Superplural Logic

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Abstract

Plural logic adds to singular logic plural variables and a two-place connective 'is/are among', written '$\preccurlyeq$'. Superplural logic adds to plural logic higher-level variables and higher-level two-place connectives 'are among', written '$\preccurlyeq^n$'. This thesis examines the linguistic, logical, and philosophical status of superplural logic. I begin by asking whether the idea of a superplural logic is intelligible at all, and whether superplurals occur in natural language. In doing so, I develop a conception of superplural logic and defend it against possible objections. I then construct a formal logic that corresponds to this conception. The discussion closes with an evaluation of the philosophical relevance of superplural logic, especially with regard to its proclaimed ontological innocence. I conclude that superplural logic is a viable alternative to existing theories and should be acknowlegded and used by linguists and philosophers alike.

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Contents

Chapter 1

Introduction

> It is haywire to think that when you have some Cheerios, you are eating a *set*—-what you're doing is: eating THE CHEERIOS.
>
> George Boolos (1984, p. 448)

'The Boswell Sisters and the Mills Brothers are the best close-harmony singers of the 20th century' is perfectly grammatical, intelligible, and perhaps even true. What's more, it also shows that plural logic cannot handle all plural phenomena in natural language. For even though 'the Boswell Sisters and the Mills Brothers' denotes more than one individual, they do not make up a single large group here (they never sang together). We will argue that 'the Boswell Sisters and the Mills Brothers' is a *superplural term*.

Superplural logic is to singular logic what higher-order logic is to first-order logic. In both cases, as we shall see, we have a hierarchy based on iterating some aspect of a logic. Where third-order logic asks: 'what are predicates of predicates?', superplural logic wants to know: 'what are terms of terms?'

A logic is not just a collection of axioms and rules formulated in some specified language. There is a context from which it emerges. There are often surprising applications, philosophical abstractions, unexpected friends and foes that gather around it. Developing superplural logic means tackling all such aspects, from technical presentation to conceptual positioning. The goal of this project is twofold: first, to compile and comment on the existing work on superplurals, relating approaches from both philosophy and linguistics; and second, to explore formal and philosophical consequences of a plural hierarchy.

One must start at the beginning, of course. We cannot talk about *superplural* logic without being clear on what *plural* logic is. Therefore, our first step in the introduction is to present plural logic from a philosophical point of view. Based on that, we will get an idea of what superplurals are. Next, we motivate the study of superplurals by outlining the ways in which it can be interesting and useful. The introduction ends with a summary of previous work on superplurals to set the stage for this thesis.

1.1 Plural Logic

We introduce plural logic by way of motivation. Suppose we are in the business of regimenting natural language, and suppose we choose to do so in classical first-order logic. How might we formalise the Geach-Kaplan sentence?

(GK) Some critics admire only one another.

The easy way is easy: we take *is such that some critics admire only one another* to be a predicate P, and write it down as $\forall x Px$. But this formalisation is too weak; it does not preserve even the minimal logical connections of the Geach-Kaplan sentence. What we want is a formal sentence that expresses, at the very least, that (i) there are some critics, (ii) and that if a critic admires anybody, it is another critic. A natural formalisation of this in fact requires second-order logic:

(GK$_2$) $\exists X(\exists x Xx \wedge \forall x \forall y(Xx \wedge Axy \rightarrow x \neq y \wedge Xy))$

Importantly, Kaplan proved that there is no first-order sentence that is true in precisely the same models as the above second-order sentence, assuming full semantics (Boolos, 1984, p. 57). While this is a bad result for first-order logic as our regimenting language, many have tried to circumvent it by making use of set theory. For a set-theoretic equivalent can be formulated as follows:

(GK$_s$) $\exists S(\exists x(x \in S) \wedge \forall x(x \in S \rightarrow Cx) \wedge \forall x \forall y((x \in S \wedge Axy) \rightarrow (x \neq y \wedge y \in S)))$

This translation requires that there are sets in the domain, which is in itself controversial. However, the ontological dispute does not necessarily pertain to whether or not sets exists, but whether a sentence such as the above should *entail that* sets exist. Be that as it may, there is an even more basic uneasiness about the set-theoretic translation, and it originates from collective predication.

Terminology first. A predicate F is **distributive** if it is analytic that F is true of some things iff it is true of each of them separately. A good example of a distributive predicate is *being in this room*, since it is analytic that some people are in this room just in case each person is in this room. F is **collective** if it is not distributive. A good example of a collective predicate is *singing in harmony*. Nobody can sing in harmony with oneself. Thus, 'Martha, Connee, and Helvetia sang in harmony' does *not* entail that Martha sang in harmony.

Here is a straightforward way to paraphrase the Geach-Kaplan sentence:

(GK$_p$) There are some critics such that, for any x and y, if x is one of them and x admires y, then $x \neq y$ and y is also one of them.

Both occurrences of 'is one of' above must be understood collectively with regard to the second argument place—otherwise we would have false statements such as 'x is one of y' where $x \neq y$. Yet there is no immediate way to capture collective predicates in classical first-order logic. Instead, we might resort to the formulation in (GK_s) and insist that the membership relation is collective. This would make the critics a *set* rather than, well, the critics.

This solution has been heavily criticised by Boolos and others, and is very much the starting point of plural logic. Oliver and Smiley dub this move 'changing the subject'. Here I will present their generalised version of Boolos's original point. The crux of the argument is that all attempts to use a singular *surrogate* (such as sets, classes, collections–but not mereological sums) are vulnerable to a general Russellian paradox (Oliver & Smiley (2013, esp. §3.5), see also Rayo (2002, §3)). It runs as follows.

Take the collective predicate 'is one of' as a class-forming relation, which we call 'constituent of' and write as '$\leq$'(its set-theoretic analogue, for instance, is membership). Further, call whatever class-like surrogate a 'collection'. First, we note that the constituent-of relation cannot be reflexive. Whitehead is a constituent of the collection of Whitehead and Russell, and so is Russell, but nothing else is. Thus, the collection of Whitehead and Russell cannot be a constituent of itself, since only Whitehead and Russell are constituents of it. Thus it is safe to say that the collection of Whitehead and Russell *is one of* the things that are not constituents of themselves, and more generally,

(R) There are some collections such that, for any y, y is one of them just in case y is a collection which is not a constituent of itself.

Now translate this sentence into collectivese:

(R') There is a collection x such that, for every y, y is a constituent of x just in case y is a collection which is not a constituent of itself.

We immediately arrive at a collection that has as constituents all and only the things that aren't constituents of themselves:

(R") $\exists x \forall y (y \leq x \leftrightarrow \neg(y \leq y))$

Since there is no such collection, as Russell pointed out, our collectivese sentence is false. This means that the translation is incorrect, since it assigns wrong truth-values to the collectivese sentences.

Thus, for any (non-trivial) choice of surrogates, we can construe a sentence that cannot be translated by appeal to those surrogates. In the words of Oliver and Smiley, 'changing the subject is simply not on' (p. 42).[1]

What, then, is the reference of 'Russell and Whitehead'? The answer is startlingly simple: Russell and Whitehead. Plural logic allows for a term to denote several things at once. 'Russell and Whitehead', 'the Boswell Sisters', and 'Saturn's moons' are all plural terms. Formally, plural logic adds to first-order logic plural variables 'xx' and a two-place connective 'is/are

[1]Note, however, that this argument does not show that there is no possible paraphrase of collective predicates in a first-order language. To this end, Rayo has provided a sentence (called Bernays's Principle) which, given that the domain ranges over absolutely anything, *seems* to resist first-order paraphrase (2002, §4).

among' or '$\preccurlyeq$'. The plural quantifiers $\exists xx$ and $\forall xx$ are interpreted as 'there are some things xx such that ...' and 'whenever there are some things xx, then ...', respectively.

Crucially, the semantic value of a plural variable is not defined as a set, but as *many values* from the domain. This thought is motivated by Boolos's famous dictum that, when you are eating Cheerios, you are not eating a *set*; you are simply eating THE CHEERIOS. It is argued that plural quantifiers do not incur (additional) ontological commitments to sets or other 'set-like' entities over and above the individuals over which we quantify. The claim that plural logic is the ontologically more palatable account of plurals is, of course, not uncontroversial (Simons, 1997; Rayo, 2007; Hawley, 2014; Florio & Linnebo, 2015). We will revisit it in chapter 4.

Once the idea of plural logic is on the table, however, it is hard not to see plural locutions everywhere in natural language. 'There are some apples' no longer translates into $(\exists xAx \wedge \exists yAy \wedge x \neq y)\ldots$ but can be much more intuitively paraphrased as $\exists xxAxx$, which is equivalent to $\exists xx\forall x(x \preccurlyeq xx \rightarrow Ax)$. Our previous definition of a distributive predicate can be formally written in plural notation as well. A monadic predicate F is distributive if

$$\forall xx(Fxx \leftrightarrow \forall x(x \preccurlyeq xx \rightarrow Fx))$$

read 'some things are F just in case each thing among them is F'. For n-place predicates, we characterise each place separately as either distributive or collective. For instance, a two-place predicate R is distributive at its second place if it is analytic that

$$\forall xx\forall yy(Rxxyy \leftrightarrow \forall y(y \preccurlyeq yy \rightarrow Rxxy))$$

It is customary to distinguish different versions of plural logic. A plural first-order language (**PFO**) is what we get from adding plural variables and quantifiers and the connective 'among' to a classical first-order language. While **PFO** is enough to capture all sentences that have 'is one of', it doesn't do justice to collective predicates. Extending **PFO** with atomic plural predicates gives us a (two-sorted) **PFO+**. Why two-sorted? Because we have collective and distributive predicates now which are distinguished by the sort of variables they can apply to.

Two-sorted **PFO+** is at odds with our intuitive grasp of predicates. Very quickly, we find examples of predicates that can be both collective and distributive. Compare 'the Boswell Sisters sang a song' and 'Martha sang a song'. Even though the sentences are understood differently (the first is plural, the second is not), they seem to involve *the same* predicate. Another example: 'Wittgenstein wrote the Tractatus, not Whitehead and Russell' (Oliver & Smiley, 2013, p. 59); clearly the sentence has one and the same predicate.

To ameliorate the situation, Boolos offers a one-sorted **PFO+** that embeds singular predicates in plural ones, since '$\preccurlyeq$' reads 'is/are among' (Boolos (1984, pp. 443-5), Rayo (2002, §11)). A term a is singular if $\forall x(x \preccurlyeq a \rightarrow x = a)$. And in general we can construct for singular predicates S their plural counterpart P, using the following condition:

$$\forall x\forall xx(\forall y(y \preccurlyeq xx \leftrightarrow y = x) \rightarrow (P(xx) \leftrightarrow S(x)))$$

This is, in a nutshell, plural logic. From a logical point of view, it is needed to regiment sentences involving collective predicates. From a philosophical point of view, it is regarded as a powerful and yet ontologically innocent formalism to characterise set-like relations. From a linguistic point of view, it offers an intuitive account of plural phenomena in natural language. Its place in these fields has been wildly debated ever since Boolos's presentation[2] (Lewis, 1991; Schein, 1993; Hazen, 1993; Higginbotham, 1998; Yi, 1999; Oliver & Smiley, 2001; Rayo, 2002; Linnebo, 2003; McKay, 2006), and our gloss here is necessarily incomplete (much like plural logic itself). But it serves as a foundation for adding the *super* to the *plural*. Where do we go from here?

Suppose I am not eating the Cheerios, but the Froot Loops. I group them according to their colours: red, blue, orange, purple, yellow, and green. Now take a distributive predicate such as 'being delicious' and say

(1) The green Froot Loops are delicious.

'Forming a circle', on the other hand, is a collective predicate. One Froot Loop, though circular in shape, cannot form a circle—arguably, at least three individuals are required to form a circle. So

(2) The red Froot Loops form a circle.

is a sentence with a collective predicate. Having carefully sorted my Froot Loops, I perform a series of scientific experiments to test my long-standing hypothesis:

(3) The green Froot Loops and the pink Froot Loops taste bad together.[3]

How is this sentence best understood? We have a collective, monadic predicate which can take in variably many arguments. Each of these arguments can in turn take in many arguments. It seems that ordinary plural logic does not suffice to explain this higher-level entity consisting of the green and the pink Froot Loops. This is where we require superplural logic.

A superplural term refers to several 'pluralities' at once, much as an ordinary plural term refers to several objects at once. We can imagine superplural variables '$xxx, xxxx, \ldots$' and many more two-place connectives '$\preccurlyeq_2, \preccurlyeq_3, \ldots$'. The same goes for superplural quantifiers. Supply this with an adequate semantics and deductive system, and we have a superplural logic to work with.

Several questions arise if we were to devise such a logic. To begin with, how cogent is the idea of a 'term of terms'? Do such things really occur in natural language? Furthermore, how should we express superplurals formally? What does the plural hierarchy, consisting of plurals, superplurals, supersuperplurals etc., look like? Lastly, are superplurals useful? If so, what are their applications? But before addressing each of these questions, let us ask *why* one should be interested in this project at all.

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[2]Though he certainly wasn't the first logician to take plurals seriously; Bertrand Russell, Max Black, Peter Simons, and Richard Sharvy all belong to the pre-historic era for plural logic.

[3]They all, as a matter of fact, taste the same: `http://www.straightdope.com/columns/read/1683/are-the-different-colors-of-froot-loops-different-flavors`

1.2 Motivation

The study of superplurals is motivated by at least three thoughts. The first is explorative. Suppose we are using plural logic as a regimenting language. The question of a hierarchy arises naturally: are there higher-level linguistic phenomena, and if so, how can plural logic capture it? But curiosity alone is not enough to justify a thesis dedicated to the subject.

One way of going beyond **PFO+** is to quantify into predicate positions, as it is done from singular first-order logic to singular second-order logic. The study of higher-*order* plural logic, however, has less to do with plural quantification than with predication in general. The study of higher-*level* plural logic, on the other hand, has everything to do with plural quantification. Higher-level plurals result from iterating the step from the singular to the plural (Linnebo & Nicolas, 2008; Uzquiano, 2004a). That is, the arguments for plural logic being required to capture certain types of reference and predication also apply to superplurals, as we will argue in 2.2. Conversely, if there are reasons to doubt the conceptual coherence of superplurals or its ontological innocence, these principles and considerations would also apply to ordinary plural logic. In other words, plural quantification stands and falls together with superplural quantification. This is the second thought that motivates a careful study of superplurals.

The final motivation comes from application. If the account of superplural quantification is successful—and superplurals are indeed ontologically innocent—then many philosophical problems related to ontological parsimony can be dealt with in terms of superplural quantification. I go into depth in chapter 4; for now, it suffices to note that superplurals could be a powerful tool in both metaphysics and logic.

1.3 Previous work

The idea of superplural quantification arguably first appears in Russell's remark that classes of classes are 'many manys', rather than singular objects (1903, p. 536). This, of course, is the direct result of his characterisation of 'classes as many' as opposed to 'classes as one':

> The distinction of a class as many from a class as a whole is often made by language: space and points, time and instants, the army and the soldiers, the navy and the sailors, the Cabinet and the Cabinet Ministers, all illustrate the distinction. . . . In a class as many, the component terms, though they have some kind of unity, have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from remaining many (§70).

Hazen (1997) also mentions 'perplurals' in an article on Lewis's theory of parts, where a perplural 'is related to plurals as plurals are to singulars' (p. 247). However, neither the theory of classes nor Lewisian mereology can be said to be the focus of the debate so far. So far, authors have mainly written on the intelligibility of superplurals.

In arguing against Boolos, Linnebo (2003, §IV) advanced the idea that the step from the singular to the plural can be *iterated*, thus creating higher-level plurals that ultimately made Boolos's position unstable. This is because superplural logic requires combinatorial and set-theoretic methods that undermine the argument for ontological innocence. Linnebo & Nicolas (2008) argue for the occurrence of superplurals in natural language, in support of the argument that the iteration to higher levels is intelligible, in principle and in practice.

In his monograph, McKay (2006) dedicates two sections on the coherence of superplural quantification. They focus on the pragmatic aspects of superplural phenomena. In natural language, McKay argues, most examples involve contextual cues that determine reference. For example, plural nouns like teams or bands have pragmatic membership standards. This leads him to conclude that the superplural machinery is not needed to explain those cases.

Superplurals prominently feature in the debate on absolute generality, the question of whether there is an all-inclusive domain of discourse. If we do semantic theorising in a metalanguage, then it is worth asking how far up this semantic hierarchy can go. Rayo (2006) asks if absolute generality is possible in light of the open-endedness of the hierarchy. For semantic adequacy, he needs a typed hierarchy that characterises an appropriate reference predicate. The formalism he provides relies on superplurals, even though he does not commit himself to their existence in natural language. He is most concerned with laying out the arguments for and against higher-*order* quantification, which he already gestured at in Rayo & Williamson (2003, §7).

Florio (2010, §4.3) investigates the relationship between plural syntax and semantics, and argues that one should opt for a plural syntax with a singular semantics. Florio treats superplurals as an alternative semantics for plural predicates. In the course of doing so, he adopts and simplifies the formal system presented in Rayo (2006). It is also in the context of semantics that Nicolas (2008) mentions superplurals; he wants to use them as semantic values for mass nouns.

There are some rival theories for superplural phenomena in linguistics, notably cover semantics (Gillon, 1987; Schwarzschild, 1996), lattice-theoretic accounts (Link, 1983) and accounts of groups (Landman, 1989). I present them in detail in 2.1.4.

Our project is not so much concerned with superplurals in the semantics; we want to look at superplurals *per se*. A treatment of superplurals *per se* can be found in the postscript of Oliver & Smiley (2013). The book is an extensive discussion of plural logic, and their presentation of superplurals is adopted here, both logically and philosophically. Their section §8.4 presents abundant examples of superplurals in natural language, and §12.2 sketches the account of superplurals and raises some open questions. We take that to be the foundation of this thesis.

The idea of superplurals has been met with increasing resistance. Ben-Yami (2013) has argued that superplural quantification is misguided, mainly because it is unnecessary to account for the corresponding phenomena in natural language. Many of the issues raised in Oliver and Smiley's book were summarised and put into a linguistic context in a review by Rieppel (2015). Among other things, he shows that existing linguistic theories can be regarded as notational variants of Oliver and Smiley's (super)plural logic. Their arguments are assessed in chapter 2.

Simons (1982) is one of the earliest authors to talk about pluralities, which he called 'manifolds'. In that discussion, which dates back to 1982, he rejected the idea of superplurals (‘manifolds of manifolds’) for lack of linguistic evidence. However, recently, he has argued that the idea of higher-order multitudes (again akin to pluralities) is both coherent, helpful, and distinct from set-theoretic equivalents (Simons, in press). This recent work provides some examples from natural language, argues for important differences to set theory, and outlines a logic of multitudes.

1.4 Synopsis

The discussion of superplurals divides roughly into three parts: language, logic, and philosophy. Chapter 2 dives right into the linguistic debate on superplurals. We look at examples in natural language and the existing linguistic accounts thereof. Is the idea of superplural logic intelligible to begin with? How does it fare in comparison to alternative accounts? Having established a linguistic foundation for superplurals, we go on to construct the superplural hierarchy. Chapter 3 presents the logic of superplurals: its syntax, semantics, and relevant features. Finally, in chapter 4, we discuss the philosophical implications of superplural logic, including first and foremost, an examination of different notions of ontological innocence. It is argued that, all things being equal, superplural logic indeed has the conceptual advantage it claims to. Chapter 5 will be the conclusion.

Chapter 2

Language

> There are, of course, no super-plural terms and quantifiers in English.
>
> Agustín Rayo (2006, p. 227)

The existing debate on superplurals has focused on the fundamentals: their intelligibility. On the one hand, there is the worry that superplurals are not meaningfully different from plural logic—superplural quantification simply collapses into ordinary plural quantification. On the other, it is unclear whether we can create a hierarchy of pluralities without the additional ontological baggage of set theory. Taken together, these criticisms put tension on superplural logic: it needs to adjudicate between the two positions they aim at. We will call this objection **intelligibility** in general.

But suppose, for a moment, that we have settled the issue of intelligibility. Does the intelligibility of a superplural hierarchy justify it? Take the thought experiment by Hazen (1997, p. 247), who uses the term 'perplural' for superplural:

> As a semi-serious example, pretend our plural endings on nouns are iterable: then we could assert the existence of infinitely many cats by saying something like:
>
> > There are some catses such that for each cats among thems there are some cats among thems including at least one more cat.
>
> (Note that the first occurrence of 'are' is perplural and the second merely plural, that 'each' is being used in construction with a plural noun, and that 'some' is used both plurally and perplurally.) Such languages are describable, and in some sense possible: we could imagine Martians speaking one, even if the genetically determined make-up of the language centres of the human brain make it impossible for Earthlings to learn them as first languages.

Of course, the mere possibility of superplurals does not translate into sufficient plausibility. For most authors, its legitimacy depends on our linguistic practice. Thus, the typical response of the superplural logician is to present examples of superplural terms in natural language. The second group of critics find these examples less convincing, and concludes that there is *no need* for superplural quantification. We will call this objection **naturalness**.

Note that these two concerns are very different. **Intelligibility** states that higher-level plural quantification is conceptually incoherent: it does not do what is required of it. **Naturalness** rejects the need for a superplural logic, even if it is perfectly coherent. I address these topics separately in this chapter. I start with a review of the arguments against **intelligibility**. In responding to the criticisms, I will develop a notion of superplurals that is, I claim, intelligible. I end the first part with an argument against the proposed alternative accounts. In the second part, I start with briefly assessing the objection from **naturalness**. I then discuss the existing examples of superplurals in natural language, and venture into other languages for further evidence.

2.1 Understanding superplurals

The word 'plurality' belongs to a cluster of words that are syntactically singular but semantically plural. That is, even though we use the grammatical singular, our semantic valuation is plural, i.e. it consists of one or more individuals. Other examples are 'group', 'collection', 'team', 'orchestra', 'gang', and 'cluster'. Traditionally, plural logic takes these to be pseudo-singular terms, and contrasts them with *sets* (both syntactically and semantically singular).

However, there are subtle differences between pseudo-singular terms as well. Consider modal and temporal contexts for groups vs. for pluralities. 'The people currently in this room' is a concept under which I fall but might not have fallen. Moreover, it is a concept under which I won't fall in the near future. By contrast, it seems that being one of some things is never contingent. Take, for instance, the 'Twelve Cellists of the Berlin Philharmonic'. It denotes an ensemble consisting of 14 cellists from the Berlin Philharmonic Orchestra—not twelve.[1] Now, this plurality exists because the 14 cellists exist. So 'being one of the 14 cellists of the Berlin Philharmonic' is necessary for the current members of the ensemble. Yet being one of the Twelve Cellists of the Berlin Philharmonic is very much a contingent matter; it requires years of training and a little bit of luck. This is also true for temporal contexts: the ensemble allows for fluctuation in members, and the current members of the Twelve Cellists of the Berlin Philharmonic have not always been part of the ensemble. In other words: being a plurality is necessary but not sufficient for concepts like orchestra and team. We may call them *groups*.

The issue of groups has generated a debate on the side (Landman, 1989; Uzquiano, 2004b; Effingham, 2010; Ritchie, 2013). While some argue that there is more to groups than an ordinary plural analysis (compare the Supreme Court and its justices), others think that groups are special kinds of sets (e.g. sets of time-indexed sets). The linguistic intuitions also diverge from country to country—compare 'Italy is playing very well' to the more British-sounding 'Italy are playing very well.' We will not digress here, and simply claim that, whatever a group is, it is *at least* a plurality.

[1] This group had twelve founding members but did not bother to change their name when they expanded.

The fact that the existence of a plurality depends solely on the existence of its members already tells us a lot about pluralities. It points to **unrestricted composition**, **determinacy**, and **extensionality**. Moreover, we observe that if the individuals are concrete, then the plurality with those individuals is concrete as well. The plurality of the people currently in this room is also in this room, unlike the set of people in this room, which is an abstract entity[2] (Simons, 2011, p. 5). Another major difference between pluralities and sets is that there are no empty pluralities, for obvious reasons—there is nothing that is no things. Based on this discussion, pluralities are understood as having the following properties:

Unrestricted Composition For any combination of individuals, there is a plurality of them.

Determinacy For a plurality P and any object a it is determinately true or determinately false that a is a member of P.

Extensionality Pluralities are identical when and only when they have the same members.

Multitude Unlike sets and sums, a plurality denotes several things at once.

Concreteness A plurality is nothing over and above its members.

Superpluralities such as 'the green Froot Loops and the pink Froot Loops' should, in the spirit of constituting a plural hierarchy, preserve these properties of pluralities.

In what follows, I will examine the objections against **intelligibility**. These are what I call singularisation and collapse, corresponding to the two worries I mentioned in the beginning of the chapter. I then formulate another objection revolving around the idea of iteration. In 2.1.4, I go over alternative accounts for superplural phenomena in language, and argue that our conception is the better one.

2.1.1 Singularisation

When moving up the hierarchy, set-theoretic instincts inevitably interfere with plurality-talk. A common interpretation of a superplurality is as a set of sets. That is, to superplurally quantify is to create singular entities that serve as pseudo-individuals for plural quantification. To illustrate, we can think of (4) (from example (3) above) as (5):

(4) The green ones and the pink ones taste bad together.

(5) *TasteBadTogether*({green Froot Loops}, {pink Froot Loops})

That is, we have formed two sets of Froot Loops and inserted them into argument places for ordinary individuals. There are two options for the next step: (a) proceed with ordinary plural quantification over these sets, or (b) form a bigger set with these two elements and continue with ordinary *singular* quantification. This gives us a set of sets. Those who propose

[2] The status of impure sets is debated in the literature (Lewis, 1986; Maddy, 1990).

this singularising move would most likely go with (b), since there is no good reason to stop at (a) if you can get rid of *both* plural and superplural quantification.

Such an account, however, 'would be a serious mistake' (Rayo, 2006, p. 227). A plurality is not another 'thing'; the argument from ontological innocence hinges on this fact (cf. chapter 4). Quantification over pluralities is understood as abbreviations of longer descriptions, and shouldn't be mistaken for singular quantification over surrogates. Linnebo (2014) has a good example: the claim that 'all pluralities are non-empty' can be rewritten as 'whenever there are some things xx, there is something u which is one of the things xx'. There is no reference to pluralities as entities themselves in this longer description.

In 1.1, we glossed over the reason any attempt to create surrogates for pluralities is misguided. If, in the course of making superplural denotation coherent, we changed the subject of plural denotations from individuals to a single entity 'composed of' individuals, it would be taking one step forward and two steps back. Therefore, both friends and enemies of superplural logic agree that singularisation must be avoided. Let us dub this the '**no surrogates**' doctrine, and move on.

2.1.2 Collapse

A plurality is not a singularity, so much is clear. But if pluralities are mere syntactic abbreviations for plural quantification over individuals, then what are superpluralities? This conceptual problem is picked up by Ben-Yami (2013, p. 83): 'if a plurality is not thought of as a singular entity but as many entities, then many pluralities are still a plurality'. In other words, how can the superpluralist reconcile the notion of a genuine plurality (**no surrogates**) with the structure needed to make superpluralities meaningfully different?

Ben-Yami thinks that superplural denotation *always* collapses to ordinary plural denotation. Compare the following sentences:

(6) My children, your children and her children played against each other.

(7) My children, your children and her children first had ice cream and then played against each other.

(6) purports to be a superplural predication. However, 'having ice cream' is considered a distributive predicate. So 'when hearing the speaker utter the second predicate in (7), i.e. [the predicate in (6)], we would need to reinterpret the noun phrase, which would thus be ambiguous; but nothing of the sort seems to be the case' (Ben-Yami, 2013, p. 87). For him, this is evidence that both sentences have the same subject: the six children.

This is an odd argument, seeing that we seem to cope well with a 'reinterpretation' in ordinary plural cases like the following:

(8) Russell and Whitehead were logicians and wrote a multivolume treatise on logic together.

Ben-Yami, who embraces ordinary plural logic, should find this equally problematic, since the second collective predicate forces the reader to 'reinterpret' the subject. Now Ben-Yami might say: in this example, both predicates apply to the same plurality, namely Russell and Whitehead. But then we can respond: very well—both 'eating ice cream' and 'playing against each other' apply to the same superplurality in (7). Unless Ben-Yami can point to a difference in the two cases, he hasn't shown anything with the example.

Ben-Yami has another argument though, taken from Linnebo & Nicolas (2008, p. 194–195): if 'my children, your children and her children' just refer to the six children, it is bound to give rise to incorrect truth conditions. For example, one could refer to my children, your children and her children as 'the boys and the girls' in the right context; but if we substitute this noun phrase for the one in (7), we get a sentence that does not necessarily have the same truth-value, i.e.:

(9) The boys and the girls played against each other.

But this argument is, again, vulnerable to a more general challenge: why should we think that intersubstitutability *salva veritate* is good in the first place? 'Superman is famous' is true. We also know that Superman is Clark Kent, and yet a substitution does not preserve truth-value. So this principle does not even apply to singular terms.

However, Ben-Yami does have a point. We need a criterion for superplural co-reference. In the superplural picture, 'my children, your children and her children' and 'the boys and the girls' are co-referring only with regard to the individuals; they are not with regard to pluralities. To illustrate, suppose only my children are wearing yellow, only your children are wearing blue, and only her children are wearing green. Then

(10) The children wearing yellow, the children wearing blue, and the children wearing green played against each other.

would be a legitimate case of co-referring terms between (6) and (10). These terms are intersubstitutable *salva veritate*. Based on this, a natural suggestion is:

Co-reference Two plural terms of level n are co-referring iff all its pluralities of level $0 \ldots n-1$ are co-referring, respectively.

Florio (2010, p. 131–33) raises a related issue with substitution *salva veritate*. Take a list of the form 'aa, bb, cc', where each of 'aa, bb, cc' stands for a plural term. We would claim that the list denotes a superplurality. But, Florio says, if aa and cc are co-referring, then the list denotes the same superplurality as 'aa and bb'. Now consider the following three sentences.

(11) The square things, the blue things and the wooden things overlap.

(12) The square things and the square things overlap.

(13) The square things overlap.

(11) is Linnebo & Nicolas's go-to example: the noun phrases cannot be understood as a single plural term, referring to the plurality of the square things, the blue things, and the wooden

things. Rather, it must be a superplural term. (12) is an instance of a superplural term with two co-referring pluralities. (13) has only one occurrences of 'the square things'.

Florio claims that (12) and (13) can differ in truth-value, and superplural denotation cannot explain that, because the two sentences have the same reference. Presumably, the reading that Florio wants to highlight in (12) is having two argument places rather than one. (12) can be read as '*xx* overlaps with itself' (always true), while (13) reads 'the *xx* overlap' (not always true). Thus, *overlap* is sensitive to the number of argument places.

Given the way we just defined co-reference, it isn't clear whether 'aa, aa' and 'aa' are co-referring. Taken at face value, one is a superplural term while the other is plural, so these should not be compared to begin with. But the problem remains with, say, 'aa, aa, aa' and 'aa, aa'. Put in more mathematical terms, we need to ask: does there need to be a bijection between the (lower-level) pluralities for two terms to be co-referring?

To answer this, we again look at the singular-plural case—this simplifies matters, and it shows us how general the problem is. Now, do 'a, a' and 'a' co-refer? While they certainly both refer to the same individual, the repetition in the first term can be relevant in a lot of cases. Among other reasons, this is why Oliver & Smiley (2013, §10) offer a strategy for the 'list treatment' of ordinary plurals. They solve the problem with indexing (pp. 169-170), which can be directly applied to superplurals: we insist that (12) is of the form '*Overlap*(aa_1, aa_2)' while (13) is of the form '*Overlap*(aa_1)'. In other words, repetition in a list is non-trivial.

This solution suggests that we indeed need bijection for higher-level co-reference to be met. We will return to this question later. For now, we have seen that Florio's worry opens up a bag of more general problems, and there is no consensus about how to solve them. But there is a standard move to be made, which we will help ourselves to. (I give a longer inspection of Florio's own analysis in 2.2.4.)

In the course of this section, Ben-Yami and Florio have each offered arguments for the collapse of superplurals, and we showed that they don't stand up to scrutiny. But there is more to say about collapse in general. First, we observe that collapse is a common phenomenon that occurs with all sentences containing distributive predicates. One feature of distributive predicates is that they have upward and downward implications (Oliver & Smiley, 2013, p. 113). Take the sentences

(14) Brouwer, Heyting, and Beth were logicians.

(15) Each of Brouwer, Heyting, and Beth was a logician.

(14) downwardly implies (15), while (15) upwardly implies (14). In fact, there are even 'intermediate' sentences between them: that Brouwer and Heyting were logicians, for example. Upwards and downwards implication are interesting phenomena on their own. In the present context, we should merely emphasise that the implication does not only go downwards, and that a lot more needs to be said if one were to regard one sentence as more basic than the other. I refer to Oliver (1999, 2010) for discussion.

A sentence with a superplural predicate that is distributive on both levels *can* collapse into a sentence with ordinary plural predicates, in this sense. As an example: 'Whitehead and

Russell, and Hilbert and Bernays were logicians.' But this collapse only means that superplurals aren't required for 'doubly distributive predicates' (Oliver & Smiley, 2013, p. 278). For a superplural sentence not to collapse, either or both levels of its denotation needs to be collective. Sentences (4), (6), and (11) all belong in this category. And more generally, for a sentence with a plural term of level n not to collapse into another sentence at all, we require at least $n/2$ collective references alternating with distributive references, starting with a collective reference at level n. This is an important insight, but it does not show that superplural denotation *has to* be collective on so-and-so many levels. To reiterate: the possibility of collapse doesn't imply that the higher-level reference is somehow less correct.

To sum up, our response here is twofold: first of all, not all cases of superplurals can collapse into ordinary plural denotation; and furthermore, even in the cases where collapse happens, it isn't clear why the collapsed sentences should be more fundamental. Let us now examine a third objection against **intelligibility**.

2.1.3 Iteration

Ben-Yami's objections against superplural logic do not stop here. The problem of superplural logic starts with a specious notion of iteration, he claims. Linnebo & Nicolas (2008, p. 186) stress that superplural quantification is the result of iterating the move from the singular to the plural. And Uzquiano (2004a, p. 438) also claims that superplural quantification 'would be a variety of quantification related to plural quantification as plural quantification is related to singular quantification.' But, Ben-Yami asks, if a singular term refers to *a single individual*, and a plural term refers to *more than a single individual*, then shouldn't a superplural term refer to *more than more than a single individual*? This kind of reference is 'either meaningless or synonymous with *more than two individuals*' (p. 83), i.e. the only meaningful way to read it is as ordinary plural reference over at least three individuals.

To be sure, the friend of superplural logic would readily admit that 'more than more than' isn't grammatical, just like Russell's term, 'many manys'. Ben-Yami finds iterating from the plural to the superplural baffling because this expression, 'more than more than', is linguistically odd. That does little to show that the concept of superplurals is incoherent—there are plenty of linguistic oddities that have no logical relevance. Nevertheless, it is worth dwelling on this criticism.

There are justified sceptical worries about Linnebo's repeated characterisation of superplurals as 'iterated plurals'. We can perhaps formulate them more precisely than this. First, is the characterisation intended as an analogy? If so, how do we apply the analogy? Second, is the move from the plural to the superplural even capable of being iterated? Would it change our conception of plurals in any way? Third, and most importantly, can the tension between collapse and singularisation, as mentioned at the beginning of this chapter, be resolved? Before we turn to each of these concerns, however, it is worth pausing to explain what sort of sceptic would raise them in the first place.

Our sceptic is one who, like Ben-Yami, accepts plural logic but finds superplural logic incoherent. The goal at this point is not to convince someone who is sceptical of plural logic at large to endorse superplural logic; we merely want to show that the position of being amenable to first-level plural logic but not to higher-level plural logic is *unstable*; that is, either the reasons to reject superplural logic are strong enough to reject plural logic also, or the reasons to accept plural logic are strong enough to also accept superplural logic.

We basically made the second argument when we introduced superplurals as a sort of 'terms of terms'. This way of thinking about superplurals can be attributed to Linnebo, who first mentions iteration in 2003. To quote,

> I argue that the considerations that allow us to add the theory of plural quantification to first-order theories are strong enough to support iterated extensions of this sort as well: These considerations allow us to add higher and higher levels of plural quantification (Linnebo, 2003, p. 84–5).

This is also how he introduces superplurals in his 2008 work with Nicolas:

> A natural question that arises is whether the step from the singular to the plural can be iterated. Are there terms that stand to ordinary plural terms the way ordinary plural terms stand to singular terms? Let's call such terms *superplurals*. A superplural term would thus, loosely speaking, refer to several 'pluralities' at once, much as an ordinary plural term refers to several objects at once (Linnebo & Nicolas, 2008, p. 186).

This passage suggests that the relationship between superplural and plural quantification is similar to the relationship between plural and singular quantification. It suggests that superplural quantification is just plural quantification over pluralities, which is bad in two ways: (i) if superplural quantification is just plural quantification, then why use it at all? and (ii) plural quantification, *qua* plural logic, ranges over individuals, so wouldn't this account singularise pluralities after all? No surprise then, that Ben-Yami finds iteration 'probably incoherent' (p. 82).

Others seem to think the same. Following Rayo (2006, p. 227), Florio (2010, p. 154) concludes that singularisation (as discussed in 2.1.1) would make superplural logic far less ontologically innocent than proclaimed:

> If plural quantification commits us to the things it plurally quantifies over, i.e., objects, then superplural quantification should commit us to the entities it plurally quantifies over, i.e., pluralities. So superplural quantification would be ontologically committing, since it would commit us to *pluralities as entities*.

Rayo suggests a second analogy: we should think of superplurals relating to plurals as third-order logic relates to second-order logic. With regard to ontological innocence, Florio (2010, ibid.) says it is 'not obviously helpful', for higher-*order* logic introduces newer forms of commitment. However, commitment or no commitment, this second analogy gets something right—and something terribly wrong.

It correctly describes superplural logic as a semantic ascent; comparing it to higher-order logics gives it its proper hierarchical characterisation. This is the shortcoming of Linnebo's singular-plural analogy, for it does not capture that superplural logic is of a higher level than plural logic. But comparing a superplural predicate to third-order logic is a mistake. It follows Boolos's tradition of lobbying for plural logic as an alternative to second-order logic. This is done by interpreting monadic predicates as pluralities (and often vice versa), a strategy we generally call 'predicative analysis'. The details of this strategy will have to wait until 3.4. Now we will simply cite the fact that monadic second-order logic is *not*, contrary to popular belief, equivalent to plural logic. Example:

$$\forall xx Fxx \therefore \forall x Fx$$

is valid in plural logic (assuming that F is distributive), but cannot be translated into second-order logic. For F, when applied to a plurality, will apply to a *predicate* and is therefore second-order. When F is applied to an individual, it becomes a plain first-order predicate. Thus there is no logical connection between the sentences, and the translation cannot preserve consequence (Oliver & Smiley, 2013, p. 239).

What, then, is the appropriate analogy? After all, *something* has to iterate for a hierarchy to exist. The answer is not difficult, but will require some rethinking.

Singular and plural terms differ in the number of things they refer to. Oliver & Smiley interpret plural terms as denoting one or more things, or zilch.[3] This definition makes a plural term 'the opposite' of a singular term—together, they are *exclusive and exhaustive*. To quote,

> How, then, to make a robust distinction between singular and plural terms? Our answer is that they may be distinguished, semantically and modally, by the number of things they are *capable* of denoting. A singular term cannot denote more than one thing on any occasion, a plural term may denote several. The interest of this classification is that it is exclusive and exhaustive: plural is the opposite of singular. This does not mean that a plural term actually denotes more than one thing: it only has to be capable of doing so (Oliver & Smiley, 2013, p. 74-5).

Higher-level plural logic does not have this distinction. Both plural and superplural terms can refer to more than one thing. This is why iteration is troublesome: we don't know what aspect to iterate.

Nonetheless, both exhaustivity and exclusivity can be preserved within the superplural hierarchy. This requires higher-level thinking, so to say. Going upwards means that we do not consider the number of referents, but the level of the reference. For this hierarchy to be exhaustive, we need the possibility to refer on each level. This will be guaranteed by terms of level n, where n is a natural number. Exclusivity is understood as being capable of referring on the next level. A level n term cannot refer on level $n+1$ or higher (but on all levels $0 \ldots n-1$). Thus, the superplural hierarchy is jointly exhaustive and pairwise exclusive with regard to the *level* of its referents. (This has repercussions for our hierarchy—but more on that later.)

[3]We will use ⊘ for 'zilch' as a term that is empty as a matter of logical necessity. In plural logic it may be defined as 'the non-self-identical things' using plural description and identity.

Does this account apply also to the singular-plural distinction? Whereas previously we took the number of referents to be the distinguishing factor, we now have to say, for the sake of consistency, that the *level* of reference distinguishes singular from plural denotation. This should not sound too strange. Singular terms are terms of level 0, so can only refer to individuals or zilch. Plural terms are terms of level 1, so can refer to pluralities, individuals or zilch. The difference between pluralities and individuals still remains unchanged, so the zero and first level distinction can be characterised by 'the number of referents' as well. This means that the traditional singular-plural distinction becomes a special case within superplural logic.

Our use of levels is—and this cannot be stressed enough—entirely a manner of speaking, convenient because 'plurality of level *n*' is shorter than 'individuals superplurally referred to at level *n*'. When it comes to expressing the intuition behind plural logic, Hazen's vernacular is perhaps most natural. Singular and plural terms differ, traditionally, in the number of *things* they can denote. And so we may say that plural and superplural terms differ in the number of *thingss* they are able to denote; analogously, superplural and supersuperplural terms differ in the number of *thingsss* they are able to denote, and so on. This way, we don't impose a level distinction onto plural logic in ordinary speech.

This concludes our explanation of Linnebo's argument for superplurals. This argument may also work in the opposite direction. If superplural logic is to be rejected (for whatever reason), then we may argue that plural logic should be rejected for the same reason. Either way, the sceptic's position is unstable.

2.1.4 Alternatives

Now it is time to address the critic who recognises the coherence of superplurals but thinks that there is a better account of them. Here are the alternative accounts of higher-level quantification. They include articulated reference, cover semantics, and sum-based semantics. At the end of this part, I offer what I call 'an argument from structure' against these accounts.

Articulated reference

Ben-Yami's solution to collapse is his own account of *articulated reference,* which is a way a plural term could refer to a plurality. He defines articulation as follows: 'the reference of an expression is articulated a certain way if the plurality is referred to by means of referring to specific sub-pluralities of it' (p. 91). This way, he can explain the different 'groupings' of individuals without having to appeal to superplurals.

To understand this view, we must first accept that articulation is determined by the *syntactic structure* of the term alone: 'a referring expression can refer to a plurality by virtue of containing other referring expressions that refer to some of that plurality' (p. 89). Now consider a plural expression containing nested lists, e.g. in the sentence

(16) Whitehead and Russell, and Hilbert and Bernays are joint authors of multivolume treatises on logic.

According to Ben-Yami, the expression 'Whitehead and Russell, and Hilbert and Bernays' contains a part referring to Russell ('Russell'), a part referring to Whitehead and Russell ('Whitehead and Russell'), but no single part referring to Russell and Hilbert: 'Russell, and Hilbert', is not a 'structural element' (read: grammatical component) in the expression. Thus, its reference is articulated into a phrase referring to the first two and one referring to the last two.

Ben-Yami thinks that articulated reference cannot be identified with higher-level plurals, because (i) ordinary plural reference can be articulated as well; (ii) it does not depend on a specious notion of iteration (see above); and (iii) we merely consider ordinary plural terms, and observe that their reference may be articulated in different ways.

This account is intriguing. First, in his definition and subsequent example of plural reference, 'contain' seems to reduce to 'explicit mention': 'Russell and Whitehead' contains 'Russell'. Two puzzles arise: (a) Which structural or grammatical rules does containment obey? For example, why exactly does 'Whitehead and Russell, and Hilbert and Bernays' not contain 'Russell, and Hilbert'? (b) How do plural terms such as 'the Boswell Sisters' refer to individuals (Martha, Connee, and Helvetia) without their explicit mention? But supposing that containment isn't explicit mention, it is still difficult to see what exactly sub-pluralities are. Are sub-pluralities pluralities? Are they related to pluralities as sets are related to a set of sets? If so, then how is this account immune to the singularisation objection mentioned earlier (2.1.1)?

It is noteworthy that Ben-Yami rejects superplural denotation as a superfluous form of reference, while proposing his own novel form of reference—it is unclear what was gained. But his proposal isn't entirely misguided. Superplural reference should reveal something about the *structure* of the individuals referred to without adding ontological burdens. As to how exactly this can be achieved, we will see at the end of this section.

Covers and sums

Let us now turn to linguistic approaches to superplurals. Two frameworks emerge as promising accounts of superplurals: the set-theoretic analysis and the sum-based analysis.

The first is a set-theoretic approach to plurals in general that can be applied to superplural phenomena. Gillon (1987, 1992) and Schwarzschild (1996) have maintained that the interpretation of a plural noun phrase always depends on the choice of a cover. A cover C of a set S is a set of sets C_i whose union is (at least)[4] S itself:

$$S \subseteq \bigcup C_i, \tag{2.1}$$

To begin, we must form a reference set from the denotation of a plural term. Given such a reference set S, we define the domain of individuals as the power set of S minus the empty set. The singleton sets are the atoms, and we use the union operation and the subset relation to express different covers of the reference set. For instance, 'Hammerstein, Rodgers and Hart wrote musicals.' is true only with respect to the cover {{Hammerstein, Rogers}, {Rogers,

[4]This is the standard definition of a cover; in our case, as it will be clear, the union must be exactly S.

Hart}}. If we take our example from (16) again, we would invoke the cover {{Russell, Whitehead}, {Hilbert, Bernays}}. In this view, superplurals are just more complex covers, and their analysis does not require more than the machinery we use for ordinary plurals.

Specifically, cover semantics postulates a distributivity operator $\mathscr{D}$ which attaches to a predicate, allowing it to apply to the elements of a contextually provided cover of the reference set. Instead of $x : Fx$, our notation for 'the things that individually F' (cf. 3.2.1), we get the following:

$$\mathscr{D}(F) = \lambda x[\forall y(y \in C_x \to F(y))] \tag{2.2}$$

For ordinary plurals, saying 'Russell and Whitehead were logicians$^{\mathscr{D}}$' amounts to saying

$$\forall y(y \in \{\text{Russell, Whitehead}\} \to \mathit{Logician}(y))$$

where the cover is just S itself. In the 'superplural' case, we need to first determine which *contextually salient* cover is invoked. Then the definition tells us that 'the men wrote musicals$^{\mathscr{D}}$' is equivalent to

$$\forall y(y \in \{\{\text{Hammerstein, Rogers}\}, \{\text{Rogers, Hart}\}\} \to \mathit{WroteMusicals}(y))$$

Another framework is a sum-based approach, proposed by Link (1983) and later advanced by Krifka (1989) and Landman (1989, 1996, 2000). As opposed to cover semantics, this framework has both plural and singular quantification share a domain, i.e. the arguments have the same type. Everything else is very similar. The *sum* of a and b is the smallest entity that has a and b as its parts. The following is in part presented in Landman (1989, §1), Nouwen (2014, §1.2), and Rieppel (2015, p. 4).

Let $\sqcup$ be a binary operator over a domain D with the properties:

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma) \quad \text{associative} \tag{2.3}$$

$$\alpha \sqcup \beta = \beta \sqcup \alpha \quad \text{commutative} \tag{2.4}$$

$$\alpha \sqcup \alpha = \alpha \quad \text{idempotent} \tag{2.5}$$

This summation operator allows us to define a partial order on D:

$$\alpha \leq \beta \text{ iff } \alpha \sqcup \beta = \beta \tag{2.6}$$

Thus, we are working in a complete atomic join semi-lattice where the summation operator is just a join operator. (The lattice is complete because we want the domain closed under arbitrary $\sqcup$, and atomic because we want all atomic parts of sums to be in the domain as well.)

Nouwen (p. 6) also offers a way to define all atoms in a domain:

$$\mathit{Atom}(\alpha) \text{ iff } \forall \beta \leq \alpha\ [\alpha = \beta] \tag{2.7}$$

$$\mathit{Atoms}(A) = \lambda \alpha. \alpha \in A \wedge \mathit{Atom}(a) \tag{2.8}$$

(We should perhaps add that α cannot be 'zilch' either.) Figure 2.1 is an illustration of the complete atomic join semi-lattice with three atomic elements. All arrows represent the $\leq$ relation.

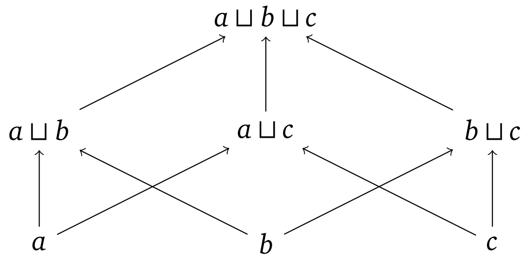

Figure 2.1: a complete atomic join semi-lattice

Unsurprisingly, this semi-lattice can be mapped onto cover semantics in a straightforward manner, where the domain is S itself, with summation the union operation, inclusion the subset relation and atoms the singleton sets.

$$\langle D, \sqcup \rangle \text{ is isomorphic to } \langle \mathscr{P}(Atoms(D)) \setminus \varnothing, \cup \rangle \tag{2.9}$$

In fact, Schwarzschild (1996) believes his set-theoretic interpretation of plural noun phrases does not significantly differ from the summation interpretation proposed by Link (1983). However, the sum-based approach is supposed to make the ontology more innocent: a domain that contains the individuals a, b, c will also contain the sums of them, $a \sqcup b,\ b \sqcup c,\ a \sqcup c,\ a \sqcup b \sqcup c$.

Importantly, Link's sum closure operator $*$ forms the closure under summation for any atoms:

$$*X \text{ is the smallest set such that: } *X \supseteq X \land \forall x, y \in *X : x \sqcup y \in *X \tag{2.10}$$

$$[\![boys]\!] = *[\![boy]\!] \tag{2.11}$$

This, in turn, allows Link to give a uniform account for singular and plural definite descriptions. 'The *F*' picks out the unique maximal individual in the extension of F, if there is one. For instance, if the extension of 'logician' is {Hilbert}, then 'the logician' will denote Hilbert. If it is {Hilbert, Russell }, then 'the logician' will be empty (since there is no unique maximal element). Now, 'the logicians', on the other hand, the extension of 'logicians' will be {Hilbert, Russell, Hilbert ⊔ Russell}, it will denote the maximal element, Hilbert ⊔ Russell. We may call this a maximalist semantics. (For a critique, see Oliver & Smiley (2013, §8) against Sharvy (1980).)

This lattice-theoretic theory, as it stands, does not offer an analysis for superplurals; but it can be easily extended to account for them based on the strategy in cover semantics. As Rieppel (2015, p. 7) points out, the maximalist could again appeal to *contextual salience*, which would help pick out a 'superplural' extension, e.g. {Russell ⊔ Whitehead, Hilbert ⊔ Bernays} for the predicate 'the most famous joint authors of multivolume logic books'.

The question, then, is whether sums are indeed more 'innocent' than sets, seeing that, in this instance, they are formally isomorphic. It seems that, just like a set, a sum is one thing rather than many. And moreover, just like a set, a sum can only be 'divided up' in one way (we are not talking about *mereological sums* here at all). In the Russellian argument against singularisation, sums are also treated as set-like entities (Schein, 2006). If anything, the isomorphism suggests that sums are merely a notational variant of cover semantics. So for our purposes, it is enough to consider the latter as the main linguistic alternative to our superplural analysis.

Even though both frameworks are popular in linguistics, neither of them were explicitly used for superplurals. It was Linnebo & Nicolas (2008) who transformed the set-theoretic framework to a real alternative against superplural analysis (which they then discharged). They explain the alternative as follows:

> For on this analysis, the plural noun phrase 'Hammerstein, Rodgers and Hart' denotes a plurality, not a superplurality. A superplurality is invoked only when the sentence is interpreted. The whole sentence makes a complex predication, whereby the property expressed by the verbal expression 'wrote musicals' is predicated of each plurality of a certain superplurality. Thus, on this analysis, plural noun phrases do not function as superplural terms, and verbal expressions do not function as superplural predicates. But the semantics of such sentences makes *covert appeal* to a superplural term and to universal quantification over the pluralities of the superplurality (i.e. the cover) that this term denotes (p.192).

Of course, covert appeal is still appeal, but Linnebo and Nicolas go on to suggest that collective predication over pluralities is immune to this sort of criticism, and therefore the best example of superplurals in English. Their example (11) has already been mentioned. Compare that with the following sentence:

(17) The things that are square, blue or wooden overlap.

According to cover semantics, the sentence induces the cover {{the square things}, {the blue things}, {the wooden things}} just as in (11). Yet it does not lend itself to a collective reading. It could be made true by one square thing overlapping with one wooden thing, or two blue things overlapping, for instance. Cover semantics fails to capture this difference in truth-value.

The argument from structure

What emerges from this discussion is an alternative picture. Instead of a hierarchy of pluralities, these alternative theories maintain only one ordinary plurality, and introduce what should be called subpluralities. In articulated reference, we are to think of them as mere groupings. In cover semantics, we have a full-on set-theoretic structure, where subpluralities are elements. In Link's lattice-theoretic semantics, sums are understood as plural objects *simpliciter*.[5]

[5] To be fair, though, Link does acknowledge at another place that the notion of a sum is 'an *inherently relative concept* ... you have to tell me first what "regular" things you are prepared to include in your domain of discourse, and then it's easy to say what the plural things are in your ontology' (Link, 1998, p. 326)

My argument against these theories is based on structural similarity. First, I will argue that articulation is a structure on a plurality, no matter what representation we use. Second, I will show that cover semantics needs a stronger structure to analyse superplural phenomena. Third, I will outline a translation between this stronger structure and the superplural hierarchy, and conclude that the superplural hierarchy is the better formalism.

Structures. Articulated reference is a new name for an old concept. Simons (1982, p. 191–2) already argued, when talking about *manifolds* (equivalent to what we call a plurality), that we need to create 'proxy objects' grouping individuals together. In this work, he does not see the need to introduce 'manifolds of manifolds' (though recently he changed his opinion). Linnebo seems to have a similar conception:

> For instance, the second-level plurality based on Cheerios organized as oo oo oo should be no more ontologically problematic than the first-level plurality based on the same objects organized as oooooo, although the former has an additional level of structure or articulation (Linnebo, 2003, p. 87–8).

These accounts suggest that higher-level denotation amounts to ordinary plural denotation with added structure. But what exactly is this added structure? Following Simons, we illustrate the difference using trees.

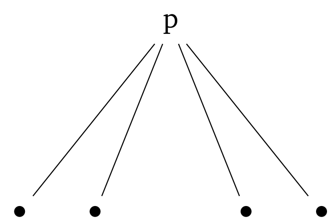

Figure 2.2: articulated reference

Figure 2.2 is the picture that articulated reference is getting at: we have plural reference *simpliciter*, yet there is some added structure to it: the distance between the dots suggest some form of grouping. Unlike an upward hierarchy, this conception is totally flat: there are only individuals arranged in a certain way. However, trees and drawings are pictorial representations of structures. Whether oo oo oo or o o o o o o represent the same thing depends on some translation between the pictures and our structural understanding of the items being shown. The mere lack of curly brackets cannot be an indication for ontological innocence! I suspect that once 'added structure' is accounted for, one cannot get around speaking of 'subpluralities' or similar structural entities. Thus, articulated reference and other accounts in this cluster are really hiding behind innocent-looking representations such as these to smuggle in structure for cheap. This is why we must treat them like other approaches that have committed themselves to more structure, albeit in the downwards direction.

Substructures. If this argument is right, then articulated reference and cover semantics share some important structural similarities (see figure 2.3). Whether we call the cells 'subpluralities' or 'subsets', we end up with one plurality that has been divided somehow. In this camp we find the linguists Gillon and Schwarzschild.[6] Others include McKay (2006, p. 47), who deems only certain groupings *salient*, and more recently Rieppel (2015), who believes that Oliver and Smiley's logic cannot accommodate for cases like 'Hammerstein, Rodgers and Hart wrote musicals'. Landman's group analysis is also an example of this sort of answer (Landman, 1989).

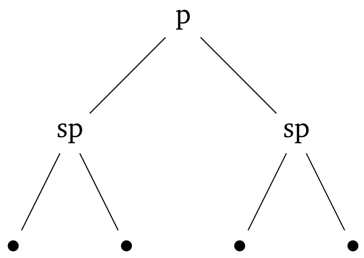

Figure 2.3: cover semantics

The point I wish to make is a theoretical one. It is true that superplural phenomena can be captured by covers. However, the formalism needs to be amended when we look at supersuperplurals. The reason is, essentially, a problem with nesting. Take the set-theoretic representation of a simple supersuperplural: {{{•,•}, {•,•}}, {{•,•}, {•,•}}}. Its union is {{•,•}, {•,•}, {•,•}, {•,•}}, which is a set of subsets, not individuals. Thus, supersuperplurals cannot be captured by covers anymore. To fix this, we need to take subsets of subsets into account, and introduce a downward hierarchy. Formally, this amounts to the following.

As before, only covers are required for second-level plurals.

$$\text{for } C^2 = \{C_i \mid x \in C_i \rightarrow x \in S\}, S = \bigcup C_i \tag{2.12}$$

For third-level plurals, we simply apply the union operator twice.

$$\text{for } C^3 = \{C_i \mid x \in C_i \rightarrow x \in \mathscr{P}(S) \smallsetminus \varnothing\}, S = \bigcup\bigcup C_i \tag{2.13}$$

And analogously for fourth-level plurals:

$$\text{for } C^4 = \{C_i \mid x \in C_i \rightarrow x \in \mathscr{P}(\mathscr{P}(S)) \smallsetminus \varnothing, \{\varnothing\}\}, S = \bigcup\bigcup\bigcup C_i \tag{2.14}$$

The rest of this downward hierarchy is straightforward. For each level we add a power set operation, and the 'higher-level covers' will need intermediate layers.[7] This concludes our rather quick characterisation of an extended cover semantics for higher-level plural phenomena.

Translation. The argument from structure is now within our reach. What is the conceptual difference between the downward 'subplural' hierarchy above, and the upward superplural

[6]For an opposing view, see Lasersohn (1995, §9).

[7]We have not even begun to look at mixed-level problems here, which would make the formalism more cumbersome to work with.

hierarchy we are defending? There is a suspicious similarity between these sub- and superstructures. In this section, we show that the frameworks are in fact interdefinable. The easiest way to do so is by using graphs.

Incurvati (2014b) has a ready framework for representing sets in pictorial terms. In particular, graphs can depict sets when we take nodes to represent sets and edges to represent (converse) membership. Standardly, the graphs will have to be directed, pointed, and accessible.[8] Every *decoration* is an assignment of set-theoretic elements to the nodes that preserves the membership relation. To illustrate, the set $A = \{\{b\}, \{b, c\}\}$ would be depicted as follows:

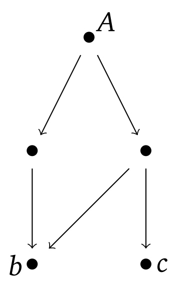

Figure 2.4: an example

In our subplural hierarchy, certain restrictions need to be put on the background set-theory. First of all, the Axiom of Foundation is required to ensure that we always have a lowest level (i.e. there are no infinite paths, no loops). This also gives us uniqueness: every well-founded graph has a unique decoration (Aczel, 1988, p. 4–5). Next, we will work in a set theory that has urelements, which constitute the reference set. Urelements are denoted by lower-case letters. Only graphs that have urelements at the end of every path are well-formed. The length of each path has to be the same, otherwise we will 'lose' shorter paths due to the standard definition of the union operator.

This is ultimately the translation we demanded from articulated reference: the graphs are interpreted in a specific way to represent structural properties, in this case set-theoretic properties. Can we do the same for the superplural hierarchy?

Things look very similar in the superplural hierarchy. Again, directed, pointed, and accessible graphs are used to depict higher-level reference. We begin with the individuals being referred to, and draw a node for every (higher-level) plurality referring to them. The edges are now interpreted as depicting the (converse) 'is/are among' predicate. This theory is also well-founded since we start with individuals. Lastly, and importantly, we will disallow so-called 'higher-level singletons', for reasons that will be explained in 3.1.1. This translates into the condition that each node must have at least two children. Other than that, the two hierarchies are very much identical.

Figure 2.5 compares the subplural ($\mathfrak{S}$) and superplural ($\mathfrak{P}$) hierarchies. It's evident that both depict the same structure, but do so in slightly different ways. We can define a transla-

[8] That is, 1) every edge has a direction, 2) every graph has one distinguished node, and 3) it is possible to reach each node of the graph by some finite path starting from said point.

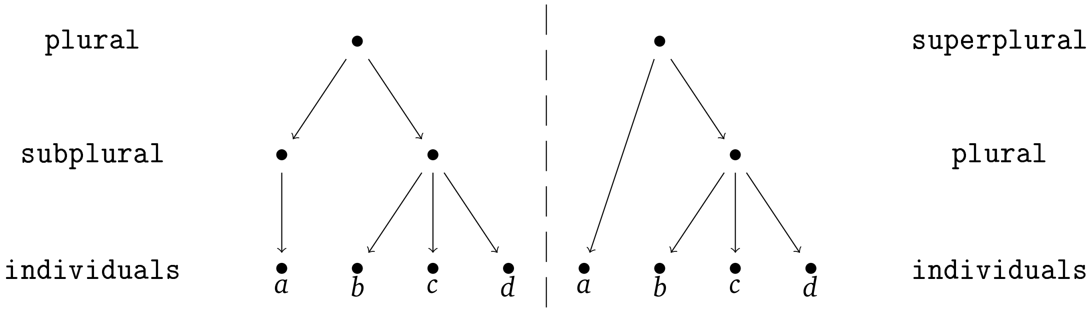

Figure 2.5: comparing the hierarchies (left: $\mathfrak{S}$, right: $\mathfrak{P}$)

tion ▹ from the language of the subplural hierarchy, $\mathscr{L}^{\mathfrak{S}}$, to $\mathscr{L}^{\mathfrak{P}}$, the language of the superplural hierarchy. ◃ will be the translation in the other direction.

Let n_i be the nodes in a graph, and $e_{i,j}$ the edges connecting nodes from i to j.[9] We take the decoration to be a function δ assigning individuals $(a, b, c, \ldots)$ or nothing to nodes. This already gives us the basic mapping of the two structures:

$$\delta \overset{\triangleright,\triangleleft}{\mapsto} \delta$$

$$n_i \overset{\triangleright,\triangleleft}{\mapsto} n_i$$

$$e_{i,j} \overset{\triangleright,\triangleleft}{\mapsto} e_{i,j}$$

The important difference is the treatment of higher-level singletons. We observe that ▹ has to delete 'singleton nodes' while ◃ has to add them at some (sometimes arbitrary) place.[10] For this we require two algorithms using some simple functions. First we need a function *#Children* that gives us the number of children for each node n_i. In the other direction, we need a function *Length* to measure the length of each path p_i, and a function *Prepend* that adds a parent to paths. (A path is a sequence of nodes each of which is a child of its predecessor, except for the first node, which is the root.)

For ▹, we add

if *#Children*(n) $! = 1$ **then**
  return $\triangleright(n)$
else
  n = child(n)
end if

[9] There is no unique enumeration of trees in general (Cayley, 1889).

[10] This means that corresponding graphs in the two hierarchies cannot not bisimilar in general; that would require reflexive closure in $\mathfrak{P}$ and transitive closure in $\mathfrak{S}$. While one could argue that $\preccurlyeq$ is in fact reflexive, $\in$ clearly cannot be transitive.

For $\triangleleft$, we add

if $Length(p) \geq max(Length(p_i))$ **then**
    return $\triangleleft(n)$
else
    repeat $Prepend(p)$
    until $Length(p) \geq max(Length(p_i))$
end if

(Intuitive gloss: whenever there is only one child for any node in a subplural graph, $\triangleright$ will simply delete the parent node and reconnect the edge. And whenever the length of paths are not equal in the superplural hierarchy, $\triangleleft$ will pad all shorter paths by adding a parent node until the maximum length is reached.) Thus, we have a way of going from one hierarchy to the other.

Non-equivalence. Finally, I want to argue for the superplural hierarchy. It might seem like the difference lies only in terminology; but ultimately, it is a point about different conceptions of the hierarchy.

The obvious points first. The subplural hierarchy has an undesirable side-effect: if second-level plurals are seen as covers on a reference set, then ordinary, first-level plurals would presumably just be the reference set.

$$\text{for } C^1 = \{x \mid x \in S\}, S = C^1 \tag{2.15}$$

This is, essentially, a singularist semantics for ordinary plural occurrences, which we wanted to avoid (see 2.1.1 above).

In addition, the extended cover semantics is a downward hierarchy, and because of that, suffers from constant name changes. Looking at the example above, the individual b, c, d can both form a plurality and a subplurality, not to mention subsubpluralities etc. On the other hand, b, c, d in the superplural hierarchy can only ever form a plurality. Their level is *fixed*, it does not depend on the context in which they occur. Whether this is preferable may be a matter of convenience, but it does seems more natural to move upwards rather than 'in between' when adding more levels.

And now to the minor structural difference. We take cover semantics to allow for higher-level singletons. (Again, the proper discussion can be found in 3.1.1, but the main idea is that it does not make sense, within superplural logic, to form superpluralities of one individual only.) The consequences are made clear again in our graph representation. Figures 2.6 and 2.7 depict two different plural references in $\mathfrak{S}$: $\{\{\{a\}, \{b, c\}\}\}$ and $\{\{\{a\}\}, \{\{b, c\}\}\}$, respectively. However, it is hard to pinpoint exactly how these references differ; as a matter of fact, the two graphs are bisimilar.

What does bisimularity mean? It is one way of comparing structural similarity between graphs such as these. In particular, it tells us that every path in one graph can be mapped to another one in the other graph. Depending on our view, bisimilarity could capture co-reference. Recall that we asked whether there needs to be a bijection between the lower-level pluralities of two co-referring terms in 2.1.2. If not, then bisimulation suffices as a criterion for co-reference.

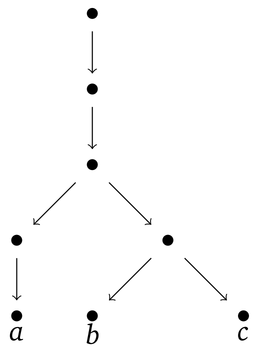

Figure 2.6: $\{\{\{a\},\{b,c\}\}\}$

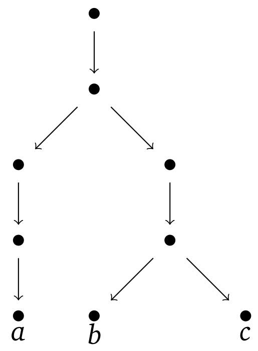

Figure 2.7: $\{\{\{a\}\},\{\{b,c\}\}\}$

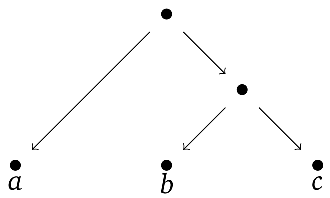

Figure 2.8: translation into $\mathfrak{P}$

Observe that the translation $\triangleright$ of both 2.6 and 2.7 yield figure 2.8 in $\mathscr{L}^{\mathfrak{P}}$. This means that $\{\{\{a\},\{b,c\}\}\}$ and $\{\{\{a\}\},\{\{b,c\}\}\}$ can both be denoted as the superplurality $a,[b,c]$ (informally put) when it comes to reference. And that would be the right move, because (i) it captures co-reference if bisimilarity is to be our criterion, and (ii) it simplifies the structure, thereby reducing the ontological commitments. This is the advantage of using $\mathscr{L}^{\mathfrak{P}}$.

To sum up, there is nothing gained by having other forms of reference. They, too, introduce hierarchies, and all worries against the superplural hierarchy also apply to them. I submit that superplural logic is better suited as a language for the plural hierarchy, as it explicitly reflects on its ontological commitments. More needs to be said on the notion of structure for this to be a satisfying answer. But for now it is important to note that the solution to ontological worries isn't to get rid of structure, but to have an adequate account for it. That we will attempt in chapter 4.

2.2 Superplurals in natural language

Let us now turn to the second objection against superplural: **naturalness.** This is the claim that **intelligibility** alone does not make superplural logic interesting. It is their locutions in natural language that matter. And, for that matter, there are no (good) examples of superplurals in natural language.

Before we look at concrete examples, however, let us think about this criticism. There are two easy ways to resist it. The first is to simply deny that we are in the business of justifying superplural logic as a logic for regimenting (parts of) natural language. Linnebo shares this sentiment. He deems the occurrence of superplurals in natural language irrelevant because we are more interested in the claim that one can iterate the step from the singular to the plural. The mere logical possibility of superplurals can already be philosophically significant. To quote,

> What really matters is presumably whether we can iterate the principles and considerations on which our understanding of ordinary first-level plural quantification is based: if we can, then higher-level plural quantification will be justified in much the same way as ordinary first-level plural quantification; and if not, then not (Linnebo, 2014, §2.4).

Of course, what he fails to mention here is that ordinary first-level plural quantification is often legitimised by its occurrence in natural language, and it isn't clear at all that iteration preserves this legitimisation. Nevertheless, the point holds: Rayo (2006) also uses the idea of a superplural hierarchy to argue that semantic theorising is unstable. For that, superplurals don't have to be natural.

Another more radical way to resist the second objection is to deny any connection between language and ontology. For example, Hazen (1993, p. 138; 1997, p. 247) made the point that we should not expect an intimate connection between natural language and logic. The idea goes back to the claim that grammatical form is not necessarily logical form. We should not preclude a superplural logic simply because superplurals are not superficially manifest in our grammar. For instance, 'GAMUT' is the author of *Logic, Language and Meaning*. It is also a collective pseudonym for five Dutch logicians who co-authored the textbook. The semantics is plural, the syntax less so. So, the argument goes, plural and superplural logic cannot rely too heavily on grammar anyway.

In fact, Linnebo (2003, p. 87) even argues that there might be independent reasons why higher-level plural locutions are scarce in natural languages. For instance, it is easier to conceptualise second-level objects rather than second-level quantification in English—we tend to plurally quantify over 'pairs' and 'groups' instead of superplurally quantifying over individuals. These proxy objects are easier to work with than 'cumbersome and unpractical' grammatical devices for higher-level quantification.

While the claim is interesting, the lack of evidence, both in Linnebo's writings and in the cognitive science and linguistics literature, makes it impossible to evaluate this claim. But we can say something about Hazen's position. 'Grammatical form misleads as to logical form' is a deeply problematic doctrine that has been abandoned by philosophical logic in the second half of the twentieth century (cf. Evans (1976); Oliver (1999) among others). The idea that a superplural analysis may reveal some 'hidden reality' in natural language is simply misguided. We have already seen that in the discussion on collapse: it makes no sense to say that one sentence is more fundamental than another (2.1.2). Likewise, it makes no sense to say that natural language doesn't have superplurals, but it *should*.

We can now turn to superplurals in English. The list of authors who deny the existence of superplurals in natural language is long. Lewis (1991, p. 70–71) thinks superplural quantification amounts to the dubious idea of 'infinite blocks of plural quantifiers'; and even that does not give us third- and higher-level plurals. Similar claims are made by Uzquiano (2004a, p. 239–40), McKay (2006, p. 46), and Rayo (2006, p. 227). Ben-Yami (2013, p. 101) writes that 'the literature failed to clarify in a satisfactory way what is meant by higher-level plural terms or to show that there exists in natural language a special form of higher-level plural predication'. Even Linnebo & Nicolas (2008), who argue for the existence of superplural terms and predicates, deny the plausibility of superplural *quantifiers*.

This is motivating. The point these authors make is not that the purported examples do not qualify as superplurals (these arguments have been dealt with in the previous section); it is rather that these examples do not warrant a new semantic category. That is, they can all be done away with using certain paraphrases.

In what follows, I first present the examples that have been suggested in the literature. I then go through a number of proposed paraphrases for them, and argue that they fail in important cases. The discussion closes with some findings in Icelandic and Finnish.

2.2.1 Examples

As Linnebo & Nicolas (2008) pointed out, we need to revisit collective predication to understand the need for superplurals. In the ordinary plural case, we argue that collective predicates range over individuals proper, and thus require a plural analysis (Oliver & Smiley, 2001). In the superplural case, we proceed with a collective predicate that ranges over multiple plural terms, which in turn form a superplural term.

(3) The green Froot Loops and the pink Froot Loops taste bad together.

is a collective predicate over two plural terms. Their example is the already mentioned

(11) The square things, the blue things and the wooden things overlap.

But there are many more. Oliver & Smiley (2013, p. 128) identify four kinds of superplural phenomena in English, which we quote here:

(a) Plurally exhaustive descriptions as just discussed, with $\mathbf{x}\colon F\mathbf{x}$ inviting the reading 'the things that alone or together with each other F'. A trio of examples are

The twin primes

The creators of a great comic opera

The authors of multivolume classics on logic

(i.e. the people who alone or together with each other created a great comic opera, or wrote multivolume logical classics, as the case may be.)

(b) Lists too may be given superplural readings, such as those whose items already include some plural terms

The odd numbers and the even numbers

The English boys and the French boys

A special case is the nested list—a list some of whose items are themselves lists—like 'Whitehead and Russell, Hilbert and Bernays, and Frege'.

(c) The predicate 'are', put between a pair of superplural terms to express identity, as in

At the celebration the compatriots were the English boys and the French boys.

Whitehead and Russell, Hilbert and Bernays, and Frege are the authors of multivolume classics on logic.

(d) The predicate 'are among', whether as a predicate holding between a plural term and a superplural one, or between a pair of superplural terms. These possibilities are illustrated by

3 and 5 are among the twin primes.

Gilbert and Sullivan, Mozart and Da Ponte, and Verdi and Boito were among the creators of a great comic opera.

These are all excellent examples of superplurals; and yet people have tried to undermine them. Assuming that there are occurrences of superplural phenomena in natural language, there is the further question of whether that warrants the introduction of a new type in our semantics (Florio, 2010, p. 130). The following is a collection of all suggested paraphrases in the literature.

2.2.2 Paraphrasing away

Partial singularisation. Many putative examples can be singularised. For example,

(18) The Beatles and the Rolling Stones gave a joint concert.

involves two band names. Whether a band denotes a single entity or multiple individuals is, as already pointed out above, controversial. The same objection applies to 'teams', 'couples', 'pairs' etc. (Linnebo & Nicolas (2008, p. 191); McKay (2006, p. 46)). A full-on group logic, as alluded to before, might be more appropriate. However, we observed that whatever a group is, it is *at least* a plurality. So this analysis does not, in fact, get rid of superplurals.

Conjunctive analysis. This applies to superplural predicates which are distributive. (16) could instead be read as

(19) Russell and Whitehead are joint authors of multivolume treatises on logic, and Hilbert and Bernays are joint authors of multivolume treatises on logic.

There are two things to be said here. First, as Oliver & Smiley (2009, p. 420) remark, this analysis wouldn't be possible if we changed the predicate to a description:

(20) Russell and Whitehead, and Hilbert and Bernays are the joint authors of multivolume treatises on logic.

But more importantly, the argument again rests on the assumption that (19) is somehow more basic than (16), and we said, in 2.1.2, that that argument does not go through.

2.2.3 Ordinary plural analysis

Linnebo and Nicolas reject the conjunctive analysis. In response, they offer a further analysis, which corresponds to the notion of collapse. According to them, the critic of superplurals could equally say

(21) Russell and Hilbert, and Whitehead and Bernays are the joint authors of multivolume treatises on logic.

Thus, (21) assumes that the grouping of the individuals doesn't matter.

This account is untenable, however. Russell was not a joint author of multivolume treatises on logic; saying this would be a gross misunderstanding of this description. (21) is simply false. However, the analysis may be applied in some cases, namely those where the superplural denotation is doubly distributive (Oliver & Smiley, 2013, p. 156):

(22) Groucho, Harpo, and Chico Marx, Abbott and Costello, and Chaplin were great comics.

Arguably, each individual mentioned above was a great comic (though some of them no doubt worked together). As with the conjunctive analysis, the mere possibility of an equivalent analysis does not make it a *better* analysis.

Linnebo & Nicolas (2008) arrive at the conclusion that collective predication over multiple pluralities is the best example of a superplural. This has been challenged recently, as we will see next.

2.2.4 Multigrade predicates

Recall the go-to example from earlier:

(11) The square things, the blue things and the wooden things overlap.

Florio (2010, p. 131–33) offers an alternative analysis of (11):

> ...a natural response to these examples involves treating the lists as strings of terms followed by multigrade predicates, rather than treating them as superplural terms followed by a superplural predicate with a fixed adicity (p. 133).

This analysis needs to be fleshed out. Multigrade predicates are predicates that can take a variable number of arguments. Oftentimes, argument places can be both singular and plural. A good example is the predicate 'cooking dinner' (Oliver & Smiley, 2004, p. 643). The following examples show how flexible it is:

(23)	Bob cooked dinner.	(1 singular term)
(24)	Bob's friends cooked dinner.	(1 plural term)
(25)	Bob and Brian cooked dinner.	(2 singular terms)
(26)	Bob's friends and Brian's friends cooked dinner.	(2 plural terms)
(27)	Bob and Brian's friends cooked dinner.	(1 singular and 1 plural term)

Oliver and Smiley call such predicates 'plural multigrade', since they can take in both singular and plural terms. Based on this understanding, almost all English predicates fit into this category.

So far, we have defined multigrade predicates syntactically. However, plural logic typically interprets the expression beyond merely having variably-many argument places; it also describes a semantic behaviour—the predicate is *true of a variable number of things*. A third way of understanding multigrade predicates is linguistic: a predicate is multigrade iff the relation expressed by it holds of a variable number of things (Oliver & Smiley, 2013, p. 158). A plural predicate is multigrade in the semantic sense. Moreover, we should add that the multigrade/non-multigrade distinction does not coincide with the collective/distributive distinction (for a more extensive discussion, I refer to ibid., p. 160).

The first thing to note is that multigrade predicates themselves don't preclude a superplural analysis. As a matter of fact, superplural predicates can be multigrade[11] as well: 'overlapping' may be applied to any number of things, as long as they are more than one. What Florio is referring to is the debate between two interpretations of lists, as presented in Oliver & Smiley (2004, p. 658; 2013, p. 165). The two rival accounts that emerge are

(i) lists as **terms**: a list is a compound term formed from its constituent terms by a multigrade function sign which takes variably many (two or more) arguments. In a predication the list itself is the argument of the predicate, which thus always takes one argument at the relevant place.

(ii) lists as **strings**: a list is a mere string of separate terms. In a predication these variably many terms are the arguments of the predicate, which is therefore multigrade.

[11] A pedantic point: Florio's multigrade predicate reading also has 'fixed adicity'—the number of *positions*, not places, varies with multigrade predicates (Oliver & Smiley 2004, §1.3; 2013, §10.4).

If each item on the list is a singular term, the compound term is a plural term. Analogously, and as indicated by Oliver & Smiley (2013, p. 128), the compound term formed from several plural terms would be a superplural term.

Let us now turn to Florio's example. All parties agree that 'the square things' is a plural term. So when we take a sentence like (11) and analyse it as a multigrade predicate over a list, we are, in effect, saying that 'the square thing, the blue thing, and the wooden things' is a list of *plural* terms, thus slightly departing from the original debate, where we form a list of singular terms. The question at hand is whether this list is a superplural term, or a string of plural terms *simpliciter*.

The answer here is similar to the conclusion at which Oliver and Smiley arrive: both accounts of lists are adequate. So we must concede that in this case, the string conception of lists doesn't harm anything. But can the 'string reading' be applied in general? Clearly, no.

Let us take a look at this hypothetical situation:

(28) The Yankees and the Red Sox, and the Giants and the Braves will be playing against each other in this year's championship series.[12]

Granting that the team names genuinely denote the players, we have two plural terms in the first argument place, and again two plural term in the second. Such a nested list seems to generate superpluralities that go into each argument place, thus making the compound term supersuperplural. It is unclear how a string interpretation of lists can handle this hierarchy without introducing more notation, or indeed concepts equivalent to superplurals.

2.2.5 Icelandic

In his survey article, Linnebo claims that Icelandic contains number words which count 'pluralities of objects that form natural groups' 2014. The example:

einn skór	one shoe
einir skór	one pair of shoes
tvennir skór	two pairs of shoes

These number words allows for talk of second-level pluralities rather than counting 'singularised' first-level objects such as *pairs*.

If this example holds, then there seems to be a good candidate for superplural quantification in natural language, something that most authors reject (Oliver & Smiley, 2013; Rayo, 2006). However, this claim has been recently challenged by Ben-Yami (2013, p. 85 fn 6):

> Icelandic in fact has plural form of number words only for the number words for one, two, three and four. Moreover, these are used mainly when the noun following them has only a plural form, even when it stands for a single object; this is the case, for instance, with 'dyr', door. The use of the plural number word system for referring to pairs is an addition to this list, and it is used to refer *only* to pairs, and not to any plurality of pluralities of objects.

[12]Two baseball teams from each league participate in the semifinals.

This calls for investigation into Icelandic languange intuitions.[13] Let us modify the example:

einn skór	one shoe
einir skór	one pair of shoes, i.e. one left shoe and one right shoe
tveir skór	two individual shoes, could form a pair or not, i.e. two for the left foot
tvennir skór	two pairs of shoes

It is true that in some cases, this form is used for singular objects, such as 'einar dyr' (one door). However, this seems to be an exception. In most cases, even when it refers to a single object, there is implicit semantic understanding that the referent is in fact plural in nature. Here are three examples:

tvenn gleraugu	two pairs of (optical) glasses
tvennar buxur	two pairs of pants
tvennir leikar	two games (in the sense of Olympic games)

In all of these examples, while the superplural quantifier refers to a singular object, it's somehow understood that they are in fact plural. You can even see that in the English in the first two examples, 'two glasses' would mean something else, and 'two pants' is just weird. Often there is a difference in meaning: 'Tvö lög' means two songs but 'tvenn lög' means two laws. It is unclear how this arose for 'dyr', though. In any case, the consensus amongst Icelandic grammarians is that these words always refer plurally.

The objection that Icelandic only has these forms for the numbers one, two, three and four is tenuous. That is an instance of a more general phenomenon, namely that Icelandic is a highly declined language, but doesn't have any variation in declension for number words higher than four at all. Those words are always the same, no matter what their syntactic role is. The phenomenon of not declining higher number words is not uncommon: it also occurs in other Indo-European languages, such as Latin and Greek. To illustrate: one could say 'Konurnar tvær sâu tvo menn gefa þremur hestum' = 'The two women saw two men feed three horses', where 'tvær' is plural, feminine and nominative, 'tvo' is plural, masculine, accusative, and 'þremur' is plural, masculine, dative. However, the corresponding sentence for higher numbers would be 'Konurnar fimm sâu fimm menn gefa sex hestum' = 'The five women saw five men feed six horses'. Here, the number words 'fimm' and 'sex' just have their nominative forms—and they always do, superplural or not.

If this is right, then it seems that Icelandic in fact contains superplural terms. Needless to say, the details will have to be examined by linguists studying Icelandic.

[13]Seeing that I lack them completely, I am very fortunate that Ásgeir Matthíasson offered great help with Icelandic grammar.

2.2.6 Finnish

Icelandic is not the only language under inspection. Oliver & Smiley (2005, p. 1063 fn) have suggested Breton, while Corbett (2000) proposed a special analyses for associate plurals in Yup'ik and other languages. The most convincing example, however, has to be Finnish (and to some extent, Estonian).

What holds for the first four numbers in Icelandic holds for all numerals and other quantifiers. Hurford (2003, §3.3.3.3) lists table 2.9:

VALUE	SINGULAR	PLURAL
1	yksi	yhdet
2	kaksi	kahdet
3	kolme	kolmet
4	neljä	neljät
5	viisi	viidet
6	kuusi	kuudet
7	seitsemän	seitsemät
8	kahdeksan	kahdeksat
9	yhdeksaän	yhdeksät
10	kymmenen	kymmenet
50	viisikymmentä	viidetkymmenet
100	sata	sadat
1000	tuhat	tuhannet
100000	satatuhatta	sadattuhannet
1000000	miljoona	miljoonat
pair, couple	pari	parit
a few	muutama	muutamat
many	moni	monet
several	usea	useat
a few, not many	harva	harvat

Figure 2.9: superplurals in Finnish

The plural numerals are translated as '*n* groups of', thus clearly involving some form of superplural reference. Both singular and plural numeral-noun phrases occur in most grammatical cases. Even though readings sometime depend on the context, the 'groups of groups' interpretation seems correct overall (p. 36). A good example is the following pair of sentences.

oppilaat	saivat	kolme	kirjaa
pupils	got	3+ACC+SG	book+PART+SG

On a collective reading, there are three books which the pupils collectively receive. On a distributive reading, each individual pupil receives three books; it might even be read as each pupil receiving copies of the same three books as the other pupils. Compare that with the following:

oppilaat	saivat	kolmet	kirjat
pupils	got	$3_{\text{+ACC+PL}}$	$\text{book}_{\text{ACC+PL}}$

Here, the special case of the distributive reading is forced. It suggests that three ‘book types’ have been handed out, with each pupil receiving one token of each type. In another contextualization, a teacher has three variously sized groups of pupils and gives each group of pupils one pile of books; the number of books in each pile in unclear, but there are exactly three piles.

No matter what interpretation we favour, however, it seems that a superplural analysis is best suited to deal with these situations. This concludes our adventure in non-English languages.

Chapter 3

Logic

> There is no such thing as a 'plurality', which is the misbegotten invention of a faulty logic.
>
> Michael Dummett (1991, p. 93)

This chapter presents the formal account of higher-level plural logic. It serves to make precise what we have been describing so far, and will be a basis for further discussion. Moreover, this will be a comprehensive account of the superplural hierarchy, which, until now, has only been attempted by Rayo (2006). With the frustrating variation in notation in the existing literature, our hope is also to set a standard for superplural talk. We proceed as follows. First, we need to address the properties of the plural hierarchy, such as cumulativity, open-endedness, and type restrictions. The rest of the chapter provides a syntax and semantics, including a deductive system to work in. Finally, we take a look at the meta-theoretic properties of superplural logic, and examine the possibility of paradoxes.

A note on notation

There is a plurality of notations proposed for higher-level quantification. We begin with a survey of the current suggestions in the literature (see 3.1, based on Oliver & Smiley (2013, §7.2)).

All authors use x, y, z for singular variables and, with the exception of Simons and Oliver & Smiley, all plural variables denote one or more things. Simons uses h, k, l specifically for two or more things. Oliver & Smiley's variables denote any number of things, which means it could also denote nothing.

A good notational choice is legible, unambiguous, and consistent with previous work. Some notations do not allow for a natural extension to higher levels (e.g. capitalisation does not have a successor). In this chapter (and beyond), we will use superscripts to indicate the level in general, where a variable of level 0 is a singular variable. Level-indication in higher-level plural first-order logic should not be confused with exponentiation. Sentences that only

	plural variables	superplural variables	plural quantifiers
Simons (1982)	$h, k, l; u, v, w$		$\exists, \forall$
(Burgess & Rosen, 1997)	xx, yy, zz		$\exists\exists, \forall\forall$
(Linnebo, 2003)	xx, yy, zz		$\exists, \forall$
Rayo (2006)	xx, yy, zz	xxx, yyy, zzz	$\exists, \forall$
McKay (2006)	X, Y, Z	XX, YY, ZZ	$\exists, \forall$
Yi (2006)	xs, ys, zs		Σ, Π
Nicolas (2008)	xs, ys, zs	xss, yss, zss	$\exists, \forall$
Oliver & Smiley (2013)	$\mathbf{x}, \mathbf{y}, \mathbf{z}$	$\mathbf{x}^2, \mathbf{y}^2, \mathbf{z}^2$	$\exists, \forall$

Table 3.1: previous notations for plural and superplural terms

feature first- and second-level plurals are commonly written with double- and triple variables to highlight the level differences. Thus, the hierarchy of plural variables is both $x^0, x^1, x^2, \ldots$ and $x, xx, xxx, \ldots$, where the latter is used only in translations for the sake of familiarity.

There is a similar indecisiveness in representing the inclusion predicate:

	predicate	reading
Simons (1982)	$\in$	is or is among
	$-\!\!\!\in$	is/are or is/are among
(Burgess & Rosen, 1997)	$==$	is or is among
(Linnebo, 2003)	$\prec$	is among
(Burgess, 2004)	α	is or is among
Rayo (2006)	$\prec$	is or is among
	$\precsim$	is/are or is/are among
McKay (2006)	K	is or is among
	A	is/are or is/are among
Yi (2006)	H	is or is among
	$\sqsubseteq$	is/are or is/are among
Nicolas (2008)	$\angle$	is/are or is/are among
Oliver & Smiley (2013)	$\preccurlyeq$	is/are or is/are among
(Florio, 2014)	$\prec$	is among
(Simons, in press)	η	is one of

Table 3.2: previous notations for the inclusion predicate

(Our version uses '$\preccurlyeq$' to express both inclusion and identity.)

Plural and superplural predicates will need to be specified as well. If we were pedantic, each predicate must be specified in four aspects: its *arity*—the number of things(s) that enter the relation, its *grade*—the number of arguments that go into each argument place, its *order*—the order in which the argument places occur, and the *level* of each argument place in the hierarchy.

Florio (2010, pp. 136) does not distinguish arity from grade, and disregards the order of argument places in his notation. He superscripts predicates with square brackets indicating how many argument places are superplural, plural, and singular, respectively. The number that precedes the square brackets designates the arity of the predicate. For example: $P^{4[2,1]}$ is a quaternary predicate, taking in two superplural, one plural, and one singular argument, in some order.

Rayo (2006, p. 234) uses a sequence to indicate arity, level, and order. If s is the sequence $n_1, \ldots, n_k$, then k is the arity of the predicate, where each n_i $(1 \leq i \leq k)$ specifies the level of the argument place. The order is determined by the sequence itself. For example: if $s = 2, 1, 0, 2$, then P^s is a quaternary predicate, taking in one superplural, one plural, one singular, and another superplural argument, in that order.

Our notation needs to be even more cumbersome: for we must allow for variation both in terms of grade and level. A natural way is to move the level indicator to the superscript. The proposal is thus: for a sequence $s = l_1^{n_1}, \ldots, l_k^{n_k}$, the predicate P^s has arity k; each l_i $(1 \leq i \leq k)$ marks the grade of the argument place, and each n_i $(1 \leq i \leq k)$ marks the level of the argument place; the order of the argument places is determined by their occurrence in the sequence.

We have seen in 2.2.4 that predicates can be both multigrade and flexible. That is, they may vary both with regard to the number of things they apply to and the level of each argument place. For our purposes, we will focus on those predicates with fixed grade and level. However, it will be straightforward to extend our syntax and semantics to include multigrade and flexible predicates. For example, we could use '$*$' for variably many argument places, and '$\bullet$' to indicate any level. An example will help: if $s = 2^0, 1^\bullet, *^2$, then P^s is a ternary predicate, relating two individuals, one plurality of any level (including 0), and variably many superpluralities, in that order. There are also cases where either level or grade is restricted but not specified. Take the predicate 'forming a circle'. It is multigrade for more than three argument places. In this case, we could write $s = l_1^\bullet$ with $l_1 > 2$.

Isn't this all too complicated? Yes, But it is the most honest approach. Recall that we have two conditions on predicates. First, a predicate that can apply to both singular and plural (and superplural) terms should be regarded as one predicate, and not two (or more) different predicates. This is the one-sorted **PFO+** we started with. Second, we do want to specify a predicate's arity and grade. For example, a sentence with a singular predicate should not apply to a plurality:

(29) Russell and Whitehead gave a soliloquy.

We want our logic to indicate that (29) isn't *well-formed* rather than declaring it plain false.[1] The best way to do so is to specify the predicate and check whether its arguments agree with the arity and grade. Thus, we ultimately cannot rid ourselves of these indices.

Examples

Here are some examples to get an idea of how sentences are regimented in superplural logic.

[1]Or at least this is a piece of logical dogma we won't challenge.

Some singers harmonised.

$\exists xx(\mathit{Singer}(xx) \wedge \mathit{Harmonise}(xx))$

Russell and Whitehead cooperated.

$\exists xx(\forall y(y \preccurlyeq xx \rightarrow (y = r \vee y = w)) \wedge \mathit{Cooperate}(xx))$

Some critics admire only one another.

$\exists xx(\forall x(x \preccurlyeq xx \rightarrow \mathit{Critic}(x)) \wedge \forall x \forall y((x \preccurlyeq xx) \wedge \mathit{Admire}(x, y)) \rightarrow (x \neq y \wedge y \preccurlyeq xx))$

To take the baseball example from above (28), we could give the following paraphrase for the supersuperplural sentence

The Yankees and the Red Sox, and the Giants and the Braves will be playing against each other in this year's championship series.

$$\exists xxxx\ (\exists xxx\ (xxx \preccurlyeq xxxx \wedge (\iota xx\ \mathit{Yankees}(xx) \preccurlyeq xxx \wedge\ \iota yy\ \mathit{RedSox}(yy) \preccurlyeq xxx))$$
$$\wedge\ \exists yyy\ (yyy \preccurlyeq xxxx \wedge (\iota uu\ \mathit{Giants}(uu) \preccurlyeq yyy \wedge \iota vv\ \mathit{Braves}(vv) \preccurlyeq yyy))$$
$$\wedge\ \mathit{PlayAgainst}(xxxx))$$

3.1 Hierarchies

In the previous chapter, we have discussed different conceptions of superplural logic. In Oliver and Smiley's logic, second-level plural logic introduces a new operator for *plurally exhaustive*. To recapitulate: we have two operators for different kinds of predication. Exhaustive denotation is used for distributive predicates, while (plurally) unique denotation is used for collective predicates. Now, let the exhaustive operator ': ' range over both singular and plural variables. A plurally exhaustive operator then gives rise to second-level plurals: $xx : Fxx$ is read as 'the things that alone or jointly with each other F' (see Oliver & Smiley, 2013, p. 275). This is our superplural term.

Thus, we've created superplurals by extending the range of the colon operator to both singular and plural. It seems natural then, that we build supersuperplurals by letting the colon operator range over singular, plural *and* superplural variables. But how do we interpret a supersuperplural term like '$xxx : Fxxx$'? Natural language does not allow us to go beyond 'the things that alone or jointly with each other F', but if the previous discussion is correct, then we should have an understanding of what '$xxx : Fxxx$' denotes: in Hazen's words, 'the thingss that alone or jointly with each other F'.

This section is concerned with building the superplural hierarchy. In particular, we focus on two aspects: cumulativity and open-endedness.

3.1.1 Cumulativity

We call a hierarchy *cumulative* when for any two terms t and s the string $t \preccurlyeq s$ is regarded as a well-formed formula just in case the level of s is strictly greater than the level of t. A higher-level language is said to be non-cumulative when the stricter requirement is imposed that the level of s must be precisely one greater than the level of t (cf. Linnebo & Rayo, 2012, p. 273).

The superplural hierarchy is cumulative: each term of level n can designate zilch or pluralities of all levels $0 \ldots n$, e.g. a superplural can denote zilch, one individual, many individuals, or 'many many' individuals. Moreover, a superplural term can have elements of *mixed levels*, such as 'Bob Marley and the Wailers' (level 0 and 1, respectively).

The main reason for adopting a cumulative hierarchy is to avoid what we may call 'higher-level zilches' and 'singletons'. One of the principles of standard set theory is that $x \neq \{x\}$ in general. This does not hold in superplural logic: every individual is also its own plurality.

If we allow for informal notation again for a moment, we would write [[Bob Marley], the Wailers] instead of [Bob Marley, the Wailers] in a non-cumulative hierarchy. This is not an option because in a plural hierarchy, [Bob Marley]=Bob Marley.

Before we move on, however, it should be mentioned that Simons (2011, p. 12-13) offers an argument against cumulativity. Consider the standard argument for why set theory distinguishes x from its singleton, $\{x\}$. x might itself be a set with more than one element. If that is so, x has several elements, whereas $\{x\}$ has only one, so by Leibniz's principle $x \neq \{x\}$. We can apply the same reasoning to pluralities (in Simons's case: 'multitudes'). Suppose $[a, b]$ is the plurality with two individual members a and b ($a \neq b$). Then $[a, b]$ has two members. Now if we allow '$[[a, b]]$' as a well-formed expression at all, there are then two possibilities: either $[[a, b]] = [a, b]$, or $[[a, b]] \neq [a, b]$.

The first option effectively says that (i) notationally, nested brackets can be reduced to a single pair, and (ii) ontologically, singletons are identical with their members, whether these are individuals or not. What can go wrong? Consider the four individuals, a, b, c, d, and the two groupings $[[a, b], [c, d]]$ versus $[[a, c], [b, d]]$. If $[[a, b]] = [a, b]$ then adding another thing, say e, to $[[a, b]]$ will give us $[[a, b], e]$ and this will have the same effect as adding e to $[a, b]$, so $[[a, b], e] = [a, b, e]$. Conversely, adding another item f to $[[c, d]]$ to give $[f, [c, d]]$ is the same as adding it to $[c, d]$ to give $[f, c, d]$. Now, let $e = [c, d]$ and $f = [a, b]$. Then we have

$$[e, f] = [[a, b], [c, d]] = [a, b, [c, d]] = [a, b, c, d]$$

But if $g = [a, c]$ and $h = [b, d]$, then we have $[g, h] = [a, b, c, d]$ also. So we arrive at

$$[e, f] = [g, h]$$

even though none of e, f, g, h are identical. This violates co-reference as we defined it earlier (2.1.2). Simons concludes that collapsing higher-level singletons isn't the right way to go, and opts for the second option, which is that $[[a, b]] \neq [a, b]$. But as it turns now, this move is a little too fast.

The fallacy lies with adding one thing to another. Borrowing set-theoretic notation for a moment, we define adding two terms as

$$a + b := \{a\} \cup \{b\} = \{a, b\} \neq a \cup b$$

The latter union operation is only equivalent to the former if a and b are atomic. However, Simons seems to have used this operation when he 'added' e to the non-atomic $[a, b]$ to get

$[a, b, e]$ (where $e = [e]$). In general, though removing redundant brackets is called for in superplural logic, it is advisable to check whether they are in fact redundant first.

Moving on to making the hierarchy cumulative. Cumulativity is expressed formally by liberalising the inclusion predicate. In a non-cumulative hierarchy, predicates are given strict level indication, and a formula is well-formed only if the level of the term(s) match the level of the predicate. In a cumulative hierarchy, we drop this restriction: a sentence is well-formed when the level of the first term is smaller or equal to the level of the second.

To borrow an example from Rayo (2006, §9.3.5): if Socrates is the one and only Socratiser, then intuitively, the sentence

(30) Socrates is the same individual as the Socratisers.

should be true, albeit ungrammatical. We already have many complaints towards ordinary grammar with regard to superplural quantification, so it shouldn't come as a surprise that we will allow for mixed levels for identity and predicates despite their grammatical deficiencies.

How can this be achieved? Identity can be reduced to the inclusion predicate: $a = b := a \preccurlyeq b \wedge b \preccurlyeq a$. This ensures that a and b are terms of the same level. To liberalise the predicate in the superplural hierarchy, we change the definition to

$$x^n = y^m := \forall z^l (z^l \preccurlyeq^n x^n \leftrightarrow z^l \preccurlyeq^m y^m)$$

This way, identity should apply to every level and across all levels. This captures the idea of collapse: higher-level terms with the exact same denotation (of whatever level) will be identical.

3.1.2 Open-endedness

Is the plural hierarchy open-ended? There has been much discussion on the open-endedness of the type-theory in the recent literature on absolute generality (Linnebo & Rayo, 2012; Florio & Shapiro, 2014; Linnebo & Rayo, 2014). Superplural logic can be viewed as a version of simple type theory interpreted plurally (see appendix A in Linnebo & Rayo). And so the arguments for and against open-endedness are familiar.

The superplural hierarchy is 'ideological' in Linnebo and Rayo's terms: it does not posit additional ontological commitment (pending further enquiry) with the addition of levels. Unlike ontological hierarchies such as set theory, the superplural hierarchy does not describe a hierarchy of independently existing sets; instead, it 'makes available an "ideological hierarchy" of stronger and stronger expressive resources' (p. 270).

There are two considerations that figure into the question of open-endedness. First, what are the expressive resources of superplural logic? Second, for the sake of ontological modesty, should we restrict the number of levels?

In a language of level n, we generally have the conceptual resources to create a language of level $n + 1$. This is the conceptual basis of the plural hierarchy: from singular terms, we construct plural terms; plural terms in turn give us superplural terms. Iterating this process *ad infinitum*, we arrive at higher-level plural logic.

The idea of iteration mirrors the principle of Semantic Optimism (Linnebo & Rayo, 2012, p. 276):

Semantic Optimism Given an arbitrary language, it should be possible to articulate a generalized semantic theory for that language.

While we are not concerned with semantic theorising, we can formulate a weaker principle for superplurals:

Plural Iteration Given a plural language of level n, it should be possible to create a plural language of level $n+1$.

Thus, it is conceivable, at least in principle, to continue this semantic ascent indefinitely.

However, do the expressive resources yield a limit level? Linnebo and Rayo show that Semantic Optimism, together with Absolute Generality and the Principle of Union, can be used to motivate ascent to languages of type α for α an arbitrary ordinal:

Absolute Generality One's (first-order) quantifiers can meaningfully be taken to range over absolutely all pluralities.

Union For any definite plurality of higher-level languages, there is a 'union' language that encompasses all of them.

Whether the argument goes through depends on whether one endorses these two principles. As Florio & Shapiro (2014) argue, the endorsement of these principles again depends on what sort of view one has on the open-endedness of the hierarchy. Here, I think we should resist the move to the transfinite for the sake of ontological modesty. Restricting the height of the hierarchy to finitary levels is echoing what Russell did for his simple type theory. Levels go up to n for an arbitrarily large n; yet they don't transcend to the transfinite. This marks an important distinction both to set theory and Linnebo and Rayo's 'liberalised' type theory.

What is the status of infinity within superplural logic in general? The above argument suggests that we don't have a 'vertical' infinity—there is no ω level. However, 'horizontal' infinity is admissible in superplural logic. One example of an infinite plurality: 'All natural numbers.' One example of an infinite superplurality: 'All natural numbers divisible by 1, and all natural numbers divisible by 2, etc.' This already allows for much expressive power (3.3).

3.2 Formal account

3.2.1 Syntax

The language of plurals and superplurals

The language of higher-level plural (first-order) logic consists of the following symbols. Corner quotes indicate quasi-quotation of the object language.

Variables of level n ($n \geq 0$)	$\ulcorner x^n \urcorner$, $\ulcorner y^n \urcorner$, ... 'x', 'xx', ..., 'y', 'yy', ...
Logical connectives	'$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\leftrightarrow$'
Quantifiers	'$\forall$', '$\exists$'
Auxiliaries	'(', ')', '[', ']'
Unique description operator	'$\imath$'
Exhaustive description operator	'$:$ '
n-level inclusion, two-place predicates	$\ulcorner \preccurlyeq^n \urcorner$
Constants	'a', 'b', ...
Predicates of degree $k \geq 1$	$F^k, G^k, \ldots$ $F^s, G^s, \ldots$
Function signs, of degree $k \geq 1$	$f, g, h\ldots$

We define an expression of higher-level plural logic as any string of the above symbols. We call the resulting language $\mathscr{L}_s$.

As discussed above, we offer a detailed specification of a predicate using sequences: for $s = l_1^{n_1}, \ldots, l_k^{n_k}$, P^s has arity k; each l_i ($1 \leq i \leq k$) is the grade, and each n_i ($1 \leq i \leq k$) is the level of the argument place; the order of the argument places is i. In 3.1.1 we talked about the two-place inclusion predicate 'is/are among', written '$\preccurlyeq$'. Since our hierarchy is cumulative, we have $s = 1^{n_1}, 1^{n_2}$ with $n_1 \leq n_2$. Based on Oliver & Smiley (2013), we also add two special operators, to be understood as unique and exhaustive description respectively. The uniqueness operator is taken directly from Whitehead. The operators are interdefinable:

Unique description $\imath x Fx := x \colon \forall y(Fy \leftrightarrow y = x)$

Exhaustive description $x \colon Fx := \imath xx \forall y(Fy \leftrightarrow y \preccurlyeq xx)$

Plurally unique description $\imath xx Fxx := xx \colon \forall yy(Fyy \leftrightarrow yy = xx)$

The two operators are also level-sensitive in the superplural hierarchy. For example, superplurals can be expressed as $xx\colon Fxx$, and supersuperplurals will be $xxx\colon Fxxx$. Likewise, we can have a superplurally unique operator such as $\iota xxFxx$. Note that we can also involve operators *inside* predicates, such as with superplural definite descriptions:

$$G(x^n\colon Fx^n) \leftrightarrow \exists x^n(\forall y^m(Fy^m \leftrightarrow y^m \preccurlyeq^n x^n) \wedge Gx^n)$$

Terms and formulas

Terms and formulas are defined in the usual fashion. Note that we will use the same symbols in the metalanguage.

a. Every variable is a term of some level n.

b. Every constant is a term of level 0.

c. If $\ulcorner F^k \urcorner$ is a k-place predicate and $\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner$ are terms, then $\ulcorner F t_1 \ldots t_k \urcorner$ is a formula. Alternatively, if $\ulcorner F^s \urcorner$ is a predicate with the sequence s=$\ulcorner l_1^{n_1} \urcorner, \ldots, \ulcorner l_k^{n_k} \urcorner$, and $\ulcorner t_1^{n_1} \urcorner, \ldots, \ulcorner t_k^{n_k} \urcorner$ are terms of level $\ulcorner n_i \urcorner$ and grade $\ulcorner l_i \urcorner$, respectively, then $\ulcorner F t_1^{n_1} \ldots t_k^{n_k} \urcorner$ is a formula.

d. If $\ulcorner x \urcorner$ is a variable of level n and ϕ a formula, then $\ulcorner \iota x^n \phi \urcorner$ and $\ulcorner x^n\colon \phi \urcorner$ are terms.

e. If $\ulcorner t_1 \urcorner$ and $\ulcorner t_2 \urcorner$ are terms of level m and n, respectively, with $m \leq n$, then $\ulcorner t_1 \preccurlyeq^n t_2 \urcorner$ is a formula.

f. If ϕ and ψ are formulas, so are $\ulcorner \neg\phi \urcorner$, $\ulcorner (\phi \wedge \psi) \urcorner$, $\ulcorner (\phi \vee \psi) \urcorner$, $\ulcorner (\phi \rightarrow \psi) \urcorner$, $\ulcorner (\phi \leftrightarrow \psi) \urcorner$.

g. If $\ulcorner x \urcorner$ is a variable of level n and ϕ a formula, then $\ulcorner \forall x \phi \urcorner$ and $\ulcorner \exists x \phi \urcorner$ are formulas.

h. Nothing else is a term or a formula.

Note that the scope of $\forall, \iota, :$ is defined as the shortest formula or term in which it occurs. Generally, an occurrence of a term t or a formula ϕ in another term or formula is bound if it is within the scope of an operator whose attached variable occurs free in t or ϕ; otherwise it is free.

Pre-defined expressions

Proper inclusion $a \prec b :=\ a \preccurlyeq b \wedge b \not\preccurlyeq a$ and $a \not\preccurlyeq b :=\ \neg(a \preccurlyeq b)$

Identity $a = b :=\ \forall c(c \preccurlyeq a \leftrightarrow c \preccurlyeq b)$ and $a \neq b :=\ \neg(a = b)$

Weak identity $a \equiv b :=\ (a = b) \vee (\neg\exists x^n \preccurlyeq a \wedge \neg\exists x^n \preccurlyeq b)$

Zilch $\oslash :=\ x : x \neq x$

3.2.2 Semantics

Preliminaries

Standard model-theoretic semantics quantifies over set-theoretic domains in its definitions of truth and consequence. Again, in maintaining ontological innocence, our metalanguage must avoid reference to such entities.

Moreover, we deviate from classical logic in our valuation of predicates. In superplural logic, predicates do not designate set-like entities, but genuine properties (and relations, as the case may be). This roughly follows a Fregean picture where predicates denote 'concepts' rather than 'objects'. The reason for this is more involved, and will be explained in 3.4.

We will use the term 'plurality of level *n*' to denote the objects we plurally or superplurally quantify over. The singularist appearance should not mislead: it is merely a shorthand to avoid cumbersome constructions in English.

Model theory

The semantics of superplural logic is given in four parts by (i) specifying a domain of quantification; (ii) mapping a semantic value to each term and predicate as an interpretation; (iii) defining satisfaction of formulas in $\mathscr{L}_{SP}$; and finally (iv) characterising logical consequence.

I. Domain

The domain of $\mathscr{L}_{SP}$ is a plurality (call it dd), and the individuals may be any objects. There may be none or one or more. We also assume the existence of properties and relations, which are denoted by predicates.

II. Interpretation

The valuation function val for the assignment of semantic values to linguistic items (terms and predicates) is defined now for variables of level n.

i. For each variable x of level n, $val\ x^n$ is either zilch, an individual, or a plurality of level $1, \ldots, n$. (For example, a superplurality is a plurality of level 2.)

ii. For each constant a, $val\ a$ is either zilch, an individual, or a plurality of some level.

iii. For each k-place predicate F, $val\ F$ is a k-place relation on the individuals; in particular, $val \preccurlyeq$ is the relation 'is/are among' or 'is/are', as the case may be.

iv. For each k-place function sign f, $val\ f$ is a k-place function on the individuals.

Note that a k-place predicate is *on the individuals* iff for any individuals $x_1 \ldots x_n$, the relation either holds or does not hold of x_1 (or zilch), …, x_n (or zilch) as argument. Thus it is determinate whether $Fa_1 \ldots a_k$ holds or not.

III. Satisfaction

i. val satisfies $Fa_1 \ldots a_k$ iff $val\ F$ holds of $val\ a_1, \ldots, val\ a_k$.

ii. $val\ f a_1 \dots a_k$ is/are the value(s), if any, of $val\ f$ for arguments $val\ a_1, \dots, val\ a_k$; otherwise it is zilch.

iii. val satisfies $\neg\phi$ iff it does not satisfy ϕ. It satisfies $\phi \to \psi$ iff it satisfies ψ or does not satisfy ϕ. And so on for the other connectives.

iv. val satisfies $\forall x^n \phi$ iff every valuation that differs from val at most in that x^n has a value and in what that value may be, satisfies ϕ.

v. $val\ \iota x^n \phi$ is/are the individual(s) $val'\ x^n$ if there is a unique valuation val' that differs from val at most in that x^n has a value and in what that value may be, satisfies ϕ; otherwise it is zilch.

vi. $val\ x\colon \phi$ is/are the individual(s) $val'\ x$ for every valuation val' that differs from val at most in that x^n has a value and in what that value may be, satisfies ϕ; if there is no such val', it is zilch.

We need to ensure that satisfaction is well-defined, that is, both sides of the above 'iff' statements are determinately true or false. For the right-hand side of (III.i), this is guaranteed by (II.iii). For the left-hand side, we may use an analogous notion for the assignment function val: A k-place function is *on the individuals* iff for any individuals $x_1 \dots x_n$, the function either has some individuals as value for the arguments x_1 (or zilch), …, x_n (or zilch), or else has no value for those arguments. Then the value of a k-place function sign should be a k-place function on the individuals for it to be well-defined. This follows Oliver and Smiley's treatment of satisfaction (pp. 195).

IV. Logical consequence

i. $\vDash C$ iff all valuations, over no matter what or how many individuals (none or one or more), satisfy C.

ii. $\Gamma \vDash C$ iff all valuations, over no matter what or how many individuals (none or one or more), satisfy C if they satisfy every one of Γ.

3.2.3 Deductive System

In this section we provide a deductive system for higher-level plural logic. The basic elements are what one would expect:

T ϕ where ϕ is a propositional tautology.

MP Modus Ponens

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

I Identity

$$t_1^0 = t_2^0 \to (\varphi(t_1^0) \to \varphi(t_2^0))$$

where t_2^0 is free for t_1^0 in φ

ED Exhaustive description operator

$$\forall y^n((y^n \preccurlyeq x^n : \varphi(x^n)) \leftrightarrow \varphi(y^n))$$

where y^n is free for x^n in φ

UD Unique description operator

$$\forall y^n((y^n = \iota x^n \varphi) \leftrightarrow \forall x^n(\varphi(x^n) \leftrightarrow x^n = y^n))$$

where y^n does not occur in x^n in $\iota x^n \varphi$

EI$^{\mathbf{H}}$ Existential Introduction

$$\exists y^n \psi(y^n) \to (\varphi(y^n : \psi(y^n)) \to \exists x^{n+1} \varphi(x^{n+1}))$$

where $y^n : \psi(y^n)$ is free for x^{n+1} in φ

EE$^{\mathbf{H}}$ Existential Elimination

$$\frac{\Gamma}{\exists y^n \psi(y^n) \to (\varphi(y^n : \psi(y^n)) \to \chi)} \quad \Rightarrow \quad \frac{\Gamma}{\exists x^{n+1} \varphi(x^{n+1}) \to \chi}$$

where ψ does not occur in Γ or χ

UE$^{\mathbf{H}}$ Universal Elimination

$$\exists y^n \psi(y^n) \to (\forall x^{n+1} \varphi(x^{n+1}) \to \varphi(y^n : \psi(y^n)))$$

where $y^n : \psi(y^n)$ is free for x^{n+1} in φ
and where there is at least two y^n

UI$^{\mathbf{H}}$ Universal Introduction

$$\frac{\Gamma}{\exists y^n \psi(y^n) \to (\chi \to \varphi(y^n : \psi(y^n)))} \quad \Rightarrow \quad \frac{\Gamma}{\chi \to \forall x^{n+1}(\varphi(x^{n+1}))}$$

where ψ does not occur in Γ or χ
and where there is at least two y^n

C Comprehension

$$\exists x^n(\varphi(x^n)) \to \exists x^{n+1} \forall y^n(y^n \preccurlyeq x^{n+1} \leftrightarrow \varphi(y^n))$$

S Strictness

$$\forall x^{n+1} \exists_{\geq 2} y^n(y^n \preccurlyeq x^{n+1})$$

(We need **S** because higher-level variables cannot take on higher-level zilches or singletons.)

3.3 Expressive power

3.3.1 Soundness

Though we only have a partially axiomatised logic, soundness has been proven for **PFO+** (Oliver & Smiley, 2013, §12 appendix). Rayo (2006, p. 235) also claims soundness for his version of higher-level logic. We will follow through here and state that higher-level **PFO+** is sound with regard to the semantics we have provided.

3.3.2 Incompleteness

Superplural logic is incomplete because full plural logic, just like full second-order logic, is both strongly and weakly incomplete. The proof for incompleteness goes via failure of both strong and weak axiomatisability. Recall that a deductive system is strongly complete iff if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$, and weakly complete iff if $\vDash \varphi$ then $\vdash \varphi$. Moreover,

Definition (Strong axiomatisability). *A logic is **strongly axiomatised** by a deductive system iff that deductive system is both strongly sound and strongly complete.*

And, analogously,

Definition (Weak axiomatisability). *A logic is **weakly axiomatised** by a deductive system iff that deductive system is both weakly sound and weakly complete.*

Thus, given our contention that plural logic is sound, a failure of axiomatisablity implies a failure of completeness (strong or weak, correspondingly). Our proof is essentially analogous to the one for second-order logic; nevertheless it is worth quoting. It draws from Oliver & Smiley (2013, §13.3) and Incurvati (2014a, §9.6).

Failure of strong axiomatisability

The failure of strong axiomatisability for full plural logic follows from the fact that it is not compact. Compactness is defined as

Definition (Compactness). *A logic is **compact** iff for every set of sentences Γ and sentence φ, if $\Gamma \vDash \varphi$, then, for some finite $\Delta \subseteq \Gamma$, $\Delta \vDash \varphi$.*

Thus, we need to formulate a valid argument in plural logic that has infinitely many premises, i.e. one where no finite subset of the set of premises entail the conclusion. This is feasible because our domain is infinite.

Theorem. *Plural logic is not compact.*

Proof. Consider the argument

There is at least one thing, there are at least two things, ...
$\therefore$ There are infinitely many things.

In symbols:

$$\begin{array}{rl} & \exists x(x = x),\ \exists x \exists y(x \neq y),\ \ldots \\ \therefore & \exists z \exists f(\forall x(z \neq f(x)) \wedge \forall x \forall y(f(x) = f(y) \rightarrow x = y)) \end{array}$$

Evidently, it has infinitely many premises, each of which can be expressed in plural logic. Its conclusion is also expressible in plural logic (recall that we have included functions in our language). The argument is valid: on any valuation in which the premises are true, the conclusion follows.

But there is no finitely long selection from these premises that entails the conclusion. Take any such selection. Then there is a greatest finite n that is described by a sentence in this selection ('There are at least n things.'). So any interpretation which has n or more things in the domain will make each member of this selection true. Some of these interpretations will have finite domains, and in these interpretations the conclusion is false. Therefore, the conclusion does not logically follow from this finite selection. Since we chose it arbitrarily, there is no finite selection of premises from which the conclusion follows. So plural logic is not compact. □

Corollary. *Plural logic is not strongly axiomatisable.*

Failure of weak axiomatisability

The proof for the fact that plural logic is not weakly axiomatisable requires heavier machinery, and we will only sketch it here. We begin by sketching a plural arithmetic, a finitely axiomatised version of Peano Arithmetic (see below). It is categorical and therefore semantically complete. By Gödel's theorems it follows that the axioms of the underlying logic cannot be effectively enumerable. Hence, plural logic is not weakly axiomatisable.

Plural Arithmetic

'y is a natural number.'

$$\forall xx(0 \preccurlyeq xx \wedge \forall x \forall z(x \preccurlyeq xx \wedge Pxz \rightarrow z \preccurlyeq xx) \rightarrow y \preccurlyeq xx)$$

Mathematical induction:

$$\forall xx(0 \preccurlyeq xx \wedge \forall x(x \preccurlyeq xx \rightarrow sx \preccurlyeq xx) \rightarrow \forall xx \preccurlyeq xx)$$

Henkin semantics

Florio & Linnebo (2015) have recently explored the option of giving plural logic a Henkin semantics. As one would expect, this way plural logic is sound and complete (and hence compact and axiomatisable). For this they simply need to specify a domain, which they call **D**, for pluralities to range over. (It is convenient for us, because **D** can be understood as a superplural.) Additionally, (Oliver & Smiley, 2013, §12) also have a mid-plural logic that is complete. It does not include plural quantification, and plural variables occur free.

Since higher-order logics can have Henkin semantics in general, it is save to assume that superplural logic can have such a semantics as well, thereby making it sound and complete. However, we won't adopt it for its limitation on expressive power—higher-order logics with Henkin semantics can be interpreted as many-sorted first-order logics (Shapiro, 1991, Theorem 3.5).

3.4 Plural Cantor

We already mentioned that in superplural logic predicate letters will refer to properties and relations rather than sets or (super)pluralities of individuals (3.2.2). This is counterintuitive for people familiar with a Boolos-style plural logic. For Boolos argues that a second-order variable X (which should denote a first-order predicate), should not be assigned a Fregean concept or a set as its value, but rather *many things*, just like plural terms. This is why one might think that predicates should be analysed in terms of pluralities, e.g. the predicate *being an elephant* is just the plurality of all elephants. And the good news is: with superplural logic we could also hope to give a semantics for *plural* predicates. One might think, for example, that the predicate *being roommates* is the superplurality of all pluralities of individuals that are living together. There is no limit to our imagination: third-order logic ranges over second-order predicates, so we need supersuperplurals for its semantics and so forth. This is essentially what Rayo (2006) proposes.

It also seems natural to include plurals and superplurals in the semantics of plural logic in general; after all, we want to stay ontologically innocent. This is why it seems strange that our project is not so much concerned with superplurals in the semantics, but superplurals *per se*. The reason is that having plurals and superplurals be the semantic values of predicates runs into deep trouble, as shown by Florio (2010, 2014).

Following Florio, we will call a logic that assigns objects (e.g. sets, pluralities) to predicates *untyped*, and by contrast a logic that assigns entities other than objects (e.g. Fregean concepts) to predicates *typed*, indicating a type distinction in the semantics. Florio offers two arguments against untyped plural logic, though we will only cite one in the interest of space.

In what follows, we want to show that some intuitive interpretations in untyped plural logic make it incompatible with a plural version of Cantor's theorem. To see this, we first show that untyped plural logic is committed to the existence of a one-to-one mapping from the pluralities to the plural properties. We adopt the proofs from Oliver & Smiley (2013, pp. 280) and Florio (2014, §5).

Take a plural predication in **PFO+**, Pxx. For any plurality in the (all-inclusive) domain, it is possible to interpret P as applying to that plurality and no other. That is, for any things xx, there should be an interpretation with an all-inclusive domain such that

$$Pxx \land \forall yy(xx \neq yy \rightarrow \neg Pyy) \tag{3.1}$$

Now add to this the account of predication in untyped plural logic. It means that for any plurality, there is an (untyped) property instantiated by xx and by no other plurality. Florio writes

$$\forall xx \exists \alpha \forall yy(\alpha(yy) \leftrightarrow yy = xx) \tag{3.2}$$

From this we get the result that there is a one-to-one mapping from the pluralities to the (plural) properties. Given plural comprehension we can even say that there are some things that map onto every plurality.

Next we want to present a plural version of Cantor's theorem. The basic idea should not come as a surprise: *there are more pluralities than objects* (even though, of course, pluralities aren't things to be counted the same way objects are). More precisely, we can say

Theorem (Plural Cantor). *For any things ss, if ss is strictly plural (i.e.* $\exists xx \prec ss$*), then there is no (possibly multivalued) function* f *such that*

$$\forall x((x \preccurlyeq ss \to f(x) \preccurlyeq ss) \land \forall xx(xx \preccurlyeq ss \to \exists y(y \preccurlyeq ss \land f(y) = x)))$$

Proof by reductio. Assume that there are some things ss, that they are more than one, and that there is such a function. Note that, were we to start with the usual 'Let zz be $x : x \preccurlyeq ss \land x \not\preccurlyeq f(x)$', we could not infer that $f(y) \equiv zz$ for some $y \preccurlyeq ss$, since zz might be empty (and hence there is no such y). Therefore, we need to distinguish three cases: the first is where zz does denote, the other two cover the two ways for zz to be empty.

(i) $\exists x(x \preccurlyeq ss \land x \not\preccurlyeq f(x))$. Now let zz be $x : x \preccurlyeq ss \land x \not\preccurlyeq f(x)$. Then $zz \preccurlyeq ss$, so $f(y) = zz$ for some $y \preccurlyeq ss$, since by hypothesis zz is one or many things. Hence $\forall x(x \preccurlyeq f(y) \leftrightarrow x \preccurlyeq zz)$. And so $\forall x(x \preccurlyeq f(y) \leftrightarrow x \preccurlyeq ss \land x \not\preccurlyeq f(x))$. In particular, we have that $y \preccurlyeq f(y) \leftrightarrow y \preccurlyeq ss \land y \not\preccurlyeq f(y))$. Since $y \preccurlyeq ss$, it follows that $y \preccurlyeq f(y) \leftrightarrow y \not\preccurlyeq f(y)$, which is absurd.

(ii) $\forall x(x \preccurlyeq ss \to x \preccurlyeq f(x))$ and yet $y \neq f(y)$ for some $y \preccurlyeq ss$. Then $f(w) = y$ for some $w \preccurlyeq ss$. Since $\forall x(x \preccurlyeq ss \to x \preccurlyeq f(x))$, then in particular $w \preccurlyeq f(w)$, whence $w \preccurlyeq y$. But $f(w) = y$ and $y \neq f(y)$ entail $w \neq y$. So we get $w \preccurlyeq y$ and $w \neq y$, which is absurd.

(iii) $\forall x(x \preccurlyeq ss \to x = f(x))$. Since $ss \preccurlyeq ss$, $f(y) = ss$ for some $y \preccurlyeq ss$. Since $\forall x(x \preccurlyeq ss \to x = f(x))$, then in particular, $y = f(y) = ss$, which is absurd, since by hypothesis there is more than one thing in ss.

□

(It is worth remarking that the proof requires the assumption that there are at least two things in the domain ss, though our semantics for higher-level plural logic does not assume the existence of anything.)

This contradicts our result (3.2) above. If we map the property corresponding to each plurality to that plurality, then clearly pluralities and properties have the same 'cardinality'. And yet they don't. To see this, consider the open formula

$$\varphi(f) \text{ iff } \exists \alpha \exists z(f(\alpha) = z \land \exists zz(z \preccurlyeq zz \land \alpha(zz))) \tag{3.3}$$

It asserts that f is a function whose output z is among some things zz which jointly instantiate its argument, α. Clearly, there is such a function: take two individuals a and b and let α be a plural property that they jointly instantiate. Then we have $f(\alpha) = a$ and so $\varphi(f)$ is instantiated. Next, we apply plural comprehension to it and obtain the plurality of objects satisfying the formula

$$\exists xx \forall y(y \preccurlyeq xx \leftrightarrow \varphi(y)) \tag{3.4}$$

Call the things picked out by (3.4) aa. We can now show that aa code every plurality.

Let yy be any plurality. It follows from (3.2) that there is a property α dependent on yy such that, for every xx, $\alpha(xx)$ if and only if $xx = yy$. By (3.3), for every y, $f(\alpha) = y \preccurlyeq aa$ just in case $y \preccurlyeq yy$. In other words, aa code the plurality yy, and α is their code. Since we chose yy arbitrarily, it follows that aa code every plurality. Contradiction!

Thus, we conclude, with Florio (2014), that predicates in a plural logic have to denote things other than objects.

What does this mean for (super)plural logic? Simply that, on pain of paradox, we have to regard predicates as entities of a different type. This means that we will need a 'genuine' second-order logic that ranges over properties and relations. We will return to this issue in 4.2.1.

3.4.1 Higher-level predicative analysis

Florio (2010, §4), in discussing the semantics of plural logic, not only considers typed and untyped plural logic, but also what he calls *superpluralism*. This is the view that superpluralities are the semantic values of plural predicates, which we will call higher-level predicate analysis (to mirror Boolos's project). Florio's arguments against superplurals are familiar by now, but it is worth summarising the main points:

(I) It is debatable whether English contains superplural terms.

(II) Even if natural language contained superplural terms, it is unclear whether they can be understood as referring to superpluralities.

(III) As a formal system, superplural logic does not have any advantages over other options, because (i) it has an unattractive treatment of empty terms, and (ii) there is no convincing argument for their ontological innocence.

We have so far rejected all points except for III. III.ii is addressed in the next chapter, and we turn to III.i now. The first thing to note is that we do not have empty (super)pluralities and higher-level singletons, which complicates the matter. For example, we could not have predicates such as *forming a round square* or *being the best predicate in the world*. What's worse, it would imply that, for any plural predicate P, $\exists xxP(xx)$ is a logical truth (Florio, 2010, §4.5.1). And, as one would expect, the fact that some predicates can fail to denote leads to an array of problems.

Theoretically, we could come up with reference rules for plural predicates similar to plural terms, e.g. a plural predicate (of level n) refers to a plurality (of level $n+1$ or lower) or whatever the property-equivalent of zilch is. (Florio says as much: 'the superpluralist could use a dummy object to serve (conventionally) as the denotation of empty predicates and would have to revise the model theory accordingly' (p. 151).) But this would be missing the point. The real issue is that such a semantics goes against the idea of plural logic, which is to provide a more natural semantics for plurals. To say that the term 'all elephants' and the predicate 'being an elephant' refer to the same plurality seems odd, to say the least. But this is exactly what Rayo (2006) suggests:

> Rather than taking '. . . is an elephant' to stand for the set of elephants, I would like to suggest that one should take it to stand for the elephants themselves. It is grammatically infelicitous to say that the semantic value of '. . . is an elephant' is the elephants. So I shall say state the view by saying that '. . . is an elephant' *refers* to the elephants (p. 225).

In my view, '. . . is an elephant' referring to the elephants is just as infelicitous. But I will not dwell on this point, for more importantly, higher-level predicative analysis faces the same problem untyped plural logic does: a plural version of Cantor's theorem.

There are two ways to get this result. We can run the above argument at a higher level (i.e. by first showing that there is a one-to-one correspondence between plural predicates and superpluralities, and then by proving that there are more superpluralities than objects, speaking informally). The more direct way is to observe that a higher-level predicative analysis extends the ordinary predicative analysis. That is, Rayo already runs into Cantor's theorem by assigning objects to the predicate *is an elephant*. The exact same argument applies here.

Chapter 4

Philosophy

> ...if not completely anodyne, then mostly harmless.
>
> Peter Simons (2011, p. 18)

The putative ontological innocence of plural logic is important in motivating superplural logic. As mentioned in the introduction, plural logic's first and foremost application still lies with Boolos's interpretation of monadic second-order logic. If superplural logic is ontologically kosher as well, then it seems that we have at our hands a good alternative semantics for higher-order languages in general, as given by Linnebo & Rayo (2012, appendix A).

The notion of ontological commitment belongs first and foremost to Quine's approach to ontology (2013, §48-§50). According to him, ontological inquiry is a two-step procedure. It involves first determining what entities (scientific) theories are committed to, and then choosing the best theories to accept. The doctrine then is to admit, and only admit, entities to which those theories are committed. To determine exactly which entities these are, we have to look at the commitment of the *sentences* that constitute our best scientific theories. And this is where truth conditions come in. In the most basic understanding, truth conditions tell us how the world must be in order for a sentence to be true. They put demands on the world which may include that certain entities exist. In short, sentences are committed to entities that must exist for them to be true.

We will assume that truth conditions are a good guide to ontology, and by that we mean truth conditions are necessary but perhaps not sufficient in accounting for what entities sentences are committed to. For example, one might think, as Rayo (2007) does, that correctness conditions are also relevant for ontology. Correctness conditions would come apart with truth conditions in cases like fictional sentences (which would be correct but not true). However, for the sake of focus, we will disregard such additional constraints.

Even so, there are numerous accounts of truth conditions in the literature (and for that matter, theories of truth), and so a precise formulation of Quine's claim will have to be more involved. Entire books have been written on the topic, and no attempt will be made here to settle the issue. Instead, we will focus on finding the correct *criterion for ontological* *commitment*, i.e. a way of extracting the commitments of a sentence given that it is true. The claim we wish to asses is a conditional: if plural logic is ontologically innocent according to some criterion, then, according to the same criterion, superplural logic is also ontologically innocent. The reason, again, is focus: the innocence of plural logic is not yet settled ((Resnik, 1988; Parsons, 1990; Hazen, 1993; Shapiro, 1993; De Rouilhan, 2002; Linnebo, 2003)), and for our purposes it is obvious that the critic who has qualms with the innocence of plural logic will not accept superplural logic either. Hence, we will need a criterion that makes plural logic come out innocent to begin with.

Again, many options present themselves as candidate for such a criterion. Theories involving truthmakers, possible worlds, or entailment relations in general are all popular in contemporary metaphysics. Quine's own approach is often dubbed 'quantifier accounts' for its emphasis on quantification. Crucially, plural logic only has a possible advantage over singular logic with regard to quantifier accounts. This is due to the fact that only quantifier accounts are sensitive to the language of regimentation. For the other accounts it generally doesn't matter what the quantifier ranges over, or how it ranges over them. Thus, our discussion on ontological commitment will remain in the Quinean orthodoxy.

The goal of this chapter is to evaluate the claim that superplural logic is ontologically innocent. First, we discuss and present an appropriate Quinean criterion for ontological commitment. Then we assess the superplural hierarchy under that criterion. Finally, we sketch the application of superplurals in meta-theories.

4.1 Quinean ontology

Quine's criterion was famously formulated in 'On What There Is' 1948, in which he argues that ontology is simply a matter of understanding the ontological commitments of our sentences. The criterion reads as follows:

> A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true (p. 33).

The prerequisite for this criterion is that the theory must be formulated in a language that makes reference more explicit. In particular, Quine's language of regimentation is a first-order predicate language. For instance,

(31) My unicorn is hungry.

translates into

(32) $\exists x \exists y (Unicorn(x) \wedge Me(y) \wedge BelongsTo(x, y) \wedge Hungry(x))$

The bound variable x ranges over unicorns (and y ranges over myself). So unicorns have to exist, among other things, for the sentence to be true. Bricker (2014) offers a more precise 'original' formulation:

Quine's Criterion A first-order sentence carries commitment to Fs just in case Fs must be counted amongst the values of the variables in order for the sentence to be true.

The criterion, as it stands now, faces a variety of criticisms. The obvious one is that, from a logical point of view, it seems arbitrary to only consider the terms and not the predicates in a sentence. To interpret Fa as a predicate applying to an object is on a par with the interpretation 'there is a property F under which the constant a falls'. This begs the question of Quinean nominalism. Then there are questions about the adequacy of the criterion. Is it sufficient, that is, in this case, does it overgenerate? Relying on quantifiers only can lead to unwanted guests in one's ontology. For example, quantifiers do not distinguish fundamental objects from derived ones, if indeed there is such a distinction to be made. If a chair should be reduced to, say, 'particles arranged chair-wise', then 'there is a chair' introduces illegitimate objects (namely: chairs). Is the criterion necessary? Perhaps not, if we look at predicates that express extrinsic properties. For example, *is a moon* applied to an object requires the existence of a further object: a planet for that object to orbit around (Rayo, 2007, p. 7). Yet extrinsic properties do not find their way into Quinean ontology. And these are only one type of 'implicit' ontological commitments. Others are part-whole related: being a whale is part of being a mammal, for example; having a car also means having wheels.

If the criterion is arbitrary and neither necessary nor sufficient, then indeed the project of assessing ontological commitment via quantifier accounts seems rather pointless. But that is not the last word on Quine's project. Here are two reasons to pursue quantifier accounts. First, Quine's criterion is part of a larger picture of ontology. In this picture, scientific theories are the arbiters of existence, and natural language has very little to do with science. Take the argument from implicit commitments. Presumably, a theory that ranges over Saturn's moons will also range over Saturn. All commitments would thus be captured by the theory as a whole, without having to consider individual sentences. Likewise, the argument from illegitimate objects loses its force if the Quinean welcomes the lack of tables and chairs in their ontology. The second reason for holding onto the Quinean account is its vast influence on other accounts of commitment in natural language. Not only is it still being developed (Michael, 2008; Peacock, 2011; Gill, 2012), many of the other accounts rely heavily on it. For example, Schaffer (2008) proposes a hybrid account of commitment (using quantifiers and truthmakers), while Raley (2007) fuses Azzouni's existence predicate with the Quinean framework.

This is not to say that the criterion is flawless—these are serious worries, but for now not our concern. What does concern us is when plural logic comes into play. As we have outlined in the introduction, Rayo (2002) argues that a first-order language cannot quantify over absolutely everything. For a criterion that is supposed to determine *what there is*, this objection is detrimental. If we take seriously the idea that the regimenting language should be a guide to ontology, then we must work within a plural first-order language for regimentation. Specifically, for our one-sorted **PFO+** language, we can have a uniform criterion for ontological commitment:

Rayo's Criterion A theory couched in a **PFO+** language is committed to the existence of objects satisfying a plural predicate if and only if some objects satisfying that predicate must be admitted as the possible values of one of the theory's plural variables in order for the theory to be true (Rayo, 2002, p. 453).

4.1.1 Plethological commitment

The first thing to observe here is that the original ontological criterion is formulated explicitly in terms of *first-order* predicate logic. The critique that Quine eventually voices, that second-order logic is *set theory in sheep's clothing* (Quine, 1986, §5), relies exactly on this point. Second-order logic quantifies over predicates, which are standardly interpreted as sets of the objects they apply to. Thus, second-order quantification commits one not just to the objects in the first-order domain, but additionally to the arbitrary subsets of these objects (assuming, of course, that we use standard semantics). For a nominalist, to posit sets when they are not part of the proper subject matter of a theory is bad, and to hide this commitment in the semantics is even worse.

Against this, Boolos famously argued that monadic second-order logic, interpreted plurally, carries no ontological commitment beyond those already carried by the first-order quantifiers. This is due to the fact that singular and plural quantifiers range over the same domain. Since Quine's criterion only consider quantifiers and their domain, we can safely say that plural logic invokes no additional ontological entities. Or can we?

For Boolos, adjusting the criterion for plural quantification is easily done. The slogan quickly became: 'to be is to be the value of some variable or the values of some variables' (Boolos, 1984). Crucially, the semantic value of a plural variable is a plurality of entities, not a plural entity. However, Rayo has anticipated a problem that was neglected by Boolos. As with plural predicates, denotation itself can be both collective and distributive, depending on the context (Oliver & Smiley, 2013, §6). As Rayo (2007, p. 435—7) points out, this leads to two distinct account of ontological commitment. Suppose that P is a predicate that applies to plural terms. If P is distributive, then a sentence like 'there are some Ps' introduces no entities beyond the things that a individually P. However, if P is collective, then 'there are some Ps' is committed to Ps in a distinctly plural way. This sort of commitment cannot be captured by Quine's criterion. Even though no more entities arise, we have a new *way* of quantifying over the same entities. Rayo calls the latter *plethological commitment*, where plethology concerns 'the realm of pluralities' (p. 454), and later revises the criterion:

Plethological Commitment A singular or plural first-order sentence carries commitment to Fs just in case Fs must be counted amongst the values of the (singular or plural) variables in order for the sentence to be true (Rayo, 2007, p. 14).

This modification of the original Quinean criterion is most welcome. While Boolos is right that plural logic is innocent with regard to ontological entities, he disregards the important fact that plural logic smuggles in the pluralities through another kind of quantification. This change needs to be accounted for in the Quinean criterion.

4.2 The innocence of higher-level plurals

This will be the criterion we use in the remainder of this chapter. Accordingly, we will define ontological innocence of a theory as carrying no ontological commitment beyond those already carried by the (singular or plural) first-order quantifiers.

Is superplural logic ontologically innocent, according to our revised criterion? The simple answer is *no*. So far, it is clear that superplural logic, while not being committed to additional set-like entities, is committed to indefinitely-many *ways* of quantifying over individuals. Plural quantification is different from singular quantification, and superplural quantification is different from both. This goes on. Thus we need to modify plethological commitment to not only include plural denotation, but also n-level superplural denotation. My proposal is therefore:

Higher-level Plethological Commitment An n-level first-order sentence carries commitment to Fs just in case Fs must be counted amongst the values of the n-level variables in order for the sentence to be true.

The principle clearly makes superplural logic ontologically kosher, since it does not range over more than what n-level variables can range over. But what is the justification for such a principle? The argument is familiar by now: if plethological commitment is acceptable, then higher-level plethological commitment is, too. The reason for Rayo to adopt a plethological account suffices for us to adopt a higher-level plethological account due to the intelligibility of iteration.

For Rayo, the new criterion can be independently motivated because sentences in natural language incur plethological commitment. Thus, **PFO+** is innocent qua capturing the subject matter of such sentences. We can say the same for higher-level plethological commitment: since superplurals do occur in natural language, it is only correct that a criterion for ontological commitment should consider higher-level plethology.

The argument here is delicate. We cannot, as responsible ontologists, wantonly posit entities to which sentences are committed, and then adjust our criterion accordingly. But if chapter 2 tried to show anything, it is that superplural quantification is a real linguistic phenomenon, and should be taken seriously. If our baseball example in (28) is correct, then supersuperplural quantification is legitimate as well. And in order to both account for the linguistic phenomena and avoid additional ontological baggage, plethological commitment seems to be an appropriate response.

One may wonder, of course, if second- and third-level quantification suffices to justify an entire hierarchy. This is not new. The same problem occurs in higher-*order* logic. Third-order logic is hardly ever used, and one could even argue that mathematics may not require more than fifth-order logic (Väänänen, in press, §7). We do not have to settle the question here. We will rest assured with the knowledge that if plural terms of level 7 should occur in natural or formal languages, then they add to our commitments in plethology only. We also do not want to rule out potentially interesting applications for the superplural hierarchy in philosophy.

4.2.1 The innocence of higher-level plurals, Take 2

Our discussion does not stop here, however. There have been at least two recent suggestions to widen our notion of ontological commitment. The first is the idea, raised by Pedersen & Rossberg (2010), that commitment *within a theory* is not enough to capture all required entities for the truth of some theory. What is missing is the ontological commitment of the meta-theory in which the original theory is couched. This is a real problem for the semantics of higher-level languages.

Let us take a closer look at our metalanguage. We used val as a general function that assigns objects to terms, properties and relations to predicates, and functions to functions signs. So our metalanguage clearly outstrips the object language. We can make the issue here more pressing. In 3.4.1, we have seen that the plural version of Cantor's theorem forces us to have a typed plural logic, where predicates are taken to refer to properties and relations of a different ontological category. Issues of whether this Fregean view is attractive or not aside, it is certainly unusual that plural logic should entail such a metaphysically significant view.

Nevertheless, it's not all bad news: these considerations only give us genuine second-order logic, and nothing else. The fact that higher-level plural logic has no additional ontological commitments beyond first-level plural logic means that we do not get a semantic ascent with regard to higher-*order* logic in the metalanguage. That is, even though plural logic requires second-order logic to some extent, superplural logic does not need third-order logic; the second order suffices here.

The second suggestion for a different notion of ontological commitment is an invitation for more creativity Florio & Linnebo (2015).

> According to this notion, ontological commitment is tied to the presence of existential quantifiers of *any logical category* in a sentence's truth conditions. If this notion is operative, then even the plurality-based semantics shows that plural locutions incur additional ontological commitments (p. 11).

On this view, merely considering the objects of a theory is not enough. Even if objects alone settle questions about the commitments of pluralities and superpluralities, there are still genuine theoretical issues that are interesting. For example, we might want to ask how many, and what kind of, pluralities a theory is committed to. For example, 'even if one believes in the metaphysical determinacy of plural quantification, one may have views about how strong, or mathematically rich, one's notion of subplurality is' (p. 13). (Cf. (Shapiro, 1993) and (Parsons, 2013) here.) Whether this needs to be part of a wider ontological criterion is, of course, a pragmatic matter. As Florio and Linnebo point out, this wider notion of ontological commitment may well be called *ideological* commitment instead.

The proposal is interesting and relevant. Evidently, superplural logic as developed here is not the only mathematically possibility for superplural phenomena. We have already highlighted some decisions we made in 3.1. It is instructive and illuminating, therefore, to compare and contrast different superplural hierarchies, as we have tried to do in 2.1.4. We now offer a more complete comparison between the existing formal accounts of superplurals.

4.2.2 Comparison

Oliver and Smiley

Superplural logic follows by large the syntax and semantics used for their full plural logic. We have integrated their two operators and pre-defined expressions. However, there is one major difference: their proposed superplural hierarchy is not cumulative. To quote,

> It would be natural to take ... that $a \preccurlyeq^2 b$ is true just when a exist(s) and $\forall x(x \preccurlyeq^2 a \rightarrow x \preccurlyeq^2 b)$, where a and b may be singular or first- or second-level plural terms. For first-level inclusion, we allowed for $a \preccurlyeq b$ where a and b are both singular terms. In such a case, $a \preccurlyeq b$ was stipulated to be true just when a and b denote the same single thing. Similarly, for second-level inclusion, we allow $a \preccurlyeq^2 b$ even where a and b are confined to singular or first-level plural terms. The formula $a \preccurlyeq^2 b$ will then be true just when a and b denote the same thing(s). Thus at the first level we have Gilbert $\preccurlyeq$ Gilbert and Sullivan, as well as Gilbert and Sullivan $\preccurlyeq$ Gilbert and Sullivan. At the second level we have Gilbert and Sullivan $\preccurlyeq^2$ Gilbert and Sullivan, Mozart and Da Ponte, and Verdi and Boito, as well as Gilbert and Sullivan $\preccurlyeq^2$ Gilbert and Sullivan, but we do not have Gilbert $\preccurlyeq^2$ Gilbert and Sullivan.

This conclusion is curious, for Oliver and Smiley take plural terms to both refer to single individuals and multiple individuals. It is therefore possible and desirable that we have 'Gilbert $\preccurlyeq$ Gilbert'. Since we have allowed a term of level n to refer to all lower levels (and I see no reason why Oliver and Smiley would not agree), it is odd that they would restrict the inclusion predicate in that way. Our way incorporates cumulativity at the cost of a higher-level collapse (or stacking, depending on one's perspective). That is, 'Gilbert $\preccurlyeq$ Gilbert and Sullivan' is equivalent to 'Gilbert $\preccurlyeq^{5137}$ Gilbert and Sullivan'. And this can be avoided if we take all inclusion predicates to be the same one, with the only restriction that the term in the second place be of a higher level than the first.

Rayo

Our deductive system, on the other hand, resembles Rayo's above all. However, we have simplified notation dramatically. And, as discussed in the previous section, superplural logic does not have a higher-level predicative analysis—that is, plural predicates denote superpluralities. Rayo himself notes that his account can also be construed as a higher-*order* logic, further reflecting the similarities to Boolos's project:

> In this paper I give no reason for favoring a hierarchy of higher and higher level predicates over a hierarchy of higher and higher order predicates. I have chosen to focus on the former because it seems to me that second-level predicates deliver a more natural regimentation of English predicates with collective readings than their second-order counterparts. But either hierarchy will do, as far as the purposes of this paper are concerned (p. 233).

Florio

Florio simplifies Rayo's system by only looking at superplurals (and not plurals of level n). Like Rayo, he adopts superpluralism (which we call higher-level predicative analysis). Ultimately, he rejects the move upwards because arguments in favour of ontological innocence rest on the assumption that plural logic is innocent to begin with. We have avoided the discussion as our main concern is the sceptic who endorses plural logic but does find superplural logic convincing. In this sense, we agree that superplural logic is of little help when arguing for the ontological innocence of plural logic (Florio, 2010, p. 155).

Simons

In his most recent work, Simons develops a logic of multitudes that is 'nominalistically acceptable' (in press, p. 20). These are the axioms Simons uses (in our notation):

Identity $a = b \leftrightarrow \forall \phi(\phi(a) \leftrightarrow \phi(b))$

Extensionality $\forall c(c \preccurlyeq a \leftrightarrow c \preccurlyeq b) \rightarrow a = b$

Existence $a \preccurlyeq b \rightarrow \exists c(c \preccurlyeq a)$

Antisymmetry $(a \preccurlyeq b \land b \preccurlyeq a) \rightarrow a = b$

Individuality $a \preccurlyeq a \rightarrow (b \preccurlyeq a \rightarrow a \preccurlyeq b)$

Supplementation $(a \preccurlyeq b \land b \neq a) \rightarrow \exists c(c \preccurlyeq b \land c \neq a)$

Regularity $\exists b \exists c(b \preccurlyeq a \land c \preccurlyeq a \land b \neq c) \rightarrow \exists b(b \preccurlyeq a \land \neg \exists c(c \preccurlyeq b \land c \preccurlyeq a \land c \neq b))$

Pair $\exists a(\forall b(b \preccurlyeq a \leftrightarrow Tc \land Td \land c \neq d \land (b = c \lor b = d)))$

Adjoint $\exists a(\forall b(b \preccurlyeq a \leftrightarrow Tb \land Td \land Ed \land \neg c \preccurlyeq d \land (b \preccurlyeq d \lor b = c)))$

Union $\exists d(\forall b(b \preccurlyeq d \leftrightarrow \exists c(c \preccurlyeq a \land b \preccurlyeq c)))$

Simons thinks that the axioms are intuitive and part of 'universal logic', and do not resemble set theory due to the lack of higher-level zilches and singletons. However, 'the relative paucity of stable intuitions about higher-order multitudes means that the correct principles for such a logic are presently not clear' (p. 17).

There are some important differences to highlight here. Simons considers 'multitudes' to be part of ontology, while we take 'pluralities' to be linguistic devices that help us talk about many things at once. Talk of 'union' and 'pair' aside, it is this metaphysical claim that will ultimately determine how similar multitude logic is to set theory. Simons argues for higher-level multitudes by considering certain geometric shapes that can be superplurally referred to, and indefinitely extends them to create a hierarchy (p. 3). While the example is artificial, it suggests that multitudes are worldly and not merely linguistic.

It seems, then, that multitude logic resembles superplural logic in some formal aspects (notably **Identity**, **Extensionality**, and **Antisymmetry**), but has a different conception of its subject matter.

4.3 Application in meta-theories

We have considered superplural logic as a default logic for regimenting superplural phenomena in natural language. But there are numerous places where a superplural logic can also be useful. As mentioned in 1.3, its main attraction is to figure in as semantics for higher-level (or higher-order) languages. This is due to the fact that traditional semantics is done in model-theoretic terms, which has been criticised for its heavy use of sets. Entities in the meta-theory quickly become problematic when we quantify over interpretations. This happens, for example, when characterising logical consequence, or when formulating the Löwenheim-Skolem-Theorems.

Plural logic gets rid of such entities. Rather than using a set of ordered pairs to capture an interpretation, we use the ordered pairs themselves. In particular, some ordered pairs (the *mm*) form a 'plural model' just in case at least one of them is of the form $\langle `\forall', x\rangle$. This makes a standard definition possible: an object $x$ can be said to belong to the domain of the *mm* just in case $\langle `\forall', x\rangle$ is one of the *mm*, and a (monadic) predicate P can be said to apply to x according to the *mm* just in case $\langle `\forall', x\rangle$ is one of the *mm*. Therefore, quantification over interpretations is simply plural quantification over ordered pairs.

Another advantage of pluralities is that they can consist of things that are too numerous to form a set. This means that we have the expressive power to capture an interpretation whose domain consists of too many individuals to form a set. Moreover, assuming a full semantics, for any interpretation of a first-order language in which the domain consists entirely of existing objects, there are some things that form a plural model capturing that interpretation.

Beyond what plural logic already accomplishes, superplural logic has the resources to figure into the meta-theory of higher-order languages. Rayo (2006) has done so in the general case, but we also find specific instances where it comes in handy. One such example is an alternative semantics for a liberalised type theory, which can be found in Linnebo & Rayo (2012, appendix A).

Another thing gained with superplural logic is that, were we to continue Boolos's project and endorse a predicative analysis, we could not only render third-order logic in second-level plural logic, but also *polyadic* second-order logic within third-level plural logic. The idea is to use a more complex supersuperplurality to denote what we normally call a set of ordered pairs, as Rayo (2006, §9.3.2) proposed. (An ordered pair $\langle a, b\rangle$ can be formalised as the superplurality consisting of the singleton plurality $[a]$ and the plurality $[a, b]$.) The point of this exercise is unclear, however. Not only are singleton pluralities disallowed in superplural logic; the pair construction makes it seem like a notational variant of set theory. Since we reject the predicative analysis, we will leave the project to those who see a value in formalising higher-order polyadic logic within a superplural hierarchy.

Chapter 5

Conclusion

> dipping one's toes in the murky waters of the superplural
>
> Alex Oliver & Timothy Smiley (2013, p. 127)

It is time to take stock. I introduced the notion of superplural quantification by first focusing on the essence of plural logic: its natural semantics, expressive power, and proclaimed ontological innocence. Then I tried to extract a more detailed picture by considering the arguments against superplurals in the literature. Two main objections are formulated, which I called **intelligibility** and **naturalness**. I argued that superplural logic is not only intelligible, it is also a better conception of the plural hierarchy. This required a notion of structural similarity. I then showed that superplurals do occur in English, and cannot be paraphrased away. Moreover, Icelandic and Finnish are languages that contain superplural quantification.

Chapter 3 began with a review of existing notation. I then made some executive decisions about the superplural hierarchy: it is cumulative and open-ended, and it has multigrade predicates. Next, I presented a unified account of superplurals that roughly follows Oliver and Smiley's full plural logic. I then showed incompleteness of superplural logic, and gave reasons for rejecting a predicative analysis based on the plural version of Cantor's theorem.

Chapter 4 evaluated the contentious claim that superplural logic is ontologically innocent. I first presented the context of ontological commitment in general, and then went through a number of suggestions for a concrete criterion. Once the idea of plethological commitment was in place, the work was laid out. I concluded that superplural logic is innocent on the assumption that plural logic is innocent. Finally, I discussed the importance of ontological innocence, and sketched the most important application of superplural logic as higher-order semantics.

Should we be satisfied? No. Firstly, our claim of ontological innocence presupposes a lot—a Quinean approach in general, truth conditions, and quantifier accounts as the measure for commitment. We also haven't discussed whether Quine would accept plethological commitment at all. But mostly we need to make an important observation. This thesis has shown that there is a way of doing higher-level reference without introducing ontologically troublesome entities. However, putting things in a different language doesn't mean that we are getting rid of the structures. Instead, we have more complex, n-level quantification, which escapes most people's imagination (latest at level 17). We might even say, following Linnebo and Florio, that this is the trade-off between ontology and ideology. We have kept our ontology sleek in exchange for an inflated ideology, full of plethological commitments of the nth level. In concluding, then, we might say that superplural logic offers natural semantics and expressive power for the ontologically consciousness. But it does not come without any ideology.

5.1 Open questions

This thesis was born out of a postscript, and so it is only appropriate that we briefly talk about the things that did not make it into this thesis.

Determinacy. In the literature on plural logic, one can find a side debate on whether plural denotation itself is collective or distributive (Higginbotham, 1998; Hossack, 2000; Rumfitt, 2005; Sainsbury, 2005; McKay, 2006). Oliver & Smiley (2013, §6) have argued that plural denotation can be both, as it produces the correct truth conditions. The resulting account, however, makes denotation indeterminate[1]: 'Martha, Connee, and Helvetia' can denote the three of them together, or any things among them. Now, we may ask whether indeterminacy applies to superplural logic. In 2.1.2, we said that it takes many alternating levels of collective and distributive denotation for one sentence not to collapse into another. This calls for further investigation on the (in)determinacy of higher-level plural denotation.

Co-reference. We encountered the problem of co-reference earlier. In 2.1.2, we defined it as

Co-reference Two plural terms of level n are co-referring iff all its pluralities of level $0 \ldots n-1$ are co-referring, respectively.

However, Florio raised the question of repeating terms, and we did not come to a conclusion besides that repetition can be 'non-trivial'. We observed that in order to account for repetition, we need a bijection between all pluralities of lower levels for two terms to co-refer. In 2.1.4, we saw that bisimilarity is a useful way to compare higher-level terms, but it forces us to drop the bijection. This means that we still need a precise formulation of co-reference.

To illustrate the implication, consider cases such as 'the Beach Boys' vs. 'the Beach Boys and the Pendletones'. The Beach Boys used to be called 'the Pendletones'. Someone who doesn't know this fact might truthfully say

(33) The Beach Boys and the Pendletones are my favourite bands.

If we drop bijection, then bisimilarity between the two corresponding graphs would make the terms co-referring. If we retain bijection, then the fact that the second plurality cannot be

[1] Note that this should not be conflated with determinacy as we have characterised pluralities in 2.1.

mapped onto anything means that co-reference fails. This dilemma generalises, and warrants a closer look at co-reference in general.

Type theory. In Linnebo & Rayo (2012), we have seen superplurals in the semantics of a liberalised type theory. Seeing that Russell entertained the idea of 'many manys', it may not be so ludicrous to compare and contrast the superplural hierarchy with simple type theory.

More applications. We have already considered using superplural logic in higher-order semantics. In fact, most of the applications proposed and/or developed in the literature are motivated by the claim of ontological innocence and formal similarities to set theory. For example, Hewitt (2012) uses superplural quantification for a serial logic that aims at capturing finite order. The goal, again, is to stay away from set theory.

Another example comes from mereology. Cotnoir (2013) has proposed, following Lewis (1991, p. 83), that we regard the part-whole relation as plural. That is, the slogan 'composition as identity' is understood as plural inclusion. In developing the semantics of his theory, Cotnoir opts for a set-theoretic approach but notes that hyperplurals (read: superplurals) would be an option for 'those with ontological qualms' (p. 301).

It is disappointing that superplural logic is not used more widely. Until now, it is mostly referred to as more innocent proxy for set theory. But it has potential. To take the above example: Cotnoir does not say whether superplurals could also figure in the syntax of his theory. However, if the part-whole relation is taken to be transitive, then using superplural quantification may produce some interesting results for parts of parts. But that is the topic of another thesis.

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