a power t of the factor.

defines how to allocate cost shares in the simple situation with only two unions. The standardness property has been defined for TU games, amongst others by Hart and Mas-Colell (1989). Union standardness states that in a two-union problem level costs have to be shared proportional to the total demands of the unions. Ortmann (2000) argues that the way to solve an arbitrary cooperative cost allocation problem is to generalize the concept of solving the two agent (or union) problem. We solely focus on the situation with two unions. So let us now consider two standard demand-proportional properties for a two-union problem.

US-I **Union Standardness I**: if there are two unions, the cost of the first level is allocated proportional to the demand of the unions and the second level also pays its own level cost.
Let $P \in \mathcal{C}_{ed}$ such that $L = \{L_1, L_2\}$ and $c = (c_{L_1}, c_{L_2})$ a two union problem. Then for all $k \in M$ we have $\mu_{L_k} = \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}$. Hence, $\mu_{L_1} = \frac{\hat{d}_{L_1}}{\bar{d}_{L_1}} \cdot c_{L_1}$ and $\mu_{L_2} = \frac{\hat{d}_{L_2}}{\bar{d}_{L_1}} \cdot c_{L_1} + c_{L_2}$.

US-II **Union Standardness II**: if there are two unions and a level cost vector where only the most upstream level (first level) cost is not zero, then the cost of the first level is allocated to both unions proportional to their demands.
Let $P \in \mathcal{C}_{ed}$ such that $L = \{L_1, L_2\}$ and $c = (c_{L_1}, 0)$ a two union problem. Then for $k \in M$ we have $\mu_{L_k} = \frac{\hat{d}_{L_k}}{\bar{d}_{L_1}} \cdot c_{L_1}$. Hence, $\mu_{L_1} = \frac{\hat{d}_{L_1}}{\bar{d}_{L_1}} \cdot c_{L_1}$ and $\mu_{L_2} = \frac{\hat{d}_{L_2}}{\bar{d}_{L_1}} \cdot c_{L_1}$.

Note that the union standardness II is stronger than union standardness I.

4.3.1 Properties of the rules

We now discuss for all the properties defined above which are satisfied and which not by the level paying rule, the equal downstream rule and the cascade rule. For the CA rule we also provide a proof, whereas the proofs for the LP and ED rule are presented in Appendix I. Finally, we summarize these results by means of a convenient table.

Theorem 4.3.1. *The level paying rule $LP(P)$*

(i) satisfies the properties EF, FE, NCP, IDC, NUC, NUCO, C-MON, ACN, DUCN-I, UCN-I, DUCN-II, UCN-II, C-ADD, C-LIN

(ii) does not satisfy the properties NDP, RAN, ET, RED, D-SYM, DR, DDPR, AN, D-ADD, US-I and US-II.

Proof. Proof can be found in Appendix I. □

We find that the level paying rule does not satisfy any of the comparison properties that take the levels into account (i.e. *RED, D-SYM, DR* and *DDPR*). This makes sense, as the rule is defined to only consider its own level. Therefore it does satisfy all the properties that imply that the cost of a union is in some way independent of upstream or downstream voltage levels (i.e. *IDC, NUC, NUCO*). So if the levels are important, this rule can be considered not so fair. On the other hand, does it satisfy the more standard properties such as *NCP, C-MON, C-ADD* and *C-LIN*. Note that also all the collusion properties are obeyed, implying that the rule is not sensitive to collusion of agents or unions.

Theorem 4.3.2. *The equal downstream rule $ED(P)$*

(i) satisfies the properties EF, FE, IDC, NUC, NUCO, C-MON, RED, D-SYM, DR, ACN, C-ADD, C-LIN

(ii) does not satisfy the properties NDP, NCP, RAN, ET, DDPR, DUCN-I, UCN-I, DUCN-II, UCN-II, AN, D-ADD, US-I and US-II.

Proof. Proof can be found in Appendix I. □

The equal downstream rule, in contrary to the level paying rule, does not satisfy any of the union collusion properties. This is due to the fact that this rule uses the cardinality of the union set to allocate the costs. Another contrast with the level paying rule is that the equal downstream rule does satisfy some of the comparison properties that impose a responsibility with respect to other levels, namely *D-SYM* and *DR*. However, none of the properties involving demand proportionality are satisfied, i.e. *DDPR, US-I* and *US-II*.

Theorem 4.3.3. *The cascade rule $CA(P)$*

(i) satisfies the properties EF, FE, NDP, IDC, NUC, NUCO, C-MON, RED, D-SYM, DDPR, ACN, DUCN-I, DUCN-II, UCN-II, C-ADD, C-LIN, US-I and US-II

(ii) does not satisfy the properties NCP, RAN, ET, DR, UCN-II, AN and D-ADD.

Proof. (i) Let $l(\max N) = m$. We prove that the following properties are satisfied by the cascade rule for all $P = (N, L, d, c) \in \mathcal{C}_{ed}$:

EF We want to show that $\sum_{k=1}^{m} CA_{L_k}(P) = C(d)$. We have

$$\begin{aligned}\sum_{k=1}^{m} CA_{L_k}(P) &= \sum_{k=1}^{m}\sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{m}\sum_{k=h}^{m} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{m} \frac{\sum_{k=h}^{m} \hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{m} \frac{\hat{d}_{L_h} + \hat{d}_{L_{h+1}} + ... + \hat{d}_{L_m}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{m} \frac{\bar{d}_{L_h}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{m} c_{L_h} \\ &= C(d). && \text{(equation (4.4))}\end{aligned}$$

FE Follows from Efficiency (EF).

NDP Assume $\hat{d}_{L_k} = 0$. It easily follows that $CA_{L_k}(P) = 0$:

$$\begin{aligned}CA_{L_k}(P) &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{k} \frac{0}{\bar{d}_{L_h}} \cdot c_{L_h} && (\hat{d}_{L_k} = 0) \\ &= 0.\end{aligned}$$

IDC Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for any $k \in M$ and $h < k$ holds that $c^1_{L_h} = c^2_{L_h}$. We want to show that for all $l < k$ we have $CA_{L_l}(P^1) = CA_{L_l}(P^2)$. Let $l < k$, we have that

$$CA_{L_l}(P^1) = \sum_{h=1}^{l} \frac{\hat{d}_{L_l}}{\bar{d}_{L_h}} \cdot c^1_{L_h}.$$

From $h \leq l$ and $l < k$ follows that $h < k$ and therefore $c^1_{L_h} = c^2_{L_h}$, hence we obtain,

$$\begin{aligned}\sum_{h=1}^{l} \frac{\hat{d}_{L_l}}{\bar{d}_{L_h}} \cdot c^1_{L_h} &= \sum_{h=1}^{l} \frac{\hat{d}_{L_l}}{\bar{d}_{L_h}} \cdot c^2_{L_h} \\ &= CA_{L_l}(P^2).\end{aligned}$$

NUC Assume that $c_{L_h} = 0$ for all $h \leq k$. We want to show that $CA_{L_k}(P) = 0$:

$$\begin{aligned} CA_{L_k}(P) &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot 0 && (h \leq k) \\ &= 0. \end{aligned}$$

NUCO Assume that $c_{L_h} = 0$ for all $h < l$. We want to show that for all $k \geq l$ we have $CA_{L_k}(P) = CA_{L_k}(N \setminus \cup_{h<k} L_h, L \setminus \{L_h\}_{h<k}, (d_i)_{i \in N \setminus \cup_{h<k} L_h}, (c_{L_f})_{L_f \in L \setminus \{L_h\}_{h<k}})$:

$$\begin{aligned} CA_{L_k}(P) &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \frac{\hat{d}_{L_k}}{\bar{d}_{L_1}} \cdot c_{L_1} + ... + \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k} \\ &= \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k} && (c_{L_h} = 0 | h < k) \\ &= \sum_{h=k}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= CA_{L_k}(N \setminus \cup_{h<k} L_h, L \setminus \{L_h\}_{h<k}, (d_i)_{i \in N \setminus \cup_{h<k} L_h}, (c_{L_f})_{L_f \in L \setminus \{L_h\}_{h<k}}). \end{aligned}$$

C-MON Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for all $h \in M$ holds that $c^1_{L_h} \leq c^2_{L_h}$. We want to show that $CA(P^1) \leq CA(P^2)$:

$$\begin{aligned} CA_{L_k}(P^1) &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c^1_{L_h} \\ &\leq \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c^2_{L_h} && (c^1_{L_h} \leq c^2_{L_h}) \\ &= CA_{L_k}(P^2). \end{aligned}$$

RED Let $P \in \mathcal{C}_{ed}$ such that $\hat{d}_{L_h} = \hat{d}_{L_k}$ and $h < k$. We want to show that for $h, k \in M$ it holds that $CA_{L_h}(P) \leq CA_{L_k}(P)$:

$$\begin{aligned} CA_{L_h}(P) &= \sum_{l=1}^{h} \frac{\hat{d}_{L_h}}{\bar{d}_{L_l}} \cdot c_{L_l} \\ &\leq \sum_{l=1}^{h} \frac{\hat{d}_{L_h}}{\bar{d}_{L_l}} \cdot c_{L_l} + \sum_{l=h+1}^{k} \frac{\hat{d}_{L_h}}{\bar{d}_{L_l}} \cdot c_{L_l} \end{aligned}$$

$$= \sum_{l=1}^{k} \frac{\hat{d}_{L_h}}{\bar{d}_{L_l}} \cdot c_{L_l}$$

$$= \sum_{l=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l} \qquad (\hat{d}_{L_h} = \hat{d}_{L_k})$$

$$= CA_{L_k}(P).$$

D-SYM Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. Assume that for $k, k' \geq l$ unions $L_k, L_{k'}$ have equal demands such that $\hat{d}_{L_k} = \hat{d}_{L_{k'}}$ for $k, k' \in M$. To prove is that $CA_{L_k}(P^l) = CA_{L_{k'}}(P^l)$:

$$CA_{L_k}(P^l) = \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}$$

$$= \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l} \qquad (k \geq l)$$

$$= \frac{\hat{d}_{L_{k'}}}{\bar{d}_{L_l}} \cdot c_{L_l} \qquad (\hat{d}_{L_k} = \hat{d}_{L_{k'}})$$

$$= \sum_{h=1}^{k'} \frac{\hat{d}_{L_{k'}}}{\bar{d}_{L_h}} \cdot c_{L_h}$$

$$= CA_{L_{k'}}(P^l).$$

DDPR Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. We want to show that for all $l \in M$ and for all $k \geq l$ that $CA_{L_k}(P^l) = \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l}$. Assume $k \geq l$, we have

$$CA_{L_k}(P^l) = \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}$$

$$= \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l}. \qquad (k \geq l)$$

ACN Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N^{ij}, L^{ij}, d^{ij}, c)$ such that agents $i, j \in L_k$ colluded. Then for $k \in M$ we want to show that: $CA_{L_k}(P^1) = CA_{L_k}(P^2)$. For $k \in M$ we have

$$CA_{L_k}(P^1) = \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}$$

$$= CA_{L_k}(P^2).$$

Note that collusion of two agents in one union does not change the demand or level cost of that union or any other union. As the cascade rule is only dependent on the demands, aggregated demands and level costs of the unions, the rule is not affected by collusion of two agents in one union.

DUCN-I Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_k L_l}, d, c^{L_k L_l})$ a problem where unions L_k and L_l colluded. Then for all $h \in M$ such that $h < k, l$, we want to show that $CA_{L_h}(P^1) = CA_{L_h}(P^2)$:

$$\begin{aligned} CA_{L_h}(P^1) &= \sum_{g=1}^{h} \frac{\hat{d}_{L_h}}{\bar{d}_{L_g}} \cdot c_{L_g} \\ &= CA_{L_h}(P^2). \qquad (h < k, l) \end{aligned}$$

DUCN-II Follows from UCN-II.

UCN-II Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_l \dots L_m}, d, c^{L_l \dots L_m})$ with $m = l(\max N)$ such that unions L_l to L_m colluded. We want to show that for all $k \in M$ we have

$$CA_{L_k}(P^2) = \begin{cases} CA_{L_k}(P^1) & \textit{if } k < l \\ \sum_{h=k}^{m} CA_{L_h}(P^1) & \textit{otherwise.} \end{cases}$$

If $k < l$, then

$$\begin{aligned} CA_{L_k}(P^1) &= \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= CA_{L_k}(P^2). \qquad (k < l) \end{aligned}$$

If $k \geq l$, then

$$\begin{aligned} \sum_{h=k}^{m} CA_{L_h}(P^1) =& CA_{L_k}(P) + CA_{L_{k+1}}(P) + \dots + CA_{L_m}(P) \\ =& \sum_{h=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} + \sum_{h=1}^{k+1} \frac{\hat{d}_{L_{k+1}}}{\bar{d}_{L_h}} \cdot c_{L_h} + \dots + \sum_{h=1}^{m} \frac{\hat{d}_{L_m}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ =& (\frac{\hat{d}_{L_k}}{\bar{d}_{L_1}} \cdot c_{L_1} + \dots + \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k}) + (\frac{\hat{d}_{L_{k+1}}}{\bar{d}_{L_1}} \cdot c_{L_1} + \dots + \frac{\hat{d}_{L_{k+1}}}{\bar{d}_{L_k}} \cdot c_{L_{k+1}}) + \dots + \\ & (\frac{\hat{d}_{L_m}}{\bar{d}_{L_1}} \cdot c_{L_1} + \dots + \frac{\hat{d}_{L_m}}{\bar{d}_{L_k}} \cdot c_{L_m}) \\ =& (\frac{\hat{d}_{L_k} + \dots + \hat{d}_{L_m}}{\bar{d}_{L_1}} \cdot c_{L_1}) + (\frac{\hat{d}_{L_k} + \dots + \hat{d}_{L_m}}{\bar{d}_{L_2}} \cdot c_{L_2}) + \dots + \\ & (\frac{\hat{d}_{L_k} + \dots + \hat{d}_{L_m}}{\bar{d}_{L_k}} \cdot c_{L_k}) + \dots + (\frac{\hat{d}_{L_m}}{\bar{d}_{L_m}} \cdot c_{L_m}) \end{aligned}$$

$$
\begin{aligned}
=&(\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_1}}\cdot c_{L_1})+(\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_2}}\cdot c_{L_2})+...+\\
&(\frac{\hat{d}_{L_{k-1}}+...+\hat{d}_{L_m}}{\bar{d}_{L_{k-1}}}\cdot c_{L_{k-1}})+\frac{\bar{d}_{L_k}}{\bar{d}_{L_k}}\cdot c_{L_k}+...+\frac{\bar{d}_{L_m}}{\bar{d}_{L_m}}\cdot c_{L_m}\\
=&\sum_{h=1}^{k-1}\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_h}}\cdot c_{L_h}+c_{L_k}+..+c_{L_m}\\
=&\sum_{h=1}^{k-1}\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_h}}\cdot c_{L_h}+c_{L_k}^{L_k...L_m}\\
=&\sum_{h=1}^{k-1}\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_h}}\cdot c_{L_h}+\frac{\bar{d}_{L_k}}{\bar{d}_{L_k}}\cdot c_{L_k}^{L_k...L_m}\\
=&\sum_{h=1}^{k}\frac{\hat{d}_{L_k}+...+\hat{d}_{L_m}}{\bar{d}_{L_h}}\cdot c_{L_h}^{L_k...L_m}\\
=&\sum_{h=1}^{k}\frac{\sum_{i\in L_k^{L_k...L_m}}d_i}{\bar{d}_{L_h}}\cdot c_{L_h}^{L_k...L_m}\\
=&CA_{L_k}(P^2).
\end{aligned}
$$

C-ADD Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, c^1 + c^2)$. For all $k \in M$ we want to show that $CA_{L_k}(P^1) + CA_{L_k}(P^2) = CA_{L_k}(P^3)$:

$$
\begin{aligned}
CA_{L_k}(P^1)+CA_{L_k}(P^2)&=\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot c_{L_h}^1+\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot c_{L_h}^2\\
&=\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot(c_{L_h}^1+c_{L_h}^2)\\
&=CA_{L_k}(P^3).
\end{aligned}
$$

C-LIN Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, \alpha\cdot c^1 + \beta \cdot c^2)$ for $\alpha, \beta \in \mathbb{R}_+$. We want to show that for all $k \in M$ we have that $CA_{L_k}(P^3) = \alpha \cdot CA_{L_k}(P^1) + \beta \cdot CA_{L_k}(P^2)$:

$$
\begin{aligned}
CA_{L_k}(P^3)&=\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot(\alpha\cdot c_{L_h}^1+\beta\cdot c_{L_h}^2)\\
&=\alpha\cdot\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot c_{L_h}^1+\beta\cdot\sum_{h=1}^{k}\frac{\hat{d}_{L_k}}{\bar{d}_{L_h}}\cdot c_{L_h}^2\\
&=\alpha\cdot CA_{L_k}(P^1)+\beta\cdot CA_{L_k}(P^2).
\end{aligned}
$$

US-I This property is the definition of the cascade rule restricted to two unions and therefore trivially holds.

US-II Follows from US-I.

(ii) Let $l(\max N) = m$. We show that the following properties are not satisfied by the cascade rule for some $P \in \mathcal{C}_{ed}$ by means of simple counterexamples:

NCP Let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For level $2 \in M$ we have that $c_{L_2} = 0$, but

$$\begin{aligned} CA_{L_2}(P) &= \sum_{h=1}^{2} \frac{\hat{d}_{L_2}}{\bar{d}_{L_h}} \cdot c_{L_h} \\ &= \frac{10}{30} \cdot 64 + 0 \\ &> 0. \end{aligned}$$

RAN Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied cascade rule to P in example 4.2.3. We have that $\hat{d}_{L_1} = 50 > 30 = \hat{d}_{L_2}$. However, $CA_{L_1}(P) = 105 < 125 = CA_{L_2}(P)$. So because of the different levels, this property is not satisfied.

ET Consider again $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied cascade rule to P in example 4.2.3. We have that $\hat{d}_{L_2} = 30 = \hat{d}_{L_3}$, whereas $CA_{L_2}(P) = 125 < 187 = CA_{L_3}(P)$. Also here because of the different levels, this property is not satisfied.

DR Take again the example defined for NCP: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$\begin{aligned} CA_{L_1}(P) &= \frac{\hat{d}_{L_2}}{\bar{d}_{L_1}} \cdot c_{L_1} \\ &= \frac{20}{30} \cdot 64 \\ &> \frac{10}{30} \cdot 64 + 0 \\ &= CA_{L_2}(P). \end{aligned}$$

UCN-I Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied cascade rule to P in example 4.2.3. Let $P^1 \in \mathcal{C}_{ed}$ be also as defined in example 4.1.1, but such that $P^1 = (N, L^{L_1 L_2}, d, c^{L_1 L_2})$ with $L^{L_1 L_2}$ the partition where unions L_1 and L_2 colluded such that $L^{L_1 L_2}_{L_1} = L_1 \cup L_2$ and $c^{L1L2}_{L_1} = c_{L_1} + c_{L_2} = 355$. We have

$$CA_{L_1}(P) + CA_{L_2}(P) = 105 + 125$$

$$
\begin{aligned}
&= 230 \\
&\neq 258 \\
&= \frac{80}{110} \cdot 355 \\
&= \frac{\sum_{i \in L_{L_1}^{L_1 L_2}} d_i}{\bar{d}_{L_1}} \cdot c_{L_1}^{L_1 L_2} \\
&= CA_{L_1}(P^1).
\end{aligned}
$$

The cascade rule applied to the problem with colluded unions ignores the levels, whereas the sum of the cascade rule does not, resulting in different shares of the second level cost and similar shares of the first level cost.

AN Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied cascade rule to P in example 4.2.3. Let π be a permutation on M such that $P^\pi = (N, \pi L, \pi d, c)$ with $\pi L = \{L_2, L_1, L_3\}$ and $\pi d = (d_3, d_4, d_1, d_2, d_5, d_6, d_7, d_8)$, so $\pi \hat{d} = (\hat{d}_{L_2}, \hat{d}_{L_1}, \hat{d}_{L_2})$. It follows that

$$
\begin{aligned}
CA_{\pi L_2}(P^\pi) &= \frac{30}{110} \cdot 230 \\
&= 63 \\
&\neq 125 \\
&= \frac{30}{110} \cdot 230 + \frac{30}{60} \cdot 125 \\
&= CA_{L_2}(P).
\end{aligned}
$$

Note that anonymity is not satisfied because the level costs are not permuted.

□

So the cascade rule satisfies most of the collusion properties, but not all. In particular, the rule is not sensitive to agent collusion or union collusion of the most downstream unions. The rule does not obey many of the standard properties, such as *NCP, RAN* and *ET*. However, the modified versions of these properties where the ordering of the levels are incorporated, are all satisfied by the cascade rule, namely *NUC, NUCO* and *RED*. Further does it not satisfy *DR*, but is does satisfy the version of this property that considers demands instead of cardinality, i.e. *DDPR*. For most of the properties that are not satisfied by the cascade rule the reason is that these properties do not consider the ordering of the levels.

4.3.2 Comparison of the rules

In table 4.1 we present an overview of all the properties discussed in Section 4.3 that are satisfied or not by the level paying rule, the equal downstream rule and the cascade rule. This table provides a summary of the theorems 4.3.1, 4.3.2 and 4.3.3.

	LP(P)	*ED(P)*	*CA(P)*
EF	+	+	+
FE	+	+	+
NDP	-	-	+
NCP	+	-	-
IDC	+	+	+
NUC	+	+	+
NUCO	+	+	+
C-MON	+	+	+
RAN	-	-	-
ET	-	-	-
RED	-	+	+
D-SYM	-	+	+
DR	-	+	-
DDPR	-	-	+
ACN	+	+	+
DUCN-I	+	-	+
UCN-I	+	-	-
DUCN-II	+	-	+
UCN-II	+	-	+
AN	-	-	-
C-ADD	+	+	+
C-LIN	+	+	+
US-I	-	-	+
US-II	-	-	+

Table 4.1: Summary of properties satisfied by the level paying rule *(LP(P))*, the equal downstream rule *(ED(P))* and the cascade rule *(CA(P))*: a plus denotes that the rule satisfies the property and a minus denotes that the rule does not satisfy the property. The red pluses represent the properties that define the axiomatic characterization of the rules (see Section 4.4).

From the results presented in the table we observe that the cascade rule satisfies most of the properties. Moreover, this rule satisfies most of the properties that take the ordering of the different levels of the unions and the demands into account. For the electricity demand problem the ordering of the levels and the demands are of great importance. Therefore, given the cost allocation rules discussed in this chapter and given the context of the electricity demand problem, the cascade rule may be considered as the ***most fair rule***. The properties that are not satisfied by this rule are mostly properties that ignore the levels of the unions. If however, in a different context, the different levels and the cardinality of the union set would be most important, then given the rules discussed in this chapter, the equal downstream rule may be considered most fair. Finally, if the levels of the unions, the cardinality of the union set and the demands would all not matter, then given the rules discussed in this chapter, the level paying rule may be considered the most fair rule.

4.4 Axiomatic characterizations of the rules

In this section we provide axiomatic characterizations for the level paying, the equal downstream and the cascade rule. The characterizations of the first two rules are inspired by the characterizations of the LRS and UES rules in Ni and Wang (2007). It should be noted that Ni and Wang (2007) use Cost-Additivity as a property, whereas in the axiomatic characterization they actually employ Cost-Linearity. Thus, the axiomatic characterizations of the first two rules are adapted to fit our problem and Cost-Additivity is replaced by Cost-Linearity. The properties that axiomatically characterize the rules are in table 4.1 highlighted by the red pluses.

Theorem 4.4.1 (Level paying rule axiomatic characterization)**.** *The level paying rule LP is the unique rule satisfying Efficiency, Null Cost Property and Cost-Linearity.*

Proof. We first prove necessity,

EF We want to show that $\sum_{k=1}^{m} LP_{L_k}(P) = C(d)$ for $P \in \mathcal{C}_{ed}$. We have

$$\begin{aligned}\sum_{k=1}^{m} LP_{L_k}(P) &= \sum_{k=1}^{m} c_{L_k} \\ &= C(d). && \text{(equation (4.4))}\end{aligned}$$

NCP Assume for some $k \in M$ that $c_{L_k} = 0$. We want to show that follows that $LP_{L_k}(P) = 0$, for all $P \in \mathcal{C}_{ed}$:

$$\begin{aligned}LP_{L_k}(P) &= c_{L_k} \\ &= 0.\end{aligned}$$

C-LIN Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, \alpha \cdot c^1 + \beta \cdot c^2)$ for $\alpha, \beta \in \mathbb{R}_+$. We want to show that for all $k \in M$ we have that $LP_{L_k}(P^3) = \alpha \cdot LP_{L_k}(P^1) + \beta \cdot LP_{L_k}(P^2)$:

$$\begin{aligned}LP_{L_k}(P^3) &= \alpha \cdot c^1_{L_k} + \beta \cdot c^2_{L_k} \\ &= \alpha \cdot LP_{L_k}(P^1) + \beta \cdot LP_{L_k}(P^2)\end{aligned}$$

Let us prove uniqueness of LP. Suppose that rule f satisfies these three properties. For any $l \in M$, define $c^l = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}_+^M$ a vector with a 1 on the l-th entry and 0 on the other entries. Let $P^l \in \mathcal{C}_{ed}$ such that $P^l = (N, L, d, c^l)$. By the ***Null Cost Property (NCP)*** it follows that $LP_{L_h}(P^l) = 0$ for all $h \neq l$. Then from ***Efficiency*** we obtain,

$$\sum_{h=1}^{m} f_{L_h}(P^l) = f_{L_l}(P^l) = 1 = \sum_{l=1}^{m} c_{L_l}.$$

Thus we have

$$f_{L_h}(P^l) = \begin{cases} 0 & \text{if } h \neq l \\ 1 & \text{if } h = l. \end{cases}$$

Any vector in $\mathbb{R}_+^M$ can be presented as a linear combination of the standard basis vectors, here given by c^l for all $l \in M$, and some $\alpha \in \mathbb{R}_+$. Thus, c can be presented as $c = \sum_{l=1}^{m} c_{L_l} \cdot c^l = (c_{L_1}, ..., c_{L_m})$, with $\alpha = c_{L_l}$. Due to ***Cost-Linearity (C-LIN)*** we obtain for all $h \in M$

$$\begin{aligned} f_{L_h}(P) &= f_{L_h}(N, L, d, c) \\ &= f_{L_h}(N, L, d, \sum_{l=1}^{m} c_{L_l} \cdot c^l) \\ &= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_h}(N, L, d, c^l) \\ &= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_h}(P^l) \\ &= 0 + ... + c_{L_h} + 0.. + 0 \\ &= c_{L_h} \\ &= LP_{L_h}(P) \end{aligned}$$

Thus rule f must be equal to rule LP and hence uniqueness is proven. □

The level paying rule does not consider the demands of the unions for the allocation and also not the number of unions. It also does not take into consideration that downstream unions make use of upstream unions. It is however the only rule of the three rules discussed in this chapter that satisfies the null cost property (NCP).

Theorem 4.4.2 (Equal downstream rule axiomatic characterization)**.** *The equal downstream rule ED is the unique rule satisfying Efficiency, Null Upstream Cost, Downstream Responsibility and Cost-Linearity.*

Proof. We first prove necessity,

EF We want to show that $\sum_{k=1}^{m} ED_{L_k}(P) = C(d)$ for $P \in \mathcal{C}_{ed}$. We have

$$\begin{aligned}
\sum_{k=1}^{m} ED_{L_k}(P) =& ED_{L_1}(P) + ED_{L_2}(P) + ... + ED_{L_m}(P) \\
=& \sum_{h=1}^{1} \frac{1}{m-h+1} \cdot c_{L_h} + \sum_{h=1}^{2} \frac{1}{m-h+1} \cdot c_{L_h} + ... + \\
& \sum_{h=1}^{m} \frac{1}{m-h+1} \cdot c_{L_m} \\
=& (\frac{1}{m} \cdot c_{L_1}) + (\frac{1}{m} \cdot c_{L_1} + \frac{1}{m-1} \cdot c_{L_2}) + ... + \\
& (\frac{1}{m} \cdot c_{L_1} + \frac{1}{m-1} \cdot c_{L_2} + ... + \frac{1}{1} \cdot c_{L_m}) \\
=& m \cdot \frac{1}{m} \cdot c_{L_1} + (m-1) \cdot \frac{1}{m-1} \cdot c_{L_2} + ... + c_{L_m} \\
=& c_{L_1} + c_{L_2} + ... + c_{L_m} \\
=& \sum_{h=1}^{m} c_{L_h} \\
=& C(d) \qquad \text{(equation (4.4))}
\end{aligned}$$

NUC Assume that $c_{L_h} = 0$ for all $h \leq k$. We want to show that $ED_{L_k}(P) = 0$:

$$\begin{aligned}
ED_{L_k}(P) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c_{L_h} \\
&= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot 0 \qquad (h \leq k) \\
&= 0
\end{aligned}$$

DR Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. We want to show that for all $h, k \geq l$ we have $ED_{L_h}(P^l) = ED_{L_k}(P^l)$:

$$\begin{aligned}
ED_{L_h}(P^l) &= \sum_{g=1}^{h} \frac{1}{m-g+1} \cdot c_{L_g} \\
&= \frac{1}{m-l+1} \cdot c_{L_l} \qquad (h \geq l)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{g=1}^{k} \frac{1}{m-g+1} \cdot c_{L_g} && (k \geq l) \\
&= ED_{L_k}(P^l)
\end{aligned}
$$

C-LIN Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, \alpha \cdot c^1 + \beta \cdot c^2)$ for $\alpha, \beta \in \mathbb{R}_+$. We want to show that for all $k \in M$ we have that $ED_{L_k}(P^3) = \alpha \cdot ED_{L_k}(P^1) + \beta \cdot ED_{L_k}(P^2)$:

$$
\begin{aligned}
ED_{L_k}(P^3) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot (\alpha \cdot c^1_{L_k} + \beta \cdot c^2_{L_k}) \\
&= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot \alpha \cdot c^1_{L_k} + \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot \beta \cdot c^2_{L_k} \\
&= \alpha \cdot \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^1_{L_k} + \beta \cdot \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^2_{L_k} \\
&= \alpha \cdot ED_{L_k}(P^1) + \beta \cdot ED_{L_k}(P^2)
\end{aligned}
$$

Let us prove uniqueness of *ED*. Suppose that rule f satisfies these four properties. For any $l \in M$, define $c^l = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^M_+$ a vector with a 1 on the l-th entry and 0 on the other entries. Let $P^l \in \mathcal{C}_{ed}$ such that $P^l = (N, L, d, c^l)$. From the ***Null Upstream Cost (NUC)*** it follows that $f_{L_k}(P^l)$=0 for all $k < l$. Then by ***Downstream Responsibility (DR)*** we know that there is a $x \in \mathbb{R}_+$ such that $f_{L_k}(P^l) = f_{L_{k'}}(P^l) = x$ for all $k, k' \geq l$. As f satisfies ***Efficiency (EF)*** we get

$$
\sum_{k=1}^{m} f(P^l) = (m-l+1) \cdot x = 1 = \sum_{l=1}^{m} c_{L_l}.
$$

Thus,

$$
f_{L_k}(P^l) = \begin{cases} 0 & \text{if } k < l \\ \frac{1}{m-l+1} & \text{if } k \geq l. \end{cases}
$$

Any vector in $\mathbb{R}^M_+$ can be presented as a linear combination of the standard basis vectors, here given by c^l for all $l \in M$, and some $\alpha \in \mathbb{R}_+$. Thus, c can be presented as $c = \sum_{l=1}^{m} c_{L_l} \cdot c^l = (c_{L_1}, ..., c_{L_m})$, with $\alpha = c_{L_l}$. Due to ***Cost-Linearity (C-LIN)*** we obtain for all $k \in M$

$$
\begin{aligned}
f_{L_k}(P) &= f_{L_k}(N, L, d, c) \\
&= f_{L_k}(N, L, d, \sum_{l=1}^{m} c_{L_l} \cdot c^l)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_k}(N, L, d, c^l) \\
&= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_k}(P^l) \\
&= c_{L_1} \cdot f_{L_k}(P^1) + c_{L_2} \cdot f_{L_k}(P^2) + ... + c_{L_k} \cdot f_{L_k}(P^k) + 0 + ... + 0 \\
&= \sum_{l=1}^{k} \frac{1}{m-k+1} c_{L_l} \\
&= ED_{L_k}(P)
\end{aligned}
$$

□

The equal downstream rule is the only rule discussed in this chapter that satisfies downstream responsibility (DR). This rule does take into account the ordering of the different levels and the responsibility of downstream levels for upstream costs, but costs are allocated according to the number of unions using a certain level. So if all unions are considered to be the same or if demands do not matter, this rule is an appropriate rule. This rule does not satisfy any of the union collusion properties. This is because this rule depends on the cardinality of M, in contrast to the level paying rule and the cascade rule, which do obey some of the union collusion properties.

Theorem 4.4.3 (Cascade rule axiomatic characterization)**.** *The cascade rule CA is the unique rule satisfying Null Upstream Cost, Downstream Demand Proportional Responsibility and Cost-Linearity.*

Proof. Necessity follows from theorem 4.3.3.

Let us prove uniqueness of CA. Suppose that rule f satisfies these three properties. For any $l \in M$, define $c^l = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}_+^M$ a vector with a 1 on the l-th entry and 0 on the other entries. Let $P^l \in \mathcal{C}_{ed}$ such that $P^l = (N, L, d, c^l)$. From ***Null Upstream Cost (NUC)*** it follows that $f_{L_k}(P^l) = 0$ for all $k < l$. By ***Downstream Demand Proportional Responsibility (DDPR)*** we have that $f_{L_k}(P^l) = \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}}$ for all $k \geq l$.

Any vector in $\mathbb{R}_+^M$ can be presented as a linear combination of the standard basis vectors, here given by c^l for all $l \in M$, and some $\alpha \in \mathbb{R}_+$. Thus, c can be presented as $c = \sum_{l=1}^{m} c_{L_l} \cdot c^l = (c_{L_1}, ..., c_{L_m})$, with $\alpha = c_{L_l}$. Due to ***Cost-Linearity (C-LIN)*** we obtain for all $k \in M$

$$
\begin{aligned}
f_{L_k}(P) &= f_{L_k}(N, L, d, c) \\
&= f_{L_k}(N, L, d, \sum_{l=1}^{m} c_{L_l} \cdot c^l)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_k}(N, L, d, c^l) \\
&= \sum_{l=1}^{m} c_{L_l} \cdot f_{L_k}(P^l) \\
&= \frac{\hat{d}_{L_k}}{\bar{d}_{L_1}} \cdot c_{L_1} + \frac{\hat{d}_{L_k}}{\bar{d}_{L_2}} \cdot c_{L_2} + ... + \frac{\hat{d}_{L_k}}{\bar{d}_{L_k}} \cdot c_{L_k} + 0 + ... + 0 \\
&= \sum_{l=1}^{k} \frac{\hat{d}_{L_k}}{\bar{d}_{L_l}} \cdot c_{L_l} \\
&= CA_{L_k}(P)
\end{aligned}
$$

□

The cascade rule considers the levels and the demands of the agents. It does however not take the cardinality of N or M into account. If the latter is important, the equal downstream rule might be a better choice. The property downstream demand proportional responsibility (DDPR) can be argued to be quite strong, it is however comparable with the downstream responsibility (DR) property and this property is in adapted form proposed for the axiomatization of the Upstream Equal Sharing rule in the polluted river game in Ni and Wang (2007). Also, this property does comprise the essence of the cascade rule, namely that downstream unions have a responsibility for the upstream costs proportional to their demands. Further can it be argued that is also a strong statement to ignore the demands, as the other two rules do. The cascade rule can also be employed for other problems, where possibly the demands can be changed to volumes. So for example in the polluted river game, the cascade can be applied with volumes of pollution instead of demands. Before we state the logical independence of the properties used in the theorems, we give an alternative axiomatic characterization of the cascade rule.

By means of the properties defined in this chapter also other axiomatic characterizations of the rules are possible. We informally describe one other possible axiomatic characterization of the cascade rule. Namely, the cascade rule is also the unique rule satisfying Null Upstream Costs Out, Union Standardness-II, Union Collusion Neutrality-II and Cost-Linearity. We provide a sketch of the proof: define again the vector $c^l \in \mathbb{R}_+^M$ with 1 on the l-th entry and 0 on the other entries. Then by NUCO all the levels lower ranked than l can be eliminated from the problem, such that the levels l to m remain. Subsequently we can collude the unions from level $l+1$ to m, such that two levels remain, namely levels l and $l+1$. By UCN-II we know that the cost shares of all the unions after the collusion do not change. By US-II we know how to allocate the costs amongst the remaining unions L_l and L_{l+1}, since this is now a two union problem. Finally, by UCN-II and C-LIN we can obtain the cost shares for all $k \in M$.

Logical Independence

The logical independence of the properties used in theorem 4.4.1 is given by

1. Consider the equal downstream allocation rule. This rules satisfies Efficiency and Cost-Linearity, but not the Null Cost Property.
2. For $P \in \mathcal{C}_{ed}$ let $f(P)$ be the rule such that $f_{L_k}(P) = 0$ for all $k \in M$. This rule satisfies Null Cost Property and Cost-Linearity, but not Efficiency.
3. Define $x := \{k \in M | c_{L_k} \neq 0\}$ the set of unions which level costs are not zero. For $P \in \mathcal{C}_{ed}$ define $f(P)$ for all $k \in M$ as follows:

$$f_{L_k}(P) = \begin{cases} 0 & \text{if } c_{L_k} = 0 \\ \frac{\sum_{h \in M} c_{L_h}}{|x|} & \text{if } c_{L_k} \neq 0. \end{cases}$$

   This rule satisfies Efficiency and the Null Cost Property, but not Cost-Linearity.

The logical independence of the properties used in theorem 4.4.2 is given by

1. Consider the level paying rule. This rules satisfies Efficiency, Null Upstream Cost and Cost-Linearity, but not Downstream Responsibility.
2. For $P \in \mathcal{C}_{ed}$ let $f(P)$ be the rule such that $f_{L_k}(P) = 0$ for all $k \in M$. This rule satisfies Null Cost Property, Cost-Linearity and Downstream Responsibility, but not Efficiency.
3. Define $x := \{k \in M | \exists l \leq k \text{ s.t. } c_{L_l} \neq 0\}$ the set of unions for which there exists an upstream level with cost non-zero or for which its own level cost is non-zero. For $P \in \mathcal{C}_{ed}$ define $f(P)$ for all $j \in M$ as follows:

$$f_{L_k}(P) = \begin{cases} 0 & \text{if } \forall h \leq k : c_{L_h} = 0 \\ \frac{\sum_{h \in M} c_{L_h}}{|x|} & \text{otherwise.} \end{cases}$$

   This rule satisfies Efficiency, Downstream Responsibility, Null Upstream Cost, but not Cost-Linearity.
4. For $P \in \mathcal{C}_{ed}$ let $f(P)$ be the rule such that $f_{L_k}(P) = \frac{\sum_{h \in M} c_{L_h}}{m}$ for all $k \in M$ and $|M| = m$. This rule satisfies Efficiency, Downstream Responsibility, Cost-Linearity, but not Null Upstream Cost.

The logical independence of the properties used in theorem 4.4.3 is given by

1. Consider the level paying rule. This rules satisfies Null Upstream Cost and Cost-Linearity, but not Downstream Demand Proportional Responsibility.

2. For $P \in \mathcal{C}_{ed}$ let $f(P)$ be the rule such that $f_{L_k}(P) = \sum_{h=1}^{m} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h}$ for all $k \in M$. This rule satisfies Cost-Linearity and Downstream Demand Proportional Responsibility, but not Null Upstream Cost.

3. For $P \in \mathcal{C}_{ed}$ define $f(P)$ for all $k \in M$ as follows:

$$f_{L_k}(P) = \begin{cases} 0 & \text{if } \forall h \leq k : c_{L_h} = 0 \\ \sum_{h=1}^{m} \frac{\hat{d}_{L_k}}{\bar{d}_{L_h}} \cdot c_{L_h} & \text{otherwise.} \end{cases}$$

   This rule satisfies Null Upstream Cost, Downstream Demand Proportional Responsibility, but not Cost-Linearity.

In summary, from this chapter we may conclude that all three rules satisfy different properties and can all uniquely be characterized by means of some of these properties. Therefore the suitability of the rules depends on the context of the problem. The electricity demand problem concerns the problem of reallocation the total electricity grid costs over unions of agents that are connected to a specific voltage level in the grid. Since for this problem the demands of the agents and the ordering of the levels of the unions are important features, the cascade rule may for this problem, given the rules discussed in this chapter, be considered most suitable and thereby the most fair rule.

Chapter 5

Electricity demand game

In the preliminaries we showed how to obtain a game from a cost allocation problem. In the last chapter we focused on the electricity demand problem and cost allocation rules directly applicable to the problem. In this chapter we define the cooperative cost game associated with the electricity demand problem. By means of this game we can apply concepts from cooperative game theory. In the preliminaries we have seen that also cooperative game theory offers many different solutions. Similar as for the problem we define properties for the game and characterize solutions by means of these properties. As there is not a one-to-one correspondence between the rules defined for the problem and the rules defined for the game, we cannot compare them based on properties. In this chapter we start by introducing a game on agent set N. For this game we analyse the characteristic function. Subsequently we construct a ***simplified version of the Shapley value*** that associates with each game a unique allocation vector for the agents in N. By means of this value we can construct two solutions on the set of unions M, namely the ***agent-*** and ***union-Shapley value***. So we define a game on agent set N, consider the Shapley value for agent set N, with the purpose to consider two forms of the Shapley value that give a solution for the unions in L. Reasons to define the problem as well as the game on N are first that it gives a more realistic representation of the situation and second because it is more convenient in case one would like to consider solutions for the set of agents in future research. Finally, we adapt several values discussed in Chapter 3 into union values and apply them to a numerical example.

5.1 The framework

Let us now define the game for problem $P = (N, L, d, c) \in \mathcal{C}_{ed}$ disclosed in Chapter 4. As for this chapter the vectors of constants $a, b \in \mathbb{R}^M_+$ are of importance, we in this chapter define P by $P = (N, L, d, a, b) \in \mathcal{C}_{ed}$. Given any $S \subseteq N$ let P_S denote the restriction of

P to S. Namely $P_S = (S, L_S, (d_i)_{i\in S}, (a_{L_k})_{L_k\in L_S}, (b_{L_k})_{L_k\in L_S})$ such that

- $S \subseteq N$ is a finite set of agents
- $L_S = \{L_k \cap S\}_{k\in M}$ is the partition L of S
- $d = (d_i)_{i\in S} \in \mathbb{R}_+^{|S|}$ is the demand vector of the agents in S
- $a = (a_{L_k})_{L_k\in L_s} \in \mathbb{R}_+^{|L_S|}$ and $b = (b_{L_k})_{L_k\in L_S} \in \mathbb{R}_+^{|L_S|}$ are the constants of the level cost vector.

Equivalently as before, we denote a greatest element of S by $\max S$ and the level of a greatest element of S by $l(\max S)$, i.e. $\max S = \max_{j\in S} j$ and $l(\max S) = \max_{j\in S} l(j)$. In this section we first consider the restriction of P to S, where S is any subset of N and thereafter the restriction where S is a subset of the unions in L. So for the first restriction we consider values and for the latter restriction union values.

Let us now define the induced cooperative cost game amongst agents in N. Our characteristic cost function v assumes that any coalition always needs to make use of the higher voltage levels, but not of the lower levels. Each coalition pays, according to the voltage levels it uses, the associated fixed costs and the variable costs related to its demand. Given the restricted electricity demand problem $P_S \in \mathcal{C}_{ed}$ we define the induced cost game on P_S by the characteristic cost function

$$v(S) = C(d_S, 0_{N\setminus S}) := \sum_{k=1}^{l(\max S)} c_{L_k} \tag{5.1}$$

$$:= \sum_{k=1}^{l(\max S)} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \tag{5.2}$$

with $v(\emptyset) = 0$, for all $S \subseteq N$.[1] Remark that $\bar{d}_{L_k} := \sum_{i\in S, l(i)\geq k} d_i$. As mentioned above, in Chapter 4 we considered the vector $c = (c_{L_k})_{k\in M}$ as a given, whereas in this chapter we consider the constants a_{L_k}, b_{L_k} for all $k \in M$ as given and deduce by means of these constants the vector c, such that $c_{L_k} = a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k}$. We denote the stand-alone cost of each agent $i \in N$ by $v(i)$ and the stand-alone cost of each union $k \in M$ by $v(L_k)$. We have a TU game with coalition structure L. Consequently, the ***electricity demand game*** is defined by the triple (N, v, L) and the class of all electricity demand games by $\mathcal{GP}_{ed}$.[2]

Example 5.1.1. *Consider a situation with three agents and a partition in three a priori unions. We define the elements of a problem $P \in \mathcal{C}_{ed}$ as follows:*

[1] Recall that $z := (d_S, 0_{N\setminus S})$ denotes a vector $z \in \mathbb{R}^N$ s.t. $z_i = d_i$ if $i \in S$ and $z_i = 0$ if $i \in N \setminus S$, defined in Section 2.3.

[2] Note that the partition L is already implicitly taken into account by the definition of v.

- $N = \{1, 2, 3\}$
- $L = \{L_1, L_2, L_3\}$ *with* $L_1 = \{1\},\ L_2 = \{2\},\ L_3 = \{3\}$
- $d = (20, 60, 100)$
- $a_{L_1} = 5,\ b_{L_1} = 200,\ a_{L_2} = 4,\ b_{L_2} = 250,\ a_{L_3} = 2$ *and* $b_{L_3} = 150.$

Before we give the cost values for all possible subsets $S \subseteq N$*, we show by means of some sample calculations how to compute the cost values.*

$$\begin{aligned} v(1) &= \sum_{k=1}^{l(\max\{1\})} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= \sum_{k=1}^{1} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= a_{L_1} \cdot d_1 + b_{L_1} \\ &= 5 \cdot 20 + 200 \\ &= 300 \end{aligned}$$

$$\begin{aligned} v(13) &= \sum_{k=1}^{l(\max\{1,3\})} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= \sum_{k=1}^{l(3)} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= \sum_{k=1}^{3} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= a_{L_1} \cdot (d_1 + d_3) + b_{L_1} + a_{L_2} \cdot d_3 + b_{L_2} + a_{L_3} \cdot d_3 + b_{L_3} \\ &= 5 \cdot 120 + 200 + 4 \cdot 100 + 250 + 2 \cdot 100 + 150 \\ &= 1800 \end{aligned}$$

$$\begin{aligned} v(123) &= \sum_{k=1}^{l(\max\{1,2,3\})} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= \sum_{k=1}^{l(3)} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \\ &= \sum_{k=1}^{3} a_{L_k} \cdot \bar{d}_{L_k} + b_{L_k} \end{aligned}$$

$$= a_{L_1} \cdot (d_1 + d_2 + d_3) + b_{L_1} + a_{L_2} \cdot (d_2 + d_3) + b_{L_2} + a_{L_3} \cdot d_3 + b_{L_3}$$
$$= 2340$$

In a similar fashion we compute the other cost values, resulting in cost shares presented in the table below.

Subsets S	**Cost** $v(S)$
$\emptyset$	0
$\{1\}$	300
$\{2\}$	990
$\{3\}$	1700
$\{1,2\}$	1090
$\{1,3\}$	1800
$\{2,3\}$	2240
$\{1,2,3\}$	2340

Table 5.1: The characteristic cost function v for $S \subseteq N$

The electricity demand game can be interpreted as the sum of a ***fixed*** electricity demand game (N, v^{fix}, L) and a ***variable*** electricity demand game (N, v^{var}, L) so that for all $S \subseteq N$ we have

$$v(S) := v^{fix}(S) + v^{var}(S) \tag{5.3}$$

with

$$v^{fix}(S) := \sum_{k=1}^{l(\max S)} b_{L_k}, \tag{5.4}$$
$$v^{var}(S) := \sum_{k=1}^{l(\max S)} a_{L_k} \cdot \bar{d}_{L_k}. \tag{5.5}$$

The game (N, v^{fix}, L) corresponds to the airport game as described in example 2.3.1, only the characteristic cost function is defined slightly different. We now show that v is concave by proving that v^{fix}, as well as v^{var} is concave.

Proposition 5.1.1. *The game (N, v^{fix}, L) is concave: for all $S, T \subseteq N$ it holds that*

$$v^{fix}(S) + v^{fix}(T) \geq v^{fix}(S \cup T) + v^{fix}(S \cap T).$$

Proof. We consider the following three cases for $S \subseteq N$:

1. $l(\max S) = l(\max T)$
2. $l(\max S) < l(\max T)$
3. $l(\max S) > l(\max T)$

In case S or T is empty, the inequality trivially holds. For $S, T \neq \emptyset$ we show that for all three cases the inequality holds:

1. Assume $l(\max S) = l(\max T) = t$. It follows that $l(\max S \cup T) = l(\max T) = t$ and $l(\max S \cap T) \leq t$. Define $l(\max S \cap T) = t'$. Then $t' \leq t$. We have

$$\begin{aligned} v^{fix}(S) + v^{fix}(T) &= \sum_{k=1}^{t} b_{L_k} + \sum_{k=1}^{t} b_{L_k} \\ &\geq \sum_{k=1}^{t} b_{L_k} + \sum_{k=1}^{t'} b_{L_k} && (t' \leq t) \\ &= v^{fix}(S \cup T) + v^{fix}(S \cap T). \end{aligned}$$

2. Assume $l(\max S) = s$, $l(\max T) = t$ such that $s < t$. It follows that $l(\max S \cup T) = t$ and $l(\max S \cap T) \leq s$. Define $l(\max S \cap T) = s'$. Then $s' \leq s$. We have

$$\begin{aligned} v^{fix}(S) + v^{fix}(T) &= \sum_{k=1}^{s} b_{L_k} + \sum_{k=1}^{t} b_{L_k} \\ &\geq \sum_{k=1}^{t} b_{L_k} + \sum_{k=1}^{s'} b_{L_k} && (s' \leq s) \\ &= v^{fix}(S \cup T) + v^{fix}(S \cap T). \end{aligned}$$

3. Due to symmetry with case 2 the proof is omitted.

□

Note that if $S \cap T \neq \emptyset$, then $\sum_{i \in S} d_i + \sum_{i \in T} d_i = \sum_{i \in S \cup T} d_i + \sum_{i \in S \cap T} d_i$. Let us now consider the game (N, v^{var}, L).

Proposition 5.1.2. *The game (N, v^{var}, L) is concave and convex: for all $S, T \subseteq N$ it holds that*

$$v^{var}(S) + v^{var}(T) = v^{var}(S \cup T) + v^{var}(S \cap T).$$

Proof. We consider the following three cases:

1. $l(\max S) = l(\max T)$
2. $l(\max S) < l(\max T)$
3. $l(\max S) > l(\max T)$

In case S or T is empty, the equality trivially holds. For $S, T \neq \emptyset$ we show that for all cases equality holds:

1. Assume $l(\max S) = t$ and $l(\max S \cap T) = r$. Then $r \leq t$. It follows that $l(\max S \cup T) = l(\max T) = t$. We have

$$\begin{aligned}
v^{var}(S) + v^{var}(T) =& \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in S, l(i)\geq k} d_i) + \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in T, l(i)\geq k} d_i) \\
=& \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in S\cup T, l(i)\geq k} d_i) + \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in S\cap T, l(i)\geq k} d_i) \\
=& \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in S\cup T, l(i)\geq k} d_i) + \sum_{k=1}^{r}(a_{L_k} \cdot \sum_{i\in S\cap T, l(i)\geq k} d_i) + \\
& \sum_{k=r+1}^{t}(a_{L_k} \cdot \sum_{i\in S\cap T, l(i)\geq k} d_i) \\
=& \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in S\cup T, l(i)\geq k} d_i) + \sum_{k=1}^{r}(a_{L_k} \cdot \sum_{i\in S\cap T, l(i)\geq k} d_i) + \\
& 0 \qquad (*) \\
=& v^{var}(S \cup T) + v^{var}(S \cap T)
\end{aligned}$$

(*) $\{i \in (S \cap T) | l(i) \geq r + 1\} = \emptyset$

2. Assume $l(\max S) = s$, $l(\max T) = t$ and $l(\max S \cap T) = r$. Then $r \leq s < t$. It follows that $l(\max S \cup T) = t$. We have

$$\begin{aligned}
v^{var}(S) + v^{var}(T) =& \sum_{k=1}^{s}(a_{L_k} \cdot \sum_{i\in S, l(i)\geq k} d_i) + \sum_{k=1}^{t}(a_{L_k} \cdot \sum_{i\in T, l(i)\geq k} d_i) \\
=& \sum_{k=1}^{s}(a_{L_k} \cdot \sum_{i\in S, l(i)\geq k} d_i) + \sum_{k=1}^{s}(a_{L_k} \cdot \sum_{i\in T, l(i)\geq k} d_i) +
\end{aligned}$$

$$
\begin{aligned}
&\sum_{k=s+1}^{t}(a_{L_k}\cdot\sum_{i\in T,l(i)\geq k}d_i)\\
=&\sum_{k=1}^{s}(a_{L_k}\cdot\sum_{i\in S\cup T,l(i)\geq k}d_i)+\sum_{k=1}^{s}(a_{L_k}\cdot\sum_{i\in S\cap T,l(i)\geq k}d_i)+\\
&\sum_{k=s+1}^{t}(a_{L_k}\cdot\sum_{i\in T,l(i)\geq k}d_i)\\
=&\sum_{k=1}^{t}(a_{L_k}\cdot\sum_{i\in S\cup T,l(i)\geq k}d_i)+\sum_{k=1}^{s}(a_{L_k}\cdot\sum_{i\in S\cap T,l(i)\geq k}d_i) \qquad (*)\\
=&\sum_{k=1}^{t}(a_{L_k}\cdot\sum_{i\in S\cup T,l(i)\geq k}d_i)+\sum_{k=1}^{r}(a_{L_k}\cdot\sum_{i\in S\cap T,l(i)\geq k}d_i)+\\
&\sum_{k=r+1}^{s}(a_{L_k}\cdot\sum_{i\in S\cap T,l(i)\geq k}d_i)\\
=&\sum_{k=1}^{t}(a_{L_k}\cdot\sum_{i\in S\cup T,l(i)\geq k}d_i)+\sum_{k=1}^{r}(a_{L_k}\cdot\sum_{i\in S\cap T,l(i)\geq k}d_i)+\\
&0 \qquad (**)\\
=&v^{var}(S\cup T)+v^{var}(S\cap T)
\end{aligned}
$$

(*) $\{i\in T|l(i)\geq s+1\}=\{i\in(S\cup T)|l(i)\geq s+1\}$
(**) $\{i\in(S\cap T)|l(i)\geq r+1\}=\emptyset$

3. Due to symmetry with case 2. the proof is omitted.

□

Proposition 5.1.3. *Every game $(N,v,L)\in\mathcal{GP}$ that is concave and convex is inessential.*

Proof. For any two disjoint sets $S,T\subseteq N$ such that $S\cap T=\emptyset$ it follows from concavity and convexity of game (N,v,L) that $v(S)+v(T)=v(S\cup T)$. Now let S,T be singletons such that $S=\{i\}$ and $T=\{j\}$. We have that for all $i\neq j\in N$, $v(i)+v(j)=v(i\cup j)$ and hence $v(S)=\sum_{i\in S}v(i)$ for all $S\subseteq N$. □

Thus it follows that the game (N,v^{var},L) is ***inessential***. Moreover, from propositions 5.1.1 and 5.1.2 we may conclude that the characteristic cost function v is concave and hence sub-additive. As a result, the Shapley value of the game $(N,v,L)\in\mathcal{GP}_{ed}$ is in the core. This is an important result, as this makes the Shapley value a good solution for the game, since now the Shapley value gives a stable cost allocation.[3] In the next section we define the Shapley value, the agent-Shapley value and the union-Shapley value and other union values for the electricity demand game. First we consider below a numerical example for three agents for which the variable and fixed characteristic function are constructed for all possible subsets S in N.

Example 5.1.2. *Consider a situation with three agents and a partition in two a priori unions. We define the elements of a problem $P \in \mathcal{C}_{ed}$ as follows:*

- $N = \{1, 2, 3\}$
- $L = \{L_1, L_2\}$ *with* $L_1 = \{1, 2\},\ L_2 = \{3\}$
- $d = (10, 5, 20)$
- $a_{L_1} = 5,\ b_{L_1} = 150,\ a_{L_2} = 4,\ b_{L_2} = 100$

By means of the demand vector we can obtain the aggregated demand vector $\bar{d} = (\bar{d}_{L_1}, \bar{d}_{L_2}) = (35, 20)$ and the level demand vector $\hat{d} = (\hat{d}_{L_1}, \hat{d}_{L_2}) = (15, 20)$. Further we have $c = (c_{L_1}, c_{L_2}) = (5 \cdot \bar{d}_{L_1} + 150, 4 \cdot \bar{d}_{L_2} + 100) = (325, 180)$. Let us now consider the variable and fixed cost values obtained by applying the characteristic function on N and subsets $S \subseteq N$, without restrictions on S. In Appendix II we show by means of some examples how to compute the characteristic functions.

Subsets S	$v^{fix}(S)$	$v^{var}(S)$	$v(S)$
$\emptyset$	0	0	0
$\{1\}$	150	50	200
$\{2\}$	150	25	175
$\{3\}$	250	180	430
$\{1, 2\}$	150	75	225
$\{1, 3\}$	250	230	480
$\{2, 3\}$	250	205	455
$\{1, 2, 3\}$	250	255	505

Table 5.2: The characteristic cost function v for $S \subseteq N$

5.2 Solutions

In Chapter 3 we defined the Shapley value, the union-Shapley value and the agent-Shapley value. In this section we take a closer look at these values for the electricity

[3]Assuming that we define a stable coalition by the core of the game.

demand game. Thereafter we define multiple union values, which were presented in Chapter 3 as values for N. As noted before, we first consider the Shapley value that is solution of the corresponding game on N and thereafter union values, which are union values of the corresponding game on M.

5.2.1 Values

Littlechild and Owen (1973) provide in their article a simple expression of the Shapley value for the class of airport games, as presented in example 3.2.1. We consider a comparable simple version of the Shapley value for the variable as well as the fixed characteristic function for the class of electricity demand games. Due to additivity of the Shapley value we may thereafter add these two values to obtain the Shapley value for game $(N, v, L) \in \mathcal{G}_{ed}$.

Proposition 5.2.1. *The Shapley value for* $(N, v^{fix}, L) \in \mathcal{GP}_{ed}$ *is given by*

$$Sh_i(N, v^{fix}, L) := \sum_{k=1}^{l(i)} \frac{b_{L_k}}{|\cup_{h=k}^{m} L_h|}, \tag{5.6}$$

for all $i \in N$ *with* $|\cup_{h=k}^{m} L_h|$ *representing the number of agents in* $\cup_{h=k}^{m} L_h$, $b_{L_k} \in \mathbb{R}_+$ *the fixed cost associated to level* L_k *and* $L = \{L_1, ..., L_m\}$.

Proof. For every $k \in M$ define game v_k^{fix} such that

$$v_k^{fix}(S) = \begin{cases} 0 & \text{if } l(\max S) < k \\ b_{L_k} & \text{if } l(\max S) \geq k. \end{cases} \tag{5.7}$$

Note that

$$l(\max S) < k \Leftrightarrow S \cap \cup_{h=k}^{m} L_h = \emptyset$$

and

$$l(\max S) \geq k \Leftrightarrow S \cap \cup_{h=k}^{m} L_h \neq \emptyset.$$

Now from equation (5.7) it follows that

$$\begin{aligned} \sum_{k=1}^{m} v_k^{fix}(S) &= \sum_{k=1}^{l(\max S)} v_k^{fix}(S) + \sum_{k=l(\max S)+1}^{m} v_k^{fix}(S) \\ &= \sum_{k=1}^{l(\max S)} b_{L_k} + 0 \\ &= \sum_{k=1}^{l(\max S)} b_{L_k} \end{aligned}$$

$$= v^{fix}(S)$$

Hence, for all $S \subseteq N$ we have $v^{fix}(S) = \sum_{k=1}^{m} v_k^{fix}(S)$. By additivity of the Shapley value we know that

$$Sh_i(N, v^{fix}, L) = \sum_{k=1}^{m} Sh_i(N, v_k^{fix}, L).$$

Consider game (N, v_k^{fix}, L). All $i \in N$ with $l(i) < k$ (i.e. $i \notin \cup_{h=k}^{m} L_h$) are dummy agents: if $l(\max S) \leq l(i) < k$, then $v_k^{fix}(S \cup i) = 0 = 0 + 0 = v_k^{fix}(S) + v_k^{fix}(i)$ and if $l(\max S) > l(i)$ and $l(\max S) > k$, then $v_k^{fix}(S \cup i) = v_k^{fix}(S) = b_{L_k}$. All i, j with $l(i), l(j) \geq k$ (i.e. $i, j \in \cup_{h=k}^{m} L_h$) are symmetric agents: for all $S \subseteq N$ it holds that $v_k^{fix}(S \cup i) = b_{L_k} = v_k^{fix}(S \cup j)$.

Since the Shapley value satisfies efficiency, dummy and symmetry we obtain:

$$Sh_i(N, v_k^{fix}, L) = \begin{cases} 0 & \text{if } l(i) < k \\ \frac{b_{L_k}}{|\cup_{h=k}^{m} L_h|} & \text{if } l(i) \geq k. \end{cases}$$

Note that also here holds that $l(i) < k$ iff $i \notin \cup_{h=k}^{m} L_h$ and $l(i) \geq k$ iff $i \in \cup_{h=k}^{m} L_h$. Consequently,

$$\begin{aligned} Sh_i(N, v^{fix}, L) &= \sum_{k=1}^{m} Sh_i(N, v_k^{fix}, L) \\ &= \sum_{k=1}^{l(i)} Sh_i(N, v_k^{fix}, L) + \sum_{k=l(i)+1}^{m} Sh_i(N, v_k^{fix}, L) \\ &= \sum_{k=1}^{l(i)} Sh_i(N, v_k^{fix}, L) + 0 \\ &= \sum_{k=1}^{l(i)} \frac{b_{L_k}}{|\cup_{h=k}^{m} L_h|} \end{aligned}$$

□

So the Shapley value of the fixed characteristic cost function allocates the fixed cost of the first level over all agents, subsequently it allocates the fixed cost of the second level over all agents of the second level and of the lower (downstream) levels and so on. Before we consider the simple expression of the Shapley value for the variable characteristic cost function, consider the following lemma.

Lemma 5.2.1. *The Shapley value of an inessential game* (N, v, L) *is given by*

$$Sh_i(N, v, L) := v(i). \tag{5.8}$$

for all $i \in N$.

The Shapley value gives the average marginal contribution over all possible joining orders of an agent and in an inessential game the marginal contribution of an agent is always its stand-alone cost.

Proposition 5.2.2. *The Shapley value for* $(N, v^{var}, L) \in \mathcal{GP}_{ed}$ *is given by*

$$Sh_i(N, v^{var}, L) := v^{var}(i) \tag{5.9}$$

$$:= d_i \cdot \sum_{k=1}^{l(i)} a_{L_k}, \tag{5.10}$$

for all $i \in N$.

Proof. By proposition 5.1.2 and 5.1.3 it follows that (N, v^{var}, L) is inessential. By lemma 5.2.1 it follows that the Shapley value of an inessential game is given by $Sh_i(N, v, L) = v(i)$. Hence, $Sh_i(N, v^{var}, L) := v^{var}(i)$. □

Proposition 5.2.3. *Due to additivity of the Shapley value, the Shapley value for* $(N, v, L) \in \mathcal{GP}_{ed}$ *is given by*

$$Sh_i(N, v, L) := Sh_i(N, v^{fix}, L) + Sh_i(N, v^{var}, L)$$

$$:= \sum_{k=1}^{l(i)} \left(d_i \cdot a_{L_k} + \tfrac{b_{L_k}}{|\cup_{h=k}^{m} L_h|} \right), \tag{5.11}$$

for all $i \in N$.

Example 5.2.1. *Consider again example 5.1.2. We construct the Shapley value* $Sh_i(N, v, L)$ *for all agents* $i \in N$ *for this game by means of the expression presented in proposition 5.2.3.*

$$\begin{aligned} Sh_1(N, v, L) &= \sum_{k=1}^{1} \left(d_1 \cdot a_{L_k} + \tfrac{b_{L_k}}{|\cup_{j=k}^{2} L_j|} \right) \\ &= d_1 \cdot a_{L_1} + \frac{b_{L_1}}{|L_1 \cup L_2|} \\ &= 10 \cdot 5 + \frac{150}{3} \\ &= 50 + 50 = 100 \end{aligned}$$

$$
\begin{aligned}
Sh_2(N,v,L) &= \sum_{k=1}^{1}\left(d_2 \cdot a_{L_k} + \tfrac{b_{L_k}}{|\cup_{j=k}^{2} L_j|}\right) \\
&= d_2 \cdot a_{L_1} + \frac{b_{L_1}}{|L_1 \cup L_2|} \\
&= 5 \cdot 5 + \frac{150}{3} \\
&= 25 + 50 = 75
\end{aligned}
$$

$$
\begin{aligned}
Sh_3(N,v,L) &= \sum_{k=1}^{2}\left(d_3 \cdot a_{L_k} + \tfrac{b_{L_k}}{|\cup_{j=k}^{2} L_j|}\right) \\
&= d_3 \cdot a_{L_1} + \frac{b_{L_1}}{|L_1 \cup L_2|} + d_3 \cdot a_{L_2} + \frac{b_{L_2}}{|L_2|} \\
&= 20 \cdot 5 + \frac{150}{3} + 20 \cdot 4 + \frac{100}{1} \\
&= 100 + 50 + 80 + 100 = 330
\end{aligned}
$$

Hence, the Shapley value gives the cost allocation vector $(100, 75, 330) \in \mathbb{R}^3_+$*, which is efficient, i.e.* $\sum_{i \in N} Sh_i(N,v,L) = 505$*.*

This example, based on proposition 5.2.3, gives the Shapley value for the agents in N. We are interested in cost allocation vectors for the unions/levels in M. However, in order to define cost allocations for the unions (union value), it can be useful to first define cost allocations for the agents. Moreover, it can be useful for future research on cost allocations for the agents. Before we define the union values, we provide three lemmas regarding symmetric agents, dummy agents and null agents.

Lemma 5.2.2. *Let* $(N,v,L) \in \mathcal{GP}_{ed}$ *a game where* $a_{L_k}, b_{L_k} > 0$ *for all* $k \in M$*, then agents* f, g *are symmetric in game* (N,v,L) *if and only if* $l(f) = l(g)$ *and* $d_f = d_g$*.*

Proof. ($\Rightarrow$) We prove this side by contraposition. Thus, first assume $l(f) = l(g) = t$ and $d_f \neq d_g$. There is a $S \setminus \{f,g\} \subset N$, namely $S = \emptyset$, such that we have

$$
\begin{aligned}
v(S \cup f) &= v(f) \\
&= \sum_{k=1}^{t}(a_{L_k} \cdot d_f + b_{L_k}) \\
&\neq \sum_{k=1}^{t}(a_{L_k} \cdot d_g + b_{L_k}) && (d_f \neq d_g) \\
&= v(g) \\
&= v(S \cup g)
\end{aligned}
$$

Now assume $s = l(f) \neq l(g) = t$ and $d_f = d_g$. There is a $S \setminus \{f, g\} \subset N$, namely $S = \emptyset$, such that we have

$$\begin{aligned}
v(S \cup f) &= v(f) \\
&= \sum_{k=1}^{s} (a_{L_k} \cdot d_f + b_{L_k}) \\
&= \sum_{k=1}^{s} (a_{L_k} \cdot d_g + b_{L_k}) && (d_f = d_g) \\
&\neq \sum_{k=1}^{t} (a_{L_k} \cdot d_g + b_{L_k}) && (s \neq t) \\
&= v(g) \\
&= v(S \cup g)
\end{aligned}$$

So by contraposition we have that if agents f, g are symmetric in a game where $a_{L_k}, b_{L_k} > 0$ for all $k \in M$, then $l(f) = l(g)$ and $d_f = d_g$.

($\Leftarrow$) Assume $l(f) = l(g)$ and $d_f = d_g$ for $f, g \in N$ in a game $(N, v, L) \in \mathcal{GP}_{ed}$ where $a_{L_k}, b_{L_k} > 0$ for all $k \in M$. If $l(f) = l(g)$ we know that $l(\max S \cup f) = l(\max S \cup g)$ for all $S \subset N \setminus \{f, g\}$. Namely $l(\max S \cup f) = l(\max S \cup g) = l(\max S)$ if $l(f) = l(g) \leq l(\max S)$ or $l(\max S \cup f) = l(f) = l(g)$ if $l(f) = l(g) > l(\max S)$. Therefore assume that $l(\max S \cup f) = l(\max S \cup g) = t$. Left to show is that $v(S \cup f) = v(S \cup g)$ for all $S \subset N \setminus \{f, g\}$:

$$\begin{aligned}
v(S \cup f) &= \sum_{k=1}^{l(\max S \cup f)} (a_{L_k} \cdot \sum_{i \in S \cup f, l(i) \geq k} d_i + b_{L_k}) \\
&= \sum_{k=1}^{t} (a_{L_k} \cdot \sum_{i \in S \cup f, l(i) \geq k} d_i + b_{L_k}) \\
&= \sum_{k=1}^{t} (a_{L_k} \cdot \sum_{i \in S, l(i) \geq k} d_i + b_{L_k}) + \sum_{k=1}^{l(f)} a_{L_k} \cdot d_f \\
&= \sum_{k=1}^{t} (a_{L_k} \cdot \sum_{i \in S, l(i) \geq k} d_i + b_{L_k}) + \sum_{k=1}^{l(g)} a_{L_k} \cdot d_g && (d_f = d_g,\ l(f) = l(g)) \\
&= \sum_{k=1}^{t} (a_{L_k} \cdot \sum_{i \in S \cup g, l(i) \geq k} d_i + b_{L_k}) \\
&= v(S \cup g).
\end{aligned}$$

Hence, if $l(f) = l(g)$ and $d_f = d_g$ then agents f, g must be symmetric in game $(N, v, L) \in \mathcal{GP}_{ed}$ where $a_{L_k}, b_{L_k} > 0$. □

Lemma 5.2.3. *All agents in game $(N, v, L) \in \mathcal{GP}_{ed}$ are dummy agents if $b_{L_k} = 0$ for all $k \in M$.*

Proof. Assume $b_{L_k} = 0$ for all $k \in M$. It follows that $v^{fix}(S) = 0$ for all $S \subseteq N$ and thus $v(S) = v^{var}(S)$. From propositions 5.1.2 and 5.1.3 it followed that the game (N, v^{var}, L) is inessential. Clearly every agent in an inessential game is a dummy agent. So it follows that all agents i in game $(N, v, L) \in \mathcal{GP}_{ed}$ are dummy agents. □

Lemma 5.2.4. *Agent i is a null agent in game $(N, v, L) \in \mathcal{GP}_{ed}$ if*

(i) $b_{L_k} = 0$ *and* $a_{L_k} = 0$ *for all* $k \in M$, *or*

(ii) $b_{L_k} = 0$, $a_{L_k} > 0$ *and* $d_i = 0$ *for all* $k \in M$.

Proof. We know that every null agent is a dummy agent with $v(i) = 0$.

(i) Assume $b_{L_k} = 0$ and $a_{L_k} = 0$ for all $k \in M$, then for all $S \subseteq N$ we have that $v(S) = 0$ and therefore clearly all agents are null agents, in particular agent i.

(ii) By lemma 5.2.3 it follows that if $b_{L_k} = 0$ for all $k \in M$, then all agents in game (N, v, L) are dummy agents. Left to show is that from $a_{L_k} > 0$ and $d_i = 0$ for all $k \in M$, follows that $v(i) = 0$ for $i \in N$. We have that $v(i) = \sum_{k=1}^{l(i)} a_{L_k} \cdot d_i = \sum_{k=1}^{l(i)} a_{L_k} \cdot 0 = 0$. And thus from $d_i = 0$ and $a_{L_k} > 0$ follows that $v(i) = 0$.

Hence agent i is a null agent in game (N, v, L) if (i) or (ii) holds. □

5.2.2 Union values

As we are interested in cost allocations for the unions of agents, we first consider two important union values, namely the ***union-Shapley value***, presented in definition 3.3.2 and ***the agent-Shapley value***, presented in definition 3.3.3 (van den Brink and Dietz, 2014). The agent-Shapley value for union L_k is the sum of the Shapley values of the agents in union L_k. Moreover we consider values discussed in Chapter 3 and adapt them to union values, namely the separable cost remaining benefit union value, the proportional union value, the egalitarian union value and the non-cooperative union value.

Definition 5.2.1. *(Agent-Shapley value) The agent-Shapley value of the electricity demand game is given by*

$$Sh^a_{L_k}(N, v, L) = \sum_{i \in L_k} Sh_i(N, v, L)$$

$$= \sum_{i \in L_k} \sum_{h=1}^{l(i)} \left(d_i \cdot a_{L_h} + \frac{b_{L_h}}{|\cup_{l=h}^{m} L_l|} \right).$$

for all $k \in M$ and $i \in N$.

This solution takes the agents as the decision makers. When an agent enters a coalition, its marginal contribution is determined by its level and its demand. The marginal contributions with respect to the fixed cost are fully determined by the level of the agent: if there is an agent in the coalition with at least the same level, its marginal contribution is zero and otherwise its marginal contributions is the sum of all fixed level costs that are not covered yet. The marginal contribution with respect to the variable costs are determined by the demand and the level of the agent. If there is already an agent in the coalition with at least the same level, the marginal contribution is only the demand of the agent times all the variable level costs depending on the levels it uses. If the agent has a higher demand, she incurs on top of the latter costs, also the variable cost part for the extra required voltage levels. If the cost of a level is determined by the marginal contributions of all agents attached to that level, the agent-Shapley value is an appropriate solution. The agent-Shapley value first allocates each agent with a share such that the variable part is dependent on the demand of the agent and all levels used by the agent and the fixed part is solely dependent on all levels used by the agent. Thereafter the agent-Shapley value sums over all the agents individual cost shares obtained by the Shapley value.

Definition 5.2.2. *(Union-Shapley value) The union-Shapley value of the electricity demand game is given by*

$$\begin{aligned} Sh^u_{L_k}(N, v, L) &= Sh_{L_k}(M, v^L) \\ &= \sum_{h=1}^{k} \left(\hat{d}_{L_k} \cdot a_{L_h} + \frac{b_{L_h}}{|m-h+1|} \right) \end{aligned}$$

with $v^L(U) = v(\cup_{h \in U} L_h)$ for all $U \subseteq M$, $L = \{L_1, ..., L_m\}$ and $M = \{1, ..., m\}$.

If unions are taken as the decision making agent, the cost of a coalition is mainly determined by the union with the highest level. Each time a union enters a coalition, its marginal contribution is determined by its level and its demand. More precisely, the marginal contribution with respect to the fixed characteristic function is fully determined by its level and the marginal contribution with respect to the variable characteristic function is determined by the demand and the level of the union. The union-Shapley value seems a reasonable solution if the cost allocation is indeed determined by the level and the demand of a union. The union-Shapley value allocates each union with a share such that the variable part is dependent on the demand of the level and the level and the fixed part is dependent on the level. It charges a union for all the levels it uses. Consider the following example to see the difference between the two union Shapley notions for the electricity demand game.

Example 5.2.2. *Consider again examples 5.1.2 and 5.2.1. Let us now construct the agent- as well as the union-Shapley, in corresponding order. The agent-Shapley value is derived as follows:*

$$\begin{aligned}
Sh^a_{L_1}(N,v,L) &= \sum_{i\in L_1}\sum_{h=1}^{l(i)}\left(d_i\cdot a_{L_h}+\tfrac{b_{L_h}}{|\cup_{l=h}^m L_l|}\right).\\
&= \sum_{h=1}^{1}\left(d_1\cdot a_{L_h}+\tfrac{b_{L_h}}{|\cup_{l=h}^2 L_l|}\right)+\sum_{h=1}^{1}\left(d_2\cdot a_{L_h}+\tfrac{b_{L_h}}{|\cup_{l=h}^2 L_l|}\right)\\
&= d_1\cdot a_{L_1}+\frac{b_{L_1}}{|L_1\cup L_2|}+d_2\cdot a_{L_1}+\frac{b_{L_1}}{|L_1\cup L_2|}\\
&= 10\cdot 5+\frac{150}{3}+5\cdot 5+\frac{150}{3}\\
&= 175
\end{aligned}$$

$$\begin{aligned}
Sh^a_{L_2}(N,v,L) &= \sum_{i\in L_2}\sum_{h=1}^{l(i)}\left(d_i\cdot a_{L_h}+\tfrac{b_{L_h}}{|\cup_{l=h}^m L_l|}\right).\\
&= \sum_{h=1}^{2}\left(d_3\cdot a_{L_h}+\tfrac{b_{L_h}}{|\cup_{l=h}^2 L_l|}\right)\\
&= d_3\cdot a_{L_1}+\frac{b_{L_1}}{|L_1\cup L_2|}+d_3\cdot a_{L_2}+\frac{b_{L_2}}{|L_2|}\\
&= 20\cdot 5+\frac{150}{3}+20\cdot 4+\frac{100}{1}\\
&= 330
\end{aligned}$$

So the agent-Shapley value is given by the allocation vector $Sh^a(N,v,L)=(175,330)\in\mathbb{R}^M_+$. *The union-Shapley value is computed as follows,*

$$\begin{aligned}
Sh^u_{L_1}(N,v,L) &= \sum_{h=1}^{1}\left(\hat{d}_{L_k}\cdot a_{L_h}+\tfrac{b_{L_h}}{|m-h+1|}\right)\\
&= \hat{d}_{L_1}\cdot a_{L_1}+\frac{b_{L_1}}{|2-1+1|}\\
&= 15\cdot 5+\frac{150}{2}\\
&= 150
\end{aligned}$$

$$Sh^u_{L_2}(N,v,L) = \sum_{h=1}^{2}\left(\hat{d}_{L_k}\cdot a_{L_h}+\tfrac{b_{L_h}}{|m-h+1|}\right)$$

$$
\begin{aligned}
&= \left(\hat{d}_{L_2} \cdot a_{L_1} + \tfrac{b_{L_1}}{|2-1+1|}\right) + \left(\hat{d}_{L_2} \cdot a_{L_2} + \tfrac{b_{L_2}}{|2-2+1|}\right) \\
&= 20 \cdot 5 + \frac{150}{2} + 20 \cdot 4 + \frac{100}{1} \\
&= 355
\end{aligned}
$$

Thus, the union-Shapley value gives the allocation vector $Sh^u(N, v, L) = (150, 355) \in \mathbb{R}_+^M$*. Note that both the union values give an efficient allocation vector:* $\sum_{k\in M} Sh^a_{L_k} = \sum_{k\in M} Sh^u_{L_k} 505 = v(N)$*.*

The agent-Shapley value in this example allocates a higher cost share to the first union (L_1) *than the union-Shapley value. This is due to the fact the agent-Shapley value allocates 100 of the fixed cost of level 1* ($b_{L_1} = 150$) *to union* L_1*, whereas the union-Shapley value allocates 75 of the fixed cost of level 1* ($b_{L_1} = 150$) *to union* L_1*: the union-Shapley value allocates this fixed level cost proportional to the number of unions and the agent-Shapley value proportional to the number of agents, i.e.* $\frac{1}{2} \cdot 150$ *versus* $\frac{2}{3} \cdot 150$ *respectively.*

In Chapter 3 we discussed some properties for the agent- and union-Shapley value. These two values differ with respect to the collusion neutrality property they obey: the agent-Shapley value satisfies union collusion neutrality and the union-Shapley value satisfies agent collusion neutrality. The agent collusion neutrality axiom implies that merging agents within one union does not change the level or the total demand of this union and therefore has no effect on the cost share of this union. The union collusion neutrality axiom implies that if two unions of agents collude such that the new union consists of all agents of the original unions, the total cost share of the new union is the same as the sum of the cost shares of the original unions.

Let us consider some other union values, that were presented as values in Chapter 3. Let the separable cost of union L_k be defined by $s_{L_k} = v(N) - v(N \setminus L_k)$ and the remaining benefit by $r_{L_k} = v(L_k) - s_{L_k}$.

Definition 5.2.3. *(Separable cost remaining benefit union value) The separable cost remaining benefit union value is given by*

$$
\begin{aligned}
SCRB^u_{L_k}(N, v, L) &= SCRB_{L_k}(M, v^L) \\
&= s_{L_k} + \frac{r_{L_k}}{\sum_{h\in M} r_{L_h}} \cdot r_N, \qquad (5.12)
\end{aligned}
$$

with $v^L(U) = v(\cup_{h\in U} L_h)$ *and* $(N, v, L) \in \mathcal{GP}_{ed}$ *for all* $k \in M$*,* $U \subseteq M$ *and with* $r_N = v(N) - \sum_{h\in M} s_{L_h}$*.*

We can further define s_{L_k}, r_{L_k}, r_N by replacing the equations with the actual characteristic functions. Then we get that for all $k < m$: $s_{L_k} = \sum_{h=1}^{k} a_{L_h} \cdot \hat{d}_{L_k}$ and $r_{L_k} = \sum_{h=1}^{k} b_{L_h}$. For $k = m$ we have that $s_{L_m} = \sum_{h=1}^{m} a_{L_h} \cdot \hat{d}_{L_m} + b_{L_m}$ and $r_{L_m} = \sum_{h=1}^{m-1} b_{L_h}$. Lastly, we have that $r_N = \sum_{h=1}^{m-1} b_{L_h}$. By means of these parts we can derive the following equations for the separable cost remaining benefit union value:

$$SCRB^u_{L_k}(N, v, L) = \sum_{h=1}^{k} a_{L_h} \cdot \hat{d}_{L_k} + \frac{\sum_{h=1}^{k} b_{L_h}}{\sum_{k=1}^{m-1} \sum_{h=1}^{k} b_{L_h} + \sum_{h=1}^{m-1} b_{L_h}} \cdot \sum_{h=1}^{m-1} b_{L_h},$$

for $k < m$ and

$$SCRB^u_{L_k}(N, v, L) = \sum_{h=1}^{m} a_{L_h} \cdot \hat{d}_{L_m} + b_{L_m} + \frac{\sum_{h=1}^{m-1} b_{L_h}}{\sum_{k=1}^{m-1} \sum_{h=1}^{k} b_{L_h} + \sum_{h=1}^{m-1} b_{L_h}} \cdot \sum_{h=1}^{m-1} b_{L_h},$$

for $k = m$. Note that even though this union value is not additive, that the $SCRB$ union value gives for the game $(N, v, L) \in \mathcal{GP}_{ed}$ the same cost share as the agent-Shapley value with respect to the variable part. Both union values allocate every union the sum of the variable costs (a_{L_k}) of the used levels times the union's demand. With regard to the fixed costs, this union value charges every union proportional to its remaining benefit. Only level m is charged extra costs, namely the fixed cost of its own level (b_{L_m}). This union value solely considers the coalitions of size 1, $M - 1$ and M.

Definition 5.2.4. *(Proportional union value) The proportional union value is given by*

$$\begin{aligned} Pr^u_{L_k}(N, v, L) &= Pr_{L_k}(M, v^L) \\ &= \frac{v(L_k)}{\sum_{l=1}^{m} v(L_l)} \cdot v(N) \\ &= \frac{\sum_{h=1}^{k} (a_{L_h} \cdot \hat{d}_{L_k} + b_{L_h})}{\sum_{l=1}^{m} (\sum_{h=1}^{l} (a_{L_h} \cdot \hat{d}_{L_l} + b_{L_h}))} \cdot \sum_{h=1}^{m} (a_{L_h} \cdot \bar{d}_{L_h} + b_{L_h}). \end{aligned} \tag{5.13}$$

with $v^L(U) = v(\cup_{h \in U} L_h)$ and $(N, v, L) \in \mathcal{GP}_{ed}$ for all $k \in M$, $U \subseteq M$.

The proportional union value only considers the coalitions of size 1 and M. It allocates the cost of the grand coalition proportional to the union's stand-alone cost. Since the proportional union value as well as the proportional union value are not additive, we can not apply the rules separately to the two cost games (N, v^{var}, L) and (N, v^{fix}, L). Therefore we can not further simplify these union values. This is an important difference with the agent- and union-Shapley value, which both satisfy additivity. The next union value does not distinguish between unions and allocates to each union the same cost share.

Definition 5.2.5. *(Egalitarian union value) The egalitarian union value is given by*

$$Eg^u_{L_k}(N, v, L) = Eg_{L_k}(M, v^L)$$

$$= \frac{v(N)}{m}$$
$$= \frac{1}{m} \cdot (\sum_{h=1}^{m} a_{L_h} \cdot \bar{d}_{L_h} + b_{L_h}). \tag{5.14}$$

with $v^L(U) = v(\cup_{h \in U} L_h)$ and $(N, v, L) \in \mathcal{GP}_{ed}$ for all $k \in M$, $U \subseteq M$.

So this union only considers coalitions of size M, namely the grand coalition. Further it uses information on the number of unions in the game. This union value is additive, as well as efficient. However, in contrast to the agent- and union-Shapley value and the separable cost remaining benefit union value and the proportional union value, the egalitarian union value does not take any marginal contributions or proportionality into account. The last union value is the non-cooperative union value, allocating each union its stand-alone cost.

Definition 5.2.6. *(Non-cooperative union value) The non-cooperative union value is given by*

$$NC^u_{L_k}(N, v, L) = NC_{L_k}(M, v^L)$$
$$= v(L_k)$$
$$= \sum_{h=1}^{k} a_{L_h} \cdot \hat{d}_{L_k} + b_{L_h} \tag{5.15}$$

for all $(N, v, L) \in \mathcal{GP}_{ed}$, all $k \in M$, with $v^L(U) = v(\cup_{h \in U} L_h)$ and $(N, v, L) \in \mathcal{GP}_{ed}$ for all $k \in M$, $U \subseteq M$

Clearly, this union value only takes into account coalitions of size 1. This non-cooperative union value is the only union value, discussed in this chapter, that is not efficient.

Example 5.2.3. *Consider again examples 5.1.2 and 5.2.1. Let us now compute the $SCRB^u(N, v, L)$, $Pr^u(N, v, L)$, $Eg^u(N, v, L)$ and $NC^u(N, v, L)$.*

$$SCRB^u_{L_1}(N, v, L) = 5 \cdot 15 + \frac{150}{150 + 150} \cdot 150$$
$$= 75 + \frac{150}{300} \cdot 150$$
$$= 150$$

$$SCRB^u_{L_2}(N, v, L) = (5 \cdot 20 + 4 \cdot 20) + 100 + \frac{150}{150 + 150} \cdot 150$$
$$= 180 + 100 + \frac{150}{300} \cdot 150$$
$$= 180 + 100 + 75$$

$$= 355$$

Thus, the SCRB-union value gives the allocation vector $SCRB^u(N, v, L) = (150, 355) \in \mathbb{R}^M_+$. *This union value is efficient, i.e.* $150 + 355 = 505 = v(N)$.

$$\begin{aligned} Pr^u_{L_1}(N, v, L) &= \frac{225}{655} \cdot 505 \\ &= 173 \end{aligned}$$

$$\begin{aligned} Pr^u_{L_2}(N, v, L) &= \frac{430}{655} \cdot 505 \\ &= 332 \end{aligned}$$

Thus, the Pr-union value gives the allocation vector $Pr^u(N, v, L) = (173, 332) \in \mathbb{R}^M_+$. *This union value is efficient, i.e.* $173 + 332 = 505 = v(N)$.

$$\begin{aligned} Eg^u_{L_1}(N, v, L) &= Eg^u_{L_2}(N, v, L) \\ &= \frac{505}{2} \\ &= 252.5 \end{aligned}$$

Thus, the Eg-union value gives the allocation vector $Eg^u(N, v, L) = (252.5, 252.5) \in \mathbb{R}^M_+$. *This union value is efficient, i.e.* $252.5 + 252.5 = 505 = v(N)$.

$$\begin{aligned} NC^u_{L_1}(N, v, L) &= 5 \cdot 15 + 150 \\ &= 225 \end{aligned}$$

$$\begin{aligned} NC^u_{L_2}(N, v, L) &= 5 \cdot 20 + 150 + 4 \cdot 20 + 100 \\ &= 430 \end{aligned}$$

Thus, the Eg-union value gives the allocation vector $Eg^u(N, v, L) = (225, 430) \in \mathbb{R}^M_+$. *This union value is not efficient, i.e.* $\sum_{k \in M} Eg^u_{L_k}(N, v, L) > v(N)$.

Note that the ordering of the partition does not matter for the union values, i.e. the union values do not take the ordering of the partition into account. However, the ordering is reflected in the way the game is defined. This is reflected by the summation charging a coalition up to the level of the greatest agent in the coalition. As we do not consider ordered union values, the properties defined for the union values defined in Chapter 3 are applicable for all the union values discussed in this chapter. If we want to consider ordered union values in the future we have to adapt these properties and the axiomatizations to take into account the ordering of the partition and thus the levels. As the values do not take the ordering into account, the agent-Shapley value seems a better choice for the electricity demand game than the union-Shapley value, because this value has no problem with merging any two voltage levels. In contrary to the union-Shapley value, for this value it is not defined what happens with the costs of the unions when two levels collude.

5.3 Simplified version of the game

In this section we briefly discuss a simplified version of the problem and corresponding game and show that the level paying rule for the problem equals the Shapley value for the game. This is comparable with the polluted river game under the LR principle and the corresponding Local Responsibility Sharing method, as defined in Ni and Wang (2007). The aim of this section is to show that other (easier) versions of the problem and game are also possible and might be useful in some situations. Also it highlights a resemblance with the polluted river game from Ni and Wang (2007). Let all elements of the electricity demand problem be as defined in Section 4.1, except for cost vector c. We redefine c in the following way and denote it by $\hat{c} \in \mathbb{R}^M_+$ such that we obtain

$$\hat{c}_{L_k} := a_{L_k} \cdot \hat{d}_{L_k} + b_{L_k},$$

with constants $a_{L_k}, b_{L_k} \in \mathbb{R}_+$ for $k \geq 1$ and 0 otherwise and $\hat{d}_{L_k} = \sum_{i \in L_k} d_i$. In this problem it is assumed that level costs are only incurred by the demands of the agents attached to that particular level. Thus there is no down- or up-streaming flow causing extra costs on higher or lower voltage levels. The corresponding game for $S \subseteq M$[4] for $M = \{1, ..., m\}$ and $L = \{L_1, ..., L_m\}$ is given by

$$v^*(S) := \sum_{k \in S} \hat{c}_{L_k}, \tag{5.16}$$

with $v^*(\emptyset) = 0$ and $v^*(S) = v(\cup_{h \in S} L_h)$. To avoid confusion we write $v(L_k)$ instead of $v(k)$ for $k \in S \subseteq M$. Consider the following proposition.

Proposition 5.3.1. *The level paying rule ($LP(P)$) for problem $P = (N, L, d, \hat{c})$ gives exactly the same allocation vector as the union-Shapley value of game v^*. Formally, $Sh^u_{L_k}(N, v^*, L) = LP_{L_k}(N, L, d, \hat{c}) = \hat{c}_{L_k}$, for all $(N, L, d, \hat{c}) \in \mathcal{C}_{ed}$, $(N, v^*, L) \in \mathcal{GP}_{ed}$ and $k \in M$.*

Proof. For $P = (N, L, d, \hat{c})$ the level paying rule is presented by

$$LP_{L_k}(P) = \hat{c}_{L_k}.$$

The game (N, v^*, L) is by definition (equation (5.16)) inessential. i.e. $v^*(S) = \sum_{k \in S} v^*(L_k)$ and $v^*(L_k) + v^*(L_h) = v^*(L_k \cup L_h)$. Thus the marginal contribution of each union joining a coalition S is exactly its union stand-alone cost in the game, so exactly $v^*(L_k)$ for

[4] Note that we take S to be a subset of M instead of N.

union L_k. As the union-Shapley value gives the average marginal contribution over all possible joining orders of a union, this value equals the stand-alone cost of the union in case of an inessential game (recall proposition 5.2.1). Also the level paying rule states that each union pays its own level cost, i.e. its union stand-alone cost. □

In summary, in this chapter we defined a cooperative cost game for the electricity demand problem: the electricity demand game. For this game we assumed that the cost of a coalition equalled the sum of the variable and fixed level costs up to the lowest (highest ranked) voltage level of the agents in the coalition. We may conclude that there are at least three important reasons why the Shapley value is an appealing solution for this game, namely

1. The Shapley value is an additive solution. Since the characteristic cost function consists of two cost parts, the Shapley value can be applied separately to both parts, which makes it an easier application. Also, if we want to extend the cost function in the future, the Shapley value only has to be found on the added part.
2. The simplified formula for the Shapley value makes the calculations considerably less demanding, and executable.
3. Since the electricity demand game is concave, the Shapley value of this game provides an allocation in the core, which ensures a form of stability.

So from all the union values discussed in this chapter, the agent- and union-Shapley value appeared the most useful solutions. However, we found that the agent-Shapley value might be the better solution of the two for this game, since it satisfies union collusion neutrality, in contrary to the union-Shapley value.

Chapter 6

Extensions of the electricity demand problem

In this chapter we present two extensions of the model of the electricity demand problem and game. The reason for these extensions is that we besides the current situation, also want to consider the possible future. We observe an important trend in the electricity market, namely the growth of decentralized sustainable production installations at every voltage level in the electricity grid. As a result of this growth, voltage levels and its consumers become less dependent on the higher voltage levels in the grid. With respect to this observation we extend the model of the electricity demand problem such that it also takes into consideration the production capacity of each voltage level. This is discussed in the first extension, namely the ***electricity demand-production problem***.

Even though the net electricity flow is downstream, there exists bilateral flow between the different voltage levels, i.e. electricity flows from downstream to upstream levels as well. In the electricity demand problem and game we considered, as is done in real-life, only the net-downstream electricity flow. The second extension in this chapter provides the opportunity to also incorporate the upstream electricity flow in a cost allocation rule and is therefore referred to as the ***bilateral flow problem***. For both of the extensions we only define the problem and its elements and consider the cascade rule. For the electricity demand-production problem we also define a corresponding game. We do not consider any properties or axiomatizations. These models are meant for future research and analyses. Further we propose for the second extension a new rule, based on the cascade rule, that compensates for upstream flow. We propose this rule as the first rule in this direction, as we found that there is need for such a rule in the future as pointed out in the literature, see for example NMA and SEO (2011, p.47), in which it is suggested to look for an alternative model to analyse the effect of the bilateral flow on the cost allocation over the levels. So although we consider production and bilateral flow in the extensions, for now we still assume that the net electricity flow is downstream. It would however be interesting to elaborate on the second extension to consider how to allocate the costs if this would not be the case.

6.1 Electricity demand-production problem

In this problem we also take into consideration that each voltage level has its own production capacity. Disregarding the electricity losses, we still assume that the production and demand capacities of the entire grid are equal. In the model of the electricity demand problem we described the downstream demand flow as the net demand of a downstream level. We assumed however that only electricity was fed into the grid at the highest level, such that the net demand of a lower level equalled the total demand of a lower level. In this section we assume each voltage level has its own production capacity, such that the net demand of a union/level is the total demand of the union minus its production capacity. So we assume that the production capacity of each level is completely used by that particular level and the rest of the required electricity is acquired from the highest voltage level.

We now formally define ***the electricity demand-production problem***, which is an extension of the electricity demand problem defined in Chapter 4. The electricity demand-production problem is defined by $P = (N, L, d, c, p)$, where N is a finite set of agents, L a partition of N, d the demand vector of the agents, c is the level cost vector and p is the production vector of the agents. Note that this entails that now the electricity producers also become agents in the problem. So the agent set now consists of agents that mainly produce, agents that mainly consume and agents that do both (e.g. households with solar panels). We denote the total production of union L_k by $\hat{p}_{L_k} = \sum_{i \in L_k} p_i$. We assume that $\hat{p}_{L_k} \leq \hat{d}_{L_k}$, for all $k > 1$. This assumption implies that most of the electricity is fed into the network at level one and the electricity flow goes downstream. By $\sum_{h \in M, h > k} \hat{p}_{L_h}$ we mean the sum of all the production capacities of the levels lower than level h. The level cost vector $c = (c_{L_k})_{k \in M}$ is presented by

$$\begin{aligned} c_{L_k} &:= a_{L_k} \cdot (\sum_{i \in N, l(i) \geq k} d_i - \sum_{i \in N, l(i) > k} p_i) + b_{L_k} \\ &:= a_{L_k} \cdot (\sum_{h \in M, h \geq k} \hat{d}_{L_h} - \sum_{h \in M, h > k} \hat{p}_{L_h}) + b_{L_k}, \end{aligned}$$

with constants $a_{L_k}, b_{L_k} \in \mathbb{R}_+$ for $k \geq 1$ and 0 otherwise. [1] Denote the class of electricity demand-production problems by $\mathcal{C}_{edp}$. So the level costs are dependent on the amount of electricity that flows through the network, which is equal to the aggregated demand of that level and lower levels minus the production of lower levels.

[1] Also for this problem we take c as given, but in order to define the game we explicitely define c.

Given the electricity demand-production problem amongst agent set N and partition L of N, we define the associated game by

$$v(S) := \sum_{k=1}^{l(\max S)} c_{L_k}$$
$$:= \sum_{k=1}^{l(\max S)} a_{L_k} \cdot \Big(\sum_{i \in S, l(i) \geq k} d_i - \sum_{i \in S, l(i) > k} p_i \Big) + b_{L_k},$$

for all $S \subseteq N$ and $v(\emptyset) = 0$.

Let us now define the cascade rule for the problem. We present the cascade rule in its recursive form as follows:

$$CA_{L_k}(P) = \frac{\hat{d}_{L_k}}{\Big(\sum_{l \in M, l \geq k} \hat{d}_{L_l} - \sum_{l \in M, l > k} \hat{p}_{L_l} \Big)} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P))$$
$$= \frac{\hat{d}_{L_k}}{\Big(\bar{d}_{L_k} - \sum_{l \in M, l > k} \hat{p}_{L_l} \Big)} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)),$$

for all $P \in \mathcal{C}_{edp}$, all $k \in M$ and $CA_{L_0}(P) = 0$. Note that in essence the cascade rule is still the same as defined in Chapter 4 with the demand of the level in the numerator and the total demand flowing through that level in the denominator. Let us now clarify by means of an example.

Example 6.1.1. *We define the elements of a problem* $P \in \mathcal{C}_{edp}$ *as follows:*

- $N = \{1, ..., 8\}$
- $L = \{L_1, L_2, L_3\}$ *with* $L_1 = \{1, 2\},\ L_2 = \{3, 4\},\ L_3 = \{5, 6, 7, 8\}$
- $d = (1, 1, 2, 3, 2, 4, 6, 10)$
- $c = (c_{L_1}, c_{L_2}, c_{L_3}) = (100, 80, 60)$
- $p = (10, 5, 2, 2, 2, 1, 3, 4)$

From these elements follows that for $\hat{p} = (15, 4, 10)$ *and* $\hat{d} = (2, 5, 22)$*. The situation is depicted in figure 6.1.*

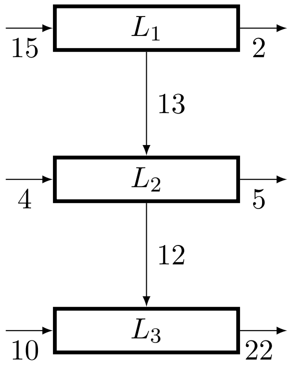

Figure 6.1: **Electricity network with production and demand capacities** Numerical example.

Let us now apply the cascade rule to problem $P \in \mathcal{C}_{edp}$.

$$\begin{aligned}
CA_{L_1}(P) &= \frac{\hat{d}_{L_1}}{(\sum\limits_{l \in M, l \geq k} \hat{d}_{L_l} - \sum\limits_{l \in M, l > k} \hat{p}_{L_l})} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)) \\
&= \frac{2}{2+5+22-4-10} \cdot c_{L_1} \\
&= \frac{2}{15} \cdot 100 \\
&= 13
\end{aligned}$$

$$\begin{aligned}
CA_{L_2}(P) &= \frac{\hat{d}_{L_2}}{(\sum\limits_{l \in M, l \geq k} \hat{d}_{L_l} - \sum\limits_{l \in M, l > k} \hat{p}_{L_l})} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)) \\
&= \frac{5}{5+22-10} \cdot (c_{L_1} + c_{L_2} - CA_{L_1}(P)) \\
&= \frac{5}{17} \cdot (100 + 80 - 13) \\
&= \frac{5}{17} \cdot (167) \\
&= 49
\end{aligned}$$

$$\begin{aligned}
CA_{L_3}(P) &= \frac{\hat{d}_{L_3}}{(\sum\limits_{l \in M, l \geq k} \hat{d}_{L_l} - \sum\limits_{l \in M, l > k} \hat{p}_{L_l})} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)) \\
&= \frac{22}{22} \cdot (c_{L_1} + c_{L_2} + c_{L_3} - CA_{L_1}(P) - CA_{L_2}(P))
\end{aligned}$$

$$= 1 \cdot (100 + 80 + 60 - 13 - 49)$$
$$= 178$$

Hence, $CA(P) = (13, 49, 178) \in \mathbb{R}_+^M$*. We have that* $\sum_{k \in M} CA_{L_k}(P) = 240$ *and thus is an efficient allocation.*

6.2 Bilateral flow problem

Let us now define the problem where we also model bilateral flow of electricity between the voltage levels. We refer to the problem as the ***bilateral flow problem***. For this problem we present one new rule, referred to as the ***downstream-cascading-upstream-discounting*** rule. For this rule we first consider the standard cascade rule, followed by the downstream-cascading-upstream-discounting rule. The reason for this extension is that the currently employed cascade rule only considers the net electricity downstream flow and ignores the upstream electricity flow. Let the elements N, L be as defined for the electricity demand problem in Chapter 4. We now assume that for each voltage level there are incoming and outgoing electricity flows. The incoming flows are caused by the production of that level and by incoming flows from lower and higher voltage levels. The outgoing flows are caused by the demand of that level and by outgoing flows to lower and higher voltage levels. So also in this extension we assume that each voltage level has its own production capacity. However, due to irregular demand patterns of the different voltage levels, it could be that even though a voltage level has enough demand capacity to foresee its own level, it still receives electricity from lower and higher voltage levels. Because maybe at peak time the production capacity is abundant, whereas in low time it is not, causing flow in both directions.

The bilateral flow problem is defined by

$$P = (N, L, f_{L_k}^{in(p)}, f_{L_k}^{in(L_{k-1})}, f_{L_i}^{in(L_{k+1})}, f_{L_k}^{out(d)}, f_{L_k}^{out(L_{k-1})}, f_{L_k}^{out(L_{k+1})}, c),$$

where N is a finite set of agents and L is a partition of N, similar as for the electricity demand problem. Let us now consider the other elements of the problem. For each voltage level the total incoming electricity flow is given by

$$F_{L_k}^{in} := f_{L_k}^{in(p)} + f_{L_k}^{in(L_{k-1})} + f_{L_k}^{in(L_{k+1})},$$

where

$f_{L_k}^{in(p)}$ denotes the electricity inflow coming from production of level L_k

$f_{L_k}^{in(L_{k-1})}$ denotes the electricity inflow coming from level L_{k-1}

$f_{L_k}^{in(L_{k+1})}$ denotes the electricity inflow coming from level L_{k+1}.

For each voltage level the total outgoing electricity flow is given by

$$F^{out}_{L_k} := f^{out(d)}_{L_k} + f^{out(L_{k-1})}_{L_k} + f^{out(L_{k+1})}_{L_k},$$

where

$f^{out(d)}_{L_k}$ denotes the electricity outflow going to the demand of level L_k

$f^{out(L_{k-1})}_{L_k}$ denotes the electricity outflow going to level L_{k-1}

$f^{out(L_{k+1})}_{L_k}$ denotes the electricity outflow going to L_{k+1}.

Note that $f^{in(L_k)}_{L_k+1} = f^{out(L_{k+1})}_{L_k}$. So every incoming flow from another level is an outgoing flow for that other level and the other way around. If a flow does not exist it equals zero. We assume that no electricity losses take place and therefore for every voltage level holds that

$$F^{in}_{L_k} = F^{out}_{L_k}.$$

The level cost vector $c = (c_{L_k})_{k \in M}$, with c_{L_k} representing the cost of voltage level k is defined by

$$c_{L_k} := a_{L_k} \cdot F^{in}_{L_k} + b_{L_k},$$

with constants $a_{L_k}, b_{L_k} \in \mathbb{R}_+$ for $k \geq 1$ and 0 otherwise and $c \in \mathbb{R}^M_+$.[2] Note that $F^{in}_{L_k}$ can be replaced by $F^{out}_{L_k}$. Let $\mathcal{C}_{bf}$ denote the class of bilateral flow problems. Let us further assume that the downstream flow between two levels is greater than the upstream flow, i.e. for all $k > 1$ it holds that $f^{in(p)}_{L_k} \leq f^{out(d)}_{L_k}$

We first consider the cascade rule for this problem. Also for this problem we employ a recursive version of the cascade rule. As in the real-life employed cascade rule, the demand of the level is considered and the net demand of the levels below. Thus, level one has to pay

$$\frac{f^{out(d)}_{L_1}}{f^{out(d)}_{L_1} + (f^{out(L_2)}_{L_1} - f^{in(L_2)}_{L_1})} \cdot c_{L_1}$$

and

$$\frac{(f^{out(L_2)}_{L_1} - f^{in(L_2)}_{L_1})}{f^{out(d)}_{L_1} + (f^{out(L_2)}_{L_1} - f^{in(L_2)}_{L_1})} \cdot c_{L_1}$$

is cascaded to level two and so on, where $(f^{out(L_2)}_{L_1} - f^{in(L_2)}_{L_1})$ represents the net downstream electricity flow from level L_1 to L_2. More generally, the cascade rule is given by the following formula:

$$CA_{L_k}(P) = \frac{f^{out(d)}_{L_k}}{f^{out(d)}_{L_k} + (f^{out(L_{k+1})}_{L_k} - f^{in(L_{k+1})}_{L_k})} \cdot \sum_{h=1}^{k} (c_{L_h} - CA_{L_{h-1}}(P)),$$

[2]Also for this problem we take c as given, but in order to define the game we explicitely define c.

for all $P \in \mathcal{C}_{bf}$, all $k \in M$ and $CA_{L_0}(P) = 0$. Also for a bilateral flow problem the cascade rule uses demand proportionality where the numerator defines the demand of the level (outgoing flow to demand) and the denominator the net downstream electricity flow.[3] So the cascade rule here does not take the upstream flow into account yet, solely to compute the net electricity downstream flow. Consider the following numerical example for a computation of the cascade rule.

Example 6.2.1. *Consider the situation depicted in figure 6.2. Let* $c = (c_{L_1}, c_{L_2}, c_{L_3}) = (100, 80, 60)$.

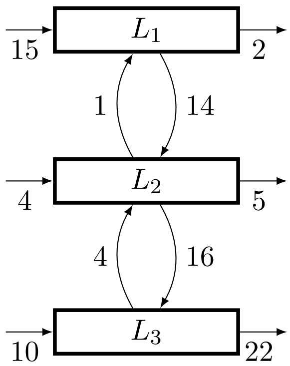

Figure 6.2: **Electricity network with bilateral flow** Numerical example.

We obtain the following allocation by application of the cascade rule,

$$\begin{aligned} CA_{L_1}(P) &= \frac{f_{L_1}^{out(d)}}{f_{L_1}^{out(d)} + (f_{L_1}^{out(L_2)} - f_{L_1}^{in(L_2)})} \cdot c_{L_1} \\ &= \frac{2}{2+14-1} \cdot 100 \\ &= \frac{2}{15} \cdot 100 \\ &= 13 \end{aligned}$$

$$\begin{aligned} CA_{L_2}(P) &= \frac{f_{L_2}^{out(d)}}{f_{L_2}^{out(d)} + (f_{L_2}^{out(L_3)} - f_{L_2}^{in(L_3)})} \cdot (c_{L_1} - CA_{L_1}(P) + c_{L_2}) \\ &= \frac{5}{5+16-4} \cdot (100 - 13 + 80) \\ &= \frac{5}{17} \cdot 167 \end{aligned}$$

[3] So the net downstream electricity flow of a level is defined by the outgoing flow to the demand of that level plus the outgoing flow to the adjacent lower level minus the incoming flow from the adjacent lower level.

$$= 49$$

$$\begin{aligned}
CA_{L_3}(P) &= \frac{f_{L_3}^{out(d)}}{f_{L_3}^{out(d)}} \cdot (c_{L_1} - CA_{L_1}(P) + c_{L_2} - CA_{L_2}(P) + c_{L_3}) \\
&= \frac{22}{22} \cdot (100 - 13 + 80 - 49 + 60) \\
&= 178
\end{aligned}$$

Hence, $CA(P) = (13, 49, 178) \in \mathbb{R}_+^M$. Note that this is the same allocation vector as obtained in example 6.1.1.

This example gives solely the cascade allocation vector. We introduce a new rule, namely the ***downstream-cascading-upstream-discounting rule***. This rule has to be executed in two steps. In the first steps an allocation is computed by the cascade rule as defined above, so first for union L_1, then for union L_2 until union L_m. Once the complete allocation vector $CA(P)$ is known, the second step follows, namely ***upstream discounting***. In this step the levels are discounted for the electricity that flows upstream. Opposite of the downward cascading, for the discounting of the cost we start with the most downstream level and proceed upstream. There are several options for the discounting part. Let m be the maximum level of M, then $x_{L_m} = CA_{L_m}(P) - \alpha_{L_m}$ with x_{L_m} the cost share for union L_m and $\alpha_{L_m} \in \mathbb{R}_+$ the discount factor for level L_m. This α_{L_m} is then charged extra to level L_{m-1}. So in the end level L_k obtaines cost allocation $x_{L_k} = CA_{L_k}(P) + \alpha_{L_{k+1}} - \alpha_{L_k}$. We consider two options for the discount factor. Let the discount factor α_{L_k} be

1. a discount factor with the upflow proportional to the production capacity:
$$\alpha_{L_k}^1 = \frac{f_{L_k}^{out(L_{k-1})}}{f_{L_k}^{in(p)}} \cdot c_{L_k}$$
2. a discount factor with the upflow proportional to the total flow through that level:
$$\alpha_{L_k}^2 = \frac{f_{L_k}^{out(L_{k-1})}}{F_{L_k}^{in}} \cdot c_{L_k}.$$

Let $\alpha_{L_0} = 0$. The rule can be defined more formally as follows.

Definition 6.2.1. *(Downward-cascading-upstream-discounting rule (CA-UD(P))) The downward-cascading-upstream-discounting rule is given by*

$$CA - UD(P)_{L_k} = CA_{L_k}(P) + \alpha_{L_{k+1}} - \alpha_{L_k}, \tag{6.1}$$

for all $P \in \mathcal{C}_{bf}$, all $k \in M$ and for α_{L_k} and $CA_{L_k}(P)$ as defined above.

Note that we propose only two options for the discount factor, but of course these two are not exhaustive. In the example below it is shown that also this rule is efficient.

Example 6.2.2. *Consider one more time the situation depicted in figure 6.2. Let* $c = (c_{L_1}, c_{L_2}, c_{L_3}) = (100, 80, 60)$*. We obtain the following allocations applying rule* $CA - UD(P)$ *with* α^1*:*

$$\begin{aligned} CA - UD^1_{L_3}(P) &= CA_{L_3} - \frac{f^{out(L_2)}_{L_3}}{f^{in(p)}_{L_3}} \cdot c_{L_3} \\ &= 178 - \frac{4}{10} \cdot 60 \\ &= 154 \end{aligned}$$

$$\begin{aligned} CA - UD^1_{L_2}(P) &= CA_{L_2} + \frac{f^{out(L_2)}_{L_3}}{f^{in(p)}_{L_3}} \cdot c_{L_3} - \frac{f^{out(L_1)}_{L_2}}{f^{in(p)}_{L_2}} \cdot c_{L_2} \\ &= 49 + \frac{4}{10} \cdot 60 - \frac{1}{4} \cdot 80 \\ &= 53 \end{aligned}$$

$$\begin{aligned} CA - UD^1_{L_1}(P) &= CA_{L_1} + \frac{f^{out(L_1)}_{L_2}}{f^{in(p)}_{L_2}} \cdot c_{L_2} \\ &= 13 + \frac{1}{4} \cdot 80 \\ &= 33 \end{aligned}$$

The resulting allocation vector is $(33, 53, 154) \in \mathbb{R}^3_+$ *and* $\sum_{k \in M} CA - UD^1_{L_k}(P) = 240$*, which is efficient. We obtain the following allocation applying rule* $CA - UD(P)$ *with* α^2*:*

$$\begin{aligned} CA - UD^2_{L_3}(P) &= CA_{L_3} - \frac{f^{out(L_2)}_{L_3}}{F^{in}_{L_3}} \cdot c_{L_3} \\ &= 178 - \frac{4}{26} \cdot 60 \\ &= 169 \end{aligned}$$

$$\begin{aligned} CA - UD^2_{L_2}(P) &= CA_{L_2} + \frac{f^{out(L_2)}_{L_3}}{F^{in}_{L_3}} \cdot c_{L_3} - \frac{f^{out(L_1)}_{L_2}}{F_{L_2}} \cdot c_{L_2} \\ &= 49 + \frac{4}{26} \cdot 60 - \frac{1}{22} \cdot 80 \\ &= 55 \end{aligned}$$

$$
\begin{aligned}
CA - UD^2_{L_1}(P) &= CA_{L_1} + \frac{f^{out(L_1)}_{L_2}}{F^{in}_{L_2}} \cdot c_{L_2} \\
&= 13 + \frac{1}{22} \cdot 80 \\
&= 16
\end{aligned}
$$

Application of this rule gives the vector $(16, 55, 169) \in \mathbb{R}^3_+$*, which is also efficient.*

In summary, in this chapter two possible extensions of the model of the electricity demand problem and game are proposed. The first extension takes production capacities of the voltage levels into account and the second extension takes production capacities as well as bilateral flow between voltage levels into account. These extensions make the models more complete than the electricity demand model and make it possible to research new solution concepts, which might be useful for the future. Moreover, the cascade rule could still be defined for both models. On the other hand, do these extra features complicate the models.

Chapter 7

Conclusion & discussion

In this chapter we start with a synopsis of the thesis, summarizing the key findings and major points of every chapter. Thereafter we discuss some possible directions for future research. During the writing of this thesis, many other models and frameworks were considered. Some appeared more suitable than others for our problem for a variety of reasons and objectives. Therefore, lastly we provide some frameworks and models that were considered and might be useful for future research.

7.1 Synopsis

In Chapter 1 we introduced the motivation and the contribution of this thesis and provided some background information on the subject. In a research project in collaboration with TNO we found that small-scale consumers are charged significantly more for their electricity consumption than large-scale consumers. One important reason for this is that grid costs are cascaded downwards to downstream voltage levels, as most of the electricity flows from up- to downstream levels.

In the preliminaries in Chapter 2 we presented an introduction into the electricity sector, in particular into the realization of the transmission tariffs and the allocation of the transmission costs. We further provided some basic notions from cooperative game theory and cost allocation. Subsequently in Chapter 3 we considered solutions concepts and related properties for cost allocation problems, TU games and TU games with coalition structure.

The main problem is defined in Chapter 4, namely the electricity demand problem. It defines the problem of allocating the total cost of the electricity grid in a fair way over the voltage levels and the associated unions of agents. For this problem we defined the level paying rule, the equal downstream rule and the cascade rule. All rules allocate a cost share to unions of agents and not to individual agents. The latter rule is currently employed and charges every union (or voltage level) proportional to its demand for the cost of its own level and all upstream levels. This rule takes into account the demands of the unions, the levels of the unions and costs of the levels. The equal downstream rule does also take the responsibility of downstream unions for their upstream incurred costs into account, but does not consider the demands of the unions, only the number of unions. Finally, the level paying rule solely takes the level of a union into consideration and charges each union its level cost. All three rules were formalised by means of axiomatic characterizations. In the context of the electricity demand problem and the rules discussed in this chapter, the cascade rule can be considered as the most fair rule.

In Chapter 5 we presented a cooperative cost game associated with the electricity demand game. The characteristic function was analysed and split in additive parts. As the Shapley value is an additive solution with nice properties it appeared a good fit for this game, also due to the simplified expression. Moreover, due to concavity of the characteristic function, the Shapley value was in the core and thereby provided a stable allocation. As we were interested in cost shares for unions, we discussed a number of union values. Most importantly the agent- and union-Shapley value and furthermore the separable cost remaining benefit, the proportional, the egalitarian and the non-cooperative union value. From these rules, the agent-Shapley value and the union-Shapley value can be considered the best solutions for this game, in particular the agent-Shapley value.

The electricity demand problem and game are a starting point in this research and therefore in Chapter 6 some possible extensions of the model were proposed and briefly discussed. These extensions incorporated production capacities of the voltage levels and also bilateral flow of electricity between the voltage levels.

7.2 Future research

For future research we propose various directions.

- **Extensions of the model:** The model defined in this thesis is a starting point for research on this topic. The model incorporates the standard, most important, features of the transmission cost allocation in the electricity grid and allows for applying appropriate cost allocation rules. By means of our model we are able to analyse the currently used cascade rule, but we are not yet able to analyse new rules that incorporate the bilateral flow between voltage levels. Due to the changes on the production side of electricity, it would be interesting to have a more inclusive model, that can also be used in view of the changes in the future. In Chapter 6 we already provided some possible extensions of the model, there are however many more extensions possible. We could for example add producers as agents. We can also endow the agents with more information, besides the demand and the voltage level, such as the peak and low demand, the demand curve, the location and so on. This extra information can provide more individual-oriented solution concepts.

- **Cost allocation at the level of individual agents:** In our model we consider solution concepts that allocate cost shares to unions of agents. The reason for this is that we focussed solely on one step in the currently employed cost allocation method (see Chapter 1). For future research it would be very interesting to consider solution concepts that allocate cost shares to individual agents. In this direction two approaches can be considered. First, a game or problem can be defined for each voltage level, such that the cost shares that are allocated to these levels can be allocated amongst the agents attached to a voltage level by means of other solution concepts. A second approach would be to redefine the entire model and analyse solution concepts for individual agents directly on the problem or game, without first allocating costs to unions of agents, e.g. Ramsey pricing. As there has to be a distinction between agents based on some features, these solution concepts could become quite complex as it would be desirable that this single rule satisfies many different properties. In the first approach (the currently employed approach) there is already a distinction between agents, namely the voltage level they are attached to, such that the next step would be to determine what properties solution concepts for each voltage level should obey. These solution concepts can differ per level and be based on the main cost drivers of that voltage level. Another possibility is to use the first approach, but redefine the partition of the agents, for example based on location instead of voltage level. Now the total grid costs can first be allocated to these new unions and thereafter be allocated amongst the agents within the unions.

- **Cooperation between producers and consumer:** In Chapter 4 and 5 we defined the problem and the cooperative game on the set of agents, representing the consumers. In the future also a problem and especially a game could be defined on the set of electricity producers, or on both. As an increasing number of consumers also become producers, referred to as prosumers, more electricity flow on a local scale occurs. In view of this it would be interesting to incorporate producers and consumers to see if a more efficient cooperation can evolve, where consumers and producers on a more local scale work together. When only considering producers, it can be analysed how to obtain an efficient collaboration between producers, that minimizes the costs.

- **Introduction of producers tariff:** By means of an extension of this model we could analyse the introduction of a producers tariff. Currently the producers tariff is set to zero and hence producers do not pay for the grid costs (Autoriteit Consument en Markt, 2013). We could analyse the effect on the costs for the unions and if we would use the model proposed in the previous item, including producers and consumers, we could also consider the effect on the cooperation between these two groups.

- **Ordered union values:** In Chapter 5 we considered union values for the electricity demand game. These union values however do not incorporate the order of the partition. The ordering of the partition is implicitly modelled in the game, but not in the unions value structure or corresponding properties. For future research it would be interesting to elaborate on this chapter and define ordered union values.

- **Other characteristic functions:** Linking up to the above mentioned item, in Chapter 5 the game could be defined differently. A characteristic function is not uniquely defined for a problem and is dependent on the interpretation of the costs for coalitions. In our game we assumed that the electricity grid is there and each agent or union is charged for all the upstream levels it uses, even if it is the only agent or union using the grid (stand-alone costs). Most likely however, the stand-alone costs for unions or agents are in real-life lower than defined by our game. For example in the situation where the electricity does not have to come from the most upstream level, but is produced by production facilities on the same level. Hence, one possibility is to redefine the game such that for example the stand-alone costs for a union are the cost of its own level plus some extra amount to foresee in the demand of that level, but without making use of other levels. An example of a simplified version of the game is presented in Chapter 5, for which the problem as well as the game were redefined.

- **Cascade rule applications:** A final suggestion for future work is to consider other applications of the cascade rule. The rule has potential to be adapted for and applied to other problems. It could for example be used in the polluted river game of Ni and Wang (2007), when the demands are replaced by volumes of pollution.

7.3 Relevant models and frameworks

In this chapter we dicuss some frameworks and models that were considered during the writing of this thesis.

- **Games with restrictions on possible coalitions:** These types of games define restrictions on the set of feasible coalitions. The idea to restrict coalitions by imposing a partial order on the agent set was first proposed by Faigle and Kern (1992). Games with permission structure, introduced by Gilles et al. (1992), impose some hierarchical order on the agents, such that there are agents who need permission from certain other agents to enter a coalition. So the position of the agent in the hierarchy determines its opportunities of joining a coalition. Related to these games are games on a graph with a communication structure. Agents in this context can only form a coalition when they are connected in the subgraph, i.e. connected subgraphs define the feasible coalitions. Thus the position of an agent in the graph determines its possibilities of joining a coalition. Communication structures were introduced by Myerson (1977) and are, amongst others, formally described in Gilles (2010). As there exists a natural hierarchy in our model, we considered these structures. For now we imposed no constraints on the feasible coalitions, as in theory all voltage levels could decide to cooperate. It would however be possible to assume that every voltage level needs at least enough production capacity and therefore is restricted in whom it may form a coalition with, since most levels need to cooperate with the highest voltage level.

- **Hub and spoke networks:** Hub and spoke networks are commonly used to describe transportation, mail delivery and telecommunication networks. A hub and spoke network consists of hub nodes that are fully interconnected and spoke nodes, that are solely connected to hub nodes. Flows from one spoke node to another solely proceed via hub nodes. By means of these hub nodes, less links are required and flows are concentrated resulting in the exploitation of economies of scale. Cooperation between the agents is required to gain the advantages of the network. The problem associated with these sort of networks concern assigning hub nodes and allocating spoke nodes to hub nodes such that an optimal flow between source and sink nodes is obtained. In the context of our problem we could interpret the voltage levels as hub nodes and the agents connected to it as spoke nodes. We could use the rules for cost allocation with respect to the use of the number of hub nodes for a flow as an inspiration for the allocation of the cost over the voltage levels, with respect to the use of the number of voltage levels. However, literature on these networks mainly focusses on optimization of the flow between the hubs and spokes to minimize costs, whereas our problem is not an optimization, but solely an allocation problem. It would however be interesting to describe the electricity network making use of the hub and spoke framework to consider flow optimization and cost allocation. One of the difficulties is however that electricity flow can not be directed. In Skorin-Kapov (1998), amongst others, hub and spoke networks are considered.

- **Network flow problems:** Network flow problems concern the flow of a product from a source to a sink, with possible restrictions along the way, such as capacitated links. Many production allocation problems are defined by means of such networks. Also in the electricity grid electricity is transmissioned over capacitated links from sources to sinks. The main objective however in these problems is to optimize the route of the flow, such that costs are minimized. In the electricity network it is hard to trace electricity flows as they do not take a fixed route, which makes this type of problems different.

- **Games with externalities (partition function form games):** In original cooperative game theory it is assumed that the worth of a coalition only depends on the members of the coalition. However, in games with externalities it is assumed that externalities (positive or negative) can affect the worth of a coalition by creating a dependency between agents in and outside the coalition. For example if there is a limited resource, which is consumed by one coalition, then it is not available to another coalition. Games with externalities can be modelled by means of games in partition function form. A partition function is a characteristic function that assigns a worth to a pair, consisting of a coalition and the corresponding coalition structure containing that coalition. In our situation, all agents are in need of electricity, regardless of whether they are in the coalition or not and hence, the cost for a coalition may be defined such that it depends on coalition formation of the agents outside the coalition. For example, if upstream levels cooperate, it might decrease the total costs such that downstream levels experience positive externalities.

- **Bankruptcy problem:** A bankruptcy problem can be used to allocate the costs of the electricity grid from another perspective. Namely, what should agents pay if we know their maximum willingness to pay to stay connected to the grid. In order to quantify agents' willingness to pay we instead state that each agent has an alternative option for electricity, which has a certain cost attached to it. For example, if an agent could be completely self-sufficient by means of solar panels for €100 per month, this could be considered as the agent's maximum willingness to pay. This problem can be modelled by means of a ***bankruptcy problem***. In a bankruptcy problem agents have a claim for a divisible resource, often referred to as the estate, that has to be allocated amongst the agents. There is however not enough of the resource to provide for all claims. The problem is how to allocate the resource amongst the agents. More formally, a bankruptcy problem is a triple (N, c, E), where $N = \{1, ..., n\}$ is the set of agents, $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ is the vector of claims and $E \in \mathbb{R}_+$ is the estate such that $0 \leq E \leq c_1 + ... + c_n$ (Aumann and Maschler, 1985). The bankruptcy problem can be adapted as follows: each agent has a demand and a claim, namely the cost of its alternative electricity source. The estate in this situation is total grid cost. In this way, the bankruptcy problem can be defined by the quadruple (N, d, A, C), where N is the set of agents, $d = (d_i)_{i \in N}$ is the vector of demands of the agents in N and $A = (A_i)_{i \in N} \in \mathbb{R}^N_+$ is the vector of costs of the alternatives for each agent $i \in N$ and $C \in \mathbb{R}_+$ is the total cost of the grid that has to be allocated, such that $0 \leq C \leq A_1 + ... + A_n$. For each voltage level there exist different alternatives. At the higher voltage levels, agents can construct a direct line[1], or agents could become involved in wind energy or engage in a closed distribution system.[2] To approach the problem in this way the alternatives for all agents should be quantified. Based on the cost of its alternative option and the demand of an agent, the aim is to find an allocation of the total cost of the electricity grid, such that no agent pays more than its alternative option. Solutions for the bankruptcy problem can be adapted and applied to this problem.

[1]A direct line is a direct connection with a producers without the need of intermediaries.

[2]A closed distribution system is a closed grid which should be within a geographically defined area, with a maximum of 500 non-households consumers.

Additions Chapter 4

Illustrative example

This example represents an illustration of the Dutch electricity grid. The numbers for the costs and the number of consumers are estimates based on the total revenues of the regional network operator per voltage level and the number of connections, respectively. For more information we refer to ACM (2014). The demands are estimates based on the total electricity consumption in the Netherlands and the findings in figure 1.2, p. 4. Note that the Extra-High Voltage is not taken into account, since we only focus on the revenues of the regional network operators and not the national network operator (TenneT).[3] The cascade method is in real-life applied separately for each of the network operators. However, we consider all regional network operators together. Thus, this illustration is not an accurate representation, but does give an indication of the range of the numbers. Moreover, the cost allocation vector obtained by the cascade rule is the original total starting revenue in 2013 for all regional network providers per voltage level. It should also be noted that in real-life not for all voltage levels the same electricity units are used to cascade the costs, but for now we use consumption (kWh) as the single electricity unit for all the levels.[4] For the original employed electricity units per level we refer to Autoriteit Consument en Markt (2013). Lastly, as we do not know the consumption (demand in kWh) per consumer, i.e. vector d, we give instead solely the level demand vector $\hat{d}$. Consider the following elements of the problem $P \in \mathcal{C}_{ed}$:

- $N = \{1, 2, ...; 10, 565, 183\}$
- $L = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ with $|L_1| = 18, |L_2| = 114, |L_3| = 72, |L_4| = 299, |L_5| = 23, 727, |L_6| = 10.5mn$
- $\hat{d}$ =(29bn, 4bn, 7bn, 18bn, 4bn, 20bn)[5]
- $\bar{c}$ =(44mn, 503mn, 395mn, 469mn, 648mn, 359mn)

[3]Recall that Tennet possesses the EHS and some of the HS networks.
[4]For some levels the capacity (kW) is used instead of the consumption.
[5]The abbreviation mn denotes million (10^6) and the abbreviation bn billlion (10^9).

The figure below gives a representation of the Dutch electricity grid and the corresponding level demand vector.

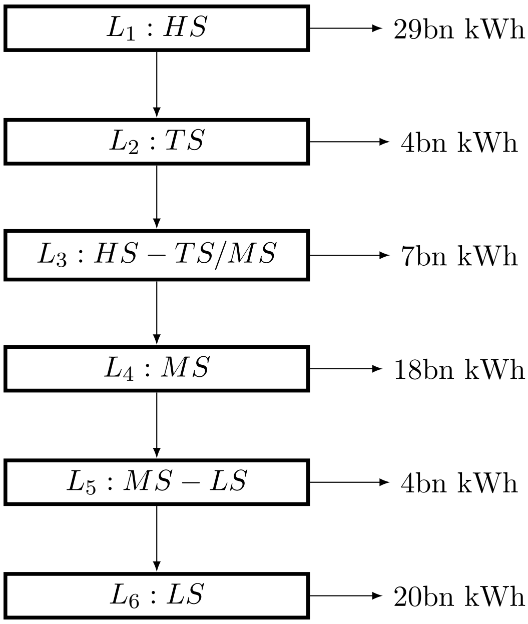

Figure 1: **The Dutch electricity grid** An illustrative example of the current cost allocation over the voltage levels in the Dutch electricity grid.

Applying the cascade rule for problem $P \in \mathcal{C}_{ed}$ gives the following allocation vector:

- $CA_{L_1}(P) =$ €15mn
- $CA_{L_2}(P) =$ €39mn
- $CA_{L_3}(P) =$ €125mn
- $CA_{L_4}(P) =$ €523mn
- $CA_{L_5}(P) =$ €224mn
- $CA_{L_6}(P) =$ €1,474mn

From this example follows that the consumers attached to the lowest voltage level LS, consume 20bn kWh, which equals 24% of the total consumption and the cost share they are charged is €1.5bn, which amounts to 61% of the total grid costs.

Proof theorem 4.3.1

Proof. (i) Let $l(\max N) = m$. We prove that the following properties are satisfied by the level paying rule for $P \in \mathcal{C}_{ed}$:

EF Proved in theorem 4.4.1.

FE Follows from Efficiency (EF).

NCP Proved in theorem 4.4.1.

IDC Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for any $k \in M$ and $h < k$ holds that $c^1_{L_h} = c^2_{L_h}$. We want to show that for all $l < k$ we have $LP_{L_l}(P^1) = LP_{L_l}(P^2)$. For all $l < k$ we have

$$\begin{aligned} LP_{L_l}(P^1) &= c^1_{L_l} \\ &= c^2_{L_l} && (l < k) \\ &= LP_{L_l}(P^2). \end{aligned}$$

NUC Assume that $c_{L_h} = 0$ for all $h \leq k$. We want to show that $LP_{L_k}(P) = 0$:

$$\begin{aligned} LP_{L_k}(P) &= c_{L_k} \\ &= 0 && (h \leq k) \end{aligned}$$

NUCO Assume that $c_{L_h} = 0$ for all $h < l$. Now for all $k \geq l \in M$ we have:

$$\begin{aligned} LP_{L_k}(P) &= c_{L_k} \\ &= LP_{L_k}(N \setminus \cup_{h<k} L_h, L \setminus \{L_h\}_{h<k}, (d_i)_{i \in N \setminus \cup_{h<k} L_h}, (c_{L_f})_{L_f \in L \setminus \{L_h\}_{h<k}}). \end{aligned}$$

As LP does not depend on any other level, this property is also satisfied.

C-MON Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for all $h \in M$ holds that $c^1_{L_h} \leq c^2_{L_h}$. We want to show that $LP(P^1) \leq LP(P^2)$:

$$\begin{aligned} LP_{L_k}(P^1) &= c^1_{L_k} \\ &\leq c^2_{L_k} \\ &= LP_{L_k}(P^2), \end{aligned}$$

for all $k \in M$.

ACN Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N^{ij}, L^{ij}, d^{ij}, c)$ such that agents $i, j \in L_k$ colluded. As $LP_{L_k}(P) = c_{L_k}$, this collusion clearly has no effect on the cost share of union L_k.

DUCN-I Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_k L_l}, d, c^{L_k L_l})$ a problem where unions L_k and L_l colluded. Then for all $h \in M$ such that $h < k, l$, we want to show that $LP_{L_h}(P^1) = LP_{L_h}(P^2)$:

$$\begin{aligned} LP_{L_h}(P^1) &= c_{L_h} \\ &= c^{L_k L_l}_{L_h} && (h \in M \setminus \{k, l\}) \\ &= LP_{L_h}(P^2). \end{aligned}$$

Note that this collusion has no effect on upstream level costs and therefore the cost shares of voltage levels higher than the colluded unions are not affected.

UCN-I Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_k L_l}, d, c^{L_k L_l})$ a problem where unions L_k and L_l colluded. Then we want to show that $LP_{L_k}(P^1) + LP_{L_l}(P^1) = LP_{L_k}(P^2)$.

$$\begin{aligned} LP_{L_k}(P^2) &= c^{L_k L_l}_{L_k} \\ &= c_{L_k} + c_{L_l} \\ &= LP_{L_k}(P^1) + LP_{L_l}(P^1) \end{aligned}$$

DUCN-II Follows from union collusion neutrality-II (UCN-II)

UCN-II Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N, L^{L_l \dots L_m}, d, c^{L_l \dots L_m})$ with $m = l(\max N)$ such that unions L_l to L_m colluded. We want to show that for all $k \in M$ we have

$$LP_{L_k}(P^2) = \begin{cases} LP_{L_k}(P^1) & \textit{if } k < l \\ \sum_{h=k}^{m} LP_{L_h}(P^1) & \textit{otherwise.} \end{cases}$$

If $k < l$, then

$$\begin{aligned} LP_{L_k}(P^2) &= c^{L_l \dots L_m}_{L_k} \\ &= c_{L_k} && (k \in M \setminus \{l, .., m\}) \\ &= LP_{L_k}(P^1) \end{aligned}$$

If $k \geq l$, then

$$\begin{aligned} LP_{L_k}(P^2) &= c^{L_l \dots L_m}_{L_k} \\ &= c_{L_k} + \dots + c_{L_m} \\ &= \sum_{h=k}^{m} LP_{L_h}(P^1). \end{aligned}$$

C-ADD Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, c^1 + c^2)$. For all $k \in M$ we want to show that $LP_{L_k}(P^1) + LP_{L_k}(P^2) = LP_{L_k}(P^3)$:

$$\begin{aligned} LP_{L_k}(P^1) + LP_{L_k}(P^2) &= c^1_{L_k} + c^2_{L_k} \\ &= LP_{L_k}(P^3) \end{aligned}$$

C-LIN Proved in theorem 4.4.1.

(ii) Let $l(\max N) = m$. We show that the following properties are not satisfied by the level paying rule for some $P \in \mathcal{C}_{ed}$ by means of simple counterexamples:

NDP Consider the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (0, 20)$ and $c = (20, 30)$. We have that $\hat{d}_{L_1} = 0$. However, $LP_{L_1}(P) = c_{L_1} = 20 \neq 0$.

RAN Consider the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (10, 20)$ and $c = (64, 0)$. We have that $\hat{d}_{L_1} = 10 < 20 = \hat{d}_{L_2}$. However, $LP_{L_1}(P) = 64 > 0 = LP_{L_2}(P)$.

ET Consider again $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied level paying rule to P in example 4.2.1. We have that $\hat{d}_{L_2} = 30 = \hat{d}_{L_3}$. However, $LP_{L_2}(P) = 125 > 62 = LP_{L_3}(P)$.

RED Similar proof as for equal treatment (ET).

Since the demands do not matter for the equal downstream rule, all rules above are not satisfied.

D-SYM Consider the following : let $P^1 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (10, 10)$ and $c^1 = (64, 0)$. For levels $1, 2 \geq 1$ with $d_1 = d_2 = 10$ we have

$$\begin{aligned} LP_{L_1}(P^1) &= 64 \\ &\neq 0 \\ &= LP_{L_2}(P^1). \end{aligned}$$

DR Consider the following example: let $P^1 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c^1 = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$\begin{aligned} LP_{L_1}(P^1) &= 64 \\ &\neq 0 \\ &= LP_{L_2}(P^1) \end{aligned}$$

DDPR Consider again the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$\begin{aligned} LP_{L_1}(P) &= 64 \\ &\neq \frac{20}{30} \cdot 64 \end{aligned}$$

AN Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied level paying rule to P in example 4.2.1. Let π be a permutation on M such that $P^\pi = (N, \pi L, \pi d, c)$ with $\pi L = \{L_2, L_1, L_3\}$ and $\pi d = (d_3, d_4, d_1, d_2, d_5, d_6, d_7, d_8)$, so $\pi\hat{d} = (\hat{d}_{L_2}, \hat{d}_{L_1}, \hat{d}_{L_2})$. It follows,

$$\begin{aligned} LP_{\pi(L_2)}(P^\pi) &= 230 \\ &\neq 125 \\ &= LP_{L_2}(P) \end{aligned}$$

Note that anonymity is not satisfied because the level costs are not permuted.

US-I Consider again the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$\begin{aligned} LP_{L_1}(P) &= 64 \\ &\neq \frac{20}{30} \cdot 64 \end{aligned}$$

US-II Follows from union standardness-I (US-I).

□

Proof theorem 4.3.2

Proof. (i) Let $l(\max N) = m$. We prove that the following properties are satisfied by the equal downstream rule for all $P \in \mathcal{C}_{ed}$:

EF Proved in theorem 4.4.2.

FE Follows from Efficiency (EF).

IDC Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for any $k \in M$ and $h < k$ holds that $c^1_{L_h} = c^2_{L_h}$. We want to show that for all $l < k$ we have $ED_{L_l}(P^1) = ED_{L_l}(P^2)$. Let $l < k$, we have that

$$ED_{L_l}(P^1) = \sum_{h=1}^{l} \frac{1}{m - h + 1} \cdot c^1_{L_h}.$$

From $h \leq l$ and $l < k$ follows that $h < k$ and therefore $c^1_{L_h} = c^2_{L_h}$, hence we obtain,

$$\sum_{h=1}^{l} \frac{1}{m-h+1} \cdot c^1_{L_h} = \sum_{h=1}^{l} \frac{1}{m-h+1} \cdot c^2_{L_h}$$
$$ED_{L_l}(P^1) = ED_{L_l}(P^2)$$

NUC Proved in theorem 4.4.2.

NUCO Assume that $c_{L_h} = 0$ for all $h < l$. For all $k \geq l \in M$ we have:

$$\begin{aligned} ED_{L_k}(P) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c_{L_h} \\ &= \frac{1}{m-k+1} \cdot c_{L_k} && (c_{L_h} = 0 | h < k) \\ &= \sum_{h=k}^{k} \frac{1}{m-h+1} \cdot c_{L_h} \\ &= ED_{L_k}(N \setminus \cup_{h<k} L_h, L \setminus \{L_h\}_{h<k}, (d_i)_{i \in N \setminus \cup_{h<k} L_h}, (c_{L_f})_{L_f \in L \setminus \{L_h\}_{h<k}}). \end{aligned}$$

C-MON Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$ and $P^2 = (N, L, d, c^2)$ such that for all $h \in M$ holds that $c^1_{L_h} \leq c^2_{L_h}$. We want to show that $ED(P^1) \leq ED(P^2)$:

$$\begin{aligned} ED_{L_k}(P^1) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^1_{L_h} \\ &\leq \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^2_{L_h} && (c^1_{L_h} \leq c^2_{L_h}) \\ &= ED_{L_k}(P^2) \end{aligned}$$

RED Let $P \in \mathcal{C}_{ed}$ such that $\hat{d}_{L_h} = \hat{d}_{L_k}$ and $h < k$. We want to show that for $h, k \in M$ it holds that $ED_{L_h}(P) \leq ED_{L_k}(P)$:

$$\begin{aligned} ED_{L_h}(P) &= \sum_{l=1}^{h} \frac{1}{m-l+1} \cdot c_{L_l} \\ &\leq \sum_{l=1}^{h} \frac{1}{m-l+1} \cdot c_{L_l} + \sum_{l=h+1}^{k} \frac{1}{m-l+1} \cdot c_{L_l} \\ &= \sum_{l=1}^{k} \frac{1}{m-l+1} \cdot c_{L_l} \\ &= ED_{L_k}(P) \end{aligned}$$

Note that RED is not dependent on the demands, but a downstream level can never be charged more than an upstream level according to this rule and therefore RED is satisfied. So the demand condition is not necessary for the ED rule to satisfy this property. The same applies to the next property.

D-SYM Let $P^l \in \mathcal{C}_{ed}$ with $P^l = (N, L, d, c^l)$ such that $c^l = (0, ..., 0, c_{L_l}, 0, ..., 0)$ for $l \in M$. We want to show that for all $h, k \geq l$ such that $\bar{d}_{L_k} = \bar{d}_{L_h}$ we have $ED_{L_h}(P^l) = ED_{L_k}(P^l)$:

$$\begin{aligned} ED_{L_h}(P^l) &= \sum_{g=1}^{h} \frac{1}{m-g+1} \cdot c_{L_g} \\ &= \frac{1}{m-l+1} \cdot c_{L_l} && (h \geq l) \\ &= \sum_{g=1}^{k} \frac{1}{m-g+1} \cdot c_{L_g} && (k \geq l) \\ &= ED_{L_k}(P^l) \end{aligned}$$

DR Proved in theorem 4.4.2.

ACN Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$ and $P^2 = (N^{ij}, L^{ij}, d^{ij}, c)$ such that agents $i, j \in L_k$ colluded. Then for $k \in M$ we want to show that: $ED_{L_k}(P^1) = ED_{L_k}(P^2)$. For $k \in M$ we have

$$\begin{aligned} ED_{L_k}(P^1) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c_{L_h} \\ &= ED_{L_k}(P^2). \end{aligned}$$

The equal downstream rule is dependent on the number of downstream unions, not on the number of downstream agents and therefore this property is satisfied.

C-ADD Let $P^1, P^2, P^3 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c^1)$, $P^2 = (N, L, d, c^2)$ and $P^3 = (N, L, d, c^1 + c^2)$. For all $k \in M$ we want to show that $ED_{L_k}(P^1) + ED_{L_k}(P^2) = ED_{L_k}(P^3)$:

$$\begin{aligned} ED_{L_k}(P^1) + ED_{L_k}(P^2) &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^1_{L_h} + \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot c^2_{L_h} \\ &= \sum_{h=1}^{k} \frac{1}{m-h+1} \cdot (c^1_{L_h} + c^2_{L_h}) \\ &= ED_{L_k}(P^3) \end{aligned}$$

C-LIN Proved in theorem 4.4.2.

(ii) Let $l(\max N) = m$. We show that the following properties are not satisfied by the equal downstream rule for some $P \in \mathcal{C}_{ed}$ by means of simple counterexamples:

NDP Consider the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (0, 10)$ and $c = (20, 30)$. We have that $\hat{d}_{L_1} = 0$. However, $ED_{L_1}(P) = \frac{1}{m} \cdot c_{L_1} = \frac{1}{2} \cdot 20 \neq 0$.

NCP Let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For level $2 \in M$ we have that $c_{L_2} = 0$, but

$$\begin{aligned} ED_{L_2}(P) &= \sum_{h=1}^{2} \frac{1}{m - h + 1} \cdot c_{L_h} \\ &= \frac{1}{2} \cdot 64 + 0 \\ &= 32 > 0. \end{aligned}$$

RAN Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied equal downstream rule to P in example 4.2.2. We have that $\hat{d}_{L_1} = 50 > 30 = \hat{d}_{L_2}$. However, $ED_{L_1}(P) = 77 < 139 = ED_{L_2}(P)$.

ET Consider again $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied equal downstream rule to P in example 4.2.2. We have that $\hat{d}_{L_2} = 30 = \hat{d}_{L_3}$, whereas $ED_{L_2}(P) = 139 < 201 = ED_{L_3}(P)$.

DDPR Consider the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$\begin{aligned} ED_{L_1}(P) &= \frac{1}{2} \cdot 64 \\ &\neq \frac{20}{30} \cdot 64. \end{aligned}$$

DUCN-I Consider the following example: Let $P^1, P^2 \in \mathcal{C}_{ed}$ with $P^1 = (N, L, d, c)$, where $N = \{1, 2, 3\}$, $L = \{\{1\}, \{2\}, \{3\}\}$, $d = (20, 10, 10)$ and $c = (10, 20, 15)$. Let $P^2 = (N, L^{L_2 L_3}, d, c^{L_2 L_3})$, where $N = \{1, 2, 3\}$, $L = \{\{1\}, \{2, 3\}\}$ and $c = (10, 35)$.

$$\begin{aligned} ED_{L_1}(P^1) &= \frac{1}{3} \cdot 10 \\ &\neq \frac{1}{2} \cdot 10 \\ &= ED_{L_1}(P^2). \end{aligned}$$

UCN-I Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied equal downstream rule to P in example 4.2.2. Let $P^1 \in \mathcal{C}_{ed}$ be also as defined in example 4.1.1, but such that $P^1 = (N, L^{L_1L_2}, d, c^{L_1L_2})$ with $L^{L_1L_2}$ the partition where unions L_1 and L_2 colluded such that $L^{L_1L_2}_{L_1} = L_1 \cup L_2$ and $c^{L_1L_2}_{L_1} = c_{L_1} + c_{L_2} = 355$. We have

$$\begin{aligned} ED_{L_1}(P) + ED_{L_2}(P) &= 77 + 139 \\ &= 216 \\ &\neq 177.5 \\ &= \frac{1}{2} \cdot 355 \\ &= \frac{1}{m} \cdot c^{L_1L_2}_{L_1} \\ &= ED_{L_1}(P^1). \end{aligned}$$

DUCN-II Consider the example as defined for DUCN-I. We have that,

$$\begin{aligned} ED_{L_2}(P^2) &= \frac{1}{2} \cdot 10 + 35 \\ &= 40 \\ &\neq 42 \\ &= (\frac{1}{3} \cdot 10 + \frac{1}{2} \cdot 20) + (\frac{1}{3} \cdot 10 + \frac{1}{2} \cdot 20 + 15) \\ &= ED_{L_2}(P^1) + ED_{L_3}(P^1) \end{aligned}$$

UCN-II Follows from downstream union collusion neutrality-II (DUCN-II).

AN Consider $P \in \mathcal{C}_{ed}$ as defined in example 4.1.1 and the applied equal downstream rule to P in example 4.2.2. Let π be a permutation on M such that $P^\pi = (N, \pi L, \pi d, c)$ with $\pi L = \{L_2, L_1, L_3\}$ and $\pi d = (d_3, d_4, d_1, d_2, d_5, d_6, d_7, d_8)$, so $\pi\hat{d} = (\hat{d}_{L_2}, \hat{d}_{L_1}, \hat{d}_{L_2})$. It follows,

$$\begin{aligned} ED_{\pi L_2}(P^\pi) &= \frac{1}{3} \cdot 230 \\ &= 77 \\ &\neq 139 \\ &= \frac{1}{3} \cdot 230 + \frac{1}{2} \cdot 125 \\ &= ED_{L_2}(P) \end{aligned}$$

Note that anonymity is not satisfied because the levels are not permuted.

US-I Consider the following example: let $P \in \mathcal{C}_{ed}$ with $P = (N, L, d, c)$, where $N = \{1, 2\}$, $L = \{\{1\}, \{2\}\}$, $d = (20, 10)$ and $c = (64, 0)$. For levels $1, 2 \geq 1$ we have

$$ED_{L_1}(P) = \frac{1}{2} \cdot 64$$

$$= 32$$
$$\neq \frac{20}{30} \cdot 64.$$

US-II Follows from union standardness-I (US-I).

□

Additions Chapter 5

Elaboration example 5.1.2

$$
\begin{aligned}
v^{fix}(2) &= \sum_{k=1}^{l(\max\{2\})} b_{L_k} \\
&= \sum_{k=1}^{1} b_{L_k} \\
&= b_{L_1} \\
&= 150
\end{aligned}
$$

$$
\begin{aligned}
v^{var}(2) &= \sum_{k=1}^{l(\max\{2\})} a_{L_k} \cdot \bar{d}_{L_k} \\
&= \sum_{k=1}^{1} a_{L_k} \cdot \bar{d}_{L_k} \\
&= a_{L_1} \cdot d_2 \\
&= 5 \cdot 5 \\
&= 25
\end{aligned}
$$

$$
\begin{aligned}
v^{fix}(2,3) &= \sum_{k=1}^{l(\max\{2,3\})} b_{L_k} \\
&= \sum_{k=1}^{l(3)} b_{L_k} \\
&= \sum_{k=1}^{2} b_{L_k} \\
&= b_{L_1} + b_{L_2} \\
&= 150 + 100 \\
&= 250
\end{aligned}
$$

$$
\begin{aligned}
v^{var}(2,3) &= \sum_{k=1}^{l(\max\{2,3\})} a_{L_k} \cdot \bar{d}_{L_k} \\
&= \sum_{k=1}^{l(3)} a_{L_k} \cdot \bar{d}_{L_k} \\
&= \sum_{k=1}^{2} a_{L_k} \cdot \bar{d}_{L_k} \\
&= a_{L_1} \cdot (d_2 + d_3) + a_{L_2} \cdot d_3 \\
&= 5 \cdot 25 + 4 \cdot 20 \\
&= 205
\end{aligned}
$$

$$
\begin{aligned}
v^{fix}(1,2,3) &= \sum_{k=1}^{l(\max\{1,2,3\})} b_{L_k} \\
&= \sum_{k=1}^{l(\max\{3\})} b_{L_k} \\
&= \sum_{k=1}^{2} b_{L_k} \\
&= b_{L_1} + b_{L_2} \\
&= 250
\end{aligned}
\qquad
\begin{aligned}
v^{var}(123) &= \sum_{k=1}^{l(\max\{1,2,3\})} a_{L_k} \cdot \bar{d}_{L_k} \\
&= \sum_{k=1}^{l(3)} a_{L_k} \cdot \bar{d}_{L_k} \\
&= \sum_{k=1}^{2} a_{L_k} \cdot \bar{d}_{L_k} \\
&= a_{L_1} \cdot (d_1 + d_2 + d_3) + a_{L_2} \cdot (d_3) \\
&= 5 \cdot 35 + 4 \cdot 20 \\
&= 255
\end{aligned}
$$

Notation

α^1, α^2	=	discounting factors
$a = (a_{L_k})_{k \in M}$	=	vector of constants for the variable part of the level cost vector
$b = (b_{L_k})_{k \in M}$	=	vector of constants for the fixed part of the level cost vector
$CA(P)$	=	the cascade rule
$CA - UD(P)$	=	the downstream-cascading-upstream-discounting rule
$c = (c_{L_k})_{k \in M}$	=	the level cost vector
$C(d)$	=	total cost of the grid
$\mathcal{C}_{bf}$	=	class of bilateral flow problems
$\mathcal{C}_{ed}$	=	class of electricity demand problems
$\mathcal{C}_{edp}$	=	class of electricity demand-production problems
$d = (d_i)_{i \in N}$	=	demand vector of the agents in N
$\hat{d} = (\hat{d}_{L_k})_{k \in M}$	=	demand vector of the unions in M
$\bar{d}$	=	aggregated demand vector with $\bar{d} \in \mathbb{R}^M_+$
$ED(P)$	=	equal downstream rule
$F^{in}_{L_k}, F^{out}_{L_k}$	=	total infow to or outflow from level L_k
$\mathcal{GP}_{ed}$	=	class of electricity demand games
L	=	partition of N
$l(i)$	=	level of agent i such that $l(i) \in M$
$l(\max S)$	=	the level of a greatest element of S
L_k	=	the k-th level/union in the network
$LP(P)$	=	level paying rule
M	=	set of unions/levels
N	=	set of agents
$p = (p_i)_{i \in N}$	=	production vector of the agents in N
$\hat{p} = (\hat{p}_{L_k})_{k \in M}$	=	production vector of the unions in M
$Sh^a(N, v, L)$	=	agent Shapley value
$Sh^u(N, v, L)$	=	union Shapley value
$v(L_k)$	=	stand-alone cost of level L_k
$v^{fix}(S)$	=	fixed characteristic function for subset $S \subseteq N$
$v^{var}(S)$	=	variable characteristic function for subset $S \subseteq N$

Bibliography

Consultatiedocument Stroom: Bijlage - Beschrijving huidige tarievensystematiek. Technical report, Economische Zaken, 2014.

R. Aalbers, D. L. F. Bressers, E. Dijkgraaf, P. J. Hoogendoorn, and S. C. de Klerk. *Een level playing field op de Nederlandse elektriciteitsmarkt. Een tariefstructuur voor het netgebruik.* 2003.

ACM. X-factoren regionaal netbeheer elektriciteit (2014-2016), 2014. URL `https://www.acm.nl/nl/onderwerpen/energie/elektriciteit/regulering-regionale-netbeheerders/x-factoren-regionaal-netbeheer-elektriciteit-2014-2016/`.

J. Alonso-Meijide, M. Álvarez Mozos, and M. Fiestras-Janeiro. Values of games with graph restricted communication and a priori unions. *Mathematical Social Sciences*, 58 (2):202–213, Sept. 2009.

R. J. Aumann and J. H. Drèze. Cooperative games with coalition structures. *International Journal of Game Theory*, 3(4):217–237, Dec. 1974.

R. J. Aumann and M. Maschler. Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 36:195–213, 1985.

Autoriteit Consument en Markt. Tarievencode Elektriciteit, 2013. URL `https://www.acm.nl/download/documenten/acm-energie/tarievencode-elektriciteit-per-13-juni-2013.pdf`.

G. Bergantiños and R. Martínez. Cost allocation in asymmetric trees. *European Journal of Operational Research*, 237(3):975–987, Sept. 2014.

J. J. M. Derks and H. H. Haller. Null players out? Linear values for games with variable supports. *International Game Theory Review*, 1(03n04):301–314, 1999. ISSN 0219-1989.

P. S. Divya, M. Murali, S. Member, M. S. Kumari, M. Sydulu, and A. Terminology. Embedded Cost Allocation Methods using Game Theory. 2012.

Y. Feng. *Extensions of the Shapley value and their characterizations.* PhD thesis, Enschede, Aug. 2013.

V. Fragnelli, I. Garcia-Jurado, H. Norde, F. Patrone, and S. Tijs. How to Share Railways Infrastructure Costs? In I. Garcia-Jurado, F. Patrone, and S. Tijs, editors, *Game practice: contributions from applied game theory*, pages 91–101. Kluwer Academic, Dordrecht, 2000.

R. P. Gilles. *The Cooperative Game Theory of Networks and Hierarchies*, volume 44 of *Theory and Decision Library C.* Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

R. P. Gilles, G. Owen, and R. van den Brink. Games with permission structures: The conjunctive approach. *International Journal of Game Theory*, 20(3):277–293, 1992.

D. Gillies. *Some Theorems on n-Person Games.* PhD thesis, Princeton University, 1953.

V. Ginsburgh and I. Zang. The museum pass game and its value. *Games and Economic Behavior*, 43(2):322–325, May 2003.

R. A. Hakvoort and A. E. H. Huygen. Sturen op het gebruik van lokale energienetten. Technical report, TNO, D-Cision, 2012.

R. A. Hakvoort, H. P. A. Knops, P. Koutstaal, A. van der Welle, and J. Gerdes. De tariefsystematiek van het elektriciteitsnet. Technical report, D-Cision, TU Delft, ECN, 2013.

H. Haller. Collusion properties of values. *International Journal of Game Theory*, 23(3): 261–281, Sept. 1994.

S. Hart and A. Mas-Colell. Potential, Value, and Consistency. *Econometrica*, 57(3): 589–614, 1989.

N.-D. Hoàng. *Algorithmic cost allocation games: theory and applications.* Springer, 2012.

A. E. H. Huygen. *Regulering bij concurrentie: de Nederlandse elektriciteitssector.* Universiteit, 1999.

M. Junqueira, L. Carlos, L. A. Barroso, S. Member, G. C. Oliveira, L. M. Thomé, and M. V. Pereira. An Aumann-Shapley Approach to Allocate Transmission Service Cost Among Network Users in Electricity Markets. 22(4):1532–1546, 2007.

M. Koster. Cost Sharing. In R. A. Meyers, editor, *Encyclopedia of complexity and systems science*, pages 1660–1689. Springer New York, New York, 2nd edition, 2009.

P. R. Koutstaal, J. Gerdes, C. H. Volkers, and W. Wetzels. Effecten invoering producentenarief. Technical report, ECN, 2012.

S. C. Littlechild and G. Owen. A Simple Expression for the Shapley Value in a Special Case. *Management Science*, 20(3):370–372, Nov. 1973.

H. J. Moulin. Axiomatic cost and surplus sharing. *Handbook of social choice and welfare*, 1:289–357, 2002.

H. J. Moulin and S. Shenker. Average Cost Pricing versus Serial Cost Sharing: An Axiomatic Comparison. *Journal of Economic Theory*, 64(1):178–201, Oct. 1994.

R. B. Myerson. Graphs and Cooperation in Games. *Mathematics of Operations Research*, 2(3):225–229, 1977.

D. Ni and Y. Wang. Sharing a polluted river. *Games and Economic Behavior*, 60(1): 176–186, July 2007.

NMA and SEO. Onderzoek naar de methodologie voor de verdeling van de kosten van netverliezen. Technical Report 74100203-NMEA/MOC 11-0529, 2011.

L. Olmos and I. J. Pérez-Arriaga. A comprehensive approach for computation and implementation of efficient electricity transmission network charges. *Energy Policy*, 37(12):5285–5295, Dec. 2009.

K. M. Ortmann. The proportional value for positive cooperative games. *Mathematical Methods of Operations Research (ZOR)*, 51(2):235–248, Apr. 2000.

G. Owen. Values of Games with a Priori Unions. In R. Henn and O. Moeschlin, editors, *Mathematical Economics and Game Theory SE - 7*, volume 141 of *Lecture Notes in Economics and Mathematical Systems*, pages 76–88. Springer Berlin Heidelberg, 1977.

I. J. Pérez-Arriaga, L. Olmos, and M. Rivier. Transmission Pricing. In J. Rosellón and T. Kristiansen, editors, *Financial transmission rights: Analysis, experiences and prospects*, volume 7, chapter 2, pages 49–77. Springer-Verlag Londen, 2013.

M. P. Rodríguez Ortega, I. J. Pérez-Arriaga, J. R. Abbad, and J. P. González. Distribution network tariffs: A closed question? *Energy Policy*, 36(5):1712–1725, May 2008.

L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games*, volume 2, pages 307–318. Princeton University Press, volume ii edition, 1953.

L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1(1): 11–26, 1971.

D. Skorin-Kapov. Hub network games. *Networks*, 31(4):293–302, July 1998.

P. Sudhölter. Axiomatizations of Game Theoretical Solutions for One-Output Cost Sharing Problems. *Games and Economic Behavior*, 24(1-2):142–171, July 1998.

R. van den Brink and C. Dietz. Union values for games with coalition structure. *Decision Support Systems*, 66:1–8, Oct. 2014.

Vereniging van Nederlandse Gemeenten. Lokaal energiek : decentrale duurzame elektriciteit. Technical report, 2013.

J. Von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1947.

A. F. Wals, E. Cross, and E. J. W. Van Sambeek. Review of current electricity policy and regulation. Technical report, ECN, 2003.

Y. Wang. A Museum Cost Sharing Problem. *American Journal of Operations Research*, 01(02):51–56, June 2011.

E. Winter. A Value for cooperative games with levels structure of cooperation. *International Journal of Game Theory*, 18(2):227–240, June 1989.

H. P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72, June 1985.

H. P. Young. Cost allocation. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications, volume 2*, volume 2, chapter 34, pages 1193–1235. Elsevier B.V., 1 edition, 1994.

J. M. Zolezzi and H. Rudnick. Transmission Cost Allocation by Cooperative Games and Coalition Formation. *IEEE Trans. Power Syst.*, 17(4):1008–1015, 2002.